The Kervaire invariant problem

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The Kervaire invariant

- A framing of a manifold M is an isomorphism F of the stable normal bundle of M with a trivial bundle.
- Suppose the dimension of M is $2n$. We can use a framing F to construct a quadratic function

$$q = q_F : H^n(M, \mathbb{Z}/2) \to \mathbb{Z}/2$$

$$q(x + y) = q(x) + q(y) + \langle x, y \rangle.$$

  Here $\langle x, y \rangle$ is the mod 2 intersection number of x and y.
- Since q is quadratic $|q^{-1}(0)| \neq |q^{-1}(1)|$.
- q has a $\mathbb{Z}/2$ invariant, its Arf invariant,

$$A(q) = 1 \iff |q^{-1}(0)| < |q^{-1}(1)|.$$

- The Kervaire invariant $K(M, F)$ is the Arf invariant of q_F.
- In dimensions $4k + 2$ the Kervaire invariant should be thought of as the analogue of the signature in dimensions $4k$.

The Kervaire invariant problem

- Problem: In what dimensions is there a framed manifold with Kervaire invariant one ?
- Answer: In dimensions 2, 6, 14, 30, 62 and possibly 126.
- The solution of the Kervaire invariant problem has a significant impact in both differential topology and homotopy theory.
- I will begin with the differential topology.

The Kervaire sphere

- $f_d(z_1, \ldots, z_{d+1}) = z_1^2 + \cdots + z_d^2 + z_{d+1}^3$
- The Kervaire sphere is the link of the singular point of f_{2n+1}

$$K^{4n+1} = f_{2n+1}^{-1}(0) \cap S^{4n+3} \subset \mathbb{C}^{2n+2}.$$

- Kervaire constructed K^{4n+1} by what is known as plumbing.
- We know that K^{4n+1} is homeomorphic to S^{4n+1}.
- Problem: When is K^{4n+1} diffeomorphic to S^{4n+1} ?
- Answer: When $4n+1$ is 1, 5, 13, 29, 61 and possibly 125.

The Kervaire invariant and the Kervaire sphere

- K^{4n+1} is the boundary of a framed $4n+2$ manifold P_0^{4n+2}.
- If K^{4n+1} is diffeomorphic to S^{4n+1} we can glue a disc onto the boundary of P_0^{4n+2} to get a smooth manifold P^{4n+2}.
- P^{4n+2} can be framed and there is a framing F such that $K(P^{4n+2}, F) = 1$.
- Kervaire then does some homotopy theory to prove that

$$K(M^{10}, F) = 0, \qquad \text{for all } (M^{10}, F).$$

- It follows that K^9 cannot be diffeomorphic to S^9.
- This argument, plus the solution of the Kervaire invariant problem, leads to the list of at most six special cases where the Kervaire sphere is diffeomorphic to the standard sphere.

Kervaire and Milnor

- A *homotopy n-sphere* is a closed manifold that is homotopy equivalent to S^n.
- How many homotopy spheres are there?
- Milnor 1956: On manifolds homeomorphic to the 7 sphere. (Annals of Math)
- Kervaire 1960: A manifold which does not admit any smooth structure. (Comment Math Helv)
- Kervaire and Milnor 1963 : Groups of homotopy spheres I. (Annals of Math)
- Kervaire and Milnor caculate the number of homotopy n-spheres in terms of the homotopy groups of spheres, modulo the Kervaire invariant problem.

Framed manifolds and homotopy theory

- Given a map $f : S^{n+k} \to S^n$ and a point $P \in S^n$ such that f is transverse to P can form

$$M^k = f^{-1}(P) \subset S^{n+k}.$$

  Then M has a natural framing F and so we get (M^k, F) a framed submanifold of S^{n+k}.
- This construction sets up an isomorphism of $\Omega^{fr}_{k,n}$, the framed cobordism classes of k dimensional closed framed submanifolds of S^{n+k}, with the group $\pi_{n+k}(S^n)$.
- This is the *Pontryagin – Thom* construction.
- Both groups are independent of n if $k < n-1$ and in this range we write

$$\Omega^{fr}_k \cong \pi^s_k.$$

Groups of homotopy spheres

- The set of homotopy n-spheres forms a group Θ_n under connected sum.
- A homotopy sphere Σ^n can be framed.
- Choose a framing to get an element

$$[\Sigma^n, F] \in \Omega_n^{fr} = \pi_n^s.$$

- Suppose F_1 and F_2 are framings of Σ^n then

$$[\Sigma^n, F_1] - [\Sigma^n, F_2] = [S^n, \Phi]$$

  for some framing Φ of S^n.
- Set $J_n \subseteq \pi_n^s$ to be the subgroup consisting of those elements $[S^n, \Phi]$ where Φ is a framing of S^n – this is the image of J.
- Then define $P(\Sigma^n) = [\Sigma^n, F] \in \pi_n^s / J_n$.
- This gives a well defined homomorphism

$$P = P_n : \Theta_n \to \pi_n^s / J_n.$$

Kervaire and Milnor

- Use the notation $bP_{n+1} = \ker P_n,\ C_n = \pi_n^s / J_n$.
- $P_{4n} : \Theta_{4n} \to C_{4n}$ is an isomorphism
- There is an exact sequence

$$0 \to bP_{4n+4} \to \Theta_{4n+3} \to C_{4n+3} \to 0$$

  and bP_{4n+4} is a cyclic group whose order is explicitly computed (in terms of Bernoulli numbers) by Kervaire and Milnor.
- There is an exact sequence

$$0 \to \Theta_{4n+2} \to C_{4n+2} \to \mathbb{Z}/2 \to \Theta_{4n+1} \to C_{4n+1} \to 0$$

  where the homomorphism $\Omega_{4n+2}^{fr} \to C_{4n+2} \to \mathbb{Z}/2$ is the Kervaire invariant.

Browder's theorem

- The Kervaire invariant of framed manifolds is zero except in dimensions of the form $2n = 2^{j+1} - 2$.
- In dimension $2^{j+1} - 2$ there is a framed manifold of Kervaire invariant one if and only if h_j^2 in the E_2 term of the classical mod 2 Adams spectral sequence is an infinite cycle.
- Browder 1969: The Kervaire invariant of framed manifolds and its generalizations (Annals of Math)

Browder's proof

- Browder uses the notion of a *Wu orientation* and the corresponding notion of *Wu cobordism*.
- There is a Kervaire invariant defined for Wu oriented (not necessarily oriented) manifolds.
- In the relevant dimensions Wu cobordism is a computable modification of unoriented cobordism.
- Can compute the image of framed cobordism in Wu cobordism in the relevant dimensions.
- There are other proofs, one due to Rees – Jones, and another due to Lannes.
- We now turn to the homotopy theory.

Mahowald

Are the homotopy groups of spheres the universal widget generated by the EHP sequence and the solution of the vector fields on spheres problem?

The EHP sequence

This is the exact sequence

$$\cdots \to \pi_j(S^n) \to \pi_{j+1}(S^{n+1}) \to \pi_{j+1}(S^{2n+1}) \to \pi_{j-1}(S^n) \to \dots$$

where the homomorphisms are:

- $E : \pi_j(S^n) \to \pi_{j+1}(S^{n+1})$ is the suspension homomorphism,
- $H : \pi_{j+1}(S^{n+1}) \to \pi_{j+1}(S^{2n+1})$ is the Hopf invariant,
- $P : \pi_{j+1}(S^{2n+1}) \to \pi_{j-1}(S^n)$ is the 'Whitehead product ';
- Also we should localize at 2.

Calculating with the EHP sequence

- This was used extensively in the late 50's and early 60's to calculate the groups $\pi_j(S^n)$ most prominently by Toda (Composition methods in the homotopy groups of spheres: Annals of Math Studies 1962) and also by Barratt and others.
- The *stem* of $\pi_j S^n$ is $j - n = k$.
- The idea is to calculate inductively on the stem.
- When we come to calculate the k-stem

  $$\pi_{k+1}(S^1), \quad \pi_{k+2}(S^2), \quad \pi_{j+3}(S^3), \quad \dots \quad , \quad \pi_{2k+2}(S^{k+1})$$

  we have already calculated the source and target of P.
- The key then is to be able to calculate P.

Back to Mahowald

- Question: Does the EHP sequence uniquely determine the homotopy groups of spheres ?
- Answer: Clearly no !
- Better Question: Are there some initial conditions we can add so that the EHP sequence plus these initial conditions does determine the homotopy groups of spheres ?
- For example, try the (silly !) assumption that $P(\iota_n) = 0$ for all n. Here $\iota_n \in \pi_n(S^n)$ is the homotopy class of the identity map of S^n.
- According to Mahowald you should then get the so-called Λ algebra – this is the E_1 term of the Adams spectral sequence.
- If we assume that $P(\iota_n) = 0$ if and only if $n = 2^j - 1$ then Mahowald predicts we will get the E_2 term of Adams spectral sequence.
- Here you start to see the key phenomenon: the interplay between calculations with the EHP sequence and the Adams spectral sequence.

Sphere of origin and Hopf invariant

- If $\alpha \in \pi_{k+n}(S^n)$ the the *sphere of origin* of α is the minimum integer $m \geq 0$ such that

$$\alpha \in \mathrm{im}(E^{n-m} : \pi_{k+m}(S^m) \to \pi_{k+n}(S^n)).$$

- The Hopf invariant of α is the set

$$H(\alpha) = \{H(\beta) : E^{n-m} = \alpha\}$$

where m is the sphere of origin of α.

The solution of the vector fields on spheres problem

- We now ask how the elements $\iota_n \in \pi_n(S^n)$ feed into the EHP sequence.
- Not difficult to see that the key is to understand

$$P(\iota_{2^i-1}) = w_i \in \pi_{2^{i+1}-3}(S^{2^i-1})$$

- w_1, w_2, w_3 are all zero and this generates the three Hopf maps which will be denoted by β_1, β_2, and β_3.
- β_1, β_2, and β_3 are the first three generators of the image of J
- On the other hand w_4 is non-zero so this generates an element β_4 which is the Hopf invariant of w_4.
- This element β_4 gives us the next generator of the image of J.
- More generally if we define β_n for $n \geq 4$ by

$$\beta_n = H(w_n)$$

  then the family of elements β_i give us generators of the image of J.

The image of J in the EHP sequence

- In this paper in Annals of Math 1982, Mahowald computes how the image of J, the family of elements β_j, behave in the EHP sequence.
- He gets complete answers modulo one problem.
- Let $s(j)$ be the stem of β_j then the problem is the computation of

$$P(\beta_j) \in \pi_{2n-1+s(j)}(S^n), \quad n + s(j) = 2^{j+1} - 2$$

- Let us look at the EHP sequence in this particular dimension

$$\pi_{2n+1+s(j)}(S^{n+1}) \to \pi_{2n+1+s(j)}(S^{2n+1}) \to \pi_{2n-1+s(j)}(S^n)$$

  Then suppose $P(\beta_j) = 0$ we get an interesting element θ_j in the $n + s(j) = 2^{j+1} - 2$ stem.
- This element θ_j should be detected by h_j^2 in the Adams spectral sequence.

θ_j and w_{j+1}

- Notice that $w_{j+1} = P(\iota_{2^{j+1}-1}) \in \pi_{2^{j+2}-3}(S^{2^{j+1}} - 1)$
- Therefore w_{j+1} and θ_j are in the same stem.
- The sphere of origin of w_{j+1} is $2^{j+1} - s(j+1)$ and its Hopf invariant is β_{j+1}
- The sphere of origin of θ_j is $2^{j+1} - s(j)$ and its Hopf invariant is β_j
- The simplest possible explanation of these facts is this:

$$w_{j+1} = 2\theta_j \in \pi_{2^{j+2}-3}(S^{2^{j+1}-1}).$$

- It takes a considerable amount of work to replace the occurrences of the phrase should be "should be" by the word theorem. See papers of Mahowald, Barratt – Jones – Mahowald and Crabb – Knapp.

I have tried to give a connected narrative, rather than a careful history, highlighting the impact of the Kervaire invariant in differential topology and homotopy theory.

Thank you for you attention