The Steenrod algebra and the automorphism group of additive formal group law

By

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1. Introduction

Let $H_*H$ be the Hopf algebra of stable co-operations of the mod 2 ordinary cohomology theory $H^*(\ )$. The structure of $H_*H$ is well known as follows. First J. P. Serre [7] determined the unstable cohomology of the Eilenberg-MacLane complex $K(n, \mathbb{Z}/2)$. He has shown the stable part of $H^*(K(n, \mathbb{Z}/2))$ is generated by iterated Steenrod operations and computed the rank of $H_i(K(n, \mathbb{Z}/2))$ in terms of excess operations. He assumed the existence of Steenrod squares $Sq_i$ but did not use the Adem relations. Using the Adem relations, we see that the algebra $S^*$ generated by Steenrod squares modulo the Adem relations is isomorphic to $H_*H$. Moreover Milnor [4] determined the Hopf algebra structure of $S^*$, the dual Steenrod algebra which is the polynomial algebra $\mathbb{F}_2[\bar{\xi}_1, \bar{\xi}_2, \ldots]$ with the coproduct $\psi(\bar{\xi}_n) = \sum_{i=0}^n \bar{\xi}_{n-i}^{2^i} \otimes \bar{\xi}_i$, and therefore we obtain the Hopf algebra structure of $H_*H$.

Now we recall strict automorphisms of the additive formal group law. Let $G_a$ be the additive formal group law, and $\text{Aut}_{\mathbb{F}_2}(G_a)(R_*)$ the set of strict automorphisms of $G_a$ over a non-negatively graded commutative $\mathbb{F}_2$-algebra $R_*$. An element $f(x)$ in $\text{Aut}_{\mathbb{F}_2}(G_a)(R_*)$ is written as a formal power series $x + \sum_{i=1}^{\infty} a_i x^{2^i}$, where $a_i \in R_{2^i-1}$. Here $\text{Aut}_{\mathbb{F}_2}(G_a)(-)$ is a functor from the category of graded algebras to the category of sets. A product of $\text{Aut}_{\mathbb{F}_2}(G_a)(R_*)$ is defined by the composition of power series, and induces the group structure. Therefore $\text{Aut}_{\mathbb{F}_2}(G_a)(-)$ is a functor to the category of groups, and is represented by the Hopf algebra $A_* = \mathbb{F}_2[\bar{\xi}_1, \bar{\xi}_2, \ldots]$ with the coproduct $\psi(\bar{\xi}_n) = \sum_{i=0}^n \bar{\xi}_{n-i}^{2^i} \otimes \bar{\xi}_i$. In other words, we have a natural group isomorphism

$$\text{Hom}_{\mathbb{F}_2}\text{-alg}(A_*, R_*) \cong \text{Aut}_{\mathbb{F}_2}(G_a)(R_*)$$

Comparing $S^*$ with $A_*$, we see that $S^* \cong A_*$ as a Hopf algebra.

We recall the Dickson algebra. Let $V^n$ be the $\mathbb{F}_2$-vector space spanned by elements $x_1, \ldots, x_n$. In the polynomial ring $\mathbb{F}_2[x_1, \ldots, x_n][t]$, consider the

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polynomial
\[ \prod_{\alpha \in V^n} (t + \alpha) = \sum_{s=0}^{n} q_{n,s} t^s, \quad \text{with } q_{n,n} = 1. \]

Then \( q_{n,s} \) is invariant under the usual action of \( GL_n(\mathbb{F}_2) \) and Dickson [2] has shown that
\[ \mathbb{F}_2[x_1, \ldots, x_n]^{GL_n(\mathbb{F}_2)} = \mathbb{F}_2[q_{n,0}, \ldots, q_{n,n-1}]. \]

Formally putting \( \deg x_i = 1 \), we have \( \deg q_{n,s} = 2^n - 2^s \). Let \( \Sigma_2^n \) be the symmetric group of degree \( 2^n \),
\[ P_n : H^*(X) \longrightarrow H^{2*} (E\Sigma_2^n \times \Sigma_2^n, X^{2^n}) \]
the extended power operations of Steenrod [8], and \( d_n : B\Sigma_2^n \times X \rightarrow E\Sigma_2^n \times \Sigma_2^n, X^{2^n} \) the diagonal map. We regard \( \Sigma_2^n \) as the group of set automorphisms of \( E^n \), and we obtain the regular embedding \( i : E^n \subset \Sigma_2^n \), which takes \( g \in E^n \) to the permutation induced by \( h \mapsto g + h \). Identify \( V^n \) by the dual of \( E^n \) over \( \mathbb{F}_2 \). Then we have canonical isomorphisms \( H^1(BE^n) \cong V^n \), and \( H^*(BE^n) \cong \mathbb{F}_2[x_1, \ldots, x_n] \). Furthermore Mûi [6] has proved \( \text{Im} i^* = \mathbb{F}_2[q_{n,0}, \ldots, q_{n,n-1}] \).

Now consider the restriction of \( d_n^* P_n \)
\[ H^*(X) \xrightarrow{d_n^* P_n} H^*(B\Sigma_2^n) \otimes H^*(X) \xrightarrow{\text{tr}} H^*(BE^n) \otimes H^*(X), \]
which is written by the same symbol \( d_n^* P_n \). Actually \( \text{Im} d_n^* P_n \subset \mathbb{F}_2[q_{n,0}, \ldots, q_{n,n-1}] \otimes H^*(X) \), and we can define an operation \( S_n : H^*(X) \xrightarrow{\text{tr}} \mathbb{F}_2[q_{n,0}, \ldots, q_{n,n-1}] \otimes H^*(X) \) by \( S_n(x) = q_{n,0}^{-\deg x} d_n^* P_n(x) \). We set \( \xi_i[n] = q_{n,i}/q_{n,0} \) and \( D[n]_a = \mathbb{F}_2[\xi_1[n], \ldots, \xi_n[n]] \subset \mathbb{F}_2[q_{n,0}, \ldots, q_{n,n-1}] \). Then by [6] we see \( S_n \) takes value in \( D[n]_a \otimes H^*(X) \).

Now we have four algebras \( H_* H, S_*, A_* \) and \( D[n]_* \). The purpose of this paper is to give a new proof of theorem of Milnor. In other words, we have showed directly that there exists a Hopf algebra isomorphism
\[ \chi_\psi : A_* \longrightarrow H_* H \]
without the usage of \( S_*. \) Since the Hopf algebra structure of \( A_* \) is easily seen, we can obtain that of \( H_* H \). Hence we have \( S_* \cong H_* H \) as a corollary. The key idea to relate those algebras is the notion of unstable multiplicative operations based on a graded ring \( R_* \)
\[ H^*(X) \longrightarrow H^*(X) \otimes R_* \]
A multiplicative operation \( \omega : H^*(X) \longrightarrow H^*(X) \otimes R_* \) induces the graded algebra homomorphism \( \omega : A_* \longrightarrow R_* \). Moreover we have the universal multiplicative operation \( \psi : H^*(X) \longrightarrow H^*(X) \otimes H_* H \). Namely, there exists a unique algebra homomorphism \( \omega : H_* H \longrightarrow R_* \) which satisfies \( (1 \otimes \omega) \circ \psi = \omega \).
for a multiplicative operation $\omega$. For the above multiplicative operation $S_n : H^*(X) \to H^*(X) \otimes D[n]_*$, we can get the following diagram:

\[
\begin{array}{ccc}
A_* & \xrightarrow{\chi_{\psi}} & H_*H \\
\downarrow{\chi_{S_n}} & & \downarrow{S_n} \\
D[n]_* & & D[n]_*
\end{array}
\]

Here $\chi_{S_n}$ is an isomorphism in low dimensional range for sufficiently large $n$. Therefore $\chi_{\psi}$ is injective and we see that $\chi_{\psi}$ is an isomorphism by Serre’s result [7]. Furthermore we can show that the algebra homomorphism $\chi_{\psi}$ is actually a Hopf algebra homomorphism.

This paper is constructed as follows. We define a multiplicative operation and construct the universal multiplicative operation $\psi$ in Section 2. In Section 3, we recall the definition of the reduced power operation in Steenrod and Epstein [8] and M"{u}i's results [5] [6], and introduce the multiplicative operation $S_n$. In Section 4, we construct $A_*$ and $\chi_{\omega} : A_* \to R_*$ from a multiplicative operation $\omega$ over $R_*$. We prove main theorem (Theorem 4.2). In appendix, we determine a coproduct of certain elements in the algebra $D_* = \prod_n D[n]_*$, and consider relations to $A_*$ and $H_*H$.

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2. Multiplicative operation

We assume that $X$ and $Y$ are spaces, and denote by $H^*(X)$ the mod 2 ordinary cohomology of $X$ in this paper.

**Definition 2.1.** Let $R_*$ be a non-negatively graded commutative algebra over $\mathbb{F}_2$, namely $R_* = 0$ for $* < 0$. We consider a graded module in which the cohomological degree $k$-part is $\prod_{n \geq 0} H^{k+n}(X) \otimes R_n$. By abuse of notation we denote the graded module by $H^*(X) \otimes R_*$. We call a natural operation $\beta : H^*(X) \otimes R_* \to H^*(X \times Y) \otimes R_*$ with cohomological degree preserving multiplicative when $\beta$ satisfies the following conditions:

(i) The diagram

\[
\begin{array}{ccc}
H^*(X) \otimes H^*(Y) & \times & H^*(X \times Y) \\
\Downarrow{\beta \otimes \beta} & & \Downarrow{\beta} \\
H^*(X) \otimes R_* \otimes H^*(Y) \otimes R_* & \xrightarrow{\mu \otimes \mu} & H^*(X \times Y) \otimes R_*
\end{array}
\]

is commutative, where $\times$ is the cross product, $\mu$ interchanges the first and second factors, and $m$ is the multiplication on $R_*$. 

(ii) $\beta(u) = u \otimes 1$ where $u$ is the generator of $H^1(S^1)$. 

Let $\tilde{H}^*(\cdot)$ be the reduced cohomology. We consider the following diagram:

$$
\begin{array}{cccccc}
0 & \longrightarrow & \tilde{H}^*(X) & \longrightarrow & H^*(X) & \longrightarrow & H^*(pt) & \longrightarrow & 0 \\
\beta & & \downarrow & & \downarrow & & \beta & & \\
0 & \longrightarrow & \tilde{H}^*(X) \otimes R_* & \longrightarrow & H^*(X) \otimes R_* & \longrightarrow & H^*(pt) \otimes R_* & \longrightarrow & 0
\end{array}
$$

with the horizontal sequences exact. Then we can define the reduced operation $\tilde{\beta}$:

$$
\tilde{\beta} : \tilde{H}^*(X) \longrightarrow \tilde{H}^*(X) \otimes R_*
$$

such that the above diagram is commutative.

Obviously $\tilde{\beta}$ is natural and the following diagram is commutative:

$$
\begin{array}{ccc}
\tilde{H}^n(X) & \longrightarrow & \tilde{H}^n(\Sigma X) \\
\tilde{\beta} & & \downarrow & \tilde{\beta} \\
[\tilde{H}^*(X) \otimes R_*]^n & \longrightarrow & [\tilde{H}^*(\Sigma X) \otimes R_*]^n+1
\end{array}
$$

where $\wedge$ is the smash product.

**Lemma 2.2.** For a multiplicative operation $\beta$, $\tilde{\beta}$ is stable. That is, the following diagram is commutative:

$$
\begin{array}{ccc}
\tilde{H}^n(X) \otimes \tilde{H}^*(Y) & \longrightarrow & \tilde{H}^n(X \wedge Y) \\
\tilde{\beta} \otimes \tilde{\beta} & & \downarrow & \tilde{\beta} \\
\tilde{H}^n(X) \otimes R_* \otimes \tilde{H}^*(Y) \otimes R_* & \longrightarrow & \tilde{H}^n(X \wedge Y) \otimes R_*
\end{array}
$$

where $\wedge$ is the smash product.

**Proof.** By the commutative diagram (1), we have the following commutative diagram:

$$
\begin{array}{ccc}
\tilde{H}^*(X) \otimes \tilde{H}^*(S^1) & \longrightarrow & \tilde{H}^*(X \wedge S^1) \\
\tilde{\beta} \otimes \tilde{\beta} & & \downarrow & \tilde{\beta} \\
\tilde{H}^*(X) \otimes R_* \otimes \tilde{H}^*(S^1) \otimes R_* & \longrightarrow & \tilde{H}^*(X \wedge S^1) \otimes R_*
\end{array}
$$

For any element $x$ in $\tilde{H}^*(X)$,

$$
\tilde{\beta}(x \wedge u) = (\wedge \otimes m) \circ (1 \otimes \mu \otimes 1) \circ (\tilde{\beta}(x) \otimes \tilde{\beta}(u))
$$

$$
= (\wedge \otimes m) \circ (1 \otimes \mu \otimes 1)(\tilde{\beta}(x) \otimes u \otimes 1)
$$

$$
= \tilde{\beta}(x) \wedge u,
$$

where $\tilde{\beta}(x) \wedge u = \sum_n (y_n \wedge u) \otimes \alpha_n$ for $\tilde{\beta}(x) = \sum_n y_n \otimes \alpha_n$. This implies that $\tilde{\beta}$ is a stable operation. □
Let $H$ be the mod 2 Eilenberg-MacLane spectrum, and $H_*H$ be $\pi_*(H \wedge H)$.
We want to introduce a multiplicative operation $\tilde{\psi}: H^*(X) \to H^*(X) \otimes H_*H$.
We define a map
\[ \tilde{\psi}: H^*(X) = [X, H]^* \longrightarrow [X, H \wedge H]^* \]
by $\tilde{\psi}(f) = i \wedge f \in [S^0 \wedge X, H \wedge H]^*$, where $i: S^0 \to H$ is the unit map.
Let $\kappa$ be the map
\[ \kappa: H^*(X) \otimes H_*H \longrightarrow [X, H \wedge H]^* \]
induced by $H \wedge (H \wedge H) \xrightarrow{m \wedge 1} H \wedge H$, where $m$ is the multiplication on $H$.

**Lemma 2.3.** $\kappa$ is an isomorphism.

*Proof.* If $X = S^n$, $\kappa$ is an isomorphism. Therefore if $H^*(X) \otimes H_*H$ is a cohomology, $\kappa$ is a cohomology operation. Because $H_*H$ is finite dimensional, we have the result.

Therefore $\kappa^{-1} \tilde{\psi}: H^*(X) \to H^*(X) \otimes H_*H$ is well-defined and it is denoted by $\psi$.

**Theorem 2.4.** The operation $\psi: H^*(X) \longrightarrow H^*(X) \otimes H_*H$ is multiplicative.

*Proof.* The map $i \wedge 1: S^0 \wedge H \to H \wedge H$ is a ring spectra map. Therefore $\tilde{\psi}: H^*(X) \to (H \wedge H)^*(X)$ preserves the external product. Since the multiplication $m: H \wedge H \to H$ is a ring spectra map, $m \wedge 1: H \wedge H \wedge H \to H \wedge H$ is so. Therefore we see that $\kappa: H^*(X) \otimes H_*H \to (H \wedge H)^*(X)$ preserves the external product. Hence $\psi$ satisfies Definition 2.1 (i).

Next we prove that $\psi$ satisfies Definition 2.1 (ii). It is enough to prove for $u = \Sigma i$. Since $\Sigma i \wedge i = i \wedge \Sigma i$ in $[S^1, \Sigma(H \wedge H)]$, we see
\[ \psi(\Sigma i) = \kappa^{-1} \circ \tilde{\psi}(\Sigma i) = \kappa^{-1}(i \wedge \Sigma i) = \kappa^{-1}(\Sigma i \wedge i) = u \otimes 1. \]

From now on, we assume any graded algebra $R_*$ is of finite type, that is $R_n$ is finite dimensional for each $n$. We define $\text{Op}(R_*)$ by the set of all multiplicative operations over $R_*$. This is a covariant functor from the category of graded algebras over $\mathbb{F}_2$ to the category of sets. We now construct a natural transformation
\[ \lambda: \text{Op}(R_*) \longrightarrow \text{Hom}_{\mathbb{F}_2}(H_*H, R_*), \]
where $\text{Hom}_{\mathbb{F}_2}(\ , \ )$ is the set of all graded linear homomorphisms.

Since $H^*(X) \otimes R_*$ is a cohomology theory in the same way as the proof of Lemma 2.3, we denote the spectrum which represents the cohomology $H^*(\ ) \otimes$
Under these conditions, we have

\[ \bar{\lambda} : (HR_\ast)^* H \xrightarrow{\cong} \text{Hom}_{F_2}^* (H_\ast H, R_\ast), \]

from \[1, \text{III, 13.5}\]. This map is defined by \( H \wedge H \xrightarrow{1 \wedge x} H \wedge HR_\ast \xrightarrow{\tau} HR_\ast \), where \( x \in [H, HR_\ast] \) and the \( H \)-module map \( \tau : H \wedge HR_\ast \to HR_\ast \). For an element \( \alpha \) in \((HR_\ast)^* H\), we write \( \bar{\lambda}(\alpha) \) as \( \bar{\alpha} \).

Let \( \beta : H^*(X) \to H^*(X) \otimes R_\ast \) be a multiplicative operation. Since \( \bar{\beta} \) is stable by Lemma 2.2, we can identify \( \beta \) as a stable cohomology operation. Therefore \( \text{Op}(R_\ast) \) is a subset in \((HR_\ast)^0 H\). \( \beta \) satisfies the following commutative diagram:

\[
\begin{array}{ccc}
H^*(X) & \xrightarrow{\psi} & H^*(X) \otimes H_\ast H \\
\downarrow_{\beta} & & \downarrow_{1 \otimes \beta} \\
H^*(X) \otimes R_\ast & & \\
\end{array}
\]

from the commutative diagram:

\[
\begin{array}{ccc}
S^0 \wedge H & \xrightarrow{1 \wedge 1} & H \wedge H \\
\downarrow_x & & \downarrow_{1 \wedge x} \\
HR_\ast & \xleftarrow{\tau} & H \wedge HR_\ast.
\end{array}
\]

Here \( x \) is a spectra map which represents \( \beta \). We define \( \lambda \) by the restriction of \( \bar{\lambda} \) to \( \text{Op}(R_\ast) \). Since \( \text{Op}(R_\ast) \subset (HR_\ast)^0 H \), the image of \( \lambda \) is actually included in \( \text{Hom}_{F_2}(H_\ast H, R_\ast) \).

**Theorem 2.5.** Let \( \text{Hom}_{F_2, \text{alg}}(\ , \ ) \) be the set of all graded algebra homomorphisms. Then there is an one to one correspondence

\[ \lambda : \text{Op}(R_\ast) \to \text{Hom}_{F_2, \text{alg}}(H_\ast H, R_\ast) \]

which is natural in \( R_\ast \).
Proof. We now prove \( \lambda(\text{Op}(R_*)) \subset \text{Hom}_{\text{alg}}(H_* H, R_*) \). It is enough to prove that the following diagram is commutative:

\[
\begin{array}{c}
H \wedge H \wedge H \wedge H \\
\downarrow 1 \wedge x \downarrow 1 \wedge x \\
H \wedge H \wedge H \wedge H \\
\downarrow \tau \wedge \tau \\
HR_* \wedge HR_*
\end{array}
\begin{array}{c}
\xrightarrow{(m \wedge m)_\alpha(1 \wedge \mu \wedge 1)} \\
\downarrow 1 \wedge x \\
\xrightarrow{(m \wedge m)_{\alpha(1 \wedge \mu \wedge 1)}} \\
\downarrow \tau \\
HR_* \wedge HR_*
\end{array}
\]

(2)

where \( \beta \) is a multiplicative operation, \( x \) represents \( \beta \), and \( m_{R_*} \) is the multiplication on \( HR_* \). Because \( \beta \) is a multiplicative operation, the following diagram is commutative:

\[
\begin{array}{c}
H \wedge H \\
\downarrow m \\
H
\end{array}
\begin{array}{c}
\xrightarrow{x \wedge x} \\
\downarrow \\
\xrightarrow{m_{R_*}} \\
HR_*
\end{array}
\]

Therefore the upper square in the diagram (2) is commutative. The lower square in (2) is commutative since \( H R_* \) is a commutative ring spectrum and \( m_{R_*} : H R_* \wedge H R_* \to H R_* \) is an \( H \)-module spectra map.

For any \( r \) in \( \text{Hom}_{\text{alg}}(H_* H, R_*) \), the operation

\[
(1 \otimes r) \circ \psi : H^*(X) \longrightarrow H^*(X) \otimes H_* H \longrightarrow H^*(X) \otimes R_*
\]

is multiplicative. This shows \( \lambda(\text{Op}(R_*)) = \text{Hom}_{\text{alg}}(H_* H, R_*) \).

\[ \Box \]

3. Construction of the reduced power

Let \( G \) be a subgroup of the symmetric group \( \Sigma_m \) of degree \( m \). For a space \( X, G \) acts on \( X^m \) as a permutation. Steenrod defined the extended power operation

\[
P_G : H^q(X) \longrightarrow H^{mq}(E_G(X)),
\]

where \( E_G(X) \) is defined by \( E_G \times_G X^m \) [8, VII]. From the diagonal map \( d_G : BG \times X \to E_G(X) \), we have the natural map \( d_G^* P_G : H^q(X) \to H^{mq}(BG \times X) \).

Let \( E^n \) be the elementary abelian 2-group with dimension \( n \) and we write \( E^n = E_1 \times \cdots \times E_n \), where \( E_i = \mathbb{Z}/2 \). Then we can identify \( \text{Aut}_{\text{Set}}(E^n) \), the set of all permutations of the set \( E^n \), as \( \Sigma_{2^n} \). Since \( E^n \) acts on itself as a vector space, there is the regular embedding \( E^n \subset \Sigma_{2^n} \). The wreath product \( E_1 \wr \cdots \wr E_n \) is a 2-Sylow subgroup of \( \Sigma_{2^n} \), and it is denoted by \( \Sigma_{2^n, 2} \). Obviously, \( \Sigma_{2^n, 2} \) contains \( E^n \). We define an inclusion \( E^{n-1} \subset E^n \) by \( E^{n-1} \cong \{0\} \times E_2 \times \cdots \times E_n \subset E^n \). Then it induces the inclusion \( \Sigma_{2^n-1, 2} \subset E_1 \wr \Sigma_{2^n-1, 2} = \Sigma_{2^n, 2} \).

From \( i_G^{G'} : G' \subset G \), we have three maps \( BG' \to BG, E_G'(X) \to E_G(X), \) and \( BG' \times X \to BG \times X \). They induce \( H^*(BG) \to H^*(BG'), H^*(E_G'(X)) \to H^*(E_G(X)) \to \)

\[ \text{...} \]
$H^*(E_{G'}(X))$ and $H^*(BG \times X) \to H^*(BG' \times X)$, which are denoted by the same symbol $i_G^*: G'.\) Since we see $i_G^*: P_G \to P_G'$ by \cite[8, VII, 2.5]{Masateru Inoue}, we obtain

\begin{equation}
\tag{3}
i_G^*: E_n d_{\Sigma n}^* P_{\Sigma n} = i_{\Sigma n}^* E_n d_{\Sigma n}^* P_{\Sigma n} = d_n^* P_n : H^q(X) \to H^{n+q}(BE^n \times X),
\end{equation}

where $d_n = d_{E^n}$ and $P_n = P_{E^n}$. We can identify $P_{E_1} P_{\Sigma_{n-1,2}}$ with $P_{\Sigma_{n,2}}$ from the following commutative diagram:

\[
\begin{array}{ccc}
H^*(X) & \xrightarrow{P_{\Sigma_{n,2}}} & H^*(E_{\Sigma_{n,2}}(X)) \\
\downarrow P_{\Sigma_{n-1,2}} & & \downarrow \cong \\
H^*(E_{\Sigma_{n-1,2}}(X)) & \xrightarrow{P_n} & H^*(E_{E_1}(E_{\Sigma_{n-1,2}}(X))) \cong H^*(E_{E_1}(E_{\Sigma_{n-1,2}}(X))).
\end{array}
\]

By the naturality of $P$, the following diagram is commutative:

\[
\begin{array}{ccc}
H^*(X) & \xrightarrow{P_{\Sigma_{n-1,2}}} & H^*(E_{\Sigma_{n-1,2}}(X)) \\
\downarrow P_{\Sigma_{n-1,2}} & & \downarrow P_n \\
H^*(E_{E_1}(E_{\Sigma_{n-1,2}}(X))) & \xrightarrow{(1 \times d_{n-1})^*} & H^*(BE^n \times X).
\end{array}
\]

Hence we see the following lemma:

**Lemma 3.1** (\cite{Masateru Inoue}). We have

\[d_n^* P_n = d_n^* P_{E_1} d_{n-1}^* P_{n-1}.
\]

Given the diagonal maps

\[\lambda: E^n \times E^n \xrightarrow{\lambda} EE^n \times EE^n \times E^n \times E^n \times X^{2^n} \times Y^{2^n},\]

and \[d': BE^n \times X \times Y \to BE^n \times BE^n \times X \times Y,\]

we obtain the following maps:

\[H^*(X) \otimes H^*(Y) \xrightarrow{P_{n} \times P_n} H^*(EE^n \times E^n \times X^{2^n} \otimes EE^n \times E^n \times Y^{2^n}) \]

and \[H^*(X) \otimes H^*(Y) \xrightarrow{d_n^* \times d_n^*} H^*(BE^n \times BE^n \times X \times Y) \xrightarrow{d'} H^*(BE^n \times X \times Y).
\]
Lemma 3.2. We have
\[ d_n^* P_n(u \times v) = d_n^* (d_n^* d_n^*) (P_n u \times P_n v) : H^*(X) \otimes H^*(Y) \longrightarrow H^*(BE^n \times X \times Y). \]

Proof. By Steenrod and Epstein [8, VII, Lemma 2.6], we obtain \( \lambda^*(P_n u \times P_n v) = P_n(u \times v) \), where \( u \in H^*(X) \) and \( v \in H^*(Y) \). The commutative diagram
\[
\begin{array}{ccc}
BE^n \times X \times Y & \xrightarrow{d_n} & EE^n \times E^n (X \times Y)^{2n} \\
\downarrow d' & & \downarrow \lambda \\
BE^n \times BE^n \times X \times Y & \xrightarrow{d_n \times d_n} & EE^n \times EE^n \times E^n \times E^n X^{2n} \times Y^{2n}
\end{array}
\]
duces
\[
H^*(BE^n \times X \times Y) \xrightarrow{d_n^*} H^*(EE^n \times E^n (X \times Y)^{2n}) \xleftarrow{\lambda^*} H^*(BE^n \times BE^n \times X \times Y) \xleftarrow{d_n^* \times d_n^*} H^*(EE^n \times EE^n \times E^n \times E^n X^{2n} \times Y^{2n}).
\]

Therefore
\[ d_n^* P_n(u \times v) = d_n^* \lambda^*(P_n u \times P_n v) = d_n^* (d_n^* d_n^*) (P_n u \times P_n v). \]

We recall that \( H^*(BE^n) = \mathbb{F}_2[x_1, \ldots, x_n] \), where each \( x_i \) is of degree 1. It is well known by M"ui [5] that
\[ \text{(4) } \text{Im}(i_{\Sigma_2^n, E^n}) = \mathbb{F}_2[x_1, \ldots, x_n]^{\text{GL}_n(\mathbb{F}_2)}, \quad \text{Im}(i_{\Sigma_2^n, 2, E^n}) = \mathbb{F}_2[x_1, \ldots, x_n]^{T_n}, \]
where \( T_n \) the upper triangular subgroup of \( \text{GL}_n(\mathbb{F}_2) \). We define \( v_{k+1} \) by
\[ v_{k+1} = \prod \left( \sum_{i=1}^{k} \lambda_i x_i + x_{k+1} \right), \]
and \( q_{n,i} \) by
\[ \prod_{\alpha \in E^n} (x + \alpha) = \sum_{s=0}^{n} q_{n,s} x^{2^s} \quad \text{with } q_{n,n} = 1. \]

Obviously \( \deg q_{n,i} = 2^n - 2^i \). Dickson [2] and M"ui [5] have shown
\[ \mathbb{F}_2[x_1, \ldots, x_n]^{\text{GL}_n(\mathbb{F}_2)} \cong \mathbb{F}_2[q_{n,0}, q_{n,1}, \ldots, q_{n,n-1}], \]
\[ \mathbb{F}_2[x_1, \ldots, x_n]^{T_n} \cong \mathbb{F}_2[v_1, \ldots, v_n]. \]

Furthermore the following relations between \( q_{n,i} \) and \( v_i \) are known.
Theorem 3.3 ([5]). We have
\[ q_{n,j} = q_{n-1,j}v_n + q_{n-1,j-1}^2, \]
where \( q_{n,j} = 0 \) for \( j < 0 \) or \( n < j \).

We need the following definition and theorem in [8].

Definition 3.4 ([8] VII 3.2). Suppose \( H^*(BE^1) = \mathbb{F}_2[x] \) and \( u \in H^q(X) \). Then we can write \( d_1^*P_1(u) = \sum k x^k \times Sq^{q-k}(u) \), where \( Sq^k : H^q(X) \to H^{q+k}(X) \).

Theorem 3.5 ([8] VII 4.3, 4.4, 3.4). For each \( k \), \( Sq^k \) is a homomorphism. If \( u \in H^q(X) \), then \( Sq^k(u) = 0 \) for \( k < 0 \), \( Sq^0(u) = u \) and \( Sq^q(u) = u^2 \).

We now consider (6)
\[ d_1^*P_1 : H^*(BE^n) \to H^*(BE^1 \times BE^n). \]

Obviously \( BE^1 \times BE^n = BE^{n+1} \). Since
\[ E^n \cong \{0\} \times E_2 \times \cdots \times E_{n+1} \subset E^{n+1} = E_1 \times \cdots \times E_{n+1}, \]
we identify \( BE^1 \) as \( BE_1 \) and \( BE^n \) as \( B(E_2 \times \cdots \times E_{n+1}) \) in (6).

Theorem 3.6 ([6] Theorem 1.5). Define an element \( v'_n \) by
\[ v'_n = \prod_{\lambda_i = 0, 1} \left( \sum_{i=2}^n \lambda_i x_i + x_{n+1} \right) \text{ in } H^*(BE^n) = \mathbb{F}_2[x_2, \ldots, x_{n+1}]. \]
Then we have
\[ d_1^*P_1(v'_n) = v_{n+1}, \text{ in } H^*(BE^{n+1}). \]

Especially
\[ d_n^*P_n x_{n+1} = v_{n+1}, \]
where \( H^*(BE_{n+1}) = \mathbb{F}_2[x_{n+1}] \) and \( d_n^*P_n : H^*(BE_{n+1}) \to H^*(B(E_1 \times \cdots \times E_n) \times BE_{n+1}) \).

Proof. By Theorem 3.5, we have \( d_1^*P_1(u) = 1 \times u^2 + x_1 \times u \) for \( u \in H^1(X) \).

By Lemma 3.2, we have
\[ d_1^*P_1(v'_n) = \prod_{\lambda_i = 0, 1} d_1^*P_1 \left( \sum_{i=2}^n \lambda_i x_i + x_{n+1} \right) = \prod_{\lambda_i = 0, 1} \left( \sum_{i=2}^n \lambda_i x_i + x_{n+1} \right) \left( x_1 + \sum_{i=2}^n \lambda_i x_i + x_{n+1} \right) = v_{n+1}. \]
The second claim is obvious by Lemma 3.1.

From (3) and (4), the image of $d^*_n P_n$ is included in $\mathbb{F}_2[x_1, \ldots, x_n]^{GL_n(\mathbb{F}_2)} \otimes H^*(X)$. For any $u \in H^q(X)$, we can denote $d^*_n P_n u$ by

\begin{equation}
(7) \quad d^*_n P_n u = \sum_{R=(r_0, \ldots, r_{n-1})} q_{r_0}^0 q_{r_1}^1 \cdots q_{r_{n-1}}^{n-1} \otimes D_R u,
\end{equation}

where $D_R : H^q(X) \to H^{2^q-|R|}(X)$ with $|R| = \sum_{s=0}^{n-1} r_s (2^n - 2^s)$.

**Lemma 3.7** ([6], Lemma 2.3). $D_R u = 0$ if $q < r_0 + r_1 + \cdots + r_{n-1}$.

**Proof.** We now prove by the induction on $n$. In the case of $n = 1$, it is obvious by Definition 3.4 and Theorem 3.5. We assume that the lemma is true for $n = k - 1$. By Lemma 3.1 we have

$$d^*_n P_k(u) = d^*_1 P_1 d^*_k P_{k-1} u = \sum_{i=0}^{2k-1} x_1^{k-1-q-i} S q^i (d^*_k P_{k-1} u).$$

So the degree of $d^*_k P_k(u)$ in $x_1$ is equal to $2^{k-1}q$. From Theorem 3.3, $\deg_{x_1} q_{k,s} = \deg_{x_k} q_{k,s} = 2^{k-1}$. From the equality (7), we must have

$$2^{k-1}(r_1 + \cdots + r_{k-1}) \leq 2^{k-1}q.$$

Therefore the lemma is true.

\[\]

Let $P_n = \mathbb{F}_2[x_1, \ldots, x_n](= H^*(B\mathbb{E}^n))$, $e_n = \prod(\sum_{i=1}^n \lambda_i x_i) \in P_n$, $\lambda_i = 0$ or $1$, $\sum \lambda_i > 0$, and $\Phi_n = P_n[e_n^{-1}]$. Then there exists the natural action of $GL_n(\mathbb{F}_2)$ on $P_n$ and $\Phi_n$. Define $\Delta_n = \Phi_n^{T_k}$ and $\Gamma_n = \Phi_n^{GL_n}$, where $\Phi_n$ is the subalgebra of the invariants of $K$ in $\Phi_n$ for $K = T_n$ or $GL_n$. We set $\omega_{k+1} = v_{k+1}/e_k$. It is easily seen that

$$\Delta_n = \mathbb{F}_2[v_1^{1 \pm 1}, \ldots, v_n^{1 \pm 1}] \cong \mathbb{F}_2[w_1^{\pm 1}, \ldots, w_n^{\pm 1}], \quad \Gamma_n = \mathbb{F}_2[q_{n,0}, q_{n,1}, \ldots, q_{n,n-1}].$$

Let $S_n : H^*(X) \to \Phi_n \otimes H^*(X)$ be the map which sends $x$ to $q_{n,0}^{−\deg(x)} d^*_n P_n(x)$. From the definition, $S_n$ preserves cohomological degree. It is the same as the definition of $S_n$ by Lomonaco [3] substantially.

Let $D[n]_*$ be the subalgebra generated by $\xi_1[n], \xi_2[n], \ldots, \xi_n[n]$ in $\Phi_n$, where $\xi_i[n] = q_{n,i}/q_{n,0}$. It is easily seen that $\xi_i[n]$ is an element in $D[n]_{2i}$ and $D[n]_* = \mathbb{F}_2[\xi_1[n], \ldots, \xi_n[n]]$.

**Corollary 3.8.** Suppose $H^*(B\mathbb{Z}/2) = \mathbb{F}_2[x]$. Then we have

$$S_n(x) = \sum_{s=0}^n \xi_s[n] x^{2^s}.$$

**Proof.** From the definition of $v_n$ and the equality (5), we have $v_{n+1} = \sum_{s=0}^n q_{n,s} x^{2^s + 1}$. By Theorem 3.6 and the definition of $S_n$, we have $S_n(x) = \sum_{s=0}^n \xi_s[n] x^{2^s}$. 

\[\]
Lemma 3.9. \( \text{Im}(S_n) \subset D[n]_* \otimes H^*(X) \).

Proof. Trivial by Lemma 3.7. \( \square \)

We consider the operation \( H^*(X) \overset{S_n}{\rightarrow} D[n]_* \otimes H^*(X) \rightarrow H^*(X) \otimes D[n]_* \), where the second map interchanges the first and second factors, and denote it by the same symbol \( S_n \).

Lemma 3.10. The cohomology operation \( S_n \) is multiplicative. That is, the following diagram is commutative:

\[
\begin{array}{ccc}
H^*(X) \otimes H^*(Y) & \times & H^*(X \times Y) \\
\downarrow \text{\( s_n \otimes s_n \)} & & \downarrow \text{\( s_n \)} \\
H^*(X) \otimes D[n]_* & \otimes H^*(Y) \otimes D[n]_* & \otimes D[n]_* \otimes D[n]_* \\
\end{array}
\]

Proof. By Lemma 3.2, it is obvious. \( \square \)

4. The relation between \( H_*H \) and \( \text{Aut}_F \mathbb{Z}_2 G_a \)

Let \( G_a \) be the additive formal group law and \( \text{Aut}_{\mathbb{F}_2}(G_a)(R_* \mathbb{F}_2) \) the set of all strict automorphisms of \( G_a \) over a graded \( \mathbb{F}_2 \)-algebra \( R_* \mathbb{F}_2 \). Then \( \text{Aut}_{\mathbb{F}_2}(G_a)(-) \) is a functor from the category of graded algebras to the category of sets. An element in \( \text{Aut}_{\mathbb{F}_2}(G_a)(R_* \mathbb{F}_2) \) is a functor from the category of graded algebras to the category of sets. An element in \( \text{Aut}_{\mathbb{F}_2}(G_a)(R_* \mathbb{F}_2) \) is a power series \( f(x) \in R_* [[x]] \) satisfying the following three conditions: (i) \( f(x + y) = f(x) + f(y) \); (ii) the coefficient of \( x \) in \( f(x) \) is equal to 1; (iii) that of \( x^k \) is an element in \( R_{2k} \). Therefore for \( f(x) \in \text{Aut}_{\mathbb{F}_2}(G_a)(R_* \mathbb{F}_2) \) we have

\[
f(x) = x + a_1x^2 + a_2x^4 + \cdots + a_mx^{2^m} + \cdots,
\]

where \( a_i \in R_{2i-1} \).

Let \( A_* \) be the graded polynomial algebra generated by \{\( \xi_1, \ldots, \xi_n, \ldots \)\} with \( \xi_i \in A_{2i-1} \). Such a power series is represented by a graded \( \mathbb{F}_2 \)-algebra homomorphism

\[
\chi : A_* = \mathbb{F}_2[\xi_1, \xi_2, \ldots] \rightarrow R_*
\]

defined by \( \chi(\xi_i) = a_i \), and we have the natural isomorphism

\[
(8) \quad \text{Hom}_{\mathbb{F}_2\text{-alg}}(A_*, R_*) \cong \text{Aut}_{\mathbb{F}_2}(G_a)(R_*), \quad \chi \mapsto \sum_{i=0}^{\infty} \chi(\xi_i)x^{2^i},
\]

where \( \xi_0 = 1 \). A product of \( \text{Aut}_{\mathbb{F}_2}(G_a)(R_* \mathbb{F}_2) \) is defined by \( (g \cdot f)(x) = f(g(x)) \). Then \( \text{Aut}_{\mathbb{F}_2}(G_a)(R_* \mathbb{F}_2) \) is a group, and thereby \( \text{Aut}_{\mathbb{F}_2}(G_a)(-) \) is a functor to the category of groups. This induces the coproduct map \( \Delta : A_* \rightarrow A_* \otimes A_* \). It is easy to check \( \Delta(\xi_0) = \sum_{i=0}^{\infty} \xi_{2^i-1} \otimes \xi_i \).

Consider a multiplicative operation \( \beta : H^*(X) \rightarrow H^*(X) \otimes R_* \). The classifying space \( B\mathbb{Z}/2 \) is an \( H \)-space and the Hopf algebra \( H^*(B\mathbb{Z}/2) \cong \mathbb{F}_2[x] \) is nothing but the additive formal group. We can identify \( \beta(x) \) as an element in \( R_* [[x]] \) and write it by \( f_\beta(x) \).
Lemma 4.1. \( f_\beta(x) \) is an element in \( \text{Aut}_{\mathbb{A}}(G_a)(R_\ast) \).

Proof. The product map \( a : B\mathbb{Z}/2 \times B\mathbb{Z}/2 \to B\mathbb{Z}/2 \) induces the commutative diagram:

\[
\begin{array}{ccc}
H^\ast(B\mathbb{Z}/2) \otimes H^\ast(B\mathbb{Z}/2) & \xrightarrow{\beta \times \beta} & H^\ast(B\mathbb{Z}/2) \otimes R_\ast \otimes H^\ast(B\mathbb{Z}/2) \otimes R_\ast \\
\downarrow & & \downarrow \left( (x) \times m_{(1 \times A \times 1)} \right) \\
H^\ast(B\mathbb{Z}/2 \times B\mathbb{Z}/2) & \xrightarrow{\beta} & H^\ast(B\mathbb{Z}/2 \times B\mathbb{Z}/2) \otimes R_\ast \\
\alpha^\ast \downarrow & & \alpha^\ast \downarrow \\
H^\ast(B\mathbb{Z}/2) & \xrightarrow{\beta} & H^\ast(B\mathbb{Z}/2) \otimes R_\ast.
\end{array}
\]

Therefore we see

\[
\beta(x \times 1+1 \times x) = \alpha^\ast \circ \beta(x) = \beta \circ \alpha^\ast (x) = \beta(x \times 1) + \beta(1 \times x) = \beta(x) \times 1 + 1 \times \beta(x).
\]

\[\Box\]

Let \( \chi_\beta : A_\ast \to R_\ast \) be the algebra homomorphism corresponding to \( f_\beta(x) \) in (8). For the multiplicative operations \( \psi \) in Section 2 and \( S_n \) in Section 3, we obtain the algebra homomorphisms \( \chi_\psi : A_\ast \to H_\ast H \) and \( \chi_{S_n} : A_\ast \to D[n]_\ast \).

The map \( H \wedge S^0 \wedge H \xrightarrow{1 \wedge \lambda^1} H \wedge H \wedge H \) induces

\[
\delta : H_\ast H = [S^0, H \wedge H]_\ast \longrightarrow [S^0, H \wedge H \wedge H]_\ast \cong H_\ast H \otimes H_\ast H,
\]

and \( H_\ast H \) is a Hopf algebra. Then \( H^\ast(X) \) is an \( H_\ast H \)-comodule with \( \psi : H^\ast(X) \to H^\ast(X) \otimes H_\ast H \).

Theorem 4.2. \( \chi_\psi : A_\ast \longrightarrow H_\ast H \) is a Hopf algebra isomorphism.

Proof. From Theorem 2.5, there exists a unique algebra homomorphism \( \hat{S}_n : H_\ast H \to D[n]_\ast \) with the commutative diagram

\[
\begin{array}{ccc}
H^\ast(X) & \xrightarrow{\psi} & H^\ast(X) \otimes H_\ast H \\
\downarrow \hat{S}_n & & \downarrow 1 \otimes \hat{S}_n \\
H^\ast(X) \otimes D[n]_\ast.
\end{array}
\]

It induces the following commutative diagram:

\[
\begin{array}{ccc}
A_\ast & \xrightarrow{\chi_\psi} & H_\ast H \\
\chi_{\hat{S}_n} \downarrow & & \downarrow \hat{S}_n \\
D[n]_\ast \rightarrow \\
\rightarrow D[n]_\ast.
\end{array}
\]

From Corollary 3.8, \( \chi_{\hat{S}_n} \) is defined by \( \chi_{\hat{S}_n}(\xi_i) = \xi_i[n] \). For sufficiently large \( n \), there exists a number \( n \) such that \( \chi_{\hat{S}_n} : H_\ast H \to D[n]_\ast \) is an isomorphism on
Therefore $\chi_\psi$ is injective. Serre [7, §18, Théorème 3] has shown that the Poincaré series of $H^* H$ and $H_* H$ is equal to $\prod_{i=1}^{\infty} 1/(1 - t^{2i} - 1)$, which is the same as that of $A_\ast$. Hence $\chi_\psi$ is bijective.

Next we prove $\chi_\psi$ is a Hopf algebra homomorphism. Since $\psi$ is an $H_\ast H$-comodule map, the following operation is multiplicative:

$$(\psi \otimes 1) \circ \psi = (1 \otimes \delta) \circ \psi : H^* (X) \to H^* (X) \otimes H_\ast H \otimes H_* H.$$  

Since $(\psi \otimes 1) \circ \psi$ is two iteration of $\psi$, we see $\chi_{(\psi \otimes 1) \circ \psi} = (\chi_\psi \otimes \chi_\psi) \circ \Delta$. Moreover we obtain $\chi_{(1 \otimes \delta) \circ \psi} = \delta \circ \chi_\psi$. Since $(\psi \otimes 1) \circ \psi = (1 \otimes \delta) \circ \psi$, we have the following commutative diagram:

$$\begin{array}{ccc}
A_\ast & \xrightarrow{\Delta} & A_\ast \otimes A_\ast \\
\chi_\psi \downarrow & & \downarrow \chi_\psi \otimes \chi_\psi \\
H_* H & \xrightarrow{\delta} & H_* H \otimes H_* H.
\end{array}$$

\[\square\]

5. Appendix

Let $D_\ast \ast$ be the bigraded algebra $\prod_{n \geq 0} D[n]_\ast$, with $D_{m,n} = D[n]_m$. In this appendix, we define a coproduct of some elements in $D_\ast \ast$, and construct algebra homomorphisms $\chi_\ast : A_\ast \to D_\ast \ast$ and $S : H_* H \to D_\ast \ast$ which preserve coproducts.

First we study a coproduct of $D[n]_\ast$. Define an algebra homomorphism $\delta_{m,n} : \Delta_{n+m} \to \Delta_m \otimes \Delta_n$ by

$$\delta_{m,n}(w_i) = \begin{cases} w_i \otimes 1 & \text{if } 0 \leq i \leq m, \\ 1 \otimes w_{i-m} & \text{if } m+1 \leq i \leq n+m. \end{cases}$$

**Lemma 5.1.** $\delta_{m,n}(\xi_j[n+m]) = \sum_{0 \leq j \leq 1} \xi_j[n+m] \otimes \xi_j[n]$. Especially $\delta_{m,n}(D[n+m]_\ast) \subset D[m]_\ast \otimes D[n]_\ast$.

**Proof.** We prove the lemma by induction on $n+m$. For $n+m=1$, it is trivial. We now assume that the lemma is true for $n+m \leq k$. For $n+m=k+1$, we consider only the map $\delta_{k,k-n+1}$ because the map $\delta_{k+1,0}$ is trivial. From Theorem 3.3 and $q_{n,0} = v_1 \cdots v_n$,

$$\xi_j[n] = q_{n-1,j}v_n + q_{n-1,j-1}^2 w_{n-1,j-1}$$

$$= \frac{q_{n-1,j}v_n + q_{n-1,j-1}^2}{v_1 v_2 \cdots v_n} v_{n-1,j-1}$$

$$= \xi_j[n-1] + \xi_{j-1}[n-1] w_n^{-1}.$$  

By this equality, we have

$$\delta_{n,k-n+1}(\xi_j[k+1])$$

$$= \delta_{n,k-n+1}(\xi_j[k] + \xi_{j-1}[k] w_{k+1}^{-1})$$

$$= \delta_{n,k-n}(\xi_j[k]) + \delta_{n,k-n}(\xi_{j-1}[k])^2 \delta_{n,k-n+1}(w_{k+1})^{-1}.$$
By the induction hypothesis, this is equal to
\[
\sum_{0 \leq i \leq j} \xi_{j-i}^2 [n] \otimes \xi_i [k-n] + \sum_{0 \leq i' \leq j-1} \xi_{j-i'}^2 [n+1] \otimes \xi_{i'}^2 [k-n] w_{k-n+1}^{-1}
\]
\[
= \sum_{0 \leq i \leq j} \xi_{j-i}^2 [n] \otimes \xi_i [k-n] + \sum_{0 \leq i' \leq j} \xi_{j-i'}^2 [n] \otimes \xi_{i'}^2 [k-n] w_{k-n+1}^{-1}
\]
\[
= \sum_{0 \leq i \leq j} \xi_{j-i}^2 [n] \otimes \xi_i [k-n+1].
\]
Therefore we have the lemma. □

From Lemma 5.1, we have obtained the coproduct \(\delta_{m,n} : D[n+m]_* \to D[n]_* \otimes D[m]_*\). Next we investigate the multiplicative operation \(S_n : H^*(X) \to H^*(X) \otimes D[n]_*\).

**Lemma 5.2.** For \(u \in H^q(X)\), we have
\[
d_n^* P_n(u) = \sum_{i_1, i_2, \ldots, i_n} c_1^{i_1} c_2^{i_2} \cdots c_n^{i_n} \times Sq^{i_1} \cdots Sq^{i_n}(u),
\]
where \(0 \leq i_k \leq q + \sum_{j=k+1}^{n} i_j\) and \(c_k = q - i_k + \sum_{j=k+1}^{n} i_j\) for any \(1 \leq k \leq n\).

**Proof.** We prove by induction on \(n\). For \(n = 1\), it is trivial by the definition of \(d_1^* P_1\). We now assume that the lemma is true for \(n \leq k\). For \(k + 1\), we use the equality \(d_{k+1}^* P_{k+1} = d_1^* P_1 d_k^* P_k\) by Lemma 3.1. Then we have
\[
d_1^* P_1 d_k^* P_k (u)
\]
\[
= d_1^* P_1 \left( \sum_{i_2, i_3, \ldots, i_{k+1}} c_2^{i_2} c_3^{i_3} \cdots c_{k+1}^{i_{k+1}} \times Sq^{i_2} \cdots Sq^{i_{k+1}}(u) \right)
\]
\[
= \sum_{i_2, i_3, \ldots, i_{k+1}} d_1^* P_1 (c_2^{i_2} \cdots c_{k+1}^{i_{k+1}}) \times d_1^* P_1 (c_2 \cdots c_{k+1}^{i_{k+1}}(u))
\]
\[
= \sum_{i_2, i_3, \ldots, i_{k+1}} (c_2^{i_2} \cdots c_{k+1}^{i_{k+1}}) \sum_{i_1} c_1^{i_1} c_2^{i_2+i_1-i_3} \cdots c_{k+1}^{i_{k+1}} \times Sq^{i_2} \cdots Sq^{i_{k+1}}(u).
\]
We have the first equality by the induction hypothesis, the second equality by Steenrod and Epstein [8, VII, 2.6] and the naturality of \(d_1\), and the third equality by Theorem 3.6. By \(\deg(Sq^{i_2} \cdots Sq^{i_{k+1}}(u)) = q + \sum_{j=2}^{k+1} i_j\), we have \(0 \leq i_1 \leq q + \sum_{j=2}^{k+1} i_j\). □

**Corollary 5.3.** For \(u \in H^q(X)\), we have
\[
S_n(u) = \sum_{i_1, i_2, \ldots, i_n} Sq^{i_1} Sq^{i_2} \cdots Sq^{i_n}(u) \times w_1^{-i_1} w_2^{-i_2} \cdots w_n^{-i_n},
\]
where \(0 \leq i_k \leq q + \sum_{j=k+1}^{n} i_j\) for any \(0 \leq k \leq n\).
Proof. By the definition of $S_n$ and Lemma 5.2, we see
\[ S_n(u) = q_n^{-q}d^*_n P_n(u) \]
\[ = (v_1 \cdots v_n)^{-q} \sum_{i_1, i_2, \ldots, i_n} v_1^{i_1} v_2^{i_2} \cdots v_n^{i_n} \times Sq^{i_1} \cdots Sq^{i_n}(u) \]
\[ = \sum_{i_1, i_2, \ldots, i_n} w_1^{-i_1} \cdots w_n^{-i_n} \times Sq^{i_1} \cdots Sq^{i_n}(u). \]

Here is a theorem which describes a relation between two iteration of $S_n$ and $\delta_{m,n}$.

**Theorem 5.4.**

\((S_m \otimes id_{D[n]_*}) \circ S_n = (id_{H^*(X)} \otimes \delta_{m,n}) \circ S_{n+m} : H^*(X) \to H^*(X) \otimes D[m]_* \otimes D[n]_*\)

Proof. Let $u$ be an element in $H^q(X)$. From the definitions of $S_n$ and $\delta_{m,n}$, and Corollary 5.3, we obtain
\[ (S_m \otimes id_{D[n]_*}) \circ S_n(u) \]
\[ = (S_m \otimes id_{D[n]_*}) \sum_{i_1, i_2, \ldots, i_n} Sq^{i_1} \cdots Sq^{i_n}(u) \times w_1^{-i_1} w_2^{-i_2} \cdots w_n^{-i_n} \]
\[ = \sum_{i_1, i_2, \ldots, i_n} S_m(Sq^{i_1} \cdots Sq^{i_n}(u)) \times w_1^{-i_1} w_2^{-i_2} \cdots w_n^{-i_n} \]
\[ = \sum_{i_1, i_2, \ldots, i_n} \left[ \left( \sum_{j_1, \ldots, j_m} Sq^{j_1} \cdots Sq^{j_m} Sq^{i_1} \cdots Sq^{i_n}(u) \right) \times w_1^{-j_1} \cdots w_m^{-j_m} \right. \]
\[ \times w_1^{-i_1} w_2^{-i_2} \cdots w_n^{-i_n} \]
\[ \times w_1^{-i_1} w_2^{-i_2} \cdots w_n^{-i_n} \]

Since
\[ S_{n+m}(u) = \sum_{i_1, i_2, \ldots, i_{n+m}} Sq^{i_1} \cdots Sq^{i_{n+m}}(u) \times w_1^{-i_1} \cdots w_{n+m}^{-i_{n+m}}, \]
\[ \delta_{m,n}(w_1^{-i_1} w_2^{-i_2} \cdots w_{n+m}^{-i_{n+m}}) = w_1^{-i_1} \cdots w_m^{-i_m} \otimes w_{m+1}^{-i_{m+1}} \cdots w_n^{-i_{n+m}}, \]
we have the result. \(\square\)

In the same way as the proof of Theorem 4.2, we have the following two commutative diagrams:
\[ \begin{array}{c}
\begin{array}{c}
A_* \\
\chi_{\partial_n+m}
\end{array} \xrightarrow{\Delta} \begin{array}{c}
A_* \otimes A_* \\
\chi_{\partial_n+m} \otimes \chi_{\partial_n+m}
\end{array} & \begin{array}{c}
H_* H \\
\delta
\end{array} \xrightarrow{\delta} \begin{array}{c}
H_* H \otimes H_* H \\
S_m \otimes S_n
\end{array}
\end{array} \]
\[ \begin{array}{c}
D[n+m]_* \xrightarrow{\delta_{m,n}} D[n+m]_* \otimes D[n]_* & \begin{array}{c}
D[n+m]_* \xrightarrow{\delta_{m,n}} D[n+m]_* \otimes D[n]_*
\end{array}
\end{array} \]
We define an element $\xi_k$ in $D_{*,*}$ by $\sum_{k \geq 0} \xi_k[n]$, where $\xi_k[n] = 0$ for $n < k$. Then we obtain the coproduct $\xi_n \mapsto \sum_{\ell \geq 0} \xi_{n-\ell} \otimes \xi_{\ell}$ of $\xi_n$ induced by $\delta_{m,n}$. We define $\chi_S : A_* \to D_{*,*}$ by $\prod_n \chi_S_n$, and $\bar{S} : H_*H \to D_{*,*}$ by $\prod_n \bar{S}_n$. Then $\chi_S$ and $\bar{S}$ preserve coproducts. Since $\chi_S : A_* \to H_*H$ is a Hopf algebra homomorphism, we get the commutative diagram of formal Hopf algebra homomorphisms

$$
\begin{array}{ccc}
A_* & \xrightarrow{\chi_S} & D_{*,*} \\
\downarrow{\chi_S} & & \downarrow{\bar{S}} \\
H_*H & & \end{array}
$$

**Remark.** Since $D_{*,*}$ is not actually a Hopf algebra, $\chi_S$ and $\bar{S}$ are not Hopf algebra homomorphisms.

**References**


