**MR0139172 (25 #2608)** 57.10
**Kervaire, Michel A.**
**A manifold which does not admit any differentiable structure.**
*Comment. Math. Helv.* **34** 1960 257–270

In this paper the author presents the first example of a triangulable manifold which admits no differentiable structure. His example may be described as follows: Let $p\colon E\to S^5$ be the tangent disc bundle over $S^5$, and let $D_5\subset S^5$ be the upper hemisphere of $S^5$. Choose a trivialization $f\colon D^5\times D^5\to p^{-1}(D^5)$ of the bundle over $D^5$. Now in the disjoint union of $E$ with itself identify $f(u,v)\in p^{-1}(D^5)$ in the first copy, with the point $f(v,u)\in p^{-1}(D^5)$ in the second copy to obtain a space $W$. After a straightening of the edges, $W$ becomes a triangulable manifold with boundary, $\partial W$, which turns out to be homeomorphic to $S^9$. The union of $W$ with the cone over $\partial W$ is the desired 10-dimensional manifold $M_0$. (This construction goes back to J. Milnor [Amer. J. Math. **81** (1959), 962–972; MR0110107 (22 #990)].)

The proof that $M_0$ carries no differentiable structure involves the following steps. Let $\Omega$ be the loop space of $S^6$ and let $e_1\in H^5(\Omega;\mathbf{Z})$, $e_2\in H^{10}(\Omega;\mathbf{Z})$ be generators. If $M$ is any 4-connected triangulable 10-manifold, and $x\in H^5(M;\mathbf{Z})$, the author first shows that there exists a map $f_x\colon M_{10}\to\Omega$ so that ${f_x}^*e_1=x$. He then defines $\varphi\colon H^5(M;\mathbf{Z}_2)\to\mathbf{Z}_2$ as follows: Let $y\in H^5(M;\mathbf{Z}_2)$. Then there exists an $x\in H^5(M;\mathbf{Z})$ such that $x\equiv y \bmod 2$, by Poincaré duality. Choose $f_x$ and evaluate ${f_x}^*\overline{e}_2$ on $M_{10}$, where $e_2$ is the mod 2 class determined by $e_2$. The resulting element of $\mathbf{Z}_2$ turns out to be independent of $x$ and $f_x$ and is defined to be the value of $\varphi$ at $y$. The author now sets $\Phi(M)=\sum\varphi(x_i)\varphi(y_i)$, where $x_1,\cdots,x_n$, $y_1,\cdots,y_n$ is a symplectic base for $H^5(M;\mathbf{Z}_2)$ (with respect to the cupproduct) and proves that the invariant $\Phi(M)$ must vanish if $M$ admits a differentiable structure.

This basic theorem depends on geometric considerations as well as on detailed information concerning the stable homotopy group $\pi_{10+n}(S^n)$. Finally, the author computes $\Phi(M_0)$ and finds that $\Phi(M_0)=1$, thereby establishing the result.

Reviewed by *R. Bott*