Constructing elements with Kervaire invariant one

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The hunt

- We now know that the only dimensions in which there are framed manifolds with Kervaire invariant one are 2, 6, 14, 30, 62, and possibly 126.
- However there is no known systematic construction which produces examples in the five dimensions where examples are known to exist.
- The aim must be to find a systematic construction which produces examples in each of these six special dimensions.
- Is this list of six special cases related to exceptional Lie groups ?
- The hunt has begun!

The three problems

- Construct framed manifolds with Kervaire invariant one.
- Construct maps between spheres detected by h_j^2 in the E_2 – term of the classical mod 2 Adams spectral sequence.
- Construct a diffeomorphism of the Kervaire sphere with the standard sphere.
- In each case there are at most six special cases where these (as we must now call them) exotic phenomena exist.
- I will summarise what is known about constructing examples in these special cases, starting with framed manifolds.

Framed manifolds in dimensions 2, 6, 14

- $K(S^1 \times S^1, F_1 \times F_1) = 1$ where F_1 is the complex framing.
- $K(S^3 \times S^3, F_3 \times F_3) = 1$ where F_3 is the quaternionic framing.
- $K(S^7 \times S^7, F_7 \times F_7) = 1$ where F_7 is the octonionic framing.
- These three examples are the three examples known to Kervaire and Milnor.
- $S^1 \times S^1$ and $S^3 \times S^3$ are Lie groups.
- $S^7 \times S^7$ is not a Lie group.
- However $Spin(8)/G_2 = S^7 \times S^7$

A framed manifold in dimension 30

- The dihedral group D_8 acts on freely on a closed orientable surface Y^2 of genus 5 with quotient $\mathbb{RP}^2 + (S^1 \times S^1)$.
- It also acts on $(S^7)^4$ via its usual permutation representation in Σ_4
- Now form

$$M^{30} = Y^2 \times_{D_8} (S^7 \times S^7 \times S^7 \times S^7).$$

- Any framing of S^7 induces a framing of M^{30}.
- Let F be the framing of M^{30} induced by the octonionic framing of S^7. Then

$$K(M^{30}, F) = 1.$$

- I proved this in my thesis :))

Some comments on the 30 dimensional case

- Note that M^{30} has an obvious framing with Kervaire invariant 0, this is the framing induced by the framing of S^7 as the boundary of the disc.
- The proof that $K(M^{30}, F) = 1$ uses the change of framing formula.
- For those who know about Toda brackets: this construction is based on geometrically modelling the Toda bracket

$$\langle \sigma, 2\sigma, \sigma, 2\sigma \rangle.$$

- This seems to be the only known explicit example in dimension 30.
- I do not know if this is really related to Lie groups but more of that later.
- There is no known explicit example in dimension 62.

An attempt to generalise

- We can consider 62 dimensional manifolds of the form
$$M^{62} = Y^6 \times_G (S^7)^8$$
where G is a subgroup of Σ_8.
- We can choose Y^6 and G so that framings of S^7 induce framings of M^{62}.
- However if we equip such a manifold with a framing induced by a framing of S^7 then it has Kervaire invariant zero.

Another attempt to generalise

- We could try to replace $S^7 \times S^7$ by a 30 dimensional framed manifold P with Kervaire invariant 1 and an involution.
- The involution gives an action of D_8 on $P \times P$ and we can form

$$M^{62} = Y^2 \times_{D_8} (P \times P)$$

  where Y^2 is the surfaces used in the 30 dimensional example.
- It is not true that any framing of P induces a framing of M; only those that are related to the involution in a particular way do so.
- If F is a framing of P which does induce a framing of M then $K(P, F) = 0$.

The problem with these attempts to generalise

- It is not easy to formulate this precisely.
- However the 30 dimensional construction uses implicitly the fact that $(S^1 \times S^7 \times S^7, F_1 \times F_7 \times F_7)$ is a framed boundary.
- However if (M^{30}, F) has Kervaire invariant one then $(S^1 \times M^{30}, F_1 \times F)$ is never a framed boundary.
- For the cognoscenti: $\eta\theta_3 = 0$ but $\eta\theta_4 \neq 0$.
- So we turn to the homotopy theory and h_j^2.

Some homotopy theory

- Let X be a space and

$$f : S^{2n+m} \to X, \quad g : X \to S^m$$

be two maps. Form the mapping cones

$$Y = X \cup_f D^{2n+m+1}, \quad Z = S^m \cup_g C(X).$$

- Assume both f and g are zero in mod 2 cohomology. Then there are isomorphisms

$$H^j(Y) \to H^j(X) \to H^{j+1}(Z), \quad \text{for } m < j < 2n + m + 1.$$

- Let $a \in H^m(Z)$ be the cohomology class corresponding to S^m.
- Let $b \in H^{2n+m+1}(Y)$ be the cohomology class corresponding to the $2n + m + 1$ disc.
- Let $\phi : H^j(Y) \to H^{j+1}(Z)$ be the above isomorphism.

More homotopy theory

- Are there triples (X, f, g) as above such that

$$Sq^{n+1}(a) = \phi(x), \quad Sq^{n+1}(x) = b.$$

- Easy to show that if so then $n+1$ must be of the form 2^j.
- When $n+1 = 2^j$ such a triple exists if and only if h_j^2 is an infinite cycle in the mod 2 Adams spectral sequence.
- When $j = 1, 2, 3$ then by Hopf invariant one we can take X to be the sphere S^{2n+1}.
- When $j = 4$ there are examples where X has 3 cells. (Mahowald – Tangora : Some differentials in the Adams spectral sequence, Topology 1967)
- When $j = 5$, in the only known example X has 9 cells. (Barratt – Jones – Mahowald: Relations amongst Toda brackets and the Kervaire invariant in dimension 62, Journal of the LMS 1984)
- The case where $j = 6$ is unknown.

The inductive approach

- We use the notation θ_j for an element in $\pi^s_{2^{j+1}-2}$ detected by h_j^2.
- So we know that θ_j exists for $j = 1, 2, 3, 4, 5$.
- The inductive approach to the Kervaire invariant problem is to assume that θ_j exists and has some more properties and show that θ_{j+1} exists.
- The most concrete result this gives is this. Suppose θ_j exists, $2\theta_j = 0$ and $\theta_j^2 = 0$. Then θ_{j+1} exists and $2\theta_{j+1} = 0$.
- This works to construct θ_4 since we can take θ_3 to be σ^2 where $\sigma \in \pi^s_7$ is the class of the Hopf map $S^{15} \to S^7$. Then it is easy to show that $2\theta_3 = 0$ a little bit more difficult to show that $\theta_3^2 = \sigma^4 = 0$ and so we see that θ_4 exist and has order 2.
- It is known that $\theta_4^2 = 0$ and so $2\theta_5 = 0$. It is not known whether $\theta_5^2 = 0$.

What does all this have to do with exceptional Lie groups?

- Honest answer: Don't really know but it is hard to believe that the answer is nothing!
- There are six special examples of homogeneous spaces with dimensions
$$4, 8, 16, 32, 64, 128$$
- Are these related to the Kervaire invariant in dimensions
$$2, 6, 14, 30, 62, 126.$$
- Here are the homogeneous spaces – you will find them playing a key role in Adams's book: Lectures on Exceptional Lie Groups.

The 6 special homogeneous spaces

- $\mathbb{P}^2(\mathbb{C}) = U(3)/(U(2) \times U(1))$
- $\mathbb{P}^2(\mathbb{H}) = Sp(3)/(Sp(2) \times Sp(1))$
- $\mathbb{P}^2(\mathbb{O}) = F_4/Spin(9)$
- $\mathbb{P}^2(\mathbb{C} \otimes \mathbb{O}) = E_6/((Spin(10) \times U(1))/\mathbb{Z}_4)$
- $\mathbb{P}^2(\mathbb{H} \otimes \mathbb{O}) = E_7/((Spin(12) \times Sp(1))/\mathbb{Z}_2)$
- $\mathbb{P}^2(\mathbb{O} \otimes \mathbb{O}) = E_8/(Spin(16)/\mathbb{Z}_2)$

Bokstedt

- In each of these special homogeneous spaces there is a middle dimensional cohomology class u such that u^2 is the fundamental class in the top dimension.
- Bokstedt proposes to use this as follows.
- P is the homogeneous space and $2n+2 = 2^{j+1}$ is its dimension.
- X is the $2n+1$ skeleton of P and $f : S^{2n+1} \to X$ is the attaching map of the $2n+2$ cell.
- Now suppose X is (stably $2n+2$ Spanier Whitehead) self dual.
- Then for some (large) m we can form the triple

$$\Sigma^{m-1} f : S^{2n+m} \to \Sigma^{m-1} X, \quad g : \Sigma^{m-1} X \to S^m$$

  where g is the Spanier Whitehead dual of f.
- The triple $(\Sigma^{m-1} X, \Sigma^{m-1} f, g)$ satisfies the conditions required to show that h_j^2 is an infinite cycle.

Bokstedt

- In the first three cases X is
$$S^2 = \mathbb{P}^1(\mathbb{C}), \quad S^4 = \mathbb{P}^1(\mathbb{H}), \quad S^8 = \mathbb{P}^1(\mathbb{O})$$
and so it is self dual.
- In the next case X is not self dual.
- However by using a combination of Morse theory and computations in homotopy theory, Bokstedt manages to find a self-dual complex of X and to compress f to this self dual complex.
- It is not known whether this approach can be made to work in dimensions 62 and 126.

The Gromoll – Meyer sphere

- This is work of Duran and Puttmann.
- The Gromoll – Meyer sphere is a Riemannian manifold whose underlying smooth manifold is Milnor's exotic 7 sphere W^7. In other words it is a Riemannian metric on W^7.
- W^7 is the quotient of a free action of S^3 on $Sp(2)$ and this defines the metric on W^7.
- We can identify W^7 with the subspace of $\mathbb{C}^5$ defined by the equations

$$z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^5 = 0.$$

$$|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 = 1.$$

- Notice that W^7 contains the Kervaire 5-sphere.

The Gromoll – Meyer sphere

- S^7 is a quotient of a different S^3 action on $Sp(2)$ and this quotient also defines the usual metric on S^7.
- Let $\pi_S : Sp(2) \to S^7$ and $\pi_W : Sp(2) \to W^7$ be the two projections.
- It is not true, in general, that fibres of π_S are fibres of π_W. However if a fibre of π_S contains a matrix A with real entries then it is also a fibre of π_W.
- Choose a real matrix $A \in Sp(2)$ with determinant 1.
- Let $Q \in S^7$ be the point $\pi_S(A)$ and let γ be a great circle in S^7 passing through Q.
- Duran and Puttmann show how to lift γ to a smooth (but not necessarily closed) curve $\tilde{\gamma}$ in $Sp(2)$ such that $\pi_W \tilde{\gamma}$ is a geodesic in W that starts and ends at $\pi_W(A)$.
- However the closed curve $\pi_W \tilde{\gamma}$ is not smooth.

The Gromoll – Meyer sphere

- From these geometric facts Duran and Puttmann construct an explicit homeomorphism of S^7 with W^7 that is smooth in the complement of a point.
- This diffeomorphism maps a copy of $S^5 \subset S^7$ diffeomorphically onto the Kervaire sphere $K^5 \subset W^7$.
- They then write down a formula for this diffeomorphism using quaternionic multiplication.
- Their formula with quaternionic multiplication replaced by octonionic multiplication gives a diffeomorphism of S^{13} with the Kervaire sphere K^{13}.
- Their diffeomorphism is G_2 (the symmetry group of the octonions) invariant.

One final point

- Using the general theory of Browder and Brown it is possible to define a quadratic form q on $H^n(M^{2n})$ using a weaker structure than a framing.
- However, this quadratic form may not be defined for all values of n.
- When it is defined it will in general take values in $\mathbb{Z}/4$ and quadratic will mean

  $$q(x+y) = q(x) + q(y) + 2\langle x, y\rangle.$$

  This $\mathbb{Z}/4$ valued quadratic form has a generalised Arf invariant $B(q) \in \mathbb{Z}/8$.
- If q takes values in $\{0, 2\} \subset \mathbb{Z}/4$ then we can identify q with a $\mathbb{Z}/2$ valued quadratic form.
- In this case $B(q) \in \{0, 4\}$ and $B(q) \neq 0$ if and only if the Arf invariant of the corresponding $\mathbb{Z}_2$ valued quadratic form is non – zero.

Codimension 1 immersions

- For example this more general theory applies if the manifold M comes equipped with an isomorphism of $TM \oplus L$ with the trivial bundle; here L is a line bundle over M.
- If L is trivial this is the same as a framing.
- Geometrically, such a structure corresponds to an immersion of M in codimension 1.
- In this context the generalised Kervaire invariant is defined in all dimensions of the form $2^{j+1} - 2$ and it is non-zero in all these dimensions.
- In dimensions 2, 6 this generalised Kervaire invariant can take any value in $\mathbb{Z}/8$.
- In the other dimensions of the form $2^{j+1} - 2$ it can take any value in $\{0, 2, 4, 6, 8\} \subset \mathbb{Z}/8$.

Oriented codimension 2 immersions

- The more general theory also applies if the manifold M comes equipped with an isomorphism of $TM \oplus P$ with the trivial bundle; here P is an oriented 2-plane bundle over M.
- This time immersion theory shows that this structure corresponds to an orientation of M and an oriented immersion in codimension 2.
- The quadratic form is defined for all dimensions of the form $2^{j+1} - 2$.
- In these dimensions the quadratic from is always $\mathbb{Z}/2$ valued so the invariant is the Kervaire invariant of a $\mathbb{Z}/2$ valued quadratic form.
- In each dimension of the form $2^{j+1} - 2$ there is an oriented codimension 2 immersion with Kervaire invariant one.
- Cohen – Jones – Mahowald 1985: The Kervaire invariant of immersions (Inventiones Math).

Back to the hunt.

Thank you once more.