A NOTE ON THE KERVAIRE INVARIANT

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In [5] M. Kervaire defined an invariant for $(4k+2)$-dimensional framed manifolds. This invariant depends only on the framed bordism class of the manifold and lies in Z_2. W. Browder [2] (see also E. H. Brown [3]) gave a generalisation of the invariant that is defined for any even dimensional manifold with a Wu orientation; in this case it depends only on the Wu bordism class. A framed manifold has a Wu orientation and using his generalisation Browder showed that the Kervaire invariant of M^n is zero unless $n = 2^r - 2$ for some $r > 1$.

In this note we reprove the above mentioned result of Browder. We use a consequence, due to Nigel Ray [7], of the theorem of D. S. Kahn and S. Priddy [4]. This allows us to avoid the computational part of the proof in [2].

Throughout, all homology and cohomology groups have Z_2 coefficients and we denote the Eilenberg–Maclane space $K(Z_2, n)$ by K_n.

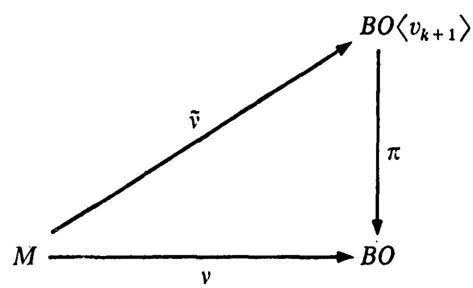
1. *Wu orientations*

Let BO denote the classifying space for stable real vector bundles and $v_{k+1} \in H^{k+1}(BO)$ the $(k+1)$-st universal Wu class. Let

$$\pi : BO\langle v_{k+1}\rangle \to BO$$

be the fibration induced by v_{k+1} from the path space fibration over K_{k+1}.

If M^{2k} is a manifold and ν denotes its stable normal bundle, then $v_{k+1}(\nu) = 0$. Hence there is a lifting $\tilde{\nu}$ making the following diagram commute.



(Notationally we confuse a bundle with its classifying map.) By general theory, the liftings are classified, in this case, by $H^k(M)$. Such a lifting is called a Wu orientation of M. It is clear that a framing of the normal bundle gives rise to a Wu orientation. All this is thoroughly discussed in [2; §4].

Given a framed manifold M and a map $\phi : M \to O$ (O is the stable orthogonal group) we may use ϕ to change the framing. Conversely, given two framings F_1, F_2

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of M, they differ by a map

$$F_2/F_1 : M \to O.$$

Analogously two Wu orientations W_1, W_2 differ by $W_2/W_1 : M \to K_k$. If the framing F_i gives the Wu orientation W_i, it follows from the naturality of the fibration sequence that W_2/W_1 is the composite $(\Omega v_{k+1})\ (F_2/F_1)$ where $\Omega v_{k+1} : O \to K_k$ is obtained by looping the map $v_{k+1} : BO \to K_{k+1}$.

Given a Wu orientation W of M an invariant $K(M, W)$ is defined. If W arises from a framing F then $K(M, W)$ can be identified with the Kervaire invariant $K(M, F)$.

2. *The Kahn–Priddy theorem and framings*

Let $i : P^\infty \to O$ denote the usual inclusion of real projective space P^∞ in the stable orthogonal group and let π_*^s denote stable homotopy. The J homomorphism $\pi_*(O) \to \pi_*^s(S^o)$ is known (see, e.g. [10]) to factor giving the stabilized J homomorphism

$$J^s : \pi_*^s(O) \to \pi_*^s(S^o).$$

THEOREM (Kahn–Priddy [4], see also [1] and [8]).

$$J^s . i_* : \pi_*^s(P^\infty) \to \pi_*^s(S^o)$$

is onto the 2*-primary component.*

By interpreting this theorem in terms of framed bordism, Nigel Ray showed

PROPOSITION. *Let M^n be a framed manifold with a framing F such that $K(M, F) = 1$. Then there exists a manifold N^n with framings F_1, F_2 such that $K(N, F_1) = 1$, $K(N, F_2) = 0$ and $F_2/F_1 : M \to O$ factors through the map $i : P^\infty \to O$.*

3. *The vanishing of the Kervaire invariant*

We now reprove

THEOREM (Browder [2]).

$$K(M^n, F) = 0 \quad \textit{for any} \quad (M^n, F) \quad \textit{if} \quad n+2 \neq 2^r.$$

We first prove

LEMMA. *$i^*(\Omega v_{k+1}) \in H^k(P^\infty)$ is zero unless $k+1 = 2^r$.*

Proof. It is equivalent to show that the suspension of this cohomology class is zero, i.e. if $\sum i : \sum P^\infty \to BO$ is the adjoint, that $(\sum i)^* v_{k+1} = 0$.

Let $\sigma : H^p(P^\infty) \to H^{p+1}(\sum P^\infty)$ be the suspension isomorphism and ξ be the bundle over $\sum P^\infty$ induced by $\sum i$ from the universal bundle over BO. Denote the

Thom class of ξ by U and its total Stiefel–Whitney and Wu classes by W, V respectively. Then if $x \in H^1(P^\infty)$ is non-zero

$$W = 1 + \sum_{n \geq 1} \sigma(x^n).$$

We will use induction to show that if $V = 1 + \sum_{n \geq 1} v_n$ then $v_n = 0$ unless $n = 2^r$ for some $r > 0$.

Clearly $v_1 = 0$ and $v_2 = \sigma(x)$. By the definition of V (e.g. [6])

$$U.v_n = (\chi Sq^n)\,U$$

and

$$\sum_{i=0}^{n} Sq^{n-i}.\chi Sq^i = 0 \qquad \text{(from [9])}$$

so

$$U.v_n = Sq^n\,U + \sum_{i=0}^{n-1} Sq^{n-i}.\chi Sq^i\,U.$$

Now $Sq^n\,U = Uw_n = U\sigma(x^{n-1})$ and by induction

$$\begin{aligned} \chi Sq^i\,U = U.v_i &= U.\sigma(x^{2^r-1}) \quad \text{if} \quad i = 2^r, \qquad r > 0 \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

If we define s by $2^s < n \leqslant 2^{s+1}$ we see that

$$U.v_n = U.\sigma(x^{n-1}) + \sum_{j=1}^{s} Sq^{n-2^j}(U.\sigma(x^{2^j-1})).$$

Since all products vanish in $H^*(\sum P^\infty)$, the Cartan formula gives

$$Sq^{n-2^j}(U.\sigma(x^{2^j-1})) = U.Sq^{n-2^j}(\sigma(x^{2^j-1})).$$

For dimensional reasons this vanishes for $j < s$, hence

$$v_n = \sigma\big(x^{n-1} + Sq^{n-2^s}(x^{2^s-1})\big).$$

But

$$Sq^{n-2^s}(x^{2^s-1}) = \binom{2^s-1}{n-2^s} x^{n-1}$$

which is non-zero for $2^s < n < 2^{s+1}$ and zero for $n = 2^{s+1}$. This completes the induction.

COROLLARY. *If F_1, F_2 are two framings of a manifold M^n such that $F_2/F_1 : M \to O$ factors through $i : P^\infty \to O$ then the induced Wu orientations are equal if $n+2 \neq 2^r$.*

(In fact a more careful analysis shows that the conclusion holds without assuming such a factorisation.)

The proposition shows that if there is a framed manifold M^n with non-zero Kervaire invariant then there is a manifold N^n satisfying the conditions of the corollary. This proves the result of Browder.

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