Math. Z. 226, 103–125 (1997)

Mathematische Zeitschrift

Im(J)-theory and the Kervaire invariant

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Received 23 August 1993

0 Introduction

Let G be the Adams summand of p-local complex periodic K-theory, l its (-1)-connected cover, i.e. $l_*(S^0) = \mathbf{Z}_{(p)}[v_1]$, $|v_1| = q := 2p-2$ and p a prime. Define the spectrum $\bar{l}$ by the cofibre sequence

$$\longrightarrow S^0 \longrightarrow l \xrightarrow{pr} \bar{l} \xrightarrow{\partial} S^1 \tag{1}$$

Since $l_*(S^0)$ is torsion free every element x in the stable homotopy groups of spheres $\pi_n^S(S^0)_{(p)}$, $n \geq 1$, has a lift $\bar{x} \in \pi_{n+1}^S(\bar{l})$ under $\partial : \bar{l} \to S^1$. In this paper we solve for $p > 3$ the problem of which elements in $\pi_*^S(\bar{l})$ can be detected by the e-invariant of Adams and Toda. It is an application of the hard computations in [12] and the main result of [13].

Instead of the e-invariant itself we shall use its refinement given by connected $Im(J)$-theory A_*. $Im(J)$-theory A_* is a generalized homology theory defined by the cofibre sequence of spectra

$$\longrightarrow A \xrightarrow{D} l \xrightarrow{Q} \Sigma^q l \xrightarrow{\Delta} \Sigma A \tag{2}$$

where Q is the l-operation with $v_1 \cdot Q = \psi^k - 1$, ψ^k is the stable Adams operation and k generates $(\mathbf{Z}/p^2)^*$ ($k = 3$ for $p = 2$). Alternatively if we choose in addition k to be a prime power, then Quillen's algebraic K-theory $\mathrm{K}\mathbf{F}_k$, localized at p, may serve as a model for A. The $Im(J)$-theory Hurewicz map

$$h_A : \pi_n^S(X)_{(p)} \to A_n(X)$$

contains all the information which the e-invariant can give. In generalizing the 2-primary case, an element $f \in \pi_n^S(S^0)_{(p)}$ is called a Kervaire invariant one element if it is detected by the secondary cohomology operation representing the class $b_i \in Ext_{\mathscr{A}}^{2,*}(\mathbf{F}_p, \mathbf{F}_p)$ for $p \neq 2$ (and $h_i^2 \in Ext_{\mathscr{A}}^{2,*}(\mathbf{F}_2, \mathbf{F}_2)$ for $p = 2$) in the

E_2-term of the classical Adams spectral sequence. For $p = 2$ such an element has well known geometric and homotopy theoretic interpretations and applications; for $p \neq 2$ some interpretations are discussed in [15] . Our main result may then be stated as follows.

Theorem 1 *There is a non trivial stably spherical element in $A_{2n-1}(\bar{l})$ if and only if there is an element of Kervaire invariant one in $\pi^S_{2n-2}(S^0)_{(p)}$.*

The negative solution of the Kervaire invariant one problem for $p > 3$ by Ravenel [13] implies then that $\mathrm{im}(h_A : \pi^S_{2n-1}(\bar{l}) \longrightarrow A_{2n-1}(\bar{l}))$ is $\mathbf{Z}/p$ for $n = p(p-1)$ and zero otherwise. The situation for $B\Sigma_p$, the classifying space of the symmetric group, is similar: As an application of Theorem 1 we show

Theorem 2 *The element of order p in $A_{2n-2}(B\Sigma_p)$ is stably spherical if and only if there is an element of Kervaire invariant one in $\pi^S_{2n-2}(S^0)_{(p)}$.*

For $p = 2$ this is a well known result of Mahowald but apparently no complete proof for one of the implications has appeared up to now. *)

In [4] the $Im(J)$-theory Chern character is defined. It is a set of natural transformations

$$ch^A_{qi-1} : A_n(X) \longrightarrow H_{n+1-qi}(X; \mathbf{Z}/i)_{(p)} \tag{3}$$

and we may ask which elements f of $\pi^S_*(S^0)_{(p)}$ are detected by the functional operation associated to it (i.e. for which f the natural transformation ch^A_{qi-1} is non trivial on the cofibre of f modulo indeterminacy). An attractive reformulation of Theorem 2 is then

Theorem 3 *An element $f \in \pi^S_n(S^0)_{(p)}$ is detected by the functional ch^A-operation if and only if f has Kervaire invariant one.*

Proofs and statements differ slightly for odd primes and $p = 2$. We have chosen to give the detailed formulation for p odd, in particular, in Theorems 1,2,3 above p is odd. But since the Kervaire invariant one problem is most interesting at $p = 2$ we have indicated the necessary changes to prove Theorem 2 for $p = 2$ in an appendix.

*) added in proof: Recently N. Minami (On the Hurewicz Image of the cokernel J spectrum, preprint 1995) has independently given a proof of Theorem 2, which is also based on [12], [16] but slightly more direct than the one given here.

1 The map e

To determine the possible spherical classes in $A_{2n-1}(\bar{l})$ we use the factorization $T : A_{2n-1}(\bar{BP}) \to A_{2n-1}(\bar{l})$ where BP is the Brown-Peterson spectrum at p, $\bar{BP}$ is the cofibre of $S^0 \to BP$ and $T : BP \to l$ the usual Todd map. The commutative diagram $(n > 1)$

$$\begin{array}{ccccccc} BP_{2n-1}(S^0) \to & \pi^S_{2n-1}(\bar{\bar{BP}}) & \xrightarrow{\cong} & \pi^S_{2n-2}(S^0) \to & BP_{2n-2}(S^0) \\ \downarrow & \downarrow & & \| & \downarrow \\ l_{2n-1}(S^0) \to & \pi^S_{2n-1}(\bar{l}) & \xrightarrow{\cong} & \pi^S_{2n-2}(S^0) \to & l_{2n-2}(S^0) \end{array} \tag{4}$$

shows that $h_A : \pi^S_{2n-1}(\bar{l}) \to A_{2n-1}(\bar{l})$ factors through

$$T \; : \; A_{2n-1}(\bar{\bar{BP}}) \longrightarrow A_{2n-1}(\bar{l})$$

Since $A_{2n-1}(\bar{\bar{BP}}) = 0$ if $n \not\equiv 0 \mod (p-1)$ we may assume $n \equiv 0 \mod (p-1)$. Also $\Delta : l_{2n-q}(\bar{\bar{BP}}) \to A_{2n-1}(\bar{\bar{BP}})$ is onto, hence every stably spherical $x \in A_{2n-1}(\bar{l})$ is in $\mathrm{im}(\Delta : l_{2n-q}(\bar{l}) \to A_{2n-1}(\bar{l}))$ by naturality. Since in general $A_{qm-1}(\bar{\bar{BP}})$ is much larger than $A_{qm-1}(\bar{l})$, we get, without further investigations, only the weak restrictions that $x \in \mathrm{im}\,\Delta$ and $n \equiv 0\ (p-1)$ above.

Let $H^s(BP_*) := Ext^{s,*}_{BP_*BP}(BP_*, BP_*)$ denote the E_2-term of the Adams-Novikov spectral sequence, based on BP-theory. We shall construct a map

$$e : H^2(BP_*) \to A_*(\bar{\bar{BP}})$$

such that any stably spherical class in $A_{qm-1}(\bar{\bar{BP}})$ lies in $\mathrm{im}(e)$. Now by the main result of [12] $H^2(BP_*)$ is explicitly known and much smaller than $A_*(\bar{\bar{BP}})$. This will give the restrictions for elements in $A_*(\bar{l})$ to be stably spherical which we shall need, namely we shall compute $T(\mathrm{im}(e))$. Whether a class in $T(\mathrm{im}(e))$ is stably spherical will then shown to be equivalent to the Kervaire invariant one problem.

In [12] the elements in $H^2(BP_*)$ are described by primitives in $BP_*/(p^\infty, v_1^\infty)$ via the universal Greek letter map η: There are short exact sequences of BP_*-comodules

$$0 \to BP_* \longrightarrow p^{-1}BP_* \longrightarrow BP_*/p^\infty \to 0 \tag{5}$$

$$0 \to BP_*/p^\infty \to v_1^{-1}BP_*/p^\infty \to BP_*/(p^\infty, v_1^\infty) \to 0 \tag{6}$$

inducing long exact Ext-sequences. The two boundary maps associated to (5) and (6) define the map η :

$$\begin{aligned} \eta \; : \; & Ext^{0,*}_{BP_*BP}(BP_*, BP_*/(p^\infty, v_1^\infty)) \xrightarrow{\dot{\partial}} Ext^{1,*}_{BP_*BP}(BP_*, BP_*/p^\infty) \\ & \xrightarrow{\ddot{\partial}} Ext^{2,*}_{BP_*BP}(BP_*, BP_*) \end{aligned} \tag{7}$$

It is shown in [12] 7.1, 7.2, 4.8, 4.2 that (for $p \neq 2$) η is an isomorphism. The short exact sequences (5) (6) belong to the defining sequences of the chromatic spectral sequence [14] and it is known that all sequences of this type may be realized geometrically. It is now clear how to proceed: We lift to filtration zero and map then to l using T. To do so, we need only the geometric realizations of (5) (6) which are well known. The sequence (5) is induced by maps between

Moore spectra. For the convenience of the reader we recall a realization of (6) (For a similar discussion see [5]). Denote by S^0/p^i, S^0/p^∞ the Moore spectra for the groups $\mathbf{Z}/p^i$ and $\mathbf{Z}/p^\infty$ and by Ad the cofibre spectrum of the stable Adams operation $\psi^k - 1$ on p-local periodic complex K-theory, i.e. Ad fits into the cofibre sequence of spectra

$$\to Ad \xrightarrow{D} G \xrightarrow{\psi^k - 1} G \xrightarrow{\Delta} \Sigma Ad \to$$

(We may equally well use the spectrum $K_{(p)}$ instead of G in this sequence, on the other wedge summands of $K_{(p)}$ the operation $\psi^k - 1$ is an equivalence). The spectrum $\bar{\bar{Ad}}$ is defined by the cofibre sequence

$$\to S^0 \xrightarrow{i} Ad \xrightarrow{pr} \bar{\bar{Ad}} \to$$

Lemma 4 *The cofibre sequence*

$$S^0/p^\infty \to Ad \wedge S^0/p^\infty \to \bar{\bar{Ad}} \wedge S^o/p^\infty \tag{8}$$

is a geometric realization of (6) i.e. if we apply BP_* to this sequence we obtain (6)

Proof. In the following commutative diagram

$$\begin{array}{ccc} BP \wedge S^0/p^\infty & \xrightarrow{1\wedge i\wedge 1} & BP \wedge Ad \wedge S^0/p^\infty \\ \downarrow & & \downarrow g_1 \\ v_1^{-1}BP \wedge S^0/p^\infty & \xrightarrow{g_2} & v_1^{-1}BP \wedge Ad \wedge S^0/p^\infty \end{array}$$

we show that g_1, g_2 are equivalences. Then we get, with $g := g_1^{-1} \circ g_2$,

$$\begin{array}{ccccc} BP_*/p^\infty & \longrightarrow & v_1^{-1}BP_*/p^\infty & \longrightarrow & BP_*/(p^\infty, v_1^\infty) \\ \| & & \cong\downarrow g_* & & \cong\downarrow \bar{g}_* \\ BP_*(S^0/p^\infty) & \longrightarrow & BP_*(Ad \wedge S^0/p^\infty) & \longrightarrow & BP_*(\bar{\bar{Ad}} \wedge S^0/p^\infty) \end{array} \tag{9}$$

proving the lemma.

a) For g_1, the map $g_{1*} : Ad_n(BP;\mathbf{Z}/p^\infty) \to Ad_n(v_1^{-1}BP;\mathbf{Z}/p^\infty)$ is the direct limit of maps $Ad_n(BP;\mathbf{Z}/p^i) \to Ad_n(v_1^{-1}BP;\mathbf{Z}/p^i)$. But $Ad_n(v_1^{-1}BP;\mathbf{Z}/p^i) \cong Ad_n(BP;\mathbf{Z}/p^i)\left[v_1^{-1}\right]$ and $v_{1*}^{p^i} = B_i$, where B_i is an Adams periodicity operator as for example constructed in [3]. To see this we use that B_i induces multiplication by $v_1^{p^i}$ in $Ad_n(BP;\mathbf{Z}/p^i) \overset{D}{\subset} G_n(BP;\mathbf{Z}/p^i)$ and $v_{1*} = p \cdot t_1 + v_1$ (see Sect. 2 below for $G_*(BP;\mathbf{Z}/p^i)$). Hence $v_{1*}^{p^i} = v_1^{p^i}$ on $G_*(BP;\mathbf{Z}/p^i)$. Since v_1 operates as an isomorphism, the same is true for $v_{1*}^{p^i}$ and g_{1*} is bijective as the direct limit of isomorphisms.

b) For g_2, we first need that the Adams periodicity operator $B_i : \Sigma^{qp^i} S^0/p^{i+1} \to S^0/p^{i+1}$ induces multiplication by $v_1^{p^i}$ (up to a unit) on $BP_*(S^0;\mathbf{Z}/p^{i+1})$.

This is well known and follows from the fact that $B_i(1) \in BP_{qp^i}(S^0;\mathbf{Z}/p^{i+1})$ must be coaction primitive. The group of primitives is cyclic and generated by $v_1^{p^i}$ (e.g. see [14]). Then $v_1^{-1}BP_*(S^0;\mathbf{Z}/p^{i+1}) = BP_*(S^0;\mathbf{Z}/p^{i+1})\left[B_i^{-1}\right]$. Now $(S^0/p^{i+1})\left[B_i^{-1}\right] \simeq Ad \wedge S^0/p^{i+1}$ by the Mahowald-Miller theorem (e.g. see [3]) and g_{2*} is the direct limit of isomorphisms.

Remark. Observe that the isomorphism $g_* : v_1^{-1}BP_*/p^\infty \cong Ad_*(BP;\mathbf{Z}/p^\infty)$ in (9) is the canonical extension of the Ad-theory Hurewicz map $h_{Ad} : \pi_*^S(BP;\mathbf{Z}/p^\infty) = BP_*/p^\infty \to Ad_*(BP;\mathbf{Z}/p^\infty)$ to $v_1^{-1}BP_*/p^\infty$. Since D: $Ad_*(BP;\mathbf{Z}/p^\infty) \to G_*(BP;\mathbf{Z}/p^\infty)$ is injective we may use the well known formulas for

$$h_G : BP_* \xrightarrow{\eta_R} BP_*BP \xrightarrow{T\wedge 1} G_*BP$$

to compute g_*. If we denote the image of $x \in BP_*$ in $G_*(BP)$ by $\bar{x}$ then

$$g_*\left(\frac{x}{p^i v_1^j}\right) = \frac{\bar{x}}{p^i\ \bar{v}_1^j}.$$

Example. If we abbreviate $T(t_i)$ by t_i then

$$\bar{v}_1 = p \cdot t_1 + v_1 \quad \text{and} \quad \bar{v}_2 = v_1 \cdot t_1^p - v_1^p \cdot t_1 \bmod p$$

in $G_*(BP) = G_*[t_1, t_2, \ldots]$ (see Sect. 2).

Denote the set of coaction primitives in $BP_n(X)$ by $P_nBP_*(X)$. We now define a map

$$e : P_nBP_*/(p^\infty, v_1^\infty) \longrightarrow A_{n-1}(\overline{BP})$$

by the following commutative diagram. Assume n is even.

$$\begin{array}{ccccccccc}
 & & & & & & P_nBP_*/(p^\infty, v_1^\infty) & & \\
 & & & & & & \cap & & \\
0 & \to & BP_n/p^\infty & \to & v_1^{-1}BP_n/p^\infty & \overset{red}{\to} & BP_n/(p^\infty, v_1^\infty) & \to & 0 \\
 & & \| & & g_* \downarrow\cong & & \downarrow\cong & & \\
0 & \to & \pi_n^S(BP;\mathbf{Z}/p^\infty) & \to & Ad_n(BP;\mathbf{Z}/p^\infty) & \to & \overline{Ad}_n(BP;\mathbf{Z}/p^\infty) & \to & 0 \\
 & & \| & & i \uparrow & & & & \\
 & & \pi_n^S(BP;\mathbf{Z}/p^\infty) & \overset{h_{Ad}}{\to} & A_n(BP;\mathbf{Z}/p^\infty) & \overset{\beta}{\to} & A_{n-1}(BP) & & \\
 & & & & pr_* \downarrow & & pr_* \downarrow & & \\
 & & & & A_n(\overline{BP};\mathbf{Z}/p^\infty) & \overset{\beta}{\to} & A_{n-1}(\overline{BP}) & &
\end{array} \tag{10}$$

($pr : BP \to \overline{BP}$ is the canonical map, β the Bockstein map and $i : A_n(X) \to Ad_n(X)$ is the map from connective $Im(J)$-theory to non-connective $Im(J)$-theory Ad, with

$$A_n(X) := im(Ad_n(X^n) \to Ad_n(X^{n+1})),$$

i is induced by inclusion of skeleta).

Definition 5 $e := \beta \circ pr_* \circ i^{-1} \circ g_* \circ red^{-1}$

In order to have e defined we must show

Lemma 6 (1) $x \in P_n BP_*/(p^\infty, v_1) \implies g_* \circ red^{-1}(x) \in im(i)$
(2) $\beta \circ h_A(\pi_n^S(BP; \mathbf{Z}/p^\infty)) = 0$

Proof. (2) is clear since $\beta \circ h_A = h_A \circ \beta$ and $\pi_{n+1}^S(BP)$ is 0 for n even.
Proof of (1): We have

$$P_n BP_*/(p^\infty, v_1) = \ker\langle(\eta_L - \eta_R) : BP_*/(p^\infty, v_1) \to BP_*BP \otimes_{BP_*} BP_*/(p^\infty, v_1)\rangle$$

An element x in $v_1^{-1}BP_n/p^\infty$ maps under red into $P_n BP_*/(p^\infty, v_1)$ if and only if $(\eta_L - \eta_R)(x)$ is in $im(BP_*BP \otimes_{BP_*} BP_*/p^\infty \longrightarrow BP_*BP \otimes_{BP_*} v_1^{-1}BP_*/p^\infty)$. Under the isomorphism g_* this translates into

$$\begin{aligned} &\left\{x \in Ad_n(BP; \mathbf{Z}/p^\infty) \mid red \circ g_*^{-1}(x) \text{ is primitive}\right\} = \\ &\quad \{x \mid (\eta_L - \eta_R)(x) = h_{ad}(z) \text{ in } Ad_n(BP \wedge BP; \mathbf{Z}/p^\infty) \\ &\quad \text{for some } z \in \pi_n^S(BP \wedge BP; \mathbf{Z}/p^\infty)\right\} \end{aligned}$$

Now $G : Ad_n(X; \mathbf{Z}/p^\infty) \to G_n(X; \mathbf{Z}/p^\infty)$ is injective for $X = BP$ or $X = BP \wedge BP$ and $\eta_L(Dx) = Dx \wedge 1$, $\eta_R(Dx) = 1 \wedge Dx$ in $G_n(BP \wedge BP; \mathbf{Z}/p^\infty)$ by the Künneth-theorem for complex K-theory. To have $(\eta_L - \eta_R)(Dx) \in im\, h_A$ implies $D(x) \in G_n(BP^{(n)}; \mathbf{Z}/p^\infty)$ since $h_A(\pi_n^S(BP \wedge BP; \mathbf{Z}/p^\infty))$ is contained in $G_n((BP \wedge BP)^{(n)}; \mathbf{Z}/p^\infty)$. This implies $x \in im(i : A_n(BP; \mathbf{Z}/p^\infty) \to Ad_n(BP; \mathbf{Z}/p^\infty))$. Here i is injective since $A_n(BP; \mathbf{Z}/p^\infty) = Ad_n(BP^{(n)}; \mathbf{Z}/p^\infty)$ $\square$.

We also need

Lemma 7 *Let n be even. Then*
(1) e is injective. (2) ∂_1 is bijective. (3) the diagram

$$\begin{array}{ccccc}
\pi_{n+2}^S(S^0/(p^\infty, v_1^\infty)) & \xrightarrow{\partial_1} & \pi_{n+1}^S(S^0/p^\infty) & \underset{\cong}{\xrightarrow{\beta}} & \pi_n^S(S^0) \\
 & & \uparrow \partial_2 & & \\
 & & & & \cong\uparrow \partial_2 \\
 & & \pi_{n+2}^S(\overline{BP}; \mathbf{Z}/p^\infty) & & \\
\downarrow h_{BP} & & & \overset{\beta}{\searrow} & \\
 & & \downarrow h_A & & \pi_{n+1}^S(\overline{BP}) \\
 & & A_{n+2}(\overline{BP}; \mathbf{Z}/p^\infty) & & \downarrow h_A \\
 & & & \overset{\beta}{\searrow} & \\
P_{n+2}(BP_*/(p^\infty, v_1^\infty)) & & \xrightarrow{e} & & A_{n+1}(\overline{BP})
\end{array}$$

commutes i.e. on stably spherical elements in $BP_{n+2}(S^0)/(p^\infty, v_1^\infty)$ *the invariant* e *is essentially the Hurewicz map* $h_A : \pi^S_{n+1}(\bar{\bar{BP}}) \to A_{n+1}(\bar{\bar{BP}})$ *(here we have written* $S^0/(p^\infty, v_1^\infty)$ *for* $\bar{\bar{Ad}} \wedge S^0/p^\infty$ *e.c.).*

Proof. (1) Choose $x_1 \in v_1^{-1}BP_n/p^\infty$ with $red(x_1) = x$. Then $e(x) = 0$ implies $g_*(x_1) \in \ker(\beta) = im(r : A_n(BP;\mathbf{Q}) \to A_n(BP;\mathbf{Q}/\mathbf{Z}))$. The commutative square

$$\begin{array}{ccc} \pi^S_n(BP;\mathbf{Z}/p^\infty) & \stackrel{h_A}{\longrightarrow} & A_n(BP;\mathbf{Z}/p^\infty) \\ \uparrow r & & \uparrow r \\ \pi^S_n(BP;\mathbf{Q}) & \stackrel{h_A}{\longrightarrow} & A_n(BP;\mathbf{Q}) \end{array}$$

then shows that x_1 is in $\ker(red)$.
(2) Since $\pi^S_{n+1}(v_1^{-1}S^0/p^\infty) = Ad_{n+1}(S^0;\mathbf{Z}/p^\infty) \cong Ad_n(S^0)$ is zero, ∂_1 is onto (n even!), and since $\pi^S_{n+2}(S^0/p^\infty) \to \pi^S_{n+2}(Ad/p^\infty)$ is onto, ∂_1 is injective.
(3) By comparing the two cofibre sequences $S^0/p^\infty \to v_1^{-1}S^0/p^\infty \to S^0/(p^\infty, v_1^\infty)$ and $S^0 \to BP \longrightarrow \bar{\bar{BP}}$ we obtain (suppressing the equivalences g, $\bar{g}$ in (10)) the following commutative diagram. It is a well known fact that $red^{-1} \circ h_{BP} \circ \partial_1^{-1} = pr_*^{-1} \circ j \circ \partial_2^{-1}$ *mod* $h_{BP}(\pi^S_{n+2}(Ad/p^\infty)) + j(BP_{n+2}(S^0/p^\infty))$ in $BP_{n+2}(Ad \wedge S^0/p^\infty)$.

$$\begin{array}{ccccc} & & \pi^S_*(Ad/p^\infty) & \to & \pi^S_*(\bar{\bar{Ad}}/p^\infty) \stackrel{\partial_1}{\to} \pi^S_*(S^0/p^\infty) \\ & & \downarrow h_{BP} & & \downarrow h_{BP} \\ BP_*(S^0/p^\infty) & \stackrel{j_*}{\to} & BP_*(Ad \wedge S^0/p^\infty) & \stackrel{red}{\to} & BP_*(\bar{\bar{Ad}} \wedge S^0/p^\infty) \\ \downarrow & & \downarrow pr_* & & \\ \bar{\bar{BP}}_*(S^0/p^\infty) & \stackrel{j_*}{\to} & \bar{\bar{BP}}_*(Ad \wedge S^0/p^\infty) & & \\ \downarrow \partial_2 & \searrow & h_{Ad} & \downarrow \cong & \\ \pi^S_*(S^0/p^\infty) & & Ad_*(\bar{\bar{BP}};\mathbf{Z}/p^\infty) & \stackrel{i}{\supset} & A_*(\bar{\bar{BP}};\mathbf{Z}/p^\infty) \stackrel{\beta}{\to} A_*(\bar{\bar{BP}}) \end{array}$$

Given $x \in \pi^S_{n+1}(S^0/p^\infty)$ choose elements x_1, x_2, x_3 with $\partial_1(x_1) = x$, $red(x_2) = h_{BP}(x_1)$, $\partial_2(x_3) = x$. Under the maps

$$\bar{\bar{BP}}_{n+2}(S^0/p^\infty) \stackrel{h_{Ad}}{\to} Ad_{n+2}(\bar{\bar{BP}};\mathbf{Z}/p^\infty) \stackrel{i}{\supset} A_{n+2}(\bar{\bar{BP}};\mathbf{Z}/p^\infty) \stackrel{\beta}{\to} A_{n+1}(\bar{\bar{BP}})$$

x_3 is mapped to $\beta \circ h_A(x_3)$. On the other hand, up to the identification

$$BP \wedge S^0/(p^\infty, v_1^\infty) \simeq BP \wedge \bar{\bar{Ad}} \wedge S^0/p^\infty$$

the definition of e reads as

$$e(h_{BP}(x_1)) = \beta \circ i^{-1} \circ pr_*(x_2)$$

But $pr_*(x_2) \equiv j_*(x_3) \bmod pr_* \circ j_*(BP_{n+2}(S^0/p^\infty))$ and under the map $\beta \circ i^{-1}$ the indeterminacy is mapped to zero. Hence $e(h_{BP}(\partial_1^{-1}(x)) = h_A(\partial_2^{-1}(\beta(x)))$. □

Remarks. Slightly simpler is the use of the two cofibre sequences

$$S^0 \to BP \to \bar{\bar{BP}} \qquad \text{and} \qquad \bar{\bar{BP}} \to \bar{\bar{BP}}\mathbf{Q} \to \bar{\bar{BP}}\ \mathbf{Q}/\mathbf{Z}$$

for the lift from Adams-Novikov filtration 2 to filtration 0. The Hattori-Stong theorem then shows that $H^2(BP_*)$ is a subgroup of $A_*(\bar{\bar{BP}})$. But in order to use the definition of the elements given in [12] we had to use (5) and (6). The approach via the Hattori-Stong theorem works for every torsion free space or spectrum (instead of $\bar{\bar{BP}}$). In our case we get the purely K-theoretic description of $Ext^{1,2n}_{BP_*BP}(BP_*, BP_*(\bar{\bar{BP}}))$ $(= H^2(BP_*))$ as $\ker(\Psi : A_{2n-1}(\bar{\bar{BP}}) \to A_{2n-1}(\bar{\bar{BP}} \wedge BP))$ where Ψ is induced from $i : S^0 \to BP$.

2 $\mathbf{A_*(BP)}$

For n even we have $A_n(BP) \cong BP_n(S^0)$. Whereas for n odd $BP_n(S^0) = \pi_n^S(BP)$ is zero, $A_{mq-1}(BP)$ is non trivial and growing very rapidly with m. So $A_{mq-1}(BP)$ may serve as a universal example for *non* stably spherical classes in $A_*(X)$. The order and the number of cyclic summands of $A_{mq-1}(BP)$ is known [9], but here we need only a certain subset of classes related to v_2. Recall

$$BP_*BP \cong BP_*[t_1, t_2, \ldots] \quad \text{and} \quad G_*BP \cong G_* \otimes_{BP_*} BP_*BP \cong G_*[t_1, t_2, \ldots]$$

where $t_i = T(t_i)$ and $T : BP \to G$ is the Todd map.

We have

$$A_{qn}(BP; \mathbf{Q}/\mathbf{Z}) = Ad_{qn}(BP^{(qn)}; \mathbf{Q}/\mathbf{Z}) \subset Ad_{qn}(BP; \mathbf{Q}/\mathbf{Z}) \overset{D}{\subset} G_{qn}(BP; \mathbf{Q}/\mathbf{Z})$$

and denote $h_G(v_i) \in G_*(BP)$ again by $\bar{v}_i$ where

$$h_G : \pi_*^S(BP) \to G_*(BP)$$

is the G-theory Hurewicz map. From $\bar{v}_1 = v_1 + p \cdot t_1$ it follows that $\bar{v}_{1*}^{p^a}$ acts on classes of order at most p^{a+1} in $G_*(BP; \mathbf{Q}/\mathbf{Z})$ as multiplication by $v_1^{p^a}$, hence $\bar{v}_{1*}$ is an isomorphism. In $G_*(BP; \mathbf{Q}/\mathbf{Z})$ we therefore have classes

$$\frac{\bar{v}_2^m}{p^i \cdot \bar{v}_1^j}$$

which are in $\ker(\psi^k - 1)$ since multiplication with $\bar{v}_i$ commutes with $\psi^k - 1$. So $\frac{\bar{v}_2^m}{p^i \cdot \bar{v}_1^j}$ defines a class in $Ad_*(BP; \mathbf{Q}/\mathbf{Z})$. To describe classes in $A_*(BP; \mathbf{Q}/\mathbf{Z})$ we need to work out the skeletal filtration of such elements:

Proposition 8 *For* $0 \leq a \leq m$ *the class*

$$\frac{\bar{v}_2^m}{p^{a+1} \cdot \bar{v}_1^{m-a}}$$

in $G_*(BP;\mathbf{Q}/\mathbf{Z})$ *is in* $\ker(\psi^k - 1)$ *and has skeletal filtration at most* $q(mp+a)$*, that is* $\frac{\bar{v}_2^m}{p^{a+1} \cdot \bar{v}_1^{m-a}}$ *defines an element in* $A_{q(mp+a)}(BP;\mathbf{Q}/\mathbf{Z})$*.*

Proof. Choose s with $s \cdot p^a - (m-a) > 0$, then

$$z = \frac{\bar{v}_2^m}{p^{a+1} \cdot \bar{v}_1^{m-a}} = \frac{\bar{v}_2^m \cdot \bar{v}_1^{s \cdot p^a - (m-a)}}{p^{a+1} \cdot \bar{v}_1^{s \cdot p^a}} = \frac{\bar{v}_2^m \cdot \bar{v}_1^{s \cdot p^a - (m-a)}}{p^{a+1} \cdot v_1^{s \cdot p^a}}$$

(since $\bar{v}_1^{s \cdot p^a} = v_1^{s \cdot p^a}$ on classes of order at most p^{a+1}). Using $\bar{v}_1 = v_1 + p \cdot t_1$ we may write z as a sum of terms

$$\frac{\binom{s \cdot p^a - (m-a)}{j} \bar{v}_2^m \cdot t_1^j}{p^{a+1-j} \cdot v_1^{m-a+j}}$$

It therefore suffices to show ($b := a - j$)

$$SF\left(\frac{\bar{v}_2^m}{p^{b+1} \cdot v_1^{m-b}}\right) \leq q \cdot (m \cdot p + b)$$

where SF abbreviates skeletal filtration. Write $\bar{v}_2 = p \cdot A + v_1 \cdot B$ where $A = t_2 - p^{p-1} \cdot t_1^{p+1}$ and $SF(A) = q \cdot (p+1)$, $SF(B) \leq q \cdot p$.

$$(\bar{v}_2 = p \cdot t_2 - p^p \cdot t_1^{p+1} + v_1 \cdot \left[1 - \binom{p+1}{1} p^{p-1}\right] \cdot t_1^p - \sum_{i=2}^{p-1} \binom{p+1}{i} t_1^{p-i-1} p^{p-i} v_1^i$$

$$- \binom{p+1}{p} t_1 \cdot v_1^p \quad \text{e.g. see [14])}$$

We get

$$\begin{aligned}
\frac{(pA+v_1B)^m}{p^{b+1} \cdot v_1^{m-b}} &= \sum_{j=0}^{m} \binom{m}{j} p^j \cdot A^j \cdot B^{m-j} \cdot v_1^{m-j} / (p^{b+1} \cdot v_1^{m-b}) \\
&\equiv \sum_{j=0}^{b} \binom{m}{j} p^j \cdot A^j \cdot B^{m-j} \cdot v_1^{m-j} / (p^{b+1} \cdot v_1^{m-b}) \\
&= \sum_{j=0}^{b} \binom{m}{b-j} A^{b-j} \cdot B^{m-b+j} \cdot v_1^j / p^{j+1}
\end{aligned}$$

Now $SF(A^{b-j} \cdot B^{m-b+j} \cdot v_1^j / p^{j+1}) \leq q \cdot (m \cdot p + b)$ and the result follows. □

Remark. All elements in $A_{qm-1}(S^0)$ are stably spherical hence the subgroup $i_*(A_{qm-1}(S^0))$ in $A_{qm-1}(BP)$ is zero. Since also $A_{qm-2}(S^0) = 0$ we have

$$A_{qm-1}(BP) \cong A_{qm-1}(\bar{\bar{BP}}) \tag{11}$$

We shall also label elements in $A_{qm-1}(\bar{\bar{BP}})$ by their names in $A_{qm-1}(BP)$, i.e. suppress the map $pr : BP \to \bar{\bar{BP}}$ in our notation.

3 $Ext^{2,*}_{BP_*BP}(BP_*, BP_*)$

In [12] the elements of $Ext^{2,n}_{BP_*BP}(BP_*, BP_*) \cong P_n BP_*/(p^\infty, v_1^\infty)$ are defined in $v_2^{-1}BP_*/(p^\infty, v_1^\infty)$ as follows: Define elements x_n in $v_2^{-1}BP_*$ by

$$\begin{aligned}
x_0 &= v_2\\
x_1 &= x_0^p - v_1^p \cdot v_2^{-1} \cdot v_3\\
x_2 &= x_1^p - v_1^{p^2-1} \cdot v_2^{p^2-p+1} - v_1^{p^2+p-1} \cdot v_2^{p^2-2p} \cdot v_3\\
&\text{and for } n \geq 3\\
x_n &= x_{n-1}^p - 2 \cdot v_1^{b_n} \cdot v_2^{c_n}
\end{aligned} \tag{12}$$

where $b_n := p^n + p^{n-1} - p - 1$, $c_n := p^n - p^{n-1} + 1$. Let $a_0 := 1$ and $a_n := p^n + p^{n-1} - 1$ for $n \geq 1$. Then for $n \geq 0$, $s \geq 1$ and $s \not\equiv 0 \mod p$, $j \geq 1$, $i \geq 0$ with $j \leq p^n$ if $s = 1$ and $p^i \mid j \leq a_{n-i}$ if $s > 1$, the elements $x_n^s / \left(p^{i+1} \cdot v_1^j\right) \in v_2^{-1}BP_*/(p^\infty, v_1^\infty)$ are in $P_*BP_*/(p^\infty, v_1^\infty)$ and define $\beta_{sp^n/j,i+1}$ via the map η in (7).

To compute the image of $\beta_{sp^n/j,i+1}$ in $A_{qm-1}(\bar{\bar{BP}})$ $(\cong A_{qm-1}(BP))$ we need a v_2^{-1}–free form of $x_n^s/(p^{i+1} \cdot v_1^j)$. For our purpose the following weak form will be sufficient

Proposition 9 *The image of* $\beta_{sp^n/j,i+1}$ *in* $A_{qm}(\bar{\bar{BP}}; \mathbf{Q}/\mathbf{Z})$ *may be written as*

$$\frac{\bar{v}_2^{sp^n}}{p^{i+1} \cdot \bar{v}_1^{j}} + \bar{v}_1^{2} \cdot z \ \textit{with}\ p \cdot z = 0$$

Proof. Step 1: We first treat the elements of order p. Calculating mod p and using $(a+b)^p \equiv a^p + b^p$ the defining equations (12) reduce to

$$\begin{aligned}
x_n \equiv\ & (-2 \cdot v_1^{b_n} \cdot v_2^{c_n} - 2 \cdot v_1^{pb_{n-1}} \cdot v_2^{pc_{n-1}} - \ldots\ldots - 2 \cdot v_1^{p^{n-3}b_3} \cdot v_2^{p^{n-3}c_3}\\
& -v_1^{p^n-p^{n-2}} \cdot v_2^{p^n-p^{n-1}+p^{n-2}} - v_1^{p^{n-2}(p^2+p+1)} \cdot v_2^{p^n-2p^{n-1}} \cdot v_3^{p^{n-2}}\\
& -v_1^{p^n} \cdot v_2^{-p^{n-1}} \cdot v_3^{p^{n-1}} + v_2^{p^n}) \mod p
\end{aligned} \tag{13}$$

If $s = 1$ then $j \leq p^n$ and (13) gives

$$\frac{x_n}{p \cdot v_1^{p^n}} = \frac{v_2^{p^n}}{p \cdot v_1^{p^n}} + \frac{v_2^{p^n-p^{n-1}+p^{n-2}}}{p \cdot v_1^{p^{n-2}}}$$

Then

$$e\left(\frac{x_n}{p \cdot v_1^{p^n}}\right) = \frac{\overline{v}_2^{p^n}}{p \cdot \overline{v}_1^{p^n}} + \bar{v}_1^{2} \cdot \frac{\overline{v}_2^{p^n-p^{n-1}+p^{n-2}}}{p \cdot \overline{v}_1^{p^{n-2}+2}}$$

in $A_*(\bar{\bar{BP}}; \mathbf{Q}/\mathbf{Z})$. Multiplication by $\overline{v}_1^{p^n-j}$ gives the conclusion for all $\beta_{sp^n/j}$. Let now $s > 1$, then $j \leq a_n = p^n + p^{n-1} - 1$ and (13) gives $\frac{x_n^s}{p \cdot v_1^j}$ as a sum of terms of the following type

$$\begin{aligned} z_{s_0,s_1,\ldots,s_n} \quad = \quad & const \cdot \left(v_2^{p^n}\right)^{s_0} \cdot \left(v_1^{p^n} \cdot v_2^{-p^{n-1}} \cdot v_3^{p^{n-1}}\right)^{s_1} \cdot \\ & \left(v_1^{p^n-p^{n-2}} v_2^{p^n-p^{n-1}+p^{n-2}} + v_1^{p^{n-2}(p^2+p+1)} v_2^{p^n-2p^{n-1}} v_3^{p^{n-2}}\right)^{s_2} \cdot \qquad (14) \\ & \cdots\cdots \left(v_1^{p^i b_{n-i}} \cdot v_2^{p^i c_{n-i}}\right)^{s_{n-i}} \cdots\cdots \left(v_1^{b_n} \cdot v_2^{c_n}\right)^{s_n} \Big/ \; p \cdot v_1^j \end{aligned}$$

Every term $z_{s_0,s_1,\ldots,s_n}$ is defined in $v_2^{-1}BP_*/(p^\infty, v_1^\infty)$ but does actually belong to $BP_*/(p^\infty, v_1^\infty)$. If $s_1 > 1$, this term contains $v_1^{2p^n}$ and so reduces to zero in $BP_*/(p^\infty, v_1^\infty)$. If $s_1 = 1$ there is an index $i_0 \neq 1$ with $s_{i_o} \geq 1$ (since $s > 1$). The negative power of v_2 in $\left(v_1^{p^n} \cdot v_2^{-p^{n-1}} \cdot v_3^{p^{n-1}}\right)^{s_1}$ is cancelled by the positive power of v_2 in the factor with exponent s_{i_0}, so the term lies in $BP_*/(p^\infty, v_1^\infty)$. In addition we have at most $p \cdot v_1^{n-1}$ in the denominator. If $i_0 > 1$ the power of v_1 contained in the factor with exponent s_{i_0} cancels $v_1^{p^{n-1}}$ in the denominator. So we are left with the cases $s_1 = 1$, $s_0 = s-1$ and $s_1 = 0$. If $s_1 = 1$, $s_0 = s-1$ we get

$$z_{s-1,1,0,0,\ldots,0} = const \cdot \frac{v_2^{p^n(s-1)-p^{n-1}} \cdot v_3^{p^{n-1}}}{p \cdot v_1^j}$$

with $j \leq p^{n-1} - 1$ and it follows (by (8)) that $e(z_{s-1,1,0,0,\ldots,0}) = \overline{v}_1^2 \cdot \dot{z}$ with $p \cdot \dot{z} = 0$. Let now $s_1 = 0$. If $s_i \geq 1$, $s_k \geq 1$ with $i, k > 2$ then $z_{s_0,0,s_2,\ldots\ldots}$ contains $v_1^{p^i b_{n-i} + p^k b_{n-k}}$ but $j \leq p^i \cdot b_{n-i} + p^k \cdot b_{n-k}$. The same conclusion follows if i or k is 2 . Hence $s_0 = s - s_{i_0}$ with $s_{i_0} \leq 1$ and $i_0 \geq 2$ and we get

$$\frac{v_2^{sp^n}}{p v_1^j} \quad \text{or} \quad \frac{v_2^{(s-1)p^n} v_2^a v_3^b}{p v_1^k}$$

with $k \leq p^{n-1} + p^{n-2} - 1$. Again by (8) the conclusion follows.

Step 2: Consider $x_n^s/(p^{i+1} \cdot v_1^j)$ with $j \equiv 0 \bmod p^i$, $j \leq a_{n-i}$, $i > 0$ and iterate on $x_k = (x_{k-1}^p - 2 \cdot v_1^{b_k} \cdot v_2^{c_k})$. Take $j_0 := p^{n-i} + p^{n-i-1} - p^i$ if $n > 2i$ or $j_0 = p^i$ if $n = 2i$ then $j \leq j_0$ and we have

$$\frac{x_n^s}{p^{i+1} \cdot v_1^{j_0}} \equiv \frac{x_{n-r}^{p^r s}}{p^{i+1} \cdot v_1^{j_0}}$$

in $BP_*/(p^\infty, v_1^\infty)$ as long as $b_{n-r+1} \geq j_0$. This is the case for $r \leq i$. The next case is

$$\begin{aligned} \frac{x_{n-i}^{p^i s}}{p^{i+1} \cdot v_1^{j_0}} &= \left(x_{n-i-1}^p - 2 \cdot v_1^{b_{n-i}} \cdot v_2^{c_{n-i}}\right)^{sp^i} \Big/ \; p^{i+1} \cdot v_1^{j_0} \\ &= \frac{x_{n-i-1}^{p^{i+1}s}}{p^{i+1} \cdot v_1^{j_0}} + \sum_{l=1} (-2)^l \cdot \binom{sp^i}{l} \cdot v_1^{l \cdot b_{n-i}} \cdot v_2^{l \cdot c_{n-i}} \cdot x_{n-i-1}^{p(sp^i - l)} / p^{i+1} \cdot v_1^{j_0} \end{aligned}$$

Only for $i = 1$ we get the extra term

$$\frac{-2s \cdot v_2^{c_{n-i}} \cdot x_{n-i-1}^{p(sp^i-1)}}{p \cdot v_1}$$

which is handled as in step 1. Proceed now by induction on k $(i < k < n-2)$. Assume

$$\frac{x_{n-k+1}^{p^{k-1}s}}{p^{i+1}\cdot v_1^{j_0}} = \frac{x_{n-k}^{p^k s}}{p^{i+1}\cdot v_1^{j_0}} + z$$

where $e(z) = \overline{v}_1^2 \cdot \hat{z}$ with $p\cdot\hat{z} = 0$. Then

$$\begin{aligned}\frac{x_{n-k}^{p^k s}}{p^{i+1}\cdot v_1^{j_0}} &= \left(x_{n-k-1}^{p} - 2\cdot v_1^{b_{n-k}}\cdot v_2^{c_{n-k}}\right)^{sp^k} \Big/ p^{i+1}\cdot v_1^{j_0}\\ &= \frac{x_{n-k-1}^{p^{k+1}s}}{p^{i+1}\cdot v_1^{j_0}} + \sum_{l=1}(-2)^l\cdot\binom{sp^k}{l}\cdot v_1^{l\cdot b_{n-k}}\cdot v_2^{l\cdot c_{n-k}}\cdot x_{n-k-1}^{p(sp^k-l)}/p^{i+1}\cdot v_1^{j_0}\end{aligned}$$

If $\nu_p(l) < k-i$, the power of p in the binomial coefficient $\binom{sp^k}{l}$ is at least $i+1$, so these summands give no contribution. Let $l = p^{k-i}\hat{l}$. If $\hat{l} > 1$, we have $\hat{l}\cdot p^{n-i}\cdot b_{n-k} \geq j_0$, so the power of v_1 is already to large. We are left with the term with $l = p^{k-i}$. Since $\nu_p\left(\binom{sp^k}{p^{k-i}}\right) = i$ the denominator reduces to $p\cdot v_1^{j_0}$ and we obtain

$$\frac{a\cdot v_2^{p^{k-i}c_{n-k}}\cdot x_{n-k-1}^{p^{k-i+1}(sp^i-1)}}{p\cdot v_1^{j_1}}$$

with $a \in \mathbf{Z}_{(p)}$ and $j_1 \leq p^{k-i+1}+p^{k-i}-p^i$ $\quad(j_1 \leq p^{k-i+1}+p^{k-i}-p^{i-1}$ if $n = 2i)$. As in step 1 it follows that the image of

$$\frac{x_{n-k+1}^{p^{k-i+1}\acute{s}}}{p\cdot v_1^{p^{k-i+1}+p^{k-i}-p^i}}$$

in $A_*(\overline{\overline{BP}})$ may be written as $\overline{v}_1^2\cdot\dot{z}$ with $p\cdot\dot{z} = 0$. This completes the induction step for $k < n-2$. The cases $k = n-2$ and $k = n-1$ have to be dealt with separately but follow exactly the same pattern. We end with

$$\frac{x_n^s}{p^{i+1}\cdot v_1^j} = \frac{v_2^{p^n s}}{p^{i+1}\cdot v_1^j} + z$$

where the image of z in $A_*(\overline{\overline{BP}})$ may be written as $\overline{v}_1^2\cdot B$ with $p\cdot B = 0$.

4 $\mathbf{A_*(\overline{l})}$ and the image of T on im(e)

Note first, that $A_{qm-1}(\overline{l}) \cong A_{qm-1}(l)$ by the same reason as for BP. In [8] it is proved that the total A-theory Chern character

$$ch^A : A_n(l) \longrightarrow W_n^A(l) := H_n(l;\mathbf{Z}_{(p)}) \oplus \bigoplus_{i\geq 1} H_{n+1-qpi}(l;\mathbf{Z}/pi)_{(p)}$$

is injective. Since $\overline{v}_1 = p\cdot m_1$ in homology, it is immediately clear that every element of order p^a in $A_n(l)$ is annihilated by $\overline{v}_1^a$. Here we shall prove a weaker form of this conclusion (with a proof which easily generalizes to $p = 2$) and use this to compute

$$T : A_{qm-1}(\bar{B}\bar{P}) \longrightarrow A_{qm-1}(\bar{l})$$

on $im(e)$.

Proposition 10 *Assume $x = \bar{v}_1^{a+1} \cdot \hat{x}$ in $A_*(l)$ with $p^a \cdot \hat{x} = 0$ and $\hat{x} = \Delta(\tilde{x})$, $\tilde{x} \in l_*(l)$, then $x = 0$.*

Proof. Recall from [1] that $h : l_*(l) \longrightarrow H_*(l \wedge l; \mathbf{Z}_{(p)})$ is injective, the torsion of $H_*(l \wedge l; \mathbf{Z}_{(p)})$ is is of order p and annihilated by $\bar{v}_{1*}$ and the description of $l_*(l)/tor$: We have

$$H_*(l \wedge l; \mathbf{Z}_{(p)})/tor \cong \mathbf{Z}_{(p)}\left[\frac{v}{p}, \frac{u}{p}\right]$$

with $u := 1 \wedge v_1 = \bar{v}_1$, $v := v_1 \wedge 1$ and a homogeneous polynomial

$$f(u, v) = \sum_i a_i \cdot \frac{u^{n-i}}{p^{n-i}} \frac{v^i}{p^i}$$

is in $im(h)$ mod tor if and only if for all integers m, s prime to p the integrality condition

$$f(m^{p-1} \cdot t, s^{p-1} \cdot t) \in \mathbf{Z}_{(p)}[t]$$

is satisfied. In the following we abbreviate m^{p-1} by $\dot{m}$ and write $c_i := (\dot{k}^i - 1)/p$. Write $h(\tilde{x}) =: f(u, v) = w_1 + \sum_{i=0} a_i \cdot u^{n-i-1} v^i / p^{n-1}$ in $H_{(n-1)q}(l \wedge l; \mathbf{Z}_{(p)})$ with $p \cdot w_1 = 0$. Since $p^a \cdot \tilde{x} \in \ker(\Delta)$ we get $p^a \cdot f(u, v) \in im(Q \wedge 1_*)$, i.e.

$$\hat{g}(u, v) := \sum_{i=0}^{n-1} \frac{a_i p^a}{p^n c_{i+1}} u^{n-i-1} v^{i+1}$$

is in $H_{nq}(l \wedge l; \mathbf{Z}_{(p)})$ with $(Q \wedge 1)_*(\hat{g}(u, v)) = p^a f(u, v)$ (since $(Q \wedge 1)_*(v_1^{i+1}/p^{i+1}) = c_{i+1} \cdot v_1^i / p^i$). Therefore $a_i \cdot p^a / c_i \in \mathbf{Z}_{(p)}$ for all i and

$$g(u, v) := \frac{u^a}{p^a} \hat{g}(u, v) - \sum_{i=0}^{n-1} \frac{a_i p^a}{c_{i+1}} \frac{u^{a+n}}{p^{a+n}}$$

is a well defined element in $H_{nq}(l \wedge l; \mathbf{Z}_{(p)})$ satisfying $(Q \wedge 1)_* g = u^a f$.

We now show that g satisfies the integrality condition for being in $im(h)$. We may write $\dot{m} = \dot{k}^c + p^\alpha e$, $\dot{s} = \dot{k}^d + p^\alpha h$ with α larger than any denominator in g. Assume also $c < d$. Then $g(\dot{m}t, \dot{s}t) \in \mathbf{Z}_{(p)}[t]$ if $g(\dot{k}^c t, \dot{k}^d t) \in \mathbf{Z}_{(p)}[t]$. Now

$$
\begin{aligned}
g(\dot{k}^c t, \dot{k}^d t) &= \sum_{i=0}^{n-1} \frac{a_i}{p^n c_{i+1}} \left[\dot{k}^{c(n-i-1)}\dot{k}^{d(i+1)} - \dot{k}^{c(a+n)}\right] \cdot t^{a+n} \\
&= \sum_{i=0}^{n-1} \frac{a_i}{p^{n-1}} \frac{\dot{k}^{c(-i-1)+d(i+1)}-1}{\dot{k}^{i+1}-1} \dot{k}^{c(a+n)} \cdot t^{a+n} \\
&= \sum_{i=0}^{n-1} \frac{a_i}{p^{n-1}} \frac{\dot{k}^{(d-c)(i+1)}-1}{\dot{k}^{i+1}-1} \dot{k}^{c(a+n)} \cdot t^{a+n} \\
&= \sum_{i=0}^{n-1} \sum_{j=1}^{d-c-1} \frac{a_i}{p^{n-1}} \dot{k}^{j(i+1)} \cdot \dot{k}^{c(a+n)} t^{a+n} \\
&= \sum_{j=1}^{d-c-1} f(t, \dot{k}^j t) \cdot \dot{k}^{j+c(a+n)} t^{a+1}
\end{aligned}
$$

which is in $\mathbf{Z}_{(p)}[t]$ since $f(t, \dot{k}^j t)$ is. Therefore there exists an element $z \in l_{nq}(l)$ with $h(z) = g(u, v) + w_2$ and $p \cdot w_2 = 0$. Multiply by $\bar{v}_1$, then $h(\bar{v}_1 z) = u \cdot g(u, v)$ since $u \cdot w_2 = 0$ and $Q(\bar{v}_1 z) = \bar{v}_1 \cdot Q(z) = \bar{v}_1^{a+1} \cdot \tilde{x}$ since

$$h(\bar{v}_1 \cdot Q(z)) = u \cdot (Q \wedge 1)_* g(u, v) = u^{a+1} f(u, v) = h(\bar{v}_1^{a+1} \tilde{x})$$

and h is injective. Therefore $\Delta(\bar{v}_1^{a+1}\tilde{x}) = 0$ and $x = 0$. □

Consider now

$$z(a) := \beta\left(\frac{\bar{v}_2^{p^{a-1}}}{p \cdot \bar{v}_1^{p^{a-1}}}\right) = e(\beta_{p^{a-1}/p^{a-1}}) \in A_{qp^a-1}(\bar{\bar{BP}})$$

and define

$$t(a) := T(z(a)) \in A_{qp^a-1}(\bar{l})$$

again suppressing $pr : BP \to \bar{\bar{BP}}$, $pr : l \to \bar{l}$ in the notation. We then know $p \cdot t(a) = 0$. We need $ch^A(t(a)) \neq 0$ on $A_*(l)$ and $ch^A(t(a)) = 0$ on $A_*(\bar{l})$. If $t_1 \in l_q(l)$ is defined as $t_1 = (\eta_L(v_1) - \eta_R(v_1))/p$ then it can be shown that $\Delta(p^{a-1}t_1^{p^a-1}) = t(a)$ in $A_{qp^a-1}(l)$. From this and Example 3 in [4] we easily get $ch^A(t(a))$. To avoid the calculation for $\Delta(p^{a-1}t_1^{p^a-1}) = t(a)$ we use (3.5) in [4] : Now $\bar{v}_2 \equiv v_1 t_1^p - v_1^p t_1$ mod p, so

$$\frac{\bar{v}_2^{p^{a-1}}}{p \cdot \bar{v}_1^{p^{a-1}}} = \frac{\left(t_1^p - v_1^{p-1} t_1\right)^{p^{a-1}}}{p} = \frac{\left(t_1^{p^a} - v_1^{(p-1)p^{a-1}} t_1^{p^{a-1}}\right)}{p}$$

in $A_{qp^a}(BP; \mathbf{Q}/\mathbf{Z})$. Hence (by (3.5) in[4])

$$ch^A_{qj-1}(z(a)) = ch^l_{qj}\left(\frac{\bar{v}_2^{p^{a-1}}}{p \cdot \bar{v}_1^{p^{a-1}}}\right) = \frac{(-1)^j \binom{p^a}{j} m_1^{p^a-j}}{p}$$

in $H_{qp^a-qj}(BP; \mathbf{Z}/j)$ since $ch^l_{qj}(v_1^{(p-1)p^{a-1}} t_1^{p^{a-1}}/p) = v_1 p\, ch^l_{q(j-1)}(v_1^{(p-1)p^{a-1}-1} t_1^{p^{a-1}}/p)$ is integral. So

$$ch^A_{qj-1}(z(a)) = \begin{cases} 0 & \text{if } j \neq p^a \\ p^a \cdot 1 \text{ in } H_0(BP; \mathbf{Z}/p^a) & \text{if } j = p^a \end{cases} \tag{15}$$

and the value for $ch^A_{qj-1}(t(a))$ follows by naturality. In particular $t(a) \neq 0$, $ch^A_{qj-1}(t(a)) \neq 0$ on $A_*(l)$ but $ch^A(t(a)) = 0$ on $A_*(\overline{l})$. Now we are ready to prove

Theorem 11 *If $z \in A_{2n-1}(\bar{l})$ is stably spherical, then $n = (p-1)p^a$, $a \geq 1$, and z is a multiple of $t(a)$.*

This follows from

Theorem 12 *The image of T on $e(Ext^{s,*}_{BP_*BP}(BP_*, BP_*)) \subset A_{2n-1}(\bar{\bar{BP}})$ is generated by the elements $t(a)$, $a \geq 1$.*

Proof. By definition $T(e(\beta_{p^{a-1}/p^{a-1}})) = t(a)$ and we have to show that all the other β's go to zero. We use Propositions (9), (10) and $A_{qm-1}(\bar{\bar{BP}}) = A_{qm-1}(BP)$, $A_{qm-1}(\bar{l}) = A_{2n-1}(l)$. If $j \geq 2$ then $T \circ e(\beta_{p^{a-1}/p^{a-1}-j}) = \overline{v}_1^j \cdot t(a) = 0$ by Proposition (10). If $j = 1$ we write

$$e(\beta_{p^{a-1}/p^{a-1}-1}) = \overline{v}_2 \cdot (\overline{v}_2^{p^{a-1}-1}/p\overline{v}_1^{p^{a-1}-1} + w) = \overline{v}_{2*}(z)$$

where we view $\overline{v}_2$ as a self map of BP. Then $T \circ e(\beta_{p^{a-1}/p^{a-1}-1}) = \overline{v}_{2*}T(\beta(z))$ but $\overline{v}_{2*} = 0$ in $A_*(l)$ (this follows from the facts that $T \circ v_2 : \Sigma^{|v_2|}BP \to BP \to l$ is zero and T is multiplicative). Next for $s < 1$ or $i > 1$ if $s = 1$ we have

$$T \circ e(\beta_{sp^a/j,i+1}) = T \circ \beta \left(\frac{\overline{v}_2^{sp^a}}{p^{i+1}\overline{v}_1^j} \right) + \overline{v}_1^2 T(z_1)$$

with $p \cdot z_1 = 0$ by Proposition (9). But in $A_{qm}(BP; \mathbf{Q}/\mathbf{Z})$ we have $\overline{v}_2^{sp^a}/p^{i+1}\overline{v}_1^j = \overline{v}_1^{i+2} \cdot z_2$ with $z_2 = \overline{v}_2^{sp^a}/p^{i+1}\overline{v}_1^{j+i+2}$ since $j + 2i + 2 \leq sp^a$ as an easy estimation shows (Proposition (8)). Hence $T(\beta(z_2 \cdot \overline{v}_1^{i+2})) = 0$ by Proposition (10) since $p^{i+1} \cdot T(\beta(z_2)) = 0$. □

The Thom reduction

$$\alpha : Ext^{2,*}_{BP_*BP}(BP_*, BP_*) \longrightarrow Ext^{2,*}_{\mathscr{A}_*}(\mathbf{F}_p, \mathbf{F}_p)$$

from the E_2-term of the Adams-Novikov spectral sequence to the E_2-term of the classical Adams spectral sequence is known by [12]. We have $\alpha(\beta_{p^a/p^a}) = -b_a$ where b_a is analogous to the class carrying a Kervaire invariant one element at $p = 2$ (if it exists). Note that in the dimension of β_{p^a/p^a} all other elements in $Ext^{2,*}_{BP_*BP}(BP_*, BP_*)$ map to zero under α, so that $\ker(\alpha) = \ker(T \circ e)$ in this case.

Corollary 13 *$t(a) \in A_{qp^a-1}(\overline{l})$ is stably spherical if and only if $b_{a-1} \in Ext^{2,*}_{\mathscr{A}_*}(\mathbf{F}_p, \mathbf{F}_p)$ is permanent (i.e. there exists an element of mod p Kervaire invariant one in dimension $q \cdot p^a - 2$).*

Proof. Note first, that the well known geometric boundary lemma ([14] 2.3.4) implies that the following diagram commutes

$$
\begin{array}{ccccc}
F^0\pi^S_{n+2}(S^0/(p^\infty, v_1^\infty)) & \xrightarrow{\partial_1} & F^1\pi^S_{n+1}(S^0/p^\infty) & \xrightarrow{\beta} & F^2\pi^S_n(S^0) \\
\downarrow & & \downarrow & & \downarrow \\
P_{n+2}BP_*/(p^\infty, v_1^\infty) & & & & \\
\| & & & & \\
Ext^{0,n+2}_{BP_*BP}(BP_*, BP_*/(p^\infty, v_1^\infty)) & \xrightarrow{\dot\partial} & Ext^{1,n+2}_{BP_*BP}(BP_*, BP_*/p^\infty) & \xrightarrow{\ddot\partial} & Ext^{2,n+2}_{BP_*BP}(BP_*, BP_*)
\end{array}
$$

Here the unnamed arrows associate to an element in Adams filtration F^i its E_2-representing set. Hence we may treat $\eta = \ddot\partial \circ \dot\partial$ as an identification and use the $Ext^{2,*}_{BP_*BP}(BP_*, BP_*)$-names for corresponding elements in $P_{n+2}BP_*/(p^\infty, v_1^\infty)$.

"$\Rightarrow$" If $t(a)$ is stably spherical then $e(\beta_{p^{a-1}/p^{a-1}} + z)$ is stably spherical with $e(z) \in \ker(T)$ (use the diagram in Lemma (7)). Then $\alpha(\beta_{p^{a-1}/p^{a-1}} + z) = -b_{a-1}$ is permanent. Conversely, if b_{a-1} is permanent, then $\beta_{p^{a-1}/p^{a-1}} + w$ with $w \in \ker(\alpha)$ is permanent, hence $T \circ e(\beta_{p^{a-1}/p^{a-1}} + w) = t(a)$ is stably spherical. □

The odd primary Kervaire invariant one problem was solved for $p > 3$ by Ravenel [13]: For $p > 3$ and $a \geq 1$ b_a is not permanent (b_0 is permanent representing β_1 ; for $p = 3$ $\beta_{3/3}$ is not permanent but $\beta_{9/9} \pm \beta_7$ is). Hence

Corollary 14 *For $p > 3$ and m odd the only stably spherical elements in $A_m(\bar l)$ are the multiples of $t(1)$.*

Remarks.

1. A purely K-theoretic proof of Theorem (12) is, in principle, possible. Since

$$Ext^{2,*}_{BP_*BP}(BP_*, BP_*) \subset A_{qm-1}(\bar{\bar{BP}}) \text{ is } \ker(\Psi : A_{qm-1}(\bar{\bar{BP}}) \to A_{qm-1}(\bar{\bar{BP}} \wedge BP))$$

(where Ψ is induced from $S^0 \to BP$), one has to compute $im(T)_{|\ker(\Psi)}$. But to compute $\ker(\Psi)$ seems to be not much easier than the work done in [12].

2. A purely K-theoretic proof of Theorem (11) is simpler: Since $ch^A : A_*(l) \to W^A(l)$ is injective [8], one only has to work out $\ker ch^A$ on $A_*(\bar l)$. The disadvantage of proving only this is, that then the relation to the Kervaire invariant one elements is harder to derive.

5 Stably spherical classes in $A_{2n}(B\Sigma_p)$ and the functional A-theory Chern character

Although there is no lift of the transfer map $\widetilde{tr}: B\Sigma_p \to S^0$ to a map $B\Sigma_p \to \Sigma^{-1}\bar l$ (since $\widetilde{tr}(1) \in l^0(B\Sigma_p)$ is non zero) there is a strong relationship between stably spherical classes in $A_*(\bar l)$ and $A_*(B\Sigma_p)$. Recall (e.g. [4])

$$A_{qm-2}(B\Sigma_p) \cong \mathbf{Z}/p^{\nu_p(m)}$$

and denote a non zero element of order p in $A_{qp^a-2}(B\Sigma_p) \cong \mathbf{Z}/p^a$ by $x(a)$. We shall show that the only possible stably spherical elements in $A_{2n}(B\Sigma_p)$ are the multiples of $x(a)$.

The cofibre sequences $S^0 \to l \to \bar{l}$ and $B\Sigma_p \wedge A \xrightarrow{\widetilde{tr}} A \xrightarrow{ch^A} W^A$ (see [4]) induce the following basic commutative diagram of exact sequences

$$\begin{array}{ccccccc}
\uparrow ch^A & & \uparrow ch^A & & & & \\
A_{qm-1}(l) & \overset{pr_*}{\underset{\cong}{\to}} & A_{qm-1}(\bar{l}) & \to & A_{qm-2}(S^0) & \to & A_{qm-2}(l) \\
\uparrow \widetilde{tr} & & \uparrow \widetilde{tr} & & \uparrow 0 & & \uparrow \\
A_{qm-1}(l \wedge B\Sigma_p) & \to & A_{qm-1}(\bar{l} \wedge B\Sigma_p) & \xrightarrow{\partial} & A_{qm-2}(B\Sigma_p) & \to & A_{qm-2}(l \wedge B\Sigma_p) \\
\uparrow & & \uparrow d & & \uparrow \cong & & \uparrow \\
W^A_{qm}(l) & \overset{pr_*}{\underset{\cong}{\to}} & W^A_{qm}(\bar{l}) & \xrightarrow{0} & W^A_{qm-1}(S^0) & \to & W^A_{qm-1}(l)
\end{array} \tag{16}$$

We first show

Proposition 15 *Suppose $x \in A_{qm-2}(B\Sigma_p)$ is stably spherical. Then $x = \partial(x_1)$ for some stably spherical element $x_1 \in A_{qm-1}(\bar{l} \wedge B\Sigma_p)$ and $\widetilde{tr}\ (x_1) \in A_{qm-1}(\bar{l})$ is non zero and stably spherical.*

Proof. Since $\pi^S_{qm-2}(B\Sigma_p) \longrightarrow l_{qm-2}(B\Sigma_p)$ is zero , any $f \in \pi^S_{qm-2}(B\Sigma_p)$ with $h_A(f) = x$ has a lift $\bar{f} \in \pi^S_{qm-1}(\bar{l} \wedge B\Sigma_p)$ with $h_A(\bar{f}) = x_1$, $\partial(x_1) = x$. Assume $\widetilde{tr}\ (x_1) = 0$, then $x_1 = d(x_2)$ but $pr_* : W^A_{qm}(l) \to W^A_{qm}(\bar{l})$ is bijective for $m \neq 0$, therefore this would imply $x = 0$. Hence $\widetilde{tr}\ (x_1) \neq 0$. □

Combining this with Theorem (11) and Corollary (13) gives

Theorem 16 *The image of $h_A : \pi^S_{2n}(B\Sigma_p) \to A_{2n}(B\Sigma_p)$ is zero for $n \neq (p-1) \cdot p^a - 1$ and contained in the subgroup of order p in $A_{qp^a-2}(B\Sigma_p) \cong \mathbf{Z}/p^a$.*

Corollary 17 *a) If $x(a) \in A_{qp^a-2}(B\Sigma_p)$ is stably spherical, then there exists a (p-primary) Kervaire invariant one class (i.e. b_{a-1} in $Ext^{2,*}_{\mathscr{A}_*}(\mathbf{F}_p, \mathbf{F}_p)$ is a permanent cycle).*
b) If $p > 3$ then $h_A : \pi^S_{2n}(B\Sigma_p) \to A_{2n}(B\Sigma_p)$ is zero except for $n = (p-1) \cdot p - 1$. For $n = (p-1) \cdot p - 1$ h_A is bijective and any generator of $\pi^S_{2n}(B\Sigma_p) = \mathbf{Z}/p$ maps to a non zero multiple of β_1 under the transfer map $\widetilde{tr}$: $\pi^S_{2n}(B\Sigma_p) \to \pi^S_{2n}(S^0)$.

We now turn to the converse of (17)a.

Theorem 18 *If the element b_{a-1} in the classical Adams spectral sequence is permanent, then $x(a) \in A_{qp^a-2}(B\Sigma_p)$ is stably spherical .*

Proof. By Corollary (13) we know $t(a) \in A_{qp^a-2}(\bar{l})$ is stably spherical if b_{a-1} is permanent. Consider the commutative diagram ($n := q \cdot p^a - 1$)

$$\begin{array}{ccccccc}
 & & A_{n-1}(B\Sigma_p) & & & & \\
 & \partial \nearrow & & \nwarrow h_A & & & \\
A_n(B\Sigma_p \wedge \bar{l}) & \stackrel{h_A}{\longleftarrow} & \pi_n^S(B\Sigma_p \wedge \bar{l}) & \stackrel{\partial}{\longrightarrow} & \pi_{n-1}^S(B\Sigma_p) & \stackrel{0}{\to} & l_{n-1}(B\Sigma_p) \\
\downarrow \widetilde{tr} & & \downarrow \widetilde{tr} & & \downarrow \widetilde{tr} & & \\
A_n(\bar{l}) & \stackrel{h_A}{\longleftarrow} & \pi_n^S(\bar{l}) & \underset{\cong}{\stackrel{\partial}{\longrightarrow}} & \pi_{n-1}^S(S^0) & &
\end{array}$$

Choose $f \in \pi_n^S(\bar{l})$ with $h_A(f) = t(a)$. Since $\widetilde{tr}$ is onto by the Kahn-Priddy-theorem we have a lift of $\partial(f)$ to an element $\overline{f} \in \pi_{n-1}^S(B\Sigma_p)$ and since $l_{n-1}(B\Sigma_p) = 0$ a lift of $\overline{f}$ to an element $\hat{f} \in \pi_n^S(B\Sigma_p \wedge \bar{l})$. Clearly $\widetilde{tr}\,(\hat{f}) = f$. Then $h_A(\hat{f}) =: x_1 \neq 0$ since $\widetilde{tr}\,(x_1) = t(a) = h_A(f)$. Assume now $\partial(x_1) = 0$ in $A_{n-1}(B\Sigma_p)$. Then there exists $x_2 \in A_n(B\Sigma_p \wedge l)$ with $pr_*(x_2) = x_1$ in (16). By commutativity in (16) we have $\widetilde{tr}\,(x_2) = t(a)$ in $A_n(l) \cong A_n(\bar{l})$ which would imply $ch^A(t(a)) = 0$ on $A_n(l)$ contradicting (15). Hence $\partial(x_1) \neq 0$ and there is a non zero stably spherical class in $A_{n-1}(B\Sigma_p)$. Then $x(a)$ must be in $im(h_A)$.

Remark. With different methods the images of $h_A : \pi_{2n}^S(B\Sigma_p) \to A_{2n}(B\Sigma_p)$ and $h_A : \pi_{2n}^S(B\mathbf{Z}/p) \to A_{2n}(B\mathbf{Z}/p)$ (for $p \neq 2$ up to the elements of order p corresponding to $x(a)$ in dimensions $n = s \cdot p^a - 1$, $0 \leq s \leq p-1$) are determined in [6].

For $f \in \ker(h_A : \pi_n^S(X) \to A_n(X))$ the functional $A-$theory Chern character ch_f^A is defined in the usual way: Let

$$S^n \stackrel{f}{\longrightarrow} X \stackrel{j}{\longrightarrow} C_f \stackrel{p}{\longrightarrow} S^{n+1}$$

be the cofibre sequence associated to f and consider the commutative diagram

$$\begin{array}{ccccccc}
0 \to & A_{n+1}(X) & \to & A_{n+1}(C_f) & \stackrel{p_*}{\to} & A_{n+1}(S^{n+1}) & \stackrel{f_*}{\to} 0 \\
 & \downarrow ch_{qr-1}^A & & \downarrow ch_{qr-1}^A & & & \\
0 \to & H_{n+2-qr}(X;\mathbf{Z}/r) & \stackrel{j_*}{\to} & H_{n+2-qr}(C_f;\mathbf{Z}/r) & \to & 0 &
\end{array}$$

If $\hat{1} \in A_{n+1}(C_f)$ is an element with $p_*(\hat{1}) = 1 \in A_{n+1}(S^{n+1})$, then $ch_{qr-1}^A(\hat{1}) = j_*(z)$ and z is well defined in $H_{n+2-qr}(X;\mathbf{Z}/r)/ch_{qr-1}^A(A_{n+1}(X))$. For $X = S^0$ we can completely describe the values which this invariant may take:

Theorem 19 *An element $f \in \pi_n^S(S^0)_{(p)}$ is detected by the functional A-theory Chern character if and only if f has Kervaire invariant one (i.e. f is represented in the classical Adams spectral sequence by b_i).*

Proof. n must be of the form $n = q \cdot r - 2$ with $\nu_p(r) > 0$. Let $\widetilde{tr}\colon B\Sigma_p \to S^0$ be the reduced transfer map and $\hat{f} \in \pi_n^S(B\Sigma_p)$ be an element with $\widetilde{tr}\,(\hat{f}) = f$ (which can be found by the Kahn-Priddy theorem). Denote the cofibre of $\hat{f}$ by $C_{\hat{f}}$ and by $t : C_{\hat{f}} \to C_f$ the fill in map between cofibres. Consider the commutative diagram

$$\begin{array}{ccccccccc}
A_{n+1}(S^0) & & & \to & & & A_{n+1}(C_f) & \to & A_{n+1}(S^{n+1}) \xrightarrow{f_*} \\
& \searrow & & & & \swarrow\ ch^A_{qr-1} & & & \\
& & H_0(S^0;\mathbf{Z}/r) & \xrightarrow{\cong} & H_0(C_f;\mathbf{Z}/r) & & & & \\
\uparrow\widetilde{tr} & & \uparrow\widetilde{tr} & & \uparrow & & \uparrow t_* & & \| \\
& & H_0(B\Sigma_p;\mathbf{Z}/r) & \to & H_0(C_{\hat{f}};\mathbf{Z}/r) & & & & \\
& \nearrow & & & & \nwarrow\ ch^A_{qr-1} & & & \\
A_{n+1}(B\Sigma_p) & & & \to & & & A_{n+1}(C_{\hat{f}}) & \to & A_{n+1}(S^{n+1}) \xrightarrow{\hat{f}_*}
\end{array}$$

Suppose $\hat{f}_* = 0$, then there exists $\approx\!\!\!\!\!1 \in A_{n+1}(C_{\hat{f}})$ with $t_*(\approx\!\!\!\!\!1) = \hat{1}$ in $A_{n+1}(C_f)$ and $ch^A_{qr-1}(\hat{1})$ factors through $H_0(C_{\hat{f}};\mathbf{Z}/r)$ and $\widetilde{tr}\colon H_0(B\Sigma_p;\mathbf{Z}/r) \to H_0(S^0;\mathbf{Z}/r)$ and must be zero. Hence if f is detected by ch^A_f, $\hat{f}_*(1) = h_A(\hat{f})$ must be non zero and the result follows from Corollary (17).

Conversely if $f \in \pi_n^S(S^0)$ is represented by b_{i-1} ($n = q \cdot p^i - 1$), then $\hat{f}_*(1) = h_A(\hat{f}) \neq 0$ (see proof of Theorem (18)). Hence $d_*(\hat{1}) \neq 0$ in $A_{n+1}(\Sigma C_t)$ where ΣC_t is the cofibre of t and $d : C_f \to \Sigma C_t$ the canonical map. But C_t is equivalent to $C_{\widetilde{tr}}$ and on $A_n(C_{\widetilde{tr}})$ the A-theory Chern character ch^A_{n+1} is an isomorphism (essentially by the identification of $C_{\widetilde{tr}} \wedge A$ with W^A, see [4], remark following (2.9)). Since

$$d_* : H_0(C_f;\mathbf{Z}/p^i) \to H_{-1}(C_{\widetilde{tr}};\mathbf{Z}/p^i) \cong H_{-1}(S^{-1};\mathbf{Z}/p^i)$$

is an isomorphism too, $ch^A_{n+1}(\hat{1})$ must be non zero (the indeterminacy is zero). □

Remark. For $p \neq 2$ the functional integral Chern character ch^I_f mod p may be interpreted as the mod p Hopf invariant.

6 Appendix: The 2-primary case

At $p = 2$ there are several versions of $Im(J)-$theory: We define complex $Im(J)-$theory by the cofibre sequences

$$\to Ad\mathbf{C} \xrightarrow{D} K_{(2)} \xrightarrow{\psi^3-1} K_{(2)} \xrightarrow{\Delta} \Sigma Ad\mathbf{C} \to \tag{17}$$

$$\to A\mathbf{C} \xrightarrow{D} bu_{(2)} \xrightarrow{Q} \Sigma^2 bu_{(2)} \xrightarrow{\Delta} \Sigma A\mathbf{C} \to \tag{18}$$

where $v_1 \cdot Q = \psi^3 - 1$. Then $A\mathbf{C}$ is the (-1)-connected cover of $Ad\mathbf{C}$. This is as for odd primes, the main difference is that not all elements in $A\mathbf{C}_n(S^0)$ are stably spherical; for $n \equiv 3, 5$ mod 8 $coker(h_{A\mathbf{C}})$ has order 2.

Real versions are defined by

$$\to Ad\mathbf{R} \xrightarrow{D} KO_{(2)} \xrightarrow{\psi^3-1} KO_{(2)} \xrightarrow{\Delta} \Sigma Ad\mathbf{R} \to \tag{19}$$

$$\to A \xrightarrow{D} bo_{(2)} \xrightarrow{Q} \Sigma^4 bsp_{(2)} \xrightarrow{\Delta} \Sigma A \to \tag{20}$$

(for bsp and Q in (20) see [11]). The spectrum A is the proper choice at $p = 2$, but differs from the (-1)-connected cover of $Ad\mathbf{R}$ in π_0 and π_1. We have a complexification map $c : Ad\mathbf{R} \to Ad\mathbf{C}$ induced by the usual complexification.

The groups $H^2(BP_*) = Ext^{2,*}_{BP_*BP}(BP_*, BP_*)$ for $p = 2$ have been determined by Mitchell and Shimomura [16]. The map η appearing in (7) is neither injective nor surjective but its kernel and cokernel are computed in [16]. Lemma (4) is true with $Ad\mathbf{R}$ instead of $Ad\mathbf{C}$, therefore the definition of the map e has to be changed slightly. We define e similarly as for $p \neq 2$ but build in complexification. With the maps from the following diagram

$$\begin{array}{ccccccccc}
 & & & & & & P_n BP_*/(2^\infty, v_1^\infty) & & \\
 & & & & & & \cap & & \\
0 & \to & BP_n/2^\infty & \to & v_1^{-1}BP_n/2^\infty & \xrightarrow{red} & BP_n/(2^\infty, v_1^\infty) & \to & 0 \\
 & & \| & & g_* \downarrow\cong & & \downarrow\cong & & \\
0 & \to & \pi_n^S(BP;\mathbf{Z}/2^\infty) & \to & Ad\mathbf{R}_n(BP;\mathbf{Z}/2^\infty) & \to & \overline{Ad}\mathbf{R}_n\,(BP;\mathbf{Z}/2^\infty) & \to & 0 \\
 & & & & c\downarrow & & & & \\
 & & \| & & Ad\mathbf{C}_n(BP;\mathbf{Z}/2^\infty) & & & & \\
 & & & & i\uparrow & & & & \\
 & & \pi_n^S(BP;\mathbf{Z}/2^\infty) & \xrightarrow{h_{Ad}} & A\mathbf{C}_n(BP;\mathbf{Z}/2^\infty) & \xrightarrow{\beta} & A\mathbf{C}_{n-1}(BP) & & \\
 & & & & pr_*\downarrow & & pr_*\downarrow & & \\
 & & & & A\mathbf{C}_n(\overline{BP};\mathbf{Z}/2^\infty) & \xrightarrow{\beta} & A\mathbf{C}_{n-1}(\overline{BP}) & &
\end{array} \tag{21}$$

we set

$$e := pr_* \circ \beta \circ i^{-1} \circ c \circ g_* \circ red^{-1}$$

and prove Lemma (6) in the same way.

We now turn to Lemma (7):

The map $\partial_1 : \pi^S_{n+2}(S^0/(2^\infty, v_1^\infty)) \longrightarrow \pi^S_{n+1}(S^0/2^\infty)$ in Lemma (7) is not onto for all n, but $\ker(\partial_1)$ and $coker(\partial_1)$ are determined by the Hurewicz map $h_{Ad\mathbf{R}} : \pi^S_m(S^0) \to Ad\mathbf{R}_m(S^0)$. Since $h_{Ad\mathbf{R}}$ is onto for m odd, $m > 1$, we find that ∂_1 is always injective but has a cokernel of order 2 in dimensions congruent 0 and 2 mod 8. We assume now that n is of the form $n = 2 \cdot 2^a - 2$, $a \geq 2$, then ∂_1 is bijective. Complexification c in (21) is injective. This may be seen as follows. It is enough to show this with $\mathbf{Z}/2^i$ coefficients, for all i. If x is in $\ker(c)$ then $B_i^m(x) \in \ker(c)$, where B_i is an Adams periodicity operator for the Moore spectrum $M(\mathbf{Z}/2^i)$, (e.g. see [3]). But $B_i^m(x)$ for m large enough comes from stable homotopy (see again [3]) and $\pi_{2r}^S(BP;\mathbf{Z}/2^i) \rightarrow Ad\mathbf{C}_{2r}(BP;\mathbf{Z}/2^i)$ is injective by the Hattori-Stong theorem. Hence $c \circ B_i^m(x) = 0$ implies $B_i^m(x) = 0$ and this gives $x = 0$. Since under the dimension assumptions made, $A\mathbf{C}_{n-1}(BP) \rightarrow A\mathbf{C}_{n-1}(\bar{\bar{BP}})$ is a monomorphism, we see that e is injective as for odd primes. Then Lemma (7) reformulated with $A\mathbf{C}_*$ is proved as for $p \neq 2$.

In Sect. 2 we have

$$\eta_R(v_2) = v_2 + 2t_2 - 5v_1t_1^2 - 4t_1^3 - 3v_1^2t_1$$

hence $A = t_2 - 2t_1^3$, $B = -5t_1^2 - 3v_1t_1$ and Proposition (8) is true for $p = 2$ without any change. Note however that $pr_* : A\mathbf{C}_{2m-1}(BP) \rightarrow A\mathbf{C}_{2m-1}(\bar{\bar{BP}})$ is still always onto but has a kernel of order 2 if $m \equiv 2, 3 \bmod 4$.

The computations in Sect. 3 have to be redone completely, but no new idea is necessary. The definition of the elements $\beta_{2^ns/j,i}$ is in [16, 14] . The computations are even simpler than for $p \neq 2$ since $x_i = x_{i-1}^2$ for $i \geq 3$ but there are more subcases to check. The simplest way to proceed then seems to be as follows. We may put in the definition of x_0, x_1, x_2 and then expand by the binomial formula. For the factor y_i^{-m} in $\beta_{2^ns/j,i+2}$ we use $(1-4v_2/v_1^3)^{-j/2}$. This gives $\beta_{2^ns/j,i+1}$ and $\beta_{2^ns/j,i+2}$ as a polynomial in v_1, v_2, v_3, v_1^{-1}, v_2^{-1}. Then one checks that every term containing a negative power of v_2 is zero if reduced mod 2^∞ and v_1^∞. To the terms left we may apply Propositions (10) and (8) directly, i.e. if $\beta_{2^ns/j,k}$ contains a summand $v_3^c \cdot v_2^m/2^a \cdot v_1^b$ with $2a + b \leq m$, $a \leq m$, then

$$e\left(\frac{v_2^m}{2^a \cdot v_1^b}\right) = \left(\frac{\bar{v}_2^m}{2^a \cdot \bar{v}_1^b}\right)$$

is divisible by $\bar{v}_1^{a+1}$ in $A\mathbf{C}_*(BP)$ and maps to zero in $A\mathbf{C}_*(l)$ by Proposition (10). The case of $\beta_{2^n/2^n-1}$ is handled as for $p \neq 2$, also some terms $v_3^c \cdot v_2^m/2^a \cdot v_1^b$ with $2a + b > m \geq a + b$ and $c \geq 1$. As for $p \neq 2$ the only $\beta_{2^ns/j,k}$ with non trivial image in $A\mathbf{C}_*(\bar{l})$ is $\beta_{2^n/2^n}$.

The proof of Proposition (10) has to be modified slightly, due to the fact that $(\mathbf{Z}/2^i)^*$ is not cyclic. The use of the Adams operation ψ^{-1} gives the remaining cases to be checked. Theorem (11) is not true for $p = 2$ as stated (since η in (7) and ∂_1 in Lemma (7) are not onto) but if $n = 2 \cdot 2^a - 1$, $a \geq 2$, any stably spherical element in $A\mathbf{C}_n(\bar{l})$ must be in $im(e)$, hence

Theorem 20 *If $z \in A\mathbf{C}_{2n-1}(\bar{l})$ is stably spherical and $n = 2^a$, $a \geq 2$, then z is a multiple of $t(a)$.*

For the Thom reduction

$$\alpha : Ext^{2,*}_{BP_*BP}(BP_*, BP_*) \longrightarrow Ext^{2,*}_{\mathscr{A}_*}(\mathbf{F}_2, \mathbf{F}_2)$$

we refer to [14] 5.4.6. In the Kervaire invariant one dimensions the kernel of α is the same as $\ker(T \circ e)$ and the proof of Corollary (13) carries over without change:

Theorem 21 *The class* $t(a) \in A\mathbf{C}_{2^{a+1}-1}(\bar{l})$ *is stably spherical if and only if* $h_a^2 \in Ext^{2,2^{a+1}}_{\mathscr{A}_*}(\mathbf{F}_2, \mathbf{F}_2)$ *is permanent.*

To carry over the results of Sect. 5 one needs the basic diagram (16) with A replaced by $A\mathbf{C}$. The 2-primary version of the complex $Im(J)$-theory Chern character $ch^{A\mathbf{C}}$ is quite analogous to the odd primary case. Let R be the cofibre of the reduced transfer map

$$B\Sigma_2 \xrightarrow{\widetilde{tr}} S^0 \longrightarrow R$$

then $bo \wedge R$ splits as $\bigvee_{i\geq 0} \Sigma^{4i} H\mathbf{Z}_{(2)}$ by [11] and from $bo \wedge \Sigma^{-2}P_2\mathbf{C} \simeq bu$ one gets $bu \wedge R \simeq \bigvee_{i\geq 0} \Sigma^{2i} H\mathbf{Z}_{(2)}$. The rest of the argument is the same as in [4] and

$$A\mathbf{C} \wedge B\Sigma_2 \xrightarrow{\widetilde{tr}} A\mathbf{C} \xrightarrow{ch^{A\mathbf{C}}} W^{A\mathbf{C}} \tag{22}$$

with $W_n^{A\mathbf{C}}(X) := H_n(X;\mathbf{Z}_{(2)}) \oplus \bigoplus_{i>0} H_{n+1-4i}(X;\mathbf{Z}/4i)_{(2)}$ is a cofibre sequence.

For $n = 2^{a+1} - 2$ we have then

1. $pr_* : A\mathbf{C}_{n+1}(l) \to A\mathbf{C}_{n+1}(\bar{l})$ is injective
2. $ch^{A\mathbf{C}}(t(a)) \neq 0$ on $A\mathbf{C}_{n+1}(l)$ and $ch^{A\mathbf{C}}(t(a)) = 0$ on $A\mathbf{C}_{n+1}(\bar{l})$
3. $pr_* : W^{A\mathbf{C}}_{n+2}(l) \longrightarrow W^{A\mathbf{C}}_{n+2}(\bar{l})$ is onto.

These facts imply as for p odd

Theorem 22 *For* $n = 2^{a+1} - 2$, $a \geq 2$, *the image of* $h_{A\mathbf{C}} : \pi_n^S(B\Sigma_2) \to A\mathbf{C}_n(B\Sigma_2)$ *is contained in the subgroup of order* 2 *and* $A\mathbf{C}_n(B\Sigma_2)$ *contains a non trivial stably spherical element if and only if* $h_a^2 \in Ext^{2,2^{a+1}}_{\mathscr{A}_*}(\mathbf{F}_2, \mathbf{F}_2)$ *is permanent, i.e. there exists an element of Kervaire invariant one in dimension* n.

We have for $n = 2^{a+1} - 2$, $a \geq 2$,

$$\begin{aligned} A\mathbf{C}_n(B\Sigma_2) &= \mathbf{Z}/2^{a+1} \quad \text{(for example by (22)) and} \\ A_n(B\Sigma_2) &= \mathbf{Z}/2^{a-1} \quad \text{(e.g. see [2,10])} \end{aligned}$$

Comparing the exact sequences giving $A\mathbf{C}_n(B\Sigma_2)$ and $A_n(B\Sigma_2)$ shows that the canonical map $A_n(B\Sigma_2) \to A\mathbf{C}_n(B\Sigma_2)$ is injective (for n as above), hence Theorem (22) may also be formulated with A-theory. In this formulation the result is due to M. Mahowald [10] (see also [2] and [7]). In [10] it is also shown that $A_*(B\Sigma_2)$ detects the transfer lifts of the Mahowald family η_j.

The reformulation of Theorem (19) is left to the reader.

References

1. J.F. Adams: Stable homotopy and generalized cohomology, University Press Chicago (1974)
2. M.G. Barratt, J.D.S. Jones, M. Mahowald: The Kervaire invariant and the Hopf invariant, in Springer Lecture Notes 1286 (1985)
3. M.C. Crabb, K. Knapp: Adams periodicity in stable homotopy. Topology **24** (1985), 475–486
4. M.C. Crabb, S. Jäschke, K. Knapp: The Im(J)-theory Chern character, preprint (1992)
5. A.S. Hilditch: On calculating the complex double transfer in homotopy, Dissertation, Manchester (1985)

6. S. Jäschke: Stabil sphärische Elemente in der Bild-J-Theory des klassifizierenden Raumes der zyklischen Gruppe mit p Elementen, Dissertation, Wuppertal (1992)
7. J. Klippenstein, V. Snaith: A conjecture of Barratt-Jones-Mahowald concerning framed manifolds having Kervaire invariant one, Topology **27** (1988) 387–392
8. K. Knapp: Connective K-theory of the Im(J)-spectrum, Proc. London Math. Soc. (3) **71** (1995) 175–196
9. K. Knapp: Some applications of K-theory to framed bordism, Habilitationsschrift, Bonn (1979)
10. M. Mahowald: The image of J in the EHP-sequence, Ann. of Math. **116** (1982) 65–112
11. M. Mahowald, R.J. Milgram: Operations which detect Sq^4 in connective K-theory and their applications, Quart. J. Math. Oxford **27** (1976) 415–432
12. H.R. Miller, D.C. Ravenel, W.S. Wilson: Periodic phenomena in the Adams- Novikov spectral sequence, Ann. of Math. **106** (1977) 469–516
13. D.C. Ravenel: The nonexistence of odd primary Arf invariant elements in stable homotopy, Math. Proc. Cambridge Phil. Soc **83** (1978) 429–443
14. D.C. Ravenel: Complex Cobordism and Stable Homotopy of Spheres, Academic Press (1986)
15. P. Selick: A reformulation of the Arf invariant one problem and applications to atomic spaces, Pacific J. of Math. **108** (1983) 431–450
16. K. Shimomura: Novikov‘s Ext^2 at the prime 2, Hiroshima Math. J. **11** (1981) 499–513