Browder's theorem and manifolds with corners

Duality and the Wu classes. The "Spanier-Whitehead dual" of a spectrum X is the function spectrum (in which S denotes the sphere spectrum) $DX = F(X, S)$ (which exists by Brown representability). It comes equipped with an "evaluation" pairing $DX \wedge X \to S$.

Write $H_*(-)$ for homology with $\mathbb{F}_2$ coefficients. If X is finite, then the induced pairing $H_{-*}(DX) \otimes H_*(X) \to \mathbb{F}_2$ is perfect, so there is an isomorphism

$$H_i(X) \to \mathrm{Hom}(H_{-i}(DX), \mathbb{F}_2)$$

which may be rewritten using the universal coefficient theorem as

$$\mathrm{Hom}(H^i(X), \mathbb{F}_2) \to H^{-i}(DX)$$

A Steenrod operation θ induces a contragredient action on the left, which coincides with the action of $\chi\theta$ on the right. Here χ is the Hopf conjugation on the Steenrod algebra. The map χ is an algebra anti-automorphism and an involution, and is characterized on the total Steenod square by the identity of operators

$$\chi \mathrm{Sq} = \mathrm{Sq}^{-1}$$

because of the form of the Milnor diagonal.

If M is a closed smooth m manifold and $X = \Sigma^\infty M_+$, then the Thom spectrum M^ν of the stable normal bundle (normalized to have formal dimension $-m$) furnishes the Spanier-Whitehead dual of X. This is "Milnor-Spanier" or "Atiyah" duality. Poincaré duality is given by the composite isomorphism

$$H^{m-i}(M) \xrightarrow{-\cup U} H^{-i}(M^\nu) \xleftarrow{\cong} H_i(M)$$

where $U \in H^{-m}(M^\nu)$ is the Thom class.

The rather boring collapse map $M_+ \to S^0$ dualizes to a much more interesting map $\iota : S^0 \to M^\nu$, which in cohomology induces the map

$$\iota^* : x \cup U \mapsto \langle x, [M] \rangle$$

By Poincaré duality, for each k there is a unique class $v_k \in H^k(M)$ such that for any $x \in H^{m-k}(M)$, $\langle \mathrm{Sq}^k x, [M] \rangle = \langle x v_k, [M] \rangle$. By separating connected components of M, it follows that in fact $\mathrm{Sq}^k x = x v_k$. Note right off that if $k > n/2$ then $v_k = 0$, by the instability of the action of the Steenrod algebra.

Wen-Tsün Wu proved that the element v_k is a characteristic class. This follows from the fact that ι^* commutes with Steenrod operations:

$$\iota^*(\mathrm{Sq}(x \cup U)) = \mathrm{Sq}\langle x, [M] \rangle = \langle x, [M] \rangle$$

since the degree zero part of Sq is 1. But by Wu's definition of the Stiefel-Whitney classes and the Cartan formula,

$$\mathrm{Sq}(x \cup U) = (\mathrm{Sq}\, x) \cup \mathrm{Sq}\, U = (\mathrm{Sq}\, x) w \cup U$$

so

$$\langle (\mathrm{Sq}\, x) w, [M] \rangle = \langle x, [M] \rangle$$

Now replace x by the class $\dfrac{x}{\mathrm{Sq}^{-1} w}$, to see that the total Wu class is

$$v = \frac{1}{\mathrm{Sq}^{-1} w}$$

When applied to the normal bundle of a manifold, the Whitney sum formula gives

$$\mathrm{Sq}\, v(\tau) = w(\nu)$$

where τ is the tangent bundle of the manifold.

Change of framing. A "framing" of a manifold M^m is an embedding $i : M \hookrightarrow \mathbb{R}^{m+k}$ together with a trivialization of the normal bundle $t : \nu_i \xrightarrow{\cong} \underline{k}_M$.

Any two embeddings of M in large codimension are isotopic, and so we can stabilize to form the set of stable framings of a manifold.

A framing t determines an isomorphism of Thom spaces $M^{\nu_i} \xrightarrow{\cong} \Sigma^k M_+$. A stable framing of M determines a homotopy equivalence $M^\nu \to \Sigma^{-m} M_+$, showing that the spectrum $\Sigma^\infty M_+$ is self-dual (up to a shift of dimension). The framing can be thought of as a fiberwise isomorphism from the normal bundle to the k plane bundle over a point, so stably we get a map $t : M^\nu \to S^{-m}$. The composite $t\iota : S^0 \to S^{-m}$ is the stable homotopy class corresponding to the framed manifold (M, t).

Let $m = 2n$. An element $x \in H^n(M)$ can be thought of as a homotopy class of maps $M_+ \to K_n$, and so determines an element

$$S^{2n} \to \Sigma^\infty M_+ \to \Sigma^\infty K_n$$

of the stable homotopy group $\pi_{2n}(K_n)$. This group is of order 2, so the framing determines a map

$$q_t : H^n(M) \to \mathbb{F}_2$$

This is the Browder-Brown definition of the quadratic refinement of the intersection pairing determined by a framing.

The "gauge group" of smooth maps from M to $O(k)$ acts transitively on framings (with respect to this embedding), and the group $K^{-1}(M) = [M, O]$ acts transitively on the set of stable framings.

Proposition. (Brown [3], 1.18) Let (M, t) be a framed $2n$ manifold, and let $f : M \to O$. Then

$$q_{ft}(x) = q_t(x) + \langle x \cdot f^* \overline{v}_{n+1}, [M] \rangle$$

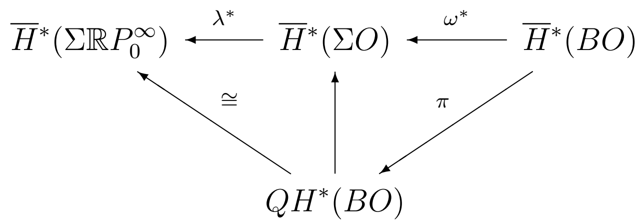
were $\overline{v}_{n+1}$ denotes the image in $H^n(O)$ of v_{n+1} under the map

$$\omega : \Sigma O \to BO$$

adjoint to the equivalence $O \to \Omega BO$.

Let $\mathbb{R}P_0^\infty$ denote projective space with a disjoint basepoint adjoined, and let $\lambda : \mathbb{R}P_0^\infty \to O$ be the (pointed) map sending a line to the reflection through the hyperplane orthogonal to that line.

Lemma. The maps $\Sigma\mathbb{R}P_0^\infty \xrightarrow{\lambda} \Sigma O \xrightarrow{\omega} BO$ induce maps fitting into the commutative diagram



Thus λ^* is bijective on the image of ω^*, and $\lambda^* \overline{w} = (1 + t)^{-1}$ where t generates $H^1(\mathbb{R}P_0^\infty)$. Since $\mathrm{Sq}\, t = t + t^2 = t(1 + t)$,

$$\mathrm{Sq}\, t^{2^k-1} = t^{2^k-1}(1 + t + \cdots + t^{2^k-1}) = t^{2^k-1} + \cdots + t^{2^{k+1}-2}$$

and hence

$$\mathrm{Sq}(1 + t + t^3 + t^7 + \cdots) = (1 + t)^{-1}$$

Now $v\mathrm{Sq}^{-1}w = 1$ gives on indecomposables $\overline{v} = \mathrm{Sq}^{-1}\overline{w}$. Thus

$$\lambda^* \overline{v} = \lambda^* \mathrm{Sq}^{-1} \overline{w} = \mathrm{Sq}^{-1}(1 + t)^{-1} = 1 + t + t^3 + t^7 + \cdots .$$

So $\overline{v}_k = 0$ unless k is a power of 2.

It follows that the quadratic form of a framed $2n$ manifold is independent of the framing unless n is of the form $2^k - 1$, and that the Kervaire invariant is too.

Theorem. In positive dimensions, every framed manifold is framed bordant to an odd multiple of a reframed framed boundary.

John Jones and Elmer Rees [4] observed the following:

Corollary. The Kervaire invariant of framed manifolds is nonzero at most in dimensions of the form $2(2^k - 1)$.

The theorem is a reformulation of the Kahn-Priddy theorem due to Nigel Ray [8]. There is a commutative diagram

$$\begin{array}{ccccc} \mathbb{R}P^\infty & \xrightarrow{\overline{\lambda}} & SO & \xrightarrow{J} & Q_0S^0 \\ \downarrow & & \downarrow & \nearrow_{\hat{J}} & \\ Q\mathbb{R}P^\infty & \longrightarrow & QSO & & \end{array}$$

where $\overline{\lambda}$ sends a line to composite of reflection through the orthogonal hyperplane with a fixed reflection. The Kahn-Priddy theorem asserts that the composite $Q\mathbb{R}P^\infty \to Q_0S^0$ has a section after localizing at 2, so the induced map $\overline{\pi}_*(SO) \to \pi_*$ is surjective in positive dimensions after tensoring with $\mathbb{Z}_{(2)}$. A map $f : S^n \to SO$ allows us to reframe the trivially framed n sphere, and $Jf \in \pi_n$ is represented by that new framed manifold. This is the "J-homomorphism." An element of $\overline{\pi}_n(SO)$ is represented by a framed boundary M^n together with a map $f : M \to SO$. $\hat{J}$ is the "stable J-homomorphism." Its image in π_n is represented by M with the new framing; so the image is the set of reframed framed boundaries.

The Adams spectral sequence. An "Adams tower" for a spectrum X is a diagram

$$\begin{array}{ccccc} & & \vdots & & \\ & & \downarrow & & \\ & & X^2 & \xrightarrow{k} & I^2 \\ & & \downarrow & & \\ & & X^1 & \xrightarrow{k} & I^1 \\ & & \downarrow & & \\ X & \xrightarrow{=} & X^0 & \xrightarrow{k} & I^0 \end{array}$$

in which each "L" is a cofiber sequence, each I^s is a mod 2 generalized Eilenberg Mac Lane spectrum, and each map labeled k induces a monomorphism in homology. The Adams spectral sequence is associated to the exact couple obtained by applying homotopy to this diagram. In it, then, under some finite type assumptions,

$$E_1^{s,t} = \pi_{s+t}(I^s) = \mathrm{Hom}^t_{A^*}(H^*(I^s), \mathbb{F}_2)$$

The long exact sequences induced in cohomology are short exact, so

$$0 \leftarrow H^*(X) \leftarrow H^*(I^0) \leftarrow H^*(\Sigma I^1) \leftarrow \cdots$$

is a projective resolution and

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{A^*}(H^*(X), \mathbb{F}_2) \Longrightarrow \pi_{t-s}(X)\hat{_2}$$

When $X = S$ we can start to compute these groups. $E_2^{0,*}$ is $\mathbb{F}_2$ concentrated in degree 0. $E_2^{1,*}$ is dual to the module of indecomposables in A^*, so is generated by classes h_i with $||h_i|| = (1, 2^i)$. $E_2^{2,*}$ was computed by Adams right away; it has as basis the set

$$h_i h_j \quad , \quad 0 \leq i \text{ and either } i = j \text{ or } i + 2 \leq j$$

Very few of these elements survive in the Adams spectral sequence. The Hopf invariant one theorem amounts to the assertion that h_i survives only for $i \leq 3$: h_0 survives to 2ι, h_1 to η, h_2 to ν, and h_3 to σ. (In fact Adams proved that for $i > 3$, $d^2 h_i = h_0 h_{i-1}^2$.)

In $s = 2$, the only survivors are:

$$h_0h_2\,, \quad h_0h_3\,, \quad h_2h_4\,, \quad h_1h_j \text{ for } j \geq 3\,, \quad \text{and possibly} \quad h_i^2$$

The class $h_1 h_j$ survives to Mahowald's class $\eta_j \in \pi_{2^j}$. For $j \leq 3$ the classes h_i^2 survive to 4ι, η^2, ν^2, and σ^2. After that things get trickier.

Theorem. (Browder [2]) Let κ denote the functional on $\mathrm{Ext}^{2,*}_{A^*}(\mathbb{F}_2, \mathbb{F}_2)$ which is nonzero on h_i^2 but zero otherwise. In dimension $2n > 0$, the Kervaire invariant can be identified with the "edge homomorphism"

$$\pi_{2n} \xleftarrow{\cong} F^2\pi_{2n} \twoheadrightarrow F^2\pi_{2n}/F^3\pi_{2n} \cong E_\infty^{2,2n+2} \hookrightarrow E_2^{2,2n+2} \xrightarrow{\kappa} \mathbb{F}_2$$

Bordism interpretation of the Adams spectral sequence.

I have defined an Adams tower in more generality than is usual because I want to give bordism interpretations of the various parts of the E^1 exact couple. Ultimately I want to express the Kervaire invariant as a characteristic number, and then identify that characteristic number with the functional κ. The Adams tower we will use is built not from the Eilenberg Mac Lane spectrum $H\mathbb{F}_2$ but rather from the Thom spectrum MO, which Thom showed to be a wedge of mod 2 Eilenberg

Mac Lane spectra. With X the sphere spectrum S and $\overline{MO} = MO/S$, there is an Adams tower of the form

$$
\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
 & & \Sigma^{-2}\overline{MO} \wedge \overline{MO} & \xrightarrow{k} & \Sigma^{-2} MO \wedge \overline{MO} \wedge \overline{MO} \\
 & & \downarrow & & \\
 & & \Sigma^{-1}\overline{MO} & \xrightarrow{k} & \Sigma^{-1} MO \wedge \overline{MO} \\
 & & \downarrow & & \\
S & \xrightarrow{=} & S & \xrightarrow{k} & MO
\end{array}
$$

All the parts of the diagram induced in homotopy admit bordism interpretations. $\pi_*(S)$ is the framed bordism ring, $\pi_*(MO)$ is the bordism ring of (unoriented) manifolds, and the map k forgets the framing. An element of $\pi_{n+1}(\overline{MO})$ represents a class of triples (N, M, t), in which N is an $n+1$ manifold with boundary, $M = \partial N$, and t is a trivialization of ν_M. Such an "(O,fr) manifold" represents zero if it is a "boundary," i.e. if there it embeds in a manifold with corner (P, N, N', M, t). This means that P is an $n+2$ manifold whose boundary is given by $N \cup_M N'$; N and N' are manifolds with boundary and $\partial N = M = \partial N'$; and t' is a trivialization of the normal bundle of N' which restricts to the given trivialization of the normal bundle of M. The map $\pi_{n+1}(\overline{MO}) \to \pi_n(S)$ sends (N, M, t) to its "boundary" (M, t), $t = t'|_M$.

Warmup: the Hopf invariant. In positive dimensions, the Hopf invariant can be described as the composite

$$\pi_n \xleftarrow{\cong} F^1\pi_n \twoheadrightarrow F^1\pi_n/F^2\pi_n \cong E_\infty^{1,n+1} \hookrightarrow \mathrm{Ext}_{A^*}^{1,n+1}(\mathbb{F}_2, \mathbb{F}_2) \xrightarrow{a_1} \mathbb{F}_2$$

in which a_1 is an element of $\mathrm{Tor}^{A^*}_{1,n+1}(\mathbb{F}_2, \mathbb{F}_2)$ (which is canonically dual to the Ext group) represented by the cycle

$$\alpha_1 = [\mathrm{Sq}^{n+1}]$$

in the bar construction. The cycle α_1 is a boundary unless $n+1$ is a power of 2 (since Sq^{n+1} is decomposable in A^* unless $n+1$ is a power of 2), so the Hopf invariant is potentially nonzero only in dimensions of the form $n = 2^k - 1$. In this case the functional a_1 sends the generator $h_k \in \mathrm{Ext}^{1,n+1}_{A^*}(\mathbb{F}_2, \mathbb{F}_2)$ to $1 \in \mathbb{F}_2$.

The short exact sequence

$$0 \leftarrow \mathbb{F}_2 \leftarrow H^*(MO) \leftarrow H^*(\overline{MO}) \leftarrow 0$$

induces a boundary map compatible with the projection map in the Adams tower:

$$\begin{array}{ccccc}
\pi_{n+1}(\overline{MO}) & \xrightarrow{\text{Hurewicz}} & \mathrm{Ext}^{0,n+1}_{A^*}(H^*(\overline{MO})), \mathbb{F}_2) & & \\
\downarrow & & \downarrow \delta & \searrow^{a_0} & \\
F^1\pi_n(S) & \xrightarrow{\text{Hopf}} & \mathrm{Ext}^{1,n+1}_{A^*}(\mathbb{F}_2, \mathbb{F}_2) & \xrightarrow{a_1} & \mathbb{F}_2
\end{array}$$

The functional a_0 here is given by the class $\partial a_1 \in \mathrm{Tor}^{A^*}_{0,n+1}(H^*(\overline{MO}), \mathbb{F}_2)$, where ∂ is the boundary map induced by the same short exact sequence. We find:

$$\begin{array}{ccccc}
[\mathrm{Sq}^{n+1}] & \leftarrow & [\mathrm{Sq}^{n+1}]U & & \\
 & & \downarrow d & & \\
 & & \mathrm{Sq}^{n+1}U & \leftarrow & w_{n+1} \cup U
\end{array}$$

so a_0 is represented by the element

$$\alpha_0 = w_{n+1} \cup U$$

The Hopf invariant is thus captured by the Hurewicz map on $\pi_{n+1}(\overline{MO})$.

The interpretation of this in terms of (O,fr) manifolds is this. Let (N, M, t) be an (O,fr) manifold. Let ν be the normal bundle of N. The trivialization t of $\nu|_M$ provides a factorization of $N \to BO$ through N/M, and hence for any $c \in \overline{H}^k(BO)$ we obtain a class $c(\nu, t) \in H^k(N, M)$; in particular, $w_{n+1}(\nu, t) \in H^{n+1}(N, M)$. Then

$$\mathrm{Hopf}(M, t) = \langle w_{n+1}(\nu_N, t), [N, M] \rangle$$

This was observed for example by Stong, [9], p. 105.

Kervaire via (O,fr) manifolds. Let me change notation, and write b_2 for the functional on $\mathrm{Ext}^{2,2n+2}_{A^*}(\mathbb{F}_2, \mathbb{F}_2)$ which detects h_i^2, $i \geq 0$. There is a convenient and explicit cycle in the bar construction which represents the element $b_2 \in \mathrm{Tor}^{A^*}_{2,2n+2}(\mathbb{F}_2, \mathbb{F}_2)$, namely

$$\beta_2 = \sum_{i=0}^{n} \binom{n+1+i}{n+1} [\mathrm{Sq}^{n+1-i} | \chi \mathrm{Sq}^{n+1+i}]$$

The fact that this is a cycle follows from the identity [1]

$$\sum_{i=0}^{n} \binom{n+1+i}{n+1} \mathrm{Sq}^{n+1-i}\, \chi\mathrm{Sq}^{n+1+i} = 0$$

This is like the defining identity for the χSq's, $\displaystyle\sum_{i=0}^{2n+2} \mathrm{Sq}^{n+1-i}\, \chi\mathrm{Sq}^{n+1+i} = 0$ but omits most of the terms. In the critical dimension, $n = 2^k - 1$, the cycle takes the form $\displaystyle\sum_{i=0}^{n} [\mathrm{Sq}^{n+1-i} | \chi\mathrm{Sq}^{n+1+i}]$.

Just as before, we have the commutative diagram

$$\begin{array}{ccccc}
F^1\pi_{2n+1}(\overline{MO}) & \xrightarrow{\text{Hopf}} & \mathrm{Ext}^{1,2n+2}_{A^*}(H^*(\overline{MO}), \mathbb{F}_2) & \overset{b_1}{\searrow} & \\
\downarrow & & \downarrow \delta & & \\
F^2\pi_{2n}(S) & \xrightarrow{\text{Kervaire}} & \mathrm{Ext}^{2,2n+2}_{A^*}(H^*(\overline{MO}), \mathbb{F}_2) & \xrightarrow{b_2} & \mathbb{F}_2
\end{array}$$

where $b_1 = \partial b_2 \in \mathrm{Tor}^{A^*}_{1,2n+2}(\mathbb{F}_2, H^*(\overline{MO}))$. Lannes computed this class to be represented by the cycle

$$\beta_1 = \sum_{i=0}^{n} [\mathrm{Sq}^{n+1-i}] v_i v_{n+1} \cup U$$

and then verified that this functional coincides with the Kervaire invariant, giving a new proof of Browder's theorem.

Codimension two. We can push this story one step further:

$$\begin{array}{ccccc}
\pi_{2n+2}(\overline{MO} \wedge \overline{MO}) & \xrightarrow{\text{Hurewicz}} & \mathrm{Ext}^{0,2n+2}_{A^*}(H^*(\overline{MO} \wedge \overline{MO}), \mathbb{F}_2) & \overset{b_0}{\searrow} & \\
\downarrow & & \downarrow \delta & & \\
F^1\pi_{2n+1}(\overline{MO}) & \xrightarrow{\text{Hopf}} & \mathrm{Ext}^{1,2n+2}_{A^*}(H^*(\overline{MO}), \mathbb{F}_2) & \overset{b_1}{\searrow} & \\
\downarrow & & \downarrow \delta & & \\
F^2\pi_{2n}(S) & \xrightarrow{\text{Kervaire}} & \mathrm{Ext}^{2,2n+2}_{A^*}(\mathbb{F}_2, \mathbb{F}_2) & \xrightarrow{b_2} & \mathbb{F}_2
\end{array}$$

where $b_0 = \partial b_1 \in \mathrm{Tor}^{A^*}_{0,2n+2}(\mathbb{F}_2, H_*(\overline{MO} \wedge \overline{MO}))$ turns out to be the class of

$$\beta_0 = \sum_{i=0}^{n} (v_{n+1-i} \cup U) \otimes (v_i v_{n+1} \cup U)$$

An element of the group $\pi_{2n+2}(\overline{MO} \wedge \overline{MO})$ is represented by a "$(O, \mathrm{fr})^2$-manifold." This consists of the data $(P, N_1, N_2, \nu_1, \nu_2, t_1, t_2)$, where P is a $(2n+2)$-manifold with boundary $N = N_1 \cup_M N_2$, $\partial N_1 = M = \partial N_2$; the normal bundle ν_P comes with a splitting $\nu_P = \nu_1 \oplus \nu_2$; t_1 is a trivialization of $\nu_1|_{N_1}$ and t_2 is a trivialization of $\nu_2|_{N_2}$. The normal bundle of the corner M thus acquires a trivialization t. The map $\pi_{2n+2}(\overline{MO} \wedge \overline{MO}) \to \pi_{2n+1}(\overline{MO})$ carries this data to (N_1, M, t).

The element β_0 gives rise to the characteristic number appearing in the following theorem.

Proposition. [6] Let $(P, N_1, N_2, \nu_1, \nu_2, t_1, t_2)$ be an $(O, \mathrm{fr})^2$ manifold. Then

$$\mathrm{Kervaire}(M, t) = \sum_{i=0}^{n} \langle v_{n+1-i}(\nu_1, t_1) \cup v_i(\nu_2) v_{n+1}(\nu_2, t_2), [P, N] \rangle$$

This gives yet another proof of Browder's theorem. A proof of the proposition is sketched below, after a reminder on quadratic forms.

Quadratic forms. Let E be a finite dimensional $\mathbb{F}_2$ vector space with a symmetric bilinear form denoted $x \cdot y$. The "perp" of a subspace $I \subseteq E$ is

$$I^\perp = \{x \in E : x \cdot y = 0 \text{ for all } y \in I\}$$

Clearly $I \subseteq I^{\perp\perp}$. The map

$$E/I^\perp \to E^* \quad , \qquad x \mapsto (y \mapsto x \cdot y)$$

is injective by definition of $I^\perp$.

Now assume that the form is nondegenerate, so that we have an "inner product space." Then this map is also surjective; any linear functional on I extends to a linear functional on E, and so is given by pairing with some element. So in this case

$$\dim I + \dim I^\perp = \dim E \qquad \text{and} \qquad I = I^{\perp\perp}$$

The monoid of isomorphism classes of inner product spaces over $\mathbb{F}_2$ (and orthogonal direct sum) is the same as the monoid of diffeomorphism classes of closed surfaces (and connected sum): The simple objects are the unique 1-dimensional inner product space $I = H^1(\mathbb{R}P^2)$, and the "hyperbolic space" $H = H^1(S^1 \times S^1)$ with inner product given by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $H \oplus I = 3I$, and any inner product space is either a multiple of I or a multiple of H.

An inner product is "even" if $x \cdot x = 0$ for all x. From the classification, this is equivalent to being a multiple of H (and corresponds to the oriented surfaces). Such spaces are necessarily even dimensional.

Note that the restriction of an inner product to a subspace is not generally nondegenerate; for example one-dimensional subspaces of even inner product spaces are always degenerate. If I is a nondegenerate subspace of the inner product space E, then $I \cap I^\perp = 0$ and so $E = I \oplus I^\perp$, the orthogonal direct sum.

At the other extreme, a subspace $I \subseteq E$ is a "Lagrangian" if $I = I^\perp$. If E admits a Lagrangian subspace I then $\dim E = 2 \dim I$ and so is even. Conversely, any $2n$ dimensional inner product space admits a Lagrangian subspace: the operation $I \mapsto I^\perp$ is an involution on the set of n-dimensional subspaces, which has odd cardinality and hence a fixed point.

A "quadratic refinement" of the inner product $x \cdot y$ on E is a map $q : E \to \mathbb{F}_2$ such that

$$q(x + y) = q(x) + q(y) + x \cdot y$$

Taking $x = y = 0$ shows that $q(0) = 0$. Taking $x = y$ shows that the inner product is even.

The hyperbolic inner product space H admits four quadratic refinements: q can be nonzero on any one of the nonzero vectors and zero otherwise; or it can be nonzero on all three nonzero vectors. The first three are permuted by automorphisms of H. Call these two quadratic spaces Q_0 and Q_1. Any quadratic space (over $\mathbb{F}_2$) is isomorphic to either nQ_0 (Arf invariant 0) or $Q_1 \oplus (n-1)Q_0$ (Arf invariant 1); dimension and Arf invariant form a complete invariant.

Since the underlying inner product of a quadratic space is even, there are Lagrangian subspaces I in E. Choose one. Since I is self-orthogonal, $q|_I$ is a linear functional, and hence there exists $u \in E$ such that $q(x) = x \cdot u$ for all $x \in I$. The set of such elements u forms a coset of $I \subseteq E$, and the calculation $q(u + x) = q(u) + q(x) + x \cdot u = q(u)$ shows that $q(u)$ is independent of choice of u. It looks like it might still depend upon the choice of Lagrangian, but it doesn't:

Proposition. (Lannes [5], 0.2.1) The Arf invariant of (E, q) is given by $q(u)$.

Let M be a $2n$ manifold which is the boundary of a $(2n+1)$-manifold N. Then, as observed by Thom, the self-duality of the exact sequence

$$H^n(N) \xrightarrow{i^*} H^n(M) \xrightarrow{\delta} H^{n+1}(N, M) \longrightarrow H^{n+1}(N)$$

implies that $I = \mathrm{Im}(i^* : H^n(N) \to H^n(M))$ is a Lagrangian in the inner product space $E = H^n(M)$. Now suppose that M is framed. Write t for the framing, and equip $E = H^n(M)$ with the quadratic form q_t. (If N admits a framing extending that of N, then the quadratic form is trivial on I, and so the Witt class of the quadratic form is a framed bordism invariant.)

In this situation, Lannes characterized the elements $u \in E$ such that $q(x) = u \cdot x$ for $x \in I$, in terms of the relative Wu class $v_{n+1}(\nu, t) \in H^{n+1}(N, M)$. This class restricts on N to $v_{n+1}(\nu) \in H^{n+1}(N)$, which vanishes since $n + 1 > (2n + 1)/2$. Let $u \in H^n(M)$ be such that $\delta u = v_{n+1}(\nu, t) \in H^{n+1}(N, M)$. It is well defined modulo I, so we may hope for the following result.

Proposition. (Lannes [5], 0.2.2) $q(x) = x \cdot u$ for any $x \in I$.

Say $x = i^* y$, for $y \in H^n(N)$. By self-duality of the sequence, this equation can be rewritten as

$$q(i^* y) = i^* y \cdot u = y \cdot \delta u = y \cdot v_{n+1}(\nu, t)$$

Sketch of proof.

Step 1. Suppose that (P, N_1, N_2) is a manifold with codimension 2 corner. The first step is to construct a self-dual diagram analogous to the (N, M) homology exact sequence. We need a space dual to P/M. Define X to be the homotopy pushout in the diagram

$$\begin{array}{ccc} P_+ & \longrightarrow & P/N_1 \\ \downarrow & & \downarrow \sigma_1 \\ P/N_2 & \xrightarrow{\sigma_2} & X \end{array}$$

and

$$V = P/N_1 \vee P/N_2 \,.$$

There is a commutative diagram of cofiber sequences

$$\begin{array}{ccccccccc}
 & & \Sigma M_+ & \longrightarrow & \Sigma P_+ & \longrightarrow & \Sigma P/M & \xrightarrow{\xi} & \Sigma^2 M_+ \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\rho} \\
V & \longrightarrow & X & \longrightarrow & \Sigma P_+ & \longrightarrow & \Sigma V & \xrightarrow{\sigma} & \Sigma X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
P/N & \longrightarrow & P/N & \longrightarrow & * & \longrightarrow & \Sigma P/N & \longrightarrow & \Sigma P/N
\end{array}$$

which enjoys a duality in cohomology across the diagonal line through ΣV and $\Sigma^2 M_+$.

Define an inner product space E as the orthogonal direct sum

$$E = H^n(M) \oplus H^{n+1}(V)$$

and let

$$J = \mathrm{Im}\left(i^* = \begin{bmatrix} \rho^* \\ \sigma^* \end{bmatrix} : H^{n+1}(X) \longrightarrow H^n(M) \oplus H^{n+1}(V) \right) \subseteq E$$

This is a Lagrangian subspace.

Step 2. Assume given a trivialization t of ν_M. Using it, impose on E the quadratic form

$$q = q_t \oplus q_h$$

where q_h is the "hyperbolic form" given using the duality between $H^{n+1}(P, N_1)$ and $H^{n+1}(P, N_2)$. Then, as in Lannes's theorem,

$$q(i^* y) = v_{n+1}(\nu_P, t) \cdot y$$

for $y \in H^{n+1}(X)$, using the duality pairing

$$H^{n+1}(P, M) \otimes H^{n+1}(X) \to \mathbb{F}_2$$

Step 3. Assume that there exist classes $u_1 \in H^{n+1}(P, N_1)$ and $u_2 \in H^{n+1}(P, N_2)$ such that

$$v_{n+1}(\nu_P, t) = \tau^*(u_1, u_2) \in H^{n+1}(P, M)$$

Then

$$\mathrm{Arf}(q_t) = u_1 \cdot u_2$$

This is a calculation using duality of the diagram:

$$q(i^* y) = \tau^*(u_1, u_2) \cdot y = (u_1, u_2) \cdot \sigma^* y = (0, u_1, u_2) \cdot i^* y$$

Therefore

$$\mathrm{Arf}(q_t) = \mathrm{Arf}(q) = q(0, u_1, u_2) = q_h(u_1, u_2) = u_1 \cdot u_2$$

Step 4. Finally, assume that we have a framed corner. Then we can take

$$\begin{aligned} u_1 &= \sum_{i=0}^{n} v_{n+1-i}(\nu_1, t_1) v_i(\nu_2) \\ u_2 &= v_{n+1}(\nu_2, t_2) \end{aligned}$$

because the Whitney sum formula for relative Wu classes shows that

$$v_{n+1}(\nu_P, t) = \sum_{i=0}^{n} v_{n+1-i}(\nu_1, t) v_i(\nu_2) + v_{n+1}(\nu_2, t) = \tau^*(u_1, u_2)$$

So by Step 3 the Arf invariant of q_t is given by

$$u_1 \cdot u_2 = \sum_{i=0}^{n} v_{n+1-i}(\nu_1, t_1) v_i(\nu_2) \cdot v_{n+1}(\nu_2, t_2)$$

$$= \sum_{i=0}^{n} \langle v_{n+1-i}(\nu_1, t_1) \cup v_i(\nu_2) v_{n+1}(\nu_2, t_2), [P, N] \rangle$$

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