

On the Telescopic Picard Group

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Abstract

We prove that for any prime p and height $n \geq 1$, the telescopic Picard group $\text{Pic}(\text{Sp}_{T(n)})$ contains a subgroup of the form $\mathbb{Z}_p \times \mathbb{Z}/a_p(p^n - 1)$, where $a_p = 1$ if $p = 2$ and $a_p = 2$ if p is odd. Using Kummer theory, we obtain an $(\mathbb{F}_{p^n}^\times \rtimes \mathbb{Z}/n)$ -Galois extension of $\mathbb{S}_{T(n)}$, obtaining the first example of a lift of a non-Abelian Galois extension of the $K(n)$ -local sphere to the telescopic world, at arbitrary positive height and prime.

Our proof proceeds by setting up a higher categorical framework for the periodicity theorem, utilizing the symmetries of this framework to construct Picard elements.

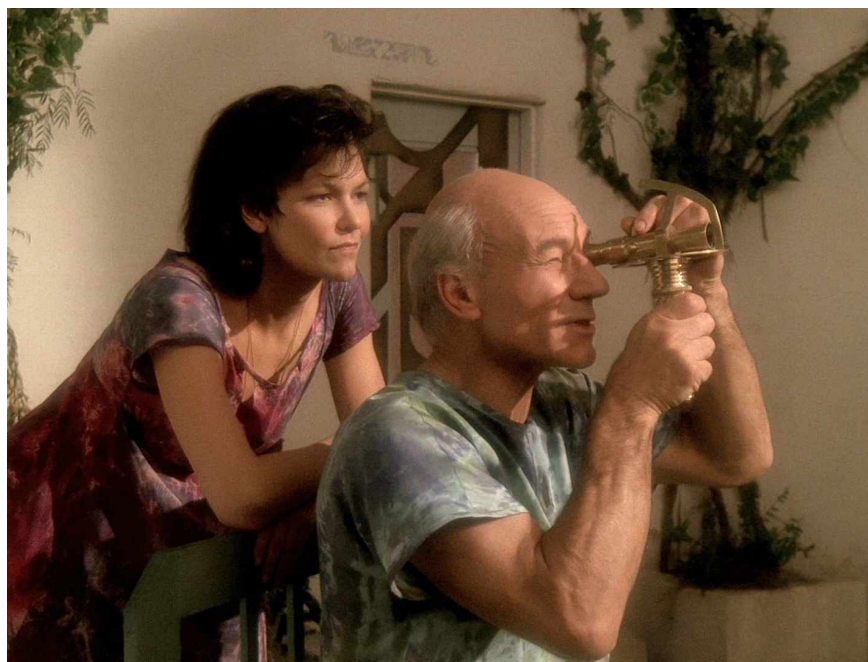


Figure 1: Patrick Stewart and Margot Rose in Star Trek: The Next Generation (1987)

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1 Introduction

Throughout this paper, we use the term category to mean an $(\infty, 1)$ -category. Similarly we use the term n -category to mean an (∞, n) -category.

1.1 Background and overview

Chromatic homotopy theory

Chromatic homotopy theory is the study of the category $\mathrm{Sp}_{(p)}$ of p -local spectra using the “height filtration” inherent in the category of compact objects. Two main approaches are taken for the extension of the filtration from compacts to the entire category, resulting in associated graded categories known as “monochromatic layers”.

The first approach filters the category according to its relationship with formal groups, using the Lubin–Tate spectrum or Morava E -theory, which pertains to formal group deformations. This yields the categories $\mathrm{Sp}_{K(n)}$ of $K(n)$ -local spectra for the monochromatic layers. This approach is more amenable to computations due to its correspondence with algebraic geometry, Dieudonné modules and Galois descent.

The second strategy, known as the telescopic approach, extends the filtration by colimits from the category of compact spectra to the category of all spectra. The monochromatic layers in this case are the categories $\mathrm{Sp}_{T(n)}$ of $T(n)$ -local spectra.

The two approaches are related by the relation $\mathrm{Sp}_{K(n)} \subseteq \mathrm{Sp}_{T(n)}$, which is known to be an equivalence for $n = 0, 1$ as shown by Miller [Mil81] and Mahowald [Mah81]. However, a recent work of Burklund, Hahn, Levy and Schlank [BHLS23] proved that the inclusion is strict for $n \geq 2$ and any prime.

The Picard group

The Picard group of a symmetric monoidal category \mathcal{C} is the group of \otimes -invertible elements in \mathcal{C} up to isomorphism. This invariant has been extensively studied, particularly in the context of modules over a ring and, more generally, quasicoherent sheaves over a scheme.

The $K(n)$ -local Picard group $\mathrm{Pic}(\mathrm{Sp}_{K(n)})$ has been the subject of extensive research, initiated by Hopkins–Mahowald–Sadofsky [HMS94], and continuing through numerous contributions, including [GHMR14, Hea15, Pst18, BBG⁺22, Mor23, BLL⁺24] leading to its complete classification for sufficiently large primes and classification up to torsion for all primes in [BSSW24]. In particular, $\mathrm{Pic}(\mathrm{Sp}_{K(n)})$ admits a subgroup of the form

$$\mathcal{Z}_n = \mathcal{Z}_{n,p} := \lim_k \mathbb{Z}/(p^k(2p^n - 2)),$$

topologically generated by $\Sigma\mathbb{S}_{K(n)}$. Note that $\mathcal{Z}_n \cong \mathbb{Z}_p \times \mathbb{Z}/(a_p(p^n - 1))$ where $a_p = 1$ if $p = 2$ and $a_p = 2$ if p is odd. Moreover, when $n \geq 2$, $\mathrm{Pic}(\mathrm{Sp}_{K(n)})$ is of rank 2 over \mathbb{Z}_p .

In contrast, much less is known about $\mathrm{Pic}(\mathrm{Sp}_{T(n)})$. In [CSY21b], Carmeli, Schlank and Yanovski show that $\mathrm{Pic}(\mathrm{Sp}_{T(n)})$ contains a subgroup isomorphic to $\mathbb{Z}/(p - 1)$. Our first main theorem is the following extension:

Theorem A (Theorem 5.2.1). The group $\mathrm{Pic}(\mathrm{Sp}_{T(n)})$ admits a subgroup isomorphic to \mathcal{Z}_n , topologically generated by $\Sigma\mathbb{S}_{T(n)}$.

In particular, it lifts the corresponding subgroup of $\text{Pic}(\text{Sp}_{K(n)})$.

Galois theory

The interplay between formal group theory and $\text{Sp}_{K(n)}$ leads to the construction of the Lubin–Tate commutative ring spectrum $E_n = E_n(\overline{\mathbb{F}}_p)$ ([GH04, Lur18]) which is a faithful and relatively computable multiplicative (co)homology theory for $K(n)$ -local spectra.

Conceptually, by the works of [DH04], [Rog05], [BR08], and [Mat16], E_n can be viewed as the Galois-closure of the $K(n)$ -local sphere $\mathbb{S}_{K(n)}$ in $\text{Sp}_{K(n)}$, with Galois group the Morava stabilizer group $\mathbf{G}_n = \text{Aut}(\Gamma) \rtimes \widehat{\mathbb{Z}}$, where Γ is a formal group law of height n over $\overline{\mathbb{F}}_p$. Using [Mat16, Section 9], the G -Galois extensions of $\mathbb{S}_{K(n)}$ are completely classified by continuous homomorphisms $\mathbf{G}_n \rightarrow G$.

This naturally raises the question of lifting Galois extensions from the $K(n)$ -local category to the $T(n)$ -local category. Carmeli, Schlank and Yanovski [CSY21b] have shown that any Abelian Galois extension of $\mathbb{S}_{K(n)}$ lifts. We present the first lift of a non-Abelian Galois extension from $\mathbb{S}_{K(n)}$ to $\mathbb{S}_{T(n)}$: $\text{Aut}(\Gamma)$ can be explicitly described as the group of units \mathcal{O}_n^\times of the order

$$\mathcal{O}_n = \mathbf{W}(\mathbb{F}_{p^n}) \langle S \mid S^n = p, Sw = w^\varphi S \ \forall w \in \mathbf{W}_n \rangle,$$

where φ is a lift of the Frobenius to $\mathbf{W}(\mathbb{F}_{p^n})$. The map sending S to 0 provides a map $\pi: \mathcal{O}_n^\times \rightarrow \mathbb{F}_{p^n}^\times$. Let Y_n be the pro-Galois extension classified by the group homomorphism

$$\mathbf{G}_n = \mathcal{O}_n^\times \rtimes \widehat{\mathbb{Z}} \xrightarrow{(\det \oplus \pi) \times \text{id}} (\mathbb{Z}_p^\times \oplus_{\mathbb{F}_p^\times} \mathbb{F}_{p^n}^\times) \rtimes \widehat{\mathbb{Z}},$$

where $\widehat{\mathbb{Z}}$ acts on $\mathbb{Z}_p^\times \oplus_{\mathbb{F}_p^\times} \mathbb{F}_{p^n}^\times$ via conjugation by S , i.e. it acts trivially on \mathbb{Z}_p and it acts as the Frobenius on $\mathbb{F}_{p^n}^\times$ (in particular n acts trivially). The group $\mathbb{Z}_p^\times \oplus_{\mathbb{F}_p^\times} \mathbb{F}_{p^n}^\times$ appearing here is the abelianization of the special Morava stabilizer group $\mathbf{S}_n = \text{Aut}(\Gamma)$.

Theorem B (Theorem 6.3.1). There exists a $((\mathbb{Z}_p^\times \oplus_{\mathbb{F}_p^\times} \mathbb{F}_{p^n}^\times) \rtimes \widehat{\mathbb{Z}})$ -pro-Galois extension Y_n^f of $\mathbb{S}_{T(n)}$, lifting Y_n .

Remark. During the writing of this project, Robert Burklund, Dustin Clausen, and Ishan Levy announced that they lifted all finite Galois extensions from $\mathbb{S}_{K(n)}$ to $\mathbb{S}_{T(n)}$, and that the Galois closure of $\mathbb{S}_{T(n)}$ is E_n , using different methods from ours.

1.2 Methods

Asymptotically defined endomorphisms of the identity

One of the fundamental theorems in chromatic homotopy theory is Hopkins and Smith’s periodicity theorem [HS98, Theorem 9]. This states that any compact spectrum of type $\geq n$ admits a v_n -self-map, which is asymptotically unique.

We set a categorical framework for asymptotically defined maps, and moreover showing that categories of asymptotically defined maps admit an asymptotically defined natural transformation $\text{id} \Rightarrow \text{id}$.

Let D be a quotient subgroup of $\widehat{\mathbb{Z}}$, and write it as a sequential limit of surjections $D = \lim_{d_i} \mathbb{Z}/d_i$.

Definition (Definition 3.4.4, Remark 3.4.7). Let $\mathcal{C} \in \text{Cat}_{\text{perf}}$. A D -asymptotically defined endomorphism of the identity α on \mathcal{C} is the data of an exhausting filtration of \mathcal{C}

$$\mathcal{C}_1 \xrightarrow{F_1} \mathcal{C}_2 \xrightarrow{F_2} \dots \rightarrow \mathcal{C},$$

equipped with natural transformations $\alpha_i: \Sigma^{d_i} \Rightarrow \text{id}_{\mathcal{C}_i}$ and isomorphisms $F_i \alpha_i^{d_{i+1}/d_i} \simeq \alpha_{i+1} F_i$ in $\text{Nat}(\Sigma^{d_{i+1}} F_i, F_i)$.

We say that α is a D -asymptotically defined **isomorphism** of the identity if α_i is an isomorphism for all i .

Categories with asymptotically defined endomorphisms and isomorphisms of the identity assemble to 2-categories $\text{Cat}_{\text{perf}}^{D\text{-end}}$ and $\text{Cat}_{\text{perf}}^{D\text{-iso}}$. There is an underlying category functor $U: \text{Cat}_{\text{perf}}^{D\text{-end}} \rightarrow \text{Cat}_{\text{perf}}$ sending $(\mathcal{C}_1 \rightarrow \mathcal{C}_2 \rightarrow \dots)$ to the colimit $\text{colim}_i \mathcal{C}_i$.

We show (Proposition 3.2.8) that U admits a right adjoint $(-)^{\bar{h}\mathbb{N}[D]}$, taking \mathcal{C} to the system with $\mathcal{C}_i = \mathcal{C}^{\bar{h}\mathbb{N}[d_i]}$ — the category all self maps in \mathcal{C} of degree d_i . The inclusion $\text{Cat}_{\text{perf}}^{D\text{-iso}} \hookrightarrow \text{Cat}_{\text{perf}}^{D\text{-end}}$ admits a left adjoint L (Corollary 3.3.10), inverting α_i at each degree.

We summarize these adjunctions in the following diagram:

$$\begin{array}{ccccc} & & L & & \\ & \swarrow & \text{⊥} & \searrow & \\ \text{Cat}_{\text{perf}}^{D\text{-iso}} & \xrightarrow{\quad} & \text{Cat}_{\text{perf}}^{D\text{-end}} & \xrightarrow{U} & \text{Cat}_{\text{perf}} \ . \\ & \nwarrow & \text{⊥} & \swarrow & \\ & & (-)^{\bar{h}\mathbb{N}[D]} & & \end{array}$$

Going back to our case of interest, we let $D = \mathcal{Z}_n = \mathbb{Z}_p \times \mathbb{Z}/(a_p(p^n - 1))$. We then construct a category with an asymptotically defined endomorphism $\text{Sp}_{v_n}^\omega \subseteq \text{Sp}^{\bar{h}\mathbb{N}[\mathcal{Z}_n]}$, by letting $\text{Sp}_{v_n, k}^\omega$ be the full subcategory of $\text{Sp}^{\bar{h}\mathbb{N}[p^k | v_n]}$ consisting of v_n -self-maps. The periodicity theorem can then be reformulated as follows:

Theorem C (Theorem 4.2.5). The category with asymptotically defined endomorphism of the identity $(\text{Sp}_{v_n}^\omega, v_n) \in \text{Cat}_{\text{perf}}^{\mathcal{Z}_n\text{-end}}$ satisfies:

- (1) The underlying category of $\text{Sp}_{v_n}^\omega$ is $\text{Sp}_{\geq n}^\omega$;
- (2) The underlying category of $L(\text{Sp}_{v_n}^\omega)$ is $\text{Sp}_{T(n)}^\omega$.

The telescopic Picard group

When \mathcal{C} is presentable, sending X to the functor $X \otimes -$ defines an isomorphism from $\text{Pic}(\mathcal{C})$ to the group of \mathcal{C} -linear automorphisms of \mathcal{C} . Specializing to $\mathcal{C} = \text{Sp}_{T(n)}$, we note that any automorphism of $\text{Sp}_{T(n)}$ is $\text{Sp}_{T(n)}$ -linear. As $\text{Sp}_{T(n)}$ is compactly generated we deduce that $\text{Pic}(\mathcal{C}) = \pi_0 \text{Aut}(\text{Sp}_{T(n)}^\omega)$. Therefore we can construct Picard elements by constructing automorphisms of $\text{Sp}_{T(n)}^\omega$.

One advantage of our categorified setting, is that it gives rise to \mathbb{E}_1 group actions on categories admitting an asymptotically defined natural isomorphism. More precisely, categories with an automorphism $\Sigma^d \xrightarrow{\sim} \text{id}$ admit a $\Omega S^1/d$ -action given by suspension. Heading towards the colimit of

such categories, we prove that these actions are compatible when varying d . We define the space

$$S_D^2 := \lim_i \Sigma(S^1/d_i)$$

and compute its second homotopy group (Lemma 3.4.13)

$$\pi_2(S_D^2) \cong D.$$

Proposition (Corollary 3.4.12). *The \mathbb{E}_1 -group $\Omega^2 S_D^2$ acts on any \mathcal{C} in $\text{Cat}_{\text{perf}}^{D\text{-end}}$. The topological generator $1 \in D \cong \pi_0 \Omega^2 S_D^2$ acts by suspension.*

In particular, $\text{Sp}_{v_n}^\omega[v_n^{-1}]$ has a $\Omega^2 S_{\mathcal{Z}_n}^2$ -action, and therefore so does its underlying category $\text{Sp}_{T(n)}^\omega$. We get a group homomorphism on homotopy groups

$$\mathcal{Z}_n = \pi_0(\Omega^2 S_{\mathcal{Z}_n}^2) \rightarrow \text{Pic}(\text{Sp}_{T(n)}).$$

To prove Theorem A it is left to show that it is injective. Recall that $\mathcal{Z}_n = \mathbb{Z}_p \times \mathbb{Z}/(a_p(p^n - 1))$. We prove it separately for each component:

- For \mathbb{Z}_p , we tensor with E_n remembering only the action of $\mathbb{Z}_p \subseteq \mathbb{Z}_p^\times \subseteq \mathbf{G}_n$. It is simple to show that the composition

$$\mathbb{Z}_p \rightarrow \text{Pic}(\text{Sp}_{T(n)}) \xrightarrow{E_n \otimes -} \text{Pic}(\text{Mod}_{E_n}(\text{Sp}_{T(n)}^{\text{B}\mathbb{Z}_p}))$$

is injective.

- For $\mathbb{Z}/(a_p(p^n - 1))$, we compute with $K(n)$ -homology, landing in invertible $K(n)_*$ -graded modules, which are exactly $K(n)_*, K(n)_{*-1}, \dots, K(n)_{*-(2(p^n-1)-1)}$. The composition sends d to $K(n)_{*-d}$ if p is odd and to $K(n)_{*-2d}$ if $p = 2$, showing that it is injective in both cases.

Kummer theory

To construct Galois extensions, we shall make use of the ∞ -categorical Kummer theory, developed by Carmeli-Schlank-Yanovski, which we now briefly recall.

Any Picard element X is in particular dualizable and therefore has a dimension $\dim(X) \in \pi_0 \mathbb{1}^\times$. By [CSY21b, Corollary 3.21], $\dim(X)^2 = 1$.

Definition ([CSY21b, Definition 3.22]). Define the even Picard group as the subgroup of *even Picard elements*:

$$\text{Pic}^{\text{ev}}(\mathcal{C}) := \{X \in \text{Pic}(\mathcal{C}) \mid \dim(X) = 1\}.$$

Theorem (Kummer theory [CSY21b, Proposition 3.23]). Let \mathcal{C} be an additive presentably symmetric monoidal category with a choice of a primitive m -th root of unity ([CSY21b, Definition 3.3]). Then there is a split short exact sequence of Abelian groups

$$0 \rightarrow (\pi_0 \mathbb{1}^\times)/(\pi_0 \mathbb{1}^\times)^m \rightarrow \pi_0 \text{CAlg}^{\mathbb{Z}/m\text{-Gal}}(\mathcal{C}) \rightarrow \text{Pic}^{\text{ev}}(\mathcal{C})[m] \rightarrow 0.$$

This enables us to lift torsion even Picard elements to Galois extensions assuming the existence of a root of unity. First, consider the $T(n)$ -localization of the spherical Witt vector of \mathbb{F}_{p^n} , denoted $\mathbf{SW}_n^f := L_{T(n)} \mathbf{SW}(\mathbb{F}_{p^n})$, which is a faithful \mathbb{Z}/n -Galois extension of $\mathbb{S}_{T(n)}$, admitting a $(p^n - 1)$ -st root of unity. [Theorem A](#) together with Kummer theory thus give an $\mathbb{F}_{p^n}^\times$ -Galois extension of \mathbf{SW}_n^f , which corresponds to an $(\mathbb{F}_{p^n}^\times \rtimes \mathbb{Z}/n)$ -Galois extension of $\mathbb{S}_{T(n)}$.

Similarly, adding higher roots of unity to our Galois extension as in [\[CSY21b\]](#) (equivalently, replacing \mathbf{SW}_n^f with $L_{T(n)}(\mathbf{SW}(\overline{\mathbb{F}}_p))[\omega_{p^\infty}^{(n)}]$ in the construction) gives a $((\mathbb{Z}_p^\times \oplus_{\mathbb{F}_p^\times} \mathbb{F}_{p^n}^\times) \rtimes \widehat{\mathbb{Z}})$ -Galois extension.

To prove [Theorem B](#) it remains to show that its $K(n)$ -localization is represented by said character. This is established by a uniqueness arguments.

1.3 Organization

[Section 2](#) provides a brief overview of (∞, n) categories and lax limits. In this section, we explicitly compute the lax limits along the directed circle $\vec{S}^1 = \mathbb{B}\mathbb{N}$ and the directed 2-sphere \vec{S}^2 , and examine their interrelations. Readers may skip this section on a first read, trusting the results that will be applied in later sections.

We use the constructions and interrelations of [Section 2](#) in [Section 3](#) in order to construct $\text{Cat}_{\text{perf}}^{D\text{-end}}$, $\text{Cat}_{\text{perf}}^{D\text{-iso}}$ and all the adjunctions described.

In [Section 4](#), we build on the previous section to construct the lift $\text{Sp}_{v_n}^\omega$ and prove [Theorem C](#). In [Section 5](#), we leverage this lift and the natural group action on $\text{Cat}_{\text{perf}}^{D\text{-iso}}$ to prove [Theorem A](#). Finally, in [Section 6](#), we use Kummer theory to lift the Galois extension from $\mathbb{S}_{K(n)}$ to $\mathbb{S}_{T(n)}$, proving [Theorem B](#).

1.4 Conventions

We use the following terminology and notation:

- (1) The category of stable categories and exact functors is denoted Cat_{st} , and the full subcategory of idempotent complete stable categories is denoted by Cat_{perf} . The idempotent completion functor is denoted by $(-)^{\text{idem}} : \text{Cat}_{\text{st}} \rightarrow \text{Cat}_{\text{perf}}$.
- (2) The category of spaces (or animae, or groupoids) is denoted by \mathcal{S} .
- (3) We denote by $\mathcal{C}^\simeq \subseteq \mathcal{C}$ the maximal subgroupoid of a category \mathcal{C} .
- (4) We denote the space of morphisms between two objects X, Y in a category \mathcal{C} by $\text{Map}_{\mathcal{C}}(X, Y)$ and omit \mathcal{C} when it is clear from context. If \mathcal{C} is stable we denote the mapping spectrum of X, Y by $\text{hom}_{\mathcal{C}}(X, Y)$ or by $\text{hom}(X, Y)$ if \mathcal{C} is clear from context.
- (5) The category of presentable categories with colimit-preserving functors is denoted by Pr^{L} .
- (6) For $\mathcal{C} \in \text{CAlg}(\text{Pr}^{\text{L}})$ and $\mathcal{D}, \mathcal{E} \in \text{Mod}_{\mathcal{C}}(\text{Pr}^{\text{L}})$ we denote the space of \mathcal{C} -linear functors from \mathcal{D} to \mathcal{E} by $\text{Map}^{\mathcal{C}}(-, -)$. Similarly we denote the space of \mathcal{C} -linear endomorphisms and \mathcal{C} -linear automorphisms by $\text{End}^{\mathcal{C}}$ and $\text{Aut}^{\mathcal{C}}$ respectively.

- (7) For a category \mathcal{C} we denote its full subcategory of compact objects by \mathcal{C}^ω .
- (8) For a symmetric monoidal category \mathcal{C} we denote its space of dualizable objects by \mathcal{C}^{dbl} .
- (9) We denote lax limits by $\overrightarrow{\text{lim}}$ and oplax limits by $\overleftarrow{\text{lim}}$. Similarly, we will denote lax operations with an arrow.
- (10) For two n -categories \mathcal{C}, \mathcal{D} , we denote by $\text{Fun}_{\text{lax}}(\mathcal{C}, \mathcal{D})$ the n -categories of functors from \mathcal{C} to \mathcal{D} and lax natural transformations. We similarly denote $\text{Fun}_{\text{op lax}}(\mathcal{C}, \mathcal{D})$ the n -category of functors and oplax natural transformations.
- (11) For an n -category \mathcal{C} and $X, Y \in \mathcal{C}$ we denote by $\text{MAP}_{\mathcal{C}}(X, Y)$ the mapping $(n - 1)$ -category. Similarly we denote the $(n - 1)$ -category of endomorphisms of X by $\text{END}_{\mathcal{C}}(X) := \text{MAP}_{\mathcal{C}}(X, X)$. We omit \mathcal{C} when it is clear from context.
- (12) For the remainder of the paper we fix a height $n > 0$ and a prime p .

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2 Higher categories and lax limits

This section is devoted for the introduction of our chosen framework of n -categories, lax limits, and their computation in two special cases:

- (1) The lax limit along the walking 1-endomorphism

$$\vec{S}^1 = \text{BN} = \begin{array}{c} \circlearrowright \\ \bullet \end{array} ,$$

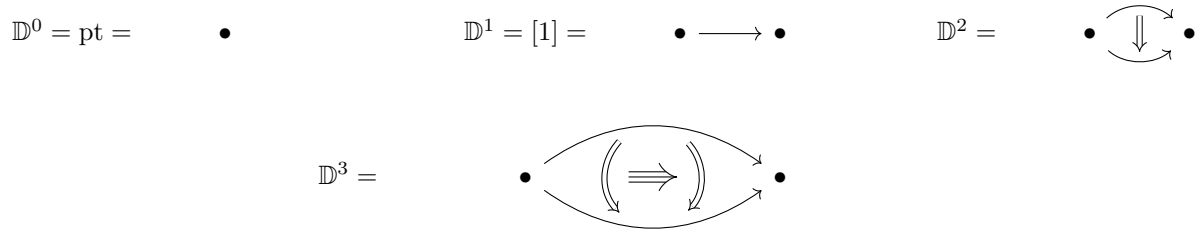
- (2) The lax limit along the walking 2-endomorphism of the identity

$$\vec{S}^2 = \begin{array}{c} \circlearrowright \\ \bullet \end{array} = \star \begin{array}{c} \xrightarrow{\text{id}_\star} \\ \Downarrow \\ \xrightarrow{\text{id}_\star} \end{array} \star .$$

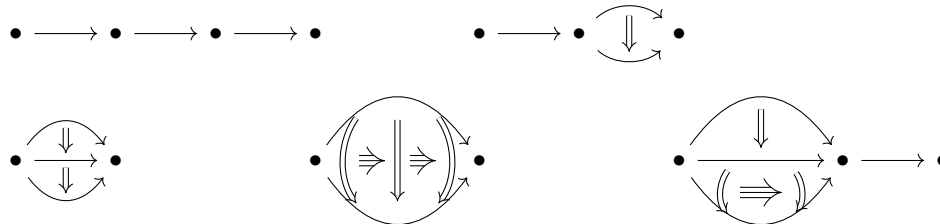
We use these computations in [Section 3](#), either choosing $\mathcal{C} \in \text{Cat}_{\text{perf}}$ and Σ^d as an endomorphism, or choosing $\text{Cat}_{\text{perf}} \in \widehat{\text{Cat}}_2$ with Σ^d as the endomorphism of $\text{id}_{\text{Cat}_{\text{perf}}}$.

2.1 Higher categories and lax natural transformations

We will model n -categories by Rezk’s complete Segal Θ_n spaces [[Rez10](#)], [[BSP11](#), § 13], [[Lou24](#), § 2], as we now briefly recall. We call a classical (1,1)-category a strict category and define inductively strict n -categories as strict categories enriched in strict $(n - 1)$ -categories. An important class of strict n -categories are the k -cells \mathbb{D}^k (also known as the walking k -morphisms, k -disks or k -globes) and their “sums”, which are finite connected colimits along inclusions of smaller cells in the boundary, corresponding to composable higher arrows. The k -cells for $k \leq 3$ are



and some examples of their cellular sums are



The collection of all cellular sums forms a (strict) category Θ_n . The category Θ_n can be formally constructed inductively as the wreath product $\Delta \wr \Theta_{n-1}$. The category Cat_n of n -categories is defined to be the full subcategory of $\text{PSh}(\Theta_n)$ on those presheaves satisfying Segal and completion conditions. There is an embedding $\Theta_n \hookrightarrow \text{Cat}_n$ representing the corresponding n -categories. The image of $a \in \Theta_n$ represents the evaluation at a functor:

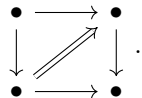
$$\text{Map}_{\text{Cat}_n}(a, \mathcal{C}) \simeq \mathcal{C}_a.$$

For an n -category $\mathcal{C} \in \text{Cat}_n$, we define its space of k -morphisms to be $\text{Map}(\mathbb{D}^k, \mathcal{C})$.

The category of ∞ -categories $\text{Cat}_\infty = \lim_n \text{Cat}_n$ was long conjectured to have a monoidal structure called the Gray tensor product which we think of as a lax cartesian product and denote \times^{lax} . The Gray tensor product, originally defined in [[Gra06](#)] for strict 2-categories and developed further in many works such as [[JFS17](#), [GR19](#), [GHL21](#), [Mae21](#), [CM23](#), [Lou24](#)]. In [[Cam23a](#)], [Campion](#) have constructed a simple universal property for the Gray tensor product and have constructed a model satisfying it. Alas, not all models have been shown to satisfy it. We will use [Lubaton’s](#)

model [Lou24, Construction 2.3.1.17], that was shown to have some computational properties we will employ.

For a typical example, the Gray tensor product of the arrow category [1] with itself is the lax square



Sometimes, wanting to stay in the realm of n -categories, we will denote also by \times^{lax} the truncated Gray tensor product

$$\text{Cat}_n \times \text{Cat}_n \xrightarrow{\times^{\text{lax}}} \text{Cat}_{2n} \xrightarrow{\tau_{\leq n}} \text{Cat}_n.$$

For $\mathcal{C} \in \text{Cat}_n$, the functors

$$\mathcal{C} \times^{\text{lax}} -: \text{Cat}_n \rightarrow \text{Cat}_n, \quad - \times^{\text{lax}} \mathcal{C}: \text{Cat}_n \rightarrow \text{Cat}_n$$

admit right adjoints denoted by

$$\text{Fun}_{\text{lax}}(\mathcal{C}, -): \text{Cat}_n \rightarrow \text{Cat}_n, \quad \text{Fun}_{\text{oplax}}(\mathcal{C}, -): \text{Cat}_n \rightarrow \text{Cat}_n$$

which are the n -categories of functors and (op)lax natural transformations (see [Lou24, Definition 4.1.4.1]). They are described as sheaves on Θ_n as follows:

$$\begin{aligned} \text{Fun}_{\text{lax}}(\mathcal{C}, \mathcal{D})_a &\simeq \text{Map}(a, \text{Fun}_{\text{lax}}(\mathcal{C}, \mathcal{D})) \simeq \text{Map}(\mathcal{C} \times^{\text{lax}} a, \mathcal{D}), \\ \text{Fun}_{\text{oplax}}(\mathcal{C}, \mathcal{D})_a &\simeq \text{Map}(a, \text{Fun}_{\text{oplax}}(\mathcal{C}, \mathcal{D})) \simeq \text{Map}(a \times^{\text{lax}} \mathcal{C}, \mathcal{D}). \end{aligned}$$

In particular, k -morphisms in $\text{Fun}_{\text{lax}}(\mathcal{C}, \mathcal{D})$ are functors $\mathcal{C} \times^{\text{lax}} \mathbb{D}^k \rightarrow \mathcal{D}$. Thus objects are just functors $\mathcal{C} \rightarrow \mathcal{D}$ and morphisms are lax natural transformations, which informally, are a collection of morphisms $\alpha_X: FX \rightarrow GX$ for every $X \in \mathcal{C}$ and a 2-morphism

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ \downarrow \alpha_X & \nearrow \alpha_f & \downarrow \alpha_Y \\ GX & \xrightarrow{Gf} & GY \end{array}$$

for every $f: X \rightarrow Y \in \mathcal{C}$.

These constructions satisfy certain useful properties:

Lemma 2.1.1. *For $\mathcal{D} \in \text{Cat}_n$ the functor $\text{Fun}_{\text{lax}}(-, \mathcal{D}): \text{Cat}_n^{\text{op}} \rightarrow \text{Cat}_n$ preserves limits (that is $\text{Fun}_{\text{lax}}(-, \mathcal{D})$ sends colimits in Cat_n to limits in Cat_n).*

Proof. Let $\mathcal{C}_{(-)}: \mathcal{I} \rightarrow \text{Cat}_n$ be a diagram and $\mathcal{D} \in \text{Cat}_n$. Using that the Gray tensor product

commutes with colimits in each variable ([Lou24, Construction 2.3.1.17]), we get for every $\mathcal{E} \in \text{Cat}_n$:

$$\begin{aligned}
\text{Map}(\mathcal{E}, \text{Fun}_{\text{lax}}(\text{colim}_i \mathcal{C}_i, \mathcal{D})) &\simeq \text{Map}((\text{colim}_i \mathcal{C}_i) \times^{\text{lax}} \mathcal{E}, \mathcal{D}) \\
&\simeq \text{Map}(\text{colim}_i (\mathcal{C}_i \times^{\text{lax}} \mathcal{E}), \mathcal{D}) \\
&\simeq \lim_i \text{Map}(\mathcal{C}_i \times^{\text{lax}} \mathcal{E}, \mathcal{D}) \\
&\simeq \lim_i \text{Map}(\mathcal{E}, \text{Fun}_{\text{lax}}(\mathcal{C}_i, \mathcal{D})) \\
&\simeq \text{Map}(\mathcal{E}, \lim_i \text{Fun}_{\text{lax}}(\mathcal{C}_i, \mathcal{D})).
\end{aligned}$$

As this is a natural isomorphism the result follows by Yoneda lemma. \square

Lemma 2.1.2. *Let $\mathcal{I}, \mathcal{J}, \mathcal{D} \in \Theta_n$. Then*

$$\text{Fun}_{\text{lax}}(\mathcal{J}, \text{Fun}_{\text{lax}}(\mathcal{I}, \mathcal{D})) \simeq \text{Fun}_{\text{lax}}(\mathcal{I} \times^{\text{lax}} \mathcal{J}, \mathcal{D}).$$

Proof. Let $\mathcal{E} \in \text{Cat}_n$. Then we compute:

$$\begin{aligned}
\text{Map}(\mathcal{E}, \text{Fun}_{\text{lax}}(\mathcal{J}, \text{Fun}_{\text{lax}}(\mathcal{I}, \mathcal{D}))) &\simeq \text{Map}(\mathcal{J} \times^{\text{lax}} \mathcal{E}, \text{Fun}_{\text{lax}}(\mathcal{I}, \mathcal{D})) \\
&\simeq \text{Map}(\mathcal{I} \times^{\text{lax}} \mathcal{J} \times^{\text{lax}} \mathcal{E}, \mathcal{D}) \\
&\simeq \text{Map}(\mathcal{E}, \text{Fun}_{\text{lax}}(\mathcal{I} \times^{\text{lax}} \mathcal{J}, \mathcal{D})).
\end{aligned}$$

The result follows by Yoneda lemma. \square

2.2 Lax limits

Let $\mathcal{C} \in \text{Cat}_n$ and $\mathcal{I} \in \text{Cat}_n$. The terminal functor $\mathcal{I} \rightarrow \text{pt}$ induces a functor

$$\Delta: \mathcal{C} \simeq \text{Fun}_{\text{lax}}(\text{pt}, \mathcal{C}) \rightarrow \text{Fun}_{\text{lax}}(\mathcal{I}, \mathcal{C})$$

which sends $X \in \mathcal{C}$ to the constant diagram on X .

Definition 2.2.1. A right adjoint for Δ at $F \in \text{Fun}_{\text{lax}}(\mathcal{I}, \mathcal{C})$ is called the lax limit of F and is denoted $\overrightarrow{\lim}_{\mathcal{I}} F$.

We call an arrow $\Delta X \rightarrow F$ in $\text{Fun}_{\text{lax}}(\mathcal{I}, \mathcal{C})$ a lax cone over F . Thus the lax limit of F is a terminal lax cone over F .

Lemma 2.2.2 (Lax Fubini). *Let $\mathcal{C} \in \text{Cat}_n$ be a category having all small lax limits. Let $\mathcal{I}, \mathcal{J} \in \text{Cat}_n$ and let $F: \mathcal{I} \times^{\text{lax}} \mathcal{J} \rightarrow \mathcal{C}$ be a diagram. Then*

$$\overrightarrow{\lim}_{\mathcal{I} \times^{\text{lax}} \mathcal{J}} F \simeq \overrightarrow{\lim}_{\mathcal{J}} \overrightarrow{\lim}_{\mathcal{I}} F.$$

That is, $\overrightarrow{\lim}_{\mathcal{I} \times^{\text{lax}} \mathcal{J}}$ is isomorphic to the composition

$$\text{Fun}_{\text{lax}}(\mathcal{I} \times^{\text{lax}} \mathcal{J}, \mathcal{C}) \simeq \text{Fun}_{\text{lax}}(\mathcal{J}, \text{Fun}_{\text{lax}}(\mathcal{I}, \mathcal{C})) \xrightarrow{\text{Fun}_{\text{lax}}(\mathcal{J}, \overrightarrow{\lim}_{\mathcal{I}}(-))} \text{Fun}_{\text{lax}}(\mathcal{I}, \mathcal{C}) \xrightarrow{\overrightarrow{\lim}_{\mathcal{I}}} \mathcal{C}.$$

Proof. The composition

$$\mathcal{C} \xrightarrow{\Delta_{\mathcal{J}}} \mathrm{Fun}_{\mathrm{lax}}(\mathcal{J}, \mathcal{C}) \xrightarrow{\mathrm{Fun}_{\mathrm{lax}}(\mathcal{J}, \Delta_{\mathcal{I}}(-))} \mathrm{Fun}_{\mathrm{lax}}(\mathcal{J}, \mathrm{Fun}_{\mathrm{lax}}(\mathcal{I}, \mathcal{C}))$$

is equivalent to $\Delta_{\mathcal{I} \times \mathrm{lax} \mathcal{J}}$. The lemma follows by taking right adjoints. \square

Lemma 2.2.3. *Let \mathcal{C} be an n -category having all colimits and all lax limits. Let*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{i_B} & \mathcal{B} \\ \downarrow i_C & & \downarrow j_B \\ \mathcal{C} & \xrightarrow{j_C} & \mathcal{D} \end{array}$$

be a pushout square in Cat_n and call $j_A: \mathcal{A} \rightarrow \mathcal{D}$ the composite map. Let $F: \mathcal{D} \rightarrow \mathcal{C}$ be a functor of n -categories. Then

$$\overrightarrow{\lim}_{\mathcal{D}} F \simeq \overrightarrow{\lim}_{\mathcal{B}} (F j_B) \times_{\overrightarrow{\lim}_{\mathcal{A}} (F j_A)} \overrightarrow{\lim}_{\mathcal{C}} (F j_C).$$

Proof. By Lemma 2.1.1, $\mathrm{Fun}_{\mathrm{lax}}(-, \mathcal{C})$ sends colimits to limits and in particular

$$\mathrm{Fun}_{\mathrm{lax}}(\mathcal{D}, \mathcal{C}) \cong \mathrm{Fun}_{\mathrm{lax}}(\mathcal{B}, \mathcal{C}) \times_{\mathrm{Fun}_{\mathrm{lax}}(\mathcal{A}, \mathcal{C})} \mathrm{Fun}_{\mathrm{lax}}(\mathcal{C}, \mathcal{C}).$$

As the map

$$\mathcal{D} \simeq \mathcal{B} \sqcup_{\mathcal{A}} \mathcal{C} \rightarrow \mathrm{pt} \sqcup_{\mathrm{pt}} \mathrm{pt} \simeq \mathrm{pt}$$

is the terminal map, the functor

$$\Delta_{\mathcal{B}} \times_{\Delta_{\mathcal{A}}} \Delta_{\mathcal{C}}: \mathcal{C} \simeq \mathcal{C} \times_{\mathcal{C}} \mathcal{C} \rightarrow \mathrm{Fun}_{\mathrm{lax}}(\mathcal{B}, \mathcal{C}) \times_{\mathrm{Fun}_{\mathrm{lax}}(\mathcal{A}, \mathcal{C})} \mathrm{Fun}_{\mathrm{lax}}(\mathcal{C}, \mathcal{C}) \simeq \mathrm{Fun}_{\mathrm{lax}}(\mathcal{D}, \mathcal{C})$$

is identified with $\Delta_{\mathcal{D}}$. The result now follows by [HY17, Theorem 5.5]. \square

When $\mathcal{C} = \mathrm{Cat}_n$ is the $(n+1)$ -category of n -categories, the lax limit of a functor $F: \mathcal{I} \rightarrow \mathrm{Cat}_n$ is the category of sections of the cartesian fibration $\int_{\mathcal{I}} F \rightarrow \mathcal{I}$ defined by the Grothendieck construction, i.e. unstraightening ([Lou24, Construction 4.1.2.5, Proposition 4.2.3.8, Example 4.2.3.13]). In particular, all lax limits in Cat_n exist. One can for example compute:

Example 2.2.4 ([Lou24, Example 4.2.3.13]). Let $\mathcal{C}, \mathcal{I} \in \mathrm{Cat}_n$. Then the lax limit of the constant functor $\Delta_{\mathcal{C}}: \mathcal{I} \rightarrow \mathrm{Cat}_n$ is

$$\overrightarrow{\lim}_{\mathcal{I}} \Delta_{\mathcal{C}} \simeq \mathrm{Fun}_{\mathrm{lax}}(\mathcal{I}, \mathcal{C}).$$

Lemma 2.2.3 allows us to compute lax limits using cell decomposition. In the next two subsections we will use it to compute some lax limits, specifically, of functors from “directed spheres” to Cat_n . Let $\partial \mathbb{D}^k$ be the maximal $(k-1)$ -subcategory of \mathbb{D}^k . We define the directed sphere as $\vec{S}^k := \mathbb{D}^k / \partial \mathbb{D}^k$.

Remark 2.2.5. The directed sphere is the walking k -endomorphism, that is, a functor $\vec{S}^k \rightarrow \mathcal{U}$ chooses an object $X \in \mathcal{U}$ and an endomorphism f of $\mathrm{id}_{\mathrm{id}}$.

More concretely, for $k = 1$ it chooses an object X and an endomorphism $f: X \rightarrow X$, for $k = 2$ it chooses an object X and an endomorphism $f: \mathrm{id}_X \rightrightarrows \mathrm{id}_X$.

A functor $\vec{S}^k \rightarrow \mathcal{U}$ chooses an invertible k -endomorphism if and only if it factors through the groupoidification $|\vec{S}^k| \simeq S^k$.

Lax limits along an arrow

The following lemma is well known. See [CDW23, Theorem 2.4.2] for a computation in $\text{Pr}_{\text{st}}^{\text{L}}$ or more generally in $\text{Mod}_{\text{Mod}_R}(\text{Pr}_{\text{st}}^{\text{L}})$. The general method for this computation is described in [CDW24].

Lemma 2.2.6. *Let $G: [1] \rightarrow \text{Cat}_n$ choosing an n -functor $G: \mathcal{C} \rightarrow \mathcal{D}$. Then its lax limit sits in a pullback diagram*

$$\begin{array}{ccc} \overrightarrow{\lim}_{[1]} G & \longrightarrow & \text{Fun}_{\text{lax}}([1], \mathcal{D}) \\ \downarrow & \lrcorner & \downarrow \text{src} \\ \mathcal{C} & \xrightarrow{G} & \mathcal{D} \end{array},$$

i.e. it is the category consisting of tuples (X, Y, g) where $X \in \mathcal{C}$, $Y \in \mathcal{D}$ and $g: GX \rightarrow Y$, with lax squares as morphisms.

Proof. The Grothendieck construction over an arrow is computed in [GHN17, Lemma 3.8]:

$$\int_{[1]} G = (\mathcal{C} \times [1]) \sqcup_{(\mathcal{C} \times \{1\})} (\mathcal{D} \times \{1\})$$

with the fibration $\int_{[1]} G \rightarrow [1]$ defined as the projection to the second coordinate. Then the category of sections $[1] \rightarrow G$ is exactly as described. \square

2.3 Lax \mathbb{N} -fixed points, or lax limits along \vec{S}^1

Recall that $\vec{S}^1 = \mathbb{D}^1 / \partial \mathbb{D}^1 = [1] / \partial [1] = \text{BN}$ is the walking endomorphism. Fix a 2-category \mathcal{V} and assume it admits lax limits along \vec{S}^1 . In this subsection we will study lax limits along \vec{S}^1 , in particular showing that for a diagram $(X, g): \vec{S}^1 \rightarrow \mathcal{V}$ the endomorphism g lifts to an endomorphism of the lax limit $\overrightarrow{\lim}_{\vec{S}^1}(X, g)$, and that there is a 2-morphism $\hat{v}: g \Rightarrow \text{id}$ of endomorphisms of the lax limit. We end the subsection by giving concrete formulas for all construction in the special case where $\mathcal{V} = \text{Cat}$.

Definition 2.3.1. Let $(X, g): \vec{S}^1 = \text{BN} \rightarrow \mathcal{V}$ be a functor. That is, the data of an object $X \in \mathcal{V}$ and an endomorphism $g: X \rightarrow X$, or equivalently a monoid-action of \mathbb{N} on X . Define the lax \mathbb{N} -fixed points of X as the lax limit

$$X^{\vec{h}\mathbb{N}} := \overrightarrow{\lim}_{\vec{S}^1}(X, g).$$

Remark 2.3.2. We always have a map from the universal cone (when it exists) to the universal lax cone

$$\lim_{\vec{S}^1}(X, g) \rightarrow \overrightarrow{\lim}_{\vec{S}^1}(X, g) = X^{\vec{h}\mathbb{N}}.$$

If g is invertible, then the functor $(X, g): \vec{S}^1 \rightarrow \mathcal{C}$ factors through $S^1 = |\vec{S}^1|$ and this map is identified with

$$X^{h\mathbb{Z}} = \lim_{S^1} X \simeq \lim_{\vec{S}^1} X \rightarrow \overrightarrow{\lim}_{\vec{S}^1} X = X^{\vec{h}\mathbb{N}}.$$

Definition 2.3.3. The commutative diagram

$$\begin{array}{ccc} \text{pt} & \longrightarrow & \vec{S}^1 \\ & \searrow X & \swarrow (X,g) \\ & & \mathcal{V} \end{array}$$

induces, by the functoriality of lax limits, a map

$$u: X^{\vec{h}\mathbb{N}} = \overrightarrow{\lim}_{\vec{S}^1}(X, g) \rightarrow \overrightarrow{\lim}_{\text{pt}} X = X$$

which we call the underlying map.

Remark 2.3.4. A lax cone over (X, g) is a map $f: Y \rightarrow X$ and a 2-morphism $\eta: gf \Rightarrow f$

$$\begin{array}{ccc} & Y & \\ f \swarrow & & \searrow f \\ X & \xrightarrow{g} & X \\ \eta \nearrow & & \end{array}$$

A map $Y \rightarrow X^{\vec{h}\mathbb{N}}$ is equivalent to such a lax cone and the composition $Y \rightarrow X^{\vec{h}\mathbb{N}} \xrightarrow{u} X$ agrees with $f: Y \rightarrow X$.

Corollary 2.3.5. *There exists a canonical 2-morphism $v: gu \Rightarrow u$.*

Proof. The universal lax cone corresponds to the identity map $X^{\vec{h}\mathbb{N}} \rightarrow X^{\vec{h}\mathbb{N}}$ and so is given as $u: X^{\vec{h}\mathbb{N}} \rightarrow X$ with a 2-morphism $v: gu \Rightarrow u$. \square

Definition 2.3.6. The map

$$g: (X, g) \rightarrow (X, g) \in \text{Fun}_{\text{lax}}(\vec{S}^1, \mathcal{V}),$$

given by $g: X \rightarrow X$ and the trivial higher cells, induces an arrow

$$g: X^{\vec{h}\mathbb{N}} = \overrightarrow{\lim}_{\vec{S}^1}(X, g) \rightarrow \overrightarrow{\lim}_{\vec{S}^1}(X, g) = X^{\vec{h}\mathbb{N}}.$$

Lemma 2.3.7. *There exists a 2-morphism*

$$\hat{v}: g \Rightarrow \text{id}_{X^{\vec{h}\mathbb{N}}}$$

of endomorphisms of $X^{\vec{h}\mathbb{N}}$ lifting v of Corollary 2.3.5, i.e. $u\hat{v} \simeq v$.

Proof. We use the universal lax cone

$$\begin{array}{ccc} & X^{\vec{h}\mathbb{N}} & \\ u \swarrow & & \searrow u \\ X & \xrightarrow{g} & X \\ v \nearrow & & \end{array}$$

to construct a diagram in $\text{Fun}_{\text{lax}}(\vec{S}^1, \mathcal{V})$ of the form

$$\begin{array}{ccc} & \Delta X^{\vec{h}\mathbb{N}} = (X^{\vec{h}\mathbb{N}}, \text{id}_{X^{\vec{h}\mathbb{N}}}) & \\ & \swarrow u & \searrow u \\ (X, g) & \xrightarrow{g} & (X, g) \end{array}$$

Specifically it is given as the diagram

$$\begin{array}{ccccc} & & X^{\vec{h}\mathbb{N}} & \xrightarrow{u} & X \\ & \swarrow u & \parallel v & \nearrow g & \parallel g \\ X & \xrightarrow{v} & X^{\vec{h}\mathbb{N}} & \xrightarrow{u} & X \\ & \parallel v & \parallel v & \parallel v & \parallel v \\ & \swarrow u & X^{\vec{h}\mathbb{N}} & \xrightarrow{u} & X \\ X & \xrightarrow{g} & X & \xrightarrow{g} & X \end{array}$$

(with the trivial front 2-cell and interior 3-cell), which gives the required 2-morphism. \square

The categorical case

We will now look at the special case $\mathcal{V} = \text{Cat}$, finding explicit formulas for each of the constructions above. First, we can describe the lax fixed points as a lax equalizer as in [NS18]:

Lemma 2.3.8. *Let $(\mathcal{C}, G): \vec{S}^1 \rightarrow \text{Cat}$. Then the lax fixed points $\mathcal{C}^{\vec{h}\mathbb{N}}$ agrees with the lax equalizer $\vec{\text{eq}}(\mathcal{C} \xrightarrow[G]{\text{id}} \mathcal{C})$. That is, it sits in a pullback diagram*

$$\begin{array}{ccc} \mathcal{C}^{\vec{h}\mathbb{N}} & \xrightarrow{p^*} & \mathcal{C}[1] \\ \downarrow u & \lrcorner & \downarrow \text{src, trgt} \\ \mathcal{C} & \xrightarrow{G, \text{id}} & \mathcal{C} \times \mathcal{C} \end{array}$$

Proof. \vec{S}^1 is the pushout

$$\begin{array}{ccc} \{0, 1\} & \longrightarrow & [1] \\ \downarrow & & \downarrow \\ \text{pt} & \longrightarrow & \vec{S}^1 \end{array}$$

so the result follows by Lemma 2.2.3. \square

Corollary 2.3.9. *An object of $\mathcal{C}^{\vec{h}\mathbb{N}}$ is equivalent to the data of (X, v_X) , where $X \in \mathcal{C}$ and $v_X: GX \rightarrow X$. The mapping space between two objects $(X, v_X), (Y, v_Y)$ is given by*

$$\mathrm{Map}_{\mathcal{C}^{\vec{h}\mathbb{N}}}((X, v_X), (Y, v_Y)) \simeq \mathrm{eq} \left(\mathrm{Map}_{\mathcal{C}}(X, Y) \begin{array}{c} \xrightarrow{-\circ v_X} \\ \xrightarrow{v_Y \circ G-} \end{array} \mathrm{Map}_{\mathcal{C}}(GX, Y) \right).$$

The underlying object functor $u: \mathcal{C}^{\vec{h}\mathbb{N}} \rightarrow \mathcal{C}$ is exact, conservative and colimit-preserving.

Proof. This is a special case of [NS18, Proposition II.1.5]. \square

Remark 2.3.10. Chasing the description of Lemma 2.2.3, the 2-morphism of Corollary 2.3.5, coming from the universal lax cone of $\mathcal{C}^{\vec{h}\mathbb{N}}$, is (u, v) where $v: Gu \Rightarrow u$ is given on an object $(X, GX \xrightarrow{v_X} X)$ by v_X .

Remark 2.3.11. Assuming G is invertible, by the description in Lemma 2.3.8, the functor $\mathcal{C}^{h\mathbb{Z}} \rightarrow \mathcal{C}^{\vec{h}\mathbb{N}}$ of Remark 2.3.2 sends a tuple $(X, GX \xrightarrow{\sim} X)$ to the tuple $(X, GX \rightarrow X)$, forgetting that the map is an isomorphism.

Lemma 2.3.12. *The morphism $G: \mathcal{C}^{\vec{h}\mathbb{N}} \rightarrow \mathcal{C}^{\vec{h}\mathbb{N}}$ of Definition 2.3.6 sends $(GX \xrightarrow{v_X} X) \in \mathcal{C}^{\vec{h}\mathbb{N}}$ to $(G^2X \xrightarrow{Gv_X} GX)$.*

Proof. Recall that a lax cone from \mathcal{D} over $(\mathcal{C}, G): \vec{S}^1 \rightarrow \mathrm{Cat}$ is the data of a functor $F: \mathcal{D} \rightarrow \mathcal{C}$ and a natural transformation $\eta: G \circ F \Rightarrow F$. The map

$$G: (\mathcal{C}, G) \rightarrow (\mathcal{C}, G) \in \mathrm{Fun}_{\mathrm{lax}}(\vec{S}^1, \mathrm{Cat})$$

sends such a lax cone $(\mathcal{D} \xrightarrow{F} \mathcal{C}, \eta)$ to the lax cone $(\mathcal{D} \xrightarrow{GF} \mathcal{C}, G\eta)$. In particular, it sends the universal lax cone $(\mathcal{C}^{\vec{h}\mathbb{N}} \xrightarrow{u} \mathcal{C}, v)$ to $(\mathcal{C}^{\vec{h}\mathbb{N}} \xrightarrow{Gu} \mathcal{C}, Gv)$ which corresponds to the required functor $\mathcal{C}^{\vec{h}\mathbb{N}} \rightarrow \mathcal{C}^{\vec{h}\mathbb{N}}$. \square

Lemma 2.3.13. *The 2-morphism $\hat{v}: G \Rightarrow \mathrm{id}_{\mathcal{C}^{\vec{h}\mathbb{N}}}$ of Lemma 2.3.7 is given on an object $(GX \xrightarrow{v_X} X)$ by*

$$\begin{array}{ccc} G^2X & \overset{Gv_X}{\dashrightarrow} & GX \\ \downarrow Gv_X & & \downarrow v_X \\ GX & \overset{v_X}{\dashrightarrow} & X \end{array}$$

Proof. By Lemma 2.3.7 under composition with u it is given by $v: Gu \rightarrow u$. The result now follows by Remark 2.3.10 and its composition with G . \square

2.4 Lax limits along \vec{S}^2

We now categorify [Section 2.3](#), finding a natural place where $(-)^{\vec{h}\mathbb{N}}$ lands, remembering the 2-morphism of [Lemma 2.3.7](#).

Let $\vec{S}^2 := \mathbb{D}^2 / \partial\mathbb{D}^2$ be the walking endomorphism of the identity, that is \vec{S}^2 consists of one object \star and is generated by a 2-morphism $\text{id}_\star \rightarrow \text{id}_\star$.

Fix a 3-category \mathcal{U} and assume lax limits of the shape \vec{S}^2 exist in \mathcal{U} and lax limits of the shape \vec{S}^1 exist in the mapping 2-categories of \mathcal{U} . A diagram

$$(X, \gamma): \vec{S}^2 \rightarrow \mathcal{U}$$

chooses $X \in \mathcal{U}$ and a 2-endomorphism $\gamma: \text{id}_X \Rightarrow \text{id}_X$. In this subsection we will consider the lax limit $\overrightarrow{\lim}_{\vec{S}^2}(X, \gamma)$.

Remark 2.4.1. As before, we have a map from the universal cone (if it exists) to the universal lax cone

$$\lim_{\vec{S}^2}(X, \gamma) \rightarrow \overrightarrow{\lim}_{\vec{S}^2}(X, \gamma).$$

If γ is invertible then $(X, \gamma): \vec{S}^2 \rightarrow \mathcal{U}$ factors through $S^2 = |\vec{S}^2|$ and this map identifies with

$$\lim_{S^2}(X, \gamma) \simeq \lim_{\vec{S}^2}(X, \gamma) \rightarrow \overrightarrow{\lim}_{\vec{S}^2}(X, \gamma).$$

Definition 2.4.2. The commutative diagram

$$\begin{array}{ccc} \text{pt} & \longrightarrow & \vec{S}^2 \\ & \searrow X & \swarrow (X, \gamma) \\ & & \mathcal{U} \end{array}$$

induces, by the functoriality of lax limits, a map

$$U: \overrightarrow{\lim}_{\vec{S}^2}(X, \gamma) \rightarrow \overrightarrow{\lim}_{\text{pt}} X \simeq X$$

which we call the underlying map.

Remark 2.4.3. A lax cone over (X, γ) is a map $f: Y \rightarrow X$ and a 3-morphism $\eta: \gamma f \Rightarrow \text{id}_f$:

$$\begin{array}{ccc} & f & \\ \curvearrowright & & \curvearrowright \\ Y & \xrightarrow{\gamma f} & X \\ \curvearrowleft & \eta & \curvearrowleft \\ & \text{id}_f & \\ & f & \end{array} .$$

Any map $Y \rightarrow \overrightarrow{\lim}_{\vec{S}^2}(X, \gamma)$ is equivalent to such a lax cone and the composition $Y \rightarrow \overrightarrow{\lim}_{\vec{S}^2}(X, \gamma) \xrightarrow{U} X$ is identified with f .

Corollary 2.4.4. *There exists a canonical 3-morphism $\alpha: \gamma U \Rightarrow U$*

$$\begin{array}{ccc}
 & \xrightarrow{\gamma U} & \\
 U & \begin{array}{c} \curvearrowright \\ \Downarrow \alpha \\ \curvearrowleft \end{array} & U \\
 & \xrightarrow{\text{id}_U} &
 \end{array} .$$

Proof. The universal lax cone over (X, γ) corresponds to the identity morphism $\overrightarrow{\lim}_{\mathcal{S}^2}(X, \gamma) \rightarrow \overrightarrow{\lim}_{\mathcal{S}^2}(X, \gamma)$ therefore it is given by (U, α) where $\alpha: \gamma U \Rightarrow U$. \square

As γ is an endomorphism of id_X , it induces a functor

$$(\text{id}_X, \gamma): \mathcal{S}^1 \rightarrow \text{END}(X).$$

Lemma 2.4.5. *The map $\text{id}_X^{\vec{h}^{\mathbb{N}}}: X \rightarrow X$ lifts to $\overrightarrow{\lim}_{\mathcal{S}^2}(X, \gamma)$. That is, there exists a map, which we also denote by $\text{id}_X^{\vec{h}^{\mathbb{N}}}: X \rightarrow \overrightarrow{\lim}_{\mathcal{S}^2}(X, \gamma)$ rendering the following diagram commutative:*

$$\begin{array}{ccc}
 X & \xrightarrow{\text{id}_X^{\vec{h}^{\mathbb{N}}}} & \overrightarrow{\lim}_{\mathcal{S}^2}(X, \gamma) \\
 & \searrow \text{id}_X^{\vec{h}^{\mathbb{N}}} & \downarrow U \\
 & & X
 \end{array} .$$

Proof. A map $X \rightarrow \overrightarrow{\lim}_{\mathcal{S}^2}(X, \gamma)$ is the same as a lax cone, i.e. a map $f: X \rightarrow X$ and a 3-morphism

$$\begin{array}{ccc}
 & \xrightarrow{\gamma f} & \\
 f & \begin{array}{c} \curvearrowright \\ \Downarrow \eta \\ \curvearrowleft \end{array} & f \\
 & \xrightarrow{\text{id}_f} &
 \end{array} .$$

Such a lax cone, composed with the underlying map U will agree with f . Therefore to construct such a lifting we must give a 3-morphism

$$\begin{array}{ccc}
 & \xrightarrow{\gamma} & \\
 \text{id}_X^{\vec{h}^{\mathbb{N}}} & \begin{array}{c} \curvearrowright \\ \Downarrow \hat{\nu} \\ \curvearrowleft \end{array} & \text{id}_X^{\vec{h}^{\mathbb{N}}} \\
 & \xrightarrow{\text{id}} &
 \end{array} ,$$

which is exactly [Lemma 2.3.7](#). \square

The categorical case

We will now look at the special case $\mathcal{U} = \text{Cat}_2$, finding explicit formulas for each of the constructions above:

Lemma 2.4.6. *Let $(\mathcal{V}, \gamma): \vec{S}^2 \rightarrow \text{Cat}_2$, i.e. a 2-category \mathcal{V} and a natural transformation $\gamma: \text{id}_{\mathcal{V}} \Rightarrow \text{id}_{\mathcal{V}}$. An object of $\overrightarrow{\lim}_{\vec{S}^2}(\mathcal{V}, \gamma)$ is equivalent to the data of $X \in \mathcal{V}$ with a 2-morphism $\alpha_X: \gamma_X \Rightarrow \text{id}_X$.*

Proof. We can write $\vec{S}^2 = \mathbb{D}^2 / \partial \mathbb{D}^2$ also as $\square / \partial \square$, where $\square := [1] \times^{\text{lax}} [1]$ is the lax square

$$\square = [1] \times^{\text{lax}} [1] = \begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & \nearrow & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

and $\partial \square$ is its maximal 1-subcategory. To see this, note that by [Cam23b, Corollary 5.12], both quotients can be computed in strict n -categories, where it is obvious. Thus, by Lemma 2.2.3, $\overrightarrow{\lim}_{\vec{S}^2}(\mathcal{V}, \gamma)$ sits in a pullback diagram

$$\begin{array}{ccc} \overrightarrow{\lim}_{\vec{S}^2}(\mathcal{V}, \gamma) & \longrightarrow & \overrightarrow{\lim}_{\square}(\mathcal{V}, \gamma) \\ \downarrow U & \lrcorner & \downarrow \\ \mathcal{V} & \longrightarrow & \overrightarrow{\lim}_{\partial \square}(\mathcal{V}, \gamma) \end{array} .$$

We can compute $\overrightarrow{\lim}_{\square}$ using Fubini (Lemma 2.2.2):

$$\overrightarrow{\lim}_{\square}(\mathcal{V}, \gamma) = \overrightarrow{\lim}_{([1] \times^{\text{lax}} [1])}(\mathcal{V}, \gamma) \simeq \overrightarrow{\lim}_{[1]} \overrightarrow{\lim}_{[1]}(\mathcal{V}, \gamma)$$

where the diagram is of the form

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\text{id}} & \mathcal{V} \\ \text{id} \downarrow & \xrightarrow{\gamma} & \downarrow \text{id} \\ \mathcal{V} & \xrightarrow{\text{id}} & \mathcal{V} \end{array} .$$

We first compute the vertical lax limit (along the identity morphism), getting the diagram

$$(\text{Fun}_{\text{lax}}([1], \mathcal{V}) \xrightarrow{\gamma} \text{Fun}_{\text{lax}}([1], \mathcal{V})) : [1] \rightarrow \text{Cat}_2,$$

where γ is the functor taking an arrow $f: X \rightarrow Y$ to the composition

$$X \xrightarrow{\gamma_X} X \xrightarrow{f} Y.$$

Computing the lax limit, using Lemma 2.2.6, we get the category

$$\text{Fun}_{\text{lax}}([1], \text{Fun}_{\text{lax}}([1], \mathcal{V})) \times_{\text{Fun}_{\text{lax}}([1], \mathcal{V})^2} \text{Fun}_{\text{lax}}([1], \mathcal{V})$$

with $(\gamma, \text{id}): \text{Fun}_{\text{lax}}([1], \mathcal{V}) \rightarrow \text{Fun}_{\text{lax}}([1], \mathcal{V})^2$. That is the category of lax squares

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ \downarrow \gamma_X & \nearrow \alpha & \downarrow f' \\ X & & Y' \\ \downarrow f & \nearrow h & \\ Y & \xrightarrow{h} & Y' \end{array} \quad (1)$$

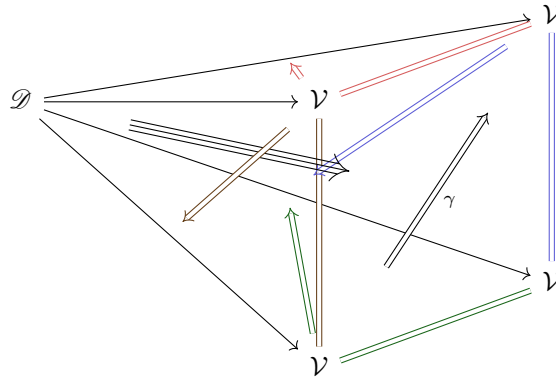
The lax limit $\overrightarrow{\lim}_{\partial \square}(\mathcal{V}, \gamma)$ is taken along the constant functor $\Delta \mathcal{V}: \partial \square \rightarrow \text{Cat}_2$. Therefore

$$\overrightarrow{\lim}_{\partial \square}(\mathcal{V}, \gamma) \simeq \text{Fun}_{\text{lax}}(\partial \square, \mathcal{V}).$$

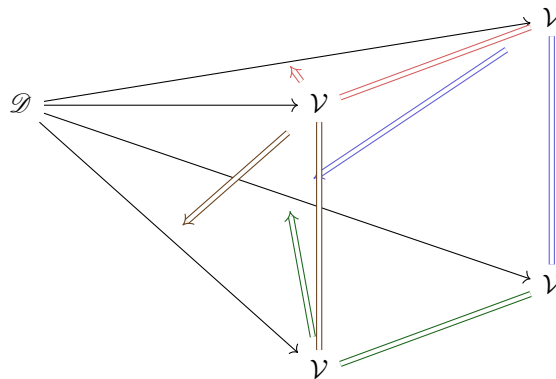
It is left showing that the functor $\overrightarrow{\lim}_{\square}(\mathcal{V}, \gamma) \rightarrow \overrightarrow{\lim}_{\partial \square}(\mathcal{V}, \gamma)$ sends a lax square of the form (1) to the (non-commutative) square

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ \downarrow f & & \downarrow f' \\ Y & \xrightarrow{h} & Y' \end{array}$$

To see this, note that a lax cone over $(\mathcal{V}, \gamma): \square \rightarrow \text{Cat}_2$ is a diagram of the form



and it is sent to the lax cone over $\Delta \mathcal{V}: \partial \square \rightarrow \text{Cat}_2$



forgetting γ and the 3-cell. The universal cone, has as a functors to \mathcal{V} , the functors sending a lax square of the form (1) to X, X', Y, Y' . The 2-morphisms are given by g, f, f', h and the 3-morphism is exactly α . The corresponding lax cone over $\partial \square$ represents the functor sending such

a lax square to

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ \downarrow f & & \downarrow f' \\ Y & \xrightarrow{h} & Y' \end{array}$$

□

Remark 2.4.7. Chasing the description of [Lemma 2.2.3](#), the 3-morphism of [Corollary 2.4.4](#), coming from the universal lax cone of $\overrightarrow{\lim}_{\vec{S}^2}(\mathcal{V}, \gamma)$, is (U, α) where $\alpha: \gamma U \Rightarrow U$ is given on an object $(X, \alpha_X: \gamma_X \Rightarrow \text{id}_X)$ as α_X .

Remark 2.4.8. Assuming γ is invertible, by the description [Lemma 2.4.6](#), the functor $\lim_{S^2}(\mathcal{V}, \gamma) \rightarrow \overrightarrow{\lim}_{\vec{S}^2}(\mathcal{V}, \gamma)$ of [Remark 2.4.1](#) sends a tuple $(X, \gamma_X \xrightarrow{\cong} \text{id}_X)$ to the tuple $(X, \gamma_X \Rightarrow \text{id}_X)$, forgetting that the map is invertible.

Reinterpreting [Lemma 2.4.5](#) for the special case $\mathcal{V} = \text{Cat}_2$, as $\gamma: \text{id}_{\mathcal{V}} \rightarrow \text{id}_{\mathcal{V}}$, any $X \in \mathcal{V}$ admits a morphism $\gamma_X: X \rightarrow X$. We can then say:

Corollary 2.4.9. *Let $X \in \mathcal{V}$, then there exists a lift $X^{\vec{h}\mathbb{N}} \in \overrightarrow{\lim}_{\vec{S}^2}(\mathcal{V}, \gamma)$. That is, there exists such an object and $U(X^{\vec{h}\mathbb{N}}) \in \mathcal{V}$ agrees with the lax fixed points of [Section 2.3](#).*

Remark 2.4.10. The 2-morphism $\hat{v}: \gamma_{X^{\vec{h}\mathbb{N}}} \Rightarrow \text{id}_{X^{\vec{h}\mathbb{N}}}$, which is part of the data of an object in $\overrightarrow{\lim}_{\vec{S}^2}(\mathcal{V}, \gamma)$ is the 2-morphism constructed in [Lemma 2.3.7](#).

The functor $\text{id}_{\mathcal{V}}^{\vec{h}\mathbb{N}}$ of [Lemma 2.4.5](#) can be thought of as the functor

$$(-)^{\vec{h}\mathbb{N}}: \mathcal{V} \rightarrow \overrightarrow{\lim}_{\vec{S}^2}(\mathcal{V}, \gamma)$$

taking $X \in \mathcal{V}$ to $X^{\vec{h}\mathbb{N}}$.

Proposition 2.4.11. *The functor $(-)^{\vec{h}\mathbb{N}}: \mathcal{V} \rightarrow \overrightarrow{\lim}_{\vec{S}^2}(\mathcal{V}, \gamma)$ is right adjoint to the underlying functor $U: \overrightarrow{\lim}_{\vec{S}^2}(\mathcal{V}, \gamma) \rightarrow \mathcal{V}$.*

Proof. In this proof we will have to distinguish between the lax fixed points landing in \mathcal{V} and the lax fixed points landing in $\overrightarrow{\lim}_{\vec{S}^2}(\mathcal{V}, \gamma)$. We will denote the latter by $(-)^{\vec{h}\mathbb{N}}$ while the former will be denoted as the composition $U((-)^{\vec{h}\mathbb{N}})$.

We prove the adjunction by constructing a unit and a counit. The map

$$u: U \circ (-)^{\vec{h}\mathbb{N}} = U((-)^{\vec{h}\mathbb{N}}) \rightarrow \text{id}_{\mathcal{V}}$$

defined in [Definition 2.3.3](#) will be a counit for the adjunction¹.

¹ u is an unfortunate name for a counit.

To construct a unit, we must construct a 2-morphism $\text{id} \xrightarrow{\lim_{\vec{S}^2}(\mathcal{V}, \gamma)} \Rightarrow U^{\vec{h}\mathbb{N}}$.² Equivalently, by the exponential adjunction, we need to provide a functor

$$\hat{\alpha}: \overrightarrow{\lim}_{\vec{S}^2}(\mathcal{V}, \gamma) \rightarrow (\overrightarrow{\lim}_{\vec{S}^2}(\mathcal{V}, \gamma))^{[1]} \simeq \overrightarrow{\lim}_{\vec{S}^2}(\mathcal{V}^{[1]}, \gamma).$$

Using [Remark 2.4.3](#), it is equivalent to a functor $\alpha: \overrightarrow{\lim}_{\vec{S}^2}(\mathcal{V}, \gamma) \rightarrow \mathcal{V}^{[1]}$ together with a 3-morphism

$$\begin{array}{ccc} & \gamma & \\ \alpha & \begin{array}{c} \curvearrowright \\ \Downarrow \eta \\ \curvearrowleft \end{array} & \alpha \\ & \text{id} & \end{array} .$$

Recall the 3-morphism $\alpha: \gamma U \Rightarrow \text{id}_U$ of [Remark 2.4.7](#). This is a lax cone from U over the diagram

$$(U, \gamma U): \vec{S}^1 \rightarrow \text{MAP}(\overrightarrow{\lim}_{\vec{S}^2}(\mathcal{V}, \gamma), \mathcal{V})$$

and so corresponds ([Remark 2.3.4](#)) to a 2-morphism $\alpha: U \Rightarrow U(U^{\vec{h}\mathbb{N}}) \simeq U \circ (-)^{\vec{h}\mathbb{N}} \circ U$, which we can think of as a functor $\alpha: \overrightarrow{\lim}_{\vec{S}^2}(\mathcal{V}, \gamma) \rightarrow \mathcal{V}^{[1]}$. Indeed, on an object X , $\alpha_X: X \rightarrow X^{\vec{h}\mathbb{N}}$ corresponds by [Remark 2.3.4](#) to a 2-morphism $\alpha_X: \gamma_X \Rightarrow \text{id}_X$.

Giving a 3-morphism η as above requires to give for any $(X, \alpha_X) \in \overrightarrow{\lim}_{\vec{S}^2}(\mathcal{V}, \gamma)$ a 2-morphism $\gamma_X \Rightarrow \text{id}_X$ in a coherent way, but such 2-morphism is given by α_X .

It is now left to check the zig-zag identities of the unit and the counit, i.e. that the compositions

$$\begin{aligned} U &\xrightarrow{\text{id} \circ \hat{\alpha}} U \circ U^{\vec{h}\mathbb{N}[d]} = U(-)^{\vec{h}\mathbb{N}[d]} \circ U \xrightarrow{u \circ \text{id}} U \\ (-)^{\vec{h}\mathbb{N}[d]} &\xrightarrow{\hat{\alpha} \circ \text{id}} U^{\vec{h}\mathbb{N}[d]} \circ (-)^{\vec{h}\mathbb{N}[d]} = (-)^{\vec{h}\mathbb{N}[d]} \circ U(-)^{\vec{h}\mathbb{N}[d]} \xrightarrow{\text{id} \circ u} (-)^{\vec{h}\mathbb{N}[d]} \end{aligned}$$

are natural isomorphisms. It is therefore sufficient to check that on objects, and this is straightforward. □

3 Categories of (asymptotically defined) self-maps

In this section we introduce a general framework for periodicity phenomena.

Let D be a quotient group of $\widehat{\mathbb{Z}}$. The goal of this section is the construction of the categories $\text{Cat}_{\text{perf}}^{D\text{-end}}$, $\text{Cat}_{\text{perf}}^{D\text{-iso}}$ of categories with asymptotically defined natural endomorphisms of the identity (of specific degrees) and of asymptotically defined natural isomorphisms of the identity (of specific degrees), along with their relations.

In the next section, we will reformulate the periodicity theorem [[HS98](#), Theorem 9] describing precisely the existence of an ‘‘asymptotically defined v_n -natural endomorphism’’ of the identity of $\text{Sp}_{\geq n}^\omega$, and deduce applications from it to the $T(n)$ -local category.

²The name $U^{\vec{h}\mathbb{N}}$ might be ambiguous, as it could refer to either $U \circ (-)^{\vec{h}\mathbb{N}}$ or the lax fixed points of U under the \mathbb{N} -action induced by γ . Fortunately, these two definitions coincide.

In [Section 3.1](#) we describe, for a finite group $D \cong \mathbb{Z}/d$ and a category $\mathcal{C} \in \text{Cat}_{\text{perf}}$, the general construction and attributes of the category of self-maps of \mathcal{C} . In [Section 3.2](#) we categorify the construction and define the category of categories with shifted endomorphism of the identity (i.e. natural transformations $\Sigma^d \Rightarrow \text{id}$), defined as the lax limit of Cat_{perf} along \vec{S}^2 as introduced in [Section 2.4](#). We then describe its properties and its connection to categories of self-maps. In [Section 3.3](#) we discuss the connection to the (non-lax) limit — the category of categories with shifted isomorphism of the identity and the related category of categories with locally nilpotent shifted endomorphism of the identity. Finally, in [Section 3.4](#), we extend to pro-finite D by taking the colimit of our constructed categories, defining categories with asymptotically defined self maps and categories of categories with asymptotically defined endomorphisms of the identity.

As we will work with group actions and (de)categorifications, we will need the following 2 simple remarks:

Remark 3.0.1. Let \mathcal{V} be a 2-category, $X \in \mathcal{V}$ and A a pointed space. An A -local system $A \rightarrow \text{END}(X)$ sending the base point to id_X , is, by the core-forgetful adjunction, a map of pointed spaces $A \rightarrow \text{END}(X)^{\simeq} \simeq \Omega_X \mathcal{V}^{\simeq}$, which in turn is the same as a map of pointed spaces $\Sigma A \rightarrow \mathcal{V}^{\simeq}$ choosing X . Again, using the core-forgetful adjunction, it is the same as a local system $\Sigma A \rightarrow \mathcal{V}$ sending the base point to X .

In particular, a G -action on id_X is the same as a functor $\Sigma BG \rightarrow \mathcal{V}$ choosing X , or equivalently, a $\Omega \Sigma BG$ -action on X .

Remark 3.0.2. Let \mathcal{V} be a 2-category and $X \in \mathcal{V}$. A G -action on X is the same as a functor $BG \rightarrow \mathcal{V}$ choosing X , and therefore induces a local system $G \rightarrow \text{END}(X)$. When G is connected, it is equivalent to a ΩG -action on id_X .

3.1 Self map categories

Note that lax limits in Cat_{perf} agree with lax limits in Cat so we may use the results of [Section 2.3](#) replacing Cat with Cat_{perf} . Let $\mathcal{C} \in \text{Cat}_{\text{perf}}$ and $d \in \mathbb{N}$. We will use [Section 2.3](#) for the specific functor

$$(\mathcal{C}, \Sigma^d): \vec{S}^1 \rightarrow \text{Cat}_{\text{perf}}.$$

Definition 3.1.1. Let $\mathcal{C} \in \text{Cat}_{\text{perf}}$ and $d \in \mathbb{N}$. Define the category $\mathcal{C}^{\vec{h}\mathbb{N}[d]}$ of objects with self-maps of degree d as the lax fixed points

$$\mathcal{C}^{\vec{h}\mathbb{N}[d]} := \overrightarrow{\lim}(\vec{S}^1 \xrightarrow{(\mathcal{C}, \Sigma^d)} \text{Cat}_{\text{perf}}).$$

Equivalently ([Lemma 2.3.8](#)) it is the lax equalizer of $\mathcal{C} \xrightleftharpoons[\text{id}]{\Sigma^d} \mathcal{C}$.

Definition 3.1.2. Let $\mathcal{C} \in \text{Cat}_{\text{perf}}$ and $d \in \mathbb{N}$. Define the category $\mathcal{C}^{h\mathbb{Z}[d]}$ of objects with self-isomorphism of degree d as the fixed points

$$\mathcal{C}^{h\mathbb{Z}[d]} := \lim(B\mathbb{Z} \xrightarrow{(\mathcal{C}, \Sigma^d)} \text{Cat}_{\text{perf}}) \simeq \lim(\vec{S}^1 \xrightarrow{(\mathcal{C}, \Sigma^d)} \text{Cat}_{\text{perf}}).$$

Equivalently, it is the equalizer of $\mathcal{C} \xrightleftharpoons[\text{id}]{\Sigma^d} \mathcal{C}$.

We summarize the main properties of of this construction from the previous section:

Proposition 3.1.3.

- (1) An object of $\mathcal{C}^{\vec{h}\mathbb{N}[d]}$ is equivalent to the data of (X, v_X) , where $X \in \mathcal{C}$ and $v_X: \Sigma^d X \rightarrow X$. The mapping space between two objects $(X, v_X), (Y, v_Y)$ is given by

$$\mathrm{Map}_{\mathcal{C}^{\vec{h}\mathbb{N}[d]}}((X, v_X), (Y, v_Y)) \simeq \mathrm{eq} \left(\mathrm{Map}_{\mathcal{C}}(X, Y) \begin{array}{c} \xrightarrow{-\circ v_X} \\ \xrightarrow{v_Y \circ \Sigma^d -} \end{array} \mathrm{Map}_{\mathcal{C}}(\Sigma^d X, Y) \right).$$

- (2) The underlying object functor $u: \mathcal{C}^{\vec{h}\mathbb{N}[d]} \rightarrow \mathcal{C}$ (induced naturally by $\mathrm{pt} \rightarrow \vec{S}^1$) is exact, conservative and colimit-preserving.
- (3) A functor $\mathcal{D} \rightarrow \mathcal{C}^{\vec{h}\mathbb{N}[d]}$ is equivalent to the data of an exact functor $F: \mathcal{D} \rightarrow \mathcal{C}$ and a natural transformation $\Sigma^d F \Rightarrow F$.
- (4) There is a natural functor $\mathcal{C}^{h\mathbb{Z}[d]} \rightarrow \mathcal{C}^{\vec{h}\mathbb{N}[d]}$ that forgets that the self-map is an isomorphism.

Proof. This follows by [Corollary 2.3.9](#) (which is just [\[NS18, Proposition II.1.5.\]](#)), [Remark 2.3.4](#), [Remark 2.3.11](#). \square

Proposition 3.1.4. *There exists a natural transformation*

$$\hat{v}: \Sigma^d \Rightarrow \mathrm{id}_{\mathcal{C}^{\vec{h}\mathbb{N}[d]}}$$

of endofunctors of $\mathcal{C}^{\vec{h}\mathbb{N}[d]}$. At an object (X, v_X) it is given by

$$\begin{array}{ccc} \Sigma^{2d} X & \xrightarrow{-\Sigma^d v} & \Sigma^d X \\ \downarrow \Sigma^d v & & \downarrow v \\ \Sigma^d X & \xrightarrow{-v} & X \end{array}$$

Proof. This follows by [Lemma 2.3.12](#) and [Lemma 2.3.13](#). \square

Power maps

We now focus on the interactions between $\mathcal{C}^{\vec{h}\mathbb{N}[d]}$ for different values of d :

Definition 3.1.5. Let $m \in \mathbb{N}$. Consider the m -composition map $\vec{S}^1 = \mathrm{BN} \xrightarrow{m} \mathrm{BN} = \vec{S}^1$ and look at the commutative diagram

$$\begin{array}{ccc} \vec{S}^1 & \xrightarrow{(\mathcal{C}, \Sigma^{md})} & \mathrm{Cat}_{\mathrm{perf}} \\ m \downarrow & \nearrow & \\ \vec{S}^1 & \xrightarrow{(\mathcal{C}, \Sigma^d)} & \end{array} .$$

The functoriality of lax limits induces the “ m -power map”

$$(-)^m: \mathcal{C}^{\vec{h}\mathbb{N}[d]} \rightarrow \mathcal{C}^{\vec{h}\mathbb{N}[md]}.$$

Lemma 3.1.6. $(-)^m$ sends a self-map $v: \Sigma^d X \rightarrow X$ to its m -power

$$v^m: \Sigma^{md} X \xrightarrow{\Sigma^{(m-1)d} v} \Sigma^{(m-1)d} X \xrightarrow{\Sigma^{(m-2)d} v} \dots \xrightarrow{\Sigma^d v} \Sigma^d X \xrightarrow{v} X.$$

Proof. Let (F, γ) be a lax cone over

$$(\mathcal{C}, \Sigma^d): \vec{S}^1 \rightarrow \text{Cat}_{\text{perf}}.$$

Thus, $F: \mathcal{D} \rightarrow \mathcal{C}$ is an exact functor and $\gamma: \Sigma^d F \Rightarrow F$ (Proposition 3.1.3(3)). Composing with $m: \vec{S}^1 \rightarrow \vec{S}^1$ induces the lax cone (F, γ^m) over

$$(\mathcal{C}, \Sigma^{md}): \vec{S}^1 \rightarrow \text{Cat}_{\text{perf}},$$

where

$$\gamma^m: \Sigma^{md} F \xrightarrow{\Sigma^{(m-1)d} \gamma} \Sigma^{(m-1)d} F \xrightarrow{\Sigma^{(m-2)d} \gamma} \dots \xrightarrow{\gamma} F.$$

In particular, the universal lax cone $(u: \mathcal{C}^{\vec{h}\mathbb{N}[d]} \rightarrow \mathcal{C}, v)$ is sent to $(u: \mathcal{C}^{\vec{h}\mathbb{N}[d]} \rightarrow \mathcal{C}, v^m)$. This represents the functor $\mathcal{C}^{\vec{h}\mathbb{N}[d]} \rightarrow \mathcal{C}^{\vec{h}\mathbb{N}[md]}$ sending (X, v_X) to (X, v_X^m) . \square

3.2 Categories with shifted endomorphisms of the identity

We now categorify the construction of Section 3.1, finding a nice category in which $(-)^{\vec{h}\mathbb{N}[d]}$, along with the natural transformation v of Proposition 3.1.4, lands.

We will use Section 2.4 with $\mathcal{V} = \text{Cat}_{\text{perf}}$ and $\gamma = \Sigma^d$. We will also replace Cat_2 by a presentable version for applications in Section 3.4.

The functor Σ^d is an (invertible) endomorphism of each object of Cat_{perf} , or equivalently, an (invertible) endomorphism of $\text{id}_{\text{Cat}_{\text{perf}}}$ in $\widehat{\text{Cat}}_2$. We shall define the category $\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}$ to be the corresponding lax limit along \vec{S}^2 . The objects of $\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}$ are categories $\mathcal{C} \in \text{Cat}_{\text{perf}}$ with a natural transformation $\Sigma^d \Rightarrow \text{id}_{\mathcal{C}}$.

Although it is possible to define $\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}$ directly, using 3-categories significantly simplifies both the construction and its universal property. Therefore, we work in the higher categorical framework to enable more streamlined formal proofs. For better readability, we provide concrete descriptions of the outcomes of the 3-categorical manipulations throughout the text.

Presentability

It will be important to us in Section 3.4 that Cat_{perf} , and our constructed categories, are presentable. Because we work in 2-categories, we would have liked to use a notion of presentable 2-categories. Unfortunately, there is not yet such an accepted notion. One candidate was suggested by Stefanich in [Ste20]. We will take a similar approach, yet simpler to define, as we do not need to require that the functor categories are also presentable. Thus we will define a 3-category \mathbb{Pr} of presentable categories enriched in categories:

By [Hei23], [BM24] there exists a 2-fully faithful embedding

$$\chi: \text{Mod}_{\text{Cat}}(\text{Pr}^{\text{L}}) \hookrightarrow (\widehat{\text{Cat}}_2)^{\text{L}}.$$

from the category of modules over Cat in Pr^{L} to the category of (large) categories enriched in categories, i.e. the category of 2-categories, and left adjoints. We can pull the 3-category structure on $\widehat{\text{Cat}}_2$ to $\text{Mod}_{\text{Cat}}(\text{Pr}^{\text{L}})$:

Definition 3.2.1. Let $\mathbb{P}\text{r}$ be the 3-full subcategory generated by the embedding $\chi: \text{Mod}_{\text{Cat}}(\text{Pr}^{\text{L}}) \hookrightarrow (\widehat{\text{Cat}}_2)^{\text{L}}$.

Notice that as a 2-category, Cat_{perf} is tensored over Cat , thus lies in the 3-category $\mathbb{P}\text{r}$.

Definitions

Definition 3.2.2. A local system $\vec{S}^2 \rightarrow \mathbb{P}\text{r}$ is equivalent to a choice of a presentable 2-category $\mathcal{U} \in \mathbb{P}\text{r}$ and an endomorphism $\gamma: \text{id}_{\mathcal{U}} \Rightarrow \text{id}_{\mathcal{U}}$.

Let $d \in \mathbb{N}$. The d -suspension functor is a natural transformation

$$\Sigma^d: \text{id}_{\text{Cat}_{\text{perf}}} \Rightarrow \text{id}_{\text{Cat}_{\text{perf}}}.$$

Definition 3.2.3. Define the category of stable, idempotent-complete categories with d -shifted natural endomorphism of the identity as the lax limit

$$\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}} := \overrightarrow{\lim}_{\vec{S}^2} (\text{Cat}_{\text{perf}}, \Sigma^d)$$

as discussed in [Section 2.4](#).

Similarly, noting that Σ^d is invertible, define the category $\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-iso}}$ of stable, idempotent-complete categories with d -shifted natural *isomorphism* of the identity as the limit

$$\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-iso}} := \lim_{S^2} (\text{Cat}_{\text{perf}}, \Sigma^d) \simeq \lim_{\vec{S}^2} (\text{Cat}_{\text{perf}}, \Sigma^d).$$

Noticing that limits and lax limits in $\mathbb{P}\text{r}$ are computed in $\widehat{\text{Cat}}_2$, we rewrite the results from [Section 2.4](#) for this case:

Corollary 3.2.4. *An object of $\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}$ is equivalent to the data of (\mathcal{C}, α) where $\mathcal{C} \in \text{Cat}_{\text{perf}}$ and $\alpha: \Sigma^d \Rightarrow \text{id}_{\mathcal{C}}$. A map between two objects $(\mathcal{C}, \alpha) \rightarrow (\mathcal{D}, \beta)$ is an exact functor $f: \mathcal{C} \rightarrow \mathcal{D}$ and a 3-isomorphism*

$$\Sigma^d f = f \Sigma^d \begin{array}{c} \xrightarrow{f\alpha} \\ \Downarrow \wr \\ \xrightarrow{\beta f} \end{array} f .$$

Proof. This is [Lemma 2.4.6](#). □

Corollary 3.2.5. A functor $\mathcal{U} \rightarrow \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}$ is the data of a functor $F: \mathcal{U} \rightarrow \text{Cat}_{\text{perf}}$ along with a 3-morphism

$$\begin{array}{ccc}
 & \xrightarrow{\Sigma^d} & \\
 F & \begin{array}{c} \curvearrowright \\ \Downarrow \eta \\ \curvearrowleft \end{array} & F \\
 & \xrightarrow{\text{id}} &
 \end{array}$$

Proof. This follows by Remark 2.4.3. □

Corollary 3.2.6. The map $\text{pt} \rightarrow \tilde{S}^2$ induces an underlying category functor $U: \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}} \rightarrow \text{Cat}_{\text{perf}}$, sending (\mathcal{C}, α) to \mathcal{C} . There is a natural functor $\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-iso}} \rightarrow \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}$, forgetting that the shifted endomorphism is an isomorphism.

Proof. The first part follows from Lemma 2.4.6, the second is Remark 2.4.7. □

Corollary 3.2.7. The functor $(-)^{\tilde{h}\mathbb{N}[d]}: \text{Cat}_{\text{perf}} \rightarrow \text{Cat}_{\text{perf}}$ admits a lift, which by abuse of notations we denote $(-)^{\tilde{h}\mathbb{N}[d]}: \text{Cat}_{\text{perf}} \rightarrow \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}$ rendering the following diagram commutative:

$$\begin{array}{ccc}
 & & \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}} \\
 & \nearrow^{(-)^{\tilde{h}\mathbb{N}[d]}} & \downarrow U \\
 \text{Cat}_{\text{perf}} & \xrightarrow{(-)^{\tilde{h}\mathbb{N}[d]}} & \text{Cat}_{\text{perf}}
 \end{array}$$

Proof. This is Lemma 2.4.5. □

Proposition 3.2.8. The functor $(-)^{\tilde{h}\mathbb{N}[d]}: \text{Cat}_{\text{perf}} \rightarrow \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}$ is right adjoint to the underlying category functor $U: \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}} \rightarrow \text{Cat}_{\text{perf}}$.

Proof. This is exactly Proposition 2.4.11. □

Power maps

Definition 3.2.9. By the functoriality of lax limits, the commutative diagram

$$\begin{array}{ccc}
 S^2 & \xrightarrow{\text{Cat}_{\text{perf}, \Sigma^{md}}} & \mathbb{P}\mathbb{r} \\
 \downarrow m & \nearrow \text{Cat}_{\text{perf}, \Sigma^d} & \\
 S^2 & &
 \end{array}$$

induces maps

$$(-)^m: \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}} \rightarrow \text{Cat}_{\text{perf}}^{\mathbb{Z}/md\text{-end}}, \quad (-)^m: \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-iso}} \rightarrow \text{Cat}_{\text{perf}}^{\mathbb{Z}/md\text{-iso}}$$

in $\mathbb{P}\mathbb{r}$, which we call the m -power maps.

Lemma 3.2.10. $(-)^m$ sends (\mathcal{C}, α) to (\mathcal{C}, α^m) , where

$$\alpha^m: \Sigma^{md} \xrightarrow{\Sigma^{(m-1)d}\alpha} \Sigma^{(m-1)d} \xrightarrow{\Sigma^{(m-2)d}\alpha} \dots \xrightarrow{\Sigma^d\alpha} \Sigma^d \xrightarrow{\alpha} \text{id}_{\mathcal{C}}.$$

Proof. Recall (Corollary 3.2.5) that lax cones over $(\text{Cat}_{\text{perf}}, \Sigma^d): \vec{S}^2 \rightarrow \mathbb{Pr}$ are the data of $\mathcal{U} \in \mathbb{Pr}$, a map $F: \mathcal{U} \rightarrow \text{Cat}_{\text{perf}}$ in \mathbb{Pr} and a 3-morphism

$$F \begin{array}{c} \xrightarrow{\Sigma^d} \\ \Downarrow \gamma \\ \xrightarrow{\text{id}} \end{array} F.$$

Composition with $m: S^2 \rightarrow S^2$, sends such a lax cone to the lax cone

$$F: \mathcal{U} \rightarrow \text{Cat}_{\text{perf}}, F \begin{array}{c} \xrightarrow{\Sigma^d} \\ \Downarrow \gamma^m \\ \xrightarrow{\text{id}} \end{array} F.$$

In particular, the universal lax cone

$$U: \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}} \rightarrow \text{Cat}_{\text{perf}}, U \begin{array}{c} \xrightarrow{\Sigma^d} \\ \Downarrow \alpha \\ \xrightarrow{\text{id}} \end{array} U$$

is sent to the lax cone

$$U: \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}} \rightarrow \text{Cat}_{\text{perf}}, U \begin{array}{c} \xrightarrow{\Sigma^d} \\ \Downarrow \alpha^m \\ \xrightarrow{\text{id}} \end{array} U$$

which implies that the corresponding functor $\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}} \rightarrow \text{Cat}_{\text{perf}}^{\mathbb{Z}/md\text{-end}}$ sends (\mathcal{C}, α) to (\mathcal{C}, α^m) . \square

Note that as a map in \mathbb{Pr} , $(-)^m: \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}} \rightarrow \text{Cat}_{\text{perf}}^{\mathbb{Z}/md\text{-end}}$ admits a right adjoint

$${}^m\sqrt{(-)}: \text{Cat}_{\text{perf}}^{\mathbb{Z}/md\text{-end}} \rightarrow \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}.$$

Proposition 3.2.11. Let $(\mathcal{C}, \alpha) \in \text{Cat}_{\text{perf}}^{\mathbb{Z}/md\text{-end}}$. Then ${}^m\sqrt{(\mathcal{C}, \alpha)}$ is equivalent to $\mathcal{C}^{\vec{h}\mathbb{N}[d]} \times_{\mathcal{C}^{\vec{h}\mathbb{N}[md]}} \mathcal{C}$ — the category of tuples (X, v, γ) where $X \in \mathcal{C}$, $v: \Sigma^d X \rightarrow X$ and γ is an isomorphism $v^m \xrightarrow{\sim} \alpha_X$.

Proof. Let (\mathcal{C}, α) be a category. Then by Proposition 3.1.3, α defines a functor $\alpha: \mathcal{C} \rightarrow \mathcal{C}^{\vec{h}\mathbb{N}[md]}$ which is a section of u . Using the m -power map $(-)^m: \mathcal{C}^{\vec{h}\mathbb{N}[d]} \rightarrow \mathcal{C}^{\vec{h}\mathbb{N}[md]}$, define

$$R(\mathcal{C}, \alpha) := \mathcal{C}^{\vec{h}\mathbb{N}[d]} \times_{\mathcal{C}^{\vec{h}\mathbb{N}[md]}} \mathcal{C}.$$

Then objects of R are self maps $\Sigma^d X \xrightarrow{v} X$ with an isomorphism $v^m \simeq \alpha_X$. It is clear that R lifts to a functor $R: \text{Cat}_{\text{perf}}^{\mathbb{Z}/md\text{-end}} \rightarrow \text{Cat}_{\text{perf}}$. Notice moreover, that for every (\mathcal{C}, α) , $R(\mathcal{C}, \alpha)$ admits a natural transformation $v: \Sigma^d \Rightarrow \text{id}$ induced by the natural transformation of [Proposition 3.1.4](#). By functoriality this lifts to a 3-map

$$\begin{array}{ccc} & \Sigma^d & \\ & \curvearrowright & \\ R & & R \\ & \Downarrow v & \\ & \curvearrowleft & \\ & \text{id} & \end{array}$$

which is a lax cone over $\vec{S}^2 \xrightarrow{(\text{Cat}_{\text{perf}}, \Sigma^d)} \widehat{\text{Cat}}_2$ and therefore induces a functor

$$R: \text{Cat}_{\text{perf}}^{\mathbb{Z}/md\text{-end}} \rightarrow \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}.$$

We will show it is right adjoint to $(-)^m$.

We will construct the unit as a map

$$u: \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}} \rightarrow (\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}})^{[1]} \simeq (\text{Cat}_{\text{perf}}^{[1]})^{\mathbb{Z}/d\text{-end}},$$

which is equivalent to, by [Corollary 3.2.5](#), to a functor $u: \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}} \rightarrow \text{Cat}_{\text{perf}}^{[1]}$ and a 3-morphism

$$\begin{array}{ccc} & \Sigma^d & \\ & \curvearrowright & \\ u & & u \\ & \Downarrow \eta & \\ & \curvearrowleft & \\ & \text{id} & \end{array}$$

Let $(\mathcal{C}, \alpha) \in \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}$. Recall that $R(\mathcal{C}, \alpha^m) \simeq \mathcal{C}^{\bar{h}\mathbb{N}[d]} \times_{\mathcal{C}^{\bar{h}\mathbb{N}[md]}} \mathcal{C}$, where the functor $\mathcal{C}^{\bar{h}\mathbb{N}[d]} \rightarrow \mathcal{C}^{\bar{h}\mathbb{N}[md]}$ is the m -power map and the functor $\mathcal{C} \rightarrow \mathcal{C}^{\bar{h}\mathbb{N}[md]}$ is α^m . We have a natural functor

$$\begin{aligned} u_{(\mathcal{C}, \alpha)}: \mathcal{C} &\rightarrow \mathcal{C}^{\bar{h}\mathbb{N}[d]} \times_{\mathcal{C}^{\bar{h}\mathbb{N}[md]}} \mathcal{C} \\ X &\mapsto (\Sigma^m X \xrightarrow{\alpha_X} X, X). \end{aligned}$$

More formally, α defines a functor $\alpha: \mathcal{C} \rightarrow \mathcal{C}^{\bar{h}\mathbb{N}[d]}$ and we have a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xlongequal{\quad} & \mathcal{C} \\ \downarrow \alpha & & \downarrow \alpha^m \\ \mathcal{C}^{\bar{h}\mathbb{N}[d]} & \xrightarrow{(-)^m} & \mathcal{C}^{\bar{h}\mathbb{N}[md]} \end{array}$$

which induces a functor $u_{(\mathcal{C}, \alpha)}: \mathcal{C} \rightarrow R(\mathcal{C}, \alpha^m)$. Moreover this construction is clearly functorial in (\mathcal{C}, α) and so gives rise to a functor

$$u: \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}} \rightarrow \text{Cat}_{\text{perf}}^{[1]}.$$

For each such (\mathcal{C}, α) there is a commutative diagram of natural transformations

$$\begin{array}{ccc}
 \mathcal{C} & \begin{array}{c} \xrightarrow{\Sigma^d} \\ \alpha \Downarrow \\ \xrightarrow{\text{id}} \end{array} & \mathcal{C} \\
 \downarrow u_{(\mathcal{C}, \alpha)} & & \downarrow u_{(\mathcal{C}, \alpha)} \\
 R(\mathcal{C}, \alpha^m) & \begin{array}{c} \xrightarrow{\Sigma^d} \\ v \Downarrow \\ \xrightarrow{\text{id}} \end{array} & R(\mathcal{C}, \alpha^m)
 \end{array}$$

which is a 2-morphism $\eta_{(\mathcal{C}, \alpha)}$ in $\text{Cat}_{\text{perf}}^{[1]}$. Again, by functoriality, it lifts to a 3-morphism

$$\begin{array}{ccc}
 & \xrightarrow{\Sigma^d} & \\
 u & \begin{array}{c} \Downarrow \eta \\ \Downarrow \eta \\ \Downarrow \eta \end{array} & u \\
 & \xrightarrow{\text{id}} &
 \end{array}$$

which induces the desired unit functor.

To define the counit we will use similar methods: Let $(\mathcal{C}, \alpha) \in \text{Cat}_{\text{perf}}^{\mathbb{Z}/md\text{-end}}$. Denote the projection functor by

$$c_{(\mathcal{C}, \alpha)}: R(\mathcal{C}, \alpha) \simeq \mathcal{C}^{\vec{h}\mathbb{N}[d]} \times_{\mathcal{C}^{\vec{h}\mathbb{N}[md]}} \mathcal{C} \rightarrow \mathcal{C},$$

which is functorial in (\mathcal{C}, α) and so gives rise to a functor $\text{Cat}_{\text{perf}}^{\mathbb{Z}/md\text{-end}} \rightarrow \text{Cat}_{\text{perf}}^{[1]}$. For each such (\mathcal{C}, α) we are given (as part of the data of $R(\mathcal{C}, \alpha)$) a natural transformations

$$\begin{array}{ccc}
 R(\mathcal{C}, \alpha) & \begin{array}{c} \xrightarrow{\Sigma^{md}} \\ v^m \Downarrow \\ \xrightarrow{\text{id}} \end{array} & R(\mathcal{C}, \alpha) \\
 \downarrow c_{(\mathcal{C}, \alpha)} & & \downarrow c_{(\mathcal{C}, \alpha)} \\
 \mathcal{C} & \begin{array}{c} \xrightarrow{\Sigma^{md}} \\ \alpha \Downarrow \\ \xrightarrow{\text{id}} \end{array} & \mathcal{C}
 \end{array}$$

which, by functoriality, lifts to a 3-morphism

$$\begin{array}{ccc}
 & \xrightarrow{\Sigma^{md}} & \\
 c & \begin{array}{c} \Downarrow \epsilon \\ \Downarrow \epsilon \\ \Downarrow \epsilon \end{array} & c \\
 & \xrightarrow{\text{id}} &
 \end{array}$$

(c, ϵ) is a data of a lax cone so it induces a functor

$$c: \text{Cat}_{\text{perf}}^{\mathbb{Z}/md\text{-end}} \rightarrow (\text{Cat}_{\text{perf}}^{[1]})^{\mathbb{Z}/md\text{-end}} \simeq (\text{Cat}_{\text{perf}}^{\mathbb{Z}/md\text{-end}})^{[1]}$$

which is the counit of the adjunction.

Verifying the zig-zag identities now is simple. □

Corollary 3.2.12. For any $\mathcal{C} \in \text{Cat}_{\text{perf}}$, $\sqrt[m]{(\mathcal{C}^{\overline{hN[md]})} \simeq \mathcal{C}^{\overline{hN[d]}}$.

Proof. The corollary follows by taking right adjoints of the commutative diagram

$$\begin{array}{ccc} \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}} & & \\ (-)^m \downarrow & \searrow U & \\ \text{Cat}_{\text{perf}}^{\mathbb{Z}/md\text{-end}} & \xrightarrow{U} & \text{Cat}_{\text{perf}} \end{array} .$$

□

3.3 Locally nilpotent and invertible natural transformations

The canonical map $\overline{S^2} \rightarrow |S^2| \simeq S^2$ induces a map in \mathbb{Pr}

$$\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-iso}} \rightarrow \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}},$$

forgetting that the natural map is an isomorphism. As a functor in \mathbb{Pr} it admits a right adjoint sending (\mathcal{C}, α) to the subcategory of \mathcal{C} on which α acts invertibly. In this subsection we construct a left adjoint $L: \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}} \rightarrow \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-iso}}$, inverting the natural transformation. L is a Bousfield localization and its kernel is the category of stable, idempotent complete categories with d -shifted, locally-nilpotent natural transformation of the identity. It will actually be simpler to first construct the colocalization functor, constructing L as its kernel.

We then show that the category $\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-iso}}$ has a natural action of the group $\Omega S^2/d$, which is equivalent to an action of $\Omega S^1/d$ on any $(\mathcal{C}, \alpha) \in \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-iso}}$. The identification $d \simeq 0$ is given by α . Moreover, the m -power map of Definition 3.2.9 is equivariant with respect to this action.

Definition 3.3.1. Let $(\mathcal{C}, \alpha) \in \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}$. We say that α is locally-nilpotent if for any $X \in \mathcal{C}$ the map $\alpha_X: \Sigma^d X \rightarrow X$ is nilpotent. That is, there exists $k \in \mathbb{Z}$ such that $\alpha_X^k: \Sigma^{kd} X \rightarrow X$ is null. We denote by $\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-nil}} \subseteq \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}$ the full 2-subcategory of $\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}$ consisting of locally nilpotent natural transformations.

Definition 3.3.2. Define $\text{Nil}: \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}} \rightarrow \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-nil}}$ as the subfunctor of $\text{id}_{\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}}$ sending (\mathcal{C}, α) to the full subcategory of \mathcal{C} consisting of objects on which α acts nilpotently.

We will write $\text{Nil}_\alpha(\mathcal{C}) := \text{Nil}(\mathcal{C}, \alpha)$.

Lemma 3.3.3. $\text{Nil}: \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}} \rightarrow \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-nil}}$ is right adjoint to the inclusion $i: \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-nil}} \rightarrow \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}$.

Proof. Note that $\text{Nil} \circ i \simeq \text{id}_{\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-nil}}}$. The counit $i \circ \text{Nil} \rightarrow \text{id}_{\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}}$ is the natural embedding. It is now simple to verify the zig-zag identities. □

As a consequence, Nil is a colocalization functor. By standard abuse of notations, we will usually denote the composition $\text{Nil} \circ i$ also by Nil .

Definition 3.3.4. Let $L: \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}} \rightarrow \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}$ be the cofiber of the counit map $\text{Nil} \rightarrow \text{id}_{\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}}$ in $\text{Fun}(\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}, \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}})$.

To understand L better, we will use presentable categories, using the fact we can embed Cat_{perf} in $\text{Pr}_{\text{st}}^{\text{L}}$ via the Ind construction.

Definition 3.3.5. Let $\mathcal{C} \in \text{Pr}_{\text{st}}^{\text{L}}$ and $\alpha: \Sigma^d \rightarrow \text{id}_{\mathcal{C}}$. Define $\mathcal{C}[\alpha^{-1}]$ to be the full subcategory consisting of objects on which α is invertible.

Remark 3.3.6. The inclusion functor $\mathcal{C}[\alpha^{-1}] \subset \mathcal{C}$ admits an essentially surjective left adjoint

$$(-)[\alpha^{-1}]: \mathcal{C} \rightarrow \mathcal{C}[\alpha^{-1}]$$

which on an object $X \in \mathcal{C}$ is given by

$$X[\alpha^{-1}] \simeq \text{colim}(X \xrightarrow{\alpha} \Sigma^{-d}X \xrightarrow{\alpha} \Sigma^{-2d}X \xrightarrow{\alpha} \dots).$$

Lemma 3.3.7. L is given on objects by

$$L(\mathcal{C}, \alpha) = ((\text{Ind } \mathcal{C})[\alpha^{-1}])^{\omega}.$$

Proof. Denote by $\text{Pr}_{\text{st}, \omega}^{\text{L}}$ the category of compactly generated stable presentable categories and colimit-preserving functors sending compact objects to compact objects. By [Lur, Lemma 5.3.2.9], there is an equivalence of categories

$$\text{Ind}: \text{Cat}_{\text{perf}} \rightleftarrows \text{Pr}_{\text{st}, \omega}^{\text{L}} : (-)^{\omega}$$

and the inclusion functor $\text{Pr}_{\text{st}, \omega}^{\text{L}} \hookrightarrow \text{Pr}^{\text{L}}$ preserves colimits. In particular

$$L(\mathcal{C}, \alpha) \simeq \text{cofib}(\text{Ind}(\text{Nil}_{\alpha}(\mathcal{C})) \rightarrow \text{Ind}(\mathcal{C}))^{\omega}$$

and the quotient functor $\text{id}_{\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}} \rightarrow L$ is given as the restriction to compacts of

$$\text{Ind}(\mathcal{C}) \rightarrow \text{cofib}(\text{Ind}(\text{Nil}_{\alpha}(\mathcal{C})) \rightarrow \text{Ind}(\mathcal{C}))$$

where the cofiber is computed in $\text{Pr}_{\text{st}}^{\text{L}}$. The cofiber can be computed as

$$\begin{aligned} & \{X \in \text{Ind}(\mathcal{C}) \mid \text{Map}_{\text{Ind}(\mathcal{C})}(Z, X) \simeq \text{pt } \forall Z \in \text{Ind}(\text{Nil}_{\alpha}(\mathcal{C}))\} \\ & \simeq \{X \in \text{Ind}(\mathcal{C}) \mid \text{Map}_{\text{Ind}(\mathcal{C})}(Z, X) \simeq \text{pt } \forall Z \in \text{Nil}_{\alpha}(\mathcal{C})\}. \end{aligned}$$

Any X on which α is invertible satisfies this condition. On the other hand, assume $\text{Map}_{\text{Ind}(\mathcal{C})}(Z, X)$ is contractible for every $Z \in \text{Nil}_{\alpha}(\mathcal{C})$. α^2 acts trivially on $X/\alpha := \text{cofib}(\Sigma^d X \xrightarrow{\alpha} X)$, and in particular $X/\alpha \in \text{Nil}_{\alpha}(\mathcal{C})$ and the cofiber sequence

$$\Sigma X/\alpha \xrightarrow{0} \Sigma^d X \xrightarrow{\alpha} X$$

splits so $X \simeq \Sigma^d X \oplus X/\alpha$. By our assumption the inclusion map $X/\alpha \rightarrow \Sigma^d X \oplus X/\alpha \simeq X$ is null, so $X/\alpha \simeq 0$ and α is invertible on X . \square

Notation 3.3.8. The above lemma justifies denoting $L(\mathcal{C}, \alpha)$ by $\mathcal{C}[\alpha^{-1}]$.

Remark 3.3.9. By Lemma 3.3.7 and Remark 3.3.6, for a category $\mathcal{C} \in \text{Cat}_{\text{perf}}$, the category $\mathcal{C}[\alpha^{-1}] = L(\mathcal{C}, \alpha)$ is equivalent to the idempotent completion of the full subcategory

$$\{X[\alpha^{-1}] \mid X \in \mathcal{C}\} \subseteq \text{Ind}(\mathcal{C}).$$

Corollary 3.3.10. $L: \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}} \rightarrow \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-iso}}$ lands in $\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-iso}}$ and is left adjoint to the inclusion $\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-iso}} \hookrightarrow \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}$.

Proof. By Lemma 3.3.7, L lands in $\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-iso}}$. Let $(\mathcal{C}, \alpha) \in \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}$, $(\mathcal{D}, \beta) \in \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-iso}}$, then a map $F: (\mathcal{C}, \alpha) \rightarrow (\mathcal{D}, \beta)$ must satisfy that the composition

$$(\text{Nil}_\alpha(\mathcal{C}), \alpha) \rightarrow (\mathcal{C}, \alpha) \xrightarrow{F} (\mathcal{D}, \beta)$$

is null, as on any object X , $F\alpha: \Sigma^d X \rightarrow X$ is both invertible and nilpotent. Thus F factors uniquely through $L(\mathcal{C}, \alpha) = \text{cofib}((\text{Nil}_\alpha(\mathcal{C}), \alpha) \rightarrow (\mathcal{C}, \alpha))$. \square

Group action

We next note that $\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-iso}}$ has a residual action by the cofiber of the map $d: \Omega S^2 \rightarrow \Omega S^2$:

Definition 3.3.11. Let S^2/d be the cofiber of the map $d: S^2 \rightarrow S^2$. Note that in the category of \mathbb{E}_1 -groups, the cofiber of the map $d: \Omega S^2 \rightarrow \Omega S^2$ is $\Omega(S^2/d)$.

Lemma 3.3.12. $\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-iso}}$ is naturally an S^2/d -local system, i.e. has a natural action by the group $\Omega S^2/d$.

Proof. Consider the pushout square

$$\begin{array}{ccc} S^2 & \xrightarrow{d} & S^2 \\ \downarrow & \lrcorner & \downarrow \pi_d \\ \text{pt} & \xrightarrow{e} & S^2/d \end{array}$$

and the diagram of local systems

$$\begin{array}{ccc} \mathbb{P}\mathbb{R}^{S^2/d} & \xrightarrow{e^*} & \mathbb{P}\mathbb{R} \\ \uparrow (\pi_d)_* & & \downarrow (S^2)_* \\ \mathbb{P}\mathbb{R}^{S^2} & \xrightarrow{d^*} & \mathbb{P}\mathbb{R}^{S^2} \end{array} \quad \begin{array}{c} \downarrow \pi_d^* \\ \downarrow (S^2)_* \end{array}$$

By Beck–Chevalley, $e^*(\pi_d)_* \simeq S^2_* d^*$. In particular, for $(\text{Cat}_{\text{perf}}, \Sigma) \in \mathbb{P}\mathbb{R}^{S^2}$,

$$e^*(\pi_d)_*(\text{Cat}_{\text{perf}}, \Sigma) \simeq S^2_* d^*(\text{Cat}_{\text{perf}}, \Sigma) = S^2_*(\text{Cat}_{\text{perf}}, \Sigma^d) = \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-iso}}.$$

The functor e^* is the underlying category functor, thus $(\pi_d)_*(\text{Cat}_{\text{perf}}, \Sigma) \in \mathbb{P}\mathbb{R}^{S^2/d}$ is a lift of $\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-iso}}$. \square

Remark 3.3.13. By Remark 3.0.1, a presentation of $\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-iso}}$ as an S^2/d -local system is equivalent to a local system $S^1/d \rightarrow \text{End}(\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-iso}})$ choosing $\text{id}_{\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-iso}}}$. That is, a $\Omega(S^1/d)$ -action on every $(\mathcal{C}, \alpha) \in \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-iso}}$. $\Omega(S^1/d)$ is the free \mathbb{E}_1 -group where $d = 0$. The action is given by the suspension functor, and $\alpha: \Sigma^d \xrightarrow{\sim} \text{id}_{\mathcal{C}}$ gives the required identification.

Recall that we have constructed an m -power map $(-)^m: \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-iso}} \rightarrow \text{Cat}_{\text{perf}}^{\mathbb{Z}/md\text{-iso}}$, and by the above lemma $\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-iso}}$ has a natural $\Omega(S^2/d)$ -action and $\text{Cat}_{\text{perf}}^{\mathbb{Z}/md\text{-iso}}$ has a natural $\Omega(S^2/md)$ -action. We have a homomorphism $q: \Omega(S^2/md) \rightarrow \Omega(S^2/d)$. Carefully reviewing the construction of the action, we can see that the m -power map is equivariant.

Lemma 3.3.14. *The m -power map $(-)^m: \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-iso}} \rightarrow \text{Cat}_{\text{perf}}^{\mathbb{Z}/md\text{-iso}}$ is $\Omega(S^2/md)$ -equivariant.*

Proof. Look at the commutative diagram

$$\begin{array}{ccc} S^2 & \xlongequal{\quad} & S^2 \\ \downarrow \pi_{md} & & \downarrow \pi_d \\ \Sigma\mathbb{B}\mathbb{Z}/md & \xrightarrow{q} & \Sigma\mathbb{B}\mathbb{Z}/d \end{array} ,$$

then the Beck–Chevalley map

$$q^*(-)^{\mathbb{Z}/d\text{-end}} \simeq q^*(\pi_d)_* \rightarrow (\pi_{md})_* \pi_{md}^* q^*(\pi_d)_* \simeq (\pi_{md})_* \pi_d^*(\pi_d)_* \rightarrow (\pi_{md})_* \simeq (-)^{\mathbb{Z}/md\text{-end}}$$

is such a lift. □

3.4 Asymptotically defined shifted endomorphisms

We will now use the constructed power maps (Definition 3.1.5, Definition 3.2.9) and take the colimit of the resulted diagrams, transitioning from a finite quotient group D to a general one. In the decategorified case, the colimit $\text{colim}_{D \rightarrow \mathbb{Z}/d} \mathcal{C}^{\vec{h}\mathbb{N}[d]}$ will be the category of d -self-maps in \mathcal{C} for any d with a surjection $D \twoheadrightarrow \mathbb{Z}/d$. Two such self-maps are identified if they are the same after some power. This category has an “asymptotically defined natural endomorphism of the identity functor” corresponding to the self-map itself.

The categorified colimit $\text{colim}_d \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}$ will be the category of categories with such asymptotically defined natural endomorphism of the identity. In particular $\text{colim}_d \mathcal{C}^{\vec{h}\mathbb{N}[d]}$ enhances to an object in there and the functor sending \mathcal{C} to $\text{colim}_d \mathcal{C}^{\vec{h}\mathbb{N}[d]}$ is right adjoint to the underlying category functor. Restricting to categories with asymptotically defined natural isomorphisms, we extend the results of Section 3.3 and get an action of the group $\Omega^2 \lim_d(S^2/d)$ on this category.

Categories of asymptotically defined self maps

Fix a quotient group D of $\widehat{\mathbb{Z}}$.

Definition 3.4.1. Let $\mathcal{C} \in \text{Cat}_{\text{perf}}$. Define the category of D -asymptotically defined self-maps $\mathcal{C}^{\vec{h}\mathbb{N}[D]}$ as the colimit in Cat_{perf} of $\mathcal{C}^{\vec{h}\mathbb{N}[d]}$ along the m -power map:

$$\mathcal{C}^{\vec{h}\mathbb{N}[D]} := \text{colim}_{D \rightarrow \mathbb{Z}/d} \mathcal{C}^{\vec{h}\mathbb{N}[d]}.$$

The underlying object functor $u: \mathcal{C}^{\vec{h}\mathbb{N}[d]} \rightarrow \mathcal{C}$ is compatible with this diagram

$$u((X, v_X)^m) \simeq u(X, v_X)$$

and so defines an underlying object functor $u: \mathcal{C}^{\vec{h}\mathbb{N}[D]} \rightarrow \mathcal{C}$.

Remark 3.4.2. Write D as a sequential limit

$$D = \lim_i \mathbb{Z}/d_i$$

of surjections, (in particular d_i divides d_{i+1}). By cofinality, $\mathcal{C}^{\vec{h}\mathbb{N}[D]}$ identifies with the sequential colimit

$$\mathcal{C}^{\vec{h}\mathbb{N}[D]} \simeq \text{colim}(\mathcal{C}^{\vec{h}\mathbb{N}[d_1]} \xrightarrow{(-)^{d_2/d_1}} \mathcal{C}^{\vec{h}\mathbb{N}[d_2]} \xrightarrow{(-)^{d_3/d_2}} \dots).$$

Remark 3.4.3. Using [Proposition 3.1.3](#), an object of $\mathcal{C}^{\vec{h}\mathbb{N}[D]}$ is equivalent to the data of $X \in \mathcal{C}$ and $v: \Sigma^d X \rightarrow X$ for some $D \rightarrow \mathbb{Z}/d$. Two such objects $(X, \Sigma^d X \xrightarrow{v} X)$ and $(X, \Sigma^{d'} X \xrightarrow{v'} X)$ are identified if they are asymptotically the same: there exists c, c' such that $cd = c'd' \in D$ and $v^c \simeq (v')^{c'}$.

Categories of asymptotically defined self maps

Definition 3.4.4. Define the category of *stable, idempotent complete categories with D -asymptotically defined shifted endomorphisms of the identity* as the colimit in $\mathbb{P}\text{r}$ of $\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}$ along the m -power maps:

$$\text{Cat}_{\text{perf}}^{D\text{-end}} := \text{colim}_{D \rightarrow \mathbb{Z}/d} \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}.$$

Similarly, define the category of *stable, idempotent complete categories with D -asymptotically defined nilpotent endomorphisms and D -asymptotically defined isomorphisms* as

$$\begin{aligned} \text{Cat}_{\text{perf}}^{D\text{-iso}} &:= \text{colim}_{D \rightarrow \mathbb{Z}/d} \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-iso}} \\ \text{Cat}_{\text{perf}}^{D\text{-nil}} &:= \text{colim}_{D \rightarrow \mathbb{Z}/d} \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-nil}} \end{aligned}$$

Remark 3.4.5. As in [Remark 3.4.2](#), $\text{Cat}_{\text{perf}}^{D\text{-end}}$, $\text{Cat}_{\text{perf}}^{D\text{-iso}}$, $\text{Cat}_{\text{perf}}^{D\text{-nil}}$ can be defined as sequential colimits.

Example 3.4.6. If $D = \mathbb{Z}/d$ is finite then $\text{Cat}_{\text{perf}}^{D\text{-end}} = \text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-end}}$.

Remark 3.4.7. Write D as a sequential limit of surjections

$$D = \lim_i \mathbb{Z}/d_i.$$

The colimit in \mathbb{Pr} is computed as the colimit in \mathbb{Pr}^L , which in turn is computed as the limit of the right adjoints. That is

$$\mathrm{Cat}_{\mathrm{perf}}^{D\text{-end}} \simeq \lim(\mathrm{Cat}_{\mathrm{perf}}^{\mathbb{Z}/d_1\text{-end}} \xleftarrow{d_2/d_1\sqrt{(-)}} \mathrm{Cat}_{\mathrm{perf}}^{\mathbb{Z}/d_2\text{-end}} \xleftarrow{d_3/d_2\sqrt{(-)}} \dots).$$

Therefore, an object in $\mathrm{Cat}_{\mathrm{perf}}^{D\text{-end}}$ is a sequence of categories \mathcal{C}_i , natural transformations $\alpha_i: \Sigma^{d_i} \rightarrow \mathrm{id}_{\mathcal{C}_i}$ and identifications $(\mathcal{C}_i, \alpha_i) \simeq {}^{d_{i+1}/d_i}\sqrt{(\mathcal{C}_{i+1}, \alpha_{i+1})}$.

Using the power-root adjunction we have a series of maps $(\mathcal{C}_i, \alpha_i) \xrightarrow{\bar{h}\mathbb{N}[d_{i+1}/d_i]} (\mathcal{C}_{i+1}, \alpha_{i+1})$, and by the above condition, the image is exactly all elements X such that $\alpha_{i+1}: \Sigma^{d_{i+1}} X \rightarrow X$ admits a root.

Definition 3.4.8. As the underlying functor category $U: \mathrm{Cat}_{\mathrm{perf}}^{\mathbb{Z}/d\text{-end}} \rightarrow \mathrm{Cat}_{\mathrm{perf}}$ is a left adjoint and commutes with the power map, it induces an underlying category functor $U: \mathrm{Cat}_{\mathrm{perf}}^{D\text{-end}} \rightarrow \mathrm{Cat}_{\mathrm{perf}}$ in \mathbb{Pr}^L . We will denote its right adjoint $R: \mathrm{Cat}_{\mathrm{perf}} \rightarrow \mathrm{Cat}_{\mathrm{perf}}^{D\text{-end}}$.

Lemma 3.4.9. *Using the notations of Remark 3.4.7, U sends the system $(\mathcal{C}_i, \alpha_i)$ to the colimit $\mathrm{colim}_i \mathcal{C}_i$ (induced by the maps $(\mathcal{C}_i, \alpha_i) \xrightarrow{d_{i+1}/d_i} (\mathcal{C}_{i+1}, \alpha_{i+1})$) and R sends \mathcal{C} to the system $(\mathcal{C}^{\bar{h}\mathbb{N}[d_i]}, v)$. In particular $U \circ R \simeq (-)^{\bar{h}\mathbb{N}[D]}$.*

Proof. R is defined as the limit of the functors $(-)^{\bar{h}\mathbb{N}[d_i]}: \mathrm{Cat}_{\mathrm{perf}} \rightarrow \mathrm{Cat}_{\mathrm{perf}}^{\mathbb{Z}/d_i\text{-end}}$ and so the formula follows. U is defined as the left adjoint of R . Using (the $(-)^{\mathrm{op}}$ version of) [HY17, Corollary 1.3], we know that U , as a left adjoint to a limit of functors, is the colimit of the left adjoints of the functors, i.e.

$$U((\mathcal{C}_i, \alpha_i)) = \mathrm{colim}_i U(\mathcal{C}_i, \alpha_i) = \mathrm{colim}_i \mathcal{C}_i.$$

□

Justified by Lemma 3.4.9, we will continue the abuse of notations of Section 3.2 and denote R by $(-)^{\bar{h}\mathbb{N}[D]}$.

Remark 3.4.10. Using Remark 3.4.7 together with Lemma 3.4.9, we will think of an object in $\mathrm{Cat}_{\mathrm{perf}}^{D\text{-end}}$ as the category $\mathcal{C} = \mathrm{colim}_i \mathcal{C}_i$ together with an “asymptotically defined natural transformation α ”, i.e. an exhausting filtration $\mathcal{C}_1 \rightarrow \mathcal{C}_2 \rightarrow \dots \rightarrow \mathcal{C}$ and natural transformations $\alpha_i: \Sigma^{d_i} \rightarrow \mathrm{id}_{\mathcal{C}_i}$ with the obvious compatibility conditions.

Repeating the argument of Lemma 3.4.9 we deduce the following:

Proposition 3.4.11. *The embedding $\mathrm{Cat}_{\mathrm{perf}}^{D\text{-iso}} \hookrightarrow \mathrm{Cat}_{\mathrm{perf}}^{D\text{-end}}$ admits a left adjoint L sending (\mathcal{C}, α) to $\mathcal{C}[\alpha^{-1}]$ (sending each $(\mathcal{C}_i, \alpha_i)$ to $(\mathcal{C}_i[\alpha_i^{-1}], \alpha_i)$). Moreover L is a Bousfield localization with kernel $\mathrm{Cat}_{\mathrm{perf}}^{D\text{-nil}}$ and the colocalization functor is $\mathrm{Nil}: \mathrm{Cat}_{\mathrm{perf}}^{D\text{-end}} \rightarrow \mathrm{Cat}_{\mathrm{perf}}^{D\text{-nil}}$ sending (\mathcal{C}, α) to $\mathrm{Nil}_\alpha(\mathcal{C})$.*

Recall that $\text{Cat}_{\text{perf}}^{\mathbb{Z}/d\text{-iso}}$ admits a natural $\Omega(S^2/d)$ -action (Lemma 3.3.12) and that the m -power map is $\Omega(S^2/md)$ -equivariant (Lemma 3.3.14). We deduce immediately:

Corollary 3.4.12. *The category $\text{Cat}_{\text{perf}}^{D\text{-iso}}$ admits an action of the group ΩS_D^2 , where*

$$S_D^2 := \lim_{D \rightarrow \mathbb{Z}/d} S^2/d.$$

Lemma 3.4.13. $\pi_2(S_D^2) \cong D$.

Proof. Write D as a sequential limit of surjections.

$$D = \lim_i \mathbb{Z}/d_i.$$

By cofinality $S_D^2 = \lim_i S^2/d_i$. Milnor's exact sequence for homotopy groups (see e.g. [BK72, Chapter IX, Theorem 3.1]), says there is an exact sequence

$$0 \rightarrow \lim_i^1 \pi_3(S^2/d_i) \rightarrow \pi_2(\lim_i S^2/d_i) \rightarrow \lim_i \pi_2(S^2/d_i) \rightarrow 0.$$

S^2/d_i is simply connected and has finite homology groups thus has finite homotopy groups. In particular, by the Mittag-Leffler criterion (see e.g. [Swi75, Theorem 7.75]), $\lim_i^1 \pi_3(S^2/d_i) = 0$. Thus

$$\pi_2(\lim_i S^2/d_i) \cong \lim_i \pi_2(S^2/d_i) = \lim_i \mathbb{Z}/d_i = D$$

as needed. □

Corollary 3.4.14. *There is an action of $\Omega^2 S_D^2$ on $\text{id}_{\text{Cat}_{\text{perf}}^{D\text{-iso}}}$ which induces, for any $(\mathcal{C}, \alpha) \in \text{Cat}_{\text{perf}}^{D\text{-iso}}$, a group homomorphism $\Sigma(-): D \rightarrow \pi_0(\text{Aut}(\mathcal{C}))$ lifting the suspension.*

Proof. There is such a group action by Corollary 3.4.12 and Remark 3.0.2. Therefore there is a group homomorphism

$$\pi_2(S_D^2) = \pi_0(\Omega^2 S_D^2) \rightarrow \pi_0(\text{Aut}(\mathcal{C})).$$

By Lemma 3.4.13 $\pi_2(S_D^2) \cong D$. By Remark 3.3.13, $1 \in D$ acts by Σ . □

4 The periodicity theorem

The main goal of this section is constructing a group homomorphism

$$\mathcal{Z}_n := \lim_k \mathbb{Z}/p^k |v_n| \rightarrow \pi_0 \text{Aut}(\text{Sp}_{T(n)}^\omega).$$

Our main tool will be Devinatz and Smith's classical periodicity theorem. We will show that it implies the existence of a category in $\text{Cat}_{\text{perf}}^{\mathcal{Z}_n\text{-iso}}$ with underlying category $\text{Sp}_{T(n)}^\omega$ (Corollary 4.2.7). Then, using Corollary 3.4.14, we get the required group homomorphism.

4.1 Telescopic localizations

We start with some preliminary well known introduction to the telescopic categories in chromatic homotopy theory. All results here are well known, but for some we did not find a proof in the literature. For a more thorough introduction to chromatic homotopy theory we refer the reader to [BB20]. For proofs of these facts for the $K(n)$ -local category we refer to [HS19].

Fix a prime p . Let $K(n)$ be the n -th Morava K -theory associated to a formal group law of height n over \mathbb{F}_p . It has homotopy groups given by

$$\pi_* K(n) \cong \mathbb{F}_p[v_n^{\pm 1}]$$

where v_n is of degree $|v_n| = 2(p^n - 1)$. It is also common to define $K(\infty) = \mathbb{F}_p$, $K(0) = \mathbb{Q}$, $K(-1) = 0$.

Definition 4.1.1. Let $\mathrm{Sp}_{\geq n}^\omega \subseteq \mathrm{Sp}_{(p)}^\omega$ be the thick subcategory of $K(n-1)$ -acyclics. That is

$$\mathrm{Sp}_{\geq n}^\omega = \{X \in \mathrm{Sp}_{(p)}^\omega \mid K(n-1) \otimes X \simeq 0\}$$

A spectrum in $\mathrm{Sp}_{\geq n}^\omega$ is said to be of type $\geq n$.

Ravenel proved in [Rav84, theorem 2.11] that $\mathrm{Sp}_{\geq n+1}^\omega \subseteq \mathrm{Sp}_{\geq n}^\omega$, Mitchell proved in [Mit85, Theorem B] that these inclusions are strict and Hopkins and Smith proved that these are all thick subcategories of $\mathrm{Sp}_{(p)}^\omega$ [HS98, Theorem 7].

Theorem 4.1.2 (Thick subcategory theorem). The thick subcategories of $\mathrm{Sp}_{(p)}^\omega$ are exactly $\mathrm{Sp}_{\geq n}^\omega$ and they assemble into a strictly ascending filtraion

$$\{0\} = \mathrm{Sp}_{\geq \infty}^\omega \subsetneq \cdots \subsetneq \mathrm{Sp}_{\geq 2}^\omega \subsetneq \mathrm{Sp}_{\geq 1}^\omega \subsetneq \mathrm{Sp}_{\geq 0}^\omega = \mathrm{Sp}_{(p)}^\omega.$$

Definition 4.1.3 ([HS98]). Let X be a p -local spectrum. A map $v: \Sigma^d X \rightarrow X$ is said to be a v_n -self-map if

- (1) $\mathrm{id}_{K(n)} \otimes v: K(n) \otimes \Sigma^d X \rightarrow K(n) \otimes X$ is an isomorphism.
- (2) $\mathrm{id}_{K(m)} \otimes v: K(m) \otimes \Sigma^d X \rightarrow K(m) \otimes X$ is nilpotent for $m \neq n$.

Hopkins and Smith's periodicity theorem states that any spectrum of type $\geq n$ admits an asymptotically unique v_n -self-map:

Theorem 4.1.4 (Periodicity theorem [HS98, Theorem 9, Corollary 3.7]). A p -local compact spectrum X admits a v_n -self-map if and only if $X \in \mathrm{Sp}_{\geq n}^\omega$. If f, g are two v_n -self-maps of X then there exist integers i, j such that $f^i \simeq g^j$. Any spectrum admitting a v_n -self-map admits also a v_n -self-map of degree $p^k |v_n| = 2p^k(p^n - 1)$ for some k .

Proposition 4.1.5 ([HS98, Corollary 3.8]). Let $(X, v), (Y, w)$ be two p -local spectra with a v_n -self-map of degrees d, e respectively. Let $f: X \rightarrow Y$ be a morphism. Then there exists i, j such that $di = ej$ and the following diagram commutes

$$\begin{array}{ccc} \Sigma^{di} X & \xrightarrow{\Sigma^{di} f} & \Sigma^{ej} Y \\ \downarrow v^i & & \downarrow w^j \\ X & \xrightarrow{f} & Y \end{array}.$$

Remark 4.1.6. We work under the convention that the v_0 -self-maps are multiplication by powers of p .

Section 4.2 is devoted to the reformulation of the periodicity theorem as an equivalence of categories using the language of Section 3.

Let $n < \infty$. We let $F(n) \in \mathrm{Sp}_{\geq n}^\omega \setminus \mathrm{Sp}_{\geq n+1}^\omega$ be any compact spectrum of type n . By the periodicity theorem there exists a v_n -self-map $v: \Sigma^d F(n) \rightarrow F(n)$. Its corresponding height n telescope is defined as $T(n) := F(n)[v^{-1}]$.

Now let $\mathrm{Sp}_{\geq n} \subseteq \mathrm{Sp}_{(p)}$ be the subcategory generated by $\mathrm{Sp}_{\geq n}^\omega$ (or equivalently by $F(n)$) under colimits. Denote by $L_n^f \mathrm{Sp}$ the Verdier quotient of $\mathrm{Sp}_{\geq n+1} \rightarrow \mathrm{Sp}_{(p)}$. This category is equivalent to the category of $(T(0) \oplus \cdots \oplus T(n))$ -local spectra. Denote the corresponding quotient functor by

$$L_n^f: \mathrm{Sp}_{(p)} \rightarrow L_n^f \mathrm{Sp}$$

which is a smashing localization.

The category $\mathrm{Sp}_{T(n)}$ of $T(n)$ -local spectra is identified with the Verdier quotient of $\mathrm{Sp}_{\geq n+1} \rightarrow \mathrm{Sp}_{\geq n}$. The quotient map is given by the restricted localization

$$L_{T(n)} = L_n^f \upharpoonright_{\mathrm{Sp}_{\geq n}}: \mathrm{Sp}_{\geq n} \rightarrow \mathrm{Sp}_{T(n)}.$$

The category of $T(n)$ -local spectra inherits a symmetric monoidal structure given by the $T(n)$ -localized tensor product of spectra. Its unit is the $T(n)$ -local sphere $\mathbb{S}_{T(n)} := L_{T(n)} \mathbb{S}$.

A related category is the monochromatic category $M_n^f \mathrm{Sp} = \ker(L_n^f \mathrm{Sp} \xrightarrow{L_{n-1}^f} L_{n-1}^f \mathrm{Sp})$. The inclusion $M_n^f \mathrm{Sp} \rightarrow L_n^f \mathrm{Sp}$ admits a right adjoint $M_n^f: L_n^f \mathrm{Sp} \rightarrow M_n^f \mathrm{Sp}$. If $X \in \mathrm{Sp}_{\geq n}$, then $L_{n-1}^f(L_n^f X) \simeq L_{n-1}^f X \simeq 0$, therefore the functor $L_n^f: \mathrm{Sp}_{\geq n} \rightarrow L_n^f \mathrm{Sp}$ factors through $M_n^f \mathrm{Sp}$.

We will use the following simple fact:

Proposition 4.1.7. *The category $\mathrm{Sp}_{\geq n}$ is compactly generated and its compacts are exactly $\mathrm{Sp}_{\geq n}^\omega$.*

Proof. As $\mathrm{Sp}_{\geq n}$ is closed under colimits in spectra, it is clear that any compact spectrum of type $\geq n$ is compact in $\mathrm{Sp}_{\geq n}$. As $\mathrm{Sp}_{\geq n}^\omega$ is closed under finite colimits, $\mathrm{Sp}_{\geq n}$ is generated from $\mathrm{Sp}_{\geq n}^\omega$ under filtered colimits. Now let $X \in \mathrm{Sp}_{\geq n}$ be compact. By the previous part, X can be written as a filtered colimit of compact spectra of type $\geq n$:

$$X \simeq \operatorname{colim}_i X_i.$$

Then, as X is compact, the identity map

$$X \xrightarrow{\operatorname{id}} X \simeq \operatorname{colim}_i X_i$$

factors through X_i for some i , so X is a retract of $X_i \in \mathrm{Sp}_{\geq n}^\omega$ whence in $\mathrm{Sp}_{\geq n}^\omega$. \square

Corollary 4.1.8. *The monochromatic category $M_n^f \mathrm{Sp}$ is compactly generated. Its compact objects are retracts of $L_n^f X$ for $X \in \mathrm{Sp}_{\geq n}^\omega$.*

Proof. Restricted to $\mathrm{Sp}_{\geq n}$, the functors M_n^f, L_n^f coincide. As L_n^f is smashing, the right adjoint $M_n^f \mathrm{Sp} \hookrightarrow \mathrm{Sp}_{\geq n}$ preserves colimits. Therefore $L_n^f: \mathrm{Sp}_{\geq n} \rightarrow M_n^f \mathrm{Sp}$ preserves compact objects. Thus any L_n^f -localization of a compact type $\geq n$ spectrum is compact in $M_n^f \mathrm{Sp}$, and whence so are their retracts.

For the other direction, let $X \in M_n^f \mathrm{Sp} \subseteq L_n^f \mathrm{Sp}$ be compact. Write X as a filtered colimit $X \simeq \mathrm{colim}_i L_n^f X_i$ for $X_i \in \mathrm{Sp}_{\geq n}^\omega$. Then, as X is compact, the identity map

$$X \xrightarrow{\mathrm{id}} X \simeq \mathrm{colim}_i L_n^f X_i$$

factors through $L_n^f X_i$ for some i , so X is a retract of $L_n^f X_i$. This also shows that $M_n^f \mathrm{Sp}$ is compactly generated. \square

The functors $L_{T(n)}: M_n^f \mathrm{Sp} \rightleftarrows \mathrm{Sp}_{T(n)}: M_n^f$ are inverses, thus we deduce:

Corollary 4.1.9. *The category $\mathrm{Sp}_{T(n)}$ of $T(n)$ -local spectra is compactly generated. Its compact objects are exactly retracts of $T(n)$ -localization of compact type $\geq n$ spectra.*

Corollary 4.1.10. *Any compact $T(n)$ -local spectrum is dualizable.*

Proof. The $T(n)$ -localization functor is symmetric monoidal, thus sends dualizable objects to dualizable objects. Any spectrum $X \in \mathrm{Sp}_{\geq n}^\omega \subseteq \mathrm{Sp}_{(p)}^\omega$ is dualizable and therefore so is its $T(n)$ -localization. The claim follows since dualizability is preserved under retracts. \square

4.2 The periodicity theorem rephrased

We now specialize to the category of compact spectra of type $\geq n$, showing the periodicity theorem is equivalent to the existence of a lift of $\mathrm{Sp}_{\geq n}^\omega$ in $\mathrm{Cat}_{\mathrm{perf}}^{\mathbb{Z}_n\text{-end}}$, corresponding to the asymptotically defined v_n -self-map. We will then show that the v_n -nilpotents gives a lift of $\mathrm{Sp}_{\geq n+1}^\omega$ and inverting v_n lifts the category of compact $T(n)$ -local spectra.

Definition 4.2.1. We say that $(X, v_X) \in (\mathrm{Sp}_{(p)}^\omega)^{\bar{h}\mathbb{N}[d]}$ is a v_n -self-map if v_X is a v_n -self-map in the sense of [Definition 4.1.3](#). We define $\mathrm{Sp}_{v_n, k}^\omega \subseteq (\mathrm{Sp}_{\geq n}^\omega)^{\bar{h}\mathbb{N}[p^k|v_n]}$ to be the full subcategory consisting of v_n -self-maps. By an abuse of notation we will denote the natural transformation of [Proposition 3.1.4](#) for this category by $v_n: \Sigma^{p^k|v_n} \rightarrow \mathrm{id}$.

Remark 4.2.2. By [Theorem 4.1.4](#) we could have defined $\mathrm{Sp}_{v_n, k}^\omega$ as the full subcategory of $(\mathrm{Sp}_{(p)}^\omega)^{\bar{h}\mathbb{N}[p^k|v_n]}$ consisting of v_n -self-maps.

Lemma 4.2.3. *The sequence $(\mathrm{Sp}_{v_n, k}^\omega, v_n)$ lifts to a compatible system $(\mathrm{Sp}_{v_n}^\omega, v_n) \in \mathrm{Cat}_{\mathrm{perf}}^{\mathbb{Z}_n\text{-end}}$.*

Proof. We need to provide isomorphisms $\sqrt[p]{\mathrm{Sp}_{v_n, k+1}^\omega} \rightarrow \mathrm{Sp}_{v_n, k}^\omega$. Recall the isomorphism

$$F: \sqrt[p]{(\mathrm{Sp}_{\geq n}^\omega)^{\bar{h}\mathbb{N}[p^{k+1}|v_n]}} \xrightarrow{\sim} (\mathrm{Sp}_{\geq n}^\omega)^{\bar{h}\mathbb{N}[p^k|v_n]}$$

of [Corollary 3.2.12](#). Recall that $\sqrt[p]{(\mathrm{Sp}_{\geq n}^\omega)^{\widehat{\mathbb{N}}[p^{k+1}|v_n|]}}$ consists of tuples $((X, v), u)$ of elements $(X, v) \in (\mathrm{Sp}_{\geq n}^\omega)^{\widehat{\mathbb{N}}[p^{k+1}|v_n|]}$ and a self-map u of it of degree $p^k|v_n|$, i.e. squares of the form

$$\begin{array}{ccc} \Sigma^{(p^k|v_n|+p^{k+1}|v_n|)} X & \xrightarrow{\Sigma^{p^k|v_n|} u} & \Sigma^{p^{k+1}|v_n|} X \\ \downarrow \Sigma^{p^{k+1}|v_n|} v & & \downarrow v \\ \Sigma^{p^k|v_n|} X & \xrightarrow{u} & X \end{array}$$

with an identification $u^p \simeq v$. The functor F then sends such a square to (X, u) . As $u^p \simeq v$ is a v_n -self-map, so is u . Thus, restricting F to $\sqrt[p]{\mathrm{Sp}_{v_n, k+1}^\omega}$ we get a fully faithful functor

$$F: \sqrt[p]{\mathrm{Sp}_{v_n, k+1}^\omega} \hookrightarrow \mathrm{Sp}_{v_n, k}^\omega.$$

It is left verifying that F is essentially surjective. Let $(\Sigma^k X \xrightarrow{u} X) \in \mathrm{Sp}_{v_n, k}^\omega$, then u is the image of

$$\begin{array}{ccc} \Sigma^{p^k|v_n|+p^{k+1}|v_n|} \Sigma^{p^k|v_n|} u \Sigma^{p^{k+1}|v_n|} X & & \\ \downarrow \Sigma^{p^{k+1}|v_n|} u^p & \downarrow u^p & \in \sqrt[p]{\mathrm{Sp}_{v_n, k+1}^\omega} \\ \Sigma^{p^k|v_n|} X & \xrightarrow{u} & X \end{array}$$

under F . □

Remark 4.2.4. We could have defined $\mathrm{Sp}_{v_n}^\omega$ living in $\mathrm{Cat}_{\mathrm{perf}}^{\widehat{\mathbb{Z}}\text{-end}}$, as the category of v_n -self-maps of arbitrary degree, and not necessarily p -typical. By [Theorem 4.1.4](#), the underlying categories in both situations identify.

Theorem 4.2.5. The composition

$$u: U(\mathrm{Sp}_{v_n, k}^\omega) \hookrightarrow U((\mathrm{Sp}_{\geq n}^\omega)^{\widehat{\mathbb{Z}}\text{-end}}) \xrightarrow{u} \mathrm{Sp}_{\geq n}^\omega$$

is an isomorphism, where u is the underlying spectrum functor as in [Proposition 3.1.3\(2\)](#).

To prove this theorem we will need the following technical lemma, which proves that a full, conservative exact functor between stable categories is already fully faithful.

Lemma 4.2.6. *Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be an exact functor between stable categories. Assume that F is full³ and conservative. Then F is fully faithful.*

Proof. It suffices to prove that for all $X, Y \in \mathcal{C}$,

$$F: \mathrm{hom}_{\mathcal{C}}(X, Y) \rightarrow \mathrm{hom}_{\mathcal{D}}(FX, FY)$$

³A functor F is full if it is surjective on path components of mapping space, i.e. for every $X, Y \in \mathcal{C}$ the map $\pi_0 F: \pi_0 \mathrm{Map}_{\mathcal{C}}(X, Y) \rightarrow \pi_0 \mathrm{Map}_{\mathcal{D}}(FX, FY)$ is surjective.

is an isomorphism. Equivalently

$$\pi_m F: \pi_m \operatorname{hom}_{\mathcal{C}}(X, Y) \rightarrow \pi_m \operatorname{hom}_{\mathcal{D}}(FX, FY)$$

is an isomorphism for all m . By exactness, $\pi_m F$ is given by

$$\begin{aligned} \pi_m \operatorname{hom}_{\mathcal{C}}(X, Y) &= \pi_0 \operatorname{hom}_{\mathcal{C}}(\Sigma^m X, Y) \xrightarrow{\pi_0 F} \pi_0 \operatorname{hom}_{\mathcal{D}}(F\Sigma^m X, FY) \\ &\simeq \pi_0 \operatorname{hom}_{\mathcal{D}}(\Sigma^m FX, FY) = \pi_m \operatorname{hom}_{\mathcal{D}}(FX, FY) \end{aligned}$$

Thus it is enough to prove that $\pi_0 F$ is an isomorphism of groups. As F is full,

$$\pi_0 F: \pi_0 \operatorname{hom}_{\mathcal{C}}(X, Y) = \pi_0 \operatorname{Map}_{\mathcal{C}}(X, Y) \rightarrow \pi_0 \operatorname{Map}_{\mathcal{D}}(FX, FY) = \pi_0 \operatorname{hom}_{\mathcal{D}}(FX, FY)$$

is surjective, so it is left to prove that its kernel is trivial, i.e. for any $f: X \rightarrow Y$, if $Ff \simeq 0$ then $f \simeq 0$. Let $f: X \rightarrow Y$ with $Ff \simeq 0$. Since $Ff \simeq 0$, the exact sequence

$$FY \rightarrow \operatorname{cofib}(Ff) \xrightarrow{Fg} \Sigma FX \simeq F\Sigma X$$

splits. Let $\bar{s}: F\Sigma X \rightarrow \operatorname{cofib}(Ff) \simeq F \operatorname{cofib}(f)$ be a splitting and $s: \Sigma X \rightarrow \operatorname{cofib}(f)$ a lifting, which exists as F is full. Denote by $g: \operatorname{cofib}(f) \rightarrow \Sigma X$ the associated map.

$$F(g \circ s) = Fg \circ \bar{s} \simeq \operatorname{id}$$

and as F is conservative, $g \circ s$ is an isomorphism. Choose $s' := s \circ (g \circ s)^{-1}$. Then

$$g \circ s' = g \circ s \circ (g \circ s)^{-1} \simeq \operatorname{id}.$$

So $Y \rightarrow \operatorname{cofib}(f) \rightarrow \Sigma X$ splits which proves $f \simeq 0$. □

Proof of Theorem 4.2.5. As u is an exact functor of stable categories, by Lemma 4.2.6 it is enough to prove u is full, conservative and essentially surjective.

Conservativity follows from Proposition 3.1.3 and as conservative functors are closed under sequential colimits. The periodicity theorem — Theorem 4.1.4 proves that u is essentially surjective.

Let $f: X \rightarrow Y$ be a map between compact spectra of type $\geq n$. As u is essentially surjective, there exist some liftings $(X, v_X), (Y, v_Y) \in \operatorname{Sp}_{v_n}^{\omega}$. Replacing the maps by large enough powers of them, we may assume $(X, v_X), (Y, v_Y) \in \operatorname{Sp}_{v_n, k}^{\omega}$ for some k . By Proposition 3.1.3

$$\pi_0 \operatorname{Map}_{\operatorname{Sp}_{v_n, k}^{\omega}}((X, v_X), (Y, v_Y)) = \{(g, H) \mid g: X \rightarrow Y, H: g \circ v_X \simeq v_Y \circ g\}$$

so to show u is full it is enough to show that (maybe for larger k) f commutes with v up to homotopy. This is exactly Proposition 4.1.5. □

Corollary 4.2.7. *The isomorphism u of Theorem 4.2.5 induces isomorphisms of categories*

$$U(\operatorname{Nil}_{v_n}(\operatorname{Sp}_{v_n}^{\omega})) \simeq \operatorname{Sp}_{\geq n+1}^{\omega}, \quad U(\operatorname{Sp}_{v_n}^{\omega}[v_n^{-1}]) \simeq \operatorname{Sp}_{T(n)}^{\omega}.$$

Proof. Recall (Proposition 3.2.8) that U is a left adjoint, therefore

$$U(\mathrm{Nil}_{v_n}(\mathrm{Sp}_{v_n}^\omega)) \simeq \mathrm{colim}_k U(\mathrm{Nil}_{v_n}(\mathrm{Sp}_{v_n,k}^\omega)).$$

The functor $U(\mathrm{Nil}_{v_n}(\mathrm{Sp}_{v_n,k}^\omega)) \rightarrow \mathrm{Sp}_{\geq n}^\omega$ lands in spectra with nilpotent v_n -self-maps, therefore of type $\geq n+1$. Thus, restricting the isomorphism of Theorem 4.2.5 to $\mathrm{Nil}_{v_n}(\mathrm{Sp}_{v_n}^\omega)$ gives a fully faithful functor

$$u: U(\mathrm{Nil}_{v_n}(\mathrm{Sp}_{v_n}^\omega)) \rightarrow \mathrm{Sp}_{\geq n+1}^\omega.$$

It is left verifying it is essentially surjective. Let $X \in \mathrm{Sp}_{\geq n+1}^\omega$, then $0: \Sigma^{p|v_n|}X \rightarrow X$ is a v_n -self map and $X \simeq u(X, 0)$.

As cofibers commute with filtered colimits, $\mathrm{Sp}_{v_n}^\omega[v_n^{-1}] = L(\mathrm{Sp}_{v_n}^\omega, v_n)$ is the cofiber of $\mathrm{Nil}_{v_n}(\mathrm{Sp}_{v_n}^\omega) \rightarrow \mathrm{Sp}_{v_n}^\omega$. Since U commutes with cofibers

$$U(\mathrm{Sp}_{v_n}^\omega[v_n^{-1}]) \simeq \mathrm{cofib}(U(\mathrm{Nil}_{v_n}(\mathrm{Sp}_{v_n}^\omega)) \rightarrow U(\mathrm{Sp}_{v_n}^\omega)) \simeq \mathrm{cofib}(\mathrm{Sp}_{\geq n+1}^\omega \rightarrow \mathrm{Sp}_{\geq n}^\omega) \simeq \mathrm{Sp}_{T(n)}^\omega.$$

□

Corollary 4.2.8. *There is a group homomorphism*

$$\Sigma^{(-)}: \mathcal{Z}_n \rightarrow \pi_0 \mathrm{Aut}(\mathrm{Sp}_{T(n)}^\omega)$$

sending the topological generator $1 \in \lim_k \mathbb{Z}/p^k|v_n|$ to the suspension automorphism.

Proof. By Corollary 3.4.14 there is a $\Omega^2 S_{\mathbb{Z}_n}^2$ -action on any category in $\mathrm{Cat}_{\mathrm{perf}}^{\mathbb{Z}_n\text{-iso}}$ where 1 acts by suspension, in particular on $\mathrm{Sp}_{v_n}^\omega[v_n^{-1}]$. As U is an exact functor, there is such a group action on $\mathrm{Sp}_{T(n)}^\omega = U(\mathrm{Sp}_{v_n}^\omega[v_n^{-1}])$. Using Lemma 3.4.13, we have a group homomorphism

$$\mathcal{Z}_n \cong \pi_0(\Omega^2 S_{\mathbb{Z}_n}^2) \rightarrow \pi_0 \mathrm{Aut}(\mathrm{Sp}_{T(n)}^\omega).$$

□

5 The telescopic Picard

We describe the $T(n)$ -local Picard group as the group of automorphisms of $\mathrm{Sp}_{T(n)}^\omega$ using Section 4.1. We then use the results of Section 4.2 to construct a large subgroup of $\mathrm{Pic}(\mathrm{Sp}_{T(n)}^\omega)$, in particular finding a subgroup of the form $\mathbb{Z}_p \times \mathbb{Z}/(p^n - 1)$ in the even Picard group.

5.1 Automorphisms and Picard

We recall the definition of the group of units. Let $\mathcal{D} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. Then \mathcal{D} admits a symmetric monoidal unit map

$$\mathbb{1}[-]: \mathcal{S} \rightarrow \mathcal{D}$$

which induces a functor in Pr^{L}

$$\mathbb{1}[-]: \mathrm{CMon} = \mathrm{CAlg}(\mathcal{S}) \rightarrow \mathrm{CAlg}(\mathcal{D}).$$

Its right adjoint is corepresented by $\mathbb{1}$, i.e. it sends R to $\text{Map}(\mathbb{1}, R)$ which admits a natural commutative monoid structure via the multiplication of R . The right adjoint to the composition

$$\text{Sp}^{\text{cn}} \simeq \text{CMon}^{\text{gp}} \subseteq \text{CMon} \xrightarrow{\mathbb{1}[-]} \text{CAlg}(\mathcal{D})$$

is called the group of units and denoted by $(-)^{\times}$. Notice that for $R \in \text{CAlg}(\mathcal{D})$, the underlying space of R^{\times} consists of the connected components of invertible elements in $\text{Map}(\mathbb{1}, R)$.

We now apply it to $\mathcal{D} = \text{Cat}$:

Definition 5.1.1. The Picard spectrum of $\mathcal{C} \in \text{CAlg}(\text{Cat})$ is defined to be $\text{pic}(\mathcal{C}) := \mathcal{C}^{\times}$. The Picard group is defined to be $\text{Pic}(\mathcal{C}) = \pi_0 \text{pic}(\mathcal{C})$.

Lemma 5.1.2. For $\mathcal{C} \in \text{CAlg}(\text{Pr}^{\text{L}})$, the Picard group $\text{Pic}(\mathcal{C})$ is isomorphic to the group of \mathcal{C} -linear automorphisms $\pi_0 \text{Aut}^{\mathcal{C}}(\mathcal{C})$.

Proof. The Picard spectrum $\text{pic}(\mathcal{C}) = \mathcal{C}^{\times}$ is given as the invertible components of $\text{Map}(\text{pt}, \mathcal{C})$ where pt is the terminal category (i.e. the unit in Cat). Now,

$$\mathcal{C}^{\times} \simeq \text{Map}(\text{pt}, \mathcal{C}) \simeq \text{Map}^{\text{L}}(\mathcal{S}, \mathcal{C}) \simeq \text{Map}^{\mathcal{C}}(\mathcal{C}, \mathcal{C}) = \text{End}^{\mathcal{C}}(\mathcal{C})$$

where Map^{L} is the space of colimit-preserving functors (i.e. functors in Pr^{L}) and $\text{Map}^{\mathcal{C}}$ is the space of \mathcal{C} -linear functors in Pr^{L} . The isomorphism $\mathcal{C}^{\times} \xrightarrow{\sim} \text{End}^{\mathcal{C}}(\mathcal{C})$ is given by $X \mapsto X \otimes -$ and is an isomorphism of monoids. We thus get

$$\text{Pic}(\mathcal{C}) = \pi_0 \text{pic}(\mathcal{C}) \cong \pi_0 \text{End}^{\mathcal{C}}(\mathcal{C})^{\times} = \pi_0 \text{Aut}^{\mathcal{C}}(\mathcal{C}).$$

□

Corollary 5.1.3. $\text{Pic}(\text{Sp}_{T(n)}) \cong \pi_0 \text{Aut}(\text{Sp}_{T(n)}^{\omega})$.

Proof. As $\text{Sp}_{T(n)}$ is a mode, i.e. an idempotent algebra in Pr^{L} (see [CSY21a, § 5]), the category of $\text{Sp}_{T(n)}$ -modules sits fully-faithfully inside Pr^{L} . Therefore

$$\text{Pic}(\text{Sp}_{T(n)}) \cong \pi_0 \text{Aut}^{\text{Sp}_{T(n)}}(\text{Sp}_{T(n)}) = \pi_0 \text{Aut}(\text{Sp}_{T(n)}) \cong \pi_0 \text{Aut}(\text{Sp}_{T(n)}^{\omega}),$$

where the last isomorphism is true since $\text{Sp}_{T(n)}$ is compactly generated (Corollary 4.1.9). □

5.2 Constructing Picard elements

We use Section 4 to define automorphisms of the compact $T(n)$ -local category. Using Corollary 5.1.3, we construct telescopic Picard elements. By Corollary 4.2.8 the suspension lifts to a group homomorphism

$$\Sigma^{(-)}: \mathcal{Z}_n \rightarrow \pi_0 \text{Aut}(\text{Sp}_{T(n)}^{\omega})$$

Note that for p odd, $|v_n|$ is prime to p and so

$$\mathcal{Z}_n = \lim_k \mathbb{Z}/p^k |v_n| \cong \lim_k \mathbb{Z}/p^k \times \mathbb{Z}/|v_n| \cong \mathbb{Z}_p \times \mathbb{Z}/|v_n|$$

and for $p = 2$

$$\mathcal{Z}_n = \lim_k \mathbb{Z}/p^k |v_n| = \lim_k \mathbb{Z}/(2^k \cdot 2(2^n - 1)) \cong \lim_k \mathbb{Z}/2^{k+1} \times \mathbb{Z}/(2^n - 1) \cong \mathbb{Z}_2 \times \mathbb{Z}/(2^n - 1).$$

Denoting

$$a_p = \begin{cases} 2, & p \text{ is odd} \\ 1, & p = 2 \end{cases}$$

we write $\mathcal{Z}_n \cong \mathbb{Z}_p \times \mathbb{Z}/(a_p(p^n - 1))$.

Theorem 5.2.1. The group homomorphism $\Sigma^{(-)}: \mathcal{Z}_n \cong \mathbb{Z}_p \times \mathbb{Z}/(a_p(p^n - 1)) \rightarrow \text{Pic}(\text{Sp}_{T(n)})$ is injective.

To prove this theorem we will look at two functors and the image of Picard elements under them. Consider the $K(n)$ homology functor

$$K(n)_*(-): \text{Sp}_{T(n)} \rightarrow \text{Mod}_{K(n)_*}(\text{GrAb}).$$

$K(n)_* \cong \mathbb{F}_p[v]$ is a graded field and satisfies a Künneth isomorphism, i.e. this functor is symmetric monoidal.

Lemma 5.2.2. A graded $K(n)_*$ -module is invertible if and only if it is of the form $K(n)_{*-d} := \pi_* \Sigma^d K(n)$ for some $d \in \mathbb{Z}$.

Proof. If $M = K(n)_{*-d}$ then $M^{-1} := K(n)_{*+d}$ is an inverse. Assume now that M is invertible. As $K(n)_*$ is a graded field, M and its inverse M^{-1} are free and can be written as

$$M = \bigoplus_{i \in I} K(n)_{*-d_i}, \quad M^{-1} = \bigoplus_{j \in J} K(n)_{*-e_j}.$$

Therefore

$$M \otimes_{K(n)_*} M^{-1} \simeq \bigoplus_{i \in I, j \in J} K(n)_{*-(d_i+e_j)}.$$

It is isomorphic to $K(n)_*$ if and only if both I, J consists of a single element $I = \{i\}, J = \{j\}$ and $d_i + e_j \equiv 0$ modulo $2(p^n - 1)$. \square

Corollary 5.2.3. The composition

$$\mathbb{Z}/(a_p(p^n - 1)) \hookrightarrow \mathbb{Z}_p \times \mathbb{Z}/(a_p(p^n - 1)) \xrightarrow{\Sigma^{(-)}} \text{Pic}(\text{Sp}_{T(n)})$$

is injective.

Proof. It is enough to check that this composition is injective after composing with the map

$$K(n)_*(-): \text{Pic}(\text{Sp}_{T(n)}) \rightarrow \text{Pic}(\text{Mod}_{K(n)_*}(\text{GrAb})).$$

By construction, this map sends $d \in \mathbb{Z}/(a_p(p^n - 1))$ to $K(n)_{*+d}$ if p is odd or to $K(n)_{*+2d}$ if $p = 2$ ⁴ and so it is injective (and an isomorphism for $p \neq 2$). \square

⁴Under the isomorphism $\lim_k \mathbb{Z}/p^k |v_n| \cong \mathbb{Z}_p \times \mathbb{Z}/(a_p(p^n - 1))$, the generator of $\mathbb{Z}/(a_p(p^n - 1))$ is 1 if p is odd and is 2 if $p = 2$.

Let Γ_n be a formal group of height n and $E_n = E_n(\Gamma_n, \overline{\mathbb{F}}_p)$ be a Morava E -theory over $\overline{\mathbb{F}}_p$. E_n admits a natural action of the (extended) Morava stabilizer group $\mathbf{G}_n = \text{Aut}(\Gamma_n, \overline{\mathbb{F}}_p)$. Let $\mathbf{W}(\overline{\mathbb{F}}_p)$ be the p -typical Witt vectors ring. Then by Lubin–Tate theory, the local ring $\pi_0 E_n = \mathbf{W}(\overline{\mathbb{F}}_p)[[u_1, \dots, u_{n-1}]]$ corepresents deformations of Γ_n . The invariant differentials ω of the universal deformation of Γ_n forms an invertible $\pi_0 E_n$ -module and $\pi_* E_n = \bigoplus_{t \in 2\mathbb{Z}} \omega^{\otimes t/2}$, where ω is of degree -2 , as commutative rings with \mathbf{G}_n -action.

The Morava stabilizer group splits as $\mathbf{G}_n = \mathbf{S}_n \rtimes \text{Gal}$ where $\text{Gal} = \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \widehat{\mathbb{Z}}$ and $\mathbf{S}_n = \text{Aut}_{\overline{\mathbb{F}}_p}(\Gamma_n)$ is isomorphic to the group of units \mathcal{O}_n^\times of the order

$$\mathcal{O}_n = \mathbf{W}_n \langle S \mid S^n = p, Sw = w^\varphi S \ \forall w \in \mathbf{W}_n \rangle$$

where $\mathbf{W}_n = \mathbf{W}(\mathbb{F}_{p^n})$ is the p -typical Witt vectors ring of \mathbb{F}_{p^n} and φ is a lift of the Frobenius to \mathbf{W}_n . The topological generator 1 of $\widehat{\mathbb{Z}}$ acts on $\mathbf{S}_n = \mathcal{O}_n^\times$ by conjugation with S . In particular we have an embedding

$$\mathbb{Z}_p \xrightarrow{1+p\mathbb{Z}} \mathbb{Z}_p^\times \subseteq \mathbf{W}_n^\times \hookrightarrow \mathcal{O}_n^\times \hookrightarrow \mathbf{G}_n.$$

Consider the symmetric monoidal functor

$$E_n \otimes -: \text{Sp}_{T(n)} \xrightarrow{(\text{B}\mathbb{Z}_p)^*} \text{Sp}_{T(n)}^{\text{B}\mathbb{Z}_p} \xrightarrow{E_n \otimes -} \text{Mod}_{E_n}(\text{Sp}_{T(n)}^{\text{B}\mathbb{Z}_p})$$

tensoring with E_n while remembering the \mathbb{Z}_p -action. As it is symmetric monoidal, it sends Picard elements to Picard elements.

Lemma 5.2.4. *The group homomorphism*

$$\mathbb{Z}_p \hookrightarrow \mathbb{Z}_p \times \mathbb{Z}/(a_p(p^n - 1)) \xrightarrow{\Sigma^{(-)}} \text{Pic}(\text{Sp}_{T(n)})$$

is injective.

Proof. By continuity it is enough to check this for $\mathbb{Z} \subseteq \mathbb{Z}_p$. It is enough to see this after composing with

$$E_n \otimes -: \text{Pic}(\text{Sp}_{T(n)}) \rightarrow \text{Pic}(\text{Mod}_{E_n}(\text{Sp}_{T(n)}^{\text{B}\mathbb{Z}_p})).$$

The composition sends $d \in \mathbb{Z} \subseteq \mathbb{Z}_p$ to $\Sigma^d E_n \in \text{Pic}(\text{Mod}_{E_n}(\text{Sp}_{T(n)}^{\text{B}\mathbb{Z}_p}))$. As $\Sigma E_n \neq E_n$ it is enough to check the injectivity for d even, i.e. that for d even $\Sigma^d E_n$ is not isomorphic to E_n in $\text{Mod}_{E_n}(\text{Sp}_{T(n)}^{\text{B}\mathbb{Z}_p})$. Note that on the level of homotopy groups with \mathbb{Z}_p -action

$$\pi_* \Sigma^d E_n \simeq \pi_* E_n \otimes \omega^{d/2} \in \text{Mod}_{\pi_* E_n}(\text{GrAb}^{\text{B}\mathbb{Z}_p})$$

(see e.g. [Hea15, Lemma 1.3.1]) therefore it is not isomorphic to $\pi_* E_n$ and in particular $\Sigma^d E_n$ is not isomorphic to E_n . \square

Proof of Theorem 5.2.1. The result follows from Corollary 5.2.3, Lemma 5.2.4 and the fact that $\mathbb{Z}/(a_p(p^n - 1))$ is torsion while \mathbb{Z}_p is not. \square

Remark 5.2.5. All computations done here hold in the $K(n)$ -local category. Thus the constructed Picard subgroup is preserved under $K(n)$ -localization.

5.3 Even Picard group

Invertible elements in \mathcal{C} are in particular dualizable. Thus every Picard element X has a dimension $\dim(X) \in \pi_0 \mathbb{1}$.

Theorem 5.3.1 ([CSY21b, Corollary 3.21]). Let \mathcal{C} be a symmetric monoidal category. For every $X \in \text{Pic}(\mathcal{C})$, we have $\dim(X)^2 = 1$. In particular, if $\pi_0 \mathbb{1}$ is a connected ring (that is, it does not contain any non-trivial idempotents) and 2 is invertible in $\pi_0 \mathbb{1}$, then $\dim(X) = \pm 1$.

Definition 5.3.2 ([CSY21b, Definition 3.22]). The even Picard group of a symmetric monoidal category \mathcal{C} , is the subgroup $\text{Pic}^{\text{ev}}(\mathcal{C}) \subseteq \text{Pic}(\mathcal{C})$ given by the kernel of the map $\text{Pic}(\mathcal{C}) \xrightarrow{\dim} \pi_0(\mathbb{1})^\times$.

Corollary 5.3.3. *There is an embedding $\mathbb{Z}_p \times \mathbb{Z}/(p^n - 1) \hookrightarrow \text{Pic}^{\text{ev}}(\text{Sp}_{T(n)})$.*

Proof. By [Theorem 5.3.1](#), the dimension of every $X \in \text{Pic}(\text{Sp}_{T(n)})$ squares to 1, thus the subgroup of squares $(\text{Pic}(\text{Sp}_{T(n)}))^2$ lies in the even Picard group. By [Theorem 5.2.1](#) there is an embedding $\Sigma^{(-)}: \mathbb{Z}_p \times \mathbb{Z}/(a_p(p^n - 1)) \hookrightarrow \text{Pic}(\text{Sp}_{T(n)})$, where

$$a_p := \begin{cases} 2, & p \text{ odd} \\ 1, & p = 2 \end{cases}.$$

Therefore we have an embedding of the group of squares:

$$\Sigma^{(-)}: 2\mathbb{Z}_p \times 2(\mathbb{Z}/(a_p(p^n - 1))) \hookrightarrow (\text{Pic}(\text{Sp}_{T(n)}))^2 \subseteq \text{Pic}^{\text{ev}}(\text{Sp}_{T(n)}). \quad ^5$$

When p is odd $2 \in \mathbb{Z}_p$ is invertible, thus $2\mathbb{Z}_p = \mathbb{Z}_p$ and $2(\mathbb{Z}/(a_p(p^n - 1))) = 2(\mathbb{Z}/(2(p^n - 1))) \cong \mathbb{Z}/(p^n - 1)$. In the case $p = 2$, $2\mathbb{Z}_2 \cong \mathbb{Z}_2$ with generator 2, and $2 \in \mathbb{Z}/(a_2(2^n - 1)) = \mathbb{Z}/(2^n - 1)$ is invertible thus $2(\mathbb{Z}/(2^n - 1)) = \mathbb{Z}/(2^n - 1)$.

In both cases the group of squares is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}/(p^n - 1)$. □

6 Galois extensions

In this section we use Kummer theory as introduced in [\[CSY21b\]](#) to find a telescopic lifting of a specific non-Abelian Galois extensions of $\mathbb{S}_{K(n)}$. We start with a brief review of Galois and Kummer theories, then move to the category of $L_{T(n)} \mathbf{SW}(\mathbb{F}_{p^n})$ -modules, which admits a $(p^n - 1)$ -st root of unity, and study its Picard group.

6.1 Kummer theory

Definition 6.1.1 ([Rog05]). Let $\mathcal{C} \in \text{CAlg}(\text{Pr}^{\text{L}})$, let G be a finite group and let $R \in \text{CAlg}(\mathcal{C}^{\text{BG}})$. We say that R is a G -Galois extension (of $\mathbb{1}$) if it satisfies the following two conditions:

- (1) The canonical map $\mathbb{1} \rightarrow R^{hG}$ is an isomorphism in \mathcal{C} .

⁵As the groups \mathbb{Z}_p and $\mathbb{Z}/(a_p(p^n - 1))$ are additive, and to avoid confusion, we denote their group of squares by $2\mathbb{Z}_p$ and $2(\mathbb{Z}/(a_p(p^n - 1))) = \{2x \mid x \in \mathbb{Z}/(a_p(p^n - 1))\}$ respectively.

(2) The canonical map $R \otimes R \rightarrow \prod_G R$ is an isomorphism in \mathcal{C}^{BG} .

A Galois extension is called *faithful* if in addition the functor $R \otimes -$ is conservative. Denote by $\text{CAlg}^{G\text{-Gal}}(\mathcal{C}) \subseteq \text{CAlg}(\mathcal{C}^{BG}) \cong$ the subspace of G -Galois extensions.

Definition 6.1.2 (Roots of unity, [CSY21b]). Let \mathcal{C} be an additive presentably symmetric monoidal category. Let $m \geq 1$. An m -th root of unity of \mathcal{C} is a map $\omega: C_m \rightarrow \mathbb{1}^\times$. We say that ω is *primitive* if $\mathbb{1}$ is m -divisible (i.e. m is divisible in $\pi_0 \mathbb{1}$) and for every d which strictly divides m , the only commutative algebra $S \in \text{CAlg}(\mathcal{C})$ for which there exists a dotted arrow rendering the diagram of connective spectra

$$\begin{array}{ccc} C_m & \xrightarrow{\omega} & \mathbb{1}^\times \\ \downarrow & & \downarrow \\ C_d & \dashrightarrow & S^\times \end{array}$$

commutative, is $S = 0$.

Theorem 6.1.3 (Kummer theory, [CSY21b, Proposition 3.23]). Let \mathcal{C} be a presentable, additive, symmetric monoidal category with a primitive m -th root of unity. There is a split short exact sequence of Abelian groups

$$0 \rightarrow (\pi_0 \mathbb{1}^\times) / (\pi_0 \mathbb{1}^\times)^m \rightarrow \pi_0 \text{CAlg}^{\mathbb{Z}/m\text{-Gal}}(\mathcal{C}) \rightarrow \text{Pic}^{\text{ev}}(\mathcal{C})[m] \rightarrow 0.$$

6.2 Galois theory of $L_{K(n)} \mathbf{SW}(\mathbb{F}_{p^n})$

Let \mathbf{W} be the Witt vectors functor and \mathbf{SW} be the spherical Witt vectors functor as defined in [Lur18, Example 5.2.7], and later in [BSY22, Section 2], [Ant23].

Notation 6.2.1. Denote the Witt vectors of \mathbb{F}_{p^n} and the $K(n)$ -localization of the spherical Witt vectors of \mathbb{F}_{p^n} by

$$\begin{aligned} \mathbf{W}_n &:= \mathbf{W}(\mathbb{F}_{p^n}) \\ \mathbf{SW}_n &:= L_{K(n)}(\mathbf{SW}(\mathbb{F}_{p^n})) \end{aligned}$$

respectively. We will also denote $\text{Mod}_{\mathbf{SW}_n}^\wedge := \text{Mod}_{\mathbf{sw}_n}(\text{Sp}_{K(n)})$.

Let $E_n = E_n(\Gamma_n, \overline{\mathbb{F}}_p)$ be a Morava E -theory.

Proposition 6.2.2 ([DH04, Theorem 5], [BR08] [Mat16, Theorem 10.9]). *The map $\mathbb{S}_{K(n)} \rightarrow E_n$ exhibits E_n as the Galois closure of $\mathbb{S}_{K(n)}$ and its Galois group is the Morava stabilizer group \mathbf{G}_n .*

Proposition 6.2.3 ([CSY21b, Proposition 5.13]). *The Galois group of \mathbf{SW}_n over $\mathbb{S}_{K(n)}$ is \mathbb{Z}/n , given as the quotient group of $\widehat{\mathbb{Z}} \cong \text{Gal} \leq \mathbf{G}_n$.*

Corollary 6.2.4. *The map $\mathbf{SW}_n \rightarrow E_n$ identifies E_n as the algebraic closure of \mathbf{SW}_n with Galois group $\mathbf{G}_n^{\mathbf{W}} := \mathbf{S}_n \times n\widehat{\mathbb{Z}} \leq \mathbf{G}_n$ ⁶.*

Using [Mat16, Section 9] we deduce the following:

⁶It is a product as the action of $n \in \widehat{\mathbb{Z}}$ on \mathbf{S}_n is given by conjugation with $S^n = p$, which is trivial

Corollary 6.2.5. *Let G be a finite group. Then there is a bijection*

$$\{\text{continuous homomorphisms } \mathbf{G}_n^{\mathbf{w}} \rightarrow G\} / \text{conj.} \longleftrightarrow \{G\text{-Galois extensions of } \mathbf{SW}_n\} / \text{iso.}$$

taking a character $\rho: \mathbf{G}_n^{\mathbf{w}} \rightarrow G$ to $C_\rho(G, E_n)^{h\mathbf{G}_n^{\mathbf{w}}}$, where

$$C_\rho(G, E_n) = \prod_G E_n \in \text{CAlg}(\text{Sp}_{K(n)})_{\mathbf{sw}_n}/$$

is equipped with the ρ -twisted $\mathbf{G}_n^{\mathbf{w}}$ -action: $\mathbf{G}_n^{\mathbf{w}}$ acts on each E_n by Galois conjugation and permutes the coordinates through ρ and the left regular action of G on itself.

Definition 6.2.6. The ring $\mathcal{O}_n = \mathbf{W}_n \langle S | S^n = p, Sw = w^\varphi S \rangle$ admits a surjection of rings

$$\pi: \mathcal{O}_n \xrightarrow{S \mapsto 0} \mathbb{F}_{p^n}$$

which induces a surjective group homomorphism

$$\mathbf{G}_n^{\mathbf{w}} = \mathbf{S}_n \times n\widehat{\mathbb{Z}} \twoheadrightarrow \mathbf{S}_n = \mathcal{O}_n^\times \xrightarrow{\pi} \mathbb{F}_{p^n}^\times.$$

Let \overline{Y}_n be the $\mathbb{F}_{p^n}^\times$ -Galois extension of \mathbf{SW}_n corresponding to this homomorphism.

Definition 6.2.7. Let S_n be the kernel of the map $\pi: \mathbf{S}_n \rightarrow \mathbb{F}_{p^n}^\times$.

The multiplicative lifts function $\mathbb{F}_{p^n}^\times \rightarrow \mathbf{W}_n^\times \subseteq \mathbf{S}_n$ splits \mathbf{S}_n as $\mathbf{S}_n = S_n \rtimes \mathbb{F}_{p^n}^\times$.

We have now introduced all ingredients used in the classification of the even Picard group of $\text{Mod}_{\mathbf{SW}_n}^\wedge$, a key step in the lifting of the Galois extension.

Proposition 6.2.8. *The group $\text{Pic}^{\text{ev}}(\text{Mod}_{\mathbf{SW}_n}^\wedge)[p^n - 1]$ is a cyclic group of order $p^n - 1$ and is generated by the image of \overline{Y}_n under the map in [Theorem 6.1.3](#).*

To prove this proposition we will need the following two lemmas, which are variants of [\[CSY21a, Proposition 2.2.6\]](#) and [\[CSY21b, Proposition 5.17\]](#). The proof of [Lemma 6.2.9](#) is almost identical, and the proof of [Proposition 6.2.8](#) using these lemmas is the same as the proof of [\[CSY21b, Proposition 5.23\]](#). We write the proofs for completeness.

Lemma 6.2.9. *The image of the unit map $u: \pi_0 \mathbf{SW}_n \rightarrow \pi_0 E_n$ is $\mathbf{W}_n \subseteq \pi_0 E_n$ and the kernel is precisely the nil-radical.*

Proof. As $\mathbf{SW}_n = E_n^{h\mathbf{G}_n^{\mathbf{w}}}$, the map u factors through the fixed points $(\pi_0 E_n)^{\mathbf{G}_n^{\mathbf{w}}} \subseteq \pi_0 E_n$. By [\[BG18, Lemma 1.33\]](#),

$$(\pi_0 E_n)^{\mathbf{G}_n^{\mathbf{w}}} = ((\pi_0 E_n)^{\mathbf{S}_n})^{n\widehat{\mathbb{Z}}} = \mathbf{W}(\overline{\mathbb{F}}_p)^{n\widehat{\mathbb{Z}}} = \mathbf{W}_n.$$

By [\[BGH22, Corollary 2.3.10\]](#), the E_∞ -page of the descent spectral sequence

$$H_c^s(\mathbf{G}_n^{\mathbf{w}}, \pi_t E_n) \Rightarrow \pi_{t-s}(\mathbf{SW}_n)$$

has a horizontal vanishing line. Since the spectral sequence is multiplicative, this implies that all elements in $\pi_0 \mathbf{SW}_n$ with positive filtration degree are nilpotent. Finally, since \mathbf{SW}_n admits a ring map from $\mathbf{SW}(\mathbb{F}_{p^n})$, the map $u: \pi_0 \mathbf{SW}_n \rightarrow \mathbf{W}_n$ is surjective. \square

Lemma 6.2.10. *The Abelianization of \mathbf{S}_n is given by*

$$\mathbf{S}_n^{\text{ab}} \cong \mathbb{Z}_p^\times \oplus_{\mathbb{F}_p^\times} \mathbb{F}_{p^n}^\times \cong \begin{cases} \mathbb{Z}_p \times \mathbb{F}_{p^n}^\times, & p > 2 \\ \mathbb{Z}_2 \times \{\pm 1\} \times \mathbb{F}_{2^n}^\times, & p = 2 \end{cases}.$$

Proof. For $n = 1$, $\mathbf{S}_n = \mathbb{Z}_p^\times$ is already Abelian and the lemma is trivial. Following [Hen17, §4], let $v: \mathcal{O}_n \rightarrow \frac{1}{n}\mathbb{Z}_{\geq 0}$ be the valuation defined by $v(S) = \frac{1}{n}$ and define a $\frac{1}{n}\mathbb{Z}_{\geq 0}$ -filtration on $\mathbf{S}_n = \mathcal{O}_n^\times$ by

$$F_{\geq i} \mathbf{S}_n = \{x \in \mathbf{S}_n \mid v(x-1) \geq i\}.$$

With this filtration $\mathbf{S}_n = F_{\geq 0} \mathbf{S}_n$ and $S_n = F_{\geq \frac{1}{n}} \mathbf{S}_n$. The group commutator and p -th power map endows $\bigoplus_{i>0} \text{gr}_i \mathbf{S}_n$ with the structure of a graded mixed Lie algebra in the sense of Lazard [Laz65] (see [Hen17, § 4.2]). As Henn, denote the corresponding operations by $[-, -]: \text{gr}_i \mathbf{S}_n \otimes \text{gr}_j \mathbf{S}_n \rightarrow \text{gr}_{i+j} \mathbf{S}_n$ and $P: \text{gr}_i \mathbf{S}_n \rightarrow \text{gr}_{\phi(i)} \mathbf{S}_n$ where $\phi(i) = \min\{i+1, pi\}$.

The mod S reduction maps

$$\begin{aligned} \text{gr}_0 \mathbf{S}_n &\rightarrow \mathbb{F}_{p^n}^\times & \text{gr}_i \mathbf{S}_n &\rightarrow \mathbb{F}_{p^n} & (2) \\ [x] &\mapsto x \pmod{S} & [1+xS^k] &\mapsto x \pmod{S} \end{aligned}$$

are isomorphisms, and it is a straightforward calculation to show that for $1+x \in \text{gr}_i \mathbf{S}_n, 1+y \in \text{gr}_j \mathbf{S}_n$,

$$[1+x, 1+y] \equiv 1+xy-yx \in \text{gr}_{i+j} \mathbf{S}_n. \quad (3)$$

For every $k \in \frac{1}{n}\mathbb{Z}_{\geq 0}$, we get a natural filtration on $(F_{\geq k} \mathbf{S}_n)^{\text{ab}} = F_{\geq k} \mathbf{S}_n / \overline{[F_{\geq k} \mathbf{S}_n, F_{\geq k} \mathbf{S}_n]}$ given by

$$\text{filt}_{\geq i}(F_{\geq k} \mathbf{S}_n)^{\text{ab}} := F_{\geq i} \mathbf{S}_n / \overline{\langle [F_{\geq a} \mathbf{S}_n, F_{\geq b} \mathbf{S}_n] \mid k \leq a, b, a+b=i \rangle}$$

which simply induces the equalities:

$$\text{gr}_i(F_{\geq k} \mathbf{S}_n)^{\text{ab}} = \text{gr}_i(F_{\geq k+\frac{1}{n}} \mathbf{S}_n)^{\text{ab}} / [\text{gr}_k \mathbf{S}_n, \text{gr}_{i-k} \mathbf{S}_n].$$

In particular, assigning $k = 0$:

$$\text{gr}_i \mathbf{S}_n^{\text{ab}} = \text{gr}_i \mathbf{S}_n^{\text{ab}} / [\text{gr}_0 \mathbf{S}_n, \text{gr}_i \mathbf{S}_n] \text{ for all } i > 0.$$

In the proofs of [Hen17, Proposition 5.2 and Proposition 5.3], it is shown that for any p ,

$$\text{gr}_i \mathbf{S}_n^{\text{ab}} = \begin{cases} \mathbb{F}_{p^n}, & i = \frac{1}{n} \\ \mathbb{F}_p, & i \in \mathbb{Z}_{\geq 1} \\ 0, & \text{otherwise} \end{cases}$$

and that the p -th power map P is

- a) an isomorphism on the integral grades for p odd;
- b) and an isomorphism on intergal grades greater than 1 and trivial for the first grade for $p = 2$.

Let $j \in \mathbb{Z}_{\geq 1}$. Using Equation (2), we may choose representatives for $\text{gr}_j \mathbf{S}_n$ of the form $1 + \omega p^j$, $\omega \in \mathbb{F}_{p^n}$, and \mathbb{F}_p^\times represent $\text{gr}_0 \mathbf{S}_n$. Let $1 + \omega p^j \in \text{gr}_j \mathbf{S}_n$, $\omega' \in \mathbb{F}_p^\times \cong \text{gr}_0 \mathbf{S}_n$. Then by Equation (3):

$$[\omega', 1 + \omega p^j] = [1 + (\omega' - 1), 1 + \omega p^j] \equiv 1 + (\omega' - 1)\omega p^j - \omega p^j(\omega' - 1) = 1.$$

Therefore $[\text{gr}_0 \mathbf{S}_n, \text{gr}_j \mathbf{S}_n] = 0$. Again, using Equation (2), we can choose representatives for $\text{gr}_{\frac{1}{n}} \mathbf{S}_n$ of the form $1 + \omega S$ for $\omega \in \mathbb{F}_{p^n}$. Let $\omega' \in \mathbb{F}_p^\times \cong \text{gr}_0 \mathbf{S}_n$ and $1 + \omega S \in \text{gr}_{\frac{1}{n}} \mathbf{S}_n$, then

$$[\omega', 1 + \omega S] = [1 + (\omega' - 1), 1 + \omega S] \equiv 1 + (\omega' - 1)\omega S - \omega S(\omega' - 1) = 1 + \omega(\omega' - \omega'^\varphi)S.$$

Choosing $\omega' \notin \mathbb{F}_p^\times$, varying over all ω we see that $[\text{gr}_0 \mathbf{S}_n, \text{gr}_{\frac{1}{n}} \mathbf{S}_n] = \text{gr}_{\frac{1}{n}} \mathbf{S}_n$. We get

$$\text{gr}_i \mathbf{S}_n^{\text{ab}} = \begin{cases} \mathbb{F}_p^\times, & i = 0 \\ \mathbb{F}_p, & i \in \mathbb{Z}_{\geq 1} \\ 0, & \text{otherwise} \end{cases}$$

and the p -th power map is

- a) an isomorphism on positive grades for p odd;
- b) and an isomorphism on grades greater than 1 and trivial for the first grade for $p = 2$.

The claim then follows. □

Corollary 6.2.11. *The Abelianization of $\mathbf{G}_n^{\mathbf{w}}$ is given as*

$$(\mathbf{G}_n^{\mathbf{w}})^{\text{ab}} \cong (\mathbb{Z}_p^\times \oplus_{\mathbb{F}_p^\times} \mathbb{F}_{p^n}^\times) \times \widehat{\mathbb{Z}}$$

Proof. Indeed, we have

$$(\mathbf{G}_n^{\mathbf{w}})^{\text{ab}} = (\mathbf{S}_n \times n\widehat{\mathbb{Z}})^{\text{ab}} = \mathbf{S}_n^{\text{ab}} \times n\widehat{\mathbb{Z}} \cong (\mathbb{Z}_p^\times \oplus_{\mathbb{F}_p^\times} \mathbb{F}_{p^n}^\times) \times \widehat{\mathbb{Z}}.$$

□

Proof of Proposition 6.2.8. Using Theorem 6.1.3 and its naturality with respect to the functor

$$\text{L}_{K(n)}: \text{Mod}_{\mathbf{sw}(\mathbb{F}_{p^n})}(\text{Sp}_p) \rightarrow \text{Mod}_{\widehat{\mathbf{sw}}_n},$$

we obtain the following morphism of short exact sequences of abelian groups

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
(\pi_0 \mathbf{SW}(\mathbb{F}_{p^n})^\times) / (\pi_0 \mathbf{SW}(\mathbb{F}_{p^n})^\times)^{p^n-1} & \xrightarrow{f} & (\pi_0 \mathbf{SW}_n^\times) / (\pi_0 \mathbf{SW}_n^\times)^{p^n-1} \\
\downarrow & & \downarrow \\
\pi_0 \text{CAlg}^{\mathbb{F}_{p^n}^\times\text{-Gal}}(\text{Mod}_{\mathbf{SW}(\mathbb{F}_{p^n})}(\text{Sp}_p)) & \xrightarrow{g} & \pi_0 \text{CAlg}^{\mathbb{F}_{p^n}^\times\text{-Gal}}(\text{Mod}_{\mathbf{SW}_n}^\wedge) \\
\downarrow & & \downarrow \\
\text{Pic}^{\text{ev}}(\text{Mod}_{\mathbf{SW}(\mathbb{F}_{p^n})}(\text{Sp}_p))[p^n-1] & \longrightarrow & \text{Pic}^{\text{ev}}(\text{Mod}_{\mathbf{SW}_n}^\wedge)[p^n-1] \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

In the top left corner we have

$$(\pi_0 \mathbf{SW}(\mathbb{F}_{p^n})^\times) / (\pi_0 \mathbf{SW}(\mathbb{F}_{p^n})^\times)^{p^n-1} = (\mathbf{W}_n^\times) / (\mathbf{W}_n^\times)^{p^n-1} \cong \mathbb{F}_{p^n}^\times.$$

The top horizontal map f is an isomorphism: Indeed, by [Lemma 6.2.9](#), the map

$$\mathbf{W}_n = \pi_0 \mathbf{SW}(\mathbb{F}_{p^n}) \rightarrow \pi_0 \mathbf{SW}_n$$

admits a retract $r: \pi_0 \mathbf{SW}_n \rightarrow \mathbf{W}_n$ whose kernel consists of nilpotent elements. In particular every element in the kernel of $r^\times: \pi_0 \mathbf{SW}_n^\times \rightarrow \mathbf{W}_n^\times$ is of the form $x = (1 + \varepsilon)$ for some nilpotent $\varepsilon \in \pi_0 \mathbf{SW}_n$. As $p^n - 1$ is invertible in $\pi_0 \mathbf{SW}_n$, every such element x has a $(p^n - 1)$ -st root, and hence r^\times induces an isomorphism after modding out the $(p^n - 1)$ -st powers. Since this induced isomorphism is a left-inverse of f it follows that f is an isomorphism as well.

Next, by [Corollary 6.2.5](#), [Corollary 6.2.11](#) and using that the absolute Abelian Galois group of $\mathbf{SW}(\mathbb{F}_{p^n})$ is $\widehat{\mathbb{Z}}$ (e.g. [\[Mat16, Theorem 6.17\]](#)), g can be identified with the inclusion

$$\text{hom}(\widehat{\mathbb{Z}}, \mathbb{F}_{p^n}^\times) \hookrightarrow \text{hom}(\widehat{\mathbb{Z}}, \mathbb{F}_{p^n}^\times) \oplus \text{hom}(\mathbb{Z}_p^\times \oplus_{\mathbb{F}_p^\times} \mathbb{F}_{p^n}^\times, \mathbb{F}_{p^n}^\times).$$

Since $\text{hom}(\widehat{\mathbb{Z}}, \mathbb{F}_{p^n}^\times) \cong \mathbb{F}_{p^n}^\times$, the entire diagram can be identified with

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathbb{F}_{p^n}^\times & \xlongequal{\quad} & \mathbb{F}_{p^n}^\times \\
\parallel & & \downarrow \\
\mathbb{F}_{p^n}^\times & \hookrightarrow & \mathbb{F}_{p^n}^\times \oplus \text{hom}(\mathbb{Z}_p^\times \oplus_{\mathbb{F}_p^\times} \mathbb{F}_{p^n}^\times, \mathbb{F}_{p^n}^\times) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \text{Pic}^{\text{ev}}(\text{Mod}_{\mathbf{SW}_n}^\wedge)[p^n - 1] \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

thus the bottom-right vertical map restricts to an isomorphism

$$\text{hom}(\mathbb{Z}_p^\times \oplus_{\mathbb{F}_p^\times} \mathbb{F}_{p^n}^\times, \mathbb{F}_{p^n}^\times) \cong \text{Pic}^{\text{ev}}(\text{Mod}_{\mathbf{SW}_n}^\wedge)[p^n - 1].$$

Moreover, $\text{hom}(\mathbb{Z}_p^\times \oplus_{\mathbb{F}_p^\times} \mathbb{F}_{p^n}^\times, \mathbb{F}_{p^n}^\times) \cong \mathbb{F}_{p^n}^\times$ (this can be checked separately for odd and even primes, using the formula in [Lemma 6.2.10](#)), so $\text{Pic}^{\text{ev}}(\text{Mod}_{\mathbf{SW}_n}^\wedge)[p^n - 1] \cong \mathbb{F}_{p^n}^\times$.

Now, \overline{Y}_n was chosen in [Definition 6.2.6](#) as corresponding to the character

$$\mathbf{G}_n^{\mathbf{w}} \rightarrow (\mathbf{G}_n^{\mathbf{w}})^{\text{ab}} \cong \widehat{\mathbb{Z}} \times (\mathbb{Z}_p^\times \oplus_{\mathbb{F}_p^\times} \mathbb{F}_{p^n}^\times) \rightarrow \mathbb{Z}_p^\times \oplus_{\mathbb{F}_p^\times} \mathbb{F}_{p^n}^\times \rightarrow \mathbb{F}_{p^n}^\times$$

which generates a cyclic subgroup of $\text{hom}(\mathbb{Z}_p^\times \oplus_{\mathbb{F}_p^\times} \mathbb{F}_{p^n}^\times, \mathbb{F}_{p^n}^\times)$ of order $p^n - 1$, and so the same holds also in $\text{Pic}^{\text{ev}}(\text{Mod}_{\mathbf{SW}_n}^\wedge)[p^n - 1]$. \square

Proposition 6.2.12. $\text{Pic}^{\text{ev}}(\text{Sp}_{K(n)})[p^n - 1]$ is a cyclic group of order $p^n - 1$.

Proof. Using [Corollary 5.3.3](#) and [Remark 5.2.5](#), there is an injective homomorphism

$$\mathbb{Z}/(p^n - 1) \hookrightarrow \text{Pic}^{\text{ev}}(\text{Sp}_{K(n)})[p^n - 1].$$

Consider the composition

$$\mathbb{Z}/(p^n - 1) \hookrightarrow \text{Pic}^{\text{ev}}(\text{Sp}_{K(n)})[p^n - 1] \xrightarrow{\mathbf{SW}_n \otimes -} \text{Pic}^{\text{ev}}(\text{Mod}_{\mathbf{SW}_n}^\wedge)[p^n - 1].$$

We will show it is injective, thus finishing by [Proposition 6.2.8](#). It is enough to show that before moving to even Picard elements, i.e. to show that the composition

$$\mathbb{Z}/(a_p(p^n - 1)) \hookrightarrow \text{Pic}(\text{Sp}_{K(n)}) \xrightarrow{\mathbf{SW}_n \otimes -} \text{Pic}(\text{Mod}_{\mathbf{SW}_n}^\wedge)$$

is injective. Assume $d \in \mathbb{Z}/(a_p(p^n - 1))$ is in the kernel and let X be the corresponding Picard spectrum. Then $\mathbf{SW}_n \otimes X \simeq \mathbf{SW}_n$ in $\text{Mod}_{\mathbb{S}\mathbf{W}_n}^\wedge$. Composing with the forgetful functor $\text{Mod}_{\mathbb{S}\mathbf{W}_n}^\wedge \rightarrow \text{Sp}_{K(n)}$, remembering that as a spectrum $\mathbf{SW}(\mathbb{F}_{p^n}) \simeq \mathbb{S}^{\oplus n}$ we get that $X^{\oplus n} \simeq \mathbb{S}_{K(n)}^{\oplus n}$ in $\text{Sp}_{K(n)}$. Tensoring now with $K(n)$ we get that $K(n) \otimes X^{\oplus n} \simeq K(n)^{\oplus n}$, therefore $K(n) \otimes X \simeq K(n)$. But we showed in the proof of [Corollary 5.2.3](#) that the composition

$$\mathbb{Z}/(a_p(p^n - 1)) \hookrightarrow \text{Pic}(\text{Sp}_{T(n)}) \xrightarrow{K(n) \otimes -} \text{Pic}(\text{BiMod}_{K(n)})$$

is injective, and it is equivalent to the composition

$$\mathbb{Z}/(a_p(p^n - 1)) \hookrightarrow \text{Pic}(\text{Sp}_{T(n)}) \xrightarrow{L_{K(n)}} \text{Pic}(\text{Sp}_{K(n)}) \xrightarrow{K(n) \otimes -} \text{Pic}(\text{BiMod}_{K(n)}).$$

Thus, as $X \otimes K(n) \simeq K(n)$, d is in the kernel of the above injective map and so is trivial as required. \square

6.3 Telescopic lifting

In [Corollary 5.3.3](#) we constructed a $(p^n - 1)$ -cyclic subgroup of $\text{Pic}^{\text{ev}}(\text{Sp}_{T(n)})$. Let $\mathbf{SW}_n^f := L_{T(n)}(\mathbf{SW}(\mathbb{F}_{p^n}))$ and denote also by $\text{Mod}_{\mathbb{S}\mathbf{W}_n^f}^\wedge := \text{Mod}_{\mathbb{S}\mathbf{W}_n^f}(\text{Sp}_{T(n)})$. By [[CSY21b](#), [Theorem 5.9](#) and [Theorem 5.27](#)] \mathbf{SW}_n^f is a faithful \mathbb{Z}/n -Galois extension of $\mathbb{S}_{T(n)}$, and $\mathbf{SW}_n = L_{K(n)}(\mathbf{SW}_n^f)$ is classified as a Galois extension of $\mathbb{S}_{K(n)}$ by the character

$$\mathbf{G}_n \twoheadrightarrow \widehat{\mathbb{Z}} \twoheadrightarrow \mathbb{Z}/n.$$

Using [Remark 5.2.5](#), [Proposition 6.2.8](#) [Proposition 6.2.12](#) we get a commutative diagram of groups

$$\begin{array}{ccc} \text{Pic}^{\text{ev}}(\text{Sp}_{T(n)})[p^n - 1] & \xrightarrow{\mathbf{SW}_n^f \otimes -} & \text{Pic}^{\text{ev}}(\text{Mod}_{\mathbb{S}\mathbf{W}_n^f}^\wedge)[p^n - 1] \\ \uparrow & & \downarrow L_{K(n)} \\ \mathbb{F}_{p^n}^\times & \xrightarrow{j} & \text{Pic}^{\text{ev}}(\text{Sp}_{K(n)})[p^n - 1] \\ \downarrow \sim & & \downarrow L_{K(n)} \\ \text{Pic}^{\text{ev}}(\text{Sp}_{K(n)})[p^n - 1] & \xrightarrow[\sim]{\mathbf{SW}_n \otimes -} & \text{Pic}^{\text{ev}}(\text{Mod}_{\mathbb{S}\mathbf{W}_n}^\wedge)[p^n - 1] \end{array} .$$

Let $P_n \in \text{Pic}^{\text{ev}}(\text{Sp}_{K(n)})[p^n - 1]$ be such that $\mathbf{SW}_n \otimes P_n \in \text{Pic}^{\text{ev}}(\text{Mod}_{\mathbb{S}\mathbf{W}_n}^\wedge)[p^n - 1]$ is the image of \overline{Y}_n under the Kummer map [Theorem 6.1.3](#). Let P_n^f be the image of $j^{-1}(P_n)$ in $\text{Pic}^{\text{ev}}(\text{Sp}_{T(n)})[p^n - 1]$, so $L_{K(n)}(P_n^f) = P_n$. Using [Theorem 6.1.3](#) again choose a lifting \overline{Y}_n^f of $\mathbf{SW}_n^f \otimes P_n^f \in \text{Pic}^{\text{ev}}(\text{Mod}_{\mathbb{S}\mathbf{W}_n^f}^\wedge)[p^n - 1]$ in

$$\text{CAlg}^{\mathbb{Z}/(p^n - 1)\text{-Gal}}(\text{Mod}_{\mathbb{S}\mathbf{W}_n^f}^\wedge) = \text{CAlg}^{\mathbb{F}_{p^n}^\times\text{-Gal}}(\text{Mod}_{\mathbb{S}\mathbf{W}_n^f}^\wedge).$$

Then the image of $L_{K(n)}\overline{Y}_n^f$ in $\text{Pic}^{\text{ev}}(\text{Mod}_{\mathbb{S}\mathbf{W}_n}^\wedge)$ is a generator of the embedded cyclic group. Thus by [Proposition 6.2.8](#) we can assume $L_{K(n)}(\overline{Y}_n^f) \simeq \overline{Y}_n$.

To summarize:

Theorem 6.3.1. There exists a $(\mathbb{F}_{p^n}^\times \rtimes \mathbb{Z}/n)$ -Galois extension \overline{Y}_n^f of $\mathbb{S}_{T(n)}$, lifting the Galois extension \overline{Y}_n of $\mathbb{S}_{K(n)}$ corresponding to the character

$$\mathbf{G}_n = \mathbf{S}_n \rtimes \widehat{\mathbb{Z}} \xrightarrow{\pi \times \text{id}} \mathbb{F}_{p^n}^\times \rtimes \widehat{\mathbb{Z}} \rightarrow \mathbb{F}_{p^n}^\times \rtimes \mathbb{Z}/n.$$

Corollary 6.3.2. *There exists an $((\mathbb{F}_{p^n}^\times \oplus_{\mathbb{F}_p^\times} \mathbb{Z}_p^\times) \rtimes \widehat{\mathbb{Z}})$ -pro-Galois extension Y_n^f of $\mathbb{S}_{T(n)}$, lifting the Galois extension Y_n of $\mathbb{S}_{K(n)}$ corresponding to the character*

$$\mathbf{G}_n = \mathbf{S}_n \rtimes \widehat{\mathbb{Z}} \xrightarrow{(\det \oplus_{\mathbb{F}_p^\times} \pi) \times \text{id}} (\mathbb{Z}_p^\times \oplus_{\mathbb{F}_p^\times} \mathbb{F}_{p^n}^\times) \rtimes \widehat{\mathbb{Z}}.$$

Proof. We saw in [Corollary 6.2.11](#) that the Abelianization of \mathbf{G}_n^w is $(\mathbf{G}_n^w)^{\text{ab}} \cong (\mathbb{Z}_p^\times \oplus_{\mathbb{F}_p^\times} \mathbb{F}_{p^n}^\times) \times n\widehat{\mathbb{Z}}$. The quotient map

$$\mathbf{G}_n^w \rightarrow (\mathbb{Z}_p^\times \oplus_{\mathbb{F}_p^\times} \mathbb{F}_{p^n}^\times) \times n\widehat{\mathbb{Z}} \rightarrow \mathbb{Z}_p^\times \times n\widehat{\mathbb{Z}}$$

corresponds to the the maximal Abelian pro-Galois extension of $\mathbb{S}_{K(n)}$, thought of as an extension of \mathbf{SW}_n :

$$\mathbf{SW}_n \rightarrow L_{K(n)} \mathbf{SW}(\overline{\mathbb{F}}_p)[\omega_{p^\infty}^{(n)}],$$

see [\[CSY21b, Proposition 5.17\]](#). The quotient

$$\mathbf{G}_n^w \rightarrow (\mathbb{Z}_p^\times \oplus_{\mathbb{F}_p^\times} \mathbb{F}_{p^n}^\times) \times n\widehat{\mathbb{Z}} \rightarrow \mathbb{F}_{p^n}^\times$$

is the character π of \overline{Y}_n . That means the maximal Abelian pro-Galois extension Y_n of \mathbf{SW}_n , corresponding to the quotient map

$$\mathbf{G}_n^w \rightarrow (\mathbb{Z}_p^\times \oplus_{\mathbb{F}_p^\times} \mathbb{F}_{p^n}^\times) \times n\widehat{\mathbb{Z}}$$

is obtained by adding all roots of unity and higher roots of unity to \overline{Y}_n , or euquivalently:

$$Y_n := \overline{Y}_n \otimes_{\mathbf{SW}_n} L_{K(n)} \mathbf{SW}(\overline{\mathbb{F}}_p)[\omega_{p^\infty}^{(n)}].$$

We can lift it to a pro-Galois extension of \mathbf{SW}_n^f , using that $L_{K(n)} \mathbf{SW}(\overline{\mathbb{F}}_p)[\omega_{p^\infty}^{(n)}]$ lifts to an extension of $\mathbb{S}_{T(n)}$ ([\[CSY21b, Theorem 5.31\]](#)):

$$Y_n^f := \overline{Y}_n^f \otimes_{\mathbf{SW}_n^f} L_{T(n)} \mathbf{SW}(\overline{\mathbb{F}}_p)[\omega_{p^\infty}^{(n)}].$$

\overline{Y}_n^f is a $((\mathbb{Z}_p^\times \oplus_{\mathbb{F}_p^\times} \mathbb{F}_{p^n}^\times) \rtimes \widehat{\mathbb{Z}})$ -pro-Galois extension of $\mathbb{S}_{T(n)}$ lifting the claimed character. \square

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