

## REFERENCES

1. J. F. ADAMS: On the structure and applications of the Steenrod algebra, *Comment. Math. Helvet.* **32** (1958), 180-214.
2. J. F. ADAMS: Lectures on generalized homology, *Lecture Notes in Mathematics*, Vol. 99, 1-138. Springer-Verlag (1969).
3. J. F. ADAMS: *Stable Homotopy and Generalized Homology*. Univ. of Chicago Press (1974).
4. M. BARR: Composite triples and derived functors, *Lectures Notes in Mathematics*, Vol. 80, 336-356. Springer-Verlag (1969).
5. M. BARRATT: Homotopy operations and homotopy groups, mimeographed notes, Seattle, Conf. (1963).
6. A. K. BOUSFIELD: Nice homology coalgebras, *Trans. Am. math. Soc.* **148** (1970), 473-489.
7. A. K. BOUSFIELD and E. B. CURTIS: A spectral sequence for the homotopy of nice spaces, *Trans. Am. math. Soc.* **151** (1970), 457-479.
8. A. K. BOUSFIELD and D. M. KAN: The homotopy spectral sequence of a space with coefficients in a ring, *Topology* **11** (1972), 79-106.
9. H. CARTAN and S. EILENBERG: *Homological Algebra*. Princeton Univ. Press (1956).
10. E. B. CURTIS: Some relations between homotopy and homology, *Ann. Math.* **83** (1965), 386-413.
11. S. EILENBERG and J. C. MOORE: Adjoint functors and triples, *III. J. Math.* **9** (1965), 381-398.
12. B. GRAY: On the sphere of origin of infinite families in the homotopy groups of spheres, *Topology* **8** (1969), 219-232.
13. D. HUSEMOLLER and J. C. MOORE: Differential graded homological algebra of several variables, *Symp. Math.*, Vol. IV (1970), *Inst. Naz. di Alta Mat.*, 397-429.
14. H. KLEISLI: On the construction of standard complexes, *J. pure appl. Algebra* **4** (1974), 243-260.
15. S. MACLANE: *Categories for the Working Mathematician*. Springer-Verlag (1971).
16. M. MAHOWALD: Description homotopy of the elements in the image of the J-homomorphism, *Manifolds-Tokyo 1973*, Univ. of Tokyo Press, 1975, 255-264.
17. H. R. MILLER, D. C. RAVENEL and W. S. WILSON: Periodic phenomena in the Adams-Novikov spectral sequence, *Ann. Math.* **106** (1977), 469-516.
18. H. R. MILLER and W. S. WILSON: On Novikov's Ext' modulo an invariant prime ideal, *Topology* **15** (1976), 131-143.
19. J. W. MILNOR: On axiomatic homology theory, *Pacif. J. Math.* **12** (1962), 337-341.
20. S. P. NOVIKOV: The methods of algebraic topology from the viewpoint of cobordism theories, *Math. U.S.S.R.-Izvestia* (1967), 827-913.
21. D. C. RAVENEL: A novice's guide to the Adams-Novikov spectral sequence, to appear.
22. D. C. RAVENEL and W. S. WILSON: The Hopf ring for complex cobordism, *J. pure appl. Algebra* **9** (1977), 241-280.
23. R. M. SWITZER: *Algebraic Topology-Homotopy and Homology*. Springer-Verlag (1975).
24. G. W. WHITEHEAD: Generalized homology theories, *Trans. Am. math. Soc.* **102** (1962), 227-283.
25. W. S. WILSON: The  $\Omega$ -spectrum for Brown-Peterson cohomology, Part I, *Comment. Math. Helvet.* **48** (1973), 45-55.

## THE KERVAIRE INVARIANT OF EXTENDED POWER MANIFOLDS

JOHN D. S. JONES

(Received 3 April 1978)

### INTRODUCTION

SUPPOSE  $M$  is a compact smooth manifold whose stable normal bundle  $\nu M$  is trivial. Let  $F$  be a specific isomorphism of  $\nu M$  with the trivial bundle. The pair  $(M, F)$  will be referred to as a framed manifold. Suppose the dimension of  $M$  is  $2n$  where  $n$  is odd, then, in [9] (see also [10]), Kervaire has defined a  $Z/2$  valued invariant  $K(M, F)$ . Browder has shown [2] that  $K(M^{2n}, F) = 0$  unless  $n = 2^k - 1$ , and has given a necessary and sufficient homotopy theoretic condition for the existence of  $2^{k+1} - 2$  dimensional framed manifolds with Kervaire invariant one. It is of much interest, both in differential topology and, via Browder's result, in homotopy theory to decide in which dimensions of the form  $2^{k+1} - 2$  such manifolds can exist.

In dimensions 2, 6 and 14,  $S^1 \times S^1$ ,  $S^3 \times S^3$  and  $S^7 \times S^7$  can be framed to have Kervaire invariant one. In dimensions 30 and 62 the necessary homotopy theory has been done: the 30-dimension case is published in [13], the 62-dimension case is due to Barratt and Mahowald but is not yet published. The problem is unsolved in dimensions greater than 62.

Extended power manifolds are constructed in the following manner: Let  $Y$  be a manifold on which the group  $G \subset \Sigma_t$  acts freely ( $\Sigma_t$  is the group of permutations of a set with  $t$  elements). Let  $N$  be another manifold, then  $G$  acts on the cartesian product  $(N)^t$  by permuting factors. We may form the "extended power"  $Y \times_G (N)^t$  which will always be denoted  $Y_G(N)$ . This is obviously based on the extended power construction in homotopy theory, a construction which has been extensively studied. Some of this work, and references to the origins of the ideas involved, may be found in Milgram's papers [16-18] and Nishida's papers [19-21].

The purpose of this paper is to examine the Kervaire invariant of extended powers of  $S^7$ , a project suggested some time ago by Barratt.

First it is necessary to decide when  $Y_G(N)$  can be framed. Define  $\bar{Y} = Y/G$  and let  $\xi$  be the  $t$ -dimensional bundle  $Y_G(R) \rightarrow \bar{Y}$ .

**THEOREM A.** *Let  $N^n$  be a framed manifold. Then  $Y_G(N)$  can be framed if and only if  $\tau \bar{Y} + n\xi$  is stably trivial. (Here  $\tau X$  stands for the tangent bundle of the manifold  $X$ .) Given a framing  $F$  of  $N$  and a stable trivialisation  $\alpha$  of  $\tau \bar{Y} + n\xi$  then there is an associated framing of  $Y_G(N)$ , denoted  $\alpha_G(F)$ .*

This theorem, in statement and proof, is a straightforward generalisation of results due to Milgram [15]. Milgram considers the case  $Y = S^m$  and  $G = \Sigma_2$ .

From now on assume  $Y, G$  and  $N$  have been chosen so that  $Y_G(N)$  can be framed. We go on to study framings of  $Y_G(N)$  induced by framings of  $N$ . Let  $g: N \rightarrow O$  be a map ( $g$  will be identified with an element of  $KO^{-1}N$ ) and let  $F$  be a framing of  $N$ . Then, as described in [23, §2] we may twist  $F$  by  $g$  and obtain a new framing  $gF$ .

**THEOREM B.** *Suppose  $G$  is a 2-group. Then there is a homomorphism  $h: KO^{-1}N \rightarrow KO^{-1}Y_G(N)$  such that*

$$\alpha_G(gF) = h(g)\alpha_G(F).$$

The homomorphism  $h$  will be described in detail in §3.

We come now to the main results of the paper, concerning the Kervaire invariant.

invariant, the Arf invariant  $A(q_F)$ , associated to  $q_F$  [1]. A thorough account of this invariant is given in [20, Appendix pp. 411–413];  $A(q_F)$  is zero if and only if  $q_F$  sends the majority of elements of  $H^m M$  to zero.

The Kervaire invariant is now defined by

$$K(M, F) = A(q_F).$$

It is a framed bordism invariant.

The first result we require is the change of framing formula due essentially to Brown [3, Thm 1.18, 4, p. 299, Thm 3.3]. A proof of this key result is also given in [6]. To state the result requires some notation. Let  $v_{n+1} \in H^{n+1} B\mathbb{O}$  be the universal  $(n+1)$ -th Wu-class (see [2, p. 164]). Let  $y_n = \Omega^{L_{n+1}}$  in  $H^n \mathbb{O}$ .

1.1 THEOREM. Let  $(M^{2n}, F)$  be a framed manifold and  $g: M \rightarrow \mathbb{O}$ . Let  $gF$  be the framing obtained by twisting  $F$  by  $g$ . Then

- (i)  $q_{gF}(x) = q_F(x) + x \cdot g^* y_n$
- (ii)  $K(M, gF) = K(M, F) + q_F(g^* y_n)$ .

The second result required concerns the evaluation of the quadratic form  $q_F$  on a class of the form  $Sq^k y$ .

1.2 THEOREM. Let  $(M^{2n}, F)$  be a framed manifold where  $n = 2^l - 1$  with  $l \geq 4$ . Suppose  $k = 1, 2$  or  $4$ . Then if  $y \in H^{n-k} M$

$$q_F(Sq^k y) = \sum_{i=0}^{k-1} (Sq^i y) \cdot (Sq^{2^k-i} y).$$

Some comments on this theorem are necessary. The hypotheses on  $n$  and  $k$  are sufficient for the use of this theorem in this paper. However, the theorem, as stated, is valid for many more values of  $n$  and  $k$ . A proof of a more general version of this theorem will be given in a forthcoming paper [8].

§2. THE EXTENDED POWER CONSTRUCTION

We will take for granted throughout this section the notation and assumptions of Theorem A. We begin with a proof of Theorem A following Milgram's proof in [15] for the special case  $Y = S^m, G = \Sigma_2$ . Let  $\pi: Y_G(N) \rightarrow \bar{Y}$  be the projection.

2.1 LEMMA.  $\tau Y_G(N)$  and  $\pi^*(\tau \bar{Y} + n\xi)$  are stably isomorphic.

Proof. Since  $N^n$  can be framed there is an embedding  $N \times R^L \subset R^{n+L}$ . This gives an embedding  $j$

$$Y_G(N) \rightarrow Y_G(N \times R^L) \rightarrow Y_G(R^{n+L}).$$

The normal bundle of this embedding is  $\nu = Y_G(N \times R^L) = \pi^* L\xi$ . Since  $Y_G(R^{L+n})$  is the total space of the bundle  $(L+n)\xi$  it follows that  $\tau(Y_G(R^{L+n})) = p^*(\tau \bar{Y} + (L+n)\xi)$  where  $p: Y_G(R^{L+n}) \rightarrow \bar{Y}$  is the projection. From the definition of the normal bundle of an embedding

$$\tau Y_G(N) + \pi^* L\xi = j^* p^*(\tau \bar{Y} + (L+n)\xi).$$

However  $pj = \pi$ , so solving this equation stably gives

$$\tau Y_G(N) = \pi^*(\tau \bar{Y} + n\xi).$$

The projection  $\pi$  has a section, namely  $\bar{Y} = Y_G(x) \rightarrow Y_G(N)$  where  $x$  is any point of  $N$ . Thus  $\tau Y_G(N)$  is stably trivial if and only if  $\tau \bar{Y} + n\xi$  is stably trivial. Therefore  $Y_G(N)$  can be framed if and only if  $\tau \bar{Y} + n\xi$  is stably trivial. This proves the first assertion of Theorem A.

From now on assume that  $Y, G$  and the framed manifold  $N$  are chosen so that  $Y_G(N)$  can be framed. We now study framings of  $Y_G(N)$  induced by framings of  $N$ . The framing  $F$  of  $N$  gives, as in the proof of 2.1, a stable isomorphism of  $\tau Y_G(N)$  with  $\pi^*(\tau \bar{Y} + n\xi)$ . Combining this with a stable trivialisation of  $\tau \bar{Y} + n\xi$  gives a stable

We will use the wreath product  $\Sigma_2 \wr \Sigma_2 \subset \Sigma_4$ . This is a Sylow-2-subgroup of  $\Sigma_4$ . It is conjugate in  $\Sigma_4$  to the dihedral group  $D_4$ , that is the full symmetry group of the square.

THEOREM C. Let  $X$  be an orientable surface of genus 5, and  $G = \Sigma_2 \wr \Sigma_2$ . Then  $G$  can act freely on  $X$  so that

- (i)  $X_G(S^7)$  can be framed.

Let  $H$  be the Cayley number framing of  $S^7$  and let  $\alpha$  be any choice of stable trivialisation of  $\tau \bar{X} + 7\xi$  (the notation is as in Theorem A). Then

- (ii)  $K(X_G(S^7), \alpha_G(H)) = 1$ .

Notes. (i) The Cayley number framing of  $S^7$  may be described as follows. Let  $F$  be a framing of  $S^7$  which extends over  $D^8$ , and let  $g: S^7 \rightarrow SO$  be the map obtained from multiplication (on the left) by unit Cayley numbers. Then  $H = gF$ .

(ii) Using the description of  $G$  as the dihedral group  $D_4$ , one may give an explicit construction of the surface  $X$ .

(iii) The theorem remains true if  $S^7$  is replaced by  $S^3$  and the Cayley number framing replaced by the Quaternionic framing. If  $S^7$  is replaced by  $S^1$  then  $X_G(S^1)$  cannot be framed.

In view of Theorem C we go on to consider the group  $G = \Sigma_2 \wr \Sigma_2 \subset \Sigma_8$  and ask whether there is a 6 manifold  $Y$  on which  $G$  acts freely so that  $Y_G(S^7)$  with framing induced by the Cayley number framing of  $S^7$  has Kervaire invariant one. The next theorem shows this cannot happen. In the statement of the theorem  $G_k \subset \Sigma_{2^k}$  is the iterated wreath product  $G_k = \Sigma_2 \wr \dots \wr \Sigma_2$  ( $k$  copies of  $\Sigma_2$ ).

THEOREM D. Let  $Y$  be a  $d$ -dimensional manifold where  $d = 2^{l+1} - 2 - 7 \cdot 2^k$ . Suppose  $G_k$  acts freely on  $Y$  and that  $Y_{G_k}(S^7)$  can be framed. Let  $F$  be any framing of  $S^7$  and  $\alpha$  a stable trivialisation of  $\tau \bar{Y} + 7\xi$ . Then if  $d \neq 2$  the Kervaire invariant of the  $2^{l+1} - 2$  dimensional manifold  $Y_{G_k}(S^7)$  equipped with the framing  $\alpha_{G_k}(F)$  is zero.

Note that Theorem D does not assert that the Kervaire invariant of  $Y_{G_k}(S^7)$  is zero for all framings of this manifold, only for the "natural" framings, that is those induced by framings of  $S^7$ . Another way of expressing this is to say that we are considering the framed manifold  $Y_{G_k}(S^7)$  as a function of the framed manifold  $S^7$ .

The line of proof of Theorem D is to use Theorem B and the change of framing formula for the Kervaire invariant to show that if  $d \neq 2$  then the Kervaire invariant of  $Y_{G_k}(S^7)$  with framing  $\alpha_{G_k}(F)$  is independent of the choice of framing  $F$  of  $S^7$ . It is easy to see from the proof of Theorem A that we may choose  $F$  so that  $K(Y_{G_k}(S^7), \alpha_{G_k}(F)) = 0$  and so Theorem D will follow.

This paper is set out as follows: §1 contains the necessary generalities on the Kervaire invariant, §2 contains a discussion of the extended power construction, including the proof of Theorem A, §3 contains the proof of Theorem B. In §4 there is a plan of the proof of Theorems C and D, and a good deal of preparatory calculation is done, §5 contains the proof of Theorem C and §6 that of Theorem D.

A large proportion of this work is contained in my Oxford D.Phil. thesis, written under the supervision of Elmer Rees. It is a great pleasure to thank Elmer Rees for introducing me to this subject and for his constant help and encouragement.

§1. GENERALITIES ON THE KERVAIRE INVARIANT

Suppose  $(M^{2n}, F)$  is a closed framed manifold. Following Pontryagin [19], Kervaire [9], Kervaire and Milnor [10], Browder [2] and Brown [3] we may use the framing to construct a quadratic function

$$q_F: H^m M \rightarrow \mathbb{Z}/2$$

(all homology groups in this paper will have  $\mathbb{Z}/2$  coefficients), that is

$$q_F(x + y) = q_F(x) + q_F(y) + x \cdot y$$

where  $x \cdot y$  stands for the mod 2 intersection number of  $x$  and  $y$ . There is a mod 2

trivialisation of  $\tau Y_G(N)$ , that is a framing of  $Y_G(N)$ . We require rather detailed information about this framing and so we give a more explicit description of it.

Since  $\xi$  is a bundle with finite structural group over a finite complex, it follows that we may choose a large integer  $L$  and an isomorphism  $\beta: L\xi \rightarrow \epsilon^{2L}$  ( $\epsilon^e$  will always denote the trivial  $q$ -dimensional bundle). The stable trivialisation  $\alpha$  of  $\tau\tilde{Y} + n\xi$  gives an isomorphism, also denoted  $\alpha$

$$\alpha: \nu^k \tilde{Y} \rightarrow \epsilon^{k-m} + n\xi$$

where  $\nu^k \tilde{Y}$  is the  $k$ -dimensional normal bundle of  $\tilde{Y}$  with  $k$  large. We now get an embedding  $i: R^{k-m} \times Y_G(R^{L+n}) \rightarrow R^{d+k+2L}$  ( $d = \dim Y$ ) as follows:

$$R^{k-m} \times Y_G(R^{L+n}) = (\epsilon^{k-m} + n\xi) + L\xi \xrightarrow{\alpha^{-1} + \beta} \nu^k \tilde{Y} + \epsilon^{2L} \subset R^{d+k+2L}. \quad (2.2)$$

Let  $F$  be a framing of  $N$ , then  $F$  gives an embedding  $j_F: N \times R^L \rightarrow R^{L+n}$ , and so an embedding

$$R^{k-m} \times Y_G(N \times R^L) \rightarrow R^{k-m} \times Y_G(R^{L+n}). \quad (2.3)$$

Note that  $Y_G(N \times R^L) = \pi^* L\xi$  so there is an isomorphism  $h$

$$R^{k-m} \times R^{2L} \times Y_G(N) \xrightarrow{1 \times (\pi^* \beta)^{-1}} R^{k-m} \times Y_G(N \times R^L). \quad (2.4)$$

The composite of these three embeddings gives a framed embedding of  $Y_G(N)$  in  $R^{d+k+2L}$ . Denote the associated framing by  $\alpha_G(F)$ .

**2.5 LEMMA.** *The framing  $\alpha_G(F)$  does not depend on the choice of trivialisation  $\beta: L\xi \rightarrow \epsilon^{2L}$ .*

*Proof.* Suppose we alter  $\beta$  by an automorphism  $g: \epsilon^{2L} \rightarrow \epsilon^{2L}$ . Stably this alters the embedding  $i$  of (2.2) by adding  $\epsilon^{2L}$  and replacing  $i$  by  $i + g$ . The embedding of (2.3) remains unaltered. The isomorphism  $h$  of (2.4) is altered by adding  $\epsilon^{2L}$  and replacing  $h$  by  $h + g^{-1}$ . The composite remains unaltered and so, stably, the embedding is independent of  $\beta$ . However  $k$  and  $L$  can be chosen large enough so that the embedding is already stable. This completes the proof.

Next we examine how the framed bordism class of  $Y_G(N)$  with framing  $\alpha_G(F)$  depends on the framed bordism class of  $(N, F)$ .

**2.6 LEMMA.** *Suppose  $(N, F)$  is framed bordant to  $(P, H)$ . Then  $(Y_G(N), \alpha_G(F))$  is framed bordant to  $(Y_G(P), \alpha_G(H))$ .*

*Proof.* Let  $Y^+$  denote  $Y$  with a disjoint base point. For any pointed space  $X$  let  $X^{(t)}$  denote the  $t$ -fold smash product of  $X$  with itself. Given  $(N, F)$ , then when the framed embedding corresponding to the framing  $\alpha_G(F)$  is compactified the resulting map between spheres factorises as follows:

$$S^{d+k+2L} \xrightarrow{A} S^{k-m} \wedge Y^+ \wedge S^{(L+n)^{(t)}} \xrightarrow{1 \wedge \beta \wedge \Phi^{(t)}} S^{k-m} \wedge Y^+ \wedge S^{(L)^{(t)}} \xrightarrow{B} S^{k-m+2L}$$

where  $A$  is the compactification of (2.2),  $\Phi$  is obtained from compactifying the embedding  $j_F: N \times R^L \rightarrow R^{L+n}$  and  $B$  is obtained from compactifying (2.4). Once we have fixed  $\alpha$  the homotopy class of the above map depends only on the homotopy class of  $\Phi$ . By transversality the lemma follows.

We next show that the framed bordism class of  $(Y_G(N), \alpha_G(F))$  depends only on a suitable bordism class of  $(Y, \alpha)$ . Let  $r: G \rightarrow O(t)$  be the permutation representation, that is the representation of  $G$  obtained by allowing  $G$  to permute the basis of  $R^t$ . Let  $\rho$  be the bundle over  $BG$  classified by  $Br$ . Giving an isomorphism  $\alpha: \nu^k \tilde{Y} \rightarrow \epsilon^{k-m} + n\xi$  is equivalent to giving a bundle map  $\nu^k \tilde{Y} \rightarrow \epsilon^{k-m} + n\rho$ , which on base spaces classifies the principal  $G$  covering  $Y \rightarrow Y/G = \tilde{Y}$ . We may classify  $d$ -dimensional manifolds whose stable normal bundle admits such a structure up to the evident bordism

relation. The resulting group is denoted by  $\Omega_d(BG; n\rho)$ . Transversality shows that  $\Omega_d(BG; n\rho) \cong \pi_{d+m}(T(n\rho))$  where  $T(\eta)$  stands for the Thom complex of the bundle  $\eta$ . The proof of the next lemma follows directly from the definitions.

**2.7 LEMMA.** *Suppose  $(Y, \alpha)$  and  $(Z, \beta)$  define the same element of  $\Omega_d(BG; n\rho)$ . Then  $(Y_G(N), \alpha_G(F))$  is framed bordant to  $(Z_G(N), \beta_G(F))$  for all framed manifolds  $(N, F)$ .*

**§3. THE PROOF OF THEOREM B**

We now examine how the framing  $\alpha_G(F)$  depends on the framing  $F$ , and hence give a proof of Theorem B. The first task is to describe the homomorphism  $h: KO^{-1}N \rightarrow KO^{-1}Y_G(N)$ . Let  $\gamma: Y \rightarrow EG$  be the equivariant map classifying the free  $G$ -action on  $Y$ . Given  $g \in KO^{-1}N$  form the map

$$Y \times_G (N)^i \xrightarrow{\gamma \times_G (i)^i} EG \times_G (N)^i \xrightarrow{1 \times_G (i)^i} EG \times_G (O)^i \xrightarrow{D} O \quad (3.1)$$

where  $D$  is the Dyer-Lashof map for the infinite loop space  $O$ , see [5, pp. 36-41]. It follows from the properties of the Dyer-Lashof map (see the diagram on page 39 of [5]) that  $h$  is a homomorphism.

To prove Theorem B we require a concrete description of the Dyer-Lashof map  $D$  when  $G$  is a 2-group. This is provided by a straightforward generalisation of an observation in [12]. Some notation is required. Let  $G_k \subset \Sigma_2^k$  be the wreath product  $G_k = \Sigma_2 \wr \Sigma_2 \dots \wr \Sigma_2$  ( $k$  copies of  $\Sigma_2$ ), and let  $\kappa_{k,L}: G_k \rightarrow O(2^{k+L})$  be the representation defined by allowing  $G_k$  to permute the factors of  $(R^{2^k})^{\oplus L}$ .

**3.2 LEMMA.** *Let  $E^i G_k$  be the  $i$ -skeleton of  $EG_k$ . Then for each  $L \geq l + 2$  there exists a map  $f_{k,L}: E^i G_k \rightarrow O(2^{k+l})$  such that*

- (i)  $f_{k,L}(gx) = f_{k,L}(x) \kappa_{k,L}(g^{-1})$  for  $g \in G_k, x \in E^i G_k$
- (ii) *The following diagram commutes, up to homotopy.*

$$\begin{array}{ccc} E^i G_k \times_{G_k} (O(2^L))^{\oplus k} & \xrightarrow{f_{k,L}} & O(2^{L+k}) \\ \cong \cap & & \cap \\ E^i G_k \times_{G_k} (O)^{\oplus k} & \xrightarrow{D} & O \end{array}$$

Here  $f_{k,L}$  is defined by

$$f_{k,L}(x: A_1, \dots, A_k) = f_{k,L}(x)(A_1 \oplus \dots \oplus A_k) f_{k,L}(x)^{-1}$$

for  $x \in E^i G_k$  and  $A_1, \dots, A_k \in O(2^L)$ .

*Proof.* The proof is by induction on  $k$ . When  $k = 1, G = \Sigma_2$  and Madsen's description of the Dyer-Lashof map, [12, pp. 237-241], shows the result is true. We summarize Madsen's work. As usual  $S^l$  with the antipodal action of  $\Sigma_2$  will be identified with  $E^l \Sigma_2$ . For each  $L \geq l + 1$ , we know that  $S^l$  is contained in the units of the real Clifford algebra  $C_L$ . As a vector space  $C_L$  has dimension  $2^L$ . The map  $f_{1,L}: S^l \rightarrow O(2^{L+l})$  is defined as follows: Regard  $R^{2^{l+1}}$  as  $C_L \oplus C_L$  and for  $(u, v) \in C_L \oplus C_L$ , and  $x \in S^l \subset C_L$ , define  $f_{1,L}(x)(u, v) = (1/\sqrt{2})(x(u+v), x(u-v))$ . Here the product on the right hand side is the product in the Clifford algebra. It is clear that  $f_{1,L}$  satisfies property (i) of the lemma, Madsen points out, in [12], that it also satisfies property (ii).

Now assume as inductive hypothesis, that for each  $s < k$  the map  $f_{s,L}: E^i G_s \rightarrow O(2^{s+l})$  has been defined for each  $L \geq l + 2$ , with the required properties. Since  $G_k = \Sigma_2 \wr G_{k-1}$  it follows that  $E^i G_k \subset S^l \times E^i G_{k-1} \times E^i G_{k-1}$ . For each  $L \geq l + 2, L + k - 1 \geq l + 2$  since  $k \geq 1$ . Therefore by the inductive hypothesis we have defined  $f_{1,L+k-1}: S^l \rightarrow O(2^{L+k})$ . Also by the inductive hypothesis we have defined

