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MORAVA K-THEORIES AND LOCALISATION

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Abstract. We study the structure of the categories of $K(n)$-local and $E(n)$-local spectra, using the axiomatic framework developed in earlier work of the authors with John Palmieri. We classify localising and colocalising subcategories, and give characterisations of small, dualisable, and $K(n)$-nilpotent spectra. We give a number of useful extensions to the theory of v_n self maps of finite spectra, and to the theory of Landweber exactness. We show that certain rings of cohomology operations are left Noetherian, and deduce some powerful finiteness results. We study the Picard group of invertible $K(n)$-local spectra, and the problem of grading homotopy groups over it. We prove (as announced by Hopkins and Gross) that the Brown-Comenetz dual of M_nS lies in the Picard group. We give a detailed analysis of some examples when $n = 1$ or 2, and a list of open problems.

Introduction

The stable homotopy category $\mathcal{S}$ is extraordinarily complicated. However, there is a set of approximations to it that are much simpler and closer to algebra. The stable homotopy category is somewhat analogous to the derived category of a ring R, except that R is replaced by the stable sphere S^0. In practice, we always consider the p-local stable homotopy category and the p-local sphere for some prime p, without changing the notation. It is very common to study R-modules using the fields over R, and in recent years there has been much work on studying the stable homotopy category via its fields. These fields are referred to as Morava K-theories, denoted by $K(n)$, and were introduced by Morava in the early 1970's. See [Mor85] for a description of Morava's earlier work.

Associated to the Morava K-theories are various (homotopy) categories of local spectra that are the approximations to the stable homotopy category mentioned above. There is the category $\mathcal{L}$ of spectra local with respect to $K(0) \vee \cdots \vee K(n)$ and the category $\mathcal{K}$ of spectra local with respect to $K(n)$. (We will always have a fixed n with $0 < n < \infty$ in mind in this paper). These categories are themselves stable homotopy categories, in the sense of [HPS95]. The purpose of this paper is to study the structure of these categories. We will show that the category $\mathcal{K}$ is in a certain sense irreducible; it has no nontrivial further localisations. On the other

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hand, there are a number of results (such as the Chromatic Convergence Theorem of Hopkins and Ravenel [Rav92a], or the proof by the same authors that suspension spectra are harmonic [HR92]) which indicate that an understanding of $\mathcal{K}$ for all n and p will give complete information about $\mathcal{S}$. Hopkins' Chromatic Splitting Conjecture [Hov95a], if true, would be a still stronger result in this direction.

The first half of the paper is mostly concerned with issues we need to resolve before beginning our study of $\mathcal{K}$. In Section 1, we define the basic objects of study: the ring spectra $E(n)$, $E=\widehat{E(n)}$ and $K=K(n)$, the categories $\mathcal{L}$ and $\mathcal{K}$ and so on. In Section 2, we study E-cohomology. We prove that E^*X is complete, using a slightly modified notion of completeness which turns out to be more appropriate than the traditional one. We show that a well-known quotient of the ring of operations in E-cohomology is a non-commutative Noetherian local ring. We also show that there are no even degree phantom maps between evenly graded Landweber exact spectra such as E, a result that has been speculated about for a long time. In Section 3, we prove some basic results about the (very simple) category of K-injective spectra. We then turn in Section 4 to the study of generalised Moore spectra, and prove a number of convenient and enlightening extensions to the theory developed by Hopkins and Smith [HS, Rav92a] and Devinatz [Dev92]. In Section 5, we assemble some (mostly well-known) results about the Bousfield classes of a number of spectra that we will need to study. In Section 6, we study the $E(n)$-local category $\mathcal{L}$. We prove some results about nilpotence, and we classify the thick subcategories of small objects, the localising subcategories and the colocalising subcategories.

In the second half of the paper, we concentrate on $\mathcal{K}$. In Section 7, we prove our most basic results about the $K(n)$-local category $\mathcal{K}$. In particular, we prove that it is irreducible, in the sense that the only localising or colocalising subcategories are $\{0\}$ and $\mathcal{K}$ itself. We also study the localisation functor $\widehat{L}=L_K$ and some related functors, describing them in terms of towers of generalised Moore spectra. In Section 8 we study two different notions of finiteness in $\mathcal{K}$, called smallness and dualisability. The (local) sphere is dualisable but not small; some rather unexpected spectra are dualisable, such as the localisation of BG for a finite group G. We prove a number of different characterisations of smallness and dualisability; in particular, we give convenient tests in terms of computable cohomology theories. We also show that dualisable spectra lie in the thick subcategory generated by E, and that $\mathcal{K}$-small spectra lie in the thick subcategory generated by K. In Section 9 we study homology and cohomology theories on $\mathcal{K}$, and prove that both are representable in a suitable sense. In Section 10 we study a version of Brown-Comenetz duality appropriate to the $K(n)$-local setting, and we prove that the Brown-Comenetz dual of the monochromatic sphere is an element of the Picard group. This result was stated in [HG94]. In Section 11, we introduce a natural topology on the groups $[X,Y]$ for X and Y in $\mathcal{K}$, and prove a number of properties. In Section 12, we return to the study of the category $\mathcal{D}$ of dualisable spectra. We prove a nilpotence theorem and an analogue of the Krull-Schmidt theorem, saying that every dualisable spectrum can be written as a wedge of indecomposables in an essentially unique way. We make some remarks about ideals in $\mathcal{D}$, but we have not been able to prove the obvious conjectures about them. In Section 13 we study K-nilpotent spectra, proving a number of interesting characterisations of them. In Section 14 we study the problem of grading homotopy groups over the Picard group of invertible spectra, rather than just over the integers. We have a satisfactory theory for homotopy groups of $\mathcal{K}$-small spectra, but the general case seems less pleasant. We show that the Picard group is profinite in a precise sense, but we do not know if it is finitely generated over the p-adics. Section 15 is devoted to the study of the simplest examples. Even when $n = 1$ and p is odd, there are simple counterexamples to plausible conjectures. We also consider the case $n = 2$ and $p > 3$, showing that the Picard graded homotopy groups of the Moore spectrum are mostly infinite. We conclude the main body of the paper with Section 16, which contains a list of interesting questions that we have been unable to answer, some of them old and others new.

We also have two appendices: the first addresses the sense in which the E-cohomology of a K-local spectrum is complete, as mentioned above, and the second shows that some other interesting localisations of the category of spectra have a rather different behaviour, in that they contain no nonzero small objects at all.

We have chosen to rely on a minimum of algebraic prerequisites; in particular, we say almost nothing about formal groups or the Morava stabiliser groups. We have preferred to use thick subcategory arguments rather than spectral sequences where possible. We have chosen our methods very carefully to avoid having to say anything special when $p = 2$. For this reason we have generally used E rather than K, as E is commutative at all primes (for example).

Our debt to Mike Hopkins will be very obvious to anyone familiar with the subject. We have been heavily influenced by his point of view and a large number of our results were previously known to him. We also thank Matthew Ando, Dan Christensen, Chun-Nip Lee, John Palmieri and Hal Sadofsky for helpful conversations about the subject matter of this paper.

Contents

1. Basic definitions

Fix a prime p and an integer $n > 0$. We shall localise all spectra at p; in particular, we shall write MU for what would normally be called $MU_{(p)}$. We write $\mathcal{S}$ for the category of p-local spectra.

1.1. The spectra $E(n)$, $\widehat{E(n)}$ and $K(n)$.

We next want to define the spectra $E = \widehat{E(n)}$ and $K = K(n)$. It is traditional to do this using the Landweber exact functor theorem and Baas-Sullivan theory. Here we will use the more recent techniques of [EKMM96] instead of the Baas-Sullivan construction.

It is well-known that the integral version of MU has a natural structure as an E_∞ ring spectrum or (essentially equivalently) a commutative S-algebra in the sense of [EKMM96]. It follows from [EKMM96, Theorem VIII.2.2] that the same applies to our p-local version. We can thus use the framework of [EKMM96, Chapters II-III] to define a topological closed model category $\mathcal{M}_{MU}$ of MU-modules. The associated homotopy category (obtained by inverting weak equivalences) is called the derived category of MU-modules and written $\mathcal{D}_{MU}$. It is a stable homotopy category in the sense of [HPS95]. There is a "forgetful" functor $\mathcal{D}_{MU} \longrightarrow \mathcal{S}$ and a left adjoint $MU \wedge (-) \colon \mathcal{S} \longrightarrow \mathcal{D}_{MU}$.

Let $w_k \in \pi_{2(p^k-1)}MU$ be the coefficient of x^{p^k} in the p-series of the universal formal group law over MU_* (so $w_0 = p$) and write $I_n = (w_0, \dots, w_{n-1})$. We can construct an object $w_n^{-1}MU/I_n$ of $\mathcal{D}_{MU}$ by the methods of [EKMM96, Chapter V] (see also [Str96]). Using [EKMM96, Theorem VIII.2.2] again, we can Bousfield-localise MU with respect to $w_n^{-1}MU/I_n$ to get a strictly commutative MU-algebra which we call $\widehat{MU}$. As explained in [GM95a], the homotopy ring of $\widehat{MU}$ is $(w_n^{-1}MU_*)^\wedge_{I_n}$.

Next, consider the graded ring

$$E(n)_* = \mathbf{Z}_{(p)}[v_1, \dots, v_n][v_n^{-1}] \qquad |v_k| = 2(p^k - 1).$$

There is a unique p-typical formal group law over this ring with the property that

$$[p]_F(x) = \exp_F(px) +_F \sum_{0<k\leq n}^{F} v_k x^{p^k}.$$

(Thus we take v_k to be a Hazewinkel generator rather than an Araki generator.) This gives a map $MU_* \longrightarrow E(n)_*$, and one can check that this makes $\mathbf{Z}_{(p)}[v_1, \dots, v_n]$ into the quotient of MU_* by a regular sequence (see [Str96, Proposition 8.15] for details). Moreover, the image of w_k is v_k modulo $I_k = (v_0, \dots, v_{k-1})$. It follows from the results of [EKMM96, Chapter V] that there is an MU-module $E(n) \in \mathcal{D}_{MU}$ with a given map $MU \longrightarrow E(n)$ inducing an isomorphism $\pi_*E(n) = E(n)_*$ of MU_*-modules. It is shown in [Str96] that this is unique up to non-canonical isomorphism under MU, and that it admits a non-canonical associative product. The resulting $E(n)_*$-module structure on $E(n)_*X$ (for any spectrum X) is nonetheless canonical, because it is derived from the MU-module structure of the spectrum $E(n) \wedge X \in \mathcal{D}_{MU}$. When $p > 2$ there is a unique commutative product on $E(n) \in \mathcal{D}_{MU}$, but we avoid using this here so that we can handle all primes uniformly.

We now define

$$E = \widehat{E(n)} = E(n) \wedge_{MU} \widehat{MU}.$$

This is clearly a module over $\widehat{MU}$ with a given map $\widehat{MU} \to E$, and one can check that this gives an isomorphism

$$E_* = (E(n)_*)^\wedge_{I_n} = \mathbf{Z}_p[v_1, v_2, \dots, v_{n-1}, v_n^{\pm 1}]^\wedge_{I_n}.$$

It is again well-defined up to non-canonical isomorphism under $\widehat{MU}$, and it admits a non-canonical associative ring structure. If $p > 2$ then there is a unique commutative product on E as an object of $\mathcal{D}_{\widehat{MU}}$.

Because $E(n)$ is an MU-module under MU, there is a canonical map

$$E(n)_* \otimes_{MU_*} MU_*X \to E(n)_*X.$$

This is of course an isomorphism, by the Landweber Exact Functor Theorem [Lan76]. Similarly, we have an isomorphism

$$E_* \otimes_{MU_*} MU_*X \to E_*X.$$

It follows that the homology theory represented by $E(n)$ is independent of the choice of object $E(n) \in \mathcal{D}_{MU}$ up to *canonical* isomorphism, and thus the underlying spectrum of $E(n)$ is well-defined up to an isomorphism that is canonical mod phantoms (see [HPS95, Section 4] for a discussion of phantoms and representability). We shall show later that the relevant group of phantoms is zero, so as a spectrum $E(n)$ is well-defined up to canonical isomorphism. We shall also show that there is a canonical commutative ring structure on this underlying spectrum. Similar remarks apply to E.

We can also define MU-modules $MU/I_k \in \mathcal{D}_{MU}$ for $0 \le k \le n$ in the evident way, and then define

$$\begin{aligned} E(n)/I_k &= MU/I_k \wedge_{MU} E(n) \\ E/I_k &= MU/I_k \wedge_{MU} E \\ K = K(n) &= E(n)/I_n = E/I_n. \end{aligned}$$

It is clear that $\pi_*(E(n)/I_k) = E(n)_*/I_k$ and so on. In particular we have $K(n)_* = E(n)_*/I_n = E_*/I_n = \mathbf{F}_p[v_n^{\pm 1}]$. These MU-modules admit (non-unique) associative products, so $(E(n)/I_k)_*X$ is canonically a module over $E(n)_*/I_k$. Similar remarks apply to E/I_k.

There are evident cofibrations

$$\Sigma^{2(p^k-1)} E/I_k \xrightarrow{v_k} E/I_k \to E/I_{k+1},$$

and similarly for $E(n)/I_k$.

We also know from [Bak91] that there is an essentially unique A_∞ structure on the spectrum E. It is widely believed that this can be improved to an E_∞ structure, and that the maps $MU \to E$ characterised by Ando [And95] (which do not include the map $MU \to E$ considered above) can be improved to E_∞ maps. Unfortunately, proofs of these things have not yet appeared. Nonetheless, just using the A_∞ structure we can still use the methods of [EKMM96] to define a derived category $\mathcal{D}_E$ of left E-modules. This is a complete and cocomplete triangulated category, with a smash product functor $\wedge\colon \mathcal{S} \times \mathcal{D}_E \to \mathcal{D}_E$.

1.2. **Categories of localised spectra.** We use the following notation.

Notation 1.1.

1. $\mathcal{S}$ is the (homotopy) category of p-local spectra. We write S for S^0.

2. $\mathcal{L} = \mathcal{L}_n$ is the category of $E(n)$-local spectra, and $L = L_n : \mathcal{S} \to \mathcal{L}$ is the localisation functor. The corresponding acyclisation functor is written C, so there is a natural cofibre sequence $CX \to X \to LX$.
3. $\mathcal{M} = \mathcal{M}_n$ is the monochromatic category. This is defined to be the image of the functor $M_n = C_{n-1}L_n \colon \mathcal{S} \to \mathcal{S}$. Note that there is a natural fibration $M_nX \to L_nX \to L_{n-1}X = L_{n-1}L_nX$.
4. $\mathcal{K} = \mathcal{K}_n$ is the category of K-local spectra, and $\widehat{L} = \widehat{L}_n : \mathcal{S} \to \mathcal{K}$ is the localisation functor. The corresponding acyclisation functor is written $\widehat{C}$.
5. $F(m)$ denotes a finite spectrum of type m, and $T(m) = v_m^{-1}F(m)$ is its telescope. Recall from [Rav92a, Chapter V] or [HS] that any two $F(m)$'s generate the same thick subcategory and have the same Bousfield class, so it usually does not matter which one we use. Note also that the Spanier-Whitehead dual of an $F(m)$ is again an $F(m)$.

There are topological closed model categories whose homotopy categories are $\mathcal{S}$, $\mathcal{L}$ and $\mathcal{K}$ [EKMM96, Chapter VIII].

1.3. **Stable homotopy categories.** In this section we collect some basic definitions from the theory of stable homotopy categories developed in [HPS95]. The reader will be familiar with most of these: we assemble them here as a convenient reference. We will not recall the definition of a stable homotopy category, except to say that a stable homotopy category is a triangulated category with a closed symmetric monoidal structure which is compatible with the triangulation and which has a set of generators in an appropriate sense. The symmetric monoidal structure is written $X \wedge Y$ and the closed structure is written $F(X, Y)$. The unit for the smash product is written S.

Definition 1.2. A full subcategory $\mathcal{D}$ of any triangulated category is *thick* if it is closed under retracts, cofibres, and suspensions. That is, $\mathcal{D}$ is thick if both of the following conditions hold.

(a) If $X \vee Y \in \mathcal{D}$, then both X and Y are in $\mathcal{D}$; and
(b) If
$$X \to Y \to Z \to \Sigma X$$
is a cofibre sequence and two of X, Y, and Z are in $\mathcal{D}$, then so is the third.

Certain kinds of thick subcategories come up frequently.

Definition 1.3. Let $\mathcal{C}$ be a thick subcategory of a stable homotopy category $\mathcal{D}$.

(a) $\mathcal{C}$ is a *localising subcategory* if it is closed under arbitrary coproducts.
(b) $\mathcal{C}$ is a *colocalising subcategory* if it is closed under arbitrary products.
(c) $\mathcal{C}$ is an *ideal* if, whenever $X \in \mathcal{D}$ and $Y \in \mathcal{C}$ we have $X \wedge Y \in \mathcal{C}$.
(d) $\mathcal{C}$ is a *coideal* if, whenever $X \in \mathcal{D}$ and $Y \in \mathcal{C}$ we have $F(X, Y) \in \mathcal{C}$.

Whenever the localising subcategory generated by S is all of $\mathcal{D}$, every (co)localising subcategory is a (co)ideal [HPS95, Lemma 1.4.6]. This is true in $\mathcal{S}$, $\mathcal{L}$, and $\mathcal{K}$ (and any other localisation of $\mathcal{S}$).

The ideal generated by a ring object is particularly important.

Definition 1.4. Let $\mathcal{C}$ be a stable homotopy category, and R a ring object in $\mathcal{C}$. We say that an object $X \in \mathcal{C}$ is *R-nilpotent* if it lies in the ideal generated by R.

We now recall some different notions of finiteness in a stable homotopy category. We have replaced "strongly dualisable" by "dualisable" for brevity. Also recall that $DZ = F(Z, S)$ is the usual duality functor.

Definition 1.5. Let $\mathcal{C}$ be a stable homotopy category, and Z an object of $\mathcal{C}$. We say that Z is

(a) *small* if for any collection of objects $\{X_i\}$, the natural map $\bigoplus[Z, X_i] \to [Z, \coprod X_i]$ is an isomorphism.
(b) *F-small* if for any collection of objects $\{X_i\}$, the natural map $\coprod F(Z, X_i) \to F(Z, \coprod X_i)$ is an isomorphism.
(c) *$\mathcal{A}$-finite* (for any family $\mathcal{A}$ of objects of $\mathcal{C}$) if Z lies in the thick subcategory generated by $\mathcal{A}$.
(d) *dualisable* if for any X, the natural map $DZ \wedge X \to F(Z, X)$ is an equivalence.

A triangulated category with a compatible closed symmetric monoidal structure is an *algebraic stable homotopy category* if there is a set $\mathcal{G}$ of small objects such that the localising subcategory generated by $\mathcal{G}$ is the whole category. An algebraic stable homotopy category is called *monogenic* if we can take $\mathcal{G} = \{S\}$.

Finally, we recall that limits and colimits generally do not exist in triangulated categories, but sometimes suitable weak versions do exist. In particular, given a sequence $X_0 \to X_1 \to \dots$, we can form the sequential colimit as the cofibre of the usual self-map of $\bigvee X_i$. We denote this by $\underrightarrow{\operatorname{holim}}\, X_i$, as it is the homotopy colimit of a suitable lift of the sequence to a model category. Similarly, we denote the sequential limit of a sequence $\dots \to X_1 \to X_0$ by $\underleftarrow{\operatorname{holim}}\, X_i$.

In case we have a more complicated functor $F : \mathcal{I} \to \mathcal{C}$ to a stable homotopy category, we say that a weak colimit X of F is a *minimal weak colimit* if the induced map $\underrightarrow{\lim}\, H \circ F \to HX$ is an isomorphism for all homology functors H. We write $X = \underrightarrow{\operatorname{mwlim}}\, F$. Minimal weak colimits are unique in algebraic stable homotopy categories when they exist, and are extremely useful. See [HPS95, Section 2] for details.

2. E theory

In this section we assemble some basic results about E theory. We begin with the fact that E^*X is L-complete in the sense of Appendix A. In Section 2.1 we show that the ring structure on E is canonical by showing there are no phantom maps between evenly graded Landweber exact spectra. We recall the modified Adams spectral sequence briefly in Section 2.2. Finally, we briefly discuss operations in E-theory in Section 2.3.

Proposition 2.1. *If X is a finite spectrum and R is one of E, $E(n)$, E/I_k, $E(n)/I_k$ or $K(n)$ then R_*X is finitely generated over R_*.*

Proof. We first recall that in each case R has an associative ring structure, so that R_*X is a module over R_*. The ring structure is not canonical but the module structure is induced by the MU_*-module structure so it is canonical. In each case R_* is Noetherian and the claim follows easily by induction on the number of cells. □

Proposition 2.2. *For any spectrum X, there is a natural topology on E^0X making it into a profinite (and thus compact Hausdorff) Abelian group. Moreover, if $\Lambda(X)$*

is the category of pairs (Y,u) where Y is finite and $u\colon Y \longrightarrow X$, then E^0X is homeomorphic to $\varprojlim_{(Y,u)\in\Lambda(X)} E^0Y$.

Proof. If X is a finite spectrum then E^*X is a finitely generated module over E^*, and it follows easily that the I_n-adic topology on E^0X is profinite. For an arbitrary spectrum X, define $F^0X = \varprojlim_{(Y,u)\in\Lambda(X)} E^0Y$, with the inverse limit topology. By [HPS95, Proposition 2.3.16], F is a cohomology theory with values in the category of profinite groups and continuous homomorphisms. There is an evident map $E^0X \longrightarrow F^0X$ which is an isomorphism when X is finite (because $(X, 1_X)$ is a terminal object of $\Lambda(X)$ in that case). It follows easily that $E^0X = F^0X$ for all X. $\square$

Corollary 2.3. *For any spectrum X, the module E^*X is L-complete in the sense of Definition A.5.*

Proof. This is immediate when X is finite. It thus follows for general X because the category of L-complete modules is closed under inverse limits (by Theorem A.6). $\square$

Proposition 2.4. *For any spectrum X, the module E^*X is finitely generated over E^* if and only if K^*X is finitely generated over K^*.*

Proof. The cofibration $\Sigma^{2(p^k-1)}E/I_k \xrightarrow{v_k} E/I_k \longrightarrow E/I_{k+1}$ gives a short exact sequence

$$(E/I_k)^*(X)/v_k \rightarrowtail (E/I_{k+1})^*(X) \twoheadrightarrow \operatorname{ann}(v_k, (E/I_k)^{*-2p^k+1}(X)).$$

It follows that if $(E/I_k)^*X$ is finitely generated then the same is true of $(E/I_{k+1})^*X$. Conversely, suppose that $(E/I_{k+1})^*X$ is finitely generated over E^*, and write $M = (E/I_k)^*X$. The above sequence shows that M/v_kM is finitely generated. This means that there is a finitely generated free module F over E^*/I_k and a map $f\colon F \longrightarrow M$ such that the induced map $F/v_kF \longrightarrow M/v_kM$ is surjective. If we let N be the cokernel of f, we conclude that N is an L-complete module over E^* with $N = v_kN$. It follows from Proposition A.8 that $N = 0$, so f is surjective and M is finitely generated.

It follows that $K^*X = (E/I_n)^*X$ is finitely generated if and only if $E^*X = (E/I_0)^*X$ is finitely generated. $\square$

Proposition 2.5. *Let X be a spectrum. Suppose E^*X is pro-free* (*in the sense of Definition A.10*)*. Then $K^*X = (E^*X)/I_n$. Conversely, if K^*X is concentrated in even degrees, then E^*X is pro-free and concentrated in even degrees.*

Proof. The first statement follows from the fact that the sequence $(v_0, \dots, v_{n-1})$ is regular on E^*X, by Theorem A.9. Conversely, suppose that $(E/I_{k+1})^*X$ is concentrated in even degrees. Consider the short exact sequence

$$(E/I_k)^*(X)/v_k \rightarrowtail (E/I_{k+1})^*(X) \twoheadrightarrow \operatorname{ann}(v_k, (E/I_k)^{*-2p^k+1}(X)).$$

It follows that $(E/I_k)^{\text{odd}}(X)/v_k = 0$. As $(E/I_k)^{\text{odd}}(X)$ is L-complete, we conclude from Proposition A.8 that it must be zero. It also follows from the sequence that $\operatorname{ann}(v_k, (E/I_k)^{\text{even}}(X)) = 0$, so that v_k acts injectively on $(E/I_k)^*(X)$. Finally, we also see from the sequence that $(E/I_{k+1})^*(X) = (E/I_k)^*(X)/v_k$.

By an evident induction we conclude that E^*X is concentrated in even degrees, that the sequence $\{v_0, \dots, v_{n-1}\}$ is regular on E^*X and that $K^*(X) =$

$E^*(X)/(v_0, \dots, v_{n-1}) = K^* \otimes_{E^*} E^*X$. It follows from Theorem A.9 that E^*X is pro-free. □

2.1. **Landweber exactness.** We next recall the theory of Landweber exact homology theories, and prove some convenient extensions. Many of the theorems we prove were proved by Franke [Fra92] in the case where π_*M is countable; our methods are a generalisation of his.

Definition 2.6. An MU_*-module M_* is said to be *Landweber exact* if the sequence $(w_0, w_1, \dots)$ is regular on M_*. We write $\mathcal{E}_*$ for the category of Landweber exact modules that are concentrated in even degrees. We also write $\mathcal{E}$ for the category of MU-module spectra M such that $\pi_*(M) \in \mathcal{E}_*$. Maps in $\mathcal{E}$ are MU-module maps. Finally, we write $\mathcal{EF}$ for the category of finite spectra X such that H_*X is free and concentrated in even degrees. We refer to such an X as an *even* finite spectrum. Note that any even finite spectrum has a finite filtration where the filtration quotients are finite wedges of even spheres.

The basic result is as follows.

Theorem 2.7 (Landweber)**.** *If $M_* \in \mathcal{E}_*$ then the functor $M_* \otimes_{MU_*} MU_*X$ is a homology theory. Thus (by Brown representability), there is a spectrum M equipped with a natural isomorphism $M_*X \simeq M_* \otimes_{MU_*} MU_*X$. This M is unique up to isomorphism, and the isomorphism is canonical modulo phantoms.*

Proof. See [Lan76, Theorem 2.6]. □

The following result summarises Proposition 2.20 and Proposition 2.19, which are proved below. It justifies the statements made in Section 1 about the uniqueness of E and $E(n)$ and their ring structures.

Theorem 2.8. *The functor $\pi_* \colon \mathcal{E} \to \mathcal{E}_*$ is an equivalence of categories. The inverse functor sends commutative MU_*-algebras to commutative MU-algebra spectra.* □

It is convenient to introduce a new category $\mathcal{E}'$ at this point; it will follow from the theorem that $\mathcal{E}' = \mathcal{E}$.

Definition 2.9. $\mathcal{E}'$ is the category of spectra M such that M_* is concentrated in even degrees, with a given MU_*-module structure on M_* and a stable natural isomorphism $M_* \otimes_{MU_*} MU_*X \to M_*X$ which is the identity when $X = S$. Morphisms of $\mathcal{E}'$ must preserve the module structure.

The converse of the Landweber exact functor theorem [Rud86] shows that if $M \in \mathcal{E}'$, then $M_* \in \mathcal{E}_*$. Theorem 2.7 says that $\pi_* \colon \mathcal{E}' \to \mathcal{E}_*$ is essentially surjective on objects.

In order to show that π_* is an equivalence of categories, we introduce the following definition.

Definition 2.10. A spectrum X is *evenly generated* if and only if, for every finite spectrum Z and every map $Z \xrightarrow{f} X$, there is an even finite spectrum W and a factorisation $Z \xrightarrow{g} W \xrightarrow{h} X$ of f.

Every even finite spectrum is evenly generated. The collection of evenly generated spectra is closed under even suspensions, coproducts, and retracts, but does not form a thick subcategory. We will see in Proposition 2.18 that evenly generated spectra are closed under the smash product.

Lemma 2.11. *MU is evenly generated.*

Proof. Any map from a finite spectrum to MU factors though a skeleton of MU. Any skeleton of MU is an even finite. □

The following result is essentially due to Hopkins.

Proposition 2.12. *Suppose $M \in \mathcal{E}'$. Then M is evenly generated.*

Proof. Suppose $f : Z \longrightarrow M$ is a map from a finite spectrum to M. Then f is a class in $M^0 Z$. Spanier-Whitehead duality implies that $M^*Z = M^* \otimes_{MU^*} MU^*Z$. We can thus write $f = \sum_{i=1}^m b_i \otimes c_i$ say. As M^* is concentrated in even degrees we see that $|c_i| = -|b_i|$ is even. Each map c_i thus has a factorisation

$$c_i = (Z \xrightarrow{g_i} W_i \xrightarrow{e_i} \Sigma^{|b_i|} MU),$$

where W_i is a skeleton of $\Sigma^{|b_i|}MU$, and so is an even finite. Write $W = W_1 \vee \ldots \vee W_m$, let $g\colon Z \longrightarrow W$ be the map with components g_i and let $h\colon W \longrightarrow M$ be the map with components $b_i \otimes e_i \in M^* \otimes_{MU_*} MU^*W_i = [W_i, M]^*$. This gives the desired factorisation. □

There are several different characterisations of evenly generated spectra.

Proposition 2.13. *The spectrum X is evenly generated if and only if X can be written as the minimal weak colimit* [HPS95, Section 2.2] *of a filtered system $\{M_\alpha\}$ of even finite spectra.*

Proof. First suppose that X can be written as such a minimal weak colimit. Then, by smallness, any map from a finite to X will factor through one of the terms in the minimal weak colimit, and so through an even finite. Thus X is evenly generated.

Conversely, suppose X is evenly generated. We replace $\mathcal{EF}$ by a small skeleton of $\mathcal{EF}$ without change of notation. Let $\Lambda_{\mathcal{EF}}(X)$ be the category of pairs (U, u), where $U \in \mathcal{EF}$ and $u\colon U \longrightarrow X$. Let $\Lambda(X)$ be the category of pairs (W, w), where W is any finite spectrum and $w\colon W \longrightarrow X$. We know from [HPS95, Theorem 4.2.4] that X is the minimal weak colimit of $\Lambda(X)$. It will therefore be enough to show that the obvious inclusion $\Lambda_{\mathcal{EF}}(X) \longrightarrow \Lambda(X)$ is cofinal.

We first show that $\Lambda_{\mathcal{EF}}(X)$, like $\Lambda(X)$, is filtered. Consider two objects (U, u) and (V, v) of $\Lambda_{\mathcal{EF}}(X)$. We need to show that there is an object (W, w) and maps $(U, u) \longrightarrow (W, w) \longleftarrow (V, v)$ in $\Lambda(X)$. Clearly we can take $W = U \vee V$, and let $w\colon W \longrightarrow X$ be the map with components u and v. We also need to show that when $f, g\colon (U, u) \longrightarrow (V, v)$ are two maps in $\Lambda_{\mathcal{EF}}(X)$, there is an object (W, w) and a map $h\colon (V, v) \longrightarrow (W, w)$ with $hf = hg$. To see this, let W' be the cofibre of $f - g$ and $h'\colon V \longrightarrow W'$ the evident map. We have $vf = u = vg$ so $v(f - g) = 0$ so $v = w'h'$ for some $w'\colon W' \longrightarrow X$. Because X is evenly generated, the map w' factors as $W' \xrightarrow{k} W \xrightarrow{w} X$ for some even finite W. We can evidently take $h = kh'$.

It is now easy to check that the inclusion $\Lambda_{\mathcal{EF}}(X) \longrightarrow \Lambda(X)$ is cofinal, as required. □

Corollary 2.14. *A spectrum X is evenly generated if and only if there is a cofibre sequence*

$$P \longrightarrow Q \longrightarrow X \xrightarrow{\delta} \Sigma P,$$

where P and Q are retracts of wedges of even finite spectra and δ is a phantom map.

Proof. If there is such a cofibre sequence, then there is a short exact sequence

$$[Z, P] \rightarrowtail [Z, Q] \twoheadrightarrow [Z, X]$$

for all finite Z, since δ is phantom. It follows easily that X is evenly generated. The converse follows from the Proposition and the construction in [CS96, Proposition 4.6]. □

Corollary 2.15. *Suppose X is evenly generated and Y is a spectrum such that Y_* is concentrated in even degrees. Let $\mathcal{P}^*(X, Y)$ be the graded group of phantom maps from X to Y. Then*

$$\mathcal{P}^{2k}(X, Y) = 0$$
$$\mathcal{P}^{2k-1}(X, Y) = [X, Y]^{2k-1}.$$

In particular, this holds if $X, Y \in \mathcal{E}'$.

Proof. Proceeding by induction, one proves easily that Y^*W is concentrated in even degrees for all even finite W. In particular, in the cofibre sequence of Corollary 2.14, Y^*Q and Y^*P are concentrated in even degrees. Any phantom map in $Y^{2k}X$ would be in the image of $Y^{2k-1}P = 0$, so there are no even phantoms. On the other hand, every class in $Y^{2k+1}X$ is in the image of $Y^{2k}P$, so is phantom. □

Although this characterisation of phantoms is the main fact that we need, we will prove something rather sharper.

Proposition 2.16. *Suppose R is a ring spectrum, S is an R-module spectrum, and X is evenly generated. Suppose as well that R_* and S_* are concentrated in even degrees. Then R_*X is flat, has projective dimension at most 1, and is concentrated in even degrees. Furthermore, we have*

$$S_*X = S_* \otimes_{R_*} R_*X$$
$$S^{2k}X = \operatorname{Hom}^{2k}_{R_*}(R_*X, S_*)$$
$$S^{2k-1}X = \operatorname{Ext}^{1,2k}_{R_*}(R_*X, S_*).$$

Proof. Choose a cofibre sequence $P \longrightarrow Q \longrightarrow X \xrightarrow{\delta} \Sigma P$ as in Proposition 2.14. If W is an even finite, it is easy to see by induction on the (even) cells that R_*W is free over R_* and concentrated in even degrees. Since $R_*X = \varinjlim_{\Lambda_{\mathcal{E}\mathcal{F}}(X)} R_*W$, we see that R_*X is also concentrated in even degrees and is a filtered colimit of free modules, so is flat. One can also check by induction on the cells, using the fact that S_* is evenly graded, that the natural map $S_* \otimes_{R_*} R_*W \longrightarrow S_*W$ is an isomorphism for W an even finite. Taking colimits, we find that $S_* \otimes_{R_*} R_*X \longrightarrow S_*X$ is also an isomorphism.

Similarly, one can check by induction on the cells that when W is an even finite, the natural map $[W, S]^* \longrightarrow \operatorname{Hom}_{R_*}(R_*W, S_*)$ is an isomorphism. As P and Q are retracts of wedges of such W, we see that R_*P and R_*Q are projective over R_*

and that $[Q, S]^* = \mathrm{Hom}_{R_*}(R_*Q, S_*)$ and $[P, S]^* = \mathrm{Hom}_{R_*}(R_*P, S_*)$. In particular, these groups are concentrated in even degrees. Because δ is phantom, we have a short exact sequence

$$R_*P \rightarrowtail R_*Q \twoheadrightarrow R_*X,$$

which is a projective resolution of R_*X. We now apply the functor $[-, S]^*$ to the cofibration $P \longrightarrow Q \longrightarrow X$ to get an exact sequence

$$\mathrm{Hom}^{*+1}_{R_*}(R_*Q, S_*) \longrightarrow \mathrm{Hom}^{*+1}_{R_*}(R_*P, S_*) \longrightarrow [X, S]^* \longrightarrow \\ \mathrm{Hom}^*_{R_*}(R_*Q, S_*) \longrightarrow \mathrm{Hom}^*_{R_*}(R_*P, S_*),$$

and thus a short exact sequence

$$\mathrm{Ext}^{1,*+1}_{R_*}(R_*X, S_*) \rightarrowtail [X, S]^* \twoheadrightarrow \mathrm{Hom}^*_{R_*}(R_*X, S_*).$$

The first term is concentrated in odd degrees and the last one in even degrees, so the sequence splits uniquely. □

Note that this proposition implies that spectra such as $P(n)$ and $K(n)$ are not evenly generated for $n > 0$. Indeed, $P(n)_*P(n)$ and $K(n)_*K(n)$ both contain a Bockstein element in degree 1. Similarly, $H\mathbf{Z}$ and $H\mathbf{F}_p$ are not evenly generated. In fact, the only MU-module spectra that are evenly generated are the even Landweber exact MU-module spectra. One can prove this by using the fact that $X \wedge F(n)$ is a retract of $(MU \wedge F(n)) \wedge X$ and is therefore evenly graded. Applying this to the spectra S/I of Section 4 shows that v_n acts injectively on X_*/I.

Corollary 2.17. *Let M and N be MU-module spectra in $\mathcal{E}'$. Then*

$$\begin{aligned}
[M, N]^{2k} &= \mathrm{Hom}^{2k}_{MU_*MU}(MU_*M, MU_*N) \\
&= \mathrm{Hom}^{2k}_{MU_*}(MU_*M, N_*) \\
[M, N]^{2k-1} &= \mathrm{Ext}^{1,2k}_{MU_*MU}(MU_*M, MU_*N) \\
&= \mathrm{Ext}^{1,2k}_{MU_*}(MU_*M, N_*).
\end{aligned}$$

If $s > 1$ then $\mathrm{Ext}^{s,*}_{MU_*MU}(MU_*M, MU_*N) = 0$.

Proof. By Landweber exactness we have $X_*N = X_*MU \otimes_{MU_*} N_*$ and in particular $MU_*N = MU_*MU \otimes_{MU_*} N_*$. This is an extended comodule, so for any comodule C_* we have

$$\mathrm{Hom}_{MU_*MU}(C_*, MU_*N) = \mathrm{Hom}_{MU_*}(C_*, N_*).$$

More generally, if we resolve N_* by injective MU_*-modules and apply the functor $MU_*MU \otimes_{MU_*} (-)$ we get a resolution of MU_*N by injective comodules. Using this it is not hard to check that

$$\mathrm{Ext}^{s,*}_{MU_*MU}(C_*, MU_*N) = \mathrm{Ext}^{s,*}_{MU_*}(C_*, N_*)$$

for all s. The rest of the corollary is proved in Proposition 2.16. □

Proposition 2.18. *Suppose X and Y are evenly generated. Then $X \wedge Y$ is evenly generated.*

Proof. It is clear that if W and Z are even finites, so is $W \wedge Z$. Now suppose X is evenly generated and W is an even finite. Then $X \wedge W$ is the minimal weak colimit of the functor on $\Lambda_{\mathcal{EF}}(X)$ which takes (U, u) to $U \wedge W$. Thus $X \wedge W$ is evenly generated. Now suppose Y is also evenly generated. Choose a cofibre sequence $P \longrightarrow Q \longrightarrow Y \xrightarrow{\delta} \Sigma P$ as in Corollary 2.14. Then $X \wedge P$ and $X \wedge Q$ are evenly generated, as they are retracts of wedges of terms of the form $X \wedge W$. Furthermore, $1_X \wedge \delta$ is still a phantom map (using the characterisation of phantom maps as those maps not seen by any homology theory). It follows that $X \wedge Y$ is evenly generated. $\square$

In fact, it is also true that $\mathcal{E}'$ is closed under the smash product, though we do not need this.

Proposition 2.19. *If $M \in \mathcal{E}'$ then M admits a canonical structure as an MU-module spectrum (in the traditional homotopical sense). If N is another spectrum in $\mathcal{E}'$ then the degree-zero MU-module maps $M \longrightarrow N$ biject with the MU_*-module maps $M_* \longrightarrow N_*$.*

Proof. For any spectrum X we have a natural map

$$\epsilon\colon MU_*(MU \wedge X) = MU_*MU \otimes_{MU_*} MU_*X \longrightarrow MU_*X$$

of left MU_*-modules, and thus a natural map

$$(M \wedge MU)_*X = M_*(MU \wedge X) = M_* \otimes_{MU_*} MU_*(MU \wedge X) \longrightarrow \\ M_* \otimes_{MU_*} MU_*X = M_*X.$$

By Brown representability, we get a map $\nu\colon M \wedge MU \longrightarrow M$ which is unique mod phantoms. It is not hard to check that this is associative and unital mod phantoms. However, Proposition 2.18 shows that $M \wedge MU$ and $M \wedge MU \wedge MU$ are evenly generated, so Corollary 2.15 tells us that there are no degree-zero phantom maps $M \wedge MU \longrightarrow M$ or $M \wedge MU \wedge MU \longrightarrow M$. Thus ν is unique, associative and unital, and $M \in \mathcal{E}$.

Now let N be another spectrum in $\mathcal{E}'$, and consider the following diagram.

$$\begin{tikzcd} {[M,N]^{MU}} \arrow[r, "\pi_*"] \arrow[d, tail] & \mathrm{Hom}_{MU_*}(M_*, N_*) \arrow[d, tail, "f"] \\ {[M,N]} \arrow[r, "\simeq", "MU_*"'] & \mathrm{Hom}_{MU_*MU}(MU_*M, MU_*N). \end{tikzcd}$$

Here $[M,N]^{MU}$ denotes the group of MU-module maps, and all the groups are groups of degree-zero maps. The bottom map is an isomorphism by Proposition 2.17. The map f sends a map $u\colon M_* \longrightarrow N_*$ to

$$1 \otimes u\colon MU_*MU \otimes_{MU_*} M_* = MU_*M \longrightarrow MU_*N = MU_*MU \otimes_{MU_*} N_*.$$

It is easy to check that this is injective and that the diagram commutes. It follows that π_* is injective. Now suppose we have a map $v\colon M_* \longrightarrow N_*$ of MU_*-modules. We then have a unique map $u\colon M \longrightarrow N$ such that $MU_*u = f(v)$. A diagram chase shows that u is a map of MU-module spectra (up to a phantom term which is zero as usual), and $f(\pi_*(u)) = f(v)$ so $\pi_*(u) = v$. Thus π_* is an isomorphism. $\square$

Proposition 2.20. *If $A \in \mathcal{E}$ and $\pi_*(A)$ is a commutative MU_*-algebra then there is a unique product on A making it into a commutative MU-algebra spectrum.*

Proof. For any X and Y we have a pairing $MU_*X \otimes_{MU_*} MU_*Y \longrightarrow MU_*(X \wedge Y)$ and thus a pairing

$$A_*(X) \otimes A_*(Y) = A_* \otimes_{MU_*} MU_*X \otimes A_* \otimes_{MU_*} MU_*Y \longrightarrow \\ A_* \otimes MU_*(X \wedge Y) = A_*(X \wedge Y).$$

This is easily seen to be commutative, associative and unital. By writing A as a minimal weak colimit of finite spectra A_α and taking $X = DA_\alpha$, $Y = DA_\beta$ we construct a map $A \wedge A \longrightarrow A$ which is well-defined, commutative, associative and unital up to phantoms. However, all the relevant phantom groups vanish by Proposition 2.18 and Corollary 2.15. □

Proposition 2.21. *If A is a Landweber exact ring spectrum and M is an A-module spectrum then there are universal coefficient spectral sequences of A_*-modules*

$$\mathrm{Ext}^{s,t}_{A_*}(A_*X, M_*) \Longrightarrow M^{t+s}X$$
$$\mathrm{Tor}^{A_*}_{s,t}(M_*, A_*X) \Longrightarrow M_{t+s}X.$$

Proof. The first spectral sequence follows from Proposition 2.12, Proposition 2.13 and [Ada74, Theorem 13.6]. The second is constructed by the same methods. □

2.2. **The E-based Adams spectral sequence.** We next briefly recall an approach to the E-based Adams spectral sequence that we learnt from Mike Hopkins, which is explained in more detail in [Dev96b]. This approach is also used by Franke [Fra96], who attributes it to Brinkmann. Let A be an even Landweber exact commutative ring spectrum; in our applications, A will be E or $E(n)$. By Proposition 2.16, A_*A is flat as a left module over A_*. Similarly, it is flat as a right module. It follows in the usual way that it is a Hopf algebroid, so we can think about comodules over A_*A. If I_* is an injective module over A_* then the extended comodule $A_*A \otimes_{A_*} I_*$ is injective. It follows that there are enough injective comodules, and that they can be used to define Ext groups. If J_* is an injective comodule then Brown representability gives a spectrum W such that $[X, W] = \mathrm{Hom}_{A_*A}(A_*X, J_*)$ for all X. The identity map of W corresponds to a map $A_*W \longrightarrow J_*$, which we claim is an isomorphism. Indeed, when $X \in \mathcal{EF}$ we know that A_*X is free over A_* and using duality we find that

$$X_*W = [DX, W] = \mathrm{Hom}_{A_*A}(A_*DX, J_*) = \mathrm{Hom}_{A_*A}(A_*, A_*X \otimes_{A_*} J_*).$$

By Proposition 2.13, we can write $A = \underrightarrow{\mathrm{mwlim}}\, A_\alpha$ for $A_\alpha \in \mathcal{EF}$. By taking $X = A_\alpha$ and passing to the limit we find that

$$A_*W = \mathrm{Hom}_{A_*A}(A_*, A_*A \otimes_{A_*} J_*) = J_*.$$

There is some inconsistency in the literature about what to call spectra such as W. We will use the following terminology.

Definition 2.22. Let A be a ring spectrum. We say that a spectrum X is *A-injective* if it is a retract of $A \wedge Y$ for some Y. Suppose in addition that A_*A is flat as a module over A_*. We say that X is *strongly A-injective* if A_*X is an injective comodule and the natural map $[Z, X] \longrightarrow \mathrm{Hom}_{A_*A}(A_*Z, A_*X)$ is an isomorphism for all Z. Note that if X is A-injective then A_*X is injective relative to A_*-split exact sequences, but not necessarily absolutely injective.

If $X = X_0$ is any spectrum then we can embed A_*X_0 in an injective comodule J_* and define $W = W_0$ as above. We then have a map $X_0 \to W_0$, and we let X_1 be the fibre. Continuing in the obvious way, we get a tower

$$X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \dots.$$

For any spectrum Y we can apply the functor $[Y, -]$ to get a spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{A_*A}(A_*Y, A_*X) \Longrightarrow [Y, L_AX]_{t-s}.$$

We call this the *modified Adams spectral sequence (MASS)* . It need not converge without additional assumptions. We will prove a convergence result in Proposition 6.5.

We will also have occasion to use the (unmodified) Adams spectral sequence. For this, we choose a complex $X \to I_0 \to I_1 \to \dots$ of E-injective spectra which becomes a split exact sequence after applying the functor $E \wedge (-)$. Such a complex is unique up to chain homotopy equivalence under X. It can be converted into a tower $X = X_0 \leftarrow X_1 \leftarrow \dots$ just as above, and by applying $[Y, -]$ we get a spectral sequence, called the *Adams spectral sequence.* If X is E-nilpotent then this converges to $[Y, X]$. If E_*Y is projective over E_* then the modified and unmodified Adams spectral sequences coincide from the E_2 page onwards [Dev96b], and in particular the E_2 page can be identified as an Ext group.

2.3. **Operations in E theory.** We now turn to the study of operations in E-theory and K-theory. There is a well-known connection between this and the study of the Morava stabiliser group, but no really adequate account of this. For this and a variety of technical reasons we have chosen to use a more traditional approach.

Write $\widehat{\Sigma}^* = E^*E$ for the (non-commutative) ring of operations in E-cohomology. Note that $K^*E = \mathrm{Hom}(K_*E, K_*)$ (by Proposition 2.16 or by the general theory of modules over a field spectrum) and $K_*E = K_*BP \otimes_{BP_*} E_*$ because E_* is Landweber exact. Because I_n is an invariant ideal, it is easy to check that K_*E is the same as the ring $\Sigma_* = \Sigma(n)_*$ studied in [Rav86, Chapter VI]. We thus have

$$K_*E = \Sigma_* = \mathbf{F}_p[t_k \mid k > 0]/(t_k^{p^n} - v_n^{p^k-1} t_k).$$

Here $|t_k| = 2(p^k - 1)$, so K_*E and $\Sigma^* = K^*E$ are in even degrees. It follows from Proposition 2.5 that $\widehat{\Sigma}^* = E^*E$ is pro-free, and that $\Sigma^* = \widehat{\Sigma}^*/I_n\widehat{\Sigma}^*$. Moreover, because I_n is an invariant ideal in BP_* one can check that $I_n\widehat{\Sigma}^* = \widehat{\Sigma}^* I_n$, and thus that Σ^* is a quotient ring of $\widehat{\Sigma}^*$.

Our main result is as follows.

Theorem 2.23. *The ring Σ^* is left Noetherian in the graded sense.*

It is also true that $\widehat{\Sigma}^*$ and K^*K are Noetherian, but we do not need this so we will not prove it.

As the theory of non-commutative Noetherian rings is less familiar than the commutative version, we start with some elementary remarks. Let R be a possibly non-commutative ring. Unless otherwise specified, we shall take "ideal" to mean "left ideal" and "module" to mean "left module". As in the commutative case, one checks easily that the following are equivalent:

(a) Every ideal $J \leq R$ is finitely generated.
(b) Every ascending chain $J_0 \leq J_1 \leq \dots$ of ideals is eventually constant.
(c) Any submodule of a finitely generated module over R is finitely generated.

(d) Every ascending chain of submodules of a finitely generated module over R is eventually constant.

If so, we say that R is (left) Noetherian. If R is a graded ring and all ideals and modules are required to be homogeneous, then the corresponding conditions are again equivalent; if they hold, we say that R is Noetherian in the graded sense.

Lemma 2.24. *Let $R \to S$ be a map of rings such that S is finitely generated and free both as a left R-module and a right R-module. Then S is left Noetherian if and only if R is left Noetherian. Similarly for the graded case.*

Proof. Suppose that R is Noetherian. Then any ascending chain of ideals in S is a chain of R-submodules of a finitely generated R-module, and thus eventually constant.

Conversely, suppose that S is Noetherian, and that $S = \bigoplus_{i=0}^{d-1} a_i R$. Then the map

$$J \mapsto SJ = \bigoplus_i a_i J \simeq \bigoplus_i J$$

embeds the lattice of ideals in R into the Noetherian lattice of ideals in S, so R is Noetherian. □

Lemma 2.25. *Let R^* be a graded algebra over K^* (so that K^* is central in R^*). Write $\overline{R} = R^*/(v_n - 1)$ and $\pi \colon R^* \to \overline{R}$ for the projection map. If $\overline{R}$ is Noetherian, then R^* is Noetherian in the graded sense.*

Proof. It is enough to show that the map $J \mapsto \pi J$ embeds the lattice of homogeneous ideals of R^* into the Noetherian lattice of ideals in $\overline{R}$, and thus enough to show that the set of homogeneous elements in $\pi^{-1}\pi J$ is just J. As R^*/J is a graded module over the graded field K^*, it is free, generated by elements e_i of degree d_i say. Suppose that $a \in R^*$ is homogeneous of degree d; then $a = \sum_i a_i v_n^{(d-d_i)/|v_n|} e_i \pmod{J}$ where $a_i \in \mathbf{F}_p$, and a_i is zero if the indicated exponent of v_n is not an integer. If $\pi(a) \in J$ then $\sum_i a_i \pi(e_i) = 0$, but it is clear that $\{\pi(e_i)\}$ is a basis for $\overline{R}/\pi J$, so that $a_i = 0$ for all i and thus $a \in J$. □

Lemma 2.26. *Let R be ring, and $\{I_s\}$ a decreasing filtration such that $I_0 = R$ and $I_s I_t \leq I_{s+t}$. Suppose that R/I_s is a finite set for all s, that $R = \varprojlim_s R/I_s$, and that the associated graded ring $R' = E_0 R = \prod_s I_s/I_{s+1}$ is Noetherian. Then R is Noetherian.*

Proof. Let J be a left ideal in R. Then $J' = \prod_s (J \cap I_s)/(J \cap I_{s+1})$ is a left ideal in R'. It is thus finitely generated, so there are elements $a_i \in J \cap I_{d_i}$ (for $i = 1, \dots, m$ say) whose images generate J'. This means that for any element $a \in J \cap I_s$ there are elements b_i such that $a = \sum_i b_i a_i \pmod{J \cap I_{s+1}}$. In other words, if $K \leq J$ is the ideal generated by $\{a_1, \dots, a_m\}$, then $J \cap I_s \leq K + J \cap I_{s+1}$. It follows easily that $J = J \cap I_0$ is contained in $\bigcap_s (K + I_s)$, which is the closure of K in the evident topology given by the ideals I_s. On the other hand, as R/I_s is finite, we see that R is I-adically compact and Hausdorff. As K is the image of an evident continuous map $R^m \to R$, we see that K is compact and thus closed. It follows that $J = K = (a_1, \dots, a_m)$, which is finitely generated as required. □

We next recall that for each $k > 0$, the ideal $(t_j \mid 0 < j < k) \lhd \Sigma_*$ is a Hopf ideal, so that $\Sigma(k)_* = \Sigma_*/(t_j \mid 0 < j < k)$ is a Hopf algebra, and $\Sigma(k)^*$ is a quotient

Hopf algebra of Σ^*. We also write $S = \Sigma_*/(v_n - 1)$ and $S(k) = \Sigma(k)_*/(v_n - 1)$. We write $S^* = \mathrm{Hom}(S, \mathbf{F}_p)$ and $S(k)^* = \mathrm{Hom}(S(k), \mathbf{F}_p)$. Note that Ravenel [Rav86] calls these objects $\Sigma(n,k)_*$, $S(n,k)$ and so on.

Proposition 2.27. *If $k > pn/(p-1)$ then the Hopf algebra $S(k)^*$ can be filtered so that the associated graded ring is a commutative formal power series algebra over $\mathbf{F}_p$ on n^2 generators.*

Proof. In this proof, all theorem numbers and so on refer to Ravenel's book [Rav86]. Our proposition is essentially Theorem 6.3.3. That theorem appears to apply to S rather than S^*, but this is a typo; this becomes clear if we read the preceding paragraph. Some modifications are necessary to replace S^* by $S(k)^*$, and anyway Ravenel does not give an explicit proof of his theorem, so we will fill in some details.

The Hopf algebra filtration of S given by Theorem 6.3.1 clearly induces a filtration on $S(k)$. It is easy to see that

$$E^0 S(k) = T[t_{ij} \mid i \geq k, j \in \mathbf{Z}/n]$$

as rings, where $T[t] = \mathbf{F}_p[t]/t^p$ and t_{ij} corresponds to $t_i^{p^j}$. There is therefore an automorphism F on $E^0 S(k)$ that takes $t_{i,j}$ to $t_{i,j+1}$ which has order n. Moreover, this is a connected graded Hopf algebra (using the grading coming from the filtration, so that the degree of t_{ij} is the integer $d_{n,i}$ of Theorem 6.3.1). The coproduct is given by Theorem 4.3.34, which says that $\Delta(t_{ij})$ is the sum of the elements in a certain unordered list $\overline{\Delta}_{ij}$. These lists are used in such a way that we may ignore any terms which have filtration less than that of t_{ij}. We start with the list

$$\overline{M}_{ij} = \{t_{ij} \otimes 1, 1 \otimes t_{ij}\}.$$

(This comes from Lemma 4.3.32, using the fact that $k > pn/(p-1)$.) We also recall the Witt polynomials w_J defined in Lemma 4.3.8. We are working modulo p so we have $w_J = w_{|J|}^{p^{\|J\| - |J|}}$. In $E^0 S(k)$ we must interpret this as $w_J = F^{\|J\| - |J|} w_{|J|}$. We have set $v_n = 1$ and killed all other v's, so $v_J = 0$ unless J has the form $J = J_r = (n, \dots, n)$ (with r terms), in which case $v_J = 1$. With these observations, Lemma 4.3.33 becomes

$$\overline{\Delta}_{ij} = \overline{M}_{ij} \cup \{F^{-r} w_r(\overline{\Delta}_{i-nr,j}) \mid r > 0\}.$$

As $\overline{M}_{ij}$ is invariant under the twist map, we see that the same holds for $\overline{\Delta}_{ij}$, and thus that $E^0 S(k)$ is cocommutative. We also see that

$$\Delta(t_{ij}) = t_{ij} \otimes 1 + 1 \otimes t_{ij} + w_1(t_{i-n,j-1} \otimes 1, 1 \otimes t_{i-n,j-1}) \pmod{t_{rs} \mid r < i - n}.$$

Here w_1 is given by

$$w_1(x_1, x_2, \dots) = (\sum_t x_t^p - (\sum_t x_t)^p)/p.$$

It is not hard to conclude that the Verschiebung is

$$V(t_{ij}) = t_{i-n,j-1} \pmod{t_{rs} \mid r < i - n}.$$

We also observe that the degree of t_{ij} is less than that of t_{kl} whenever $i < k$ (this follows easily from the definition in Theorem 6.3.1). It follows by induction on i that each t_{ij} lies in the image of V, so that V is surjective.

We now dualise, and conclude that $E_0 S(k)^*$ is a bicommutative connected graded Hopf algebra for which the Frobenius map is injective. We can thus apply the Borel structure theory [Spa66, Section 5.8][Bor53] to conclude that $E_0 S(k)^*$ is a formal power series algebra. One can check that there must be n^2 generators, corresponding to the $t_{k+i,j}$ for $0 \leq i < n$. □

We can now prove as promised that Σ^* is Noetherian.

Proof of Theorem 2.23. According to Lemma 2.25, it is enough to check that S^* is Noetherian. By Lemma 2.26, it is enough to check that E_0S^* is Noetherian. We have an extension of Hopf algebras

$$S'(k) = \mathbf{F}_p[t_1, \dots, t_{k-1}]/(t_j^{p^n} - t_j) \rightarrowtail S \twoheadrightarrow S(k),$$

and thus an injective map of connected graded Hopf algebras $E_0S(k)^* \rightarrowtail E_0S^*$. It follows from the Milnor-Moore theorem (the dual of [Rav86, Corollary A1.1.20]) that E_0S^* is a free (as a left or right module) over $E_0S(k)^*$. The rank is just $\dim(S'(k)) < \infty$. Thus, by Lemma 2.24, it is enough to check that $E^0S(k)^*$ is Noetherian, and this follows from Proposition 2.27. □

We next show that Σ^* is local in a suitable sense. Let I be the kernel of the augmentation map $\Sigma^* \longrightarrow K^*$.

Proposition 2.28. $\Sigma^* = \varprojlim_k \Sigma^*/I^k$.

Proof. This follows easily from the fact [Rav86, Theorem 6.3.3] that S^* has a filtration whose associated graded ring is a graded connected Hopf algebra of finite type. □

This gives us a version of Nakayama's lemma.

Corollary 2.29. *Let M be a finitely generated graded module over Σ^*. If $IM = M$ then $M = 0$.*

Proof. If $M = \Sigma^*m$ is cyclic and $IM = M$, then $m = am$ for some $a \in I$. This means that $(1-a)m = 0$, but the sum $\sum_k a^k$ converges to an inverse for $(1-a)$, so $M = 0$ as required.

More generally, suppose $M = \Sigma^*\{m_1, \dots, m_k\}$ and $IM = M$. Write $M' = \Sigma^*\{m_1, \dots, m_{k-1}\}$, so that $N = M/M'$ is cyclic and $IN = N$. It follows that $N = 0$, so $M = M'$, and the claim follows by induction on k. □

This means that the usual theory of minimal projective resolutions applies.

Corollary 2.30. *Let M be a finitely generated graded module over Σ^*. Then M has a resolution $P_* \longrightarrow M$ by finitely generated free modules over Σ^*, such that the maps $P_k/IP_k \longrightarrow P_{k-1}/IP_{k-1}$ are zero. This is a retract of any other projective resolution, and is unique up to non-canonical isomorphism.*

Proof. Suppose that the first k stages

$$M \longleftarrow P_0 \longleftarrow \dots \longleftarrow P_{k-1}$$

have been constructed. Let N be the kernel of the differential $P_{k-1} \longrightarrow P_{k-2}$. As P_{k-1} is finitely generated and Σ^* is Noetherian, we see that N is finitely generated. We can thus choose finitely many elements $n_1, \dots, n_r$ in N giving a basis for N/IN over K^*. Let P_k be a direct sum of r copies of Σ^* (suitably shifted in degree), and let $P_k \longrightarrow N \longrightarrow P_{k-1}$ be the obvious map. We leave it to the reader to check that this does the job. □

3. K-INJECTIVE SPECTRA

We say that a spectrum X is *K-injective* if it is equivalent to a wedge of suspensions of copies of K. We recall a number of facts about such spectra, and prove a few more.

Remark 3.1. It will follow from the results below that this is consistent with Definition 2.22.

We start by considering K-module spectra. If $p > 2$ then K is commutative, and if $p = 2$ it has a canonical antiautomorphism χ with $\chi^2 = 1$ (see [Nas96]), so we can convert freely between left and right modules.

Proposition 3.2. *Suppose that R is a ring spectrum, M is an R-module spectrum, and X is arbitrary. Then there are natural R-module structures on $M \wedge X$, $F(X, M)$ and $F(M, X)$. Moreover, if $R = K$, then M is K-injective.*

Proof. For the first part, let $\mu\colon R \wedge M \longrightarrow M$ be the R-module structure map. We can make $M \wedge X$ into a R-module with structure map $\mu \wedge 1_X$. We can use the following composite as a structure map for $F(X, M)$:

$$R \wedge F(X, M) = F(S, R) \wedge F(X, M) \xrightarrow{\wedge} F(X, R \wedge M) \xrightarrow{\mu_*} F(X, M).$$

Finally, the structure map $R \wedge F(M, X) \longrightarrow F(M, X)$ is adjoint to the following composite:

$$M \wedge R \wedge F(M, X) \xrightarrow{\tau\wedge 1} R \wedge M \wedge F(M, X) \xrightarrow{\mu\wedge 1} M \wedge F(M, X) \xrightarrow{\text{eval}} X.$$

For the last part, note that K_* is a graded field, so M_* is necessarily a free module over K_*, say $M_* = K_*\{e_i \mid i \in I\}$. Each map $e_i\colon S^{|e_i|} \longrightarrow M$ gives a K-module map $\Sigma^{|e_i|}K \longrightarrow M$, so we have a map $\bigvee_i \Sigma^{|e_i|}K \longrightarrow M$ which is an equivalence by construction. □

The following proposition (most of which appears in [HS]) summarises the main facts that we need.

Proposition 3.3.

(a) *For any spectrum X, the smash product $K \wedge X$ is K-injective.*
(b) *A spectrum X is K-injective if and only if it admits a K-module structure.*
(c) *Any retract of a K-injective spectrum is K-injective.*
(d) *If X is K-injective, then π_*X has a natural structure as a K_*-module. Any map $f : X \longrightarrow Y$ of K-injective spectra induces a K_*-linear map $f_* : \pi_*X \longrightarrow \pi_*Y$.*
(e) *If $f : X \longrightarrow Y$ is a map of K-injective spectra such that f_* is a split monomorphism of K_*-modules, then f is a split monomorphism and the cofibre of f is K-injective. If f is compatible with given module structures on X and Y, then the splitting may also be chosen compatibly. Similarly for epimorphisms.*
(f) *If $\{X_i\}$ is a family of K-injective spectra, then the natural map $\bigvee_i X_i \longrightarrow \prod_i X_i$ is a split monomorphism.*

Proof. (a) $K \wedge X$ is clearly a K-module, and thus K-injective by Proposition 3.2.
(b) It is clear that a K-injective spectrum admits a module structure; the converse is just Proposition 3.2 again.
(c) This is proved after (d).

(d) Any choice of K-module structure on X gives a K_*-module structure on $\pi_* X$. Suppose that we have two different module structures, given by maps $\alpha, \beta\colon K \wedge X \longrightarrow X$. We then have a map

$$\gamma = (K \wedge K \wedge X \xrightarrow{1\wedge\alpha} K \wedge X \xrightarrow{\beta} X).$$

The two different v_n-multiplications are given by composing γ with the two maps

$$\alpha' = \eta \wedge v_n \wedge 1\colon \Sigma^{|v_n|} X \longrightarrow K \wedge K \wedge X$$
$$\beta' = v_n \wedge \eta \wedge 1\colon \Sigma^{|v_n|} X \longrightarrow K \wedge K \wedge X$$

and applying π_*. If $X = K$ then a well-known calculation shows that $\pi_*(\alpha') = \pi_*(\beta')$, so this remains true for any K-injective spectrum X. It follows that the two K_*-module structures coincide.

Before proving the rest of (d), we need to prove a part of (f). Namely, given a set I and integers d_i for $i \in I$, we show that the natural map $\bigvee_i \Sigma^{d_i} K \longrightarrow \prod_i \Sigma^{d_i} K$ is a split monomorphism. Note that this map is a map of K-module spectra, so induces the usual monomorphism of K_*-modules $\bigoplus_i \Sigma^{d_i} K_* \longrightarrow \prod_i \Sigma^{d_i} K_*$. Extend the usual basis $\{e_i\}$of $\bigoplus_i \Sigma^{d_i} K_*$ to a basis of $\prod_i \Sigma^{d_i} K_*$ by adding new basis elements f_j for $j \in J$. Just as in the proof of Proposition 3.2, we get an equivalence $\bigvee_i \Sigma^{d_i} K \vee \bigvee_j \Sigma^{|f_j|} K \longrightarrow \prod_i \Sigma^{d_i} K$. This gives an obvious left inverse to the inclusion of the wedge, as required.

Now consider a map $f\colon X \longrightarrow Y$ of K-injective spectra. We claim that $f_*\colon \pi_* X \longrightarrow \pi_* Y$ is K_*-linear. Note that X is a wedge of K's, and we have just shown that Y is a retract of a product of K's; this reduces us easily to the case $X = Y = K$ (up to suspension). The usual theory of flat ring spectra shows that the graded map $r_*\colon E_* \longrightarrow E_*$ induced by a graded map $r\colon E \longrightarrow E$ (that is, an element $r \in E^*E = \operatorname{Hom}_{E_*}(E_*E, E_*)$) is the composite $E_* \xrightarrow{\eta_R} E_*E \xrightarrow{r} E_*$. Normally, η_R is a homomorphism of right E_*-modules, but for $E = K$ we have $\eta_R = \eta_L$. Thus, r_* is K_*-linear.

(c) Let X be K-injective, and $e\colon X \longrightarrow X$ an idempotent. By (d) we know that e_* is K-linear, and thus $\pi_* X$ splits as a direct sum $\pi_* eX \oplus \pi_*(1-e)X$ of K_*-modules. By choosing a basis for $\pi_* X$ adapted to this splitting and proceeding as in the proof of Proposition 3.2, we see that eX is K-injective.

(e) Let $f\colon X \longrightarrow Y$ be a map of K-injective spectra such that $f_*\colon X_* \longrightarrow Y_*$ is a monomorphism. Note that the image is a K_*-submodule by (d). Choose elements $\{e_i \in Y_* \mid i \in I\}$ giving a basis for Y_*/f_*X_*. Choose a K-module structure on Y, and use it to convert the maps $e_i\colon S^{|e_i|} \longrightarrow Y$ into module maps $\Sigma^{|e_i|} K \longrightarrow Y$, and thus a map $Z = \bigvee_i \Sigma^{|e_i|} K \longrightarrow Y$. By construction, the evident map $X \vee Z \longrightarrow Y$ is an equivalence, so that $X \longrightarrow Y$ is a split monomorphism with cofibre Z. Clearly, if f is compatible with given module structures on X and Y, then the splitting will also be compatible. The proof for epimorphisms is similar.

(f) This follows immediately from (e).

□

Recall that $\Sigma_* = K_*E = (E_*E)/I_n$, using the invariance of I_n. There is thus a natural coaction of Σ_* on $(E_*X)/I_n$ induced from the coaction of E_*E. In particular, if X is such that E_*X is annihilated by I_n then E_*X is a comodule over Σ_*.

Proposition 3.4. *Let M be a K-module spectrum, and X a spectrum. Then there are natural isomorphisms*

$$\begin{aligned} M_*X &\simeq M_* \otimes_{K_*} K_*X \\ [X, M]_* &\simeq \mathrm{Hom}_{K_*}(K_*X, M_*) \\ E_*M &\simeq \Sigma_* \otimes_{K_*} M_* \\ M_* &\simeq \mathrm{Prim}_{\Sigma_*} E_*M \end{aligned}$$

Proof. The map

$$M \wedge K \wedge X \xrightarrow{\mathrm{mult}\wedge 1} M \wedge X$$

induces a natural map $M_* \otimes_{K_*} K_*X \longrightarrow M_*X$, which is visibly an isomorphism when X is a sphere. Both sides are homology theories because M_* is free over K_*, so the map is an isomorphism for all X. The second part is similar.

The third part follows from the first part, since we have

$$E_*M = M_*E = M_* \otimes_{K_*} K_*E = M_* \otimes_{K_*} \Sigma_*.$$

A diagram chase shows that this is actually an isomorphism of comodules (where M_* has trivial coaction) from which the fourth part is immediate. $\square$

4. Generalised Moore spectra

We now discuss generalised Moore spectra. The most important ideas are due to Hopkins and Smith [HS], and are also explained in [Rav92a, Chapter 6]. Here we offer some convenient technical improvements.

In this section, if we write $w\colon X \longrightarrow Y$ then we always allow the possibility that w has nonzero degree, but we always insist that the degree should be even.

We shall consider v_n self maps with two extra properties. The first is strong centrality in the following sense:

Definition 4.1. Let X be a finite spectrum. We say that a self map $w\colon X \longrightarrow X$ is *strongly central* if $1_X \wedge w$ is central in $\mathrm{End}(X \wedge X)_*$.

It seems that this condition was first considered by Ethan Devinatz [Dev92]. It turns out to have surprisingly strong implications. To explain this, we need the following definition.

Definition 4.2. If X is a finite spectrum, we let $\mathfrak{J}_X$ be the full subcategory of (possibly infinite) spectra Y such that Y may be written as a retract of some spectrum of the form $X \wedge Z$. (Note that we do not take such a retraction as part of the structure of Y.)

Remark 4.3. Let R be the finite ring spectrum $\mathrm{End}(X) = F(X, X) = DX \wedge X$. If Y is an R-injective spectrum then Y is a retract of $R \wedge Y = X \wedge (DX \wedge Y)$, so $Y \in \mathfrak{J}_X$. Conversely, if $Y \in \mathfrak{J}_X$ then Y is a retract of some spectrum $X \wedge Z$, which is an R-module (because X is). It follows that $\mathfrak{J}_X$ is precisely the category of R-injective spectra, and thus that it is closed under products, coproducts, and retracts. Moreover, if $Y \in \mathfrak{J}_X$ and U is an arbitrary spectrum then $Y \wedge U$, $F(U, Y)$ and $F(Y, U)$ are all in $\mathfrak{J}_X$ (by Proposition 3.2).

Later in this section we will prove the following result.

Proposition 4.4. *Let v be a strongly central self-map of a finite spectrum X. Then there is a unique natural transformation $v_Y : Y \longrightarrow Y$ for $Y \in \mathcal{J}_X$, such that $v_{X\wedge Z} = v \wedge 1_Z$ for all spectra Z. Furthermore, $v_{Y\wedge Z} = v_Y \wedge 1_Z$ for all spectra Z.* □

This means that when we work with spectra in $\mathcal{J}_X$ we can pretend that v is an element of π_*S. We will often write v instead of v_Y.

Our second extra condition involves the group of MU-module self maps of $MU \wedge X$. This can be interpreted in two different ways. On the one hand, we can work entirely in the homotopy category of spectra. The spectrum MU is a ring object in this category, so we can consider the category of modules over it; we shall call this $\mathcal{C}_{MU}$. On the other hand, we can work in the derived category $\mathcal{D}_{MU}$ of strict MU-modules, as explained in Section 1. In the present context it does not matter which interpretation we use, because

$$\mathcal{C}_{MU}(MU \wedge X, M)_* = \mathcal{D}_{MU}(MU \wedge X, M)_* = [X, M]_*.$$

In particular,

$$\mathcal{C}_{MU}(MU \wedge X, MU \wedge X)_* = [X, X \wedge MU]_* = MU_*(DX \wedge X).$$

Definition 4.5. Let X be a p-local finite spectrum of type at least $n > 0$. We say that $w\colon \Sigma^{p^k|v_n|}X \longrightarrow X$ is a *good v_n self map* if

1. $1 \wedge w = v_n^{p^k} \wedge 1$ in $\mathcal{C}_{MU}(MU \wedge X, MU \wedge X)$.
2. w is strongly central.

Every good v_n self map is a v_n self map in the sense of Hopkins and Smith [HS]. Indeed, we shall eventually prove the following result:

Theorem 4.6. *Any finite spectrum X of type at least $n > 0$ admits a good v_n self map, and if v and w are two such maps then $v^{p^i} = w^{p^j}$ for some i and j. Moreover, for any MU-module spectrum $M \in \mathcal{C}_{MU}$ we have*

$$1 \wedge w = v_n^{p^k} \wedge 1 \colon M \wedge X \longrightarrow M \wedge X.$$

If M is an object of $\mathcal{D}_{MU}$, then the above equation also holds when interpreted in $\mathcal{D}_{MU}$. In any case, it follows that the induced map $w_*\colon M_*X \longrightarrow M_*X$ is just multiplication by $v_n^{p^k}$. □

We now start work on the proofs.

Lemma 4.7. *Let w be a strongly central self map of a finite spectrum X. Then $w \wedge 1_X = 1_X \wedge w$ in $\mathrm{End}(X \wedge X)_*$. Moreover, for any spectra Y, Z and any map $u\colon X \wedge Y \longrightarrow X \wedge Z$, the following diagram commutes:*

$$\begin{tikzcd} X \wedge Y \arrow[r, "u"] \arrow[d, "w\wedge 1"'] & X \wedge Z \arrow[d, "w\wedge 1"] \\ X \wedge Y \arrow[r, "u"'] & X \wedge Z \end{tikzcd}$$

(*In particular, by taking $Z = Y$ we see that $w \wedge 1$ is central in* $\mathrm{End}(X \wedge Y)_*$*; in particular, by taking $Y = S$ we see that w is central in* $\mathrm{End}(X)_*$.)

Proof. Let w be a strongly central self map of a finite spectrum X. Let $\tau \in \mathrm{End}(X \wedge X)$ be the twist map. As $1_X \wedge w$ is central, we have

$$1_X \wedge w = \tau \circ (1_X \wedge w) \circ \tau = w \wedge 1_X.$$

Now consider a map $u\colon X \wedge Y \to X \wedge Z$, and write

$$g = u \circ (w \wedge 1_Y) - (w \wedge 1_Z) \circ u,$$

so the claim is that $g = 0$. By rewriting $1_X \wedge w$ as $w \wedge 1_X$, we see that both composites in the following diagram are $w \wedge u$, and thus that the diagram commutes.

$$\begin{tikzcd} X \wedge X \wedge Y \arrow[r, "1\wedge u"] \arrow[d, "1\wedge w\wedge 1"'] & X \wedge X \wedge Z \arrow[d, "1\wedge w\wedge 1"] \\ X \wedge X \wedge Y \arrow[r, "1\wedge u"'] & X \wedge X \wedge Z \end{tikzcd}$$

This means that $1_X \wedge g = 0$; *a fortiori* we have $1_{DX\wedge X} \wedge g = 0$. On the other hand, $X \wedge Z$ is a module-spectrum over $DX \wedge X = \mathrm{End}(X)$, so the map

$$\eta \wedge 1_{X\wedge Z}\colon X \wedge Z \to DX \wedge X \wedge X \wedge Z$$

is a split monomorphism. By considering the following diagram, we conclude that $g = 0$ as claimed.

$$\begin{tikzcd} X \wedge Y \arrow[r, "g"] \arrow[d, tail, "\eta\wedge 1"'] & X \wedge Z \arrow[d, tail, "\eta\wedge 1"] \\ DX \wedge X \wedge X \wedge Y \arrow[r, "1\wedge g=0"'] & DX \wedge X \wedge X \wedge Z \end{tikzcd}$$

$\square$

Proof of Proposition 4.4. Let Y be a spectrum in $\mathcal{J}_X$. We can choose a spectrum Z and maps $Y \xrightarrow{j} X \wedge Z \xrightarrow{q} Y$ such that $qj = 1_Y$. By naturality, we must define $v_Y = q(v \wedge 1)j\colon Y \to Y$. Suppose that we have a different spectrum Z' and maps $Y \xrightarrow{j'} X \wedge Z' \xrightarrow{q'} Y$ with $q'j' = 1_Y$. We claim that $q'(v \wedge 1)j' = q(v \wedge 1)j$, so that our definition is independent of the choice of Z, j and q. To see this, note that Lemma 4.7 gives $j'q(v \wedge 1_Z) = (v \wedge 1_{Z'})j'q$. We thus have

$$q(v \wedge 1)j = q'j'q(v \wedge 1)j = q'(v \wedge 1)j'qj = q'(v \wedge 1)j'$$

as required. It follows that $v_{Y\wedge Z} = v_Y \wedge 1_Z$ as well.

Now suppose that we have a morphism $f\colon Y_0 \to Y_1$ in $\mathcal{J}_X$. We need to show that v is natural with respect to f, or in other words that the central square in the following diagram commutes:

$$\begin{tikzcd} X \wedge Z_0 \arrow[r, two heads, "q_0"] \arrow[d, "v\wedge 1"'] & Y_0 \arrow[r, "f"] \arrow[d, "v_{Y_0}"'] & Y_1 \arrow[r, tail, "j_1"] \arrow[d, "v_{Y_1}"] & X \wedge Z_1 \arrow[d, "v\wedge 1"] \\ X \wedge Z_0 \arrow[r, two heads, "q_0"'] & Y_0 \arrow[r, "f"'] & Y_1 \arrow[r, tail, "j_1"'] & X \wedge Z_1 \end{tikzcd}$$

Because Y_0, Y_1 are in $\mathcal{J}_X$, we can choose spectra Z_0, Z_1 and a split monomorphism j_1 and split epimorphism q_0 as shown. By the definition of v given above, the outer squares commute. The total rectangle commutes by Lemma 4.7. It follows easily that the middle square commutes, as required. $\square$

Definition 4.8. A *μ-spectrum* is a spectrum X equipped with maps $S \xrightarrow{\eta} X \xleftarrow{\mu} X \wedge X$ such that

$$\mu \circ (\eta \wedge 1) = 1 \ : X \to X.$$

(We reserve the term *ring spectrum* for examples in which μ is associative and η is a two-sided unit.) If X is a μ-spectrum, then $\pi_* X$ is a (possibly non-associative) graded ring, with possibly only a left unit. A *module spectrum* over X is a spectrum Y equipped with a map $\nu : X \wedge Y \to Y$ such that $\nu \circ (\eta \wedge 1) = 1 : Y \to Y$. Note that this is independent of μ. If $X \to Y$ is a map of μ-spectra, then Y has an obvious structure as an X-module; in particular, X is an X-module. We shall say that a μ-spectrum X is *atomic* if every map $f\colon X \to X$ such that $f\eta = \eta$ is an isomorphism.

Definition 4.9. Given a finite μ-spectrum X and an element $v \in \pi_d X$, we define

$$\lambda(v) = (\Sigma^d X \xrightarrow{v \wedge 1} X \wedge X \xrightarrow{\mu} X) \in [X, X]_d.$$

If $\lambda(v)$ is a strongly central self map of X, we say that v is a *strongly central element* of $\pi_* X$. If in addition the Hurewicz image of v in $MU_* X$ is (the image under the unit map of) $v_n^{p^k}$ for some k, we say that v is a *good v_n element* . Given a self map $w\colon \Sigma^d X \to X$, we define $\eta^*(w) = w \circ \eta \in \pi_d X$.

Proposition 4.10. *Let X be a finite, atomic μ-spectrum. Then every strongly central self map of X is a map of X-modules. Moreover, the set of such maps forms a commutative and associative ring under composition, which is isomorphic to the ring of strongly central elements of $\pi_* X$ via λ and η^*. This also induces a bijection between good v_n self maps and good v_n elements.*

Proof. Write A_* for the set of strongly central self maps of X; this is clearly a commutative and associative graded ring under composition. Write B_* for the set of strongly central elements of $\pi_* X$, so we have a map $\lambda\colon B_* \to A_*$.

Consider $w \in A_*$. We first claim that w is a map of X-modules, in other words that the following diagram commutes.

$$\begin{tikzcd} X \wedge X \arrow[r, "1 \wedge w"] \arrow[d, "\mu"'] & X \wedge X \arrow[d, "\mu"] \\ X \arrow[r, "w"'] & X \end{tikzcd}$$

This is just Lemma 4.7 (with $Y = X, Z = S, u = \mu$).

Next, we claim that $w = \lambda(\eta^* w)$. To see this, write $v = \eta^* w$ and consider the following diagram.

$$\begin{tikzcd} X \arrow[rr, "w"] \arrow[d, "\eta \wedge 1"'] \arrow[drr, "v \wedge 1 = \eta \wedge w"'] & & X \arrow[d, "\eta \wedge 1"] \arrow[drr, "1"] & & \\ X \wedge X \arrow[rr, "w \wedge 1 = 1 \wedge w"'] & & X \wedge X \arrow[rr, "\mu"'] & & X. \end{tikzcd}$$

We remarked earlier that $w \wedge 1$ commutes with the twist map and thus $w \wedge 1 = 1 \wedge w$. By thinking of this map as $1 \wedge w$ we see that the square commutes and that the diagonal map is $\eta \wedge w$. By thinking of it as $w \wedge 1$ instead we see that the diagonal can also be described as $v \wedge 1$. The right hand triangle commutes by the axiom for a μ-spectrum. By looking at the long composite in the diagram we see that $w = \lambda(\eta^* w)$ as claimed.

This means that $v \in B_*$, so we have maps

$$A_* \xrightarrow{\eta^*} B_* \xrightarrow{\lambda} A_*$$

with $\lambda\eta^* = 1$.

Next, consider two strongly central self maps w, w', and write $v = \eta^* w$ and $v' = \eta^* w'$. We claim that $\eta^*(ww') = vv'$ (which means $\mu \circ (v \wedge v')$). This follows by inspecting the following diagram:

$$\begin{tikzcd}
S \arrow[rr, "v \wedge v'"] \arrow[d, "\eta"'] \arrow[dr, "v'"] & & X \wedge X \arrow[d, "\mu"] \\
X \arrow[r, "w'"'] & X \arrow[ur, "v \wedge 1"] \arrow[r, "w"'] & X.
\end{tikzcd}$$

Thus, η^* is a ring map from A_* to B_*.

Next, consider the map $\theta = \mu \circ (1 \wedge \eta) \colon X \longrightarrow X$. As we do not assume that η is a two-sided unit for μ, this need not be the identity. However, it is easy to see that $\theta\eta = \eta$; as X is atomic, we conclude that θ is an isomorphism. For any $v \in B_*$, we have a commutative diagram as follows:

$$\begin{tikzcd}
S \arrow[r, "v"] \arrow[d, "\eta"'] & X \arrow[d, "1 \wedge \eta"'] \arrow[dr, "\theta"] & \\
X \arrow[r, "v \wedge 1"'] & X \wedge X \arrow[r, "\mu"'] & X
\end{tikzcd}$$

This shows that $\eta^*\lambda(v) = \theta \circ v$, so that $\lambda \colon B_* \longrightarrow A_*$ is injective. As $\lambda\eta^* = 1$, it is not hard to see that λ and η^* are mutually inverse isomorphisms.

We leave it to the reader to check that good v_n self maps go to good v_n elements and *vice versa*. $\square$

In view of the above proposition, we allow ourselves to write v for $\lambda(v)$, when v is a strongly central homotopy element for a finite, atomic μ-spectrum.

Proposition 4.11. *Let X be a finite atomic μ-spectrum, and $v \in \pi_* X$ a strongly central element. Suppose that $k > 1$, and let X/v^k be the cofibre of $v^k \colon X \longrightarrow X$. Then X/v^k can be made into a μ-spectrum in such a way that the map $q \colon X \longrightarrow X/v^k$ is a map of μ-spectra and the maps*

$$X \xrightarrow{v^k} X \xrightarrow{q} X/v^k \xrightarrow{d} \Sigma X$$

are maps of X-modules.

Proof. This is essentially due to Ethan Devinatz [Dev92]. His Theorem 1 states that if $w \colon X \longrightarrow X$ is a self map of a finite μ-spectrum and

(i) w is strongly central
(ii) w^2 is a map of X-modules

then X/w^2 can be made into a μ-spectrum, with properties essentially above. Our Proposition 4.10 shows that hypothesis (ii) is redundant. We assume that X is atomic so that we can translate cleanly between (powers of) strongly central self maps and homotopy elements. This proves the case $k = 2$ of our claim. The general case is essentially the same; in Devinatz' Lemma 5, one simply has to apply Verdier's axiom to the maps $X \xrightarrow{v^{k-1}} X \xrightarrow{v} X$ rather than $X \xrightarrow{v} X \xrightarrow{v} X$. $\square$

Definition 4.12. Consider an ideal $I \leq MU_*$ of the form

$$I = (v_0^{a_0}, \dots , v_{n-1}^{a_{n-1}}),$$

where each a_i is a power of p. We call n the height of I. We give a recursive definition of "generalised Moore spectra of type S/I"; such a thing will be a certain kind of finite, atomic μ-spectrum. The only spectrum of type $S/0$ is S, equipped with the obvious structure. Consider an ideal I as above, and write $J = (v_0^{a_0}, \dots , v_{n-2}^{a_{n-2}})$. A μ-spectrum X has type S/I if there is a μ-spectrum Y of type S/J with a good v_{n-1} self map v of degree $|v_{n-1}^{a_{n-1}}|$ and a cofibration $X \xrightarrow{v} X \xrightarrow{q} Y \xrightarrow{d} \Sigma X$ such that q is a map of μ-spectra and q and d are maps of X-modules.

If a spectrum of type S/I exists, we shall refer to it as S/I; this is an abuse, as there may be many non-isomorphic spectra of type S/I. It is easy to see that the unit map $S \longrightarrow S/I$ induces an isomorphism $MU_*/I \simeq MU_*(S/I)$. It follows that any map $f \colon S/I \longrightarrow S/I$ with $f \circ \eta = \eta$ satisfies $MU_* f = 1$ and thus is an isomorphism; this means that S/I is atomic. We can also construct an object $MU/I \in \mathcal{D}_{MU}$ by the methods of [EKMM96, Chapter V] or [Str96], and it is easy to check that $MU \wedge S/I$ is isomorphic in $\mathcal{D}_{MU}$ to MU/I.

Proposition 4.13. *Suppose that there is a spectrum X of type S/I; then it has a good v_n element, and thus $S/(I, v_n^{p^j})$ exists for $j \gg 0$.*

Proof. This is based on results in [HS]; we refer to Ravenel's account [Rav92a] as it is more readily available.

By [Rav92a, Theorem 1.5.4], there exists $k \geq 0$ and a map $w \in \mathrm{End}(X)_*$ such that $K(m)_* w = 0$ for $m \neq n$, and $K_* w = K(n)_* w$ is an isomorphism. By [Rav92a, Lemma 6.1.1], we may assume that $K_* w = v_n^r$ for some r. By examining the proofs, we see that r may be assumed to have the form p^k. It follows that $1 \wedge w \in \mathrm{End}(X \wedge X)_*$ also has $K(m)_*(1 \wedge w) = 0$ for $m \neq n$ and $K_*(1 \wedge w) = v_n^{p^k}$. Thus, by [Rav92a, Lemma 6.1.2], the map $1 \wedge w^{p^i}$ is central in $\mathrm{End}(X \wedge X)_*$ for large i. After replacing w by w^{p^i}, we may assume that w is a strongly central self map of X. Write $v = \eta^* w \in \pi_* X$, so that v is a strongly central element. Note that v maps to a primitive element v' in the MU_*MU-comodule $MU_* X = MU_*/I$. As $\mathrm{Prim}(MU_*/I_n) = \mathbf{F}_p[v_n]$, we conclude that $v' = \alpha v_n^{p^k} \pmod{I_n}$ for some $\alpha \in \mathbf{F}_p$. Recall that $K_* v = v_n^{p^k}$; by comparing the unit maps, we conclude that $\alpha = 1$. Using [HS, Lemma 3.4], we see that $(v')^{p^i} = v_n^{p^{k+i}}$ for $i \gg 0$; we may replace v by v^{p^i} and thus assume that $v' = v_n^{p^k}$. This means that v is a good v_n element, as required. Using Proposition 4.11, we see that $S/(I, v_n^{p^j}) = X/v^{p^{j-k}}$ exists when $j > k$. $\square$

Corollary 4.14. *For any ideal $J \leq MU_*$ with radical I_n, there exists an ideal $I = (v_0^{a_0}, \dots , v_{n-1}^{a_{n-1}})$ such that $I \leq J$ and S/I exists.* $\square$

Corollary 4.15. *Any finite spectrum X of type at least n admits a good v_n self map.*

Proof. Let $\mathcal{C}$ be the category of those X that admit a good v_n self map. Spectra of type S/I (where I has height n) lie in $\mathcal{C}$, so it will suffice to check that $\mathcal{C}$ is thick. It is clearly closed under suspensions. Suppose that $X \in \mathcal{C}$ and that $e\colon X \longrightarrow X$ is idempotent. Write $Y = eX$ and $Z = (1-e)X$, so that $X = Y \vee Z$. Choose a good v_n self map $w\colon X \longrightarrow X$. As w is central, it commutes with e, so it must have the form $u \vee v$ where $u\colon Y \longrightarrow Y$ and $v\colon Z \longrightarrow Z$. It is easy to check that u and v are good v_n self maps, so $\mathcal{C}$ is closed under retracts.

Now consider a cofibration $X \xrightarrow{f} Y \xrightarrow{g} Z$ with $X, Z \in \mathcal{C}$. For $k \gg 0$, we can choose good v_n self maps $u\colon X \longrightarrow X$ and $w\colon Z \longrightarrow Z$ with $|u| = |w| = |v_n^{p^k}|$. We may also choose a compatible fill-in map $v\colon Y \longrightarrow Y$. After replacing u, v and w by suitable iterates, we may assume that v is strongly central. Indeed, u and w are strongly central, so $1 \wedge v - v \wedge 1$ is nilpotent. It follows from [HS, Lemma 3.4] that v^{p^j} is strongly central for large j. For typographical convenience, we write $x = v_n^{p^k}\colon MU \longrightarrow MU$. We thus have a commutative diagram

$$\begin{array}{ccccc}
MU \wedge X & \xrightarrow{1\wedge f} & MU \wedge Y & \xrightarrow{1\wedge g} & MU \wedge Z \\
{\scriptstyle x\wedge 1}\downarrow & & {\scriptstyle 1\wedge v}\downarrow & & \downarrow{\scriptstyle x\wedge 1} \\
MU \wedge X & \xrightarrow[1\wedge f]{} & MU \wedge Y & \xrightarrow[1\wedge g]{} & MU \wedge Z
\end{array}$$

Write $d = 1\wedge v - x\wedge 1 \in \operatorname{End}(MU\wedge Y)_*$. Note that $d\circ(1\wedge f) = 0$ and $(1\wedge g)\circ d = 0$, so that d can be written in the form $(1\wedge g)\circ r$ and also in the form $s\circ(1\wedge f)$. Thus d^2 factors through $(1\wedge f)(1\wedge g) = 0$, so $d^2 = 0$. As $x \wedge 1$ commutes with $1\wedge v$, the same is true of d. Thus we have

$$1 \wedge v^{p^i} = x^{p^i} \wedge 1 + p^i (x^{p^i - 1} \wedge 1) d.$$

Moreover, we may assume that $n > 0$ so that $\operatorname{End}(MU \wedge Y)_*$ is a torsion group. It follows that

$$1 \wedge v^{p^i} = x^{p^i} \wedge 1 = v_n^{p^{k+i}} \wedge 1$$

for $i \gg 0$, as required. $\square$

Proof of Theorem 4.6. Let X be a finite spectrum of type at least n. We know from Corollary 4.15 that X admits a good v_n self map v, with $|v| = |v_n^{p^k}|$ say. Let w be another such map. By the asymptotic uniqueness of v_n self maps [Rav92a, Lemma 6.1.3], we know that $v^N = w^M$ for some N and M. By inspecting the proof (bearing in mind that $MU_* v$ and $MU_* w$ are powers of v_n and not merely unit multiples of powers of v_n) we see that M and N may be taken to be powers of p.

Now let M be a module-spectrum over MU in the classical sense, so $M \in \mathcal{C}_{MU}$; let $\nu\colon M \wedge MU \longrightarrow M$ be the structure map. Using the bijection $\mathcal{C}_{MU}(MU\wedge X, MU \wedge X) = [X, MU \wedge X]$, we find that

$$\eta \wedge v = v_n^{p^k} \wedge 1\colon X \longrightarrow MU \wedge X.$$

By applying $M \wedge (-)$ and composing with $\nu \wedge 1_X$, we find that

$$1 \wedge v = v_n^{p^k} \wedge 1\colon M \wedge X \longrightarrow M \wedge X$$

as claimed.

On the other hand, let N be an MU-module in the strict sense, so $N \in \mathcal{D}_{MU}$. We know that

$$1 \wedge v = v_n^{p^k} \wedge 1 \in \mathcal{C}_{MU}(MU \wedge X, MU \wedge X) = \mathcal{D}_{MU}(MU \wedge X, MU \wedge X).$$

We can apply the functor $N \wedge_{MU} (-) \colon \mathcal{D}_{MU} \longrightarrow \mathcal{D}_{MU}$ to this equation and conclude that

$$1 \wedge v = v_n^{p^k} \wedge 1 \in \mathcal{D}_{MU}(N \wedge X, N \wedge X)$$

as claimed. □

Proposition 4.16. *If S/I is a generalised Moore spectrum and X is an S/I-module then $S/I \wedge X$ is a wedge of finitely many suspended copies of X.*

Proof. We may assume that I has nonzero height, so $S/I = S/(J, v)$ for some generalised Moore spectrum S/J and some good v_n element v. Note that there is a map $S/J \longrightarrow S/I$ of μ-spectra so X is a module over S/J. Thus $X \in \mathcal{J}_{S/J}$ and v_X is defined. By induction we may assume that $W = X \wedge S/J$ is a wedge of suspended copies of X, and thus that $X \wedge S/I$ is the cofibre of $v \colon W \longrightarrow W$. However, it is clear that $v_{S/I} = 0$ and X is a retract of $S/I \wedge X$ so $v_X = 0$ so $v_W = 0$. It follows that $X \wedge S/I$ is a wedge of suspended copies of X, as claimed. □

Proposition 4.17. *Let X be a spectrum. Then the following are equivalent:*

(a) *X is a retract of some spectrum of the form $Y \wedge Z$, where Z is a finite spectrum of type at least n.*
(b) *There is a generalised Moore spectrum S/I of type n such that X is a module over S/I.*

Moreover, the category of such X is an ideal.

Proof. (a)⇒(b): It is enough to show that Z is a module over some S/I of type n. By induction, we may assume that it is a module over some S/J of type $n-1$. Let v be a good v_{n-1} self map of S/J. As $K(n-1)_* Z = 0$ and $K(m)_* v = 0$ for all $m \neq n-1$, the Nilpotence Theorem tells us that $v \wedge 1 \colon S/J \wedge Z \longrightarrow S/J \wedge Z$ is nilpotent, say $v^{p^N} \wedge 1 = 0$. It follows that Z is a module over $S/(J, v^{p^N})$.

(b)⇒(a): Clear.

Suppose that $X \longrightarrow Y \longrightarrow Z$ is a cofibration and that X and Z satisfy (b). We claim that Y satisfies (b) as well. By induction, we may assume that X, Y and Z are all modules over some S/J of type $n-1$. Let v be a good v_{n-1} self map of S/J. By the argument above we see that v_X and v_Z are nilpotent, and it follows easily that v_Y is nilpotent, say $v_Y^{p^N} = 0$. It follows that Y is a module over $S/(J, v^{p^N})$. This shows that the category in question is thick. Using (a) it is trivial to see that it is an ideal. □

It is sometimes convenient to know that S/I is self-dual.

Proposition 4.18. *Let X be a spectrum of type S/I, and let d be the dimension of its top cell. Then there is an isomorphism $\Sigma^d DX = X$.*

Proof. Write $I = (J, v_n^{p^k})$, so there is a generalised Moore spectrum Y of type S/J and a good v_n self map v of Y such that $X = Y/v$. We can assume by induction that Y is self-dual. Given any map $f \colon U \longrightarrow V$, Df is the composite

$$DV \xrightarrow{1 \wedge \eta} DV \wedge U \wedge DU \xrightarrow{1 \wedge f \wedge 1} DV \wedge V \wedge DU \xrightarrow{\epsilon \wedge 1} DU$$

In particular, $1_{DY} = D(1_Y) = (\epsilon \wedge 1) \circ (1 \wedge \eta)$, so DY is in the category $\mathcal{J}_Y$ and we have a self map v_{DY} of DY. Using the naturality of v with respect to $1 \wedge \eta$ and $\epsilon \wedge 1$ we see that $v_{DY} = Dv$. Now let $t : Y \longrightarrow DY$ be a self-duality isomorphism, where we have omitted the suspension. By naturality, we have $t \circ v = Dv \circ t$. Hence t induces an isomorphism from Y/v to (a suspension of) $DY/Dv = D(Y/v)$, as required. □

4.1. **Towers of generalised Moore spectra.** We next try to build a tower of spectra $S/I(j)$, where the $I(j)$ all have height n, their intersection is trivial, and the maps are compatible with the unit maps $S \longrightarrow S/I(j)$. We start with some generalities about towers (or more general pro-systems) of finite spectra. These ideas also appear in [CS96].

Let $\mathcal{F}$ be the category of finite spectra. Its pro-completion is the dual of the category of those functors $F\colon \mathcal{F} \longrightarrow \text{Sets}$ that can be written as a small filtered colimit of representable functors. Consider two such functors, say $FZ = \varinjlim_{i \in I}[X_i, Z]$ and $GZ = \varinjlim_{j\in J}[Y_j, Z]$. Using the Yoneda lemma, we see that

$$\text{Pro}(\mathcal{F})(F,G) = [\mathcal{F},\text{Sets}](G,F) = [\mathcal{F},\text{Sets}](\varinjlim_j [Y_j,-], \varinjlim_i [X_i,-]) = \varprojlim_j \varinjlim_i [X_i, Y_j].$$

An alternate definition of $\text{Pro}(\mathcal{F})$ is as the category of pairs (I, X), where I is a small filtered category, $\{X_i\}$ is an I^{op}-indexed system of finite spectra, and the morphisms from (I, X) to (J, Y) are given by the above formula. This clearly gives a category equivalent to the one described previously.

Note that the Yoneda functor $X \longrightarrow [X, -]$ gives a full and faithful covariant embedding of $\mathcal{F}$ in $\text{Pro}(\mathcal{F})$, which we think of as an inclusion. Every object in $\text{Pro}(\mathcal{F})$ is an inverse limit of objects of $\mathcal{F}$ (because inverse limits in $\text{Pro}(\mathcal{F})$ are the same as direct limits in the functor category $\text{Pro}(\mathcal{F})^{\text{op}}$).

There is a popular full subcategory $\text{Tow}(\mathcal{F})$ of $\text{Pro}(\mathcal{F})$ consisting of limits of towers (or equivalently, countable filtered systems with countably many morphisms) of representable functors. In this case we can be more explicit about the morphisms. Let $\text{Seq}(\mathcal{F})$ be the category of inverse sequences of finite spectra, like

$$X_0 \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \dots .$$

The maps from $X_\bullet$ to $Y_\bullet$ are just systems of maps $f_k\colon X_k \longrightarrow Y_k$ making the evident diagrams commute. Let U be the directed set of strictly increasing maps $\mathbf{N} \longrightarrow \mathbf{N}$. If $u \in U$ and $X_\bullet \in \text{Seq}(\mathcal{F})$ then we define $u^*X_\bullet \in \text{Seq}(\mathcal{F})$ by $(u^*X_\bullet)_k = X_{u(k)}$. The maps $X_{u(k)} \longleftarrow X_{u(k+1)}$ are the evident composites of the maps $X_j \longleftarrow X_{j+1}$. It is not hard to check that

$$\text{Tow}(\mathcal{F})(\varprojlim_i X_i, \varprojlim_j Y_j) = \varinjlim_U \text{Seq}(\mathcal{F})(u^*X_\bullet, Y_\bullet).$$

In particular, a map $f\colon X_\bullet \longrightarrow Y_\bullet$ becomes zero in the Pro category if and only if there exists $u \in U$ such that the composite

$$X_{u(k+1)} \longrightarrow X_{u(k)} \xrightarrow{f_{u(k)}} Y_{u(k)}$$

is zero for all k. It is equivalent to say that for all i there exists $j > i$ such that the composite $X_j \longrightarrow X_i \xrightarrow{f_i} Y_i$ vanishes. In particular (taking $f = 1$) we see that $\varprojlim_i X_i = 0$ in $\text{Pro}(\mathcal{F})$ if and only if for all i there exists $j > i$ such that the map

$X_j \longrightarrow X_i$ is zero. Note that this certainly implies that $\operatorname{ho}\varprojlim X_i = 0$ in $\mathcal{S}$, but is a much stronger statement.

It turns out that $\operatorname{Pro}(\mathcal{F})$ is a rather familiar category in disguise.

Proposition 4.19. $\operatorname{Pro}(\mathcal{F})$ *is the dual of the category of homology theories* (*or equivalently, spectra mod phantoms*)*. Moreover,* $\operatorname{Tow}(\mathcal{F})$ *corresponds to the subcategory of homology theories with countable coefficients.*

Proof. Any homology theory defined on finite spectra has an essentially unique extension to the category of all spectra, so it doesn't matter which we meant.

As filtered colimits are exact, it is clear that any filtered colimit of representable functors is a homology theory. Conversely, let H be a homology theory. It is not hard to check that the category I of pairs (X, u) (where $X \in \mathcal{F}$ and $u \in HX$) is filtered, and it is formal that $H = \varinjlim_I [X, -]$. Moreover, I is essentially small because $\mathcal{F}$ is.

If H has countable coefficients, then I is easily seen to be countable, so there is a cofinal functor $\mathbf{N} \longrightarrow I$, so $H \in \operatorname{Tow}(\mathcal{F})$. The converse is also clear. □

Remark 4.20. As the smash product of a phantom map and any other map is phantom, there is an induced smash product on the category of spectra mod phantoms. We can transfer this to the equivalent category of homology theories, or the dual category $\operatorname{Pro}(\mathcal{F})$. One can easily check that with this definition we have

$$(\varprojlim_i X_i) \wedge (\varprojlim_j Y_j) = \varprojlim_{i,j} X_i \wedge Y_j = \varprojlim_i \varprojlim_j X_i \wedge Y_j.$$

Remark 4.21. One can see either directly from the definitions or from Proposition 4.19 that an object $X \in \operatorname{Pro}(\mathcal{F})$ is zero if and only if $[X, W] = 0$ for all $W \in \mathcal{F}$, and that a map $f \colon X \longrightarrow Y$ in $\operatorname{Pro}(\mathcal{F})$ is an isomorphism if and only if the induced map $[X, W] \longleftarrow [Y, W]$ is an isomorphism for all $W \in \mathcal{F}$.

It follows from the above proposition that $\operatorname{Pro}(\mathcal{F})$ is not a triangulated category. Indeed, in a triangulated category every monomorphism is split, but any map of spectra with phantom fibre gives a non-split monomorphism of represented homology theories. Nonetheless, some features of triangulated categories remain. In particular, suppose that we have an inverse system of cofibre sequences $X_i \xrightarrow{f_i} Y_i \xrightarrow{g_i} Z_i \xrightarrow{h_i} \Sigma X_i$. Write $X = \varprojlim_i X_i \in \operatorname{Pro}(\mathcal{F})$ and so on, so we have maps $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$. For any $W \in \mathcal{F}$ we have an exact sequence $[X_i, W] \longleftarrow [Y_i, W] \longleftarrow [Z_i, W] \longleftarrow [\Sigma X_i, W]$, and passage to direct limits gives an exact sequence $[X, W] \xleftarrow{f^*} [Y, W] \xleftarrow{g^*} [Z, W] \xleftarrow{h^*} [\Sigma X, W]$. In view of Remark 4.21, we see that f is an isomorphism if and only if $Z = 0$.

We next construct some special objects of $\operatorname{Pro}(\mathcal{F})$. Fix an integer n and let $\mathcal{F}_n$ be the category of finite spectra of type at least n. Let Λ_n be the category of pairs (X, u) where $X \in \mathcal{F}_n$ and $u \colon S \longrightarrow X$. It is easy to see that Λ_n^{op} is a filtered category, so we have an object $T_n = \varprojlim_{(X,u) \in \Lambda_n} X$ of $\operatorname{Pro}(\mathcal{F}_n)$. It is formal to check that this is the initial object in $S \downarrow \operatorname{Pro}(\mathcal{F}_n)$.

Proposition 4.22. *Fix an integer n. Then there exists a tower*

$$\dots \xrightarrow{g_3} X_2 \xrightarrow{g_2} X_1 \xrightarrow{g_1} X_0$$

such that

(a) *X_i is a generalised Moore spectrum of type $S/J(i)$ for some ideal $J(i)$ of height n.*
(b) *We have $g_i \circ \eta_i = \eta_{i-1}$ for all i, where $\eta_i \colon S \longrightarrow X_i$ is the unit map of the μ-spectrum X_i.*
(c) $\bigcap_i J(i) = 0$.
(d) *For any $Z \in \mathcal{F}_n$ we have $Z = \varprojlim_i Z \wedge X_i$ in $\mathrm{Pro}(\mathcal{F}_n)$.*

Note that the unit maps η_i give a map from S to $\varprojlim_i X_i$ in $\mathrm{Pro}(\mathcal{F})$, which is implicitly used in (d). For any such tower, the corresponding object of $S \downarrow \mathrm{Pro}(\mathcal{F}_n)$ is isomorphic to T_n.

Remark 4.23. We shall often talk about towers of the form

$$\ldots \longrightarrow S/J(3) \longrightarrow S/J(2) \longrightarrow S/J(1) \longrightarrow S/J(0),$$

by which we mean a tower with the properties mentioned in the above proposition. We will also use notation like $\operatorname{holim}_{\overleftarrow{J}} S/J$ to refer to the homotopy inverse limit of any such tower. This depends only on the isomorphism class of the tower in $\mathrm{Pro}(\mathcal{F})$ and thus is independent of the choices made.

Proof. We assume that $n > 1$ and that the result holds for $n-1$; small modifications are required for $n = 1$, but we leave them to the reader. By induction, there is a tower $\{W_i\}$ of the required type for the $n-1$ case. Choose a spectrum Y of type n. Write W_i' for the cofibre of the map $S \longrightarrow W_i$. These cofibres can be assembled into a tower. Because $Y \longrightarrow \varprojlim_i Y \wedge W_i$ is an isomorphism, we see that $\varprojlim_i Y \wedge W_i' = 0$. After replacing $\{W_i\}$ and $\{W_i'\}$ by cofinal subtowers, we may assume that the maps $Y \wedge W_i' \longrightarrow Y \wedge W_{i-1}'$ are all zero.

We shall recursively build a tower $\{X_i\}$ and cofibrations as shown below, such that w_i is a good v_{n-1} self map, and $1_Y \wedge w_i = 0 \colon Y \wedge W_i \longrightarrow Y \wedge W_i$.

$$\begin{tikzcd}
W_i \arrow[r, "w_i"] \arrow[d, "w_{i-1}^{p^k-1}\circ f_i"'] & W_i \arrow[r] \arrow[d, "f_i"'] & X_i \arrow[r] \arrow[d, "g_i"] & \Sigma W_i \arrow[d, "w_{i-1}^{p^k-1}\circ f_i"] \\
W_{i-1} \arrow[r, "w_{i-1}"'] & W_{i-1} \arrow[r] & X_{i-1} \arrow[r] & \Sigma W_{i-1}
\end{tikzcd}$$

Suppose we have constructed all this up to the $(i-1)$'th level. Choose a good v_n self map w_i of W_i. After replacing w_i by $w_i^{p^j}$ for some $j > 0$, we may assume that w_i is compatible with w_{i-1}, say $f_i \circ w_i = w_{i-1}^{p^k} \circ f_i$ for some k. We may also assume that $k > 0$ and $1_Y \wedge w_i = 0$. We let X_i be the cofibre of w_i; the usual comparison of cofibrations gives a map g_i as shown. Moreover, X_i can be made into a μ-spectrum of type $S/J(i)$ for a suitable ideal $J(i)$, as in Proposition 4.11. Thus (a) holds. The maps $W_i \longrightarrow X_i$ are maps of μ-spectra and thus carry units to units. By the induction hypothesis, f_i carries the unit of W_i to that of W_{i-1}. It follows that g_i carries the unit of X_i to that of X_{i-1}, so (b) holds. The assumption $k > 0$ above ensures that $|w_i| \longrightarrow \infty$ as $i \longrightarrow \infty$, and thus that $\bigcap_i J(i) = 0$, so (c) holds.

Clearly, when we smash the towers at either end of the above diagram with Y, all the maps become zero. It follows that $\varprojlim_i Y \wedge X_i = \varprojlim_i Y \wedge W_i = Y$ in $\mathrm{Pro}(\mathcal{F})$, as claimed.

It is not hard to check that $\{Z \mid Z = \varprojlim_i Z \wedge X_i\}$ is a thick subcategory of $\mathcal{F}$, so it contains all of $\mathcal{F}_n$. This proves (d).

Similarly, for $Z \in \mathcal{F}_n$ we have

$$\mathrm{Pro}(\mathcal{F})(\varprojlim_i X_i, Z) = \mathrm{Pro}(\mathcal{F})(\varprojlim_i DZ \wedge X_i, S) = \mathrm{Pro}(\mathcal{F})(DZ, S) = [S, Z].$$

It follows that for any object $Y = \varprojlim_j Z_j \in \mathrm{Pro}(\mathcal{F}_n)$ we have

$$\mathrm{Pro}(\mathcal{F})(\varprojlim_i X_i, Y) = \varprojlim_j [S, Y_j] = \mathrm{Pro}(\mathcal{F})(S, Y).$$

This means that $\varprojlim_i X_i$ is initial in $S \downarrow \mathrm{Pro}(\mathcal{F}_n)$, so $\varprojlim_i X_i = T_n$. $\square$

Corollary 4.24. *We have $T_n \wedge T_n = T_n$ in $\mathrm{Pro}(\mathcal{F})$ (because $T_n \wedge T_n = \varprojlim_I S/I \wedge T_n$ and $S/I \wedge T_n = S/I$).* $\square$

Remark 4.25. In any tower as above, the spectrum $X_i = S/J(i)$ can be made into a module over X_j for $j > i$. Indeed, as X_i is a μ-spectrum we know that the map $X_i \xrightarrow{\eta_i \wedge 1} X_i \wedge X_i$ is a split monomorphism. This can be factored as $X_i \xrightarrow{\eta_j \wedge 1} X_j \wedge X_i \xrightarrow{g \wedge 1} X_i \wedge X_i$, so $\eta_j \wedge 1$ is a split monomorphism, as required.

5. Bousfield classes

In this section, we recall some known results about Bousfield classes that we need in the rest of this paper. Recall that in a stable homotopy category $\mathcal{C}$, the *Bousfield class* $\langle X \rangle$ of an object X is the collection of X-local objects. We order Bousfield classes by inclusion. (It is more traditional to define $\langle X \rangle$ to be the category of X-acyclic objects, and to order Bousfield classes by reverse inclusion. Either approach gives the same lattice of Bousfield classes, but the former has some conceptual advantages.) We begin with the crucial result of Hopkins and Ravenel.

Theorem 5.1 ([Rav92a, Theorem 7.5.6])**.** *The localisation functor L is smashing. That is, the natural map $LS \wedge X \longrightarrow LX$ is an isomorphism.* $\square$

Note that the Bousfield class in $\mathcal{L}$ of an L-local spectrum X is the same as the Bousfield class in $\mathcal{S}$, since any X-local spectrum is L-local. Indeed, since L is a smashing localisation functor, if W is L-acyclic, then $X \wedge W = X \wedge LS \wedge W = X \wedge LW = 0$.

If X and Y have the same Bousfield class, we write $X \sim Y$.

Recall that C_j is the acyclisation functor corresponding to L_j, and $M_j = C_{j-1} L_j$ is the fibre of the natural map $L_j \longrightarrow L_{j-1} = L_{j-1} L_j$.

Lemma 5.2. *We have $\widehat{L} E(n) = E$.*

Proof. Let F be the fibre of the evident map $E(n) \longrightarrow E$. Choose a generalised Moore spectrum S/I of type n. By Landweber exactness we have $E(n)_*(S/I) = E(n)_*/I$ and $E_*(S/I) = E_*/I$. But $E(n)_*/I = E_*/I$ so $F \wedge S/I = 0$. Thus $K_*(F) \otimes_{K_*} K_*(S/I) = 0$ and $K_*(S/I) \neq 0$ so $K_* F = 0$. Thus, the map $E(n) \longrightarrow E$ is a K-equivalence. On the other hand, one can use the techniques of [EKMM96, Chapter V] to build a tower of spectra $E/I \in \mathcal{D}_{MU}$ with $\pi_*(E/I) = E_*/I$. There is a natural map from E to the sequential limit of this tower, which is easily seen to be an isomorphism. The spectra E/I lie in the thick subcategory generated by

$E/I_n = K$, so the sequential limit is K-local. Thus E is a K-local spectrum that is K_*-equivalent to $E(n)$, so $E = \widehat{L}E(n)$. □

Proposition 5.3. *We have the following equalities of Bousfield classes:*

$$\begin{aligned}\langle LS\rangle &= \langle \widehat{L}S\rangle = \langle E(n)\rangle = \langle E\rangle = \langle K(0)\rangle \vee \ldots \vee \langle K(n)\rangle \\ \langle L_j S\rangle &= \langle K(0)\rangle \vee \ldots \vee \langle K(j)\rangle \\ \langle C_j LS\rangle &= \langle LF(j+1)\rangle = \langle K(j+1)\rangle \vee \ldots \vee \langle K(n)\rangle \\ \langle M_j S\rangle &= \langle K(j)\rangle = \langle LT(j)\rangle\end{aligned}$$

Proof. It is proved in [Rav84, Theorem 2.1] that $\langle E(n)\rangle = \langle K(0)\rangle \vee \ldots \vee \langle K(n)\rangle$. By Theorem 5.1, $LX = LS \wedge X$. It is immediate from this that $\langle LS\rangle = \langle E(n)\rangle$. By Landweber exactness we have $E_*X = E_* \otimes_{E(n)_*} E(n)_*X$, and E_* is faithfully flat over $E(n)_*$ so $\langle E\rangle = \langle E(n)\rangle$. We have ring maps $LS \longrightarrow \widehat{L}S \longrightarrow \widehat{L}E(n) = E$ so $\langle E\rangle \leq \langle \widehat{L}S\rangle \leq \langle LS\rangle$ but we have seen that $\langle E\rangle = \langle LS\rangle$ so $\langle \widehat{L}S\rangle = \langle LS\rangle$ also.

Next, we claim that $\langle C_j LS\rangle = \langle K(j+1)\rangle \vee \ldots \vee \langle K(n)\rangle$. Indeed, because L_j is smashing, we know that

$$C_j LS = C_j S \wedge LS \sim C_j K(0) \vee \ldots \vee C_j K(n)$$

If $i \leq j$ then $K(i)$ is $E(j)$-local. Indeed, $K(i)$ is self-local, and $\langle K(i)\rangle \leq \langle E(j)\rangle$, so $K(i)$ is also $E(j)$-local. Hence $C_j K(i) = 0$. If $j < i \leq n$ then

$$L_j K(i) = L_j S \wedge K(i) \sim (K(0) \wedge K(i)) \vee \ldots \vee (K(j) \wedge K(i)).$$

We know from [Rav84, Theorem 2.1] that $K(k) \wedge K(i) = 0$ unless $i = k$. Thus, if $j < i \leq n$, then $L_j K(i) = 0$ and $C_j K(i) = K(i)$. It follows that $C_j LS \sim K(j+1) \vee \ldots \vee K(n)$ as claimed. Clearly $M_j S = C_{j-1} L_j S$, so $\langle M_j S\rangle = \langle K(j)\rangle$.

By the definition of a type j spectrum, we have $K(i) \wedge F(j) = 0$ if and only if $i < j$. Moreover, if $i \geq j$ then $K(i) \wedge F(j)$ is a nontrivial wedge of suspensions of $K(i)$, hence Bousfield equivalent to $K(i)$. It follows that

$$LF(j) \sim (K(0) \wedge F(j)) \vee \ldots \vee (K(n) \wedge F(j)) \sim K(j) \vee \ldots \vee K(n)$$

Now let $v : \Sigma^d F(j) \longrightarrow F(j)$ be a v_j-self map, so that $v^{-1}F(j) = T(j)$. By definition, v induces a nilpotent endomorphism of $K(i)_*F(j)$ when $i \neq j$, so that $K(i)_*T(j) = 0$. It follows easily that $\langle LT(j)\rangle \leq \langle K(j)\rangle$. On the other hand, the Telescope Lemma [Rav84, Lemma 1.34] says that $LF(j) \sim LT(j) \vee LF(j+1)$. By smashing with $K(j)$ we deduce that $\langle K(j)\rangle \leq \langle LT(j)\rangle$ and thus that $K(j) \sim LT(j)$. □

6. The $E(n)$-local category

In this section, we discuss the general properties of $\mathcal{L}$. Some of this material also appears in [Dev96a] or [HS95].

Proposition 6.1. *$\mathcal{L}$ is a monogenic stable homotopy category in the sense of* [HPS95]. *In particular, it is a triangulated category with coproducts and compatible smash products and function spectra, giving a closed symmetric monoidal structure. Coproducts, minimal weak colimits, smash products and function spectra are the same as in $\mathcal{S}$. The unit for the smash product is LS, which is a small graded* *weak generator. This last statement means that for any family of spectra $X_i \in \mathcal{L}$ we have*

$$[LS, \bigvee X_i] = \bigoplus [LS, X_i],$$

and that if $X \in \mathcal{L}$ satisfies $[LS, X]_* = 0$ then $X = 0$.

Proof. Apply [HPS95, Theorems 3.5.1 and 3.5.2]. □

Theorem 6.2.

(a) *Homology and cohomology functors $\mathcal{L} \to \mathbf{Ab}$ are representable.*
(b) *Any spectrum $X \in \mathcal{L}$ is the minimal weak colimit of $L\Lambda(X)$, where $\Lambda(X)$ is the category of finite ordinary spectra over X.*
(c) *Suppose that $X \in \mathcal{L}$. The following are equivalent:*
   1. *X is small.*
   2. *X is F-small.*
   3. *X is LS-finite.*
   4. *X is dualisable.*
   5. *X is a retract of LY for some finite spectrum $Y \in \mathcal{S}$.*

   See Definition 1.5 for definitions.
(d) *Every spectrum in $\mathcal{L}$ is $E(n)$-nilpotent (Definition 1.4).*

Proof. Part (a) follows from [HPS95, Theorem 3.5.2] and [HPS95, Theorem 4.1.5]. For part (b), recall that X is the minimal weak colimit in $\mathcal{S}$ of $\Lambda(X)$, by [HPS95, Theorem 4.2.4] or [Mar83, Chapter 5]. It follows from part (e) of [HPS95, Proposition 2.2.4] that $X = LS \wedge X$ is the minimal weak colimit in $\mathcal{S}$ of $LS \wedge \Lambda(X) = L\Lambda(X)$. Moreover, minimal weak colimits are the same in $\mathcal{S}$ or $\mathcal{L}$. For part (c), Theorem 2.1.3 of [HPS95] implies that the first four conditions are equivalent. Moreover, if $Y \in \mathcal{S}$ is finite, then it is easy to see that LY (and hence, any retract of LY) is LS-finite. Conversely, suppose that $X \in \mathcal{L}$ is small, so that $[X, -]$ is a homology theory on $\mathcal{L}$. It follows from part (b) that $[X, X] = \varinjlim_{Y \in \Lambda(X)} [X, LY]$, so the identity map $1_X \in [X, X]$ factors through LY, in other words, X is a retract of LY. Part (d) is proved in [HS95]. □

Proposition 6.3. *Suppose that $X, Y \in \mathcal{L}$ are small. Then $[X, Y]_*$ is countable.*

Proof. By an evident thick subcategory argument, we need only show that $\pi_* LS$ is countable. This is proved in [Hov95a, Lemma 5.5] (by showing that the E_1 term of the Adams-Novikov spectral sequence converging to $\pi_* LS$ is countable in each bidegree, and that the E_∞ term has only finitely many lines). □

For calculations when $n = 1$, see [Bou79] or [Rav84]. For the case $n = 2$, see the many papers by Katsumi Shimomura and his coworkers, for example [SY95]. See also Section 15.

We can now prove a convergence theorem for the modified Adams spectral sequence described in Section 2.2.

Lemma 6.4. *Let $\mathcal{C}$ be a stable homotopy category, and R a ring object in $\mathcal{C}$. Then the ideal of R-nilpotent objects is the same as the thick subcategory generated by the R-injective objects.*

Proof. Let $\mathcal{N}$ be the category of R-nilpotent spectra, which is by definition the ideal generated by R. Let $\mathcal{I}$ be the category of R-injectives, and $\mathcal{N}'$ the thick subcategory generated by $\mathcal{I}$. Write

$$\mathcal{N}'' = \{X \mid X \wedge Y \in \mathcal{N}' \text{ for all } Y\}.$$

Clearly $\mathcal{I} \subseteq \mathcal{N}$ and $\mathcal{N}$ is thick so $\mathcal{N}' \subseteq \mathcal{N}$. Clearly $R \in \mathcal{N}''$ and $\mathcal{N}''$ is an ideal so $\mathcal{N} \subseteq \mathcal{N}''$. By putting $Y = S$ in the definition we see that $\mathcal{N}'' \subseteq \mathcal{N}'$. Thus $\mathcal{N}' = \mathcal{N} = \mathcal{N}''$ as required. $\square$

Proposition 6.5. *There are constants r_0, s_0 such that for all spectra X and Y, in the modified Adams spectral sequence*

$$\mathrm{Ext}^{s,t}_{E(n)_*E(n)}(E(n)_*X, E(n)_*Y) \Longrightarrow [X, LY]_{t-s}$$

(*see Section 2.2*) *we have $E^{s,t}_r = 0$ whenever $r \geq r_0$ and $s \geq s_0$. Moreover, the spectral sequence converges.*

Proof. Let $i\colon \overline{E(n)} \longrightarrow LS$ be the fibre of the unit map $LS \longrightarrow E(n)$. It is easy to see that $i \wedge 1_{E(n)}\colon \overline{E(n)} \wedge E(n) \longrightarrow E(n)$ is null, and thus that $i \wedge 1_I = 0$ for any $E(n)$-injective spectrum I. It follows using Lemma 6.4 that for any $E(n)$-nilpotent spectrum X we have $i^{(N)} \wedge 1_X = 0\colon \overline{E(n)}^{(N)} \wedge X \longrightarrow X$ for some N. However, we know that LS is $E(n)$-nilpotent, so that $i^{(N)}\colon \overline{E(n)}^{(N)} \longrightarrow LS$ is null for some N. Now suppose that X is such that $E(n)_*X$ is a projective module over $E(n)_*$. We then have an ordinary Adams spectral sequence for $[X, LY]_*$ obtained by mapping X to the tower

$$LY \longleftarrow \overline{E(n)} \wedge LY \longleftarrow \overline{E(n)}^{(2)} \wedge LY \longleftarrow \dots .$$

As every composite in the tower of length at least N vanishes, one can show that the spectral sequence converges, and that it stops at the E_N page with a flat vanishing line. One also knows from [Dev96b, Proposition 1.9 and Remark 1.10] that this spectral sequence is isomorphic to the modified Adams spectral sequence from the E_2 page onwards, and that the resulting filtrations of $[X, LY]_*$ are also the same. In particular, in the filtration arising from the MASS we have $F^s = 0$ for $s \geq N$.

Now choose a tower $LY = Y^0 \longleftarrow Y^1 \longleftarrow Y^2 \longleftarrow \dots$ of the type used in the MASS. By the above, we see that when $E(n)_*X$ is projective, the map $[X, Y^s]_* \longrightarrow [X, Y^0]_*$ is null when $s \geq N$. Moreover, if we truncate the tower below Y^t we get a tower of the type that computes the MASS for $[X, Y^t]_*$ so we can apply the same logic and deduce that the map $[X, Y^{t+s}]_* \longrightarrow [X, Y^t]_*$ is null for $s \geq N$.

Now consider a general spectrum X. By the methods of [Ada74, Theorem 13.6] (taking account of Proposition 2.13) we can choose a tower

$$X = X_0 \longrightarrow X_1 \longrightarrow \dots \longrightarrow X_n$$

and triangles $\Sigma^{-1}X_{k+1} \longrightarrow P_k \longrightarrow X_k$ such that $E(n)_*P_k$ is projective and the map $E(n)_*P_k \longrightarrow E(n)_*X_k$ is surjective. As $E(n)_* = \mathbf{Z}_{(p)}[v_1, \dots, v_n^{\pm 1}]$ has global dimension n (in the graded sense) we see that $E(n)_*X_n$ is projective. It is not hard to show by induction that $[X_{n-j}, Y^{t+(j+1)N}]_* \longrightarrow [X_{n-j}, Y^t]_*$ is null and thus that $[X, Y^{t+(n+1)N}]_* \longrightarrow [X, Y^t]_*$ is null. It follows that the MASS for $[X, LY]_*$ converges, and that it stops with a flat vanishing line at the $E_{(n+1)N}$ page. $\square$

Jens Franke has shown [Fra96, Theorem 10] that when $2p - 2 > n^2 + n$, the category $\mathcal{L}$ is equivalent to a purely algebraically defined derived category of periodic complexes of comodules. He shows that (in contrast to Quillen's algebraicisation of rational unstable homotopy) the equivalence does not capture all the higher-order homotopical phenomena of the model category underlying $\mathcal{L}$, but that the approximation improves as p increases. His result implies for example that there is an $\mathcal{L}$-small spectrum X with $E(n)_*X = K(n)_*$. There is some reason to suspect that this is not of the form LY for any finite spectrum Y (although it must be a retract of some such LY) but the situation is rather unclear at the moment. Ethan Devinatz has also constructed the spectrum X by a more direct argument.

6.1. **Nilpotence and thick subcategories.** In this situation, we have the simple nilpotence theorem proved in [HPS95, Section 5]. Recall that a map $f : X \longrightarrow Y$ is called *smash nilpotent* if $f^{\wedge m} : X^{\wedge m} \longrightarrow Y^{\wedge m}$ is null for large enough m.

Corollary 6.6 (Nilpotence theorem)**.**

(a) *Let F be an $\mathcal{L}$-small spectrum, and X an arbitrary spectrum in $\mathcal{L}$. A map $f : F \longrightarrow X$ is smash nilpotent if and only if $K(i)_*f = 0$ for all $i \leq n$.*
(b) *Let X be an $\mathcal{L}$-small spectrum. A map $f : X \longrightarrow X$ is nilpotent if and only if $K(i)_*f$ is nilpotent for all $i \leq n$.*
(c) *Let R be a ring spectrum in $\mathcal{L}$. An element $\alpha \in \pi_*R$ is nilpotent if and only if $K(i)_*\alpha$ is nilpotent in the ring $K(i)_*R$ for all $i \leq n$.*

Proof. Most of this is proved in [HPS95, Section 5]. The only thing left to prove is that if $f : F \longrightarrow X$ is a smash nilpotent map from an $\mathcal{L}$-small spectrum F to X, then $K(i)_*f = 0$. This follows easily from the Künneth theorem since $K(i)_*(f^{\wedge N}) = (K(i)_*f)^{\otimes N}$. □

Note that this nilpotence theorem follows directly from the Bousfield decomposition of LS, but that Bousfield decomposition depends on the smashing theorem of Hopkins-Ravenel, whose proof requires the much more difficult nilpotence theorem of Devinatz-Hopkins-Smith [DHS88].

The nilpotence theorem can be used to classify thick subcategories (Definition 1.2) of $\mathcal{L}$-small spectra. For this, we need a definition of the support of a spectrum or full subcategory of $\mathcal{L}$.

Definition 6.7. If $\mathcal{X}$ is any class of spectra (in particular if $\mathcal{X} \subseteq \mathcal{L}$), we write

$$K(m) \wedge \mathcal{X} = \{K(m) \wedge X \mid X \in \mathcal{X}\}$$

$$\operatorname{supp}(\mathcal{X}) = \{m \mid K(m) \wedge \mathcal{X} \neq \{0\}\}.$$

We refer to $\operatorname{supp}(\mathcal{X})$ as the *support* of $\mathcal{X}$. Similarly, the *cosupport* of $\mathcal{X}$ is

$$\operatorname{cosupp}(\mathcal{X}) = \{m \mid F(K(m), \mathcal{X}) \neq \{0\}\}.$$

We have to classify the possible supports of $\mathcal{L}$-small spectra. Since not every such spectrum is the localisation of an ordinary finite spectrum, this does not immediately follow from knowledge of possible supports in $\mathcal{S}$. However, the same result does hold.

Proposition 6.8. *If X is small in $\mathcal{L}$ and $j < n$, then $\dim K(j)_*X \leq \dim K(j+1)_*X$. Thus $\operatorname{supp}(X) = \{m, m+1, \ldots, n\}$ for some m with $0 \leq m \leq n+1$ (where the case $m = n+1$ means $\operatorname{supp}(X) = 0$ and thus $X = 0$).*

Proof. The proof is the same as that of [Rav84, Theorem 2.11]. □

Now, given an integer j such that $0 \leq j \leq n+1$, let $\mathcal{C}_j$ denote the thick subcategory of $\mathcal{L}$ consisting of all small spectra X such that $K(i)_*X = 0$ for all $i < j$. Then we have

$$\mathcal{C}_0 \supseteq \mathcal{C}_1 \supseteq \cdots \supseteq \mathcal{C}_{n+1} = \{0\}.$$

Note that $\mathcal{C}_0$ consists of all $\mathcal{L}$-small spectra. All of the inclusions are strict, since $LF(j)$ is in $\mathcal{C}_j$ but not $\mathcal{C}_{j+1}$.

We now have the following theorem, which is a corollary of the general thick subcategory theorem proved in [HPS95, Section 5] and the classification of supports proved above.

Theorem 6.9 (Thick Subcategory Theorem). *If $\mathcal{C}$ is a thick subcategory of $\mathcal{L}$, then $\mathcal{C} = \mathcal{C}_j$ for some j such that $0 \leq j \leq n+1$.* □

The telescope conjecture also holds in $\mathcal{L}$. More explicitly, consider the following three functors:

$$\begin{aligned}
L_i : \mathcal{L} \to \mathcal{L} \quad &= \quad \text{localisation with respect to } K(0) \vee \ldots \vee K(i) \\
L_i^{lf} : \mathcal{L} \to \mathcal{L} \quad &= \quad \text{finite localisation away from } LF(i+1) \\
L_i^f : \mathcal{S} \to \mathcal{S} \quad &= \quad \text{finite localisation away from } F(i+1)
\end{aligned}$$

(See [HPS95, Section 3.3] for discussion of finite localisation). The following corollary is the telescope conjecture for $\mathcal{L}$.

Corollary 6.10. *If L' is a smashing localisation functor on $\mathcal{L}$, then $L' = L_i$ for some i. Moreover, $L_i = L_i^{lf} = LL_i^f$.*

Proof. Since L' is smashing, L' is Bousfield localisation with respect to $L'S$. Note that if $L'S \wedge K(j)$ is nonzero then it is a wedge of suspensions of $K(j)$ and thus has the same Bousfield class as $K(j)$. We thus have

$$\langle L'S \rangle = \langle L'S \rangle \wedge \bigvee_{j=0}^{n} \langle K(j) \rangle = \bigvee_{j \in \operatorname{supp}(L'S)} \langle K(j) \rangle.$$

Now, if $j \in \operatorname{supp}(L'S)$, then $L_{K(j)}S$ is a module over $L'S$ so we have

$$\langle L'S \rangle \geq \langle L_{K(j)}S \rangle = \langle K(0) \rangle \vee \ldots \vee \langle K(j) \rangle$$

(using Proposition 5.3). Thus $L' = L_i$ for the largest i in the support of $L'S$.

Next, recall that there is a cofibre sequence $C_i^f S \to S \to L_i^f S$, in which $C_i^f S$ has a cellular tower built from $F(i+1)$ (see [HPS95, Theorem 3.3.3]). Moreover, $[F(i+1), L_i^f S]_* = 0$, so $DF(i+1) \wedge L_i^f S = 0$. It follows that $LC_i^f S$ has a cellular tower built from $LF(i+1)$, so it is L_i^{lf}-acyclic. Moreover,

$$[LF(i+1), LL_i^f S]_* = [LS, L(DF(i+1) \wedge L_i^f S)]_* = 0.$$

It follows that $LL_i^f S$ is L_i^{lf}-local, so $LL_i^f = L_i^{lf}$.

We know that L_i^{lf} is again a smashing localisation. Thus, to see that $L_i^{lf} = L_i$, it is enough to check that $\operatorname{supp}(L_i^{lf} S) = \{0, \ldots, i\}$. We know from Proposition 5.3 that $L_i^f S \sim T(0) \vee \ldots \vee T(i)$, so

$$L_i^{lf} S \sim LT(0) \vee \ldots \vee LT(i) \sim K(0) \vee \ldots \vee K(i)$$

The claim follows. □

Corollary 6.11. *If $X \in \mathcal{C}_j$ then X is a retract of LY for some finite spectrum Y of type at least j.*

Proof. We have seen that X lies in the thick subcategory generated by $LF(i)$, so $X = C_i^{lf} X$, but we have also seen that this is the same as $LC_i^f X$, which is a minimal weak colimit of terms of the form LY. As X is small, we conclude that it is a retract of one of these terms. □

We conclude this section by showing that the theory of v_j self-maps (defined by the evident generalisation of Definition 4.5) works as expected in $\mathcal{L}$.

Theorem 6.12. *Every spectrum $X \in \mathcal{C}_j$ admits a good v_j self map .*

Proof. Choose a finite spectrum Y of type at least j such that X is a retract of $LY = LS \wedge Y$, so that X lies in the category $\mathcal{J}_Y$ of Definition 4.2. Choose a good v_j self map $v\colon \Sigma^d Y \to Y$. Proposition 4.4 gives an induced map $v_X\colon \Sigma^d X \to X$, and it is easy to check that this is a good v_j self map. □

6.2. **Localising and colocalising subcategories.** In this section, we classify the localising and colocalising subcategories (Definition 1.3) of $\mathcal{L}$. Every such category is determined by the Morava K-theories which it contains.

We write $\operatorname{loc}\langle \mathcal{X}\rangle$ and $\operatorname{coloc}\langle \mathcal{X}\rangle$ for the localising and colocalising subcategories generated by a class $\mathcal{X}$ of spectra. We also write

$$\mathcal{X}^\perp = \{Y \mid [\mathcal{X}, Y]_* = 0\}$$

$$^\perp\mathcal{X} = \{Y \mid [Y, \mathcal{X}]_* = 0\}$$

(where $[\mathcal{X}, Y]_* = \{[X, Y]_* \mid X \in \mathcal{X}\}$ and so on). Note that $\mathcal{X}^\perp$ is a colocalising subcategory and $^\perp\mathcal{X}$ is a localising subcategory. Also note that every (co)localising subcategory is a (co)ideal, since $\mathcal{L}$ is monogenic.

Definition 6.13.

(a) We write $\mathcal{P}$ for the lattice of subsets of $\{0, \dots, n\}$. For any $S \in \mathcal{P}$ we write $S^c = \{0, \dots, n\} \setminus S$.
(b) $K(S) = \bigvee_{m \in S} K(m)$
(c) $\mathcal{C}_S = \{X \in \mathcal{L} \mid K(S^c)_* X = 0\} = \{K(S^c)\text{–acyclic spectra in } \mathcal{L}\}$
(d) $\mathcal{D}_S = \{K(S)\text{–local spectra}\} \subseteq \mathcal{L}$

Theorem 6.14.

(a) *The lattice of localising subcategories is isomorphic to $\mathcal{P}$, via the maps $S \mapsto \mathcal{C}_S$ and $\mathcal{C} \mapsto \operatorname{supp}(\mathcal{C})$.*
(b) *The lattice of colocalising subcategories is isomorphic to $\mathcal{P}$, via the maps $S \mapsto \mathcal{D}_S$ and $\mathcal{D} \mapsto \operatorname{cosupp}(\mathcal{D})$.*
(c) $\mathcal{C}_S = \operatorname{loc}\langle K(S)\rangle = \{X \in \mathcal{L} \mid \langle X\rangle \le \langle K(S)\rangle\}$.
(d) $\mathcal{D}_S = \operatorname{coloc}\langle K(S)\rangle = \{X \in \mathcal{L} \mid \langle X^*\rangle \le \langle K(S)^*\rangle = \langle K(S)\rangle\}$.
(e) $\mathcal{C}_{S^c} = {}^\perp\mathcal{D}_S$ *and* $\mathcal{D}_{S^c} = \mathcal{C}_S^\perp$.

The notation $\langle X^*\rangle$ denotes the cohomological Bousfield class of X, the collection of X^*-local spectra. A spectrum Y is said to be X^*-local if $[Z, Y]_* = 0$ for all spectra Z such that $[Z, X]_* = 0$.

This theorem is similar to the analogous theorem proven as [HPS95, Corollary 6.3.4] for a Noetherian stable homotopy category (with some hypotheses). However, we have been unable to find a worthwhile common generalisation of the two theorems. We shall prove the theorem after some intermediate results.

Proposition 6.15. *If X is a finite spectrum of type at least n, then LX is K-nilpotent.*

Proof. By a thick subcategory argument, it is sufficient to verify this for $X = S/I$. We know from [HS95] that LS is $E(n)$-nilpotent, and $\{Y \mid Y \wedge S/I \in \mathrm{ideal}\langle E(n)/I\rangle\}$ is an ideal containing $E(n)$, so it contains LS. Thus, LS/I lies in the ideal generated by $E(n)/I$, which is easily seen to be the same as the ideal generated by K. □

Remark 6.16. We shall show in Corollary 8.12 that the localisation of a finite spectrum of type n actually lies in the thick subcategory generated by K.

Proposition 6.17. *For any $X \in \mathcal{S}$, the spectrum $M_m X = M_m S \wedge X$ lies in the localising subcategory generated by $K(m)$, and $F(M_m S, X)$ lies in the colocalising subcategory generated by $K(m)$.*

Proof. Observe that $M_m X = C_{m-1} L_m X$, which is L_{m-1}-acyclic. Recall that the functor $L_{m-1} : \mathcal{L}_m \to \mathcal{L}_{m-1}$ is simply finite localisation away from $L_m F(m)$ (Corollary 6.10), so that the L_{m-1}-acyclics in $\mathcal{L}_m$ are precisely $\mathrm{loc}\langle L_m F(m)\rangle$. Because $L_m F(m)$ is $K(m)$-nilpotent (Proposition 6.15), this is contained in $\mathrm{loc}\langle K(m)\rangle$, so $M_m X \in \mathrm{loc}\langle K(m)\rangle$.

The spectrum $F(K(m), X)$ is a $K(m)$-module, hence a wedge of suspensions of $K(m)$. By part (f) of Proposition 3.3, the wedge splits off as a retract of the product, so it lies in $\mathrm{coloc}\langle K(m)\rangle$. The subcategory $\{Y \mid F(Y, X) \in \mathrm{coloc}\langle K(m)\rangle\}$ is localising and contains $K(m)$, so it contains $M_m S$. Thus $F(M_m S, X) \in \mathrm{coloc}\langle K(m)\rangle$. □

Proposition 6.18. *For any spectrum $X \in \mathcal{L}$ we have*

$$X \in \mathrm{loc}\langle X\rangle = \mathrm{loc}\langle K(\mathrm{supp}(X))\rangle$$

and

$$X \in \mathrm{coloc}\langle X\rangle = \mathrm{coloc}\langle K(\mathrm{cosupp}(X))\rangle.$$

Proof. We have a tower of spectra as follows, in which the vertical maps are the fibres of the horizontal maps.

$$\begin{array}{ccccccccc} M_n S & & M_{n-1} S & & & M_1 S & & M_0 S & \\ \downarrow & & \downarrow & & & \downarrow & & \downarrow & \\ L_n S & \longrightarrow & L_{n-1} S & & \ldots & L_1 S & \longrightarrow & L_0 S & \longrightarrow 0 \end{array}$$

After smashing this with X, we see that $X \in \mathrm{loc}\langle M_0 X, \ldots, M_n X\rangle$. Because $\langle M_m S\rangle = \langle K(m)\rangle$ (by Proposition 5.3), we see that $M_m X = 0$ unless $m \in \mathrm{supp}(X)$. On the other hand, when $m \in \mathrm{supp}(X)$ we have $M_m X \in \mathrm{loc}\langle K(m)\rangle$ by Proposition 6.17. It follows easily that $X \in \mathrm{loc}\langle K(m) \mid m \in \mathrm{supp}(X)\rangle = \mathrm{loc}\langle K(\mathrm{supp}(S))\rangle$, so $\mathrm{loc}\langle X\rangle \subseteq \mathrm{loc}\langle K(\mathrm{supp}(S))\rangle$. Moreover, if $m \in \mathrm{supp}(S)$ then $\mathrm{loc}\langle X\rangle$ contains $K(m) \wedge X$, which is a nonzero wedge of suspensions of $K(m)$, so $\mathrm{loc}\langle X\rangle$ contains $K(m)$. It follows that $\mathrm{loc}\langle X\rangle = \mathrm{loc}\langle K(\mathrm{supp}(S))\rangle$. For the colocalising case, we apply the functor $F(-, X)$ to the above tower and argue in the same way. □

Proof of Theorem 6.14. It is clear that $S \mapsto \mathcal{C}_S$ and $S \mapsto \mathcal{D}_S$ are order preserving maps from $\mathcal{P}$ to the lattices of localising and colocalising subcategories. Next, recall from Proposition 5.3 that $K(m)_*K(n) \neq 0$ if and only if $m = n$. Using this, we see that $K(S) \in \mathcal{C}_S$, and that $\mathrm{supp}(\mathcal{C}_S) = S$. Moreover, because $K(m)^*X = \mathrm{Hom}_{K(m)_*}(K(m)_*X, K(m)_*)$, we see that $K(m)^*K(n) \neq 0$ if and only if $m = n$. Using this and the fact that $K(S) \in \mathcal{D}_S$, we see that $\mathrm{cosupp}(\mathcal{D}_S) = S$.

It is immediate from Proposition 6.18 that any localising subcategory $\mathcal{C} \subseteq \mathcal{L}$ satisfies $\mathcal{C} = \mathrm{loc}\langle K(\mathrm{supp}(\mathcal{C}))\rangle$, and similarly for colocalising subcategories. In particular, $\mathcal{C}_S = \mathrm{loc}\langle K(S)\rangle$. Thus, for any $\mathcal{C}$, we have $\mathcal{C} = \mathcal{C}_{\mathrm{supp}(\mathcal{C})}$. This proves part (a) of the theorem, and part (b) is similar.

We have already proved the first equality in part (c). For the second, write $\mathcal{C} = \{X \in \mathcal{L} \mid \langle X\rangle \leq \langle K(S)\rangle\}$. It is easy to see that this is a localising subcategory. Suppose that $\langle X\rangle \leq \langle K(S)\rangle$. As $K(S)\wedge K(S^c) = 0$, we conclude that $X\wedge K(S^c) = 0$, so that $X \in \mathcal{C}_S$. Thus $\mathcal{C} \subseteq \mathcal{C}_S$. On the other hand, it is clear that $K(S) \in \mathcal{C}$, so $\mathcal{C}_S = \mathrm{loc}\langle K(S)\rangle \subseteq \mathcal{C}$. Thus $\mathcal{C} = \mathcal{C}_S$ as claimed. The proof of part (d) is similar.

For the first part of (e), observe that $K(S)^*X = 0$ if and only if $K(S)_*X = 0$, so $\mathcal{C}_{S^c} = \{X \mid [X, K(S)]_* = K(S)^*X = 0\}$. In other words,

$$\mathcal{C}_{S^c} = {}^{\perp}\{K(S)\} = {}^{\perp}\,\mathrm{coloc}\langle K(S)\rangle = {}^{\perp}\mathcal{D}_S.$$

For the second part, recall that by definition

$$\mathcal{D}_{S^c} = \{K(S)\text{–local spectra}\} = \{K(S)\text{–acyclics}\}^{\perp} = \mathcal{C}_S^{\perp}.$$

□

Note that Theorem 6.14 implies that we can localise with respect to any localising subcategory of $\mathcal{L}$ (because we can localise with respect to the homology theory $K(S)_*$).

6.3. **The monochromatic category.** As mentioned in the introduction, there are two different ways to investigate the difference between the $E(n)$-local category and the $E(n-1)$-local category. The first is to consider the fibre $MX = M_nX$ of the map $L_nX \longrightarrow L_{n-1}X$, and the second is to observe that $E(n) \sim E(n-1)\vee K(n)$ (where $\sim$ means Bousfield equivalence, as in Section 5) and thus consider $L_{K(n)}X = \widehat{L}X$. The purpose of this section is to demonstrate that these two approaches are essentially equivalent.

First, we observe that M_n (considered as a functor $\mathcal{L}_n \longrightarrow \mathcal{L}_{n-1}$) is just $E(n-1)$-acyclisation, so it is an idempotent exact functor.

Theorem 6.19. *For any $X \in \mathcal{S}$, there are natural equivalences $M\widehat{L}X = MX$ and $\widehat{L}MX = \widehat{L}X$. It follows that the functors $\mathcal{M} \xrightarrow{\widehat{L}} \mathcal{K} \xrightarrow{M} \mathcal{M}$ are mutually inverse equivalences between the monochromatic category $\mathcal{M}$ and the K-local category $\mathcal{K}$.*

Proof. We start by observing that for any $X \in \mathcal{S}$, we have $MX = MLX$ and $\widehat{L}X = \widehat{L}LX$.

Next, we know that $L_{n-1}S \sim K(0) \vee \ldots \vee K(n-1)$, so that $K \wedge L_{n-1}S = 0$, so $\widehat{L}L_{n-1}S = 0$. By applying $\widehat{L}$ to the cofibration $MX \longrightarrow LX \longrightarrow L_{n-1}X$, we conclude that $\widehat{L}MX = \widehat{L}X$. Next, using [Hov95a, Lemma 4.1] (or part (e) of Proposition 7.10) and the cofibration $MS \longrightarrow LS \longrightarrow L_{n-1}S$, we obtain a cofibration

$$F(L_{n-1}S, X) \longrightarrow LX = F(LS, LX) \longrightarrow \widehat{L}X = F(MS, LX).$$

We claim that $F(L_{n-1}S, X)$ is $E(n-1)$-local. Indeed, if Z is $E(n-1)$-acyclic then

$$[Z, F(L_{n-1}S, X)]_* = [L_{n-1}S \wedge Z, X]_* = [L_{n-1}Z, X]_* = 0$$

because $L_{n-1}Z = 0$. It follows easily that $MF(L_{n-1}S, X) = 0$, and thus that $MX = M\widehat{L}X$.

If $X \in \mathcal{K}$ then $\widehat{L}X = X$ so $\widehat{L}MX = X$. If $X \in \mathcal{M}$ then $MX = X$ so $M\widehat{L}X = X$. It follows that M and $\widehat{L}$ are mutually inverse equivalences between $\mathcal{M}$ and $\mathcal{K}$. □

7. General properties of the $K(n)$-local category

This section begins the second part of the paper, in which we will study the K-local category $\mathcal{K}$. Unlike many other categories of localised spectra (see Appendix B) this category has a good supply of small objects. These are the K-localisations of finite type n spectra. We will show that these $\mathcal{K}$-small spectra can be used as replacements for the sphere in many cases. That is, many of the constructions commonly used in stable homotopy categories, such as the cellular tower, have analogues in the K-local category. It is just that the cells are no longer spheres, but suspensions of $LF(n)$.

Theorem 7.1. *$\mathcal{K}$ is an algebraic stable homotopy category in the sense of* [HPS95]*. The spectrum $LF(n)$ is a graded weak generator, and $\operatorname{loc}\langle LF(n)\rangle = \mathcal{K}$. The coproduct, smash product and function objects in $\mathcal{K}$ are*

$$\bigvee_{\mathcal{K}} X_i = \widehat{L}(\bigvee_{\mathcal{S}} X_i)$$

$$X \wedge_{\mathcal{K}} Y = \widehat{L}(X \wedge_{\mathcal{S}} Y)$$

$$F_{\mathcal{K}}(X, Y) = F_{\mathcal{S}}(X, Y).$$

Proof. This all follows from [HPS95, Theorem 3.5.1], except for the fact that $LF(n)$ is a graded weak generator, which will be proved as Theorem 7.3. (It follows from this that $\operatorname{loc}\langle LF(n)\rangle = \mathcal{K}$, as in [HPS95, Theorem 1.2.1]). □

We shall usually not write the subscript $\mathcal{K}$ on the symbols for the coproduct and smash product. Instead, we shall say "in $\mathcal{K}$, we have $X = Y \wedge Z$" (rather than "$X = Y \wedge_{\mathcal{K}} Z$").

Lemma 7.2. *If X is a finite spectrum of type at least n and $Y \in \mathcal{S}$ then $L(X \wedge Y) = \widehat{L}(X \wedge Y) = X \wedge \widehat{L}Y$.*

Proof. For any finite spectrum X, we have $\widehat{L}(X \wedge Y) = X \wedge \widehat{L}Y$, and similarly for the functor L, as we see easily by induction on the number of cells. To complete the proof, it suffices to show that $X \wedge LY$ is already K-local. Suppose Z is K-acyclic. Then $Z \wedge DX$ is E-acyclic, since X, and thus DX, has type at least n. Hence $[Z, X \wedge LY] = [Z \wedge DX, LY] = 0$, as required. □

Theorem 7.3. *If X is a finite type n spectrum then $LX = \widehat{L}X$ is a small graded weak generator in $\mathcal{K}$. It follows that any spectrum $Y \in \mathcal{K}$ is the sequential colimit of a sequence $0 = Y_0 \to Y_1 \to Y_2 \to \ldots Y$, in which the cofibre of $Y_k \to Y_{k+1}$ is a coproduct of suspensions of X (and therefore $Y \in \operatorname{loc}\langle X\rangle$).*

Proof. Suppose that $\{Y_i\}$ is a family of K-local spectra. We have

$$\begin{aligned}
[\widehat{L}X, \widehat{L}(\bigvee Y_i)] &= [X, \widehat{L}(\bigvee Y_i)] \\
&= [S, \widehat{L}(\bigvee Y_i) \wedge DX] \\
&= [S, L(\bigvee Y_i \wedge DX)] \\
&= [S, \bigvee L(Y_i \wedge DX)] \\
&= \bigoplus [S, L(Y_i \wedge DX)] \\
&= \bigoplus [S, Y_i \wedge DX)] \\
&= \bigoplus [\widehat{L}X, Y_i].
\end{aligned}$$

Here DX is the Spanier-Whitehead dual of X. The third and sixth equalities use Lemma 7.2, and the fact that $Y_i = \widehat{L}Y_i$. The fourth equality uses the fact that L is smashing. We conclude that LX is small.

Now suppose that $Y \in \mathcal{K}$ has $[LX, Y]_* = 0$. Observe that

$$[LX, Y]_* = [X, Y]_* = \pi_*(DX \wedge Y).$$

It follows that $DX \wedge Y = 0$, so

$$0 = K_*(DX \wedge Y) = \mathrm{Hom}_{K_*}(K_*X, K_*Y).$$

This means that $K_*Y = 0$, and thus (because Y is K-local) that $Y = 0$. In other words, LX is a graded weak generator.

The remaining claims follow from [HPS95, Theorem 1.2.1]. $\square$

7.1. **K-nilpotent spectra.** This short section concerns a small technicality. The class of K-nilpotent spectra is by definition the ideal generated by K. Because there are different smash products in $\mathcal{L}$ and $\mathcal{K}$, it might *a priori* make a difference whether we interpret this definition in $\mathcal{K}$ or $\mathcal{L}$.

Lemma 7.4. *The category $\mathcal{N}$ of K-nilpotent spectra is the thick subcategory of $\mathcal{K}$ generated by the K-injective spectra. It does not make a difference whether we interpret the definition in $\mathcal{S}$, $\mathcal{L}$ or $\mathcal{K}$. If $X \in \mathcal{N}$ and $Y \in \mathcal{K}$ then $X \wedge_{\mathcal{K}} Y = X \wedge_{\mathcal{S}} Y$. Moreover, $\mathcal{N} \subseteq loc_{\mathcal{K}}\langle K\rangle$.*

Proof. It is easy to see that $K \wedge_{\mathcal{S}} X \in \mathcal{K}$ for all X, and thus that the K-injectives are the same in $\mathcal{S}$, $\mathcal{L}$ or $\mathcal{K}$. It follows immediately from Lemma 6.4 (applied in $\mathcal{S}$, $\mathcal{L}$ and $\mathcal{K}$) that the K-nilpotents are the same in all three categories and that they are always the same as the thick subcategory generated by the K-injectives.

Now write $\mathcal{C} = \mathrm{loc}_{\mathcal{K}}\langle K\rangle$, and

$$\mathcal{D} = \{X \in \mathcal{S} \mid \forall Y \in \mathcal{S} \quad X \wedge_{\mathcal{S}} Y \in \mathcal{C}\}.$$

It is easy to see that $\mathcal{D}$ is an ideal in $\mathcal{S}$ containing K, so $\mathcal{N}_{\mathcal{S}} \subseteq \mathcal{D}$. On the other hand $\mathcal{D} \subseteq \mathcal{C}$ (take $Y = S$ in the definition of $\mathcal{D}$). $\square$

7.2. **Localising and colocalising subcategories.**

Theorem 7.5. *The only localising or colocalising subcategories of $\mathcal{K}$ are $\{0\}$ and $\mathcal{K}$.*

Proof. It is easy to see that $\operatorname{loc}_{\mathcal{K}}\langle \widehat{L}S\rangle = \mathcal{K}$ (because the analogous thing holds in $\mathcal{L}$), and thus that any (co)localising subcategory is a (co)ideal [HPS95, Lemma 1.4.6].

Suppose that $\mathcal{C} \subseteq \mathcal{K}$ is a nontrivial localising subcategory, say $0 \neq X \in \mathcal{C}$. Then $K \wedge_{\mathcal{K}} X = K \wedge_{\mathcal{S}} X$ lies in $\mathcal{C}$ and is a nontrivial wedge of suspensions of K. Thus $K \in \mathcal{C}$, and therefore every K-nilpotent spectrum lies in $\mathcal{C}$. In particular, $LF(n) \in \mathcal{C}$. By Theorem 7.1, we conclude that $\mathcal{C} = \mathcal{K}$.

Now suppose that $\mathcal{D} \subseteq \mathcal{K}$ is a nontrivial colocalising subcategory, say $0 \neq X \in \mathcal{K}$. Because $\mathcal{D}$ is a coideal, the K-injective spectrum $F(K, X)$ lies in $\mathcal{D}$. Observe that $\{Y \in \mathcal{K} \mid [Y, X]_* = 0\}$ is a localising subcategory of $\mathcal{K}$ which does not contain X, so it is zero. It follows that $[K, X]_* \neq 0$, so $F(K, X) \neq 0$. Thus K is a retract of $F(K, X)$, so $K \in \mathcal{D}$. Any K-injective spectrum Z is a wedge of suspensions of K, and the wedge is a retract of the product by part (f) of Proposition 3.3, so $Z \in \mathcal{D}$. It follows by Lemma 7.4 that any K-nilpotent spectrum lies in $\mathcal{D}$.

Suppose that $Y \in \mathcal{K}$. Consider $\mathcal{C} = \{U \in \mathcal{K} \mid F(U, Y) \in \mathcal{D}\}$. This is clearly a localising subcategory. Because $F(LF(n), Y) = LDF(n) \wedge Y$ is K-nilpotent, we see that $F(n) \in \mathcal{C}$. We know that $\operatorname{loc}_{\mathcal{K}}\langle F(n)\rangle = \mathcal{K}$, so $\mathcal{C} = \mathcal{K}$. In particular, $\widehat{L}S \in \mathcal{C}$, so $Y \in \mathcal{D}$. Thus $\mathcal{D} = \mathcal{K}$. □

Corollary 7.6. *If $X, Y \in \mathcal{K}$ are both nontrivial, then $F(X, Y) \neq 0$ (equivalently, $[X, Y]_* \neq 0$) and $X \wedge Y \neq 0$.*

Proof. $\mathcal{C} = \{Z \mid [Z, Y]_* = 0\}$ is a localising subcategory of $\mathcal{K}$ not containing Y, so $\mathcal{C} = \{0\}$, so $X \notin \mathcal{C}$. Similarly, $\{Z \mid Z \wedge Y = 0\}$ is a localising subcategory which does not contain $\widehat{L}S$, so it is trivial. More directly, $K_*X \neq 0 \neq K_*Y$, so $K_*(X \wedge Y) = K_*X \otimes_{K_*} K_*Y \neq 0$. □

As an aside, we point out that from this theorem it is very easy to prove that the Bousfield class of K is minimal among all cohomological Bousfield classes, as was conjectured in [Hov95b].

Theorem 7.7. *Suppose $\tau : H \longrightarrow K$ is a natural transformation of homology or cohomology functors on $\mathcal{K}$. If there is a nontrivial X such that τ_X is an isomorphism, then τ is a natural equivalence.*

Proof. $\{X \mid \tau_X$ is an isomorphism $\}$ is a nontrivial localising subcategory, hence is all of $\mathcal{K}$. □

Corollary 7.8. *If H is a homology or cohomology functor on $\mathcal{K}$ such that $H_*(X) = 0$ or $H^*(X) = 0$ for some nontrivial X, then H is trivial.* □

For cohomology functors, this corollary follows immediately from Corollary 7.6 and the fact that cohomology functors are representable (discussed in Section 9).

We do not yet have a complete classification of ideals in $\mathcal{K}$, but we do have the following simple result.

Proposition 7.9. *Suppose that $\mathcal{D}$ is a nonzero ideal of $\mathcal{K}$. Then $\mathcal{D}$ contains the ideal of K-nilpotent spectra.*

Proof. Suppose X is a nontrivial element of $\mathcal{D}$. Then $X \wedge K \in \mathcal{D}$, but $X \wedge K$ is a wedge of suspensions of K. Therefore $K \in \mathcal{D}$, so $\mathcal{D}$ contains the ideal of K-nilpotent spectra. $\square$

7.3. **The localisation functor $\widehat{L}$.** In this section we recall some facts about the functor $\widehat{L} : \mathcal{S} \to \mathcal{K}$. Let $L^f : \mathcal{S} \to \mathcal{S}$ be the finite localisation away from $F(n)$, and C^f the corresponding acyclisation functor [Mil92] [HPS95, Section 3.3]. Thus, we have a cofibration

$$C^f X \to X \to L^f X$$

in which $C^f X$ has an $F(n)$-cellular tower and $[F(n), L^f X]_* = 0$.

Proposition 7.10. *In the claims below, all homotopy colimits are calculated in $\mathcal{S}$. (The homotopy limits are the same in $\mathcal{S}$, $\mathcal{L}$ or $\mathcal{K}$). They are taken over a tower of generalised Moore spectra, as in Proposition 4.22.*

(a) $C^f X = \underset{\longrightarrow I}{\operatorname{holim}} F(S/I, X) = \underset{\longrightarrow I}{\operatorname{holim}} D(S/I) \wedge X.$

(b) $LL^f X = L_{n-1} X = L^f LX.$

(c) $LC^f X = MX = C^f LX = \underset{\longrightarrow I}{\operatorname{holim}} D(S/I) \wedge LX.$

(d) $L_{F(n)} X = \underset{\longleftarrow I}{\operatorname{holim}} X \wedge S/I = F(C^f S, X).$

(e) $\widehat{L} X = L_{F(n)} LX = \underset{\longleftarrow I}{\operatorname{holim}} LX \wedge S/I = F(MS, LX).$

Proof. By the Thick Subcategory Theorem [HS, Theorem 7] [Rav92a, Theorem 3.4.3], we know that $D(S/I)$ lies in the thick subcategory generated by $F(n)$. It follows that $\underset{\longrightarrow I}{\operatorname{holim}} D(S/I) \wedge X \in \operatorname{loc}\langle F(n) \rangle$. On the other hand,

$$[F(n), \underset{\overrightarrow{I}}{\operatorname{holim}}\, D(S/I) \wedge X] = \underset{\overrightarrow{I}}{\lim} [F(n) \wedge S/I, X] = [F(n), X]$$

(the second equality by Proposition 4.22). It follows that $\underset{\longrightarrow I}{\operatorname{holim}} D(S/I) \wedge X \to X$ has universal property of $C^f X \to X$, which gives (a). For part (b), we know that L and L^f are both smashing, so that $LL^f = L^f L$, and Corollary 6.10 tells us that $L^f L = L_{n-1}$. Part (c) follows immediately, as MX is by definition the fibre of $LX \to L_{n-1} X$.

We now prove (d) (which is also proved as [Hov95a, Theorem 2.1]). The second equality is clear from (a). Suppose that Z is $F(n)$-acyclic. As $C^f S \in \operatorname{loc}\langle F(n) \rangle$, we see that $[Z, F(C^f S, X)]_* = [Z \wedge C^f S, X]_* = 0$. Thus $F(C^f S, X)$ is $F(n)$-local. We have a cofibration

$$F(L^f S, X) \to X = F(S, X) \to F(C^f S, X).$$

We know that $[U, L^f S]_* = 0$ whenever U is a finite type n spectrum. In particular $\pi_*(F(n) \wedge L^f S) = [DF(n), L^f S]_* = 0$, so $F(n) \wedge L^f S = 0$. It follows that $[F(n), F(L^f S, X)]_* = [F(n) \wedge L^f S, X]_* = 0$, so that $X \to F(C^f S, X)$ is an $F(n)$-equivalence. Claim (d) follows.

Finally, we prove (e). By part (c), we know that $MS = \underset{\longrightarrow I}{\operatorname{holim}} LD(S/I)$, which implies that $F(MS, LX) = \underset{\longleftarrow I}{\operatorname{holim}} LX \wedge S/I$, and this is the same as $L_{F(n)} LX$ by (d). Because $X \to LX$ is an LS-equivalence, and $LX \to L_{F(n)} LX$ is an $F(n)$-equivalence, we see that $X \to L_{F(n)} LX$ is an $LS \wedge F(n)$-equivalence. Because $LS \wedge F(n) = LF(n)$ is Bousfield equivalent to K, it is a K-equivalence. Thus, we need only show that $L_{F(n)} LX$ is K-local. This follows immediately from the equivalence $L_{F(n)}LX = F(MS, LX)$ and the fact that MS is Bousfield equivalent to K. □

Corollary 7.11. *If $X \in \mathcal{K}$ and we compute sequential (co)limits in $\mathcal{K}$ then*

$$X = \underset{\overrightarrow{I}}{\operatorname{holim}}\, F(S/I, X) = \underset{\overleftarrow{I}}{\operatorname{holim}}\, X \wedge S/I$$

Proof. Note that $X = LX = \widehat{L}X$. By part (c) of Proposition 7.10, we see that

$$MX = \underset{\overrightarrow{I}}{\operatorname{holim}}^{\mathcal{S}} D(S/I) \wedge X.$$

By Theorem 6.19, we have $X = \widehat{L}MX = \underset{\longrightarrow}{\operatorname{holim}}^{\mathcal{K}}_{I}\, D(S/I) \wedge X$ as claimed. It is immediate from part (e) of Proposition 7.10 that $X = \underset{\longleftarrow}{\operatorname{holim}}_{I}\, X \wedge S/I$. □

8. Smallness and duality

We next consider various notions of smallness in $\mathcal{K}$. To show that things are not as simple as in $\mathcal{S}$ or $\mathcal{L}$, we have the following lemma.

Lemma 8.1. *The functor $\widehat{L}$ is not smashing, and $\widehat{L}S$ is not small in $\mathcal{K}$.*

Proof. We know from Proposition 5.3 that $\langle \widehat{L}S \rangle = \langle E(n) \rangle > \langle K \rangle$. On the other hand, if $\widehat{L}$ were smashing we would have

$$\widehat{L}S \wedge X = 0 \Leftrightarrow \widehat{L}X = 0 \Leftrightarrow K_*X = 0 \Leftrightarrow K \wedge X = 0$$

It would follow that $\langle \widehat{L}S \rangle = \langle K \rangle$, a contradiction. If $\widehat{L}S$ were small then $\widehat{L}$ would preserve smallness, so it would be smashing by [HPS95, Theorem 3.5.2], another contradiction. □

In spite of this, we have a good understanding of two different notions of smallness, as indicated by the theorems below. We start with some definitions.

Definition 8.2. We say that a graded Abelian group A_* is *finite* if A_k is a finite set for each k. (This is the most natural definition given that most of our graded Abelian groups are periodic.)

Definition 8.3. For any spectrum X we define $E^\vee_* X = \pi_* \widehat{L}(E \wedge X)$.

We will see below that $E^\vee_* X$ is a better covariant analogue of E^*X than E_*X for $X \in \mathcal{K}$, though $E^\vee_* X$ is not a homology theory. For now we note the following analogue of some of the results in Section 2.

Proposition 8.4.

(a) *The E_*-module $E^\vee_* X$ is L-complete.*
(b) *If X is finite in $\mathcal{S}$ or in $\mathcal{L}$, then $E^\vee_* X = E^\vee_* \widehat{L}X = E_*X$.*
(c) *If E_*X is a free E_*-module, then $E^\vee_* X = (E_*X)^\wedge_{I_n}$ (which is a pro-free E_*-module).*
(d) *$E^\vee_* X$ is finitely generated if and only if K_*X is finite-dimensional.*
(e) *If $E^\vee_* X$ is pro-free, then $K_*X = (E^\vee_* X)/I_n$.*
(f) *If K_*X is concentrated in even dimensions, then $E^\vee_* X$ is pro-free and concentrated in even dimensions.*

Proof. (a) Proposition 7.10 gives a Milnor exact sequence

$$\varprojlim_I{}^1 (E/I)_{*+1}(X) \rightarrowtail E^\vee_* X \twoheadrightarrow \varprojlim_I (E/I)_* X.$$

Theorem A.6 then implies that $E^\vee_* X$ is L-complete.

(b) If X is finite in $\mathcal{S}$ or $\mathcal{L}$ then $E \wedge X$ is already K-local, so $E^\vee_* X = E^\vee_* \widehat{L} X = E_* X$.

(c) This follows from the Milnor sequence and the fact that $(E/I)_* X = (E_* X)/I$, so there is no $\varprojlim^1$ term.

(d) This is very similar to the proof of Proposition 2.4, except that we replace $(E/I_k)^* X$ by $(E/I_k)^\vee_* X = \pi_* \widehat{L}(E/I_k \wedge X)$. Note that $(E/I_n)^\vee_* X = K_* X$ and that $(E/I_k)^\vee_* X$ is L-complete (by induction on k).

(e) If $E^\vee_* X$ is pro-free, then the sequence $\{v_0, \dots, v_{n-1}\}$ is regular on $E^\vee_* X$, so $K_* X = (E/I_n)^\vee_* X = (E^\vee_* X)/I_n$.

(f) Proceed just as in Proposition 2.5 using the theories $(E/I_k)^\vee_* X$.

□

Recall that we have different notions of finiteness in $\mathcal{K}$, or any stable homotopy category (Definition 1.5). Throughout the rest of this paper, we will denote the category of small spectra in $\mathcal{K}$ by $\mathcal{F}$ and the category of dualisable spectra in $\mathcal{K}$ by $\mathcal{D}$. Note that if X is dualisable, we have $E^* DX = \pi_*(E \wedge_{\mathcal{K}} X) = E^\vee_* X$.

Theorem 8.5. *Suppose that $X \in \mathcal{K}$. Consider the following statements:*

(a) *X is small.*
(b) *X is $LF(n)$-finite.*
(c) *X is a retract of $LX' = \widehat{L} X'$ for some finite spectrum X' of type at least n.*
(d) *$E^\vee_* X$ is finite.*
(e) *$E^* X$ is finite.*
(f) *X is dualisable and K-nilpotent.*

Then (a),(b) and (c) are equivalent, and they imply (d),(e) and (f). (We shall show in Corollary 12.16 that (d),(e) and (f) are also equivalent to (a),(b) and (c).)

The category $\mathcal{F} \subseteq \mathcal{D}$ is thick. Moreover, if $X \in \mathcal{F}$ and $Y \in \mathcal{D}$ then $X \wedge Y$, $F(X,Y)$ and $F(Y,X)$ lie in $\mathcal{F}$. In particular, $DX = F(X, \widehat{L} S) \in \mathcal{D}$.

Proof. It follows from part (c) of [HPS95, Theorem 2.1.3] that (a)⇔(b), and that small implies dualisable. Because every finite spectrum of type at least n lies in the thick subcategory generated by $F(n)$, we see that (c)⇒(b).

We now prove that (a)⇒(c). Suppose that X is small. By Corollary 7.11, we see that $X = \operatorname{holim}_{\overrightarrow{I}} D(S/I) \wedge X$, so $[X,X] = \varinjlim_I [X, D(S/I) \wedge X]$, so X is a retract of $Y = D(S/I) \wedge X$ for some I. We claim that Y is small in $\mathcal{L}$. To see this, consider a family $\{Z_i\}$ of spectra in $\mathcal{K}$. By Proposition 5.3 we have $L(S/I) \sim K$, which implies that

$$L(S/I) \wedge \bigvee_i^{\mathcal{L}} Z_i = L(S/I) \wedge \bigvee_i^{\mathcal{K}} Z_i = \bigvee_i^{\mathcal{K}} S/I \wedge Z_i$$

It follows that

$$[Y, \bigvee_i^{\mathcal{L}} Z_i]_* = [X, \bigvee_i^{\mathcal{K}} S/I \wedge Z_i]_* = \bigoplus_i [X, S/I \wedge Z_i]_* = \bigoplus_i [Y, Z_i]_*$$

as claimed. Since $K(i)_* Y = 0$ for $i < n$, Corollary 6.11 implies that Y, and hence X, is a retract of $LZ = \widehat{L} Z$ for some finite spectrum Z of type at least n.

We now show that (c) implies (d),(e), and (f). By an evident thick subcategory argument, we need only show that $E^\vee_* S/I$ and $E^* S/I$ are finite, and that S/I is dualisable and K-nilpotent. As S/I is finite, we see that $E^\vee_* S/I = E_* S/I = (E/I)_*$, which is clearly finite. Similarly, $E^*(S/I) = (E/I)^*$ which is finite. Moreover, Proposition 6.15 shows that S/I is K-nilpotent, and it is easy to see that S/I is dualisable.

From (b) it is clear that $\mathcal{F}$ is thick, and from (c) it is clear that $\mathcal{F}$ is closed under the Spanier-Whitehead duality functor D. Part (a) of [HPS95, Theorem 2.1.3] says that $\mathcal{F}$ is an ideal in $\mathcal{D}$. This also means that if $X \in \mathcal{F}$ and $Y \in \mathcal{D}$ then $F(X,Y) = DX \wedge Y$ and $F(Y,X) = X \wedge DY$ lie in $\mathcal{F}$. $\square$

Theorem 8.6. *Suppose that $X \in \mathcal{K}$. The following are equivalent:*

(a) *X is F-small.*
(b) *X is dualisable.*
(c) *K_*X is finite.*
(d) *K^*X is finite.*
(e) *$E^\vee_* X$ is finitely generated.*
(f) *$E^* X$ is finitely generated.*

The category $\mathcal{D}$ of dualisable spectra is thick. Moreover, $\widehat{L}S \in \mathcal{D}$ and if $X, Y \in \mathcal{D}$ then $X \wedge Y \in \mathcal{D}$ and $F(X,Y) \in \mathcal{D}$. In particular, $DX = F(X, \widehat{L}S) \in \mathcal{D}$.

Proof. It follows from part (c) of [HPS95, Theorem 2.1.3] that (a)$\Leftrightarrow$(b), and also that the subsidiary claims after (f) hold. Because

$$K^*X = \mathrm{Hom}_{K_*}(K_*X, K_*),$$

we see that (c)$\Leftrightarrow$(d). Proposition 2.4 shows that (d)$\Leftrightarrow$(f), and Proposition 8.4 shows that (c)$\Leftrightarrow$(e).

We now prove that (a)$\Rightarrow$(c). Suppose that X is F-small. We know from Theorem 8.5 that $LF(n)$ is small, and thus that $F(n) \wedge X$ is small, and thus that $F(n) \wedge X$ is a retract of LY for some finite type spectrum Y of type at least n. For any finite spectrum Y, it is easy to see that $K_*Y = K_*LY$ is finite (by induction on the number of cells). It follows that $K_*(F(n) \wedge X) = K_*F(n) \otimes_{K_*} K_*X$ is finite. As $K_*F(n) \neq 0$, we conclude that K_*X is finite.

Finally, we prove that (c)$\Rightarrow$(b). Suppose that K_*X is finite. There is a natural map

$$X \wedge F(X,U) \wedge Y \xrightarrow{\mathrm{ev} \wedge 1} U \wedge Y$$

with adjoint

$$\rho_{U,Y} : F(X,U) \wedge Y \to F(X, U \wedge Y).$$

Write

$$\mathcal{C} = \{U \in \mathcal{K} \mid \forall Y \in \mathcal{K} \quad \rho_{U,Y} \text{ is an isomorphism } \}.$$

we claim that every K-module M lies in $\mathcal{C}$. Indeed, using Proposition 3.4 repeatedly, we see that

$$\pi_*(F(X,M) \wedge Y) = \mathrm{Hom}_{K_*}(K_*X, M_*) \otimes_{K_*} K_*Y$$

$$\pi_*(F(X, M \wedge Y)) = \mathrm{Hom}_{K_*}(K_*X, M_* \otimes_{K_*} K_*Y).$$

One can check that the map induced by $\rho_{M,Y}$ is the obvious one, which sends $f \otimes y$ to $x \mapsto f(x) \otimes y$. Given that K_*X is finite, we see that this map is an isomorphism.

As $\mathcal{C}$ is clearly thick, we see from Lemma 7.4 that all K-nilpotent spectra lie in $\mathcal{C}$, in particular $LF(n) \in \mathcal{C}$. On the other hand, because $LF(n)$ is dualisable, we can identify $F(X, LF(n)) \wedge Y$ with $LF(n) \wedge (F(X, \widehat{L}S) \wedge Y)$ and $F(X, LF(n) \wedge Y)$ with $LF(n) \wedge F(X, Y)$. Under these identifications, $\rho_{LF(n),Y}$ becomes $1_{LF(n)} \wedge \rho_{\widehat{L}S,Y}$. It follows that the smash product of $LF(n)$ with the fibre of $\rho_{\widehat{L}S,Y}$ is zero, so (by Corollary 7.6) $\rho_{\widehat{L}S,Y}$ is an isomorphism. This means that X is dualisable. □

Corollary 8.7. *Let G be a finite group, and X a finite G-complex. Then $\widehat{L}(EG \wedge_G X)$ is dualisable. Moreover, we have $D\widehat{L}BG_+ = \widehat{L}BG_+$.*

Proof. It is shown in [GS96b, Corollary 5.4] that $K^*(EG \wedge_G X)$ is finite, so by the preceding theorem, $\widehat{L}(EG \wedge_G X)$ is dualisable. We next use the Greenlees-May Tate construction [GM95b] (see also [GS96a]). We can think of any spectrum X as a G-equivariant spectrum with trivial action indexed on a G-fixed universe, apply a change of universe functor to get to a complete universe and perform the Tate construction to get a G-spectrum $t_G X$. We write $P_G X$ for the Lewis-May fixed-point spectrum $(t_G X)^G$. It fits into a natural cofibre sequence

$$BG_+ \wedge X \to F(BG_+, X) \to P_G(X).$$

If we let X be $\widehat{L}S$ and localise, we get a cofibre sequence

$$\widehat{L}BG_+ \to D\widehat{L}BG_+ \to \widehat{L}P_G\widehat{L}S.$$

Now in [HS96] it is shown that the Bousfield class of $P_G\widehat{L}X$ is the same as that of $L_{n-1}X$ if X is finite. In particular, its K-localisation is trivial. Thus $\widehat{L}BG_+ = D\widehat{L}BG_+$. □

Corollary 8.8. *If X is a connected p-local loop space such that $\pi_k(X)$ is finite for all k and zero for almost all k. Then $\widehat{L}\Sigma^\infty X$ is dualisable.*

Proof. It is shown in [RW80] that $K_*K(A, q)$ is finite whenever A is a finite Abelian group, so it follows from the results of [HRW96] that K_*X is finite. □

Another important example of a dualisable spectrum is the Brown-Comenetz dual of MS; see Section 10.

Theorem 8.9. *Every dualisable spectrum in $\mathcal{K}$ is E-finite.*

The proof depends on the following lemmas.

Lemma 8.10. *Let $X \in \mathcal{K}$ be a spectrum such that*
 (a) *E^*X is pro-free as a module over E.*
 (b) *K^*X is a finitely generated module over Σ^*, of projective dimension $0 \leq d < \infty$.*

Then there is a cofibration $X \to J \to Y$ such that
 (i) *J is a finite wedge of suspensions of E.*
 (ii) *K^*J is the Σ^*-projective cover of K^*X.*
 (iii) *Y satisfies (a) and (b), with projective dimension $d-1$ (and $Y = 0$ if $d = 0$).*

Proof. Since E^*X is pro-free, we have $K^*X = (E^*X)/I_n$ by Proposition 2.5. In particular, $\Sigma^* = (E^*E)/I_n$ acts on K^*X, so (b) makes sense. Let I be the augmentation ideal in Σ^*. Choose a (finite) basis $\{\overline{m}_i\}$ for $(K^*X)/I$ over K^*. Since $K^*X = (E^*X)/I_n$, we can lift the $\overline{m}_i$ to get elements $m_i \in E^*X$. Define

$J = \bigvee_i \Sigma^{|m_i|}E$, so the m's give an evident map $m\colon X \longrightarrow J$. Define Y to be the cofibre. It is immediate, using the arguments of Corollary 2.30, that $K^*J \longrightarrow K^*X$ is the Σ^*-projective cover, and thus that the kernel K^*Y has projective dimension $d-1$. In the exceptional case $d = 0$, we see that $K^*X = K^*J$ and thus $K^*Y = 0$; as Y is K-local, we conclude that $Y = 0$. In any case, because Σ^* is Noetherian, we see that K^*Y is finitely generated over Σ^*.

By construction, the image of E^*J in $(E^*X)/I_n = K^*X$ is K^*X. As E^*J and E^*X are L-complete, we conclude from Theorem A.6 that $E^*J \longrightarrow E^*X$ is epi. As E^*X is pro-free, we conclude that this epimorphism splits. As E^*J is also pro-free, we see that E^*Y is again pro-free. □

In the next lemma, we use the derived category $\mathcal{D}_E$ of E-modules defined as in [EKMM96]; see Section 1 for more discussion.

Lemma 8.11. *Let M be an object of $\mathcal{D}_E$ such that π_*M is a finitely generated module over E_*. Then M is E-finite in $\mathcal{D}_E$ (and thus also in the category of spectra).*

Proof. This is very similar to the last result. We can recursively construct finitely generated free modules P_k over the ring spectrum E and cofibrations $M_{k+1} \longrightarrow P_k \longrightarrow M_k$ (with $M_0 = M$), such that the modules π_*M_k form a minimal projective resolution for π_*M over E_*. As E_* has finite global dimension n, we see that $M_{n+1} = 0$, and thus M is E-finite as claimed. □

We can now prove the theorem.

Proof of Theorem 8.9. Jeff Smith has constructed a finite p-local spectrum X with only even cells such that

$$\mathrm{Ext}^k_{\Sigma^*}(K^*X, K^*) = \mathrm{Ext}^k_{\Sigma_*}(K_*, K_*X) = 0 \text{ for } k > n^2.$$

See [Rav92a, Section 8.3] for a proof of this. Ravenel actually shows that

$$\mathrm{Ext}^k_{BP_*BP}(BP_*, v_n^{-1}BP_*X/I_n) = 0$$

for large k, but the Miller-Ravenel change of rings theorem [Rav86, Theorem 6.6.1] and the fact that X has only even cells implies that this is equal to $\mathrm{Ext}_{\Sigma_*}(K_*, K_*X)$. That we can choose the horizontal vanishing line to be n^2 does not appear to be in the published literature, but is not important for this argument. As X has only even cells, the Atiyah-Hirzebruch spectral sequence collapses to show that E^*X is free of finite rank over E^*. It follows *a fortiori* that $K^*X = E^*X/I_n$ is finitely generated over Σ^*. Now choose a minimal projective resolution P_* of K^*X as in Corollary 2.30. We find that the complex $\mathrm{Hom}_{\Sigma^*}(P_*, K^*)$ has zero differential, so the projective dimension of K^*X is at most n^2. Thus Lemma 8.10 applies to $X_0 = \widehat{L}X$ to give a sequence of cofibrations $X_k \longrightarrow J_k \longrightarrow X_{k+1}$, where K^*X_k has projective dimension at most $n^2 - k$ over R, and $X_k = 0$ for $k > n^2$. As each J_k is a finite wedge of E's, we conclude that $X_0 = \widehat{L}X$ is E-finite. By a thick subcategory argument, we even see that $\widehat{L}S$ is E-finite.

Now let $Y \in \mathcal{K}$ be dualisable. By the above, $Y = \widehat{L}S \wedge_{\mathcal{K}} Y$ is in the thick subcategory generated by $M = E \wedge_{\mathcal{K}} Y$. Moreover, because Y is dualisable, we know that π_*M is a finitely generated module over π_*E. It follows by Lemma 8.11 that M is E-finite in $\mathcal{D}_E$ (and thus in the category of spectra). It follows that everything in the thick subcategory generated by M is E-finite, and thus that Y is E-finite. □

Corollary 8.12. *If $X \in \mathcal{K}$ is small, then X is K-finite, and in particular $\pi_k X$ is finite for all k.*

Proof. Because X is small it is easy to see using Theorem 8.5 and Proposition 4.17 that the unit map $X \longrightarrow S/I \wedge X$ is a split monomorphism for some generalised Moore space S/I of height n. As $\widehat{L}S$ is E-finite, we see that $S/I \wedge X$ lies in the thick subcategory generated by $E \wedge S/I \wedge X = E/I \wedge X$. It is easy to see that this lies in the thick subcategory generated by $K \wedge X$, which is a finite wedge of suspensions of K. This implies that X is K-finite. □

9. Homology and cohomology functors

In this section, we discuss the representability of (co)homology functors from $\mathcal{K}$ to the category **Ab** of Abelian groups. We write $\mathcal{K}_\bullet$ and $\mathcal{K}^\bullet$ for the categories of homology and cohomology functors.

Theorem 9.1. *Every cohomology functor $H : \mathcal{K} \longrightarrow \mathbf{Ab}$ is uniquely representable. More precisely, the Yoneda functor $X \mapsto [-, X]$ is an equivalence $\mathcal{K} \simeq \mathcal{K}^\bullet$.*

Proof. Observe that $H \circ \widehat{L}$ is a cohomology theory on $\mathcal{S}$, and apply the Brown representability theorem for $\mathcal{S}$. It is easy to check that the representing spectrum automatically lies in $\mathcal{K}$. □

Before we discuss the more complicated representability of homology functors, we discuss the closely related subject of realizing any object of $\mathcal{K}$ as a minimal weak colimit of small spectra. Recall that colimits do not usually exist in triangulated categories, but that weak colimits always exist but are not unique. Minimal weak colimits, discussed in [HPS95, Section 2.2], often exist and are always unique (in algebraic stable homotopy categories).

We define

$$\Lambda'(X) = \{X' \xrightarrow{u} X \mid X' \in \mathcal{K} \text{ is small } \}$$

$$\Lambda''(X) = \{X'' \xrightarrow{u} X \mid X'' \in \mathcal{S} \text{ is a finite spectrum of type at least } n\}.$$

As usual, we make $\Lambda'(X)$ and $\Lambda''(X)$ into categories, in which the morphisms are commutative triangles. It is not hard to see that they are filtered (compare [HPS95, Proposition 2.3.9]). Recall that every $\mathcal{K}$-small spectrum is a retract of LY for some finite spectrum Y of type at least n, and that there are only countably many isomorphism classes of finite spectra. It follows that $\Lambda'(X)$ and $\Lambda''(X)$ both have only a set of isomorphism classes, in other words that they are essentially small.

Proposition 9.2. *For any $X \in \mathcal{K}$ we have*

$$X = \underset{\Lambda'(X)}{\underrightarrow{\mathrm{mwlim}}}\, X' = \underset{\Lambda''(X)}{\underrightarrow{\mathrm{mwlim}}}\, \widehat{L}X''$$

(*where the minimal weak colimits are computed in $\mathcal{K}$*)*. Moreover, the evident functor $\widehat{L}\colon \Lambda''(X) \longrightarrow \Lambda'(X)$ is cofinal.*

Proof. Let $L^f X$ be the finite localisation of X away from $F(n)$ and $C^f X$ the corresponding acyclisation. We then have a cofibration $C^f X \longrightarrow X \longrightarrow L^f X$, and $[F(n), L^f X] = 0$, so $F(n) \wedge L^f X = 0$. As $K_* F(n) \neq 0$ and we have a Künneth isomorphism, we see that $K_* L^f X = 0$, so $\widehat{L}L^f X = 0$ and $\widehat{L}C^f X = \widehat{L}X = X$.

By [HPS95, Proposition 2.3.17], we know that $C^f X = \underrightarrow{\mathrm{mwlim}}^{\mathcal{S}}_{\Lambda''(X)} X''$. As localisation functors preserve minimal weak colimits [HPS95, Theorem 3.5.1], we find that $X = \widehat{L}C^f X = \underrightarrow{\mathrm{mwlim}}^{\mathcal{K}}_{\Lambda''(X)} \widehat{L}X''$.

We now prove that $\widehat{L} : \Lambda''(X) \longrightarrow \Lambda'(X)$ is cofinal. Suppose that $(X' \xrightarrow{u} X) \in \Lambda'(X)$. Because X' is small, $[X', -]$ is a homology functor on $\mathcal{K}$. It follows that

$$[X', X] = \varinjlim_{\Lambda''(X)} [X', \widehat{L}X''].$$

We can apply this fact to the element $u \in [X', X]$. The conclusion is that there is a map $X' \longrightarrow \widehat{L}X''$ over X, for some $X'' \in \Lambda''(X)$. Moreover, given two such maps $X' \longrightarrow \widehat{L}X''_i$ (for $i = 0, 1$), there are maps $X''_0 \longrightarrow X''_2 \longleftarrow X''_1$ in $\Lambda''(X)$ such that the two composites $X' \longrightarrow \widehat{L}X''_2$ are the same. This means precisely that $\widehat{L} : \Lambda''(X) \longrightarrow \Lambda'(X)$ is cofinal. It follows easily that $X = \underrightarrow{\mathrm{mwlim}}_{\Lambda'(X)} X'$. $\square$

Recall from [HPS95, Section 4.1] that a homology functor $H : \mathcal{K} \longrightarrow \mathbf{Ab}$ is *representable* if there is a spectrum $X \in \mathcal{K}$ and a natural isomorphism $H(Y) \simeq [S, X \wedge Y]$ for $\mathcal{K}$-small spectra Y. (It does not matter whether the smash product is interpreted in $\mathcal{S}$ or $\mathcal{K}$, because Y is small). Recall also that for any X and Y in $\mathcal{K}$ we have

$$M(X \wedge_{\mathcal{K}} Y) = M(X \wedge_{\mathcal{S}} Y) = MS \wedge_{\mathcal{S}} X \wedge_{\mathcal{S}} Y.$$

Lemma 9.3. *A natural isomorphism $H(Y) \simeq [S, X \wedge Y]$ for small spectra Y gives rise to a natural isomorphism $H(Y) \simeq [S, M(X \wedge Y)] = MX_0 Y$ for all $Y \in \mathcal{K}$.*

Proof. Write $\{Y_i\}$ for the objects of $\Lambda''(Y)$ (as in Proposition 9.2). Thus each Y_i is a finite spectrum of type at least n, so $LY_i = MY_i = \widehat{L}Y_i$ is K-nilpotent by Proposition 6.15. Moreover, $Y = \underrightarrow{\mathrm{mwlim}}^{\mathcal{K}}_i LY_i$ and thus $HY = \varinjlim_i [S, X \wedge Y_i]$.

Next, because $X \in \mathcal{K}$ we see that $X = LX$, so there is a cofibration $MX \longrightarrow X \longrightarrow L_{n-1}X$. Because $L_{n-1}X \wedge_{\mathcal{S}} Y_i = X \wedge_{\mathcal{S}} L_{n-1}Y_i = 0$, we see that

$$X \wedge Y_i = MX \wedge Y_i.$$

We know that $\underrightarrow{\mathrm{mwlim}}^{\mathcal{S}}_i Y_i = C^f Y$, using the notation of the proof of Proposition 9.2. Moreover $[F(n), C^f Y]_* = [F(n), Y]_*$, so $DF(n) \wedge C^f Y = DF(n) \wedge Y$, so $MC^f Y = MY$ (since $MZ = MS \wedge Z$ and $MS \sim LS \wedge DF(n)$). It follows that

$$\underrightarrow{\mathrm{mwlim}}^{\mathcal{S}}_i MX \wedge Y_i = MX \wedge C^f Y = MX \wedge Y.$$

This gives an isomorphism

$$[S, MX \wedge Y] \simeq \varinjlim_i [S, MX \wedge Y_i] \simeq \varinjlim_i [S, X \wedge Y_i] \simeq H(Y). \quad \square$$

Before stating the representability theorem for homology functors, it is convenient to record the definition of phantom maps.

Definition 9.4. A map $f : X \longrightarrow Y$ in $\mathcal{K}$ is *phantom* if $(Z \xrightarrow{u} X \xrightarrow{f} Y) = 0$ for all small Z and all $u : Z \longrightarrow X$. We also say that f is *cophantom* if $(X \xrightarrow{f} Y \xrightarrow{u} Z) = 0$ for all small Z and all $u : Y \longrightarrow Z$.

Theorem 9.5. *Any homology functor $H\colon \mathcal{K} \longrightarrow \mathbf{Ab}$ is representable. More precisely, there is a spectrum $X \in \mathcal{K}$ such that $H(Y) = X_0Y = \pi_0(X \wedge Y)$ for all $\mathcal{K}$-small Y, and $H(Y) = MX_0Y = \pi_0(MX \wedge Y)$ for all $Y \in \mathcal{K}$. The group of natural maps $MX_0 \longrightarrow MY_0$ is naturally isomorphic to $[X,Y]/$phantoms. Any composite of a phantom with another map is phantom, and any composite of two phantoms is zero.*

Proof. This follows easily by combining the results of [HPS95, Section 4.1] with Lemma 9.3. □

10. Brown-Comenetz duality

In this section, we define and study a good notion of Brown-Comenetz duality on the category $\mathcal{K}$. We have been strongly influenced by the outline in [HG94]. Recall the usual definition of Brown-Comenetz duality. In keeping with our convention that everything is p-local, we write $\mathbf{Q}/\mathbf{Z}$ for $\mathbf{Q}/\mathbf{Z}_{(p)}$. Because π_0X is a homology functor on $\mathcal{S}$ and $\mathbf{Q}/\mathbf{Z}$ is an injective Abelian group, we see that $\mathrm{Hom}(\pi_0Y, \mathbf{Q}/\mathbf{Z})$ is a cohomology functor. By the Brown representability theorem, there is a spectrum I unique up to canonical isomorphism, with a natural isomorphism

$$\mathrm{Hom}(\pi_0Y, \mathbf{Q}/\mathbf{Z}) \simeq [Y, I].$$

We also write $IX = F(X, I)$, so there is a natural isomorphism $\mathrm{Hom}(X_0Y, \mathbf{Q}/\mathbf{Z}) \simeq [Y, IX]$. Recall from Section 9 that any homology functor on $\mathcal{K}$ has the form $Y \mapsto \pi_0(MX \wedge Y)$ for some $X \in \mathcal{K}$. This motivates the following definition:

Definition 10.1. If $X \in \mathcal{K}$, the *Brown-Comenetz dual* of X is

$$\widehat{I}X = IMX = F(MX, I).$$

We also write $\widehat{I} = \widehat{I}S$, so that $\widehat{I}X = F(X, \widehat{I})$.

Theorem 10.2. (a) *There is a natural isomorphism*

$$[Y, \widehat{I}X] \simeq \mathrm{Hom}(MX_0Y, \mathbf{Q}/\mathbf{Z}).$$

(b) *The correspondence $X \mapsto \widehat{I}X$ is a contravariant exact functor $\mathcal{K}^{\mathrm{op}} \longrightarrow \mathcal{K}$.*
(c) *$\widehat{I}K \simeq K$.*
(d) *The natural map $X \longrightarrow \widehat{I}^2X$ is an isomorphism when $\pi_*(F(n) \wedge X)$ is finite in each degree (in particular, when X is dualisable, for example if $X = S$).*
(e) *The spectrum $\widehat{I}$ is invertible under the smash product.*
(f) *The functor $X \mapsto \widehat{I}X$ preserves the categories $\mathcal{F}$, $\mathcal{D}$, $\mathcal{I}$ and $\mathcal{N}$ of small, dualisable, K-injective and K-nilpotent spectra.*

The isomorphism classes of invertible spectra form the Picard group Pic, studied in Section 14 below and in [HMS94, Str92]. There is a natural homomorphism from Pic to an algebraically defined Picard group, as explained in [HMS94]. It is natural to ask where $\widehat{I}$ goes under this map. The answer is stated in [HG94], but we will not discuss this issue here.

We will prove Theorem 10.2 after two lemmas.

Lemma 10.3. *Let $\{A_k\}$ be an inverse system of finite-dimensional vector spaces of dimension at most d. Then $\varprojlim_k A_k$ has dimension at most d (and in particular is finite-dimensional), and $\varprojlim^1_k A_k = 0$.*

Proof. This follows easily from the standard Mittag-Leffler construction. Explicitly, we define

$$A'_k = \bigcap_{m>k} \operatorname{image}(A_m \to A_k).$$

As A_k has finite dimension we must have $A'_k = \operatorname{image}(A_m \to A_k)$ for $m \gg 0$. The groups A'_k form a subtower of A, and the quotient A/A' is pro-trivial so $\varprojlim_i^s A_i = \varprojlim_i^s A'_i$ for $s = 0, 1$. The maps $A'_k \to A'_{k-1}$ are surjective so $\varprojlim_i^1 A'_i = 0$. The dimension of A'_i is a nondecreasing function of i that is bounded by d, so for large i it is constant and thus the map $A'_i \to A'_{i-1}$ is an isomorphism. This means that $\varprojlim_i A'_i = A'_m$ for large m, so $\varprojlim_i A'_i$ has dimension at most d. □

The next lemma is crucial.

Lemma 10.4. *Let S/I be a localised generalised Moore spectrum of type n. Then $[E, S/I]_*$ is finite.*

Proof. We can form the standard E-based Adams resolution of S/I and apply the functor $[E, -]$ to get a spectral sequence whose E_1 term is the group

$$C^{st}(I) = E_1^{s,t} = [E, E \wedge \Sigma^{-s}\overline{E}^{(s)} \wedge S/I]_{t-s} = [E, \Sigma^{-s}\overline{E}^{(s)} \wedge E/I]_{t-s}.$$

Here $\overline{E}$ is the cofibre of the unit map $S \to E$. As E_*E is not a projective module over E_*, the standard theorems are not enough to identify the E_2 page as an Ext group but this turns out not to be important. What is important is that S/I is K-nilpotent and thus E-nilpotent, so the resulting spectral sequence converges to $[E, S/I]_*$ (by the argument of Proposition 6.5). It will thus be enough to show that the E_2 page is finite in each total degree.

To do this, we define $C^{st} = [E, E \wedge \Sigma^{-s}\overline{E}^{(s)}]_{t-s}$. We make C^{**} into a module over E^* using the second copy of E. Using Proposition 8.4 (f) we see that $\pi_*\widehat{L}(E \wedge \overline{E}^{(s)})$ is pro-free and in even degrees, so that $\widehat{L}(E \wedge \overline{E}^{(s)})$ is isomorphic as an E-module to a spectrum of the form $\widehat{L}(\bigvee_i \Sigma^{2d_i} E)$. This is a retract of $\prod_i \Sigma^{2d_i} E$ so C^{s*} is a retract of $\prod_i [E, \Sigma^{2d_i} E]^*$ and thus is pro-free. It is not hard to deduce that $C^{**}(I) = C^{**}/IC^{**}$ and that this admits a finite filtration with quotients $C^{**}/I_n C^{**} = C^{**}(I_n)$. It will thus be enough to show that $H^*C^{**}(I_n)$ is finite in each total degree. But we have $E/I_n = K$, so by Proposition 3.4 we have

$$E_1^{s,t} = \operatorname{Hom}_{K_*}^{t-s}(K_*E, \Sigma^{-s}K_*(\overline{E}^{(s)})) = \operatorname{Hom}_{K_*}^t(\Sigma_*, \overline{\Sigma}_*^{\otimes s}) = \operatorname{Hom}_{\Sigma_*}^t(\Sigma_*, \Sigma_* \otimes \overline{\Sigma}_*^{\otimes s})$$

where $\overline{\Sigma}_*$ is the coaugmentation coideal of Σ_*. It follows that

$$H^*C^{**}(I_n) = \operatorname{Ext}_{\Sigma_*}^{**}(\Sigma_*, K_*).$$

Consider the subring $\Sigma'(k)_* < \Sigma_*$ generated by $t_1, \ldots, t_{k-1}$. It is easy to check that this is a sub Hopf algebra, and thus a subcomodule of Σ_*. Moreover, we have $\Sigma_* = \bigcup_k \Sigma'(k)_*$, which gives a short exact sequence of comodules

$$\bigoplus_k \Sigma'(k)_* \rightarrowtail \bigoplus_k \Sigma'(k)_* \twoheadrightarrow \Sigma_*.$$

This in turn gives a Milnor sequence relating $\operatorname{Ext}_{\Sigma_*}(\Sigma_*, K_*)$ to $\varprojlim^i \operatorname{Ext}_{\Sigma_*}(\Sigma'(k)_*, K_*)$ for $i = 0, 1$. As $\Sigma'(k)$ is a finite-dimensional cocommutative Hopf algebra, it is self-dual (up to a dimension shift) as a comodule over itself [LS69] and thus as a comodule over Σ_*. We thus have

$$\mathrm{Ext}_{\Sigma_*}(\Sigma'(k)_*, K_*) = \mathrm{Ext}_{\Sigma_*}(K_*, \Sigma'(k)_*) = \mathrm{Ext}_{\Sigma_*//\Sigma'(k)_*}(K_*, K_*).$$

We know from [Rav86, Theorem 6.3.7] that for $k \gg 0$ this is an exterior algebra over K^* on n^2 generators, and thus has dimension 2^{n^2} independently of k. It follows from Lemma 10.3 and the Milnor sequence that $\mathrm{Ext}^{**}_{\Sigma_*}(\Sigma_*, K_*)$ has finite dimension over K_*, as required. □

Corollary 10.5. *$[K, S]_*$ is finite.*

Proof. It follows from the lemma that $[E \wedge DS/I, S]_*$ is finite, and it is easy to see that K lies in the thick subcategory generated by $E \wedge DS/I$. □

Proof of Theorem 10.2. (a): This is clear.

(b): We first need to show that $\widehat{I}X$ is K-local. Suppose that Z is K-acyclic. As $MS \sim K$ and $MX = MS \wedge X$, we see that $MX \wedge Z = 0$, so

$$[Z, \widehat{I}X]_* = \mathrm{Hom}(\pi_*(MX \wedge Z), \mathbf{Q}/\mathbf{Z}) = 0$$

as required. Because $\widehat{I}X = F(X, \widehat{I})$, the correspondence $X \mapsto \widehat{I}X$ is clearly a contravariant exact functor.

(c): Note that $MK = K$. Think of $\pi_0 K = \mathbf{F}_p$ as a subgroup of $\mathbf{Q}/\mathbf{Z}$ in the obvious way. We know that $[X, K]_* \simeq \mathrm{Hom}_{K_*}(K_*X, K_*)$. Looking in degree zero, we see that

$$[X, K] \simeq \mathrm{Hom}(MK_0X, \mathbf{F}_p) = \mathrm{Hom}(MK_0X, \mathbf{Q}/\mathbf{Z})$$

the first equality because v_n is a unit, and the second equality because MK_0X is a vector space over $\mathbf{F}_p$. It follows that $\widehat{I}K = K$.

(d): The evaluation map $X \wedge F(X, \widehat{I}) \longrightarrow \widehat{I}$ gives by adjunction a natural map $\kappa : X \longrightarrow F(F(X, \widehat{I}), \widehat{I}) = \widehat{I}^2 X$. Write $F = F(n)$ and $Y = F \wedge X$, so that $\pi_* Y$ is finite in each degree. Because F is dualisable, we have $\widehat{I}Y = DF \wedge \widehat{I}X$ and $\widehat{I}^2 Y = F \wedge \widehat{I}^2 X$. Under this identification, we have $\kappa_Y = \kappa_{F\wedge X} = 1_F \wedge \kappa_X$, so it is enough to show that κ_Y is an isomorphism. Because F and DF are K-nilpotent, we see that $MY = Y$ and $M\widehat{I}Y = \widehat{I}Y$. It follows that $\pi_* \widehat{I}^2 Y = \mathrm{Hom}(\mathrm{Hom}(\pi_* Y, \mathbf{Q}/\mathbf{Z}), \mathbf{Q}/\mathbf{Z})$, which is the same as $\pi_* Y$ because $\pi_* Y$ is finite in each degree. Thus κ_Y is an isomorphism, as required.

If X is dualisable then $F \wedge X$ is small by Theorem 8.5, so $\pi_*(F \wedge X)$ is finite in each degree by Corollary 8.12, so the above applies.

(e): We know from Lemma 10.5 that $[K, S]_*$ is finite in each degree. We also know from (c) and (d) that κ_K and κ_S are isomorphisms, from which it follows that $\widehat{I} : [K, S]_* \longrightarrow [\widehat{I}, \widehat{I}K]_* = K^*\widehat{I}$ is an isomorphism. Thus $K^*\widehat{I}$ is finite, which implies by Theorem 8.6 that $\widehat{I}$ is dualisable. This implies that $F(\widehat{I}, \widehat{I}) = D\widehat{I} \wedge \widehat{I}$, but we also know that $F(\widehat{I}, \widehat{I}) = \widehat{I}^2 S = S$. Thus $D\widehat{I} \wedge \widehat{I} \simeq S$, which means that $\widehat{I} \in \mathrm{Pic}$.

(f): By part (c), $\widehat{I}K = K$. Any K-injective spectrum X is a wedge of suspensions of K, so $\widehat{I}X$ is a product of suspensions of K, which is still K-injective by Proposition 3.3. Thus $X \mapsto \widehat{I}X$ preserves $\mathcal{I}$, and so also $\mathcal{N}$, by Lemma 7.4. Part (e) implies that $\widehat{I} \in \mathcal{D}$, so it follows from Theorems 8.6 and 8.5 that the functor $X \mapsto \widehat{I}X = DX \wedge \widehat{I}$ preserves $\mathcal{D}$ and $\mathcal{F}$. □

Corollary 10.6. *If X is an arbitrary spectrum in $\mathcal{K}$, then $\widehat{I}X \simeq DX \wedge \widehat{I}$.*

Proof. Because $\widehat{I}$ is dualisable, we have $F(X,\widehat{I}) \simeq F(X,S)\wedge \widehat{I}$. □

Recall the definition of phantom and cophantom maps from Definition 9.4.

Theorem 10.7. *Suppose that $X, Y \in \mathcal{K}$.*

(a) *There are no nonzero phantom maps $X \longrightarrow IY$.*
(b) *If $\pi_*(F(n)\wedge Y)$ is finite in each degree (in particular, if Y is dualisable, for example if $Y = S$) then there are no nontrivial phantom maps $X \longrightarrow Y$.*
(c) *A map $u : X \longrightarrow Y$ is phantom if and only if it is cophantom.*

Proof. (a): Suppose that $u : X \longrightarrow IY$ is phantom. This means that the corresponding map $u^\# : \pi_0(X \wedge MY) = \pi_0(MX \wedge Y) \longrightarrow \mathbf{Q}/\mathbf{Z}$ vanishes on $\pi_0(Z \wedge MY)$, for every small spectrum Z equipped with a map $Z \longrightarrow X$. On the other hand, X is the minimal weak colimit in $\mathcal{K}$ of these Z's (Proposition 9.2), and $\pi_0(-\wedge MY)$ is a homology functor, so $\pi_0(X\wedge MY) = \varinjlim_Z \pi_0(Z\wedge MY)$. It follows that $u^\# = 0$, and thus $u = 0$.

(b): By part (d) of Theorem 10.2, we know that $Y = \widehat{I}^2 Y$. It therefore follows from (a) that there are no nonzero phantoms $X \longrightarrow \widehat{I}^2 Y = Y$.

(c): In the following statements, Z runs over $\mathcal{K}$-small spectra. Thus $MDZ = DZ$ and

$$\mathrm{Hom}([Z,X],\mathbf{Q}/\mathbf{Z}) = [X, Z\wedge \widehat{I}].$$

$$\begin{aligned}
u \text{ is phantom} \quad &\Leftrightarrow \quad \forall Z \quad [Z,X] \longrightarrow [Z,Y] \text{ is zero}\\
&\Leftrightarrow \quad \forall Z \quad \mathrm{Hom}([Z,X],\mathbf{Q}/\mathbf{Z}) \longleftarrow \mathrm{Hom}([Z,Y],\mathbf{Q}/\mathbf{Z}) \text{ is zero}\\
&\Leftrightarrow \quad \forall Z \quad [X, Z\wedge\widehat{I}] \longleftarrow [Y, Z\wedge \widehat{I}] \text{ is zero}\\
&\Leftrightarrow \quad \forall Z \quad [X,Z] \longrightarrow [Y,Z] \text{ is zero}\\
&\Leftrightarrow \quad u \text{ is cophantom}
\end{aligned}$$

(The fourth implication uses the fact that $\widehat{I} \in \mathrm{Pic}$, so that $Z \mapsto Z\wedge\widehat{I}$ is an automorphism of the category of small spectra). □

11. The natural topology

As in [HPS95, Section 4.4], we can give $[X,Y]$ a natural topology. For any map $f : F \longrightarrow X$ with F small, we write $U_f = \{u : X \longrightarrow Y \mid u\circ f = 0\}$, and then give $[X,Y]$ the linear topology for which the sets U_f form a basis of neighbourhoods of 0. We recall from [HPS95, Proposition 4.4.1] the basic properties of this topology:

Proposition 11.1.

(a) *The composition map $[X,Y]\times[Y,Z] \longrightarrow [X,Z]$ is continuous.*
(b) *Any pair of maps $X' \longrightarrow X$ and $Y \longrightarrow Y'$ induces a continuous map $[X,Y] \longrightarrow [X',Y']$.*
(c) *If X is small then $[X,Y]$ is discrete.*
(d) *The closure of 0 in $[X,Y]$ is the set of phantom maps, so $[X,Y]$ is Hausdorff if and only if there are no phantoms from X to Y.*
(e) *$[X,Y]$ is always complete.*
(f) *$[\bigvee_i X_i, Y]$ is homeomorphic to $\prod_i [X_i,Y]$ with the product topology, but the natural topology on $[X,\prod_i Y_i]$ is finer than the product topology in general.*

Corollary 11.2. *If we give* $\mathrm{Hom}(\pi_k X, \pi_k Y)$ *the compact-open topology, then it follows from* (a) *that the evident map*

$$[X,Y] \longrightarrow \mathrm{Hom}(\pi_k X, \pi_k Y)$$

is continuous.

We next prove some additional properties.

Proposition 11.3. *The smash product map* $[W,X]\times[Y,Z]\longrightarrow[W\wedge Y, X\wedge Z]$ *is continuous.*

Proof. Suppose we have maps $u\colon W\longrightarrow X$ and $v\colon Y\longrightarrow Z$, a $\mathcal{K}$-small spectrum A and a map $A\xrightarrow{f}W\wedge Y$, so that

$$N=\{w\colon W\wedge Y\longrightarrow X\wedge Z \mid wf=(u\wedge v)f\}$$

is a basic neighbourhood of $u\wedge v$ in $[W\wedge Y, X\wedge Z]$. We know from Proposition 9.2 that W can be written as a minimal weak colimit $W=\underset{\longrightarrow\,\alpha}{\mathrm{mwlim}}\, W_\alpha$ with W_α small, and similarly $Y=\underset{\longrightarrow\,\beta}{\mathrm{mwlim}}\, Y_\beta$. Note that $[A, W\wedge(-)]$ is a homology theory, so that $[A, W\wedge Y]=\underset{\longrightarrow\,\beta}{\lim}\,[A, W\wedge Y_\beta]$, so f factors through a map $g\colon A\longrightarrow W\wedge Y_\beta$ for some β. By a similar argument, g factors through a map $h\colon A\longrightarrow W_\alpha\wedge Y_\beta$ for some α. Write i and j for the given maps $W_\alpha\longrightarrow W$ and $Y_\beta\longrightarrow Y$. Write

$$N'=\{(u',v')\in[W,X]\times[Y,Z] \mid u'i=ui \text{ and } v'j=vj\}.$$

Then N' is a neighbourhood of (u,v) and if $(u',v')\in N'$ then $u'\wedge v'\in N$. This means that the smash product is continuous at (u,v), as required. □

Corollary 11.4. *The adjunction map* $[X, F(Y,Z)]\longrightarrow[X\wedge Y, Z]$ *is a continuous bijection, and a homeomorphism if* Y *is dualisable.*

Proof. Let $\epsilon\colon F(Y,Z)\wedge Y\longrightarrow Z$ be the evaluation map. Then the adjunction map is the composite

$$[X,F(Y,Z)]\xrightarrow{(-)\wedge 1_Y}[X\wedge Y, F(Y,Z)\wedge Y]\xrightarrow{\epsilon_*}[X\wedge Y, Z],$$

which is continuous by Proposition 11.3 and Proposition 11.1(a). It is also a bijection, by the defining property of $F(Y,Z)$. If Y is dualisable and $\eta\colon S\longrightarrow Y\wedge DY=F(Y,Y)$ is the unit map then we can identify the inverse of the adjunction map with the composite

$$[X\wedge Y, Z]\xrightarrow{(-)\wedge 1}[X\wedge Y\wedge DY, Z\wedge DY]\xrightarrow{(1\wedge\eta)^*}[X, Z\wedge DY],$$

which is again continuous. □

Proposition 11.5. *If* $[F(n),Y]_*$ *is finite, then for any* $X\in\mathcal{K}$ *we have*

$$[X,Y]=\underset{\longleftarrow}{\lim}\{[X',Y]\mid(X'\to X)\in\Lambda'(X)\},$$

and this is a compact Hausdorff group.

Remark 11.6. This applies when Y is E-finite, and thus (by Theorem 8.9) when Y is dualisable.

Proof. Because X is the minimal weak colimit of $\Lambda'(X)$, the map from $[X,Y]$ to the inverse limit is surjective. The kernel is the set of phantom maps $X\longrightarrow Y$, which is zero by part (b) of Theorem 10.7. □

Corollary 11.7. *If X is dualisable and Y is small then $[X, Y]$ is both finite (by Corollary 8.12) and compact Hausdorff, hence discrete.* □

Corollary 11.8. *If X and Y are dualisable then the map $D\colon [X, Y] \to [DY, DX]$ is a homeomorphism.*

Proof. As $D^2 = 1$ it is enough to check that D is open. Consider a $\mathcal{K}$-small spectrum Z and a map $f\colon Z \to X$, so that $U_f = \{u\colon X \to Y \mid uf = 0\}$ is a basic open neighbourhood of 0 in $[X, Y]$. The image under D is $\{v\colon DY \to DX \mid (Df)\circ v = 0\}$. This is open in $[DY, DX]$ because $(Df)_*$ is continuous and $[DY, Z]$ is discrete. □

Proposition 11.9. *If X is dualisable, then the natural topology on $[X, E]$ is the I_n-adic topology. This is also the same as the topology defined by the kernels of the maps $[X, E] \to [X, E/I]$ as I runs over a tower as in Proposition 4.22. Similar remarks apply to $E^\vee_* X$.*

Proof. The natural topology is profinite by Proposition 11.5. The I_n-adic topology is profinite because E^*X is finitely generated over E^* by Theorem 8.6 and Proposition 2.4. Using Theorem 8.6 again and a thick subcategory argument we see that $(E/I)^*X$ is finite. We also know from Corollary 7.11 that $E = \operatorname{holim}_{\leftarrow I} E/I$ and there are no $\lim^1_{\leftarrow}$ terms because everything is finite so $[X, E] = \lim_{\leftarrow I} [X, E/I]$ and the topology defined by the kernels is again profinite. As the maps $[X, E] \to [X, E/I]$ are continuous, we see that the natural topology is at least as large as the topology defined by the E/I, and this is at least as large as the I_n-adic topology because the groups $[X, E/I]$ are I_n-torsion. By a well-known lemma, comparable compact Hausdorff topologies are always equal. The result carries over to $E^\vee_* X$ because of Corollary 11.4. □

12. Dualisable spectra

In this section we prove some more results about the category $\mathcal{D}$ of dualisable spectra in $\mathcal{K}$, and its subcategory $\mathcal{F}$ of $\mathcal{K}$-small spectra.

First observe that $\mathcal{F}$ is the same as the category $\mathcal{C}_n \subseteq \mathcal{L}$ considered in Section 6.1. The following three results thus follow from Theorem 6.9, Theorem 6.12 and Corollary 6.6.

Proposition 12.1. *The only thick subcategories of $\mathcal{F}$ are $\{0\}$ and $\mathcal{F}$.* □

Proposition 12.2. *Every small spectrum X has a good v_n self map $v\colon \Sigma^d X \to X$ (which is an isomorphism).* □

Proposition 12.3. *Let $u\colon \Sigma^d X \to X$ be a self map of a $\mathcal{K}$-small spectrum such that $K_* u$ is nilpotent. Then u is nilpotent.* □

For self maps of spectra that are dualisable but not small it is natural to consider self maps that are topologically nilpotent rather than nilpotent. We first need to define topological nilpotence.

Proposition 12.4. *Let X be a spectrum in $\mathcal{K}$ and $u\colon \Sigma^d X \to X$ a map. Let $u^{-1}X$ denote the sequential colimit in $\mathcal{K}$ of the sequence*

$$X \xrightarrow{u} \Sigma^{-d} X \xrightarrow{u} \Sigma^{-2d} X \to \dots .$$

Consider the following statements, in which Z and W run over $\mathcal{K}$-small spectra.

(a) $u^{-1}X = 0$.

(b) *For all Z and all $z\colon Z \longrightarrow X$ we have $u^N \circ z = 0$ for $N \gg 0$.*
(c) *For all Z we have $1_Z \wedge u^N = 0$ for $N \gg 0$.*
(d) *For all W and all $w\colon W \longrightarrow S$, the following composite vanishes for $N \gg 0$:*

$$\Sigma^{Nd} W \wedge X \xrightarrow{w \wedge 1} \Sigma^{Nd} X \xrightarrow{u^N} X.$$

Equivalently, the following adjoint map vanishes:

$$\Sigma^{Nd} W \xrightarrow{w} S^{Nd} \xrightarrow{(u^{\#})^N} F(X, X).$$

(e) *$d = 0$ and $u^N \longrightarrow 0$ in the natural topology on $[X, X]$.*

Then (a)⇔(b)⇐(c) and (b)⇐(d). If X is dualisable then (b)⇔(c)⇔(d). If $d = 0$ then (c)⇔(e).

Definition 12.5. If (a) and (b) hold in the above proposition, we say that u is *topologically nilpotent.*

Proof of Proposition 12.4. (a)⇔(b): We know that $u^{-1}X = 0$ if and only if we have $[Z, u^{-1}X] = 0$ for all Z, and as Z is small we have $[Z, u^{-1}X] = u^{-1}[Z, X]$. The claim follows easily.

(a)⇐(c): If (c) holds then we see that $Z \wedge u^{-1}X = 0$ and thus $u^{-1}X = 0$, so (a) holds. Conversely, suppose (a) holds and X is dualisable. Then $Z \wedge X$ is small by Theorem 8.5, and $(1 \wedge u)^{-1}(Z \wedge X) = 0$. It follows that (c) holds.

(d)⇒(b): By Corollary 7.11 we have $X = \underrightarrow{\mathrm{holim}}_I\, D(S/I) \wedge X$, so any map $z\colon Z \longrightarrow X$ factors through $D(S/I) \wedge X \longrightarrow X$ for some I. If (d) holds then $D(S/I) \wedge X \longrightarrow X \xrightarrow{u^N} \Sigma^{-Nd} X$ vanishes for $N \gg 0$ and thus so does $Z \longrightarrow X \xrightarrow{u^N} \Sigma^{-Nd} X$, so (b) holds.

We next assume that X is dualisable; then if (b) holds we can put $Z = W \wedge X$ and deduce that (d) holds.

Finally, if $d = 0$ then (e) is a direct translation of (b). □

Lemma 12.6. *If $u\colon \Sigma^d X \longrightarrow X$ is topologically nilpotent, then the same is true of $u \wedge 1_Y$ for any Y.*

Proof. The functor $(-) \wedge Y$ preserves sequential colimits, so the claim is obvious from criterion (a). □

Recall that a self map $u : A \longrightarrow A$ of a topological Abelian group A is said to be *topologically nilpotent* if the sequence $\{u^n x\}$ converges to 0 uniformly in x. We will apply this definition to E^*X and $E^\vee_* X$ for dualisable X, using the I_n-adic topology. In this case, both E^*X and $E^\vee_* X$ are finitely generated, so there is no difference between pointwise and uniform convergence.

Proposition 12.7. *Let $u\colon \Sigma^d X \longrightarrow X$ be a self map of a dualisable spectrum. Then the following are equivalent:*

(a) *$K_* u$ is nilpotent.*
(b) *$K^* u$ is nilpotent.*
(c) *$E^\vee_* u$ is topologically nilpotent.*
(d) *$E^* u$ is topologically nilpotent.*
(e) *u is topologically nilpotent.*

Proof. (a)⇔(b): Duality.

(a)⇔(e): Recall that u is topologically nilpotent if and only if $1_Z \wedge u$ is nilpotent for all small Z. This is equivalent to (a) by the Künneth theorem and Proposition 12.3.

(d)⇒(b): We check by induction on k that u induces a topologically nilpotent self map of $(E/I_k)^*X$. Indeed, consider the short exact sequence

$$(E/I_k)^*(X)/v_k \rightarrowtail (E/I_{k+1})^*(X) \twoheadrightarrow \operatorname{ann}(v_k, (E/I_k)^{*-2p^k+1}(X)).$$

The subspace topology on $\operatorname{ann}(v_k, (E/I_k)^*(X))$ is the same as the I_n-adic topology by the Artin-Rees lemma. It is then easy to see that u induces a topologically nilpotent map on $(E/I_{k+1})^*X$ if it does on $(E/I_k)^*X$. In the case $k = n$ we have $(E/I_n)^*X = K^*X$ and this is finite and thus discrete so the self map is nilpotent.

(b)⇒(d): It is easy to check that for any generalised Moore spectrum S/I of height n, the spectrum E/I is K-finite so $(E/I)^*u$ is nilpotent. It follows easily that if $a \in E^*X$ then the composite $\Sigma^{Nd}X \xrightarrow{u^N} X \xrightarrow{a} E \to E/I$ vanishes for large N. We know from Proposition 11.9 that the natural topology is the same as that defined by the kernels of the maps $E^*X \to (E/I)^*X$. It follows that $a \circ u^N \to 0$ in the I_n-adic topology. Thus the sequence $(u^N)^* \in \operatorname{End}_{E^*}(E^*X)$ converges to zero pointwise, and hence uniformly.

This shows that all our statements are equivalent except for (c). However, $E^\vee_* u = E^*Du$ and by the above applied to Du this is topologically nilpotent if and only if Du is, and Du is topologically nilpotent if and only if u is (by Corollary 11.8). Thus (c) is equivalent to (e) also. □

Lemma 12.8. *Let $u\colon \Sigma^d X \to X$ be a self map of a small spectrum. Then there is a unique idempotent $e\colon X \to X$ such that u is the wedge of an isomorphism $\Sigma^d eX \to eX$ with a nilpotent map $\Sigma^d(1-e)X \to (1-e)X$.*

Proof. Let $v\colon \Sigma^t X \to X$ be a good v_n self map. We can choose $a > 0$ and $b \in \mathbf{Z}$ such that $w = u^a v^b$ has degree zero. By Corollary 8.12, we know that $[X, X] = \pi_0(DX \wedge X)$ is a finite ring. It follows that there exist $r < s$ such that $w^r = w^s$. We then find that $w^{k(s-r)} = w^{(k+1)(s-r)}$ as long as $k(s-r) > r$. Thus, if t is a multiple of $s - r$ which is greater than r, we find that $e = w^t = u^{at}v^{bt}$ satisfies $e^2 = e$. Thus e is an idempotent which commutes with u, which means that u respects the splitting $X = eX \vee (1-e)X$. As $u^{at}v^{bt} = 1$ on eX we see that $u\colon \Sigma^d eX \to eX$ is an isomorphism. On the other hand, we have $u^{at}v^{bt} = 0$ on $(1-e)X$, so that $u\colon (1-e)X \to (1-e)X$ is nilpotent.

If e' is a different idempotent with the required properties then $e = u^{at}v^{bt}$ is the wedge of an isomorphism on $e'X$ with a nilpotent self map on $(1-e')X$. As e is idempotent, it is easy to deduce that $e' = e$. □

Proposition 12.9. *Let $u\colon \Sigma^d X \to X$ be a self map of a dualisable spectrum in $\mathcal{K}$. Then there is a unique idempotent $e\colon X \to X$ such that u is the wedge of an isomorphism $\Sigma^d eX \to eX$ with a topologically nilpotent map $\Sigma^d(1-e)X \to (1-e)X$.*

Proof. Choose a tower of generalised Moore spectra $S/I(j)$ as in Proposition 4.22. We will write $X/I(j)$ for $X \wedge S/I(j)$. For each j Lemma 12.8 gives a unique idempotent $e_j\colon X/I(j) \to X/I(j)$ such that $u \wedge 1$ is a wedge of an isomorphism and a nilpotent in terms of the splitting given by e_j. We claim that the following diagram commutes:

$$\begin{tikzcd} X/I(j) \arrow[r, "e_j"] \arrow[d, "1\wedge g_j"'] & X/I(j) \arrow[d, "1\wedge g_j"] \\ X/I(j-1) \arrow[r, "e_{j-1}"'] & X/I(j-1). \end{tikzcd}$$

To see this, let v be a good v_n self map of $S/I(j)$. Using Remark 4.25 and Proposition 4.4, we see that v induces compatible v_n self maps of all spectra in the diagram (all of which we call v) and that $v \circ (1 \wedge g_j) = (1 \wedge g_j) \circ v$. Note also that $(u \wedge 1) \circ (1 \wedge g_j) = (1 \wedge g_j) \circ (u \wedge 1)$. It follows from the proof of Lemma 12.8 that there exist integers a, b such that $e_j = (u \wedge 1)^a v^b$ (interpreted as a self map of $X/I(j)$) and $e_{j-1} = (u \wedge 1)^a v^b$ (interpreted as a self map of $X/I(j-1)$). It follows easily that the diagram commutes.

We next recall from Proposition 7.10 that $X = \operatorname{holim}_{\overleftarrow{k}} X \wedge S/I(k)$. Moreover, Corollary 8.12 shows that $[X, X \wedge S/I(k)] = \pi_0(DX \wedge X \wedge S/I(k))$ is finite, so there is no $\lim^1$ term and $[X, X] = \lim_{\overleftarrow{k}} [X, X \wedge S/I(k)]$. It follows that there is a unique map $e\colon X \to X$ such that the following diagram commutes for each j:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "1\wedge \eta_j"'] & X \arrow[d, "1\wedge \eta_j"] \\ X/I(j) \arrow[r, "e_j"'] & X/I(j). \end{tikzcd}$$

The map e^2 also has this property, so $e^2 = e$. We also know that

$$(1 \wedge \eta_j) \circ (ue - eu) = ((u \wedge 1)e_j - e_j(u \wedge 1)) \circ (1 \wedge \eta_j) = 0$$

for all j, and thus that $ue = eu$, so u is compatible with the splitting $X = eX \vee (1-e)X$. Similar considerations show that u is an isomorphism on eX and that $u \wedge 1$ is nilpotent on $(1-e)X/I(j)$ for all j. It follows by Proposition 12.7 that u is topologically nilpotent on $(1-e)X$. □

It follows from the previous proposition that the Krull-Schmidt theorem holds in $\mathcal{D}$. (It was shown by Freyd [Fre66] that the Krull-Schmidt theorem holds for finite torsion spectra, and our argument is much the same.)

Definition 12.10. We say that a spectrum $X \in \mathcal{D}$ is *indecomposable* if $X \neq 0$ and there do not exist nontrivial spectra $Y, Z \in \mathcal{D}$ with $X \simeq Y \vee Z$.

We will see in Proposition 12.17 that there are only a set of isomorphism classes of dualisable spectra, and hence of indecomposable spectra.

Proposition 12.11. *Let $X \in \mathcal{D}$ be an indecomposable spectrum, and write $R = [X, X]$. Let $I \subseteq R$ be the set of topologically nilpotent self maps of X. Then I is a two-sided ideal in R and R/I is a finite field.*

Remark 12.12. Adams and Kuhn have shown [AK89] that in a slightly different context, all possible finite fields can occur. Presumably this is true for us too.

Proof. First, it follows immediately from Proposition 12.9 that every $u \in R$ is either invertible or topologically nilpotent, so $R = R^{\times} \amalg I$. If $a \in I$ and $b \in R$ then $a \notin R^{\times}$ so $ab, ba \notin R^{\times}$ so $ab, ba \in I$. Now suppose that $a, b \in I$. We claim that $c = a + b \in I$. If not, then c is invertible. By our previous observation we have $c^{-1}b \in I$ so $c^{-1}b$ is topologically nilpotent. Hence $\sum_k (c^{-1}b)^k$ converges to an inverse for $1 - c^{-1}b = c^{-1}a$, so a is invertible. This contradicts the assumption that $a \in I$. Thus I is a two-sided ideal. It is clear that every nonzero element of R/I is invertible, so R/I is a division ring. Next, let J be the kernel of the evident map $R \longrightarrow \operatorname{End}(K_*X)$. This is clearly a two-sided ideal, and Proposition 12.7 tells us that $J \leq I$. Because K_*X has finite dimension over K_*, we see that R/J is finite and thus that R/I is finite. A well-known theorem of Wedderburn says that a finite division ring is a field. $\square$

Proposition 12.13. *For any spectrum $X \in \mathcal{D}$ there are unique integers $n_Y = n_Y(X) \geq 0$, as Y runs through isomorphism classes of indecomposable spectra, such that $n_Y = 0$ for all but finitely many Y, and X is isomorphic to $\bigvee_{Y \in \mathcal{J}} n_Y Y$. (Here $n_Y Y$ means the wedge of n_Y copies of the spectrum Y).*

Proof. We may assume that this holds for all X' such that $\dim(K_*X') < \dim(K_*X)$. It is clear that it also holds if X is indecomposable. If X is decomposable we can write $X = V \vee W$ with $V \neq 0 \neq W$ and thus $\dim(K_*V) > 0$ and $\dim(K_*W) > 0$ and thus $\dim(K_*V) < \dim(K_*X)$ and $\dim(K_*W) < \dim(K_*X)$. By the induction hypothesis we have $V \simeq \bigvee_Y n_Y(V)Y$ and $W \simeq \bigvee_Y n_Y(W)Y$. It follows that $X \simeq \bigvee_Y (n_Y(V) + n_Y(W))Y$, so we may take $n_Y(X) = n_Y(V) + n_Y(W)$.

We still need to prove uniqueness. For this, we change notation and assume that we have indecomposables $U_1, \dots, U_r$ and $V_1, \dots, V_s$ such that there exists an isomorphism $f \colon U_1 \vee \dots \vee U_r \simeq V_1 \vee \dots \vee V_s$. We need to show that $r = s$ and that after reordering the V's we have $U_i \simeq V_i$ for all i. Write $g = f^{-1}$, and f_{ij} for the component of f mapping U_i to V_j, and g_{ji} for the component of g mapping V_j to U_i. We then have $\sum_j g_{ji} f_{kj} = \delta_{ik} \colon U_k \longrightarrow U_i$. In particular, we have $\sum_j g_{j1} f_{1j} = 1 \colon U_1 \longrightarrow U_1$. Recall that the maps $U_1 \longrightarrow U_1$ that are not isomorphisms form a two-sided ideal (and thus an additive subgroup) in $[U_1, U_1]$; it follows that $g_{j1} f_{1j}$ is invertible for some j. After renumbering the V's, we may assume $j = 1$. This means that $f_{11} \colon U_1 \longrightarrow V_1$ is a split monomorphism but V_1 is indecomposable so f_{11} is an isomorphism. Similarly g_{11} is an isomorphism. We now write $U' = U_2 \vee \dots \vee U_r$ and $V' = V_2 \vee \dots \vee V_s$ and redefine our notation slightly so that f_{12} is the component of $f \colon U_1 \vee U' \longrightarrow V_1 \vee V'$ mapping U_1 to V' and so on. We thus have a matrix equation

$$\begin{pmatrix} f_{11} & f_{21} \\ f_{12} & f_{22} \end{pmatrix} \begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{pmatrix} = \begin{pmatrix} 1_{V_1} & 0 \\ 0 & 1_{V'} \end{pmatrix}.$$

One can deduce directly that $f_{22} \colon U' \longrightarrow V'$ is an isomorphism, with inverse given by $g_{22} - g_{12} g_{11}^{-1} g_{21}$. By induction on the number of indecomposables, we can assume that $r - 1 = s - 1$ and that after reordering we have $U_i \simeq V_i$ for $i > 1$. Thus $r = s$ and $U_i \simeq V_i$ for all i. $\square$

We next exhibit some ideals in $\mathcal{D}$; we conjecture that they are all the ideals.

Definition 12.14. Given $k \leq n$ we write $\mathcal{D}_k$ for the category of spectra $X \in \mathcal{D}$ such that X is a retract of $Y \wedge Z$ for some $Y \in \mathcal{D}$ and some finite spectrum Z of type at least k. It is easy to see that $\mathcal{D}_n = \mathcal{F}$.

Proposition 12.15. *$\mathcal{D}_k$ is an ideal in $\mathcal{D}$ and is closed under D. Moreover, given $X \in \mathcal{D}$, the following are equivalent.*

(a) *$X \in \mathcal{D}_k$.*
(b) *X is a module over some generalised Moore spectrum S/I of height k.*
(c) *$E^\vee_* X$ is I_k-torsion.*
(d) *$E^* X$ is I_k-torsion.*

Proof. (a)$\Leftrightarrow$(b): This is Proposition 4.17.

(b)$\Rightarrow$(c): It is easy to see using the cofibrations which define S/I that $E^\vee_*(X \wedge S/J)$ is I_k-torsion. It follows that $E^\vee_* X$ is I_k-torsion.

(c)$\Rightarrow$(b): By induction we may assume that there is a generalised Moore spectrum S/J of type $k-1$ such that X is a module over S/J. Let v be a good v_{k-1} self map of S/J. Let $\mathcal{C}$ be the category of those spectra Y such that for each $a \in \pi_*(S/J \wedge X \wedge Y)$ we have $v^N a = 0$ for $N \gg 0$. One sees easily that $\mathcal{C}$ is thick and $E \in \mathcal{C}$. Thus $D(S/J \wedge X) \in \mathcal{C}$ by Theorem 8.9, and v acts nilpotently on $1 \in \pi_0(S/J \wedge X \wedge D(S/J \wedge X)) = \mathrm{End}(S/J \wedge X)$. This means that v is nilpotent as a self map of $S/J \wedge X$, so that the evident map $S/J \wedge X \to S/(J, v^{p^N}) \wedge X$ is a split monomorphism for large N. As $X \to S/J \wedge X$ is also a split monomorphism, we see that X is a module over $S/(J, v^{p^N})$.

We also know from Proposition 4.17 that $\mathcal{D}_k$ is an ideal. If X is a retract of $Y \wedge Z$ for dualisable Y and Z, then DX is a retract of $D(Y \wedge Z) = DY \wedge DZ$. Thus $\mathcal{D}_k$ is closed under D. It is then immediate from the isomorphism $E^* X = E^\vee_* DX$ that (c)$\Leftrightarrow$(d). □

Corollary 12.16. *If $X \in \mathcal{K}$ then the following are equivalent:*

(a) *X is small.*
(b) *$E^* X$ is finite.*
(c) *$E^\vee_* X$ is finite.*
(d) *X is dualisable and K-nilpotent.*

Proof. Using Theorem 8.6 we see that any of (a) ... (d) implies that X is dualisable, so we may assume this throughout. We saw in Theorem 8.5 that (a) implies (b),(c) and (d). By applying Proposition 12.15, we see that (b) implies (a) and (c) implies (a).

All that is left is to prove that (d) implies (a). Suppose that X is dualisable and K-nilpotent. Let $\mathcal{J}$ be the category of those spectra $Z \in \mathcal{K}$ such that Z is a module over S/I for some I of height n. We know from Proposition 4.17 that $\mathcal{J}$ is an ideal, and thus from Proposition 7.9 that it contains all K-nilpotent spectra. In particular, we see that X is a retract of some $S/I \wedge X$, and $S/I \wedge X$ is the smash product of a small spectrum with a dualisable one so it is small, so X itself is small. □

12.1. **The semiring $\pi_0\mathcal{D}$.** Write $\pi_0\mathcal{D}$ for the set of isomorphism classes of spectra in $\mathcal{D}$. We start by verifying that this is really a set.

Proposition 12.17. *The category $\mathcal{D}$ has $2^{\aleph_0}$ isomorphism classes.*

Proof. There are only countable many finite spectra Y of type at least n, and for each one we know that $[LY, LY]$ is finite, so LY has only finitely many retracts. It follows that $\pi_0(\mathcal{F})$ is countable. Moreover, if $U, V \in \mathcal{F}$ then $[U, V]$ is finite. It follows that there are at most $\aleph_0^{\aleph_0} = 2^{\aleph_0}$ different towers of $\mathcal{K}$-small spectra. If $X \in \mathcal{D}$ then

$X \wedge S/I$ is small and $X = \underset{\longleftarrow I}{\text{holim}}\, X \wedge S/I$, so X is the inverse limit of one of these towers. Thus $|\pi_0 \mathcal{D}| \leq 2^{\aleph_0}$.

Conversely, it is shown in [HMS94, Proposition 9.3] that the p-adic integers embed in the Picard group, which in turn embeds in $\pi_0 \mathcal{D}$. Thus $|\pi_0 \mathcal{D}| \geq 2^{\aleph_0}$. □

The set $\pi_0 \mathcal{D}$ is a commutative semiring under $\vee$ and $\wedge$. The group of units is the aforementioned Picard group of $\mathcal{K}$. See [HMS94, HS95, Str92] for discussion and calculation of Picard groups. See also Section 14 and Section 15.1.

The subsets $\pi_0 \mathcal{D}_k$ are ideals. The maps $n_Y : \pi_0 \mathcal{D} \longrightarrow \mathbf{N}$ give an additive isomorphism

$$\pi_0 \mathcal{D} = \bigoplus_{Y \in \mathcal{I}} \mathbf{N}$$

where $\mathcal{I}$ denotes the set of isomorphism classes of indecomposable spectra. There is also an interesting semiring homomorphism

$$d \colon \pi_0 \mathcal{D} \longrightarrow \mathbf{N}[t]/(t^{|v_n|} - 1)$$

defined by

$$d(X) = \sum_{i=0}^{|v_n|-1} t^i \dim_{\mathbf{F}_p} K_i(X).$$

We can of course set $t = 1$ to get a semiring map $\pi_0 \mathcal{D} \longrightarrow \mathbf{N}$, or set $t = -1$ to get a map $\pi_0 \mathcal{D} \longrightarrow \mathbf{Z}$. This factors through the ring $K_0 \mathcal{D}$, which is the quotient of $\pi_0 \mathcal{D}$ in which we set $Y = X + Z$ whenever there is a cofibre sequence $X \longrightarrow Y \longrightarrow Z$.

When $X \in \mathcal{F}$ is small, we have two other measures of the "size" of X.

Definition 12.18. Given a finite Abelian p-group A we define the *length* to be $\text{len}(A) = \log_p |A|$. Given a graded Abelian p-group A_* which is periodic of period $|v_n^{p^K}|$ for some K, we define

$$\text{len}(A_*) = \sum_{0 \leq k < |v_n^{p^K}|} \text{len}(A_k)/p^K \in \mathbf{Z}[\tfrac{1}{p}].$$

We have normalised this so that $\text{len}(K_*) = 1$, and thus $\text{len}(A_*) \in \mathbf{N}$ when A_* is a finite module over E_*.

The length $\text{len}(K_* X)$ is defined whenever X is dualisable. It is not a very sensitive measure of the size of X, because $\text{len}(K_*(S/I)) = 2^n$ for all I. If X is small we can also define $\text{len}(E_* X)$ and $\text{len}(\pi_* X)$ (we have the required periodicity because of the existence of v_n self maps). The following result shows that $\text{len}(E_* X)$ is a good measure of size.

Theorem 12.19. *For any integer $k \geq 0$, the set of isomorphism classes of $\mathcal{K}$-small spectra X with $\text{len}(E_* X) \leq k$ is finite.*

Proof. We will work with $E(n)$ instead of E: since we are only considering finite modules and small spectra, this makes no difference. Note that there are only finitely many $E(n)_*$-modules M with $\text{len}\, M \leq k$. Now suppose we have a finite $E(n)_*$-module M. If $M \cong E(n)_* X$ for some K-local spectrum X, then Y is necessarily small. The filtration theorem proved in [HS95, Section 2] then tells us that M has a finite comodule filtration $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_t$ where $t \leq k$ and $M_i/M_{i-1} \cong \Sigma^{j(i)} K_*$ for some $0 \leq j(i) < |v_n|$. Given M_{i-1}, the number of possibilities for M_i is then determined by $\operatorname{Ext}^{1,j(i)}_{E(n)_*E(n)}(K_*, M_{i-1})$. It follows that, as long as the groups $\operatorname{Ext}^{s,t}_{E(n)_*E(n)}(K_*, K_*)$ are finite for all s and t, the number of realisable comodule structures on M is finite.

Now fix a realisable $E(n)_*E(n)$-comodule structure on M. Consider the category $\mathcal{C}(M)$ of pairs (X, x), where X is a small spectrum and $x\colon M \to E(n)_*X$ is an isomorphism of comodules. It is enough to show that $\mathcal{C}(M)$ has only finitely many isomorphism classes. Recall from Proposition 6.5 that there is an Adams spectral sequence

$$E_2^{st}(X, Y) = \operatorname{Ext}^{s,t}_{E(n)_*E(n)}(E(n)_*X, E(n)_*Y) \Longrightarrow [X, Y]_{t-s},$$

with $E_N = E_\infty$ for some constant N independent of X and Y. Moreover, given maps $E(n)_*X \xrightarrow{f} E(n)_*Y \xrightarrow{g} E(n)_*Z$ which survive to E_r, we have $d_r(gf) = g_*d_r(f) + f^*d_r(g)$ (see [Mos68, Theorem 2.1]. We say that objects (X, x) and (Y, y) in $\mathcal{C}(M)$ are E_r-equivalent if the element $yx^{-1} \in \operatorname{Hom}_{E(n)_*E(n)}(E(n)_*X, E(n)_*Y) = E_2^{0,0}(X, Y)$ survives to E_r. Our formula for $d_r(gf)$ shows that this is an equivalence relation. Clearly any two objects in $\mathcal{C}(M)$ are E_2-equivalent. If they are E_{r+1}-equivalent then they are E_r-equivalent, and if they are E_N-equivalent then they are isomorphic. It is thus enough to show that each E_r-equivalence class splits into finitely many E_{r+1}-equivalence classes.

If (X, x) and (Y, y) are E_r-equivalent, we define

$$\delta(X, x, Y, y) = y_*(x^{-1})^*d_r(y^{-1}x) \in \operatorname{Ext}^{r,r-1}_{E(n)_*E(n)}(M, M).$$

As x and y are isomorphisms, we have $\delta(X, x, Y, y) = 0$ if and only if (X, x) and (Y, y) are E_{r+1}-equivalent. Moreover, the formula for $d_r(gf)$ implies that

$$\delta(X, x, Z, z) = \delta(X, x, Y, y) + \delta(Y, y, Z, z)$$

whenever this makes sense. We conclude that the set of E_{r+1}-equivalence classes in a given E_r-equivalence class bijects with a subgroup of $\operatorname{Ext}^{r,r-1}_{E(n)_*E(n)}(M, M)$, so it will be enough to show that this group is finite. Using our comodule filtration of M, we find once again that it suffices to show that $\operatorname{Ext}^{s,t}_{E(n)_*E(n)}(K_*, K_*)$ is finite for all s and t.

It is easy to check using cobar resolutions that

$$\operatorname{Ext}^{s,t}_{E(n)_*E(n)}(E(n)_*, K_*) = \operatorname{Ext}^{s,t}_{\Sigma_*}(K_*, K_*).$$

This is finite in each bidegree by the proof of of [Rav86, Theorem 6.2.10(a)]. It follows inductively that $\operatorname{Ext}^{s,t}_{E(n)_*E(n)}(E(n)_*/I_k, K_*)$ is finite and in particular that $\operatorname{Ext}^{s,t}_{E(n)_*E(n)}(K_*, K_*)$ is finite, as required. □

13. K-nilpotent spectra

In this section, we give a number of characterisations of K-nilpotent spectra.

Theorem 13.1. *Suppose X is a spectrum in $\mathcal{K}$. Then the following are equivalent.*

(a) *X is a retract of $X \wedge Z$ for some finite spectrum Z of type at least n.*
(b) *X is K-nilpotent.*
(c) *X lies in the thick subcategory generated by the K-injectives.*
(d) *$X \wedge_{\mathcal{S}} Y$ is K-local for all $Y \in \mathcal{S}$.*
(e) *The functor on $\mathcal{K}$ which takes Y to $\pi_0(X \wedge_{\mathcal{K}} Y)$, is a homology functor.*

Before proving this, we need to address a different problem. Let X be a finite spectrum of type $m < n$, with a good v_m self map $v\colon \Sigma^d X \to X$. We need to understand what happens to X and v under K-localisation.

As usual, if Z is a spectrum with a self-map f, we denote the cofibre of f by Z/f.

Proposition 13.2. *If $Y \in \mathcal{K}$ and X is a finite spectrum of type $m < n$ with $v\colon \Sigma^d X \to X$ a good v_m self map of X, then the natural map*

$$X \wedge Y \to \underleftarrow{\operatorname{holim}}(X \wedge Y)/v^k$$

is an equivalence. Furthermore we have a short exact sequence

$$0 \to \underleftarrow{\lim}^1 \ker(v^k, \pi_{*-dk-1}(X \wedge Y)) \to \pi_*(X \wedge Y) \to \pi_*(X \wedge Y)^\wedge_v \to 0$$

Proof. First note that it does not matter whether we work in $\mathcal{K}$ or in $\mathcal{S}$. Indeed, X is finite, so $X \wedge Y$ is already K-local, so we have $X \wedge Y = \widehat{L}(\widehat{L}X \wedge Y)$. Also, the products in $\mathcal{K}$ and in $\mathcal{S}$ are the same. The homotopy inverse limit of a tower can be constructed as the fibre of a standard self map of the product of its terms, so it is also the same in $\mathcal{K}$ and $\mathcal{S}$.

We claim that the map $X \wedge Y \xrightarrow{g} \underleftarrow{\operatorname{holim}}(X \wedge Y)/v^k$ is an X/v-equivalence. Indeed, as v is strongly central, the map $X/v \wedge X \xrightarrow{1 \wedge v^k} X/v \wedge X$ is trivial for $k \geq 2$. Thus the map $X/v \wedge X \wedge Y \to X/v \wedge X/v^k \wedge Y$ is a split monomorphism for $k \geq 2$, with cofibre a suspension of $X/v \wedge X \wedge Y$. Thus we have a short exact sequence

$$\pi_i(X/v \wedge X \wedge Y) \rightarrowtail \pi_i(X/v \wedge X/v^k \wedge Y) \twoheadrightarrow \pi_{i-dk-1}(X/v \wedge X \wedge Y)$$

and a corresponding six-term long exact sequence of $\underleftarrow{\lim}$ and $\underleftarrow{\lim}^1$ terms. However, the maps of the tower induce multiplication by $1 \wedge v$ on the cokernel terms, and using strong centrality again, we find that the tower of cokernels is pro-isomorphic to the zero tower. The tower of kernels is constant. Hence the map g is an X/v-equivalence.

The map g is therefore an $F(m+1)$-equivalence between $K(n)$-local, and hence $F(n)$-local, spectra. Since $m+1 \leq n$, we see that g is an equivalence.

We now consider the Milnor exact sequence

$$0 \to \underleftarrow{\lim}^1 \pi_{*+1}(X \wedge Y/v^k) \to \pi_*(X \wedge Y) \to \underleftarrow{\lim}\, \pi_*(X \wedge Y/v^k) \to 0.$$

To identify the terms in this sequence, note that we have short exact sequences

$$0 \to (\pi_*(X \wedge Y))/v^k \to \pi_*(X \wedge Y/v^k) \to \ker(v^k, \pi_{*-dk-1}(X \wedge Y)) \to 0.$$

Taking inverse limits gives us the usual 6-term exact sequence

$$0 \to \pi_*(X \wedge Y)^\wedge_v \to \underleftarrow{\lim}\, \pi_*(X \wedge Y/v^k) \to \underleftarrow{\lim} \ker(v^k, \pi_{*-dk-1}(X \wedge Y))$$

$$\to \underleftarrow{\lim}^1 \pi_*(X \wedge Y)/v^k \to \underleftarrow{\lim}^1 \pi_*(X \wedge Y/v^k) \to \underleftarrow{\lim}^1 \ker(v^k, \pi_{*-dk-1}(X \wedge Y)) \to 0.$$

Note that the sequence $\pi_*(X \wedge Y)/v^k$ is Mittag-Leffler, so its $\underleftarrow{\lim}^1$ term vanishes. Thus $\underleftarrow{\lim}^1 \pi_*(X \wedge Y/v^k) = \underleftarrow{\lim}^1 \ker(v^k, \pi_{*-dk-1}(X \wedge Y))$, so the first term in the Milnor sequence is as claimed.

It remains to prove that $\pi_*(X \wedge Y)^\wedge_v = \underleftarrow{\lim}\, \pi_*(X \wedge Y/v^k)$, or in other words that the first map in the six term sequence is surjective. This is true because the surjective map $\pi_*(X \wedge Y) \to \underleftarrow{\lim}\, \pi_*(X \wedge Y/v^k)$ factors through it. $\square$

Proof of Theorem 13.1. (a)$\Rightarrow$(b): We know by Proposition 6.15 that LZ is K-nilpotent. It follows that any retract of $X \wedge Z = X \wedge LZ$ is also K-nilpotent.

(b)$\Leftrightarrow$(c): this is Lemma 7.4.

(b)$\Rightarrow$(d): If X is K-nilpotent, so is $X \wedge_{\mathcal{S}} Y$ for any Y, so in particular $X \wedge_{\mathcal{S}} Y$ is K-local.

(d)$\Rightarrow$(e): Suppose that $X \wedge_{\mathcal{S}} Y$ is K-local for all Y, so that $X \wedge_{\mathcal{K}} Y = X \wedge_{\mathcal{S}} Y$. It also follows that for any $Z \in \mathcal{S}$, the spectrum $X \wedge_{\mathcal{S}} \widehat{C}Z$ is both K-local and K-acyclic, hence is zero. This implies that $X \wedge_{\mathcal{K}} \widehat{L}Z = X \wedge_{\mathcal{S}} \widehat{L}Z = X \wedge_{\mathcal{S}} Z$. In particular, we find that

$$X \wedge_{\mathcal{K}} \bigvee_i^{\mathcal{K}} Y_i = X \wedge_{\mathcal{S}} \bigvee_i^{\mathcal{S}} Y_i.$$

Given this, it is trivial to verify that $Y \mapsto \pi_0(X \wedge_{\mathcal{K}} Y)$ is a homology functor on $\mathcal{K}$.

(e)$\Rightarrow$(a): This is the most difficult part of the argument. For the rest of the proof, all smash products, limits and so on are taken in $\mathcal{K}$. The plan is to start with the obvious fact that X is a retract of $X \wedge \widehat{L}S$. We then show that if Z is a finite type m spectrum with a v_m self-map v, then $X \wedge \widehat{L}Z$ is a retract of $(X \wedge \widehat{L}Z)/v^k$ for some sufficiently large k. It will then follow by induction that X is a retract of $X \wedge \widehat{L}Z$ for some finite type n spectrum Z.

So consider the functor H on $\mathcal{K}$ defined by $H(Y) = \pi_0(\widehat{L}Z \wedge X \wedge Y)$. By (e), H is a homology functor on $\mathcal{K}$. Our first goal is to show that there is a k such that $v^k H_*(Y)$ is v-divisible for all Y. Indeed, suppose not. For each k, choose a spectrum Y_k and an element $a_k \in H_*(Y_k)$ such that $b_k = v^k a_k$ is not v-divisible. We can assume that b_k is in degree 0. Let $Y = \bigvee Y_k$, so we have $H_*(Y) = \bigoplus H_*(Y_k)$. We know from Proposition 13.2 that $H(Y)$ maps onto its v-completion. However, (b_k) defines an element of the v-completion of $H(Y)$, since only finitely many are not in the image of any v^l. Furthermore, since no b_k is v-divisible, each maps non-trivially to the v-completion, so that (b_k) can't be in the image of $H(Y)$, which is a contradiction.

Hence there is a k such that $v^k H(Y)$ is v-divisible for all Y. We will now show that $v^k H(Y)$ is in fact 0 for all Y. Indeed, by Proposition 13.2, the kernel of the surjective map from $H(Y)$ to its v-completion is $\varprojlim^1 \ker(v^l, H(Y))$, where we have left out the dimension shift. But we claim this tower is in fact Mittag-Leffler, so that it has no $\varprojlim^1$ term. Indeed, suppose we have a class x in $\ker(v^l, H(Y))$ which is in the image of $\ker(v^{l+k}, H(Y))$. Then in particular, x is in $v^k H(Y)$ so is v-divisible. Then for all $i \geq k$, there is a w such that $v^i w = x$. In particular, w is in $\ker(v^{i+l}, H(Y))$, so x is in the image of $\ker(v^{l+i}, H(Y))$ for all $i \geq k$. Thus the tower is Mittag-Leffler, and we find that $H(Y)$ is v-complete, for all Y. In particular, $v^k H(Y)$ is a v-divisible subgroup of a v-complete group, so is trivial.

Now, take $Y = DW$ for a finite type n spectrum W. Using Spanier-Whitehead duality in $\mathcal{K}$, we find that for any map $W \longrightarrow \widehat{L}Z \wedge X$, the composite

$$W \longrightarrow \widehat{L}Z \wedge X \xrightarrow{v^k \wedge 1} \widehat{L}Z \wedge X$$

is null. (Remember the smash product is taken in $\mathcal{K}$). Thus,

$$v^k \wedge 1 : \widehat{L}Z \wedge X \longrightarrow \widehat{L}Z \wedge X$$

is phantom, so by the last part of Theorem 9.5, $v^{2k} \wedge 1$ is trivial. Thus, $\widehat{L}Z \wedge X$ is a retract of $\widehat{L}(Z/v^{2k}) \wedge X$. Since Z/v^{2k} is a finite spectrum of type $m+1$, we are done. $\square$

Proposition 13.3. *If X is K-nilpotent and Y is arbitrary then $X \wedge Y$, $F(X,Y)$ and $F(Y,X)$ are K-nilpotent.*

Proof. Let $\mathcal{C}$ be the category of those X such that $X \wedge Y$, $F(X,Y)$ and $F(Y,X)$ are K-nilpotent; this is clearly thick. If X is a K-module, then $X \wedge Y$, $F(X,Y)$ and $F(Y,X)$ are K-modules and therefore K-nilpotent. This shows that $\mathcal{C}$ contains all K-injective spectra, and hence (by part (c) of Theorem 13.1) all K-nilpotent spectra. $\square$

It follows from Proposition 13.3 that the K-nilpotent spectra form the coideal generated by K as well as the ideal generated by K. One might then guess that if X is in both the localising subcategory generated by K (equivalently, $MX = X$) and the colocalising subcategory generated by K (equivalently $\widehat{L}X = X$), then X is K-nilpotent. This is false, however: the spectrum Y considered in Section 15.1 is a counterexample.

14. Grading over the Picard group

Recall, from for example [HMS94, Str92], that the Picard group $\mathrm{Pic} = \mathrm{Pic}(\mathcal{K})$ consists of isomorphism classes of spectra in $\mathcal{K}$ which are invertible under the smash product. In this section we will need to be very careful about the distinction between objects and isomorphism classes, so we make formal definitions as follows.

Definition 14.1. We say that a spectrum $P \in \mathcal{K}$ is invertible if there is a spectrum $Q \in \mathcal{K}$ such that $P \wedge Q \simeq S$. We write $\mathcal{P}$ for the category of invertible spectra. Given $P \in \mathcal{P}$ we write $[P]$ for the isomorphism class of P. We write Pic for the collection of these isomorphism classes, which is a set (rather than a proper class) by [HMS94, Proposition 7.6] or by Proposition 12.17.

The following result is proved as [HMS94, Theorem 1.3] but we give a different argument here for the sake of completeness.

Proposition 14.2. *Given $P \in \mathcal{K}$, the following are equivalent:*

(a) $P \in \mathrm{Pic}$.
(b) $K_*P \simeq K_*$ *(up to suspension).*
(c) $E^*P \simeq E^*$ *(up to suspension).*
(d) $E^\vee_*P \simeq E_*$ *(up to suspension).*

Proof. (a)$\Rightarrow$(b): If $P \wedge Q = S$ then $K_*(P) \otimes_{K_*} K_*(Q) = K_*$, so $K_*(P)$ has rank one and is isomorphic to K_* up to suspension.

(b)$\Leftrightarrow$(c): We may assume that $K_*P \simeq K_*$ so that K_*P is in even degrees. Then E^*P is pro-free by Proposition 2.5, and $(E^*P)/I_n = K^*$. It follows easily that $E^*P = E^*$. The converse also follows from Proposition 2.5.

(b)$\Leftrightarrow$(d): This is similar to (b)$\Leftrightarrow$(c), using Proposition 8.4.

(b)$\Rightarrow$(a): Since K_*P is finite dimensional, we know that P is dualisable. Furthermore, the group $K_*DP = \mathrm{Hom}_{K_*}(K_*P, K_*)$ is also one-dimensional. Hence the unit $S \to DP \wedge P$ of the ring spectrum $DP \wedge P$ is a K-equivalence, and thus an isomorphism. $\square$

There is an evident surjective homomorphism $\deg\colon \operatorname{Pic} \longrightarrow \mathbf{Z}/|v_n|$ which sends $[P]$ to the degree in which K_*P is concentrated.

The general properties of the Picard group are given in the following proposition.

Proposition 14.3.

(a) *The homomorphism* $\mathbf{Z} \longrightarrow \operatorname{Pic}$ *that takes* m *to* $\widehat{L}S^{m|v_n|}$ *extends to an injective homomorphism* $\mathbf{Z}_p \longrightarrow \operatorname{Pic}$.
(b) $|\operatorname{Pic}| = 2^{\aleph_0}$.
(c) *If* X *is small, the orbit of* X *under the action of* Pic *is finite.*
(d) Pic *is a profinite Abelian group, and the kernel of* $\deg\colon \operatorname{Pic} \longrightarrow \mathbf{Z}/|v_n|$ *is a pro-p-group of finite index in* Pic.

Proof. Part (a) is proved in [HMS94, Section 9], and part (b) is an immediate corollary of part (a) and Proposition 12.17. Part (c) follows from Proposition 14.2 and Theorem 12.19. Part (d) can also be deduced from the results of [HMS94]. Here we will give an independent proof that Pic is profinite, but we rely on [HMS94] to tell us that ker(deg) is p-local.

To prove that Pic is profinite, choose a tower $\{S/J(i)\}$ as in Proposition 4.22. Let $G(i)$ be the stabiliser in Pic of $S/J(i)$. Part (c) says that $\operatorname{Pic}/G(i)$ is finite. We claim first that $G(i+1) \subseteq G(i)$. Indeed, suppose $P \in G(i+1)$, so that $P \wedge S/J(i+1) \simeq S/J(i+1)$. Then we have the composite

$$P \longrightarrow P \wedge S/J(i+1) \xrightarrow{\simeq} S/J(i+1) \longrightarrow S/J(i)$$

induced by the unit. Since $S/J(i)$ is a μ-spectrum, there is an induced map $P \wedge S/J(i) \xrightarrow{f} S/J(i)$. The map E_*f is easily seen to be an isomorphism, so f is an equivalence.

To complete the proof, we need only show that if $P \in G(i)$ for all i, then $P \simeq S$. Let M_i be the set of maps $P \longrightarrow S/J(i)$ such that the induced map $E^\vee_* P \longrightarrow E_*/J(i)$ is surjective. Because $P \wedge S/J(i) \simeq S/J(i)$, one can check that M_i is nonempty. It is also finite, and the sets M_i form an inverse system. It follows that the inverse limit is nonempty. The Milnor sequence gives us a map $P \longrightarrow \operatorname{holim}_{\overleftarrow{i}} S/J(i) = S$. It is easy to see that this induces an isomorphism $E^\vee_* P \simeq E^\vee_* S$, and thus that $P \simeq S$. ☐

Note that Proposition 14.3 does not rule out the possibility that Pic contains an infinite product of copies of $\mathbf{Z}_p$. The main unanswered question about Pic is whether it is finitely generated as a profinite group.

We next address two related problems. Firstly, we have seen that most of Pic is a module over $\mathbf{Z}_p$. However, given $P \in \operatorname{Pic}$ and $a \in \mathbf{Z}_p$ we would like to be able to define $P^{(a)}$ as an object rather than just an isomorphism class, or at least to define it up to *canonical* isomorphism. Secondly, given an element $a \in \operatorname{Pic}$ it is natural to choose $P \in \mathcal{P}$ with $[P] = a$ and define $\pi_a(X) = [P, X]$. One would like to choose the spectra P compatibly for all a and collect the groups $\pi_a(X)$ into a graded object with commutative and associative pairings $\pi_a(X) \otimes \pi_b(Y) \longrightarrow \pi_{a+b}(X \wedge Y)$. This is not automatically possible: there is an obstruction in $H^3(\operatorname{Pic}; \operatorname{Aut}(S))$, and if it vanishes then the solutions form a principal homogeneous space for $H^2(\operatorname{Pic}; \operatorname{Aut}(S))$ (here Pic is acting trivially on $\operatorname{Aut}(S)$). We believe that this fact is in the literature, perhaps in the theory of Tannaka categories, but we have not managed to find a reference. Rather than using this general theory, we will proceed more directly.

We have not been able to construct a grading over Pic itself, but Theorem 14.11 provides a tolerable substitute.

We start with a discussion of signs, most of which is taken from [Riv72, Chapter I]. We write $R = [S, S]$, which is a commutative ring under composition (which is the same as the smash product). For any $X \in \mathcal{K}$, the smash product gives a natural map $R \longrightarrow [X, X]$. If $X = P \in \mathcal{P}$ then this map is an isomorphism (because $[P, P] = [S, P^{-1} \wedge P] = [S, S]$). We write $t = t_P : [P, P] \longrightarrow R$ for the inverse. One can check that $t_{P \wedge Q}(u \wedge v) = t_P(u)t_Q(v)$ and $t_P(v \circ w) = t_P(v)t_P(w)$. Moreover, if we have maps $P \xrightarrow{u} Q \xrightarrow{v} P$ then $t_P(vu) = t_Q(uv)$.

Given a spectrum $P \in \mathcal{P}$, we have a twist map $\tau_P : P \wedge P \longrightarrow P \wedge P$ and thus an element $\epsilon_P = t(\tau_P) \in R$. Of course for $n \in \mathbf{Z}$ we have $\epsilon_{S^n} = (-1)^n$.

Lemma 14.4. *If $p > 2$ then $\epsilon_P = (-1)^{\deg[P]}$. Even if $p = 2$, we have $\epsilon_P^2 = 1$ and the map $[P] \mapsto \epsilon_P$ is a homomorphism* $\mathrm{Pic}/2 \longrightarrow R^\times$.

Proof. As $\tau^2 = 1$ we have $\epsilon_P^2 = 1$. It is not hard to check that $\epsilon_{P \wedge Q} = \epsilon_P \epsilon_Q$ and that $\epsilon_P = \epsilon_{P'}$ if $P \simeq P'$. We thus get a homomorphism $\epsilon \colon \mathrm{Pic}/2 \longrightarrow R^\times$. Now suppose that $p > 2$. Then K is commutative so external products in K homology are commutative up to the usual sign; it follows that $K_* \epsilon_P = (-1)^{\deg[P]}$. For notational convenience we will assume that $\deg[P]$ is even so that $K_* \epsilon_P = 1$. It is not hard to see that $e = (1 - \epsilon_P)/2$ is an idempotent in R and $K_* e = 0$. Thus $K_* eS = 0$, so $e = 0$, so $\epsilon_P = 1$ as required. □

We write $\mathcal{P}^0$ for the category of those $P \in \mathcal{P}$ such that $\deg[P] = 0$ and $\epsilon_P = 1$ (the second condition being redundant when $p > 2$). Because the symmetric group on k letters is generated by adjacent transpositions, we see that it acts trivially on $P^{(k)}$ if $P \in \mathcal{P}^0$. We write $\mathcal{P}'$ for the category of pairs (P, u) such that $P \in \mathcal{P}^0$ and u generates $K_0 P$. The morphisms $(P, u) \longrightarrow (Q, v)$ are required to send u to v. We write Pic^0 and Pic' for the groups of isomorphism classes in $\mathcal{P}^0$ and $\mathcal{P}'$.

If $(P, u) \in \mathcal{P}'$ and $X \in \mathcal{F}$, we define

$$\pi_k(P, u)(X) = \begin{cases} [P^{(k)}, X] & \text{if } k \geq 0 \\ [DP^{(|k|)}, X] & \text{if } k < 0 \end{cases}$$

These are easily seen to be finite groups. It is not hard to construct associative and commutative (without signs) pairings

$$\pi_k(P, u)(X) \otimes \pi_l(P, u)(Y) \longrightarrow \pi_{k+l}(P, u)(X \wedge Y).$$

We would like to interpolate these groups for p-adic values of k. The main problem is to find a natural formulation of this not depending on any choices. Our solution involves the following definition.

Definition 14.5. Given a compact Hausdorff space X, let $\mathcal{B}(X)$ be the category of locally-trivial bundles over X whose fibres are finite Abelian p-groups with the discrete topology. The morphisms are continuous bundle maps that are homomorphisms on each fibre.

Given a morphism $f \colon A \longrightarrow B$ in $\mathcal{B}(X)$, it is easy to see that for each $x \in X$ there is a neighbourhood U of x and groups A', B' and a homomorphism $f' \colon A' \longrightarrow B'$ such that the restriction of f over U is isomorphic to $f' \times 1 \colon A' \times U \longrightarrow B' \times U$. It follows from this that the kernel, cokernel and image of f all lie in $\mathcal{B}(X)$ and thus that $\mathcal{B}(X)$ is an Abelian category.

A map $\phi\colon X \to Y$ gives a pullback functor $\phi^*\colon \mathcal{B}(Y) \to \mathcal{B}(X)$ in an evident way. In particular, if G is an compact Hausdorff Abelian topological group and $\sigma\colon G \times G \to G$ is the addition map, we have a functor $\sigma^*\colon \mathcal{B}(G) \to \mathcal{B}(G \times G)$. We also have an external tensor product functor $\otimes\colon \mathcal{B}(G) \times \mathcal{B}(G) \to \mathcal{B}(G \times G)$. In this context, a *pairing* from A and B to C (where $A, B, C \in \mathcal{B}(G)$) means a map $A \otimes B \to \sigma^* C$.

We write $F_x\colon \mathcal{B}(X) \to \mathbf{Ab}$ for the functor which sends A to the fibre A_x of A over x.

Proposition 14.6. *Given $(P, u) \in \mathcal{P}'$ there is a homology theory $\pi(P, u)\colon \mathcal{F} \to \mathcal{B}(\mathbf{Z}_p)$, together with natural pairings*

$$\pi(P, u)(W) \otimes \pi(P, u)(Z) \to \sigma^* \pi(P, u)(W \wedge Z)$$

that are commutative and associative in a suitable sense. Moreover, for $m \in \mathbf{Z}$ there are natural isomorphisms $F_m \pi(P, u)(Z) = \pi_m(P, u)(Z)$ which are compatible with the above pairings.

This will be proved after Proposition 14.8. We will denote $F_m \pi(P, u)(Z)$ by $\pi_m(P, u)(Z)$ even for non-integer values of m. Note that, if Z is a specific small spectrum and $m \in \mathbf{Z}_p$, then the local triviality of $\pi(P, u)(Z)$ implies that $\pi_m(P, u)(Z) = \pi_i(P, u)(Z)$ for some integer i sufficiently close to m in the p-adic topology. However, the i in question depends on Z.

Note also that $\pi(P, u)$ is a homology theory defined only on small spectra. The category $\mathcal{B}(\mathbf{Z}_p)$ does not have colimits, so in order to interpret $\pi(P, u)$ as a homology theory on $\mathcal{K}$, we would need to consider sheaves over $\mathbf{Z}_p$ rather than locally trivial bundles.

Definition 14.7. Let $X \in \mathcal{F}$ be a small spectrum, and (P, u) an object of $\mathcal{P}'$. We say that a map $v\colon P^{(p^k)} \wedge X \to X$ is a *good* (P, u) *self map* of X if

(a) The map $v_*\colon K_*(P)^{\otimes p^k} \otimes K_*(X) \to K_*(X)$ is given by $v_*(u^{\otimes p^k} \otimes x) = x$ for all $x \in K_* X$.
(b) $v \wedge 1$ is central in the graded ring $\pi_*(P, u) F(X \wedge X, X \wedge X)$.

Note that there are no suspensions involved in the definition of a good (P, u) self map. Note also that a good (P, u) self map is an equivalence.

Proposition 14.8. *Every small spectrum X admits a good (P, u) self map. If v and v' are two such maps then $v^{p^i} = (v')^{p^j}$ for some i and j. If v is a good (P, u) self map of X and w is a good (P, u) self map of Y then there exist integers i and j such that $w^{p^j} f = f v^{p^i}$ for all maps $f\colon X \to Y$.*

Proof. This is proved in a way which is closely parallel to the proof for integer-graded v_n self maps [Rav92a, Chapter 6]. If v is a map satisfying part (a) of the definition, we form the ring spectrum $R = F(X, X)$. The map v defines a map $v : P^{(p^k)} \to R$, so we can form the map $\mathrm{ad}(v)\colon P^{(p^k)} \wedge R \to R$ which measures the difference between left multiplication and right multiplication by v. By our assumption, the map $K_* v : K_* P^{(p^k)} \to K_* R = \mathrm{Hom}_{K_*}(K_* X, K_* X)$ takes the generator $u^{\otimes p^k}$ to the identity map, which is obviously in the centre. Hence $K_* \mathrm{ad}(v) = 0$, so the telescope of the sequence

$$R \xrightarrow{\mathrm{ad}(v)} DP^{(p^k)} \wedge R \xrightarrow{\mathrm{ad}(v)} DP^{(2p^k)} \wedge R \xrightarrow{\mathrm{ad}(v)} \dots$$

is zero. It follows using the smallness of R that $\mathrm{ad}(v)$ is nilpotent, and in the usual way we deduce that some power v^{p^j} is central in $\pi_*(P,u)F(X,X)$. We can apply the same argument to $v\wedge 1$ and thus show that $v^{p^j}\wedge 1$ is central in $\pi_*(P,u)F(X^{(2)},X^{(2)})$ for large j, and thus that v^{p^j} is a good (P,u) self map.

Next, suppose that we have two good (P,u) self maps, say v and v'. We then have $K_*(v-v')=0$ and thus $v-v'$ is nilpotent, by an argument with telescopes as above. Just as in [Rav92a], we deduce that $v^{p^i}=(v')^{p^j}$ for some i and j. Asymptotic naturality follows from asymptotic uniqueness, again just as in [Rav92a]. Given the asymptotic naturality, it is not hard to see that the category of small spectra that admit a good (P,u) self map is thick. It is thus enough so show that a generalised Moore spectrum S/I of type n admits a good (P,u) self map, or even just a map satisfying condition (a).

It follows from Proposition 14.3 that $P^{(p^k)}\wedge S/I\simeq S/I$ for some k. Choose an isomorphism v. Choose an element $\tilde{u}\in E^\vee_*P^{(p^k)}$ lifting $u^{\otimes p^k}$. After replacing v by a p-adic unit multiple of v, we may assume that $v_*\tilde{u}=1 \pmod{I_n}$. It follows using the cofibrations $\Sigma^{|v_k|}E/I_k\xrightarrow{v_k}E/I_k\to E/I_{k+1}$ that v^{p^n} satisfies condition (a), as required. □

Proof of Proposition 14.6. Let (P,u) be an object of $\mathcal{P}'$, and $X\in\mathcal{F}$ a small spectrum. Let C be a coset of $p^k\mathbf{Z}_p$ in $\mathbf{Z}_p$ for some k. A good (P,u) self map $v:P^{(p^k)}\wedge X\to X$ induces an isomorphism $\pi_{j+p^k}(P,u)(X)\to\pi_j(P,u)(X)$. Let $\pi_C(P,u)(X)$ be the set of elements $a\in\prod_{j\in C\cap\mathbf{Z}}\pi_j(P,u)(X)$ such that there exists a good (P,u) self map $v\colon P^{(p^k)}\wedge X\to X$ such that $a_j=va_{j+p^k}$ for all $j\in C\cap\mathbf{Z}$. If C' is another coset and $C'\subseteq C$ then there is an evident restriction map $\pi_C(P,u)(X)\to\pi_{C'}(P,u)(X)$. For $i\in\mathbf{Z}_p$ we define

$$\pi_i(P,u)(X)=\varinjlim_{C\ni i}\pi_C(P,u)(X).$$

We shall see shortly that this is consistent with our earlier definition when $i\in\mathbf{Z}$. If $a\in\pi_C(P,u)(X)$ and $a'\in\pi_{C'}(P,u)(X)$ and $i\in C\cap C'$ then we see using the asymptotic uniqueness of (P,u) self maps that there is a coset C'' with $i\in C''\subseteq C\cap C'$ such that the restriction of $a+a'$ lies in $\pi_{C''}(P,u)(X)$. Using this, we make $\pi_i(P,u)(X)$ into an Abelian group. We define

$$\pi(P,u)(X)=\coprod_{i\in\mathbf{Z}_p}\pi_i(P,u)(X),$$

so there is an evident map $q\colon\pi(P,u)(X)\to\mathbf{Z}_p$ with fibres $\pi_i(P,u)(X)$. Given an element $a\in\pi_C(P,u)(X)$, we get an element $a_i\in\pi_i(P,u)(X)$ for each $i\in C$. We define $U(C,a)=\{(i,a_i)\mid i\in C\}\subseteq\pi(P,u)(X)$, and we give $\pi(P,u)(X)$ the smallest possible topology for which all sets of this form are open.

Suppose we have integers j,k with $k\geq 0$ and a good (P,u) self map $v\colon P^{(p^k)}\wedge X\to X$. There is an evident map

$$\theta=\theta_{j,k,v}\colon\pi_j(P,u)(X)\to\pi_{j+p^k\mathbf{Z}_p}(P,u)(X),$$

sending $a\in\pi_j(P,u)(X)$ to the system of elements $(v^m a\mid m\in\mathbf{Z})$. Using asymptotic uniqueness again, it is not hard to check that this is an isomorphism. One can also deduce that $\pi(P,u)(X)$ is a locally trivial bundle over $\mathbf{Z}_p$, or in other words an object of the category $\mathcal{B}(\mathbf{Z}_p)$.

Given a triangle $X \to Y \to Z$ of small spectra, we can choose good (P, u) self maps of X, Y and Z that are compatible with the maps of the triangle. Having done so, it is easy to check that $\pi(P, u)(X)$ is an exact functor of X.

Similar arguments show that when $i \in \mathbf{Z}$ there is a canonical isomorphism between our old definition of $\pi_i(P, u)(X)$ and our new one.

Suppose that $i, j \in \mathbf{Z}_p$. We would like to construct a natural map

$$\mu_{i,j} \colon \pi_i(P, u)(X) \otimes \pi_j(P, u)(Y) \to \pi_{i+j}(P, u)(X \wedge Y).$$

Suppose that $v \colon P^{(p^k)} \wedge X \to X$ and $w \colon P^{(p^k)} \wedge Y \to Y$ are good (P, u) self maps with $v \wedge 1 = 1 \wedge w$ on $X \wedge Y$, and that i', j' are integers congruent to i, j modulo p^k. The smash product gives a pairing

$$\nu_{i',j'} \colon \pi_{i'}(P, u)(X) \otimes \pi_{j'}(P, u)(Y) \to \pi_{i'+j'}(P, u)(X \wedge Y).$$

The maps $\theta_{i',k,v}$ and so on give isomorphisms

$$\begin{aligned} \pi_{i'}(P, u)(X) &\simeq \pi_i(P, u)(X) \\ \pi_{j'}(P, u)(Y) &\simeq \pi_j(P, u)(Y) \\ \pi_{i'+j'}(P, u)(X \wedge Y) &\simeq \pi_{i+j}(P, u)(X \wedge Y). \end{aligned}$$

It is natural to require that $\mu_{i,j}$ should be compatible with $\nu_{i',j'}$ under these isomorphisms. It is not hard to check using the strong centrality of good (P, u) self maps that there is a unique map $\mu_{i,j}$ which has this compatibility for all choices of k, i', j', u and v. Given this, one can deduce the expected naturality, commutativity and associativity properties. □

Corollary 14.9. *Given an object $(P, u) \in \mathcal{P}'$, there are objects $(P^{(k)}, u^{\otimes k})$ in $\mathcal{P}'$ for all $k \in \mathbf{Z}_p$, defined up to canonical isomorphism, such that there is a natural isomorphism*

$$\pi_k(P, u)(X) = [P^{(k)}, X]$$

for all small spectra X. Moreover, there are canonical and coherent isomorphisms $P^{(k)} \wedge P^{(l)} = P^{(k+l)}$. The spectrum $P^{(k)}$ is independent of u up to unnatural isomorphism.

Proof. The functor $X \mapsto \pi_{-k}(P, u)(X)$ is a homology theory on small spectra, so by [HPS95, Corollary 2.3.11] it extends canonically to a homology theory on all of $\mathcal{K}$. By Theorem 9.5 there is a spectrum $P^{(k)}$ representing this functor, so in particular $\pi_{-k}(P, u)(X) = \pi_0(P^{(k)} \wedge X)$ whenever X is small. This spectrum is unique up to isomorphism, and the isomorphism is unique up to phantoms. However, for any small spectrum Z we have $[Z, P^{(k)}] = \pi_{-k}(P, u)DZ$ which is finite, so Theorem 10.7 tells us that there are no phantom maps into $P^{(k)}$. This shows that $P^{(k)}$ is unique up to canonical isomorphism.

Let X be a nontrivial small spectrum, and $v \colon P^{(p^j)} \wedge X \to X$ a good (P, u) self map. Choose an integer i congruent to k mod p^j. Using v we construct a natural isomorphism

$$\pi_{-k}(P, u)(X \wedge Y) = \pi_{-i}(P, u)(X \wedge Y) = \pi_0(P^{(i)} \wedge X \wedge Y)$$

for all small Y, and thus (by unique representability of homology theories) an isomorphism $P^{(k)} \wedge X \simeq P^{(i)} \wedge X$. We know that K_*P has dimension one over K_*, and it follows that $K_*P^{(k)}$ has dimension one, and thus that $P^{(k)} \in \mathcal{P}$.

We still need to produce a canonical generator of $K_0P^{(k)}$, however. To do this, suppose that we have small spectra X and Y, a map $f\colon X \longrightarrow Y$, and good self maps v, w of X and Y such that $wf = fv$. If we construct isomorphisms $P^{(k)} \wedge X \simeq P^{(i)} \wedge X$ and $P^{(k)} \wedge Y \simeq P^{(i)} \wedge Y$ by the procedure outlined above, then we find that they commute with the maps $1 \wedge f$. In particular, we can take $X = S/I$ and $Y = S/I \wedge S/I$, choose a good self map v of S/I and take $w = v \wedge 1 = 1 \wedge v$. We can then take $f = \eta \wedge 1$ or $1 \wedge \eta$, where $\eta\colon S \longrightarrow S/I$ is the unit map. This gives us a commutative diagram

$$\begin{array}{ccc}
P^{(k)} \wedge S/I & \overset{\eta\wedge 1}{\underset{1\wedge\eta}{\rightrightarrows}} & P^{(k)} \wedge S/I \wedge S/I \\
\big\downarrow & & \big\downarrow \\
P^{(i)} \wedge S/I & \overset{\eta\wedge 1}{\underset{1\wedge\eta}{\rightrightarrows}} & P^{(i)} \wedge S/I \wedge S/I.
\end{array}$$

If we apply K_*, then it is easy to see (just by calculating everything) that the equaliser of the top line is $K_*P^{(k)}$ and the equaliser of the bottom line is $K_*P^{(i)}$, so we get an isomorphism $K_*P^{(i)} \simeq K_*P^{(k)}$. One can check that this is independent of the choices involved. We define $u^{\otimes k} \in K_*P^{(k)}$ to be the image of $u^{\otimes i}$ under this isomorphism.

A similar argument shows that $\epsilon_{P^{(k)}} = 1$. Indeed, one can see as above that the image of $\epsilon_{P^{(k)}}$ in $[S, S/I]$ is the same as the image of $\epsilon_{P^{(i)}}$ for some integer i depending on I. Since this image is the unit map η of S/I, and $[S, S] = \varprojlim [S, S/I]$, we find that $\epsilon_{P^{(k)}} = 1$.

We next produce an isomorphism $P^{(k)} \wedge P^{(l)} \simeq P^{(k+l)}$. Suppose that we have finite spectra X and Y of type at least n, and maps $S \longrightarrow X$ and $S \longrightarrow Y$. This gives a map $P^{(k)} \longrightarrow P^{(k)} \wedge X$, or equivalently an element of $\pi_{-k}(P, u)F(P^{(k)}, X)$. Similarly, we have an element of $\pi_{-l}(P, u)F(P^{(l)}, Y)$. Using the pairings on the groups $\pi_*(P, u)(-)$ and the dualisability of $P^{(k)}$ and $P^{(l)}$ we get an element of $\pi_{-k-l}(P, u)F(P^{(k)} \wedge P^{(l)}, X \wedge Y)$, and thus a map $S \longrightarrow P^{(k+l)} \wedge F(P^{(k)} \wedge P^{(l)}, X \wedge Y)$, or equivalently a map $P^{(k)} \wedge P^{(l)} \longrightarrow P^{(k+l)} \wedge X \wedge Y$. One can show that these maps are compatible as X and Y run over the category of finite spectra of type n equipped with a map in of S. We can therefore pass to the inverse limit, which is just $P^{(k+l)}$ by Remark 4.20 and Proposition 7.10. This gives a map $P^{(k)} \wedge P^{(l)} \longrightarrow P^{(k+l)}$. There are no $\lim^1$ terms because all groups involved are finite, so the map is unique. One can show that it induces our earlier pairing $\pi_{-k}(P, u)Z \otimes \pi_{-l}(P, u)W \longrightarrow \pi_{-k-l}(P, u)(Z \wedge W)$. Using this, one can check that the maps $P^{(k)} \wedge P^{(l)} \longrightarrow P^{(k+l)}$ have the expected coherence properties.

Finally, we show that our definition of $P^{(k)}$ is independent of u up to unnatural isomorphism. If j is an integer then $P^{(j)}$ is just an iterated smash power of P or DP and clearly does not depend on u. For any generalised Moore spectrum S/I we can choose a good self map $P^{(p^i)} \wedge S/I \longrightarrow S/I$ and an integer j congruent to k modulo p^i. We then find that $P^{(k)} \wedge S/I$ is isomorphic to $P^{(j)} \wedge S/I$ independently of u. We are thus reduced to showing that if $P \in \mathrm{Pic}$ and $P \wedge S/I \simeq S/I$ for all I then $P \simeq S$. We have shown this in the proof of Proposition 14.3. $\square$

We also want to be able to grade things over subgroups of Pic' of rank greater than one. The following lemma will help us to do this.

Lemma 14.10. *Let (P, a) and (Q, b) be objects of $\mathcal{P}'$. Let X be a small spectrum, with good self maps $v\colon P^{(p^k)} \wedge X \to X$ and $w\colon Q^{(p^l)} \wedge X \to X$. Then the following diagram commutes.*

$$\begin{tikzcd}
Q^{(p^l)} \wedge P^{(p^k)} \wedge X \arrow[r, "\tau\wedge 1"] \arrow[d, "1\wedge v"'] & P^{(p^k)} \wedge Q^{(p^l)} \wedge X \arrow[r, "1\wedge w"] & P^{(p^k)} \wedge X \arrow[d, "v"] \\
Q^{(p^l)} \wedge X \arrow[rr, "w"'] & & X
\end{tikzcd}$$

Proof. We may assume that $k = l = 0$, to simplify the notation. It is easy to generalise Proposition 4.4 to show that w induces a natural transformation $w'_Y\colon Q \wedge Y \to Y$ on the category of spectra Y which can be written as a retract of a spectrum of the form $X \wedge Z$. By applying naturality to the twist map $\tau\colon P \wedge X \to X \wedge P$, we see that $w'_{P\wedge X}$ is the composite

$$Q \wedge P \wedge X \xrightarrow{1\wedge\tau} Q \wedge X \wedge P \xrightarrow{w\wedge 1} X \wedge P \xrightarrow{\tau^{-1}} P \wedge X.$$

General nonsense about symmetric monoidal categories tells us that this is the same as the top line of the diagram in the lemma (think about the analogous statement with vector spaces and tensor products). By applying naturality of w' to the map $v\colon P \wedge X \to X$, we conclude that the diagram commutes. □

We finish with a theorem about grading over groups of rank greater than one.

Theorem 14.11. *Given objects $(P_1, u_1), \dots , (P_r, u_r)$ of $\mathcal{P}'$, there is a homology theory $F\colon \mathcal{F} \to \mathcal{B}(\mathbf{Z}_p^r)$ whose fibre over a point $\underline{a} = (a_1, \dots , a_r)$ is*

$$F_{\underline{a}}(X) = [P_1^{(a_1)} \wedge \dots \wedge P_r^{(a_r)}, X].$$

There are also natural pairings from $F(X)$ and $F(Y)$ to $F(X \wedge Y)$.

Note that this gives a grading over $\mathbf{Z}_p^r$ and not over the image of $\mathbf{Z}_p^r$ in Pic'. Moreover, our construction uses a choice of basis for $\mathbf{Z}_p^r$; it is not clear whether this can easily be avoided.

Proof. We simply define $F(X)$ to be the disjoint union of the sets $F_{\underline{a}}(X)$ as $\underline{a}$ runs over $\mathbf{Z}_p^r$. If we choose good (P_i, u_i) self maps for each i, these give us isomorphisms $F_{\underline{a}}(X) \simeq F_{\underline{b}}(X)$ for all $\underline{b}$ sufficiently close to $\underline{a}$. These isomorphism appear to depend on the order in which we use the different self maps, but Lemma 14.10 tells us that this is not the case. Moreover, if we use a different system of self maps then by asymptotic uniqueness we find that the isomorphisms $F_{\underline{b}}(X) \simeq F_{\underline{a}}(X)$ do not change provided that $\underline{b}$ is sufficiently close to $\underline{a}$. Thus, we get a bundle structure on $F(X)$. We leave the rest of the details to the reader. □

15. Examples

15.1. **The case $n = 1$.** The height one case is quite straightforward and amenable to calculation. However, we warn the reader that it rather atypical and often misleading as a guide to the case $n > 1$. The material in this section is well-known. See [Rav84] or [Bou79] for the closely related theory of localisation with respect to K theory.

We shall assume that $p>2$ for simplicity. The spectrum E is the $2(p-1)$-periodic Adams summand of $KU_p^\wedge$, and $K=E/p$. For each $a\in 1+p\mathbf{Z}_p<\mathbf{Z}_p^\times$ we have an Adams operation $\psi^a\colon E\to E$, with $\psi^a\psi^b=\psi^{ab}$. These are ring maps, and their action on $E_*=\mathbf{Z}_p[v_1^{\pm 1}]$ is given by $\psi^a(v_1)=a^{p-1}v_1$. We choose a topological generator a of the group $1+p\mathbf{Z}_p\simeq\mathbf{Z}_p$ and define $T=\psi^a-1\in E^0E$. It turns out that $E^0E=\mathbf{Z}_p[\![T]\!]$; in particular, this ring is commutative. More generally, the ring E^*E is the non-commutative power series ring $E^*[\![T]\!]$ in which $\psi^b v_1=b^{p-1}v_1\psi^b$ and thus $Tv_1=a^{p-1}v_1T+(a^{p-1}-1)v_1$.

For any $d\in\mathbf{Z}_p$ we define X_d to be the fibre of the map $\psi^a-a^d=T+1-a^d\colon E\to E$. One checks that $E^*X_d=E^*[\![T]\!]/(T+1-a^d)=E^*$ and that ψ^b acts on E^0X_d with eigenvalue b^d for all $b\in 1+p\mathbf{Z}_p$. We can also determine π_*X_d. Note that ψ^a-a^d acts on $\pi_{2(p-1)k}E$ with eigenvalue $a^{(p-1)k}-a^d$, which is a unit multiple of $1-a^{d-(p-1)k}$ or of $p(d-(p-1)k)$. It follows that π_*X_d has a summand $\mathbf{Z}_p/(p(d-(p-1)k))$ in dimension $2k(p-1)-1$, and a summand $\mathbf{Z}_p$ in dimension $2d$ if d is an integer divisible by $p-1$. One can check that in this case the generator of $\pi_{2d}X_d$ is a K-equivalence $S^{2d}\to X_d$, so that $X_d=\widehat{L}S^{2d}$. More generally, it is not hard to show that X_d is just the p-adic smash power $(S^{2p-2})^{(d/(p-1))}$ considered in Section 14. Note that the fibration $\widehat{L}S^{2d}\to E\xrightarrow{\psi^a-a^d}E$ shows that $\widehat{L}S^{2d}$ is E-finite, as predicted by Theorem 8.9. We know from [HMS94] that the Picard group consists of the spectra Σ^kX_d for $d\in\mathbf{Z}_p$ and $0\le k<2p-2$, and that $\mathrm{Pic}\simeq\varprojlim_N\mathbf{Z}/|v_1^{p^N}|=\mathbf{Z}/(2p-2)\times\mathbf{Z}_p$.

We next define Y to be the fibre of $T^2-p\colon E\to E$. This spectrum is a counterexample to a number of plausible conjectures, as pointed out to us by Mike Hopkins. Note that $E^*Y=E^*[\![T]\!]/(T^2-p)=E^*\oplus E^*$, so Y is dualisable. Because T acts on each homotopy group π_dE (for $d\in\mathrm{Pic}$) with eigenvalue in $p\mathbf{Z}_p$ we find that the eigenvalue of T^2-p is a unit multiple of p and thus that $\pi_dY=\mathbf{F}_p$ when $d=-1\pmod{2p-2}$ and $\pi_dY=0$ otherwise. Thus $p\colon Y\to Y$ is a map of dualisable spectra which induces the zero map of all Pic-graded homotopy groups. However, $p\neq 0$ because $E^*Y=E^*\oplus E^*$ is torsion-free. Moreover, using the fibration $Y/p^2\to E/p^2\to E/p^2$ one finds that $p\pi_d(Y/p^2)=0$ for all d and thus that $p\colon Y/p^2\to Y/p^2$ is a map of small spectra that is nontrivial but acts trivially on all Pic-graded homotopy groups. This shows that all reasonable analogues of Freyd's generating hypothesis [Fre66] in $\mathcal{K}$ are false. See [Dev96a] for more discussion of the generating hypothesis from the chromatic point of view. Appendix 2 of that paper contains a different kind of counterexample to a certain analogue of the hypothesis. It appears that Devinatz does not consider this to be evidence against the original conjecture, and he is the expert.

We next show that Y does not lie in the thick subcategory generated by the Picard group. Indeed, for any Z in that category we see easily that $[Z,Y]$ is a torsion group, but the identity map has infinite order in $[Y,Y]$. Another instructive proof is to observe that when Z lies in the thick subcategory generated by the Picard group, the group $\mathbf{Q}\otimes E^0Z$ is a finite-dimensional vector space over $\mathbf{Q}_p$ on which T acts with eigenvalues in $\mathbf{Q}_p$. However, the eigenvalues of T acting on E^0Y are $\pm\sqrt{p}\notin\mathbf{Q}_p$.

One might expect that a dualisable spectrum with finite homotopy groups would be small, but again Y is a counterexample.

15.2. **The case $n=2$.** We next analyse some calculations of Shimomura [Shi86] in the light of the theory developed above. See also [Sad93]. We take $n=2$ and

$p \geq 5$. Amongst other things, we obtain the following thought-provoking result. Recall from [HMS94] that the map from $\mathbf{Z}$ to Pic sending n to S^n extends over the completion $\widehat{\mathbf{Z}} = \varprojlim_N \mathbf{Z}/|v_2^{p^N}| = \mathbf{Z}/(2p^2-2) \times \mathbf{Z}_p$. Recall also that there is a unique translation-invariant measure on $\widehat{\mathbf{Z}}$ with $\mu(\widehat{\mathbf{Z}}) = 1$, called the .

Theorem 15.1. *For all $a \in \mathbf{Z}$ and all finite torsion spectra X the group $\pi_a \widehat{L}X$ is finite. However,*

(i) *$\pi_a \widehat{L}S/p$ is finite for all $a \in \widehat{\mathbf{Z}}$ such that $a = 0 \pmod{|v_1|}$.*
(ii) *If $b = -1, -2, -3$ or -4 then $\{a \in \widehat{\mathbf{Z}} \mid a = b \pmod{|v_1|}$ and $\pi_a \widehat{L}S/p$ is finite$\}$ has Haar measure zero.*
(iii) *If $a \notin \{0, -1, \ldots, -4\} \pmod{|v_1|}$ then $\pi_a \widehat{L}S/p = 0$.*

We have not yet understood the more recent calculation of $\pi_* L_2 S$ due to Shimomura and Yabe [SY95].

Note that $E(2)_*$ and $E(2)_*E(2)$ are concentrated in degrees divisible by $2p-2$. Thus if $E(2)_*X$ is generated in degrees divisible by $2p-2$ then the whole cobar complex which calculates $H^{**}E(2)_*X = \mathrm{Ext}^{**}_{E(2)_*E(2)}(E(2)_*, E(2)_*X)$ is concentrated in those degrees, as is the Ext group itself. Moreover, we know from [HS95] that $H^{s*}E(2)_* = 0$ when $s > n^2 + n = 6$. The argument used there shows *a fortiori* that $H^{s*}E(2)_*X = 0$ when $s > 6$ and X is any of the spectra $S/(p, v_1^N)$ used below, where N may be infinite. As $2p-2 > 6$ we see that the $E(2)$-based Adams spectral sequence collapses and $\pi_k LX = H^{s,k+s}E(2)_*X$ for the unique s with $0 \leq s < 2p-2$ and $s = -k \pmod{2p-2}$.

Shimomura calculates $H^{**}M_1^1$, where

$$M_1^1 = E_*/(p, v_1^\infty) = E(2)_*S/(p, v_1^\infty) = E_*M_2S^1,$$

and thus $H^{**}M_1^1 = \pi_*M_2S^1$. We would prefer to work with $\widehat{L}S$, so we start with some remarks about the necessary translation. We know that $\pi_*LS/(p, v_1^N) = H^{**}E_*/(p, v_1^N)$ is finite in each total degree, as the homotopy of any small object is finite, so there are no $\varprojlim^1$ terms and $\pi_*\widehat{L}S/p = \varprojlim_N H^{**}E_*/(p, v_1^N)$. We shall write $H^{**}E_*/(p, v_1^N)$ as a direct sum of groups $A_{i,N}$, where for fixed i the groups $A_{i,N}$ form a tower as N varies. As $H^{**}E_*/(p, v_1^N)$ is finite we see that $A_{i,N} = 0$ for almost all i so the direct sum is the same as the product and thus $\pi_*\widehat{L}S/p = \varprojlim_N \prod_i A_{i,N} = \prod_i \varprojlim_N A_{i,N}$.

Let Ω^{**} be the cobar resolution for $E(2)_*/p$. We have a finite number of summands in $H^{**}M_1^1$ that are isomorphic to $\mathbf{F}_p[v_1]/v_1^\infty$. Each such summand is generated by elements x_r/v_1^r where $x_r \in \Omega^{s,t}$ say and $d(x_r)$ is divisible by v_1^r and x_{r+1}/v_1^r is homologous to x_r/v_1^r. We refer to these as summands of type (∞, s, t). We also have summands isomorphic to $\mathbf{F}_p[v_1]/v_1^\alpha$, where $\alpha > 0$. Each of these is generated by an element y/v_1^α where $y \in \Omega^{s,t}$ say and $d(y)$ is divisible by v_1^α. We refer to these as summands of type (α, s, t). We now use the short exact sequence

$$\Sigma^{-N|v_1|}E(2)_*/(p, v_1^N) \xrightarrow{v_1^{-N}} E(2)_*/(p, v_1^\infty) = M_1^1 \xrightarrow{v_1^N} \Sigma^{-N|v_1|}M_1^1.$$

This gives a short exact sequence

$$U_N = H^{**}(M_1^1)/v_1^N \longrightarrow H^{**}E(2)_*/(p, v_1^N) \longrightarrow \mathrm{ann}(v_1^N, H^{**}M_1^1) = V_N,$$

and thus a short exact sequence

$$\varprojlim_N U_N \rightarrowtail \pi_* \widehat{L}S/p \twoheadrightarrow \varprojlim_N V_N.$$

Each summand of type (∞, s, t) in $H^{**}M_1^1$ contributes a summand of V_N generated by x_N and isomorphic to $\mathbf{F}_p[v_1]/v_1^N$. As N varies these form a tower isomorphic to the evident tower

$$\mathbf{F}_p[v_1]/v_1 \twoheadleftarrow \mathbf{F}_p[v_1]/v_1^2 \twoheadleftarrow \mathbf{F}_p[v_1]/v_1^3 \twoheadleftarrow \mathbf{F}_p[v_1]/v_1^4 \twoheadleftarrow \dots,$$

so the (graded) inverse limit is $\mathbf{F}_p[v_1]$. It is generated by a class $x_\infty \in H^{s,t}$.

On the other hand, consider a summand of type (α, s, t) generated by y/v_1^α. Write $\beta = \min(N, \alpha)$. We then get a summand of U_N generated by $d(y)/v_1^\beta$ and isomorphic to $\mathbf{F}_p[v_1]/v_1^\beta$. We also get an isomorphic summand in V_N generated by $v_1^{N-\beta}y$. As N varies, the summands in U_N form a pro-constant tower and the summands in V_N form a pro-trivial tower. In the inverse limit we are left with a summand in $\varprojlim_N U_N$ isomorphic to $\mathbf{F}_p[v_1]/v_1^\alpha$, generated by $d(y)/v_1^\alpha \in H^{s+1,t-\alpha|v_1|}$.

Feeding Shimomura's calculations into this analysis gives the following description of $\pi_* \widehat{L}S/p$. Firstly, there is a class $\zeta \in \pi_{-1}\widehat{L}S/p$ and an isomorphism $\pi_*\widehat{L}S/p = E[\zeta] \otimes A_*$ for some other graded group A_*. This group contains two copies of $\mathbf{F}_p[v_1]$, generated by $1 \in \pi_0$ and $h_0 \in \pi_{2p-3}$. We write B_* for the direct sum of the other summands in A_*, which is a v_1-torsion group.

We need some new language to give a manageable description of B_*. Firstly, we will write numbers in terms of their base p expansions, for example $[123] = p^2 + 2p + 3$. We write $\overline{k}$ for the digit $p - k$ so that $[\overline{1}\,2] = (p-1)p + 2$ for example. We indicate repetitions by exponents, so $[1^n] = \sum_{i=0}^{n-1} p^i = (p^n - 1)/(p-1)$. We write $*$ for an undetermined string of digits, so that $[*11]$ means any integer congruent to $p + 1$ modulo p^2. We handle negative numbers by allowing infinite expansions, for example $-p^n = [\overline{1}^\infty 0^n]$. We write $x \uparrow n$ for x^n where typographically convenient.

Proposition 15.2 (Shimomura)**.** *The generators of B_* are as follows. The entries will be explained and justified after the table.*

	generator	*parameters*	*width*	*period*	*degree*
(a)	$v_2^{[*\,t]}h_1$	$t \neq 0$	$[1]$	p	$1-2p$
(b)	$v_2^{[*\,t\,\overline{1}\,0^{n-1}]}h_0$	$t \neq \overline{1}\ ,\ n > 0$	$[1\,0\,\overline{1}^{n-1}]$	p^{n+1}	$2p-3$
(c)	$v_2^{[*\,\overline{1}]}g_0$		$[\overline{1}]$	p	$2p-4$
(d)	$v_2^{[*\,t\,\overline{2}^n]}g_1 v_2$	$t \neq \overline{1}, \overline{2}\ ,\ n \geq 0$	$[1\,2^n] + 1$	p^{n+1}	$-2p$
(e)	$v_2^{[*\overline{212}^n]}g_1 v_2$	$n \geq 0$	$[1\,0\,0\,2^n] + 1$	p^{n+2}	$-2p$
(f)	$v_2^{[*\,t\,\overline{2}^n]}\rho v_2$	$t \neq \overline{2}\ ,\ n \geq 0$	$[1\,0\,\overline{1}^{n-1}]$	p^{n+1}	-3

In part (f) the width is 1 *when* $n = 0$.

What all this means is as follows. The group B_* is a direct sum of modules isomorphic to $\mathbf{F}_p[v_1]/v_1^w$ generated by various elements x. These generators x come in v_2-periodic families. In the first column we identify the generator x by giving its image in $H^{**}E_*/I_2$. The entries in the first column depend on various parameters which are listed in the second column (except for the parameter "$*$" which always occurs). It is implicit that t is an integer with $0 \leq t < p$. The third column is the integer w such that the summand in question is isomorphic to $\mathbf{F}_p[v_1]/v_1^w$. If the entry in the fourth column is p^k, then the generators in question form a $v_2^{p^k}$-periodic family as $*$ varies. The last column is the degree of the generators modulo $|v_2| = 2(p^2-1)$.

We now deduce our table from Shimomura's calculation of $H^{**}M_1^1$ in [Shi86].

Proof. We start by remarking that our degrees are the usual ones, which are $2(p-1)$ times larger than Shimomura's degrees (see his (3.5.8)).

(a): This comes from Shimomura's (4.1.5). He has summands $\mathbf{F}_p[v_1]\langle x_n^s/v_1^{a_n}\rangle$ in $H^{**}M_1^1$ for $s \neq 0 \pmod p$. Our part (a) is the case $n=0$. In (3.6.4) he defines $a_0 = 1$. By our earlier discussion, we get a generator in $\pi_*\widehat{L}S/p$ of width $a_0 = 1$ by taking $d(x_0^s)/v_1$. In (4.1.5) he states that $d(x_0^s)/v_1$ is a unit multiple of $v_2^s h_1 \bmod I_2$. In (3.5.6) he defines $h_1 = t_1^p/v_2$ which has cohomological degree $s=1$ and internal degree $t = 2(p-1)p - 2(p^2-1)$ and thus total degree $t-s = 1-2p$. Our part (a) follows easily.

(b): This comes from the case $n > 0$ in Shimomura's (4.1.5). We then have $a_n = p^n + p^{n-1} - 1 = [10\overline{1}^{n-1}]$ by (3.6.4). By (4.1.5) we see that $d(x_n^s/v_1^{a_n})$ is a unit multiple of $v_2^{sp^n - p^{n-1}} h_0$, so these elements are generators of $\pi_*\widehat{L}S/p$ of width a_n. As $s \neq 0 \pmod p$ we have $t \in \{0, \dots, p-2\}$ such that $s = t+1 \pmod p$ and thus $sp^n - p^{n-1} = [*t0^n] + p^n - p^{n-1} = [*t\overline{1}0^{n-1}]$. In (3.6.5) h_0 is defined as $t_1 \in \Omega^{1,2p-2}$ so the degree is $2p-3$.

(c): This comes from the summands $\mathbf{F}_p[v_1]\langle v_2^t V/v_1^{p-1}\rangle$ in Shimomura's (4.1.6). Here $t \in p\mathbf{Z}$ and $d(v_2^t V)/v_1^{p-1} = -v_2^{t+p-1} g_0 \bmod I_2$. We thus have generators $v_2^{t+p-1} g_0$ of width $p-1$, and the general form of $t+p-1$ is $[*\overline{1}]$. We learn from (3.5.7) that $g_0 \in \Omega^{2,2p-2}$ so the total degree is $2p-4$.

(d): Shimomura has summands $\mathbf{F}_p[v_1]\langle y_m/v_1^{A(m)}\rangle$ for numbers m of the form sp^n where either $s \notin \{0,-1\} \pmod p$ or $s = -1 \pmod{p^2}$. Our part (d) is the case $s \notin \{0,-1\} \pmod p$, so in Shimomura's notation we have $\epsilon(s) = 0$ and thus $A(m) = 2 + (p+1)(p^n-1)/(p-1)$. He shows that $d(y_m)/v_1^{A(m)}$ is a unit multiple of $v_2^{e(m)} g_1 \bmod I_2$, where in the case $\epsilon(s) = 0$ we have $e(m) = m - (p^n-1)/(p-1)$. The general form of $m - p^n$ is $[*t0^n]$ with $t = 0, \dots, p-3$, so the general form of $e(m)$ is $[*t0^n] + [10^n] - [1^n] = [*t\overline{2}^n] + 1$, so the generators are $v_2 \uparrow [*t\overline{2}^n](g_1 v_2)$. (We have separated out one factor of v_2 to avoid special behaviour when $n=0$.) We learn from (3.5.7) that $g_1 \in \Omega^{2,2-2p}$ so the total degree is $-2p$.

(e): This comes from the summands $\mathbf{F}_p[v_1]\langle y_m/v_1^{A(m)}\rangle$ where $m = sp^n$ with $s = -1 \pmod{p^2}$ so $\epsilon(s) = 1$ and so $A(m) = 2 + p^n(p^2-1) + (p+1)(p^n-1)/(p-1)$. Once again $d(y_m)/v_1^{A(m)}$ is a unit multiple of $v_2^{e(m)} g_1 \bmod I_2$, but now $e(m) = m - p^n(p-1) - (p^n-1)/(p-1)$. The general form of m is $[*\overline{1}\overline{1}0^n]$ so $e(m) = [*\overline{1}\overline{1}0^n] - [\overline{1}0^n] - [1^n] = [*\overline{2}\overline{1}\overline{2}^n] + 1$. The total degree is $-2p$ again.

(f): This comes from the summands $\mathbf{F}_p[v_1]\langle x_n^s G_n/v_1^{a_n}\rangle$ in Shimomura's Theorem 4.4. Here $n \geq 0$ and $s \neq -1 \pmod p$ and $a_n = p^n + p^{n-1} - 1$ as before, except that $a_0 = 1$. His Proposition 4.3 shows that $d(x_n^s G_n)/v_1^{a_n}$ is a unit multiple of $v_2^{e(n,s)}\rho$, where $e(n,s) = sp^n - (p^n-1)/(p-1)$ (see his Lemma 4.2). The general form of $(s-1)p^n$ is $[*t0^n]$ with $t \neq \overline{2}$ and then $e(n,s) = [*t0^n] + [10^n] - [1^n] = [*t\overline{2}^n] + 1$, so the generators are $v_2 \uparrow [*t\overline{2}^n](\rho v_2)$. We know from (3.5.7) that $\rho \in \Omega^{3,0}$ so the total degree is -3. □

Remark 15.3. Our table only gives the $\mathbf{Z}$-graded homotopy groups of $\widehat{L}S/p$. However, if $a \in \mathbf{Z}$ and $b \in \mathbf{Z}_p$ then we still have $\pi_{a+b|v_2|}\widehat{L}S/p = \varprojlim_N \pi_{a+b|v_2|}LS/(p, v_1^N)$, and the terms in the inverse limit and the maps between them are locally constant as b varies. One can deduce that our table also describes $\pi_{a+b|v_2|}\widehat{L}S/p$ if we allow the symbol $*$ to denote an *arbitrary* infinite sequence of p-adic digits.

15.3. **Finiteness questions.** We next study the finiteness or otherwise of the groups $\pi_{a+b|v_2|}\widehat{L}S/p$. (These are of course vector spaces over $\mathbf{F}_p$ so that "finite" means the same as "finitely generated".) It is convenient to formulate the problem more generally. Let M be a group graded over $\mathbf{Z}_p$, for example $M_b = \pi_{a+b|v_2|}\widehat{L}S/p$. We assume that it is a graded module over $\mathbf{F}_p[u]$, where u has degree one; in our applications $u = v_1^{p+1}$.

Definition 15.4. Suppose that $\lambda \geq 0$ and $0 \leq \phi, w < p^\lambda$. Let $A(\phi, \lambda, w)$ be the $\mathbf{Z}_p$-graded group which is a direct sum of copies of $\mathbf{F}_p[u]/u^w$ with one copy generated in each degree congruent to ϕ mod p^λ. Thus $A(\phi, \lambda, w)$ is a p-adically interpolated version of $\Sigma^{\phi|v_2|}\mathbf{F}_p[u, v_2^{\pm p^\lambda}]/u^w$. Given a with $0 \leq a < w$ and $b \in p^\lambda \mathbf{Z}_p$ we write $u^a v_2^b x$ for the generator of $A(\phi, \lambda, w)_{a+b+\phi}$. The mnemonic is that ϕ, λ and w are the phase, wavelength and width of the group. We also write $\operatorname{supp}(A(\phi, \lambda, w)) = \{b \in \mathbf{Z}_p \mid A(\phi, \lambda, w)_b \neq 0\}$.

Definition 15.5. Given integers $0 \leq r \leq s$ and $\phi \in \mathbf{Z}_p$ we let ϕ_t be the t-th p-adic digit of ϕ (so $\phi = \sum_{t \geq 0} \phi_t p^t$ and $0 \leq \phi_t < p$) and define

$$B(r, s; \phi) = \{\psi \in \mathbf{Z}_p \mid \psi_t = \phi_t \text{ for } r \leq t < s\}.$$

We say that a set of this form is an (r, s)-*block*.

Lemma 15.6. *If $2p^\mu \leq w$ then $B(\mu, \lambda; \phi + p^\mu) \subseteq \operatorname{supp}(A(\phi, \lambda, w))$.*

Proof. Write $\theta = \phi + p^\mu$. If $\psi \in B(\mu, \lambda, \theta)$ then we define $k = p^\mu + \sum_{j=0}^{\mu-1}(\psi_t - \phi_t)p^t$. It is not hard to check that $1 \leq k \leq 2p^\mu - 1 < w$. Moreover, we have

$$\psi - k - \phi = \psi - \sum_{t=0}^{\mu-1}(\psi_t - \phi_t) - \theta = \sum_{t<\mu}(\phi_t - \theta_t)p^t + \sum_{t \geq \mu}(\psi_t - \theta_t)p^t.$$

It is clear that $\theta_t = \phi_t$ when $t < \mu$, and by assumption we have $\psi_t = \theta_t$ when $\mu \leq t < \lambda$. It follows that $\psi - k - \phi$ is divisible by p^λ so that $u^k v_2^{\psi-k-\phi}x$ defines a nonzero element of $A(\phi, \lambda, w)_\psi$. □

Proposition 15.7. *Suppose that $M = \bigoplus_i A(\phi_i, \lambda_i, w_i)$, where the numbers λ_i are unbounded and the numbers p^{λ_i}/w_i are bounded above. Then $\{a \in \mathbf{Z}_p \mid M_a$ is finite $\}$ has Haar measure zero.*

Proof. As the numbers p^{λ_i}/w_i are bounded above, we can find $\alpha \geq 0$ such that $2p^{\lambda_i - \alpha} \leq w_i$ for all i. As the numbers λ_i are unbounded, we may assume (after passing to a subgroup and reindexing if necessary) that our direct sum is indexed by the natural numbers and that $\lambda_i > \lambda_{i-1}+\alpha$ (where λ_{-1} means 0). We now write

$$B_i = B(\lambda_i-\alpha,\lambda_i,\phi_i+p^{\lambda_i-\alpha}) \subseteq \operatorname{supp}(A(\phi_i,\lambda_i,w_i)).$$

It will be enough to show that $C = \{a \mid a \text{ lies in infinitely many of the sets } B_i\}$ has measure one. Note that the conditions $a \in B_i$ depend on disjoint subsets of the digits of a (because $\lambda_i > \lambda_{i-1}+\alpha$). It follows that given any finite set I of integers we have $\mu(\bigcap_{j\in I} B_j^c) = (1-p^{-\alpha})^{|I|}$, and thus that the corresponding measure is zero if I is infinite. In particular if J is finite then $\mu(\bigcap_{j\notin J} B_j^c) = 0$. There are only countably many such sets J, so we have $\mu(\bigcup_J \bigcap_{j\notin J} B_j) = 0$. This union is just the complement of C, so $\mu(C) = 1$. $\square$

Proposition 15.8. *Suppose that $M = \bigoplus_i A(\phi_i,\lambda_i,w_i)$ and define $\phi_i' = p^{\lambda_i}-(w_i-1)-\phi_i$. If $\phi_i \to \infty$ and $\phi_i' \to \infty$ then M_a is finite for all $a \in \mathbf{Z}$.*

Proof. If $a \in \mathbf{Z}$ then for almost all i we have $\phi_i > |a|$ and $\phi_i' > |a|$, and in particular $\phi_i' > 0$. It is clear that $\phi_i = |x_i|$ is the lowest positive degree in which $A(\phi_i,\lambda_i,w_i)$ is nonzero, and $-\phi_i' = |v_2^{-1}u^{w_i-1}x_i|$ is the highest negative degree in which $A(\phi_i,\lambda_i,w_i)$ is nonzero, so $A(\phi_i,\lambda_i,w_i)_a = 0$ for all but finitely many i. It is easy to see that $A(\phi_i,\lambda_i,w_i)_a$ is finite for all i, so M_a is finite. $\square$

We now want to transfer these results to modules that are graded in the more usual way. For each $a \in \mathbf{Z}$ and $b \in \mathbf{Z}_p$ we can define $\pi_{a+b|v_2|}\widehat{L}S/p = \pi_{b|v_2|}\widehat{L}S^{-a}/p$ using the p-adic powers of $S^{|v_2|} \in \mathcal{P}_0$ that we considered earlier. We can patch these groups together to get a group graded over $\widehat{\mathbf{Z}} = \varprojlim_N \mathbf{Z}/|v_2^{p^N}| = \mathbf{Z}/2(p^2-1)\times\mathbf{Z}_p$. (We do not need to worry about making coherent choices of isomorphisms here, as we are simply counting group orders.) We now define $A'(\phi,\lambda,w)$ to be the obvious interpolation of the $\mathbf{Z}$-graded group $\Sigma^\phi \mathbf{F}_p[v_1,v_2^{\pm p^\lambda}]/v_1^w$. Note that when w is large this is almost $|v_1|$-periodic. In this context we write $\phi' = p^\lambda|v_2|-(w-1)|v_1|-\phi = -|v_2^{-p^\lambda}v_1^{w-1}x|$, where x is the generator of $A'(\phi,\lambda,w)$.

It is not hard to deduce the following from Propositions 15.7 and 15.8.

Proposition 15.9. *Suppose that $M = \bigoplus_i A'(\phi_i,\lambda_i,w_i)$, where*

(a) *The integers ϕ_i are all congruent to a fixed number ϕ modulo $|v_1| = 2p-2$.*
(b) *The sequences λ_i, w_i, ϕ_i and ϕ_i' all tend to infinity.*
(c) *The numbers p^{λ_i}/w_i are bounded above.*

Then M_a is finite for all $a \in \mathbf{Z}$, but $\{a \in \widehat{\mathbf{Z}} \mid a = \phi \pmod{|v_1|}$ and M_a is finite $\}$ has measure zero. $\square$

We can now prove our result about $\pi_*\widehat{L}S/p$.

Proof of Theorem 15.1. By a thick subcategory argument it suffices to verify the finiteness statement for $X = S/p$, so we work with S/p throughout.

Let C_* be the direct sum of the entries (b), (d), (e) and (f) in Proposition 15.2. The omitted entries, together with the $\mathbf{F}_p[v_1]$-free summands generated by 1 and h_0, are all concentrated in degrees $0,\dots,-3$ modulo $|v_1|$ and are finite in all degrees. We also have $\pi_*\widehat{L}S = E[\zeta]\otimes A_*$ with $|\zeta| = -1$. It will thus suffice to show that C_* is finite in all integer degrees, but the set of degrees a congruent to -1, -2 or -3 mod $|v_1|$ such that C_a is finite has measure zero. This will follow from Proposition 15.9 once we have determined the parameters λ, ϕ, ϕ' and w for the families in entries (b), (d), (e) and (f). This is a straightforward calculation. It is easy to see that in each case the period λ and the width w tend to infinity as n does, but that $p^\lambda/w \leq p$. Moreover, if we set $* = 0$ we get a generator x whose degree satisfies $0 \leq |x| < |v_2^{p^\lambda}|$, which implies that $\phi = |x|$. Given this, it is immediate that ϕ tends to infinity as n does. A little more calculation is required to find the numbers ϕ'. For example, in entry (b) we have $\phi = 2(pt + p - 1)p^{n-1}(p^2 - 1) + 2p - 3$ and $w = p^n + p^{n-1} - 1$ and $\lambda = n + 1$ so

$$\begin{aligned}\phi' &= p^\lambda|v_2| - (w-1)|v_1| - \phi \\ &= 2p^{n+1}(p^2-1) - 2(p^n + p^{n-1} - 2)(p-1) - 2(pt + p - 1)p^{n-1}(p^2-1) - 2p + 3 \\ &= 2(p - 1 - t)p^{n-1}(p^2 - 1) + 2p - 1.\end{aligned}$$

As $0 \leq t < p - 1$ we see that this tends to infinity as well. The other cases are similar (although the case $t = p - 1$ in entry (f) needs to be done separately). $\square$

16. Questions and conjectures

In this section we assemble a long list of questions that we have not been able to answer. Some of them have a long history, but others seem not to have been asked before.

Problem 16.1. When X is a small spectrum in $\mathcal{K}$, is the number $\mathrm{len}(\pi_* X)$ defined in Definition 12.18 always an integer? If not, is there a universal constant M such that $p^M \mathrm{len}(\pi_* X)$ is always an integer?

Problem 16.2. Is $\pi_k \widehat{L}S$ always a finitely generated module over $\mathbf{Z}_p$ for all $k \in \mathbf{Z}$? Theorem 15.1 suggests that this is true but subtle.

Problem 16.3. It would be pleasant to have a natural topology on $\pi_0\mathcal{D}$ making it a topological semiring. One idea would be to define $U_Z(X) = \{Y \mid Z \wedge Y \simeq Z \wedge X\}$ and declare the sets $U_Z(X)$ (as Z runs over $\mathcal{F}$) to be a basis of neighbourhoods of X. This has some good properties and some bad ones. For example, take $n = 1$ and $p > 3$. Then we can define a v_1 self map $v\colon S^{2p-2}/p \longrightarrow S/p$, which becomes an isomorphism in $\mathcal{K}$. Write $X = \bigvee_{i=0}^{p-2} S^{2i}/p$, so that $\Sigma^2 X \simeq X \not\simeq \Sigma X$. It seems that X cannot be separated from ΣX, so the topology is not Hausdorff, and the subspace $\pi_0\mathcal{F}$ is not discrete. We hope nonetheless that some minor modifications will yield a satisfactory theory.

Problem 16.4. We can define a group $K_0(\mathcal{F})$ as the monoid of isomorphism classes of small spectra under the wedge operation, modulo relations $[Y] = [X] + [Z]$ for every cofibre sequence $X \longrightarrow Y \longrightarrow Z$. There is a homomorphism $\chi\colon K_0(\mathcal{F}) \longrightarrow \mathbf{Z}$ defined by

$$\chi[X] = \sum_{0 \leq k < |v_n|} (-1)^k \mathrm{len}(E_k X).$$

There is also a homomorphism $\xi\colon K_0(\mathcal{F}) \longrightarrow \mathbf{Z}[\frac{1}{p}]$, defined by

$$\xi[X] = \mathrm{len}(\pi_{\mathrm{even}}(X)) - \mathrm{len}(\pi_{\mathrm{odd}}(X)).$$

What is the relationship between χ and ξ? Is χ an isomorphism? If not, is $\mathbf{Q} \otimes \chi$ an isomorphism? Can one say anything about the higher Waldhausen K-theory of $\mathcal{F}$? There is a paper by Waldhausen about this [Wal84] but he assumed that the Telescope Conjecture would turn out to be true. Ravenel discusses this briefly in [Rav93] in the light of his disproof of the Telescope Conjecture [Rav92b].

Problem 16.5. Is Hopkins' chromatic splitting conjecture true? The simplest form of this says that $K(n-1)_*L_{K(n)}S$ has rank two over $K(n-1)_*$. A much more elaborate version is explained in [Hov95a], where some interesting consequences are deduced. Part of it can be rephrased as saying that

$$L_{n-1}L_{K(n)}S = L_{n-1}S_p^\wedge \wedge \bigwedge_{j=1}^{n}(S \vee L_{n-j}S^{1-2j}).$$

One possible approach is to consider the functor $v_k^{-1}((E/I_k)^*X)$ for $k < n$. This is a cohomology theory on the category of E-finite spectra; it would be interesting to know how it relates to $K(k)^*X$.

Problem 16.6. A problem related to the previous one is as follows: in the language of Definition 6.7, what is the relationship between the support and cosupport of an E-local spectrum X? In particular, X is K-local if and only if $\operatorname{cosupp}(X) = \{n\}$, and the examples suggest that in this case we have $\operatorname{supp}(X) = \{m, m+1, \dots, n\}$ for some m. Is this always the case?

Problem 16.7. Is there a cofinal set of ideals I for which there is a generalised Moore spectrum of type S/I which is a commutative and associative ring spectrum? Given a map $BP_*/I \to BP_*/J$ of BP_*BP-comodules, is there a map $S/I \to S/J$ inducing it? What about connecting maps for short exact sequences of comodules? How large a diagram of such maps can be made to commute on the nose? The first step here is clearly to find a convenient and conceptual formulation of the problem. A good answer would be helpful for the theory of unstable v_n periodicity, amongst other things.

Problem 16.8. What are the invariant prime ideals in E^*? The obvious conjecture is that they are just the ideals $I_k = (v_j \mid j < k)$ for $k \leq n$. Many people believe, as we once did, that this is an easy consequence of Landweber's classification of invariant prime ideals in BP_*, but this does not seem to be the case. We also conjecture that every invariant radical ideal is prime. This would also help to prove that the categories $\mathcal{D}_k$ are the only nontrivial ideals in $\mathcal{D}$.

Appendix A. Completion

In this appendix, we study the theory of (usually infinitely generated) modules over a regular local ring, that satisfy certain completeness properties. In particular, this applies to the modules E^*X and $E^\vee_*X$ over E_*, for $X \in \mathcal{K}$. Much of the theory presented here is obtained by specialising the results of [GM92] to our simpler context. Presumably a lot of it is known in some form to commutative algebraists.

Let R be a Noetherian graded ring with a unique maximal homogeneous ideal $\mathfrak{m}$, which is generated by a regular sequence of homogeneous elements $(x_0, \dots, x_{n-1})$. All modules, ideals and so on are required to be graded, and all maps and elements are required to be homogeneous.

A.1. **Ordinary completion.** Recall that the completion of a module M at $\mathfrak{m}$ is

$$M^\wedge_{\mathfrak{m}} = \varprojlim_k M/\mathfrak{m}^k M.$$

Theorem A.1.

(a) $(M^\wedge_{\mathfrak{m}})^\wedge_{\mathfrak{m}} = M^\wedge_{\mathfrak{m}}$.
(b) *If M is finitely generated then $M^\wedge_{\mathfrak{m}} = R^\wedge_{\mathfrak{m}} \otimes_R M$, and this is an exact functor of M.*
(c) *$R^\wedge_{\mathfrak{m}}$ is flat over R.*
(d) *If $f\colon M \longrightarrow N$ is epi, then so is the induced map $M^\wedge_{\mathfrak{m}} \longrightarrow N^\wedge_{\mathfrak{m}}$.*

Proof. (a): Because $M^\wedge_{\mathfrak{m}}$ is defined as the inverse limit of $\{M/\mathfrak{m}^k M\}$, there is a map $\pi_k\colon M^\wedge_{\mathfrak{m}} \longrightarrow M/\mathfrak{m}^k M$. Because the maps in the inverse system are epi, the map π_k is also epi. It is clear that π_k factors through an epimorphism $M^\wedge_{\mathfrak{m}}/\mathfrak{m}^k M^\wedge_{\mathfrak{m}} \longrightarrow M/\mathfrak{m}^k M$. On the other hand, the obvious map $M \longrightarrow M^\wedge_{\mathfrak{m}}$ induces a map $M/\mathfrak{m}^k M \longrightarrow M^\wedge_{\mathfrak{m}}/\mathfrak{m}^k M^\wedge_{\mathfrak{m}}$. One can check that these maps are mutually inverse, so that

$$(M^\wedge_{\mathfrak{m}})^\wedge_{\mathfrak{m}} = \varprojlim_k M^\wedge_{\mathfrak{m}}/\mathfrak{m}^k M^\wedge_{\mathfrak{m}} = \varprojlim M/\mathfrak{m}^k M = M^\wedge_{\mathfrak{m}}$$

(b) and (c) are well-known consequences of the Artin-Rees lemma.

(d): It is easy to see that the induced maps $M/\mathfrak{m}^n M \longrightarrow N/\mathfrak{m}^n N$ and $\mathfrak{m}^n M \longrightarrow \mathfrak{m}^n N$ are epi, and thus that $\mathfrak{m}^n M/\mathfrak{m}^{n+1} M \longrightarrow \mathfrak{m}^n N/\mathfrak{m}^{n+1} N$ is also epi. Let K_n be the kernel of the map $M/\mathfrak{m}^n M \longrightarrow N/\mathfrak{m}^n N$. We get a diagram as follows, in which the columns and the last two rows are exact.

$$\begin{tikzcd}
L \arrow[r] \arrow[d, tail] & \mathfrak{m}^n M/\mathfrak{m}^{n+1} M \arrow[r, two heads] \arrow[d, tail] & \mathfrak{m}^n N/\mathfrak{m}^{n+1} N \arrow[d, tail] \\
K_{n+1} \arrow[r, tail] \arrow[d] & M/\mathfrak{m}^{n+1} M \arrow[r, two heads] \arrow[d, two heads] & N/\mathfrak{m}^{n+1} N \arrow[d, two heads] \\
K_n \arrow[r, tail] & M/\mathfrak{m}^n M \arrow[r, two heads] & N/\mathfrak{m}^n N
\end{tikzcd}$$

It follows from the snake lemma that the map $K_{n+1} \longrightarrow K_n$ is epi, and thus that $\varprojlim^1_n K_n = 0$. We therefore have a short exact sequence $\varprojlim K_n \longrightarrow \varprojlim M/\mathfrak{m}^n M \longrightarrow \varprojlim N/\mathfrak{m}^n N$, so $M^\wedge_{\mathfrak{m}} \longrightarrow N^\wedge_{\mathfrak{m}}$ is epi as claimed. $\square$

Note that completion does not preserve monomorphisms and is not right exact. For example, we can take $R = \mathbf{Z}_{(p)}$ and $\mathfrak{m} = (p)$. Then the completion of $\mathbf{Z}_{(p)} \longrightarrow \mathbf{Q}$ is $\mathbf{Z}_p \longrightarrow 0$, so completion does not preserve monomorphisms. Next, consider $M = \bigoplus_k \mathbf{Z}$ and $f = \bigoplus_k p^k : M \longrightarrow M$. One can show that the element $(p, p^2, \dots)$ is nonzero in the cokernel of $f : M^\wedge_{\mathfrak{m}} \longrightarrow M^\wedge_{\mathfrak{m}}$ but is zero in the completion of the cokernel. Thus, completion is not right exact.

A.2. **Local homology.** We next recall the theory of local (co)homology and the derived functors of the completion functor. Recall that one can define the left-derived functors of any additive functor F. One only needs F to be right exact in order to prove that the zeroth derived functor of F is F (which is false in the present context). Our response is to take the zeroth derived functor as the "correct" definition of completion.

For any $x \in R$ we let $K^\bullet(x)$ denote the complex $R \to R[1/x]$, with R in degree 0 and $R[1/x]$ in degree 1. We also write $K^\bullet(\mathfrak{m}) = K^\bullet(x_0) \otimes \ldots \otimes K^\bullet(x_{n-1})$; there is then a natural map $K^\bullet(\mathfrak{m}) \to R$. Note that $K^\bullet(\mathfrak{m})$ is a complex of flat modules. In fact, $K^\bullet(\mathfrak{m})$ is the finite acyclisation of R determined by $R/\mathfrak{m}$ in the derived category of R, so it is determined by the ideal $\mathfrak{m}$ up to canonical quasiisomorphism over R. To see this, write $K^\bullet(x)$ as the colimit of the complexes $R \xrightarrow{x^k} R$ and tensor these complexes together. We also write $PK^\bullet(x)$ for the complex of projectives $R \oplus R[t] \xrightarrow{(1,tx-1)} R[t]$, which is quasiisomorphic to $K^\bullet(x)$. Here the map $tx - 1 : R[t] \to R[t]$ is the $R[t]$-module map which takes 1 to $tx - 1$. We write $PK^\bullet(\mathfrak{m}) = PK^\bullet(x_0) \otimes \ldots \otimes PK^\bullet(x_{n-1})$, which is quasiisomorphic to $K^\bullet(\mathfrak{m})$ by flatness. The local homology and cohomology groups of a module M are

$$L_* M = H_*^{\mathfrak{m}}(M) = H_*(\mathrm{Hom}(PK^\bullet(\mathfrak{m}), M))$$

$$H^*_{\mathfrak{m}}(M) = H^*(PK^\bullet(I) \otimes M) = H^*(K^\bullet(I) \otimes M).$$

The last equality again uses the flatness of $K^\bullet(\mathfrak{m})$. Because R is a regular local ring, we have

$$H^k_{\mathfrak{m}}(R) = \begin{cases} k < n & 0 \\ k = n & R/(x_0^\infty, \ldots, x_{n-1}^\infty) \end{cases}$$

There are important questions to address about the extent to which this description of $H^n_{\mathfrak{m}}(R)$ is natural; however, we shall not address them here.

It follows from the above that $K^\bullet(\mathfrak{m})$ is actually quasiisomorphic to $\Sigma^n H^n_{\mathfrak{m}}(R)$ (in other words, the complex whose only nonzero term is $H^n_{\mathfrak{m}}(R)$, concentrated in degree n). Note that this is a complex of injectives.

It turns out that the functors $L_k M$ are the left derived functors of completion, in the following sense.

Theorem A.2.

(a) *There are natural maps $M \xrightarrow{\eta_M} L_0 M \xrightarrow{\epsilon_M} M^\wedge_{\mathfrak{m}}$. Moreover, ϵ is an epimorphism, and the composite $M \to M^\wedge_{\mathfrak{m}}$ is the obvious map.*

(b) *There is a short exact sequence*

$$\lim_{\leftarrow k}{}^1 \mathrm{Tor}^R_{s+1}(R/\mathfrak{m}^k, M) \rightarrowtail L_s M \twoheadrightarrow \lim_{\leftarrow k} \mathrm{Tor}^R_s(R/\mathfrak{m}^k, M).$$

In particular, there is a short exact sequence

$$\lim_{\leftarrow k}{}^1 \mathrm{Tor}^R_1(R/\mathfrak{m}^k, M) \rightarrowtail L_0 M \twoheadrightarrow M^\wedge_{\mathfrak{m}}.$$

(c) *For any short exact sequence $M' \to M \to M''$ there is a long exact sequence*

$$L_{k+1} M'' \to L_k M' \to L_k M \to L_k M'' \to L_{k-1} M'.$$

(d) *There is a natural isomorphism*

$$L_s M = \mathrm{Ext}_R^{n-s}(H^n_{\mathfrak{m}}(R), M).$$

Moreover, both sides vanish if $s < 0$ or $s > n$. (Thus L_0 is right exact and L_n is left exact).

(e) *If M is projective then ϵ_M is an isomorphism.*

Proof. In [GM92] Greenlees and May construct functors L_k and a natural map $\epsilon : L_0M \longrightarrow M$, satisfying (c) and (e). In Theorem 2.5, they prove that their groups L_kM are the same as $H_k^{\mathfrak{m}}(M)$ as defined above. They prove (b) as Proposition 1.1; it follows that ϵ_M is an epimorphism, as stated in (a). The map $K^\bullet(\mathfrak{m}) \longrightarrow R$ induces a map

$$\eta\colon M = \mathrm{Hom}(R, M) \longrightarrow H_0(\mathrm{Hom}(K^\bullet(\mathfrak{m}), M)) = L_0M.$$

One can check that the composite $M \xrightarrow{\eta} L_0M \xrightarrow{\epsilon} M^\wedge_{\mathfrak{m}}$ is the obvious map.

In our case, we know that $PK^\bullet(\mathfrak{m})$ is a cochain complex of projectives, whose only cohomology is $H^n_{\mathfrak{m}}(R)$, concentrated in degree n. Moreover, $PK^\bullet(\mathfrak{m})$ vanishes above degree n. It follows that $P_k = PK^{n-k}(\mathfrak{m})$ defines a projective resolution of $H^n_{\mathfrak{m}}(R)$, and thus that $L_kM = H_k(\mathrm{Hom}(PK^\bullet(\mathfrak{m}), M)) = \mathrm{Ext}^{n-k}_R(H^n_{\mathfrak{m}}(R), M)$. This proves (d). □

Proposition A.3. *There is a natural map* $\mu\colon L_sM \otimes L_tN \longrightarrow L_{s+t}(M \otimes N)$*, which is commutative, associative and unital in the obvious sense.*

Proof. The universal property of $K^\bullet(\mathfrak{m})$ as the finite acyclisation of R gives a quasiisomorphism $\psi\colon PK^\bullet(\mathfrak{m}) \longrightarrow PK^\bullet(\mathfrak{m}) \otimes_R PK^\bullet(\mathfrak{m})$ compatible with the maps $PK^\bullet(\mathfrak{m}) \longrightarrow R \longleftarrow PK^\bullet(\mathfrak{m}) \otimes_R PK^\bullet(\mathfrak{m})$, which is unique up to homotopy. This gives a map of chain complexes

$$\mathrm{Hom}_R(PK^\bullet(\mathfrak{m}), M) \otimes_R \mathrm{Hom}_R(PK^\bullet(\mathfrak{m}), N) \longrightarrow \mathrm{Hom}_R(PK^\bullet(\mathfrak{m}), M \otimes N).$$

By taking homology, we get the advertised pairing. Using the uniqueness of ψ (and the analogous map $PK^\bullet(\mathfrak{m}) \longrightarrow PK^\bullet(\mathfrak{m})^{\otimes 3}$), we see that the pairing is commutative, associative, and unital. □

Proposition A.4. *If M is finitely generated then the map* $\mu\colon M \otimes_R L_0N \longrightarrow L_0M \otimes_R L_0N \longrightarrow L_0(M \otimes_R N)$ *is an isomorphism, and* $L_0M = M^\wedge_{\mathfrak{m}}$*. In particular,* $R/\mathfrak{m}^k \otimes L_0N = N/\mathfrak{m}^kN$*.*

Proof. Because L_0 is additive, it is clear that $R^k \otimes_R L_0N = (L_0N)^k = L_0(N^k) = L_0(R^k \otimes N)$. One can check that the map is just μ. For an arbitrary finitely generated module, choose an exact sequence $R^j \longrightarrow R^k \longrightarrow M \longrightarrow 0$, and use the use the right exactness of L_0 to conclude that $M \otimes_R L_0N = L_0(M \otimes_R N)$. In particular,

$$M^\wedge_{\mathfrak{m}} = M \otimes_R R^\wedge_{\mathfrak{m}} = M \otimes_R L_0R = L_0(M \otimes_R R) = L_0M.$$

□

A.3. L-complete modules.

Definition A.5. An R-module M is *L-complete* if $\eta_M : M \longrightarrow L_0M$ is an isomorphism. We write $\mathcal{M}$ for the category of R-modules, and $\widehat{\mathcal{M}}$ for the subcategory of L-complete modules.

Theorem A.6.

(a) *For any $M \in \mathcal{M}$, the modules $M^\wedge_{\mathfrak{m}}$ and L_kM lie in $\widehat{\mathcal{M}}$. In particular, $L_0^2M = L_0M$.*
(b) *If $M \in \widehat{\mathcal{M}}$ then $L_kM = 0$ for $k > 0$.*
(c) $L_0M = 0 \Leftrightarrow M^\wedge_{\mathfrak{m}} = 0 \Leftrightarrow M = \mathfrak{m}M$.
(d) *If $M \in \widehat{\mathcal{M}}$ and $M = \mathfrak{m}M$ then $M = 0$.*
(e) *$\widehat{\mathcal{M}}$ is an Abelian subcategory of $\mathcal{M}$, which is closed under extensions.*

(f) $L_0 : \mathcal{M} \to \widehat{\mathcal{M}}$ *is left adjoint to the inclusion* $\widehat{\mathcal{M}} \to \mathcal{M}$.

(g) *If* $\{M_k\}$ *is a collection of* L*-complete modules, then* $\prod_k M_k$ *is* L*-complete. If they form an inverse system then* $\varprojlim_k M_k$ *is* L*-complete. If they form a tower then* $\varprojlim^1_k M_k$ *is also* L*-complete.*

Proof. It is proved as Theorem 4.1 in [GM92] that if $N = M^\wedge_\mathfrak{m}$ or $N = L_k M$ then $L_0 N = N$ and $L_k N = 0$ for $k > 0$. It follows that $M^\wedge_\mathfrak{m}$ and $L_k M$ are L-complete. It also follows that if M is L-complete and $k > 0$ then $L_k M = L_k L_0 M = 0$. This proves (a) and (b).

(c): We have epimorphisms $L_0 M \to M^\wedge_\mathfrak{m} \to M/\mathfrak{m}M$, so $L_0 M = 0 \Rightarrow M^\wedge_\mathfrak{m} = 0 \Rightarrow M = \mathfrak{m}M$. Suppose that $M = \mathfrak{m}M$. We shall prove that $L_0 M = 0$, using an argument supplied by John Greenlees. It follows that $M = \mathfrak{m}^k M$, so that $R/\mathfrak{m}^k \otimes M = 0$ for all k. Using the short exact sequences

$$\mathfrak{m}^k/\mathfrak{m}^{k+1} \to R/\mathfrak{m}^{k+1} \to R/\mathfrak{m}^k$$

and the fact that $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ is a free module over $R/\mathfrak{m}$, we get an exact sequence

$$\operatorname{Tor}_1^R(\mathfrak{m}^k/\mathfrak{m}^{k+1}, M) \to \operatorname{Tor}_1^R(R/\mathfrak{m}^{k+1}, M) \to \operatorname{Tor}_1^R(R/\mathfrak{m}^k, M) \to 0.$$

From this we see that the maps in the tower

$$\operatorname{Tor}_1^R(R/\mathfrak{m}^k, M) \leftarrow \operatorname{Tor}_1^R(R/\mathfrak{m}^{k+1}, M) \leftarrow \dots$$

are surjective, so that $\varprojlim^1_k \operatorname{Tor}_1^R(R/\mathfrak{m}^k, M) = 0$. Using part (b) of Theorem A.2, we see that $L_0 M = 0$ as claimed.

(d): If $M \in \widehat{\mathcal{M}}$ and $M = \mathfrak{m}M$ then $M = L_0 M = 0$ by (c).

(e): First, we claim that the image of any map $f\colon M' \to M''$ of L-complete modules is L-complete. To see this, factor f as a composite $M' \xrightarrow{q} M \xrightarrow{j} M''$, with q epi and j mono, so that M is the image of f. We have a diagram as follows:

$$\begin{tikzcd}
M' \arrow[r, two heads, "q"] \arrow[d, "\eta'"', "\simeq"] & M \arrow[r, tail, "j"] \arrow[d, "\eta"'] & M'' \arrow[d, "\simeq"', "\eta''"] \\
L_0 M' \arrow[r, two heads, "L_0 q"'] & L_0 M \arrow[r, "L_0 j"'] & L_0 M''
\end{tikzcd}$$

Note that $L_0 q$ is epi because L_0 is right exact. Because the left square commutes, we see that η is epi; because the right square commutes, we see that it is mono. Thus η is an isomorphism, and M is L-complete.

Next, suppose that $N' \to N \to N''$ is a short exact sequence, and that any two of the terms are L-complete. Part (c) of Theorem A.2 gives a long exact sequence relating the L_k-groups of N', N and N'', from which it is easy to see that the third of these is also L-complete. Thus $\widehat{\mathcal{M}}$ is closed under extensions, and under kernels and cokernels of epimorphisms and monomorphisms. For any map $f\colon M' \to M''$ as in the previous paragraph, we have $\ker(f) = \ker(q)$ and $\operatorname{cok}(f) = \operatorname{cok}(j)$; it follows that $\widehat{\mathcal{M}}$ is closed under kernels and cokernels, and thus that it is Abelian.

(f): Suppose that N is L-complete and M is arbitrary. We need to show that $\eta : M \to L_0 M$ induces an isomorphism $\eta_M^*\colon \operatorname{Hom}(L_0 M, N) \to \operatorname{Hom}(M, N)$. There is a map $\lambda\colon \operatorname{Hom}(M, N) \to \operatorname{Hom}(L_0 M, N)$, defined by the commutativity of the following diagram:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\eta_M"'] & N \arrow[d, "\eta_N", "\simeq"'] \\ L_0 M \arrow[r, "L_0 f"'] \arrow[ur, "\lambda(f)"] & L_0 N \end{tikzcd}$$

It is clear that $\eta_M^* \circ \lambda = 1$, so that η_M^* is epi. Suppose that $f \in \ker(\eta_M^*)$, so that f can be factored as $L_0 M \xrightarrow{q} M' \xrightarrow{g} N$, where M' is the cokernel of η_M. By the argument given above, we have an epimorphism $\eta_{M'}^* : \mathrm{Hom}(L_0 M', N) \longrightarrow \mathrm{Hom}(M', N)$. However, because L_0 is right exact and idempotent, we see that $L_0 M' = 0$. It follows that $g = 0$ and thus $f = 0$. Thus $\eta_M^* \colon \mathrm{Hom}(L_0 M, N) \longrightarrow \mathrm{Hom}(M, N)$ is an isomorphism, as required.

(g): It is easy to see from the definitions that $H_s^{\mathfrak{m}}(\prod_k M_k) = \prod_k H_s^{\mathfrak{m}}(M_k)$. It follows that a product of L-complete modules is L-complete. If the modules $\{M_k\}$ form an inverse system involving various maps $u \colon M_k \longrightarrow M_l$ then $\varprojlim_k M_k$ is the kernel of a map $\prod_k M_k \longrightarrow \prod_u M_l$, so it is L-complete by (d). If the inverse system is a tower, then $\varprojlim_k^1 M_k$ is the cokernel of a map $\prod_k M_k \longrightarrow \prod_k M_k$, and thus is L-complete. $\square$

Corollary A.7. *$\widehat{\mathcal{M}}$ is a symmetric monoidal category with tensor product $M \overline{\otimes} N = L_0(M \otimes N)$.*

Proof. It suffices to show that the map $L_0\mu \colon L_0(L_0 M \otimes L_0 N) \longrightarrow L_0(M \otimes N)$ induced by the pairing of Proposition A.3 is an isomorphism. Consider two modules $M, N \in \mathcal{M}$. Let M' be the cokernel of $M \longrightarrow L_0 M$. Because L_0 is idempotent and right exact, we have $L_0 M' = 0$. It follows that $M' = \mathfrak{m} M'$, or in other words that the natural map $\mathfrak{m} \otimes_R M' \longrightarrow M'$ is surjective. This implies that $\mathfrak{m} \otimes_R M' \otimes_R N \longrightarrow M' \otimes_R N$ is also surjective, so that $L_0(M' \otimes_R N) = 0$. Using the right exactness of L_0, we see that the map $L_0(\eta \otimes 1) \colon L_0(M \otimes_R N) \longrightarrow L_0(L_0 M \otimes_R N)$ is surjective. Similarly, the map $L_0(L_0 M \otimes_R N) \longrightarrow L_0(L_0 M \otimes_R L_0 N)$ is surjective, and thus $L_0(\eta \otimes \eta) \colon L_0(M \otimes_R N) \longrightarrow L_0(L_0 M \otimes_R L_0 N)$ is surjective. On the other hand, one sees from the definitions that $L_0\mu \circ L_0(\eta \otimes \eta) = 1$. It follows that $L_0\mu$ is an isomorphism, with inverse $L_0(\eta \otimes \eta)$. $\square$

We next prove a useful criterion for modules to be L-complete.

Proposition A.8. *If M is L-complete and $a \in \mathfrak{m}$ then $\varprojlim^0 M = \varprojlim^1 M = 0$, where the limits refer to the tower*

$$M \xleftarrow{a} M \xleftarrow{a} M \xleftarrow{a} \dots .$$

Conversely, if $(x_0, \dots, x_{n-1})$ is a regular system of parameters and the above holds with $a = x_i$ for each i, then M is L-complete.

Proof. Let $C^\bullet$ be the complex (of projectives)

$$R[t] \xrightarrow{at-1} R[t].$$

One checks directly that $H^s \operatorname{Hom}(C^\bullet, M) = \varprojlim^s M$. Moreover, there is an obvious short exact sequence $C^\bullet \to PK^\bullet(a) \to R$. It follows that $\varprojlim^0 M = \varprojlim^1 M = 0$ if and only if $\operatorname{Hom}(PK^\bullet(a), M)$ is quasiisomorphic to $M = \operatorname{Hom}(R, M)$.

Suppose that M is L-complete, so that $M \simeq \operatorname{Hom}(PK^\bullet(\mathfrak{m}), M)$. The complex $K^\bullet(\mathfrak{m})$ is quasiisomorphic to $K^\bullet(a) \otimes K^\bullet(\mathfrak{m})$. Indeed, $K^\bullet(\mathfrak{m}) \otimes R[1/a]$ is acyclic, since $K^\bullet(\mathfrak{m})$ is finitely acyclic and $R[1/a]$ is finitely local. Thus $PK^\bullet(\mathfrak{m})$ is homotopy equivalent to $PK^\bullet(a) \otimes PK^\bullet(\mathfrak{m})$. Therefore

$$\begin{aligned}\operatorname{Hom}(PK^\bullet(a), M) &\simeq \operatorname{Hom}(PK^\bullet(a), \operatorname{Hom}(PK^\bullet(\mathfrak{m}), M)) \\ &\simeq \operatorname{Hom}(PK^\bullet(a) \otimes PK^\bullet(\mathfrak{m}), M) \\ &\simeq M.\end{aligned}$$

It follows that $\varprojlim^0 M = \varprojlim^1 M = 0$.

Conversely, let $(x_0, \ldots, x_{n-1})$ be a regular system of parameters, and suppose that $M \simeq \operatorname{Hom}(PK^\bullet(x_i), M)$ for each i. As $PK^\bullet(\mathfrak{m}) = PK^\bullet(x_0) \otimes \ldots \otimes PK^\bullet(x_{n-1})$, we see easily that $M \simeq \operatorname{Hom}(PK^\bullet(\mathfrak{m}), M)$, so M is L-complete. □

A.4. **Pro-free modules.**

Theorem A.9. *Let M be an L-complete R-module. The following are equivalent:*

1. *Every regular sequence of parameters $\underline{x}$ is regular on M.*
2. *Some regular sequence of parameters $\underline{x}$ is regular on M.*
3. $\operatorname{Tor}^R_s(M, R/\mathfrak{m}) = 0$ *for all* $s > 0$.
4. $\operatorname{Tor}^R_1(M, R/\mathfrak{m}) = 0$.
5. $M = L_0 F = F^\wedge_\mathfrak{m}$ *for some free module F.*
6. *M is a projective object of $\widehat{\mathcal{M}}$.*

Definition A.10. If $M \in \widehat{\mathcal{M}}$ satisfies the conditions of Theorem A.9, we say that M is *pro-free.*

Proof of Theorem A.9. (1)⇒(2): clear.

(2)⇒(3): Suppose that $\underline{x} = (x_0, \ldots, x_{n-1})$ is a regular system of parameters for R, and that $\underline{x}$ is also regular on M. Write $I_m = (x_0, \ldots, x_{m-1})$, so that $I_n = \mathfrak{m}$. We therefore have short exact sequences

$$R/I_m \xrightarrow{x_m} R/I_m \to R/I_{m+1}$$

and

$$M/I_m M \xrightarrow{x_m} M/I_m M \to M/I_{m+1} M.$$

The first of these also gives a long exact sequence

$$\begin{aligned}\operatorname{Tor}^R_{s+1}(R/I_{m+1}, M) \to \operatorname{Tor}^R_s(R/I_m, M) \xrightarrow{x_m} \operatorname{Tor}^R_s(R/I_m, M) \to \\ \operatorname{Tor}^R_s(R/I_{m+1}, M) \to \operatorname{Tor}^R_{s-1}(R/I_m, M).\end{aligned}$$

By comparing this with the second short exact sequence, we see inductively that $\operatorname{Tor}^R_s(R/I_m, M) = 0$ for all $s > 0$ and all m, in particular $\operatorname{Tor}^R_s(R/\mathfrak{m}, M) = 0$ for $s > 0$.

(3)⇒(4): clear.

(4)⇒(5): Suppose that $\operatorname{Tor}^R_1(M, R/\mathfrak{m}) = 0$. Choose a basis $\{\overline{e}_i \mid i \in I\}$ for $M/\mathfrak{m}M$ over $R/\mathfrak{m}$, and choose elements $e_i \in M$ lifting $\overline{e}_i$. These give a map $F = \bigoplus_I R \to M$, and thus a map $f \colon L_0 F \to M$. By proposition A.4, we see that

$R/\mathfrak{m} \otimes L_0 F = R/\mathfrak{m} \otimes F \simeq R/\mathfrak{m} \otimes M$, so that $R/\mathfrak{m} \otimes \operatorname{cok}(f) = 0$. By part (d) of Theorem A.6, we see that $\operatorname{cok}(f) = 0$, so f is epi. We therefore get an exact sequence

$$0 = \operatorname{Tor}_1^R(R/\mathfrak{m}, M) \to R/\mathfrak{m} \otimes_R \ker(f) \to R/\mathfrak{m} \otimes_R L_0 F \xrightarrow{\simeq} R/\mathfrak{m} \otimes_R M \to 0.$$

It follows that $R/\mathfrak{m} \otimes_R \ker(f) = 0$, so that $\ker(f) = 0$. Thus f is an isomorphism, as required.

(5)$\Rightarrow$(1): Suppose that $\underline{x} = (x_0, \dots, x_{n-1})$ is a regular system of parameters for R. Write $I_m = (x_0, \dots, x_{m-1})$. Let S be a set, and $a = (a_s)_{s \in S}$ an element of $\prod_{s \in S} R/I_m$. We shall say that a converges to zero (or write $a \to 0$) if for all k we have $a_s \in \mathfrak{m}^k R/I_m$ for all but finitely many s. One can check directly that

$$\left(\bigoplus_{k \in S} R/I_m \right)^{\wedge}_{\mathfrak{m}} = \{ a \in \prod_{s \in S} R/I_m \mid a \to 0 \}.$$

We write F_m for this module. Using the above description, one can check directly that the sequence

$$F_m \xrightarrow{x_m} F_m \to F_{m+1}$$

is short exact. It follows that $\underline{x}$ is regular on F_0 (which is the same as $L_0(\bigoplus_S R)$ by part (e) of Theorem A.2).

(5)$\Rightarrow$(6): Suppose that F is free and that $M = L_0 F$. Then for $N \in \widehat{\mathcal{M}}$ we have $\widehat{\mathcal{M}}(M, N) = \mathcal{M}(F, N)$ by part (f) of Theorem A.6. This is clearly an exact functor of N, as required.

(6)$\Rightarrow$(4): Suppose that P is a projective object of $\widehat{\mathcal{M}}$. Choose a free module F and an epimorphism $F \to P$. This gives an epimorphism $L_0 F \to L_0 P = P$, which must split because P is projective. It follows that $\operatorname{Tor}_1^R(R/\mathfrak{m}, P)$ is a summand of $\operatorname{Tor}_1^R(R/\mathfrak{m}, L_0 F)$, which vanishes because (5)$\Rightarrow$(4). □

Corollary A.11. *The product of any family of pro-free modules is pro-free.*

Proof. This follows easily from condition (1). □

Corollary A.12. $\widehat{\mathcal{M}}$ *has enough projectives.*

Proof. Given $M \in \widehat{\mathcal{M}}$, choose a free module F and an epimorphism $F \to M$. This gives an epimorphism $L_0 F \to L_0 M = M$. □

Proposition A.13. *If $f \colon P \to Q$ is a map of pro-free modules such that the induced map $P/\mathfrak{m}P \to Q/\mathfrak{m}Q$ is a monomorphism, then f is a split monomorphism. In particular, if $\{P_k\}$ is a family of pro-free modules, then the natural map $L_0 \bigoplus_k P_k \to \prod_k P_k$ is a split monomorphism.*

Proof. For the first claim, choose elements $e_s \in P$ for $s \in S$ giving a basis for $P/\mathfrak{m}P$ as an $R/\mathfrak{m}$-vector space, and then choose elements $e'_t \in Q$ for $t \in T$ such that $\{fe_s \mid s \in S\} \amalg \{e_t \mid t \in T\}$ gives a basis for $Q/\mathfrak{m}Q$. This gives an obvious diagram

$$\begin{array}{ccc}
\bigoplus_S R & \longrightarrow & \bigoplus_{S \amalg T} R \\
\Big\downarrow & & \Big\downarrow \\
P & \longrightarrow & Q
\end{array}$$

As in the proof of (4)⇒(5) above, we see that the vertical maps are isomorphisms after applying L_0. The top horizontal map is a split monomorphism, so the bottom one is also.

Now consider a family $\{P_k\}$ of pro-free modules. By proposition A.4, we know that

$$R/\mathfrak{m} \otimes_R L_0 \bigoplus_k P_k = R/\mathfrak{m} \otimes_R \bigoplus_k P_k = \bigoplus_k R/\mathfrak{m} \otimes_R P_k.$$

It is easy to see that $R/\mathfrak{m} \otimes_R \prod_k P_k = \prod_k R/\mathfrak{m} \otimes_R P_k$. Thus, the first part of this proposition applies. □

Appendix B. Small objects in other categories

In this appendix we investigate when various other categories of local spectra have no small objects. We assume that all spectra and Abelian groups are p-local. We write I for the Brown-Comenetz dual of the sphere, so that $[X, I] = \mathrm{Hom}(\pi_0 X, \mathbf{Q}/\mathbf{Z})$ and $IX = F(X, I)$. Here $\mathbf{Q}/\mathbf{Z}$ denotes $\mathbf{Q}/\mathbf{Z}_{(p)}$ in keeping with our conventions. If $L\colon \mathcal{S} \longrightarrow \mathcal{S}$ is a localisation functor, we say that a spectrum X is *L-small* if it is a small object in the category of L-local spectra. Localisation functors split naturally into two categories: we say that a localisation functor *L has a finite local* if there is a nonzero finite spectrum X such that $LX = X$. Similarly, we say that *L has a finite acyclic* if there is a nonzero finite X such that $LX = 0$. It is an old conjecture of the first author that every localisation functor either has a finite local or a finite acyclic: it certainly cannot have both.

Definition B.1. We say that a spectrum X is *finite-dimensional* if it admits a finite filtration $0 = X_a \leq X_{a+1} \leq \ldots X_b = X$ for some integers $a \leq b$, such that X_k/X_{k-1} is a wedge of copies of S^k. Clearly if Y is a finite-dimensional CW complex then $\Sigma^a \Sigma^\infty Y$ is a finite-dimensional spectrum. We define the *width* of X to be the minimum possible value of $b - a$.

Remark B.2. It is not hard to show that X has a filtration as above if and only if $\pi_k X = 0$ for $k \leq a$ and $H_b X$ is free and $H_k X = 0$ for $k > b$. It follows that any retract of X admits a filtration of the same type.

Lemma B.3. *Let L be a localisation functor on the category of spectra, and $\mathcal{S}_L$ the category of L-local spectra. If X is an L-small spectrum then it is a retract of $\Sigma^{-N} LY$ for some integer N and some finite-dimensional suspension spectrum Y.*

Proof. Let X_k denote the k'th space of X. We assume some foundational setting in which this is an infinite loop space with $\pi_j X_k = \pi_{j-k} X$ for $j \geq 0$. It is well-known that X is the telescope of the spectra $\Sigma^{-k}\Sigma^\infty X_k$. It is not hard to conclude that X is also the telescope of the spectra $\Sigma^{-k}\Sigma^\infty Y_k$, where Y_k is the $2k$-skeleton of a CW complex weakly equivalent to X_k. As localisation functors preserve telescopes, we see that $X = LX$ is the telescope of the spectra $L\Sigma^{-k}\Sigma^\infty Y_k$. By smallness, it must be a retract of one of these terms. □

Lemma B.4. *For any set A, the spectrum $(\bigvee_{i\in A} S)^\wedge_p$ is a retract of $\prod_{i\in A} S^\wedge_p$.*

Proof. By Proposition A.13 (with $R = \mathbf{Z}_p$) we see that there is a set $B \supseteq A$ such that $\prod_{i\in A}\mathbf{Z}_p = (\bigoplus_{j\in B}\mathbf{Z}_p)^\wedge_p$. It follows that for any finitely generated Abelian group M we have $\prod_{i\in A} M^\wedge_p = (\bigoplus_{j\in B} M)^\wedge_p$. We can use the set B to define a map $(\bigvee_{j\in B} S^\wedge_p)^\wedge_p \to \prod_{i\in A} S^\wedge_p$, and by applying the previous remark to $M = \pi_k S$ we conclude that this map is an isomorphism. It follows that $W^\wedge_p = (\bigvee_{i\in A} S^\wedge_p)^\wedge_p$ is a retract of $\prod_{i\in A} S^\wedge_p$ as claimed. □

Lemma B.5. *If X is a nontrivial finite spectrum then $\operatorname{coloc}\langle X\rangle = \operatorname{coloc}\langle S^\wedge_p\rangle$, and this contains $Z^\wedge_p$ for all finite-dimensional spectra Z. Moreover, $\operatorname{coloc}\langle X, H\mathbf{Q}\rangle = \operatorname{coloc}\langle S\rangle$, and this contains all finite-dimensional spectra.*

Proof. Consider a generalised Moore spectrum $S/I = S/(J, v)$. It is easy to see that $S/(J, v^k)$ lies in the thick subcategory generated by S/I, and thus that $(S/J)^\wedge_v = \operatorname{holim}_{\overleftarrow{k}} S/(J, v^k) \in \operatorname{coloc}\langle X\rangle$. If $J \neq 0$ then $|v| > 0$ and S/J is (-1)-connected and simple connectivity arguments show that $(S/J)^\wedge_v = S/J$. When $J = 0$ we know that v is just a power of p and $(S/J)^\wedge_v = S^\wedge_p$. By induction on the height we conclude easily that $\operatorname{coloc}\langle S/I\rangle = \operatorname{coloc}\langle S^\wedge_p\rangle$. If X is an arbitrary nontrivial finite spectrum then X generates the same thick subcategory as some generalised Moore spectrum, so $\operatorname{coloc}\langle X\rangle = \operatorname{coloc}\langle S^\wedge_p\rangle$. This part of the argument is essentially [Hov95a, Lemma 3.7].

Next, let X be a finite-dimensional spectrum, with filtration as in Definition B.1. Lemma B.4 tells us that $(X_k/X_{k-1})^\wedge_p \in \operatorname{coloc}\langle S^\wedge_p\rangle$, and it follows easily that $X^\wedge_p \in \operatorname{coloc}\langle S^\wedge_p\rangle$.

Next, recall that $Z^\wedge_p = F(S^{-1}/p^\infty, Z)$, so we have a fibration $F(S\mathbf{Q}, Z) \to Z \to Z^\wedge_p$, in which the fibre is a rational spectrum. It is easy to see that whenever V is a nontrivial rational spectrum, we have $\operatorname{coloc}\langle V\rangle = \operatorname{coloc}\langle H\mathbf{Q}\rangle = \{\text{all rational spectra}\}$. By taking $Z = S$, we conclude that $S \in \operatorname{coloc}\langle S^\wedge_p, H\mathbf{Q}\rangle$. We also conclude that $H\mathbf{Q} \in \operatorname{coloc}\langle S, S^\wedge_p\rangle = \operatorname{coloc}\langle S\rangle$. This shows that $\operatorname{coloc}\langle S\rangle = \operatorname{coloc}\langle S^\wedge_p, H\mathbf{Q}\rangle$ as claimed. Finally, we can take Z to be a finite-dimensional spectrum. We have seen that $Z^\wedge_p \in \operatorname{coloc}\langle S^\wedge_p\rangle$ and that $F(S\mathbf{Q}, Z)$ is rational and thus in $\operatorname{coloc}\langle H\mathbf{Q}\rangle$, so it follows that $Z \in \operatorname{coloc}\langle S^\wedge_p, H\mathbf{Q}\rangle = \operatorname{coloc}\langle S\rangle$. □

Theorem B.6. *Suppose L is a localisation functor with a finite local.*

(a) *If $LH\mathbf{Q} \neq 0$, then L is the identity on finite-dimensional spectra. If $LH\mathbf{Q} = 0$, then L is p-completion on finite-dimensional spectra.*
(b) *Every L-small spectrum is finite, and is torsion if $LH\mathbf{Q} = 0$.*

Proof. (a): Let Z be a finite-dimensional spectrum. The category $\mathcal{S}_L$ of L-local spectra is a colocalising category which contains a nontrivial finite spectrum, so by Lemma B.5 it contains $Z^\wedge_p$. If $LH\mathbf{Q} \neq 0$ then $\mathcal{S}_L$ contains a nontrivial rational spectrum so it contains all rational spectra, and in particular it contains $F(S\mathbf{Q}, Z)$. We have a fibration $F(S\mathbf{Q}, Z) \to Z \to Z^\wedge_p$ so we conclude that $Z \in \mathcal{S}_L$ and thus $LZ = Z$. On the other hand, suppose that $LH\mathbf{Q} = 0$. If V is a rational spectrum we must have $LV = 0$, for otherwise $\mathcal{S}_L$ would contain a nontrivial rational spectrum and thus would contain $H\mathbf{Q}$. Thus $LF(S\mathbf{Q}, Z) = 0$ so our fibration shows that $LZ = LZ_p^\wedge = Z_p^\wedge$.

(b): We begin by showing that any L-small spectrum is finite-dimensional. In case $LH\mathbf{Q} \neq 0$, this follows from part (a) and Lemma B.3. If $LH\mathbf{Q} = 0$, then the telescope $\underrightarrow{\operatorname{holim}}(p, X)$ of an L-small spectrum X computed in $\mathcal{S}_L$ is zero. Applying smallness to the canonical map $X \longrightarrow \underrightarrow{\operatorname{holim}}(p, X)$, we find that $0 = p^m \colon X \longrightarrow X$ when $m \gg 0$. Thus X is a retract of X/p^m, which is a retract of $\Sigma^{-n} Y/p^m$ (because $LY/p^m = Y/p^m$ by part (a)). It follows that X is finite-dimensional and torsion.

We next claim that we have $[X, \bigvee_i S] = \bigoplus_i [X, S]$ for any set of indices i. If $LH\mathbf{Q} \neq 0$ then this is clear, because X is small in $\mathcal{S}_L$ and the wedge in $\mathcal{S}$ is finite-dimensional and thus L-local and thus the same as the coproduct in $\mathcal{S}_L$. In the case $LH\mathbf{Q} = 0$ we know that $D(S/p^m) \wedge \bigvee_i S = \bigvee_i D(S/p^m)$ is L-local so $[X \wedge S/p^m, \bigvee_i S] = \bigoplus_i [X \wedge S/p^m, S]$ and we have seen that X is a retract of $X \wedge S/p^m$ for some m so the claim follows.

Let $\mathcal{C}$ be the full subcategory of finite-dimensional spectra Y such that

$$[Y, \bigvee_{k=0}^{\infty} S]_* = \bigoplus_k [Y, S]_*.$$

This is clearly thick, and contains both X and all finite spectra. We shall prove by induction on the width of Y that if $Y \in \mathcal{C}$ then Y is finite. We may assume that Y has width greater than zero, so there is a cofibration $Z \longrightarrow Y \longrightarrow W$ in which W is a wedge of spheres of the same dimension and Z has width strictly less than that of Y. By the defining property of $\mathcal{C}$, there is a finite subwedge V of W such that the map $Y \longrightarrow W$ factors through V. Let T be the cofibre of the map $Y \longrightarrow V$, so that $T \in \mathcal{C}$. By applying the octahedral axiom to the maps $Y \longrightarrow V \rightarrowtail W$ we get a cofibration $T \rightarrowtail \Sigma Z \twoheadrightarrow U$ in which the first map is a split monomorphism, so T is a retract of Z and thus has width less than that of Y. It follows that T is finite, and thus that Y is finite. We conclude that our L-small spectrum X must be finite. $\square$

We now investigate when $LF(n)$ is L-small. It is convenient to use the notation $\langle L \rangle$ for the collection of L-local spectra, and to think of $\langle L \rangle$ as a generalised Bousfield class. If $L = L_E$, then $\langle L \rangle = \langle E \rangle$. We can then use notation such as $\langle L \wedge F(n) \rangle$, which is the local objects corresponding to the acyclics for the functor $X \mapsto LX \wedge F(n) = L(X \wedge F(n))$.

Proposition B.7. *Suppose L is a localisation functor on $\mathcal{S}$. Then the following are equivalent.*

(a) *$LF(n)$ is L-small.*
(b) *If $\{X_\alpha\}$ is a set of L-local spectra, then $\bigvee_\alpha X_\alpha \wedge F(n)$ is L-local.*
(c) *The natural map $LS \wedge X \wedge F(n) \longrightarrow L(X \wedge F(n))$ is an isomorphism for all X.*
(d) *$\langle LS \wedge F(n) \rangle = \langle L \wedge F(n) \rangle$.*

Furthermore, $LF(n)$ is a small generator of $\mathcal{S}_L$ if and only if $\langle L \rangle = \langle LS \wedge F(n) \rangle$.

Proof. (a)⇒(b): Suppose $LF(n)$ is L-small, and $\{X_\alpha\}$ is a set of L-local spectra. Then we have

$$\begin{aligned}
[F(n), L\bigvee X_\alpha]_* &= [LF(n), L\bigvee X_\alpha]_* \\
&= \bigoplus [LF(n), X_\alpha]_* \\
&= \bigoplus [F(n), X_\alpha]_* \\
&= [F(n), \bigvee X_\alpha]_*.
\end{aligned}$$

It follows using duality that the map $\bigvee X_\alpha \wedge F(n) \longrightarrow L(\bigvee X_\alpha \wedge F(n))$ is an equivalence.

(b)⇒(c): Consider the collection of all X such that the map $LS \wedge X \wedge F(n) \longrightarrow L(X \wedge F(n))$ is an equivalence. This category is clearly thick and contains S. Part (b) implies that it is localising, so it is all of $\mathcal{S}$.

(c)⇒(d): This is clear.

(d)⇒(a): We first show that if X_α is L-local for all α, then $\bigvee X_\alpha \wedge F(n)$ is also L-local. To see this, choose $F(n)$ to be a ring spectrum (take $F(n) = DY \wedge Y$ for a finite type n Y). Then $\bigvee X_\alpha \wedge F(n)$ is an $LS \wedge F(n)$-module spectrum, so in particular is $LS \wedge F(n)$-local. Since we are assuming part (d), we see that $\bigvee X_\alpha \wedge F(n)$ is $L \wedge F(n)$-local, and in particular is L-local.

We therefore have

$$\begin{aligned}
[LF(n), L\bigvee X_\alpha] &= [F(n), L\bigvee X_\alpha] \\
&= [S, L(\bigvee X_\alpha \wedge DF(n))] \\
&= [S, \bigvee X_\alpha \wedge DF(n)] \\
&= [F(n), \bigvee X_\alpha] \\
&= \bigoplus [F(n), X_\alpha] \\
&= \bigoplus [LF(n), X_\alpha]
\end{aligned}$$

as required.

To complete the proof, note that $LF(n)$ is a generator of $\mathcal{S}_L$ if and only if $\pi_* L(X \wedge F(n)) = [DF(n), LX] = 0$ implies that $LX = 0$. This is true if and only if $\langle L \rangle = \langle L \wedge F(n) \rangle$. □

Corollary B.8. *Suppose L is a localisation functor with a finite local. Then $\mathcal{S}_L$ has a nonzero small object if and only if $\langle L \rangle \geq \langle F(n) \rangle$ for some n, and in this case the small objects in $\mathcal{S}_L$ are $\mathcal{C}_n$ for the smallest such n. The category $\mathcal{S}_L$ is an algebraic stable homotopy category if and only if $L = L_{F(n)}$ for some n, and in this case $F(n)$ is a small generator.*

Proof. The small objects in $\mathcal{S}_L$ are all finite by Theorem B.6. Furthermore, LS is either S or the p-completion of S, and in either case we have $\langle LS \rangle = \langle S \rangle$. The corollary then follows from Proposition B.7. □

Proposition B.7 also applies to localisations with finite acyclics. For example, we get the following corollary.

Corollary B.9. *Let $E = K(m) \vee K(m+1) \vee \dots K(n)$ for some $0 \leq m \leq n < \infty$. Then $\mathcal{S}_E$ is an algebraic stable homotopy category with small generator $L_E F(m)$.*

It would be interesting to investigate these categories, as well as $\mathcal{S}_{F(n)}$.

In order to apply Corollary B.8, we need a criterion to determine when $\langle L\rangle \geq \langle F(n)\rangle$ for some n.

Lemma B.10. *Suppose $\langle L\rangle \geq \langle F(n)\rangle$ for some n. Then $LI \neq 0$.*

Proof. It suffices to show that $DF(n)\wedge I \neq 0$ for all n. But $DF(n)\wedge I = IF(n)$ and IX is never zero unless X is. $\square$

Lemma B.10 is more useful than it appears: in fact, we conjecture that its converse holds as well. There are very few spectra that detect I.

Lemma B.11. $BP\wedge I = 0$.

Proof. This follows from Ravenel's results in [Rav84]. He proves in Lemma 3.2 that $[P(1), S]_* = 0$, where $P(1) = BP/p = BP\wedge S/p$. As $D(S/p) = S^{-1}/p$ we see that $[BP, S/p]_* = 0$. As S/p has finite homotopy groups, we see that $S/p = I^2(S/p)$ so $[BP, S/p]_* = \mathrm{Hom}(\pi_*(BP\wedge I(S/p)), \mathbf{Q}/\mathbf{Z}) = 0$. Thus $BP\wedge S/p\wedge I = 0$. We also have $S[\frac{1}{p}]\wedge I = 0$ because $\pi_* I$ is a torsion group. As $\langle S\rangle = \langle S/p\rangle \vee \langle S[\frac{1}{p}]\rangle$, we conclude that $BP\wedge I = 0$. $\square$

Corollary B.12. *We have $X\wedge I = 0$ if X is a BP-module. In particular $K(n)$, $H\mathbf{Z}$, $H\mathbf{F}_p$ and $H\mathbf{Q} = S\mathbf{Q}$ are I-acyclic. Furthermore, I itself is I-acyclic.*

Proof. The only thing to prove is that I is I-acyclic. But the homotopy groups of I are concentrated in non-negative dimensions and are all finite except for $\pi_0 I = \mathbf{Q}/\mathbf{Z}$. It follows that I is in the localising subcategory generated by $H\mathbf{F}_p$. Since $H\mathbf{F}_p$ is I-acyclic, so is I. $\square$

Now recall that a spectrum is said to be *harmonic* if it is W-local, where $W = \bigvee_{0\leq n<\infty} K(n)$.

Corollary B.13. *There are no nonzero small objects in the following categories.*

(a) *The BP-local category.*
(b) *The harmonic category.*
(c) *The $H\mathbf{Z}$-local category.*
(d) *The $H\mathbf{F}_p$-local category.*
(e) *The I-local category.*

Proof. All of these localisation functors kill I, so it suffices to check that they do have a finite local. It is well-known that all finite spectra are BP-local and $H\mathbf{Z}$-local, and that all finite torsion spectra are $H\mathbf{F}_p$-local ([Bou79, Theorem 3.1]). Similarly, all finite spectra are harmonic by [Rav84, Corollary 4.5]. To see that S/p is I-local, note that $I^2(S/p) = S/p$ since the homotopy groups of S/p are finite. When Z is I-acyclic, we have

$$[Z, S/p] = [Z, F(I(S/p), I)] = [Z\wedge I(S/p), I] = [Z\wedge D(S/p)\wedge I, I] = 0$$

as required. $\square$

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