$K(1)$-LOCAL E_∞ RING SPECTRA

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1. Certain $K(1)$-local spectra

Let $\mathcal{C}$ be the topological model category of $K(1)$-local spectra. Some useful examples of objects of $\mathcal{C}$ are p-adic K-theory, K, the Adams "summand" B of K, and the sphere S. At the prime 2, the spectrum B is KO, and is not a summand of K.

The group $\mathbb{Z}_p^\times$ of p-adic units acts on K via the Adams operations. If λ is a p-adic unit we will denote $\psi_\lambda : K \to K$ the corresponding Adams operation. Let $\mu \subset \mathbb{Z}_p^\times$ be the maximal finite subgroup. When p is odd, μ is the group of $(p-1)^{\text{st}}$ roots of unity, and when p is 2 it is the group $\{\pm 1\}$. The spectrum B is the homotopy fixed point spectrum of the action of μ on K. The action of the Adams operations on K restricts to an action of $\mathbb{Z}_p^\times/\mu \approx \mathbb{Z}_p$ on B.

2. Homotopy groups of $K(1)$-local spectra

Let g be a topological generator of $\mathbb{Z}_p^\times/\mu$, and

$$\psi_g : B \to B$$

the corresponding map. For any object X of $\mathcal{C}$, there is a fibration

$$X \to B \wedge X \xrightarrow{(\psi_g - 1)\wedge 1} B \wedge X.$$

This makes it easy to compute the homotopy groups of X in terms of the homotopy groups of $B \wedge X$. Take for example X to be the sphere spectrum. The action of ψ_g on $\pi_0 B = \mathbb{Z}_p$ is trivial. This means that the element 1 in the rightmost $\pi_0 B$ comes around to give a non-trivial element

$$\zeta \in \pi_{-1} S^0.$$

We will see that ζ plays an important role in things to come. Though not mentioned in the notation, the element ζ depends on the choice of generator g.

Given an element $f \in \pi_0 K \wedge K$ and $\lambda \in \mathbb{Z}_p^\times$, let $f(\lambda)$ be the element of $\pi_0 K = \mathbb{Z}_p$ which is the image of f under the map induced by the composite

$$K \wedge K \xrightarrow{\psi_\lambda \wedge 1} K \wedge K \to K.$$

Thinking of λ as a variable, this defines a map

$$\pi_0 K \wedge K \to \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathbb{Z}_p).$$

Proposition 1. *The above map is an isomorphism. It gives rise to isomorphisms*

$$\begin{aligned} \pi_* K \wedge K &\approx \mathrm{Hom}_{cts}(\mathbb{Z}_p^\times, \pi_* K) \\ \pi_* K \wedge B &\approx \mathrm{Hom}_{cts}(\mathbb{Z}_p^\times/\mu, \pi_* K) \\ \pi_* B \wedge B &\approx \mathrm{Hom}_{cts}(\mathbb{Z}_p^\times/\mu, \pi_* B). \end{aligned}$$

□

With respect to the above isomorphism, the actions of $\psi_g \wedge 1$ and $1 \wedge \psi_g$ are given by

$$\begin{aligned} (\psi_q \wedge 1\, f)(\lambda) &= f(\lambda g) \\ (1 \wedge \psi_q\, f)(\lambda) &= \psi_q f(g^{-1}\lambda). \end{aligned}$$

Let

$$M_\zeta = S^0 \underset{\zeta}{\cup} e^0$$

be the mapping cone of ζ, and fix a generator g of $\mathbb{Z}_p^\times/\mu$. By definition, we have a diagram

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d, equal] & M_\zeta \arrow[r, "\delta"] \arrow[d, "\iota"'] & S^0 \arrow[d, "1"] \\ S^0 \arrow[r] & B \arrow[r, "\psi_g - 1"'] & B \end{tikzcd}$$

The maps

$$1 \circ \delta,\ \iota : M_\zeta \to B$$

form a basis of $\mathrm{ho}\,\mathcal{C}(M_\zeta, B)$. From the above diagram it follows that

$$\psi_g \iota = \iota + 1 \circ \delta.$$

This will be more useful when written in homology. Thus define

$$\{a, b\} \subset \pi_0 B \wedge M_\zeta$$

by

$$\begin{aligned} \langle \iota, a \rangle &= 1 & \langle \iota, b \rangle &= 0 \\ \langle 1 \circ \delta, a \rangle &= 0 & \langle 1 \circ \delta, b \rangle &= 1. \end{aligned}$$

One easily checks that

Lemma 2. *Under the map*

$$\pi_0 B \wedge M_\zeta \to \pi_0 B \wedge B \to \mathrm{Hom}_{cts}(\mathbb{Z}_p^\times/\mu, \mathbb{Z}_p)$$

the element a goes to the constant function 1, and the element b goes to the unique homomorphism sending g to 1. □

3. The category of E_∞ ring spectra

The topological model category of E_∞ ring spectra in $\mathcal{C}$ will be denoted $\mathcal{C}^{E_\infty}$. The spectra K have unique E_∞ structures, and the Adams operations act by E_∞ maps. This gives the spectrum B an E_∞ structure as well.

Let $B\Sigma_{p+}$ be the image in $\mathcal{C}$ of the unreduced suspension spectrum of the classifying space of Σ_p. There are two natural maps

$$B\Sigma_{p+} \to S^0.$$

One is derived from the map $\Sigma_p \to \{e\}$ and will be denoted ϵ. The other is the transfer map

$$B\Sigma_{p+} \to S^0$$

and will be denoted Tr.

Lemma 3. *The map*

$$B\Sigma_{p+} \xrightarrow{(\epsilon, \mathrm{Tr})} S^0 \times S^0$$

is a weak equivalence in $\mathcal{C}$. □

Define maps in $\operatorname{ho}\mathcal{C}$

$$\theta, \psi : S^0 \to B\Sigma_{p_+}$$

by requiring

$$\begin{aligned} \operatorname{Tr}(\theta) &= \tfrac{-1}{(p-1)!} & \operatorname{Tr}(\psi) &= 0 \\ \epsilon(\theta) &= 0 & \epsilon(\psi) &= 1. \end{aligned}$$

The map $B\{e\} \to B\Sigma_p$ gives rise to a map

$$e : S^0 \approx B\{e\}_+ \to B\Sigma_{p_+}.$$

It follows from the definition that

$$\epsilon \circ e = 1,$$

and from the double coset formula that

$$\operatorname{Tr} \circ e = p!.$$

It follows that

$$e = \psi - p\theta \tag{4}$$

Let $E \in \mathcal{C}^{E_\infty}$, and $x \in \pi_0 E$. The E_∞ structure associates to x a map

$$P(x) : B\Sigma_{p_+} \to X$$

with the property that

$$P(x) \circ e = x^p.$$

We define operations

$$\theta, \psi : \pi_0 E \to E$$

by

$$\theta(x) = P(x) \circ \theta \qquad \psi(x) = P(x) \circ \psi.$$

In view of (4) we have

$$\psi(x) - x^p = p\,\theta(x).$$

Thus the operation ψ is determined by θ. One easily checks that ψ is a ring homomorphism, and that θ does what it has to so that

the above equation will remain true:

$$\begin{aligned} \theta(x+y) &= \theta(x) + \theta(y) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i y^{p-i} \\ \theta(x\,y) &= \theta(x)\theta(y) + \theta(x)y^p + \theta(y)x^p. \end{aligned}$$

Since the Adams operations are E_∞ maps, they commute with the operations ψ and θ when acting on

$$\pi_0 K \wedge R.$$

4. Some algebra

We now work in the category of p-complete abelian groups, and we want to consider comutative algebras with operations θ and ψ as described above. Let's call them *Frobenius algebras* (even though this collides with another use of the phrase). I think the forgetful functor from Frobenius algebras to p-complete abelian has a left adjoint. I should also check that the category of Frobenius algebras has limits and colimits.

In any case, there is a free Frobenius algebra on one generator. The underlying ring is

$$\mathbb{Z}_p[x, x_1, x_2, \dots].$$

One defines θ by setting $\theta(x_i) = x_{i+1}$ ($\theta(x) = x_1$), and extending it to the whole ring by requiring that $p\theta(x) + x^p$ be a ring homomorphism. One needs to check that this works, and it does. We'll call the free Frobenius algebra on one generator x, $T\{x\}$. The following result plays an important role in everything we do.

Theorem 5 (McClure)**.** *For any $K(1)$-local E_∞ ring spectrum E, the natural map of Frobenius algebras*

$$E_* \otimes T\{x\} \to \pi_* E \wedge T\{S^0\}$$

is an isomorphisms.

There is another, perhaps more useful description of this free algebra. Let $\mathbb{W} = \mathbb{Z}_p[a_0, a_1, \dots]$ be the p-completion of the Witt Hopf algebra. The classical result says that if one defines elements $w_i \in \mathbb{W}$ by

$$w_n = a_0^{p^n} + pa_1^{p^{n-1}} + \cdots + p^n a_n,$$

then $\mathbb{W}$ has a unique Hopf algebra structure for which the w_i are primitive. One defines ψ by $\psi(w_i) = w_{i+1}$, and checks that this extends uniquely to a Frobenius structure on $\mathbb{W}$. This is the map which co-represents the classical Frobenius map.

Define a ring homomorphism

$$\mathbb{Z}_p[w_0, w_1, \dots] \to \mathbb{Z}_p[x_0, x_0, \dots]$$

by sending w_0 to $x = x_0$, and requiring that it be compatible with the map ψ.

Lemma 6. *The above map extends uniquely to an isomorphism of Frobenius algebras*

$$\mathbb{W} \to \mathbb{Z}_p[x_0, x_1, \dots].$$

□

The proof of the above lemma makes use of the following result of Dwork:

Lemma 7. *Let A be a ring with a ring homomorphism $\phi : A \to A$ satisfying*

$$\phi(a) \equiv a^p \mod p$$

(thus if A is torsion free, then A is a Frobenius algebra). Let $w_0, w_1, \dots$ be a sequence of elements of A. In order that the system of equations

$$\begin{aligned} a_0 &= w_0 \\ a_0^p + pa_1 &= w_1 \\ a_0^{p^2} + pa_1^p + p^2 a_2 &= w_2 \\ a_0^{p^n} + \dots + p^n a_n &= w_n \end{aligned}$$

have a solution, it is necessary and sufficient that for each n

$$w_n \equiv \phi(w_{n-1}) \mod p^n.$$

This gives the map. The isomorphism follows easily from the fact that, modulo indecomposables,

$$\begin{aligned} w_n &= p^n a_n \\ \psi^n x &= p^n x_n. \end{aligned}$$

This gives the free Frobenius algebra on one generator the structure of a Hopf algebra. We knew it had one anyway, since it was free.

5. Continuous functions on $\mathbb{Z}_p$

Another important example of a Frobenius algebra is the ring C of continuous functions

$$\mathbb{Z}_p \to \mathbb{Z}_p,$$

with $\psi(f) = f$. The point of this section is to describe a small projective resolution of C.

Each p-adic number λ can be written uniquely in the form

$$\lambda = \sum_{i \ge 0} \alpha_i p^i$$

with each $\alpha_i = \alpha_i(\lambda)$ equal to 0 or a $(p-1)^{\text{st}}$ root of unity. The α_i are continuous functions from $\mathbb{Z}_p$ to $\mathbb{Z}_p$, and satisfy

$$\alpha_i^p = \alpha_i.$$

For $i < n$, the α_i can be regarded as functions from $\mathbb{Z}/p^n \to \mathbb{Z}_p$.

Proposition 8. *For each m, n, the map*

$$\mathbb{Z}/p^n[\alpha_0, \alpha_1, \dots, \alpha_{m-1}]/(\alpha_i^p - \alpha_i) \to \operatorname{Hom}_{cts}(\mathbb{Z}/p^m, \mathbb{Z}/p^n)$$

is an isomorphism.

Corollary 9. *The map*

$$\mathbb{Z}_p[\alpha_0, \alpha_1, \dots]/(\alpha_i^p - \alpha_i) \to \operatorname{Hom}_{cts}(\mathbb{Z}_p, \mathbb{Z}_p)$$

is an isomorphism. □

Proof of Proposition 8: When $m = 1$ the injectivity follows from the linear independence of characters (of μ_{p-1}), and surjectivity follows from the fact that both groups are finite of the same order. For the general case, note that for finite (discrete) sets S, and T, the natural map

$$\operatorname{Hom}_{\text{cts}}(S, \mathbb{Z}/p^n) \otimes \operatorname{Hom}_{\text{cts}}(T, \mathbb{Z}/p^n) \to \operatorname{Hom}_{\text{cts}}(S \times T, \mathbb{Z}/p^n)$$

is an isomorphism. Apply this to the (set-theoretic) isomorphism

$$\mathbb{Z}/p^m \xrightarrow{\prod \alpha_i} \prod \left(\{0\} \cup \mu_{p-1}\right).$$

□

Lemma 10. *Under the map of Frobenius algebras*

$$\iota : \mathbb{W} \to \mathrm{Hom}_{cts}(\mathbb{Z}_p, \mathbb{Z}_p)$$

sending a_0 to the identity map, one has

$$\iota(a_i) \equiv \alpha_i \mod p.$$

Proof: This map sends each w_i to the identity function. To work out the values of $a_i \mod p$ we work by induction on i. Suppose we have proved the result for $i < n$. The inductive step is provided by the congruence

$$w_n = a_0^{p^n} + pa_1^{p^{n-1}} + \cdots + p^n a_n \mod p^{n+1}.$$

Since for $i < n$, we have

$$a_i \equiv \alpha_i \mod p$$

it follows that

$$a_i^{p^i} \equiv \alpha_i^{p^i} = \alpha_i \mod p^{i+1},$$

and so

$$p^{n-i} a_i^{p^i} \equiv p^{n-i} \alpha_i \mod p^{n+1}.$$

Solving for a_n then gives the result. □

Map

$$T\{x\} \to C$$

by sending x to the identity map $\mathbb{Z}_p$. The element $\psi(x) - x$ then goes to zero.

Lemma 11. *The commutative diagram*

$$\begin{tikzcd} T\{y\} \arrow[r, "y \mapsto 0"] \arrow[d, "y \mapsto \psi(x)-x"'] & \mathbb{Z}_p \arrow[d] \\ T\{x\} \arrow[r] & C \end{tikzcd}$$

is a pushout square in the category of Frobenius algebras. The left vertical map is étale. □.

6. Structure of B

Define T_ζ by the pushout (in $K(1)$-local E_∞ ring spectra)

$$\begin{tikzcd} TS^{-1} \arrow[r, "*"] \arrow[d, "\zeta"'] & S^0 \arrow[d] \\ S^0 \arrow[r] & T\zeta. \end{tikzcd}$$

By definition of ζ and the fact that B is E_∞, there is a canonical map

$$T\zeta \to B.$$

Since $B \wedge \zeta$ is null, we have

$$\pi_0 B \wedge T_\zeta = T\{b\},$$

where b was defined earlier. The same thing holds with B replace by K, only in that case, the odd homotopy groups are zero. Under the map of Frobenius algebras

$$\pi_0 K \wedge T_\zeta = T\{b\} \to \pi_0 K \wedge B = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times/\mu, \mathbb{Z}_p)$$

the element b goes to the unique abelian group homomorphism sending g to 1. Let's use this homomorphism to identify $\mathbb{Z}_p^\times/\mu$ with $\mathbb{Z}_p$ and hence $\pi_0 K \wedge B$ with $\mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$ After doing this, we find that b maps to the identity map of $\mathbb{Z}_p$.

Now consider the element $\psi(b) - b$. This is fixed under ψ_g, since

$$\begin{aligned} \psi_g(\psi(b) - b)) &= \psi_g \psi(b) - \psi_g(b) \\ &= \psi \psi_g(b) - \psi_g(b) \\ &= \psi(b+1) - (b+1) = \psi(b) - b. \end{aligned}$$

Lemma 12. *The maps*

$$\pi_* K \wedge T\zeta \xrightarrow{(\psi_g - 1)\wedge 1} \pi_* K \wedge T\zeta$$

$$\pi_* B \wedge T\zeta \xrightarrow{(\psi_g - 1)\wedge 1} \pi_* B \wedge T\zeta$$

are surjective. The map

$$\pi_* T\zeta \to \pi_* B \wedge T_\zeta$$

is therefore a monomorphism, with image the invariats under ψ_g.

We therefore have a unique element f in $\pi_0 T\zeta$ whose image in $\pi_0 B \wedge T\zeta \approx \pi_0 K \wedge T\zeta$ is $\psi(b) - b$. This leads to the diagram

$$\begin{array}{ccc} TS^0 & \xrightarrow{*} & S^0 \\ {\scriptstyle f}\downarrow & & \downarrow \\ T\zeta & \longrightarrow & B. \end{array}$$

Our main result is

Proposition 13. *The map $K_* f$ is étale. The above diagram is a pushout in $\mathcal{C}^{E_\infty}$.* □

7. $K(1)$-local E_∞-elliptic spectra

We first recall the geometric interpretation of the operation ψ. In general, an E_∞ structure on a complex oriented cohomology theory E gives the following structure. Given a map

$$f : \pi_0 E \to R$$

and a closed finite subgroup $H \subset f^* G$, one gets a new map

$$\psi_H : \pi_0 E \to R$$

and an isogeny $f^* G \to \psi^* G$ with kernel H. When the formal group is isomorphic, locally in the flat topology, to the formal multiplicative group, one can take f to be the identity map, and H the "canonical subgroup" of order p. Since it is so canonical, it is invariant under all automorphisms of G, and one doesn't even need a formal group in this case.

An *elliptic spectrum* is a ring spectrum E with $\pi_{\text{odd}} E = 0$, and for which there exists a unit in $\pi_2 E$, (hence E is complex orientable and we get a canonical formal group G over $\pi_0 E$), together with an elliptic curve over $\pi_0 E$ and an isomorphism of the formal completion of this elliptic curve with the formal group G.

An E_∞ *elliptic spectrum* is an E_∞ spectrum E which is elliptic, and for which the isogenies described above come equipped with extensions to isogenies of the elliptic curve.

Suppose that E is a $K(1)$-local E_∞ elliptic spectrum with

$$\pi_1 K \wedge E = 0$$

(this is automatic if $\pi_0 E$ is torsion free). Then the sequence

$$\pi_0 E \to \pi_0 B \wedge E \xrightarrow{(\psi_g - 1)\wedge 1} \pi_0 B \wedge E$$

is short exact. There is therefore an element b in $\pi_0 B \wedge E$ with $\psi_g b = b + 1$. Such a b is well-defined up to translation by an element in the image of $\pi_0 E$. Here is another description. Choose an extension of the unit to a map

$$\iota : M_\zeta \to E.$$

The map ι is unique up to translation by an element in $\pi_0 E$. Now look at the image of the element $b \in \pi_0 K \wedge M_\zeta$ in $\pi_0 K \wedge E$. This class, which we shall also call b is well-defined up to translation by an element in the image of $\pi_0 E$. The fact that it satisfies $\psi_g b = b+1$ follows from the commutative diagram

$$\begin{array}{ccc} \pi_0 B \wedge M_\zeta & \xrightarrow{(\psi_g - 1)\wedge 1} & \pi_0 B \wedge M_\zeta \\ \downarrow & & \downarrow \\ \pi_0 B \wedge E & \xrightarrow[(\psi_g - 1)\wedge 1]{} & \pi_0 B \wedge E. \end{array}$$

Suppose we have chosen such a b. Since ψ and ψ_g commute (ψ_g is an E_∞ map) one easily checks that $\psi(b)$ is another such element. It follows that $\psi(b) - b$ lies in

$$\pi_0 E \subset \pi_0 B \wedge E.$$

We are interested in choosing such a b as canonically as possible, and looking at the element $\psi(b) - b$. This will be the obstruction to making an E_∞ map from B to E, but it also tells us quite a bit about the E_∞ structure of E. Since the map

$$\pi_0 B \wedge E \to \pi_0 K \wedge E$$

is an isomorphism, it suffices to make this calculation in $\pi_0 K \wedge E$.

Let's first do this at the prime 2. Let $c_4 \in \pi_8 E$ be the normalized modular form of weight 4. The q-expansion of c_4 is given by

$$c_4 = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n,$$

where

$$\sigma_3(n) = \sum_{d|n} d^3.$$

Let $g \in \mathbb{Z}_2^\times$ be an element which projects to a topological generator of $\mathbb{Z}_2^\times/\{\pm 1\}$. Let $u \in \pi_2 K$ be the Bott class. We have

$$g^4 \equiv 1 \mod 16,$$

and checking the q-expansion gives

$$u^4 \equiv c_4 \mod 16.$$

Now consider the element

$$b = -\frac{\log(c_4/u^4)}{\log g^4},$$

where

$$\log(1+x) = \sum_{n\geq 0} (-1)^n \frac{x^{n+1}}{n+1}$$

is regarded as a 2-adic analytic function. Then, since $\psi_g c_4 = c_4$ (it is in the image of $\pi_8 E$), and $\psi_g u^4 = g^4 u^4$, we have

$$\begin{aligned}\psi_g b &= -\frac{\log(c_4/g^4u^4)}{\log g^4} \\ &= -\frac{\log(c_4/u^4)}{\log g^4} + \frac{\log g^4}{\log g^4} \\ &= b + 1,\end{aligned}$$

so b is an *Artin-Schrier* element.

Now let $f = \psi(b) - b$. The element f is a a modular function (since $\psi_g f = f$), and so is an element of

$$\mathbb{Z}_2[j^{-1}],$$

where $j = \frac{c_4^3}{\Delta}$ is the modular function.

Lemma 14. *The map*

$$\mathbb{Z}_2[f] \to \mathbb{Z}_2[j^{-1}]$$

is an isomorphism.

Proof: It clearly suffices to do this modulo 2.

Working mod 2 we have

$$b \equiv \sum_{n\geq 1} \sigma_3(n) q^n \mod 2$$

Writing

$$n = 2^{m_0} p_1^{m_1} \dots p_k^{m_k}$$

One easily checks that the number of divisors of n is

$$(1+m_0) \times \cdots \times (1+m_k),$$

and that the number of *odd* divisors of n is

$$(1+m_1) \times \cdots \times (1+m_k).$$

It follows that $\sigma_3(n)$ is even unless n is the product of a power of 2 and an odd square. This gives

$$b \equiv \sum_{m,d\geq 0} q^{2^m(2d+1)^2} \mod 2,$$

and so

$$\begin{aligned}\psi(b) - b &\equiv \sum_{d\geq 0} q^{(2d+1)^2} \\ &= q \sum_{d\geq 0} q^{8(d(d+1)/2)}\end{aligned}$$

since the operation ψ is given in terms of q-expansions by

$$\psi(q) = q^2.$$

As for j^{-1} we have

$$j^{-1} = \frac{\Delta}{c_4^3} \equiv \Delta \mod 2,$$

and

$$\Delta = q \prod (1-q^n)^{24} \equiv q \prod (1-q^{8n})^3 \mod 2.$$

The congruence

$$\psi(b) - b \equiv j^{-1} \mod 2$$

is then a consequence of the following special case of the Jacobi triple product identity

$$\sum_{d\geq 0} (-1)^d (2d+1) z^{d(d+1)/2} = \prod_{k\geq 0} (1-z^k)^3.$$

$\square$

At the prime 3 we can attempt the analogous thing with c_6 (which represents v_1^3)

$$c_6 = 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n.$$

It seems as if the following identity holds

$$\frac{\log c_6}{9} \equiv j^{-1} \mod 3.$$

I have checked out to the $q = 300$ term. Oddly, the analogous result for $p > 3$ seems not to hold, though I'm not really sure what "analogous" means.

Remark 1. Fred Diamond and someone else (whose name I didn't catch) have told me that the mod 3 result is also true, and that both follow from known congruences for the Ramanujan τ function as described in Serre's course in arithmetic.

In fact there is a better approach which will give us more information. Note that

$$b = -\frac{\log(c_4/u^4)}{\log g^4} \equiv -\sum_{n \geq 1} \sigma_3(n) q^n \mod 8,$$

and so

$$\psi b - b \equiv \sum_{n \geq 1} (\sigma_3(n) - \sigma_3(n/2)) q^n \mod 8,$$

where we adopt the convention that $\sigma_3(n) = 0$ if n is not an integer. Now it's easy to check that

$$\sigma_3(2n) \equiv \sigma_3(n) \mod 8,$$

so that

$$\psi b - b \equiv \sum_{n \text{odd}} (\sigma_3(n)) q^n \mod 8.$$

On the other hand,

$$\frac{1}{j} = \frac{\Delta}{c_4^3} \equiv q \prod_{n \geq 1} (1 - q^n)^{24} \equiv \sum \tau(n) q^n \mod 8.$$

It follows from a congruence of Ramanujan (see [1, page 4]) that

$$\tau(n) \equiv \begin{cases} 0 \mod 8 & \text{if } n \text{ is even} \\ \sigma_3(n) \mod 8 & \text{if } n \text{ is odd.} \end{cases}$$

This means that we have the congruence

$$\psi b - b \equiv j^{-1} \mod 8.$$

Returning to the prime 2 we will now build a canonical $K(1)$-local E_∞ ring spectrum mapping to any $K(1)$-local elliptic spectrum. Since ψ_g and θ commute, the element

$$\theta(f) \in \pi_0 E$$

is a modular function, and hence is can be written as a 2-adically convergent power series in j^{-1}. By lemma 14 there is a 2-adically convergent power series h with

$$\theta(f) = h(f).$$

This gives a universal relation in the homotopy groups of any $K(1)$-local E_∞ ring spectrum.

Returning to T_ζ and continuing our horrible practice of not choosing good notation, let $b \in \pi_0 B \wedge T_\zeta$ once again denote the universal "b," let $f \in \pi_0 T_\zeta$ be the unique element whose image in $\pi_0 B \wedge T_\zeta$ is

$$f = \psi(b) - b,$$

and, finally, set

$$y = \theta(f) - h(f) \in \pi_0 T_\zeta.$$

This gives the vertical map in the following diagram. The requirement that it be a pushout defines the $K(1)$-local E_∞ ring spectrum M.

$$\begin{tikzcd} TS^0 \arrow[r, "*"] \arrow[d, "y"] & S^0 \arrow[d] \\ T_\zeta \arrow[r] & M. \end{tikzcd}$$

by construction it is clear that there is a pretty canonical map from M to any $K(1)$-local E_∞-ring spectrum.

Proposition 15. *The map*

$$K_* y : K_* TS^0 \to K_* T_\zeta$$

is smooth of relative dimension 1 (it is also flat, but I suppose that is a consequence of smoothness). Therefore

$$K_* M = K_* T_\zeta \underset{K_* TS^0, y}{\otimes} \mathbb{Z}_2$$

and

$$\pi_* M = KO_*[j^{-1}].$$

Proof: The map $TS^0 \to T_\zeta$ comes about as the composite

$$TS^0 \xrightarrow{\theta(x)-h(x)} TS^0 \xrightarrow{f} T_\zeta.$$

Since $K_* f$ is etale (Lemma 11) it suffices to show that the map $K_*(\theta(x) - h(x))$ is smooth of relative dimension 1. If we write out the rings, we are looking at the map of F-algebras

$$\begin{aligned} \mathbb{Z}_2[y_0, y_1, \dots] &\to \mathbb{Z}_2[x_0, x_1, \dots] \\ y_0 &\mapsto x_1 - h(x_0). \end{aligned}$$

It is easy to check that $h(x_0) = x_0^2 + \dots$, so that our map is of the form

$$y_n \mapsto x_{n+1} \quad \text{mod decomposables.}$$

This probably shows that the map

$$\begin{aligned} \mathbb{Z}_2[y_0, y_1, \dots][x_0] &\to \mathbb{Z}_2[x_0, x_1, \dots] \\ y_0 &\mapsto x_1 - h(x_0) \dots \\ x_0 &\mapsto x_0 \end{aligned}$$

is an isomorphism. I've checked out to the q^{100} term and found that $h(x_0) \equiv 0 \mod 8$. If this is true, then it also shows that the above map is isomorphism.

I suppose that the smartest thing to do is to consider the increasing filtration by the x_i. One easily checks that the map is of the form

$$y_i \mapsto x_{i+1} + t(x_0, \dots, x_i),$$

which gives an easy inductive proof of surjectivity. Injectivity follows from "iso mod indecomposables" since the intersection of the powers of the "augmentation ideal" is zero. □

8. Homotopy groups of T_ζ

Recall that $g \in \mathbb{Z}_p^\times$ is chosen so that g projects to a topological generator of $\mathbb{Z}_p^\times/\mu$. Define $h \in \mathbb{Z}_p$ by

$$1 + h = \begin{cases} g^{p-1} & p > 2 \\ g^2 & p = 2. \end{cases}$$

Then for p odd, $h \equiv 0 \mod p$ and for $p = 2$, $h \equiv \mod 8$. In both cases

$$(1+h)^{(-b)} = \sum \binom{-b}{n} h^n.$$

defines an element of $\mathbb{Z}_p[b]$ (recall that this ring is p-complete.

Define a multiplicative map

$$i : B_* \to B_* T_\zeta = B_* \otimes T\{b\},$$

by

$$i(v_1) = v_1 g^{(p-1)^{(-b)}} = v_1 (1+h)^{(-b)},$$

for p odd, and

$$\begin{aligned} i(2v_1^2) &= 2v_2^2 g^{2^{(-b)}} = 2v_1^2(1+h)^{(-b)} \\ i(v_1^4) &= v_1^4 g^{4^{(-b)}} = v_1^4 (1+h)^{(-2b)}. \end{aligned}$$

At the prime 2, the image of the element $\eta \in \pi_1 KO$ is forced, since it is in the image of the homotopy groups of spheres. Note that in all cases, $i(v) \equiv v \mod 2$. This makes it easy to check that the map at the prime 2 is multiplicative (where one needs, perhaps, to worry about the elements in dimension $8k+1$ and $8k+2$, since they are in the image of the homotopy groups of spheres).

We have a surjective map

$$B_* T_\zeta \to B_* B$$

of F-algebras with an action of $\mathbb{Z}_p^\times$. We are going to define an additive section which is compatible with the action of $\mathbb{Z}_p^\times$. For this we need to refer to the "big Witt vectors."

Consider the algebra

$$A = \mathbb{Z}[a_1, a_2, \dots]$$

and define, for each $n \geq 1$

$$w^B{}_n = \sum_{d|n} d a_d^{n/d}.$$

The algebra has a unique Hopf algebra structure for which the $w^B{}_n$ are primitive. In fact, the group functor represented by A is the functor

$$R \mapsto (1 + x\, R[\![x]\!])^\times.$$

The universal series in A is the series

$$a(x) = \prod (1 - a_n x^n),$$

and one easily checks that

$$x d \log a(x) = -\sum w^B{}_n x^n.$$

There is the following an analogue of Dwork's lemma for the big Witt vectors.

It is helpful to write $a = (a_1, \dots)$ and define

$$(1-x)^a = a(x).$$

The "group law" is then defined so that

$$(1-x)^{a+b} = (1-x)^a (1-x)^b.$$

Lemma 16. *Suppose that A is a ring with endomorphisms*

$$\phi_p : A \to A \qquad p \text{ prime}$$

satisfying

$$\phi_p(a) \equiv a^p \mod p.$$

Then, given a sequence $w^B{}_1, w^B{}_2, \dots \in A$, one can find a sequence $a_1, a_2, \dots \in A$ if an only if for each prime p, each $k \geq 1$, and each $(m,p) = 1$, one has

$$w^B{}_{p^k m} \equiv \phi_p(w^B{}_m) \mod p^k.$$

If A is torsion free then such a solution is unique.

The proof is very similar to the p-typical one.

One application of this is that there is a unique map from the ring of big Witt vectors to the ring of functions from $\mathbb{Z}$ to $\mathbb{Z}$ sending each $w^B{}_n$ to the identity map. One easily checks that this maps sends the series $(1-x)^a$ to the map

$$n \mapsto (1-x)^n.$$

One useful consequence of this is that if we define elements $c_k \in A$

$$c_k = (-1)^k a_k + \text{monomials in } a_i,\ i < k$$

by writing

$$(1-x)^a = \sum c_k (-x)^k,$$

then under the map described above, c_k is sent to the binomial function

$$n \mapsto \binom{n}{k}.$$

In fact, the image of A in the ring of functions on $\mathbb{Z}$ is the ring of "numerical polynomials," and has basis these binomial functions.

The section we are describing results from lifting the "Pascal's triangle" identity

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

to A.

Lemma 17. *There is a unique map of Hopf algebras*

$$T : A \to A$$

with the property that for all n,

$$T w^B{}_n = w^B{}_n + 1.$$

With respect to the map to the ring of functions on $\mathbb{Z}$, we have

$$T f(n) = f(n+1).$$

Finally, the following "Pascal's triangle" identity holds:

$$T c_k = c_k + c_{k-1}.$$

Proof: Let's define the elements $T a_n$ by writing

$$(1-x)(1-x)^a = \prod (1 - T a_n\, x^n).$$

The effect of T on the w_i is easily checked by taking the log of both sides. The rest of the lemma also follows easily. $\square$

We now map the ring A to the ring $\mathbb{W}$.

Lemma 18. *There is a unique map of Hopf algebras*

$$f : A \to \mathbb{W}$$

with

$$f(w^B{}_{p^k m}) = w_k \qquad (m,p) = 1.$$

This map is compatible with the action of T on $\mathbb{W}$ and on A.

Proof: We define ring homomorphisms

$$\phi_l : \mathbb{W} \to \mathbb{W} \qquad l \text{ prime},$$

by setting $\phi_l = 0$ if $l \neq p$, and by setting

$$\phi_p a_n = a_n^p.$$

The result then follows easiy from Dwork's lemma. The compatibility with T follows easily from the effect on primitives. □

The entire point of all of this was to define the binomial classes in the Witt world. We abuse the heck out of the notation to do this.

Definition 19. Let $c_n \in \mathbb{W}$ be the image of the classes $c_n \in A$ under the map defined above.

We can finally define our ψ_g-equivariant section.

Lemma 20. *The map*

$$s : \mathrm{Hom}_{cts}(\mathbb{Z}_p^\times/\mu, \mathbb{Z}_p) \to \pi_0 B \wedge T_\zeta$$

defined by

$$\begin{tikzcd}
\mathrm{Hom}_{cts}(\mathbb{Z}_p^\times/\mu, \mathbb{Z}_p) \arrow[r] \arrow[d, "g"'] & \pi_0 B \wedge T_\zeta \arrow[d, "\approx"] \\
\mathrm{Hom}_{cts}(\mathbb{Z}_p, \mathbb{Z}_p) \arrow[r, "\binom{n}{k} \mapsto c_k"] & \mathbb{W}.
\end{tikzcd}$$

is ψ_g equivariant.

Proof: This follows from the above when one notes that under the vertical isomorphisms in the diagram, the map ψ_g is sent to T. □

Finally, we can return to our computation of the homotopy groups of T_ζ. We have defined a ring homomorphism

$$i : \pi_* B \otimes T\{f\} \to \pi_* B \wedge T_\zeta$$

which is compatible with the action of ψ, and whose image is fixed under the action of ψ_g.

Lemma 21. *The sequence*

$$0 \to \pi_* B \otimes T\{f\} \to \pi_* B \wedge T_\zeta \xrightarrow{\psi_g - 1} \pi_* B \wedge T_\zeta \to 0$$

is exact.

Proof: Consider the *additive* map

$$\mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times/\mu, \mathbb{Z}_p) \otimes \pi_* B \otimes T\{f\} \xrightarrow{\mu \circ s \otimes i} \pi_* B \wedge T_\zeta. \tag{22}$$

We will see below that it is an isomorphism. Granting this, the lemma then reduces to showing that the sequence

$$0 \to \mathbb{Z}_p \xrightarrow{\text{constants}} \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times/\mu, \mathbb{Z}_p) \xrightarrow{\psi_g - 1} \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times/\mu, \mathbb{Z}_p) \to 0$$

is exact. But this is easy to check. Just use the basis of binomial functions, and write down the map. □

We have used

Lemma 23. *The map* (22) *is an isomorphism.*

This really belongs with the discussion on $\mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$.

Proof: This is the standard Milnor–Moore argument. We have an exact sequence of (p-compolete) Hopf-algebras

$$T\{y\} \xrightarrow{\psi(b) - b} T\{b\} \to \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p)$$

and a section

$$s : \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p) \to T\{b\},$$

which is a map of co-algebras. It is formal to check that the maps

$$s \otimes \iota : \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p, \mathbb{Z}_p) \otimes T\{y\} \to T\{b\}$$

and

$$T\{b\} \xrightarrow{\text{coproduct}} T\{b\} \otimes T\{b\}$$

$\square$

Remark 2. The binomial functions have the "Cartan" coproduct. This makes the section a map of co-algebras. This is needed for the above proof (which I'd better write out to make sure.) In the Milnor-Moore argument their hypothesis of "connected-graded" makes this automatic.

Gee though, I don't think it needs to be this hard.

References

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