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The homology of BPO

Po Hu and Igor Kriz

1. Introduction

The goal of this note is to calculate $H_*BPO = H_*(BPO, \mathbb{Z}/2)$ where $BPO = BP\mathbb{R}^{\mathbb{Z}/2}$. Here, $BP\mathbb{R}$ is the 2-local $\mathbb{Z}/2$-equivariant irreducible summand of Landweber's Real cobordism spectrum $M\mathbb{R}$ (see **[9]**,**[3]**). The spectrum $M\mathbb{R}$ was investigated in **[6]**, where we also give a calculation of its coefficients, and hence also of π_*BPO. Our main interest in these spectra is motivated by the possibility of investigating an Adams-type spectral sequence based on them. This will be pursued elsewhere. However, the calculation of H^*BPO contains some surprizing complications, and is of independent interest. This will be our main focus here.

To describe H^*BPO, recall the cofibration

$$(BP\mathbb{R} \wedge E\mathbb{Z}/2_+)^{\mathbb{Z}/2} \to BPO \to \Phi^{\mathbb{Z}/2}BP\mathbb{R} \tag{1.1}$$

where $\Phi^{\mathbb{Z}/2}$ denotes geometric fixed points (**[10]**). This gives a long exact sequence of the form

(1.2)

$$\begin{array}{ccccc} H_*((BP\mathbb{R} \wedge E\mathbb{Z}/2_+)^{\mathbb{Z}/2}) & \longrightarrow & H_*BPO & \longrightarrow & H_*(\Phi^{\mathbb{Z}/2}BPO) \\ & & & & \downarrow \partial \\ & & & & H_{*-1}((BP\mathbb{R} \wedge E\mathbb{Z}/2_+)^{\mathbb{Z}/2}). \end{array}$$

THEOREM 1.3. *$H_*BPO \cong Ker(\partial) \oplus Coker(\partial)$ as comodules over the dual A_* of the Steenrod algebra. Moreover,*

$$Ker(\partial) \cong \mathbb{Z}/2,$$

with generator 1.

To interpret this result, we must describe the map ∂ explicitly. We have

$$H_*(\Phi^{\mathbb{Z}/2}BP\mathbb{R}) \cong H_*H \cong A_* = \mathbb{Z}/2[\zeta_1, \zeta_2, ...].$$

We will recall in the next section that we may write

$$\Sigma H_*((BP\mathbb{R} \wedge E\mathbb{Z}/2_+)^{\mathbb{Z}/2}) \cong \mathbb{Z}/2[b_1, b_2, ...][r^{-1}]\{r^{-1}\} \tag{1.4}$$

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where $dim(r^{-1}) = 1$, $dim(b_i) = 2(2^i - 1)$. By our convention, square brackets denote polynomial generators, while braces denote an additive generator, typically used for comparison with other structures, and dimensional shift.

THEOREM 1.5. *The map ∂ can be characterized by the formula*

$$\partial(\zeta_i) = \sum_{j=0}^{i} b_{i-j}^{2^j} r^{2^i - 2^{j+1}+1}$$

where the formula is to be interpreted as multiplicative on the generators, while in the result, all non-negative powers of r are set to 0.

Remark: In particular, we actually have $\partial(\zeta_i) = r^{1-2^i}$.

The result can be extended to calculate also the homology of

$$H_*((\Sigma^{n\alpha} BP\mathbb{R})^{\mathbb{Z}/2})$$

as an A_*-comodule, for an arbitrary $n \in \mathbb{Z}$, where α is the real sign representation of $\mathbb{Z}/2$. In this case, the long exact sequence (1.2) takes on the form

$$\begin{array}{ccc} H_*((\Sigma^{n\alpha} BP\mathbb{R} \wedge E\mathbb{Z}/2_+)^{\mathbb{Z}/2}) & \longrightarrow & H_*((\Sigma^{n\alpha} BP\mathbb{R})^{\mathbb{Z}/2}) \\ & & \downarrow \\ & & H_*(\Phi^{\mathbb{Z}/2} BPO) \\ & & \downarrow \partial_n \\ & & H_{*-1}((\Sigma^{n\alpha} BP\mathbb{R} \wedge E\mathbb{Z}/2_+)^{\mathbb{Z}/2}) \end{array}$$

since $\Phi^{\mathbb{Z}/2} E = \Phi^{\mathbb{Z}/2} \Sigma^{\alpha} E$. It turns out that

$$\Sigma H_*((\Sigma^{n\alpha} BP\mathbb{R} \wedge E\mathbb{Z}/2_+)^{\mathbb{Z}/2}) = \mathbb{Z}/2[b_1, b_2, ...][r^{-1}]\{r^{-n-1}\},$$

so ∂_n is given by the same formula as ∂, where in the result, however, all powers of r greater or equal than $-n$ are set to 0.

THEOREM 1.6. *$Ker(\partial_n)$ is 0 if $n < 0$, and is spanned by monomials in the conjugates $\overline{\zeta_1}$, $\overline{\zeta_2}$,... of degree $\leq n$ (where each $\overline{\zeta_i}$ is counted as having degree 1). Moreover, $H_*((\Sigma^{n\alpha} BP\mathbb{R})^{\mathbb{Z}/2}) \cong Ker(\partial_n) \oplus Coker(\partial_n)$ as A_*-comodules.*

To make use of the statements about A_*-comodules structure, we need to understand the A_*-comodule structure on $H_*((BP\mathbb{R} \wedge E\mathbb{Z}/2_+)^{\mathbb{Z}/2})$. When done directly, this is surprizingly complicated. First of all, we have

$$\Sigma H_* \mathbb{R}P^{\infty} = \mathbb{Z}/2[r^{-1}]\{r^{-1}\} \subset H_*((BP\mathbb{R} \wedge E\mathbb{Z}/2_+)^{\mathbb{Z}/2}).$$

Recall that the A_*-comodule structure on $H_*\mathbb{R}P^{\infty}$ is given by

$$r^{-k} \mapsto (\sum_{\ell \geq 0} r^{2^\ell} \otimes \zeta_\ell)^{-k} \tag{1.7}$$

with non-negative powers of r set to 0. As it turns out, the formula for b_i is

$$b_i \mapsto$$

$$(1.8)\qquad \sum_{j=0}^{i} b_j^{2^{i-j}} \left(\sum_{k=0}^{i-j} (r^{2^i-2^{i-j+1}+2^k} \otimes \zeta_k) \cdot \sum_{\ell\geq 0} (r^{2^\ell} \otimes \zeta_\ell) + \left(\sum_{k<i-j} r^{2^i-2^{i-j+1}+2^k} \otimes \zeta_k^2 \right) \right) \cdot \left(\sum_{\ell \geq 0} r^{2^\ell} \otimes \zeta_\ell \right)^{-2^i}.$$

This is, again, understood as a multiplicative formula together with (1.7), with non-negative powers of r set to 0. The formula is analogous for the case of suspension by $\Sigma^{n\alpha}$, with the difference that we set to 0 all powers of r greater or equal to $-n$.

These formulas, however, seem almost too complicated to be useful. We will show in the next section how they can be understood in a different light by using the equivariant language, and considering all the suspensions $\Sigma^{n\alpha}$, $n \in \mathbb{Z}$, at the same time.

2. The equivariant language

The Adams isomorphism gives

$$(2.1)\qquad (BP\mathbb{R} \wedge E\mathbb{Z}/2_+)^{\mathbb{Z}/2} = BP\mathbb{R} \wedge_{\mathbb{Z}/2} E\mathbb{Z}/2_+.$$

Thus,

$$(2.2)\qquad \begin{aligned} H_*((BP\mathbb{R} \wedge E\mathbb{Z}/2_+)^{\mathbb{Z}/2}) &= (H \wedge (BP\mathbb{R} \wedge_{\mathbb{Z}/2} E\mathbb{Z}/2_+))_* \\ &= ((H_{\mathbb{Z}/2} \wedge BP\mathbb{R}) \wedge_{\mathbb{Z}/2} E\mathbb{Z}/2_+)_* = (H_{\mathbb{Z}/2} \wedge BP\mathbb{R})_*^f. \end{aligned}$$

On the right hand side, $H_{\mathbb{Z}/2}$ denotes $\mathbb{Z}/2$-equivariant Mackey cohomology with constant coefficients, but could alternately denote for example the Borel cohomology theory.

The right hand side notation refers to the *Tate cofibration sequence* [**5**]

$$(2.3)\qquad E^f \to E^c \to E^t$$

for any $\mathbb{Z}/2$-equivariant $RO(\mathbb{Z}/2)$-graded spectrum E. The first map (also known as the *norm map*) is the map

$$E^f = E \wedge E\mathbb{Z}/2_+ \simeq F(E\mathbb{Z}/2_+, E) \wedge E\mathbb{Z}/2_+ \to F(E\mathbb{Z}/2_+, E) = E^c$$

induced by the collapse $E\mathbb{Z}/2_+ \to S^0$. The Tate spectrum E^t is then defined simply as the cofiber of this map.

Now for $E = H_{\mathbb{Z}/2} \wedge BP\mathbb{R}$, the long exact sequence associated with the cofibration sequence (2.3) has been completely calculated in [**6**]. Recall our coefficient convention: for a $\mathbb{Z}/2$-equivariant spectrum E, $E_* = (E^{\mathbb{Z}/2})_*$, while $E_\star$ denotes the complete system of coefficients

$$E_{k+\ell\alpha} = [S^{k+\ell\alpha}, E].$$

Then, for $E = H_{\mathbb{Z}/2} \wedge BP\mathbb{R}$, (2.3) gives a short exact sequence

$$(2.4)\qquad 0 \to (H_{\mathbb{Z}/2} \wedge BP\mathbb{R})_\star^c \to (H_{\mathbb{Z}/2} \wedge BP\mathbb{R})_\star^t \to \Sigma(H_{\mathbb{Z}/2} \wedge BP\mathbb{R})_\star^f \to 0$$

which has the form

(2.5)

$$0 \to \mathbb{Z}/2[\xi_i, \rho, \rho^{-1}, a]_a^\wedge \to \mathbb{Z}/2[\xi_i, \rho, \rho^{-1}, a]_a^\wedge[a^{-1}] \to a^{-1}\mathbb{Z}/2[\xi_i, \rho, \rho^{-1}, a^{-1}] \to 0.$$

Here the element ρ is of dimension $\alpha - 1$, the element a is of dimension $-\alpha$ and the element ξ_i is of dimension $(2^i - 1)(1+\alpha)$. To compute the right A_*-comodule structure, we further compare (2.5) with the analogous sequence for $H_{\mathbb{Z}/2}$ using the Thom reduction

$$BP\mathbb{R} \to H_{\mathbb{Z}/2}. \tag{2.6}$$

Thus, we obtain a diagram of the form

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathbb{Z}/2[\xi_i, \rho, \rho^{-1}, a]^{\wedge}_{a} & \longrightarrow & \mathbb{Z}/2[\zeta_i, \sigma, \sigma^{-1}, a]^{\wedge}_{a} = (H \wedge H)^c_* \\
\downarrow & & \downarrow \\
\mathbb{Z}/2[\xi_i, \rho, \rho^{-1}, a]^{\wedge}_{a}[a^{-1}] & \xrightarrow{f} & \mathbb{Z}/2[\zeta_i, \sigma, \sigma^{-1}, a]^{\wedge}_{a}[a^{-1}] \\
\downarrow & & \downarrow \\
a^{-1}\mathbb{Z}/2[\xi_i, \rho, \rho^{-1}, a^{-1}] & \xrightarrow{f_1} & a^{-1}\mathbb{Z}/2[\zeta_i, \sigma, \sigma^{-1}, a^{-1}] \\
\downarrow & & \downarrow \\
0 & & 0
\end{array} \tag{2.7}$$

Here the A_*-comodule structure on the bottom row is determined by the (a)-complete $\mathbb{Z}/2[a]$-Hopf algebroid structure on $(H \wedge H)^c_*$. This is described in [**6**]. We have

$$\psi(\zeta_i) = \sum_{j=0}^{i} \zeta_j^{2^{i-j}} \otimes \zeta_{i-j} \tag{2.8}$$

and

$$\eta_R(\sigma) = \sum_{i \geq 0} \zeta_i \sigma^{2^i} a^{2^i - 1}. \tag{2.9}$$

Similarly, to determine the vertical maps (2.7), it suffices to consider the middle map f.

Referring to [**6**], we have

$$f(\rho) = \eta_R(\sigma),$$

and the images of the ξ_i's are given by the recursion

$$\begin{array}{l} f(\xi_0) = 1 \\ f(\xi_i) = \frac{1}{a^{2^i}} \left(\frac{\zeta_{i-1}^2}{\eta_R(\sigma)} + \zeta_i a + \frac{f(\xi_{i-1})}{\sigma^{2^i - 1}} \right), \; i \geq 1. \end{array} \tag{2.10}$$

Remark: If we are only interested in the coefficients in dimensions $k + 0\alpha$, this already gives the formulas from the previous section: Put

$$\begin{array}{l} b_i = \xi_i \rho^{1-2^i}, \\ r = \rho a. \end{array} \tag{2.11}$$

This gives (1.4). To compute the Steenrod coaction, we shall use the map f, and the fact that

$$\psi(\eta_R(\sigma) = 1 \otimes \eta_R(\sigma),$$
$$\psi(f(\xi_i)) = \sum_{j=0}^{i} f(\xi_{i-j})^{2^j} \otimes f(\xi_j)$$

(see [6]). Along with (2.11), by expanding the formula (2.10) to explicit form in the $f(\xi_i)$'s, we obtain the formula (1.8).

It will also be useful for us to make formula (2.10) explicit in the ζ_i's.

LEMMA 2.12. *Identifying $f(\xi_i)$ with ξ_i and ρ with $\eta_R(\sigma)$, we have*

$$\zeta_i = \sum_{j=0}^{i} \xi_j^{2^{i-j}} \rho^{-2^{i-j}+1} a^{2^i-2^{i-j+1}+1} + \sigma^{-2^{i-1}} \sum_{j=0}^{i-1} \xi_j^{2^{i-i-1}} \rho^{-2^{i-j-1}+1} a^{1-2^{i-j}}.$$

Proof: Induction. For $i = 0$, the statement is obviously correct. Processing the second formula of (2.10) gives

$$\zeta_i = a^{2^i-1}\xi_i + \xi_{i-1}\sigma^{-2^{i-1}} a^{-1} + \zeta_{i-1}^2 \rho^{-1} a^{-1}.$$

By the induction hypothesis, the right hand side is

$$\begin{aligned} &a^{2^i-1}\xi_i + \xi_i\sigma^{-2^{i-1}} a^{-1} + \\ &\left(\textstyle\sum_{j=0}^{i-1} \xi_j^{2^{i-j}} \rho^{-2^{i-j}+2} a^{2^i-2^{i-j+1}+2}\right) \rho^{-1} a^{-1} + \\ &\left(\sigma^{-2^{i-1}} \textstyle\sum_{j=0}^{i-1} \xi_j^{2^{i-j-1}} \rho^{-2^{i-j-1}+2} a^{2-2^{i-j}}\right) \rho^{-1} a^{-1}. \end{aligned}$$

This is the induction step. □

Now (2.10) completely determines the map f. We shall use this to completely determine the connecting map ∂ of (1.2). As a first step, we will find the following statement useful:

LEMMA 2.13. *Let E be a spectrum (non-equivariant) and let X be a G-spectrum indexed over the G-complete universe. Let i be the inclusion from trivial to complete G-universe. Then we have a natural equivalence*

$$(i_* E_{fixed} \wedge X)^G \simeq E \wedge (X^G),$$

where E_{fixed} denotes E thought of as a fixed G-equivariant spectrum indexed over the trivial universe.

Proof: We will construct a map

$$\phi : E \wedge (X^G) \to (i_* E_{fixed} \wedge X)^G.$$

By adjunction, this is equivalent to having a map

$$i_*(E \wedge (X^G))_{fixed} \to i_* E_{fixed} \wedge X.$$

We have

$$i_*((E \wedge (X^G)_{fixed}) = i_*(E_{fixed}) \wedge i_*(X^G)_{fixed},$$

so it suffices to have a map

$$i_*((X^G)_{fixed}) \to X.$$

We choose this to be the adjoint to the identity $X^G \to X^G$. To show that ϕ is an equivalence, it suffices to consider the case $E = S^0$. In this case the procedure we described starts with the identity $X^G \to X^G$ and adjoins it forth and back, to get back the identity. $\square$

By Lemma 2.13, our goal is equivalent to computing

$$(i_* H\mathbb{Z}/2 \wedge BP\mathbb{R})_*. \tag{2.14}$$

Remark: In [**6**], it was incorrectly claimed that $i_* H\mathbb{Z}/2$ is Real-oriented (which would trivialize this problem). However, the statement about $i_* H\mathbb{Z}/2$ is given only as a curiosity in [**6**], while the homology theory used in the applications is $H_{\mathbb{Z}/2}$. Consequently, other results of [**6**] are unaffected by the mistake.

To calculate (2.14), we will consider the diagram

$$\begin{array}{ccc} \Phi^{\mathbb{Z}/2}(i_* H\mathbb{Z}/2 \wedge BP\mathbb{R}) & \xrightarrow{\partial} & \Sigma(i_* H\mathbb{Z}/2 \wedge BP\mathbb{R} \wedge E\mathbb{Z}/2_+) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle =} \\ \widehat{i_* H\mathbb{Z}/2 \wedge BP\mathbb{R}} & \longrightarrow & \Sigma(i_* H\mathbb{Z} \wedge BP\mathbb{R} \wedge E\mathbb{Z}/2_+). \end{array}$$

On coefficients, this is

$$\begin{array}{ccc} \mathbb{Z}/2[\zeta_i, a, a^{-1}] & \xrightarrow{\partial} & \Sigma a^{-1}\mathbb{Z}/2[\xi_i, \rho, \rho^{-1}, a^{-1}] \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle =} \\ \mathbb{Z}/2[\xi_i, \rho, \rho^{-1}, a]^{\wedge}_{a}[a^{-1}] & \longrightarrow & \Sigma a^{-1}\mathbb{Z}/2[\xi_i, \rho, \rho^{-1}, a^{-1}]. \end{array}$$

Theorems (1.3), (1.5) can now be restated as follows:

THEOREM 2.15. *The map g (and hence the map ∂) is given by*

$$g(\zeta_i) = \sum_{j=0}^{i} \xi_{i-j}^{2^j} \rho^{-2^j+1} a^{2^i - 2^{j+1}+1}.$$

Moreover, in twist 0, *the kernel of the map ∂ is $\mathbb{Z}/2\{1\}$. (The maps f, g are maps of rings.)*

The proof of the Theorem hinges on the following result:

LEMMA 2.16. $gf(\zeta_i) = \sum_{j=0}^{i} (\sigma a)^{-2^i+2^j} \zeta_j.$

Proof that Lemma 2.16 implies Theorem 2.15: Similarly as in the proof of Lemma 2.12, we suppress, for the moment, f from the notation (this is OK since f is injective). ¿From the statement of Lemma 2.16, we now notice that

$$g(\zeta_i) = (\sigma a)^{-2^{i-1}} g(\zeta_{i-1}) + \xi_i.$$

Thus, by induction, to prove the formula of Theorem 2.15, it suffices to prove that

$$\sum_{j=0}^{i} \xi_{i-j}^{i} \rho^{-2^j+1} a^{2^i-2^{j+1}+1} = (\sigma a)^{2^{i-1}} \sum_{j=0}^{i-1} \xi_{i-j-1}^{2^{i-1}} \rho^{-2^j+1} a^{2^{i-1}-2^{j+1}+1} + \xi_i.$$

This is Lemma 2.12.

To prove the injectivity of ∂, let $r = \sigma a$, $u_i = gf(\zeta_i)$. Thus, we have

$$u_i = gf(\zeta_i) = \sum_{j=0}^{i} r^{-2^i+2^j}\zeta_i.$$

Notice that on the right hand side, all summands involve a negative power of r, except the summand ζ_i. Our statement is now equivalent to the following result. □

CLAIM 2.17. *Suppose* $p(u_1, u_2, ...) = q(\zeta_1, \zeta_2, ...)$ *where* p, q *are polynomials with coefficients in* $\mathbb{Z}/2$. *Then* $p = q = 1$ *or* 0.

Proof of Claim: Let $x = r^{-1}$. We have

$$u_n = \zeta_n + u_{n-1}x^{2^{n-1}}.$$

Thus, the equation of the Claim reads

$$(2.18) \qquad p(u_1, ..., u_n) = q(u_1 + x, u_2 + u_1x^2, ..., u_n + u_{n-1}x^{2^{n-1}}),$$

since p, q are polynomials with coefficients in $\mathbb{Z}/2$. Separating the terms free of x on the right, we see that

$$p = q \in \mathbb{Z}/2[x_1, ..., x_n].$$

Now grade $\mathbb{Z}/2[x_1, ..., x_n]$ by giving x_i degree 2^{i-1}. A monomial is given the sum of degrees of its factors. Let p_{top} be the sum of monomials of p of top degree. We see that the sum of monomials of

$$p(u_1 + x, u_2 + u_1x^2, ..., u_n + u_{n-1}x^{2^{n-1}})$$

carrying the highest power of x is

$$p_{top}(x, u_1x^2, ..., u_{n-1}x^{2^{n-1}}).$$

Thus, (2.18) implies that

$$p_{top}(x, u_1x^2, ...u_{n-1}x^{2^{n-1}}) = 0,$$

while we know that

$$p_{top} \neq 0.$$

This is impossible, since $x, u_1x^2, ..., u_{n-1}x^{2^{n-1}}$ are algebraically independent. □

3. Real Morava K-theories and the image of ζ_i

The purpose of this secion is to prove Lemma 2.16. Let $r = \sigma a$. Note that for any $\mathbb{Z}/2$-equivariant spectrum E, we have an obvious natural map

$$i_*H \wedge F(E\mathbb{Z}/2_+, E) \to F(E\mathbb{Z}/2_+, i_*H \wedge E)$$

(indeed, this is adjoint to the map

$$E\mathbb{Z}/2_+ \wedge i_*H \wedge E \simeq E\mathbb{Z}/2_+ \wedge i_*H \wedge F(E\mathbb{Z}/2_+, E) \to i_*H \wedge E$$

given by the collapse $E\mathbb{Z}/2_+ \to S^0$). Thus, in the right hand column of (2.7), $(H \wedge H)^c$ can be replaced by $i_* H \wedge H^c$. Passing to twist 0, the bottom square of (2.7) can be replaced by

$$\text{(3.1)} \qquad \begin{array}{ccc} H_*\Phi^{\mathbb{Z}/2}BP\mathbb{R} & \xrightarrow{\partial} & \Sigma H_*(E\mathbb{Z}/2_+ \wedge BP\mathbb{R}^{\mathbb{Z}/2}) \\ \downarrow{\scriptstyle \overline{f}} & & \downarrow{\scriptstyle \overline{f}_1} \\ H_*\Phi^{\mathbb{Z}/2}(H^c) & \xrightarrow{\overline{\partial}} & \Sigma H_*((E\mathbb{Z}/2_+ \wedge H^c)^{\mathbb{Z}/2}) \end{array}$$

which on coefficients (in twist 0) is

$$\text{(3.2)} \qquad \begin{array}{ccc} \mathbb{Z}/2[\zeta_i] & \xrightarrow{\partial} & (a^{-1}\mathbb{Z}/2[\xi_i, \rho, \rho^{-1}, a^{-1}])_{*+0\alpha} \\ \downarrow{\scriptstyle \overline{f}} & & \downarrow \\ \mathbb{Z}/2[\zeta_i, r, r^{-1}] & \xrightarrow{\overline{\partial}} & r^{-1}\mathbb{Z}/2[\zeta_i, r^{-1}]. \end{array}$$

Since (3.1) is obtained by applying H_* to a diagram of (non-equivariant) spectra, it is a diagram of right A_*-comodules. On the other hand, notice that the bottom row of (3.2) consists of extended right A_*-comodules, that the elements r, r^{-1} are primitive (since they come from $\pi_*\Phi^{\mathbb{Z}/2}H^c$) and that, in fact, the augmentation is induced by the map

$$i_* H \wedge H^c \longrightarrow H^c \wedge H^c \xrightarrow{\phi} H^c.$$

The augmentation takes the form

$$\text{(3.3)} \qquad \begin{array}{ccc} \mathbb{Z}/2[\zeta_i, r, r^{-1}] & \xrightarrow{\overline{\partial}} & r^{-1}\mathbb{Z}/2[\zeta_i, r^{-1}] \\ {\scriptstyle \epsilon}\downarrow & & \downarrow{\scriptstyle \epsilon_1} \\ \mathbb{Z}/2[r, r^{-1}] & \xrightarrow{proj} & r^{-1}\mathbb{Z}/2[r^{-1}] \end{array}$$

where $\epsilon(\zeta_i) = 0$, $\epsilon(\zeta_i) = 0$ for $i > 0$. It is then a general fact about extended comodules that

$$\overline{f}(\zeta_i) = \phi(\epsilon \otimes 1)\psi(\zeta_i) = \sum_{j=0}^{i} \epsilon\overline{f}(\zeta_i)^{2^{i-j}} \otimes \zeta_{i-j}.$$

Thus, we need to show that

$$\epsilon\overline{f}(\zeta_i) = r^{-2^i+1}.$$

Since, however, projection in (3.3) is an isomorphism in positive dimensions, we have reduced the proof of Lemma 2.16 to the following statement.

LEMMA 3.4. *$\epsilon_1\overline{\partial f}(\zeta_i) \neq 0$ for $i > 0$.*

To prove Lemma 3.4, we will recall certain constructions from [**6**]. Specifically, we have constructed a 'quotient spectrum' $BP\mathbb{R}/(x_1, ..., x_n)$ if $x_1, ...x_n \in BP_*$ (non-equivariant!) is a regular sequence. Let

$$k\mathbb{R}(n) = BP\mathbb{R}/(v_i, i \neq n),$$
$$k\mathbb{R}(n)/v_n^2 = BP\mathbb{R}/(v_i, i \neq n, v_n^2).$$

We constructed a cofibration sequence of the form

$$\begin{array}{c}
\Sigma^{(2^n-1)(1+\alpha)}H\mathbb{Z}/2\wedge E\mathbb{Z}/2_+\\
\Big\downarrow t\\
k\mathbb{R}(n)/v_n^2\wedge E\mathbb{Z}/2_+\\
\Big\downarrow\\
H\mathbb{Z}/2\wedge E\mathbb{Z}/2_+\\
\Big\downarrow \mathbb{Q}_n\\
\Sigma^{(2^n-1)(1+\alpha)+1}H\mathbb{Z}/2\wedge E\mathbb{Z}/2_+.
\end{array}$$

($H\mathbb{Z}/2$ is Mackey cohomology with constant coefficients, but could be $i_*H\mathbb{Z}/2$ or $H\mathbb{Z}/2^c$.)

LEMMA 3.5. *Let δ be the composition*

$$BP\mathbb{Z}\wedge S^{\infty\alpha}\xrightarrow{\overline{\delta}}\Sigma BP\mathbb{R}\wedge E\mathbb{Z}/2_+\longrightarrow\Sigma H\mathbb{Z}/2\wedge E\mathbb{Z}/2_+.$$

Then there exists a lift

$$\begin{array}{ccc}
 & & \Sigma^{-(2^n-1)(1+\alpha)}H\mathbb{Z}/2\wedge E\mathbb{Z}/2_+\\
 & \overset{\delta_0}{\nearrow} & \Big\downarrow \mathbb{Q}_n\\
BP\mathbb{R}\wedge S^{\infty\alpha} & \xrightarrow[\delta]{} & \Sigma H\mathbb{Z}/2\wedge E\mathbb{Z}/2_+.
\end{array}$$

Furthermore, if $1:S^0\to BPO\wedge S^{\infty\alpha}$ is the unit, then $\delta_0(1)\neq 0$.

Proof: We consider the diagram

(3.6)

$$\begin{array}{ccccc}
C & \xrightarrow{\iota} & C\wedge S^{\infty\alpha} & \xrightarrow{\overline{\delta}_1} & \Sigma^{-(2^n-1)(1+\alpha)}H\mathbb{Z}/2_f\\
\Big\downarrow\overline{\pi} & & \Big\downarrow\pi & & \Big\downarrow\mathbb{Q}_n\\
BP\mathbb{R} & \xrightarrow{\overline{\iota}} & BP\mathbb{R}\wedge S^{\infty\alpha} & \xrightarrow{\delta} & \Sigma H\mathbb{Z}/2_f\\
\Big\downarrow v_n & & \Big\downarrow v_n\wedge S^{\infty\alpha} & & \Big\downarrow t\\
\Sigma^{-(2^n-1)(1+\alpha)}BP\mathbb{R} & \longrightarrow & \Sigma^{-(2^n-1)(1+\alpha)}BP\mathbb{R}\wedge S^{\infty\alpha} & \xrightarrow{\delta_1} & \Sigma^{1-(2^n-1)(1+\alpha)}(k\mathbb{R}(n)/v_n^2)_f.
\end{array}$$

The vertical lines of this diagram are cofibration sequences, which defines C. The map δ_1 is constructed as follows: By general module-theoretical considerations, we have a diagram

$$\begin{array}{ccc}
\Sigma^{(2^n-1)(1+\alpha)}k\mathbb{R}(n)_f & \longrightarrow & \Sigma^{(2^n-1)(1+\alpha)}(k\mathbb{R}(n)/v_n)_f\\
\Big\downarrow v_n & & \Big\downarrow t\\
k\mathbb{R}(n)_f & \xrightarrow{\theta} & (k\mathbb{R}(n)/v_n^2)_f.
\end{array}$$

The map δ_1 is the composition

$$BP\mathbb{R} \wedge S^{\infty\alpha} \xrightarrow{\overline{\delta}} \Sigma BP\mathbb{R}_f \longrightarrow \Sigma k\mathbb{R}(n)_f \xrightarrow{\theta} \Sigma(k\mathbb{R}(n)/v_n^2)_f$$

where the second map is the $M\mathbb{R}$-module theoretic quotient map.

We now turn our attention to the diagram (3.6). In [6], it was proved that

$$v_n \wedge S^{\infty\alpha} = 0. \tag{3.7}$$

Thus, $t\delta = 0$, which immediately implies the existence of the map δ_0.

Now consider the unit $1 \in \pi_0 BP\mathbb{R}$. Let $\overline{1} = 1 \wedge S^{\infty\alpha} \in \pi_0 BP\mathbb{R} \wedge S^{\infty\alpha}$. Note that since $v_n \wedge S^{\infty\alpha} = 0$, we have

$$C \wedge S^{\infty\alpha} = (BP\mathbb{R} \wedge S^{\infty\alpha}) \vee (\Sigma^{-(2^n-1)(1+\alpha)-1} BP\mathbb{R} \wedge S^{\infty\alpha}).$$

Thus, $\overline{1}$ lifts to $1' \in \pi_0 C \wedge S^{\infty\alpha}$. Note that we may define $\overline{\delta}_1 = \delta_0 \pi$. Thus, the non-vanishing statement is equivalent to

$$(\overline{\delta}_1)_* 1' \neq 0.$$

This is equivalent to saying that $1' \notin Im(\iota_*) : \pi_0 C \to \pi_0 C \wedge S^{\infty\alpha}$. Assume this was not true, i.e.

$$1' = \iota_* 1'', \; 1'' \in \pi_0 C. \tag{3.8}$$

Recall that $\overline{\iota} : \pi_0 BP\mathbb{R} \to \pi_0 BP\mathbb{R} \wedge S^{\infty\alpha}$ is a mod 2 isomorphism, so we know that

$$\overline{\pi}_*(1'') \equiv 1 \mod 2.$$

But note that this is not possible, since by the calculation of $\pi_* BP\mathbb{R}$ ([6]),

$$v_n \cdot e \neq 0$$

for any class $e \in \pi_0 BP\mathbb{R}$ congruent to $1 \mod 2$. Thus, (3.8) leads to a contradiction and Lemma 3.5 is proved. □

Now recall that we can identify

$$\pi_*(H\mathbb{Z}/2 \wedge E\mathbb{Z}/2_+) = \Sigma^{-1} a^{-1} \mathbb{Z}/2[\sigma, \sigma^{-1}, a^{-1}] \tag{3.9}$$

via the quotient map

$$\widehat{H\mathbb{Z}/2} \to \Sigma H\mathbb{Z}/2_f$$

where $\widehat{H\mathbb{Z}/2}_* = \mathbb{Z}/2[\sigma, \sigma^{-1}, a, a^{-1}]$. Using (3.9), Lemma 3.5 implies that

$$(\delta_0)_*(1) = \sigma^{-2^n} a^{2-2^{n+1}}. \tag{3.10}$$

Now consider the map

$$\begin{array}{ccc} (i_* H\mathbb{Z}/2 \wedge \Sigma^{-(2^n-1)(1+\alpha)} H\mathbb{Z}/2 \wedge E\mathbb{Z}/2_+)_* & \xrightarrow{(1\wedge Q_n)_*} & (i_* H\mathbb{Z}/2 \wedge \Sigma H\mathbb{Z}/2 \wedge E\mathbb{Z}/2_+)_* \\ & & \downarrow{\scriptstyle \phi\wedge Id} \\ & & (i_* H\mathbb{Z}/2 \wedge \Sigma E\mathbb{Z}/2_+)_*. \end{array}$$

In twist 0, this is, by Lemma 2.13,

$$\begin{array}{ccc} H\mathbb{Z}/2_*(\Sigma^{-(2^n-1)(1+\alpha)} H\mathbb{Z}/2 \wedge E\mathbb{Z}/2_+) & \longrightarrow & H\mathbb{Z}/2_*(\Sigma H\mathbb{Z}/2 \wedge E\mathbb{Z}/2_+) \\ & \searrow^{\lambda_1} & \downarrow \\ & & H\mathbb{Z}/2_* E\mathbb{Z}/2_+. \end{array} \tag{3.11}$$

On the upper left corner of (3.11), $\pi_*(X) \subset H_*(X)$ (where $X = \Sigma^{-(2^n-1)(1+\alpha)}H\mathbb{Z}/2\wedge E\mathbb{Z}/2_+$) sits as the submodule of primitives, and

$$H\mathbb{Z}/2_*X = \pi_*X \otimes A_*. \tag{3.12}$$

By (3.10) and Lemma 3.5, the statement of Lemma 3.4 in the language of (3.12) would follow from

$$\lambda_1(\sigma^{-2^n}a^{2-2^{n+1}} \otimes \zeta_{n+1}) \neq 0. \tag{3.13}$$

Using the short exact sequence

$$0 \to \pi_*H\mathbb{Z}/2^c \to \pi_*\widehat{H\mathbb{Z}/2} \to \pi_*\Sigma H\mathbb{Z}/2_f \to 0,$$

we may pull $\sigma^{-2^n}a^{2-2^{n+1}}$ back to $\widehat{H\mathbb{Z}/2}_*$, divide by $a^{2-2^{n+1}}$, and pull back to $(H\mathbb{Z}/2^c)_*$. Considering the composition

$$\begin{array}{ccc} (i_*H\mathbb{Z}/2\wedge\Sigma^{-(2^n-1)(1+\alpha)}H\mathbb{Z}/2^c)_* & \xrightarrow{1\wedge F(E\mathbb{Z}/2_+,\mathbb{Q}_n)} & (i_*H\mathbb{Z}/2\wedge\Sigma H\mathbb{Z}/2^c)_* \\ & & \downarrow\phi \\ & & (\Sigma H\mathbb{Z}/2^c)_*, \end{array} \tag{3.14}$$

$$\lambda = \phi\circ(1\wedge F): (i_*H\mathbb{Z}/2\wedge\Sigma^{-(2^n-1)(1+\alpha)}H\mathbb{Z}/2^c)_* \to (\Sigma H\mathbb{Z}/2^c)_*$$

the inequality (3.13) would now follow from

$$\lambda(\sigma^{-2^n}\otimes\zeta_{n+1}) \neq 0. \tag{3.15}$$

However, we have forgetful maps to non-equivariant coefficients, where (3.14) is simply the cap product

$$A_* \xrightarrow{\cap Q_n} A_* \xrightarrow{\epsilon} \mathbb{Z}/2.$$

Thus, (3.15) follows from the fact that non-equivariantly,

$$(\zeta_{n+1}\cap Q_n) \neq 0.$$

This concludes the proof of Lemma 3.4.

4. The twisted dimensions

We shall now calculate $H_*(\Sigma^{n\alpha}BP\mathbb{R})^{\mathbb{Z}/2}$ for $n\in\mathbb{Z}$. Proceeding analogously as in the twist 0 case, we have the map

$$\Phi^{\mathbb{Z}/2}(i_*H\mathbb{Z}/2\wedge BP\mathbb{R}) \xrightarrow{\partial_n} \Sigma(\Sigma^{n\alpha}i_*H\mathbb{Z}/2\wedge BP\mathbb{R}\wedge E\mathbb{Z}/2_+)^{\mathbb{Z}/2} \tag{4.1}$$

and an extension of the form

$$0 \to Ker(\partial_n) \to H_*(\Sigma^{n\alpha}BP\mathbb{R})^{\mathbb{Z}/2} \to Coker(\partial_n). \tag{4.2}$$

Furthermore,

$$\begin{aligned}&\Sigma(\Sigma^{n\alpha}i_*H\mathbb{Z}/2\wedge BP\mathbb{R}\wedge E\mathbb{Z}/2_+)^{\mathbb{Z}/2} \equiv \\ &\Sigma(\Sigma^{n\alpha}BP\mathbb{R})_f^{\mathbb{Z}/2} = r^{-(n+1)}\mathbb{Z}/2[\xi_1,\xi_2,\ldots][r^{-1}].\end{aligned}$$

the map ∂_n is determined by the map g of Theorem 2.15. Thus, the question reduces to determining the behaviour of the map ∂_n, and the extension (4.2) in the category of A_*-comodules.

To this end, let $P(n)\subset A_*$ denote the subgroup generated by all monomials of order $\leq n$ in the variables $\overline{\zeta_1}$, $\overline{\zeta_2}$, ... (the conjugates of ζ_1, ζ_2,...) where each of the variables $\overline{\zeta_i}$ is given the degree 1. Note that $P(n) \subset A_*$ is a right A_*-submodule. Thus, we get a short exact sequence of right A_*-modules

$$0 \to P(n) \to A_* \to Q(n) \to 0. \tag{4.3}$$

Proof of Theorem 1.6: We will begin by considering the geometrically defined cofibration sequence

$$\Sigma^{-1}MO \xrightarrow{\partial} (\Sigma^{n\alpha}M\mathbb{R})_f^{\mathbb{Z}/2} \xrightarrow{\kappa} (\Sigma^{n\alpha}M\mathbb{R})^{\mathbb{Z}/2} \xrightarrow{\lambda} MO \tag{4.4}$$

which in homotopy induces a long exact sequence

$$\to \mathbb{Z}/2[a_1, a_2, ...] \to \Sigma H_*(\Sigma^{n\alpha}M\mathbb{R})_f^{\mathbb{Z}/2} \to \Sigma H_*(\Sigma^{n\alpha}M\mathbb{R})^{\mathbb{Z}/2} \to \dots . \tag{4.5}$$

Recall that the generators a_i of H_*MO are obtained as elements in the image of

$$H_*\Sigma^{-1}(BO(1))^{\gamma_1} \to H_*MO. \tag{4.6}$$

We have, of course,

$$BO(1)^{\gamma_1} \cong B\mathbb{Z}/2.$$

Now consider the map

$$i_n : \Sigma^{-n}BO(n)^{\gamma_n} \to MO. \tag{4.7}$$

By a general principle, $\partial i_n = 0$, so i_n lifts to a map

$$\iota_n : \Sigma^{-n}BO(n)^{\gamma_n} \to (\Sigma^{n\alpha}M\mathbb{R})^{\mathbb{Z}/2},\ i_n = \lambda\iota_n. \tag{4.8}$$

(Geometrically, ι_n is adjoint to the Thomification of the classifying map of the complexification of γ_n - which is a Real bundle on the fixed space $BO(n)$ - with the inclusion of fixed points.)

Thus, we have proven

LEMMA 4.9. *The kernel of $H_*\partial$ contains $H_*\Sigma^{-n}BO(n)^{\gamma_n}$, which consists of monomials of degree $\leq n$ in the variables $a_1, a_2, ...$ (where each a_i is given degree 1).*

□

To relate this to the statement of Theorem 1.6, we need to apply the Quillen idempotent to (4.4), (4.5), and determine the images of the a_i's in terms of the ζ_i's. This is done as follows:

Recall that the a_i's are the coefficients of the exponential function of the universal Formal Group Law on MO_* (classifying the FGL's G which satisfy $[2]_G = 0$). In other words, the series

$$\sum_{i \geq 0} a_i x^{i+1} \tag{4.10}$$

is the functional inverse of the series

$$\sum_{i \geq 0} m_i x^{i+1} \tag{4.11}$$

where m_i are the coefficients of the logarithm of the universal FGL on MO_*. Now on the coefficients of the logarithm, the Quillen idempotent

$$e : MO \to MO \tag{4.12}$$

has the effect

$$\begin{aligned} m_{2^k-1} &\mapsto \zeta_k \\ m_i &\mapsto 0,\ i \neq 2^k - 1. \end{aligned} \tag{4.13}$$

We conclude that in terms of the ζ_k's, the a_i's are the coefficients of the functional inverse of the series

$$\sum_{k\geq 0} \zeta_k x^{2^k}. \tag{4.14}$$

In other words, (4.12) sends

$$\begin{aligned} a_{2^k-1} &\mapsto \overline{\zeta_k},\ k = 0, 1, 2, \ldots \\ a_i &\mapsto 0,\ i \neq 2^k - 1. \end{aligned} \tag{4.15}$$

Thus, applying now

$$e : M\mathbb{R} \rightarrow M\mathbb{R} \tag{4.16}$$

to (4.4), (4.5), (4.15) implies that

$$P(n) \subseteq Ker(\partial_n). \tag{4.17}$$

On the other hand, when examining the image of $\overline{\zeta_k}$ under (4.1), we see from the ζ_k summand of $\overline{\zeta_k}$ that the coefficient of the lowest power of r (=highest power of r^{-1}), which is r^{-1}, contains the summand

$$a_{k-1} r^{-1}. \tag{4.18}$$

Since the elements (4.18) are algebraically independent (over $\mathbb{Z}/2$), we see that the image of any polynomial in the $\overline{\zeta_i}$'s of degree $> n$ will be of degree $> n$ in r^{-1}. Therefore,

$$Ker(\partial_n) \subseteq P(n). \tag{4.19}$$

Furthermore, (4.8) shows that $P(n)$ is a right A_*-subcomodule of $H_*(\Sigma^{n\alpha} BP\mathbb{R})^{\mathbb{Z}/2}$, which proves the splitting claimed in the statement of Theorem 1.6. □

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