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Conformal field theory and elliptic cohomology

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Abstract

In this paper, we use conformal field theory to construct a generalized cohomology theory which has some properties of elliptic cohomology theory which was some properties of elliptic cohomology. A part of our presentation is a rigorous definition of conformal field theory following Segal's axioms, and some examples, such as lattice theories associated with a unimodular even lattice. We also include certain examples and formulate conjectures on modular forms and Monstrous Moonshine related to the present work.

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1. Introduction

The purpose of the present paper is to address an old question (posed by Segal [37]) to find a geometric construction of elliptic cohomology. This question has recently become much more pressing due to the work of Hopkins and Miller [19], who constructed exactly the "right", or universal, elliptic cohomology, called *TMF* (the theory of topological modular forms). In the present paper, however, we do not propose a construction which would give *TMF*. We do propose what could be called the "first rigorous reasonable attempt" of constructing geometrically any generalized cohomology theory which could be called elliptic cohomology. To explain what this

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means, we must say more about what is expected of a theory which could be called elliptic cohomology, and also what qualifies as a geometric construction: as it turns out, quite a few general desiderata of such theory can be written down, and finding simply a candidate which would satisfy them all is a non-trivial goal.

First of all, elliptic cohomology should be related to, and ideally better explain, Borcherds' proof [2] of the Moonshine conjectures [5] on the modularity of the Thompson series, which are character series of a certain graded representation of the Monster called the Moonshine module $V^\natural$ [13]. (More precisely, the statement of the conjectures is that the Thompson series are Hauptmoduln. Much work has been done on this and related topics; see e.g. [21,25,32,42–44] for reference.) The Thompson series are characters, i.e. trace series, of a certain graded module of the Monster F_1 [17], known as the Moonshine module [13]

$$V^\natural = \bigoplus_{n \geqslant -1} V_n q^n.$$

Thus, if one wants to work in homotopy theory, one can interpret $V^\natural$ as a map

$$BF_1 \to K[[q]][q^{-1}], \tag{1}$$

where K is K-theory. This might suggest $K[[q]][q^{-1}]$ as the first candidate for elliptic cohomology. This approach was indeed pursued by Ando [1], and leads to some valuable conclusions; in particular, $K[[q]][q^{-1}]$ is the "homotopical counterpart" of the Tate curve. However, we want to go further: the coefficient ring of the Tate curve does not consist of modular forms of any kind, so this approach does not explain the modularity of the Thompson series. Also, homotopy-theoretically, $K[[q]][q^{-1}]$ brings no new information beyond K-theory. One way to say in which direction we want to go is that the q in (1) corresponds to an S^1-parameter in an elliptic curve, and we would like a candidate for elliptic cohomology which would be modular in the sense that it would not need an a priori specification of such parameter.

We must go back to the geometry to see where such structure could come from; the most substantial idea [2], which is at the heart of the very construction of the Monster [17], is the fact [2,13] that $V^\natural$ is a vertex operator algebra (VOA). Indeed, F_1 is the group of automorphisms of the VOA $V^\natural$. VOA is a mathematical structure which is also the first rigorous mathematical encoding of the physical concept of (chiral perturbative) conformal field theory (CFT). On the other hand, a more "maximalistic" approach to CFT [36] builds in modularity in the form of correlation functions on elliptic curves. This is not directly visible to VOAs, and the proof [2] takes a different route. But the modularity of CFT suggests trying to replace $K[[q]][q^{-1}]$ with some type of theory which would be based on the structure of CFT. More specifically, (1) should factor as

$$BF_1 \xrightarrow{\alpha} B_{\mathrm{ell}} V^\natural \xrightarrow{\beta} E \xrightarrow{\gamma} K[[q]][q^{-1}], \tag{2}$$

where $B_{\text{ell}}V^{\natural}$ is some type of classifying space associated with CFT, whose construction would, similarly as the CFT itself, be modular, and E would be the elliptic cohomology, i.e. some type of generalized cohomology theory based on $B_{\text{ell}}V^{\natural}$, whose construction would therefore also be modular; in particular, the map γ in (2), on coefficients, would have a modular image. This is the route we take in the present paper, although we must point out that we do not quite get (2), since that involves certain technical aspects of exponentiating vertex operators, which we cannot resolve. We do, however, get certain simpler analogues, for example one where F_1 is replaced by E_8.

To describe this construction, we must first talk about the space $B_{\text{ell}}V^{\natural}$ of (2). This should be some type of classifying space associated with a CFT (more precisely 1-CFT—see Section 2 below). Therefore, it is appropriate to talk about $B_{\text{ell}}\mathscr{H}^{\natural}$ where $\mathscr{H}^{\natural}$ is the 1-CFT completion of $V^{\natural}$—see Section 4 below.

In this paper, we give a rigorous construction of $B_{\text{ell}}\mathscr{H}$ for every 1-CFT $\mathscr{H}$. This construction is, in some sense, analogous to the construction of a classifying space of a group. The question of existence of the map α of (2) can be phrased in general terms, whether there always exists a map

$$\alpha : B(Aut(\mathscr{H})) \to B_{\text{ell}}\mathscr{H}.$$

This is at present only a conjecture, but we will give some examples where it is true (see Sections 6 and 7 below).

Now E should be "elliptic cohomology" in the sense of [37] (see also [41]). Our approach to the map β of (2) is to define a choice of E based on $B_{\text{ell}}\mathscr{H}$. In fact, it is a "free construction", obtained by taking the suspension spectrum and formally inverting, in a suitable sense, certain elements $\omega \in \pi_* B_{\text{ell}}\mathscr{H}$. Evidence in favor of such approach is given by a well known result in K-theory, where inverting the Bott class in $\Sigma^{\infty}\mathbb{C}P^{\infty}_{+}$ gives K [39,40]. With this approach to E, the construction of the map γ of (2) becomes a non-trivial problem, more difficult than with homotopy theory-based definitions of E. The definition of E and construction of the map γ are given in Section 5 below.

To give the reader a preview, the main idea of constructing $B_{\text{ell}}\mathscr{H}$ is to adapt the idea of a bundle on X (say, X is a compact complex curve) to give a notion of 'stringy' bundle B on the loop space LX. The main feature of a stringy bundle should be that a holomorphic embedding of a rigged surface A into X should induce, up to scalar multiple, a map

$$\bigotimes_{in} \mathscr{H}_{c_i} \to \bigotimes_{out} \mathscr{H}_{d_j}$$

where c_i (resp. d_j) are the inbound (resp. outbound) boundary components of A, and $\mathscr{H}_c$ is the fiber of the bundle B over c.

An ordinary bundle on X can be trivialized when pulled back to a cover $\mathscr{U}$ of X, which can be thought of as a 0-equivalence

$$\coprod_{\mathscr{U}} U \to X.$$

In case of a stringy bundle, 0-equivalence should be replaced by 1-equivalence. In the case of a complex curve X, this essentially amounts to

$$X - S$$

where $S \subset X$ is some finite set of "punctures". The precise definition of stringy bundle specifies the data at the punctures, and will be given in Section 5 below. Next, we define $B_{\text{ell}}\mathscr{H}$ as the space of stringy bundles with fiber $\mathscr{H}$ on an elliptic curve E_τ which are equivariant with respect to the translation action of E_τ. We call such stringy bundles elliptic bundles.

A large part of this paper is in doing certain calculations which allow us to come up with candidates of the class ω. This requires reconciling certain standard computations of characters of CFT and modular forms with the new construction. The first examples of elements ω which can be used to define E are given in Section 6. These elements are given by ratios of theta functions of suitable lattices. In Section 7, we give more advanced examples. We give a map

$$BG \to B_{\text{ell}}\mathscr{H}_G$$

for any simply connected simply laced group G where $\mathscr{H}_G$ is the conformal field theory on the basic level 1 representation of the loop group LG [33]. For $G = E_8$, this affords a choice of an element ω whose image in $K_*[[q]]$ is the discriminant form Δ (at least up to a multiplicative constant, i.e. localized, in the sense of homotopy theory, away from finitely many small primes). We also discuss the example of the Leech lattice, and formulate a general conjecture about theta functions of lattices. We also discuss the Moonshine module and the Monster. We show how parts of Borcherds' calculations [2] lead to a possible higher homotopy analogue of the Moonshine conjecture.

Finally, before any of this discussion can begin, we must address the question of a rigorous definition of conformal field theory (and its variations, such as 1-CFT), which our theory inherently needs. While the idea of a rigorous definition of CFT is firmly contained in [36], incredibly, details were never published, or perhaps even completely worked out, during the last 20 years. In the next three Sections 2–4 we must undertake the formidable task of, at least partially, remedying this situation. We divide this task as follows: In Section 2, we give the complete axiomatic definition of chiral CFT (and related notions needed). This greatly exceeds the rather limited step taken in [26]. The completely rigorous axioms are quite complicated, and involve substantially the language of 2-categories and stacks. In Section 3, we give, also in substantial detail, the construction of 1-CFTs associated with even lattices, as well as the full CFTs associated with even unimodular lattices. In Section 4, we give, in somewhat less detail, the construction of the 1-CFT structure on the Hilbert-completed Moonshine module: this example is important for our motivation, but much of the main technical discussion of the rest of the paper can be carried out without it.

Having described the positive features of the theory proposed in this paper, it is important to also point out its shortcomings. As we said in the beginning of the introduction, the theory is only a first attempt, and the properties we control would, in some sense, have to be shared by any reasonable attempt. Many properties which would be desired from a more definitive theory, however, are unimplemented in the present one. This includes, of course, the issue of picking the "right" theory, i.e. one which would construct *TMF*, or at least one whose coefficients could be completely calculated. Further, there should be a better geometric reason for choosing the classes ω: in the present theory, the only evidence in favor of picking particular classes is that they are certain distinguished modular forms on coefficients. Ideally, however, one should have an index theory on loop space, as proposed in [36], a geometric interpretation of the Witten genus, and its twisted form, which would explain the classes ω. Finally, to do that, presumably one needs a better additive theory than the "free" theory (suspension spectrum).

On some of these points, there have been recent clues. For example, a candidate additive theory is suggested in [20] via 2-vector spaces, and this theory makes contact with Rognes' K-theory of K-theory [34]. A fascinating program has been also recently revealed by Stolz and Teichner, whereby the elliptic cohomology infinite loop space should be constructed directly as a "moduli space of CFTs", which would be directly delooped, without use of additive loop space theory. These topics, however, exceed the scope of the present paper, and will not be discussed here.

2. Conformal field theory

In this paper, a *rigged surface* is a two-dimensional smooth manifold with boundary X and a parametrization diffeomorphism

$$f_c : S^1 = \{z \in \mathbb{C} \mid ||z|| = 1\} \to c \tag{3}$$

for every boundary component c of X, together with a complex structure on X with respect to which each of parametrizations (3) is analytic. The complex structure determines an orientation, and with respect to that parametrizations (3) have two possible orientations, which we will call *inbound* and *outbound*. By convention, we call, for

$$D = \{z \in \mathbb{C} \mid ||z|| \leqslant 1\},$$

the identity boundary parametrization inbound.

Our first task is to capture fully the structure present on the set $\mathscr{C}$ of all rigged surfaces. The essential point is that there are two operations on rigged surfaces: disjoint union and gluing. Disjoint union $\coprod$ is obvious. Gluing means that if X is a rigged surface with one chosen inbound boundary component c and one chosen outbound boundary component c', then there is a canonical rigged surface structure on

$$\check{X} = X/\sim,$$

where $\sim$ is the smallest equivalence relation on X which identifies

$$f_c(z) \sim f_{c'}(z)$$

for every $z \in S^1$.

In addition, however, one can also consider *families* of rigged surfaces. Let B be a complex manifold. Then a family of rigged surfaces over B is, roughly speaking, a transverse map

$$p : X \to B,$$

where X is a complex manifold with analytic boundary, where each fiber is a rigged surface, where the parametrizations vary holomorphically.

The most convenient way to make this precise is to consider the manifold Y obtained by gluing, locally, solid cylinders to the boundary components of X. Then, a holomorphic family of rigged surfaces X over a finite dimensional complex manifold B is a holomorphic map

$$q : Y \to B$$

transverse to every point, such that $dim(Y) = dim(B) + 1$ and B is covered by open sets U_i for each of which there are given holomorphic regular inclusions

$$s_{i,c} : D \times U_i \to Y$$

with

$$q \circ s_{i,c} = Id_{U_i},$$

where c runs through some indexing set C_i. Further, if $U_i \cap U_j \neq \emptyset$, we require that there be a bijection $\iota : C_i \to C_j$ such that

$$s_{i,c}|_{D \times (U_i \cap U_j)} = s_{j,\iota(c)}|_{D \times (U_i \cap U_j)}.$$

Then we let

$$X = Y - \left(\bigcup_i \bigcup_{c \in C_i} s_{i,c}((D - S^1) \times U_i) \right).$$

Then the fiber of X over each $b \in B$ is a rigged surface, which vary holomorphically in b, in the sense we want. (Note that the reason the maps s_c cannot be defined globally in B is that it is possible for a non-trivial loop in $\pi_1(B)$ to permute the boundary components of X.)

Capturing the mathematical structure contained in the operations $\coprod$, $\check{X}$ and the notion of holomorphic family of rigged surfaces is a formidable task. To some level of detail, this was done in [10,26]. Note, for example, that the operation $\coprod$ is not strictly commutative and associative. Rather, we must consider the set $\mathscr{C}$ as a groupoid, where the isomorphisms are diffeomorphisms compatible with complex structure and boundary component parametrizations. This groupoid is then a symmetric monoidal category with respect to the operation $\coprod$.

More generally, this leads to the notion of a *lax* algebraic structure. To define that notion, however, we need to understand *strict* algebraic structures completely. In the most simple (=classical) case, this is accomplished through the notion of a *theory* according to Lawvere [27]. A theory is essentially a universal algebra, i.e. the structure given by a set of algebraic operations, which satisfy certain relations (identities) between the operations. More precisely, Lawvere defines a theory as a category with objects $\mathbb{N}$ such that n is the product of n copies of 1. It is beneficial to let, for a theory T,

$$T(n) = Hom_T(n, 1),$$

and write down specifically the axioms for $T(n)$. In this (equivalent) sense, a theory T is a functor from the category whose objects are natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ and morphisms from k to m are maps of sets $\{1, \dots, k\} \to \{1, \dots, m\}$ together with a distinguished element $1 \in T(1)$, a composition operation

$$\gamma : T(k) \times T(n_1) \times \cdots \times T(n_k) \to T(n_1 + \cdots + n_k).$$

The operation γ is associative and unital, and equivariant with respect to the functorial structure, in the obvious sense. For a set X, we have the *endomorphism theory* $End(X)$ where $End(X)(n) = Map(X^{\times n}, X)$. (In fact, Lawvere's approach tells us that any theory is an endomorphism theory, if we replace the category of sets by a suitable category.) Then a structure of a T-*algebra* on X is given by a map of theories

$$T \to End(X).$$

Note that a universal algebra type is often given by a set of operations Ω each of which has an *arity*, i.e. specified number of input variables, and relations E between (compositions of) the operations Ω. This amounts simply to taking the *free theory* on the sequence $\Omega(n)$ consisting of operations of arity n, factored out by the smallest congruence (in the category of theories) containing the relation E.

Defining a *lax algebra* X over a theory T is not difficult. X is a groupoid, and we set

$$End_{lax}(X)(n) = Functors(X^n, X).$$

Then a structure of a *lax* T-*algebra* on X is given by a map

$$\phi : T \to End_{lax}(X)$$

and natural isomorphisms

$$\phi(1) \cong Id,$$

$$\phi(\gamma(g; g_1, \ldots, g_k)) \cong \gamma(\phi(g); \phi(g_1), \ldots, \phi(g_k)),$$

$$\phi(T(f)(g)) \cong (End_{lax}(X)(f))(\phi(g)).$$

Note that these *coherence isomorphisms* correspond to *operations* of a general theory (i.e. unit, composition and functoriality). The coherence isomorphisms are subject to *coherence diagrams* which must commute. These diagrams correspond to *relations* which are satisfied by the operations of a general theory, i.e. the associativity, unit and equivariance relations mentioned above. We shall not draw these diagrams explicitly.

We will, however, note that we have a notion of *lax morphism*

$$f : X \to Y \tag{4}$$

of lax algebras X, Y over a theory T which is a functor (4) where for each operation α (identity, composition and functoriality) of a theory, we have an isomorphism

$$\alpha(f, \ldots, f) \cong f\alpha. \tag{5}$$

For each relation of a general theory (composition associativity, unit, equivariance and functoriality associativity) we then have a *coherence diagram* closed by isomorphisms (5) and the coherence isomorphisms of the lax T-algebras X, Y; we require of a lax morphism that all such diagrams commute.

In fact, lax T-algebras form a 2-category [3] in which lax morphisms are 1-morphisms. By a 2-*morphism* between (4) and

$$g : X \to Y \tag{6}$$

we shall mean a natural isomorphism

$$f \cong g, \tag{7}$$

we require of isomorphisms (7) to form commutative diagrams with the coherence isomorphisms of the 1-morphisms f, g. Therefore, such coherence diagrams will be indexed over operations of a general theory.

Note that in the 2-category of lax T-algebras, every 2-morphism is an isomorphism. Furthermore, it is known that this 2-category has lax limits. (We refer the reader to [10]. A classical reference about 2-categories, which however contains only some of the relevant results, and uses a slightly different terminology, is [3].) For any 2-category $\mathscr{F}$ which satisfies these two conditions, and any Grothendieck topology $\mathscr{B}$, we can speak of $\mathscr{F}$-stacks over $\mathscr{B}$. These are contravariant lax functors $\mathscr{B} \to \mathscr{F}$ which turn Grothendieck covers into lax limits. Thus, we can speak of stacks of lax T-algebras. We should remark that in CFT, stacks play an important role for classification (otherwise it appears one could construct a lot of artificial examples), but a marginal role from the point of view of characteristics of the structure itself: stacks simply describe how our structure varies over spaces which are objects of the indexing site. For example, they provide a proper axiomatization of the notion of "holomorphic family of rigged surfaces". A reader uninterested in this detail may simply restrict attention to sections over a point, and disregard stacks entirely.

Rigged surfaces form a stack of lax commutative monoids over the site of complex manifolds with Grothendieck topology of open covers, with respect to the operation of disjoint union. Unfortunately, however, the formalism of lax algebras over a theory is not general enough to describe the gluing operation $\check{X}$, which is indexed by a choice of pair of inbound and outbound boundary component of X; we must consider the set of all inbound and outbound components of X as an attribute of X, and be able to index operations by such attributes. The framework of algebras over a theory, lax or strict, does not allow for that.

What we need is the notion of a 2-*theory* Θ *fibered over another theory* T. We begin by discussing the strict structure.

Using Lawvere's language, a 2-theory consists of a natural number k, a theory T and a (strict) contravariant functor Θ from T to the category of categories (and functors) with the following properties. Let T^k be a category with the same objects as T, and $Hom_{T^k}(m,n) = Hom_T(m,n)^{\times k}$. Then

$$Obj(\Theta(m)) = \coprod_n Hom_{T^k}(m,n),$$

for $\phi : m \to n$ in T, the map $Obj(\Theta(n)) \to Obj(\Theta(m))$ which is a part of $\Theta(\phi)$ is given by precomposition with $(\phi, \ldots, \phi)$ (this axiom was originally missing and the mistake was found by T. Fiore) and

$$\gamma \in Hom_{T^k}(m,n)$$

is the product, in $\Theta(m)$, of the n-tuple

$$\gamma_1, \ldots, \gamma_n \in Hom_{T^k}(m,1)$$

with which it is identified by the fact that T is a theory. (We need to allow $k \neq 1$ because in the example we are interested in, $k = 2$.)

Again, it is beneficial to write down the axioms of a 2-theory explicitly, by letting

$$\Theta(w; w_1, \ldots, w_n) = Hom_{\Theta(m)}((w_1, \ldots, w_n), w)$$

for w_i, $w \in T(m)^{\times k}$ where the n-tuple $(w_1, \ldots, w_n)$ is identified with the corresponding element of $Hom_{T^k}(m,n)$.

Let T be a theory, and let $k \in \mathbb{N}$ be a fixed number. Then, as remarked, for a number $m \in \mathbb{N}$ and for any elements

$$w_1, \ldots, w_n, w \in T(m)^{\times k},$$

we have a set

$$\Theta(w; w_1, \ldots, w_n). \tag{8}$$

There are the following operations on a 2-theory:

1. A unit $1 \in \Theta(w; w)$ where $w \in T(m)^{\times k}$ is any element.
2. Θ-composition

$$\gamma : \Theta(w; w_1, \ldots, w_n) \times \Theta(w_1; w_{11}, \ldots, w_{1p_1}) \times \cdots \times \Theta(w_n; w_{n1}, \ldots, w_{np_n})$$
$$\to \Theta(w; w_{11}, \ldots, w_{np_n})$$

   where all w, w_i, $w_{ij} \in T(m)^{\times k}$.
3. Θ-functoriality: for a map

$$\imath : \{1, \ldots, q\} \to \{1, \ldots n\},$$

   a map

$$\Theta(w; w_{\imath(1)}, \ldots, w_{\imath(q)}) \to \Theta(w; w_1, \ldots, w_n).$$

4. T-functoriality: for a map

$$\imath : \{1, \ldots, m\} \to \{1, \ldots, q\},$$

   a map

$$\Theta(w; w_1, \ldots, w_n) \to \Theta(\imath^{\times k} w; \imath^{\times k} w_1, \ldots, \imath^{\times k} w_n).$$

5. T-substitution: For $u_i \in T(k_i)$, $i = 1, \ldots, m$, and $v_i = \gamma^{\times k}(w_i; u_1^{\times k}, \ldots, u_m^{\times k})$, $v = \gamma^{\times k}(w; u_1^{\times k}, \ldots, u_m^{\times k})$, a map

$$\Theta(w; w_1, \ldots, w_n) \to \Theta(v; v_1, \ldots, v_n).$$

The axioms (relations) required of a 2-theory are: associativity and unitality of Θ-composition, associativity of Θ-functoriality, Θ-equivariance, associativity of T-functoriality and T-substitution, T-equivariance, and commutativity between T-substitution and T-functoriality and Θ-composition and Θ-functoriality. The meaning of these axioms is clear, and will not be given in detail here (since one can always use the categorical definition for guidance). Similarly, it is clear what one means by (strict) morphism

$$(\Theta, T) \to (\Sigma, S),$$

where Θ is a 2-theory fibered over T and Σ is a 2-theory fibered over S.

Now consider a set I and a map

$$X : I^k \to Sets.$$

To such data there is assigned a 2-theory $End(X)$ fibered over the theory $End(I)$: let

$$\Theta(w; w_1, \dots, w_n)$$

consist of the set of all possible simultaneous choices of maps

$$X(w_1(i_1, \dots, i_m)) \times \cdots \times X(w_n(i_1, \dots, i_m)) \to X(w(i_1, \dots, i_m)), \tag{9}$$

where i_j range over elements of I. A structure of an *algebra* over the 2-theory Θ fibered over T is given by a morphism

$$(\Theta, T) \to (End(X), End(I)).$$

To define a *lax algebra over* (Θ, T), let I be a groupoid, and let X be a *strict functor* from I to groupoids. We have already defined $End_{lax}(I)$. To define

$$End_{lax}(X)(w; w_1, \dots, w_n),$$

we take the set of simultaneous choices of *functors* (9) for each $(i_j)_j \in I^m$, which are *strictly natural transformations* (where $X(w_i(i_1, \dots, i_m))$ is a functor in I^m using the strict functoriality of X, and the usual functoriality of *Hom*'s).

Now a lax algebra over (Θ, T) consists of a lax algebra I over T (i.e. in particular a map $\phi : T \to End_{lax}(I)$), a strict functor X from I to groupoids, and a map

$$\Theta(w; w_1, \dots, w_n) \to End_{lax}(X)(\phi(w); \phi(w_1), \dots, \phi(w_n)) \tag{10}$$

together with a natural coherence isomorphism for each operation $1, \dots, 5$ of a 2-theory, and a commutative coherence diagram for each relation among the operations of a general 2-theory (see above for the list of such relations).

Note that, similarly as above, lax algebras over a 2-theory in this sense form a 2-category where every 2-morphism is an iso and lax limits exist. Therefore, we can talk about stacks of (Θ, T)-algebras.

Note that it is possible to talk about (Θ, T)-algebras in an even more lax sense, which, however, would lead us into the realm of 3-categories and 2-stacks. We shall not pursue this here, although another remark in this direction will be made later.

The example is related to the following 2-theory Θ fibered over T, which we will call the 2-theory of *commutative monoids with cancellation* (CMC): T is the theory of commutative monoids with an operation $+$. We set $k = 2$. The 2-theory Θ has three generating operations, addition (or disjoint union)

$$+ : X_{a,c} \times X_{b,d} \to X_{a+b,c+d},$$

unit

$$0 \in X_{0,0}$$

and cancellation (or gluing)

$$\check{?} : X_{a+c,b+c} \to X_{a,b}.$$

(In this notation, X is a general CMC, i.e. algebra over the 2-theory of CMCs.) The axioms are commutativity, associativity and unitality for $+$, 0, transitivity for $\check{?}$

$$\begin{array}{ccc} X_{a+c+d,b+c+d} & \xrightarrow{\check{?}} & X_{a+c,b+c} \\ & \searrow^{\check{?}} & \downarrow \check{?} \\ & & X_{a,b} \end{array}$$

(note that the cancellation operation of $c+d$ uses T-substitution) and distributivity of $\check{?}$ under $+$:

$$\begin{array}{ccc} X_{a+c,b+c} \times X_{e,f} & \xrightarrow{+} & X_{a+c+e,b+c+f} \\ \downarrow \check{?} \times Id & & \downarrow \check{?} \\ X_{a,b} \times X_{e,f} & \xrightarrow{+} & X_{a+e,b+f} \end{array}$$

(similarly, note that in this diagram, T-substitution is used).

Now we are not interested in any actual examples of CMCs, but we will be interested in *lax* CMCs (LCMCs). In our basic example, I is the category of finite sets and isomorphisms, where $+$ is disjoint union, and $X_{a,b}$ is the set of all rigged surfaces x together with bijections

$$\{\text{inbound boundary components of } x\} \cong a,$$

$$\{\text{outbound boundary components of } x\} \cong b.$$

As usual, morphisms of rigged surfaces are diffeomorphisms preserving complex structure and boundary parametrizations. The reader can check that with this definition, $X_{a,b}$ is a strict functor in a, b, and the other axioms of LCMC are also easily verified.

Even further, we are interested in the fact that $\mathscr{C}$ has, in fact, the structure of a *stack of LCMCs*. Here the Grothendieck topology is the category $\mathscr{B}$ of finite-dimensional complex manifolds, where coverings are coverings by open subsets. To describe a stack of LCMCs, note that we have to first describe the underlying stack of lax T-algebras, in our case lax commutative monoids. This is simply the stack of covering spaces: the set of sections over $B \in \mathscr{B}$ is the set of all covering spaces of B with locally finitely many sheets. Note that if we look at the map of

Grothendieck topologies

$$i : * \to \mathscr{B},$$

this stack of covering spaces can be, in the appropriate sense, described as a universal construction, which we may call

$$Q = lax\, i_{\#} S,$$

where S is the lax commutative monoid of finite sets (this notation has precise meaning as lax adjoints in 2-categories, see [10]).

Now the set of sections of the stack of LCMCs $\mathscr{C}$ over $B \in \mathscr{B}$ and over a pair s, t of covering spaces of B is the set of all holomorphic families x of rigged surfaces over B together with a choice of isomorphisms between the covering spaces of B consisting of inbound and outbound boundary components of x, and s, t, respectively. Again, it is easily checked that $\mathscr{C}$ with this structure is an LCMC, and this is the total structure on $\mathscr{C}$ we are interested in.

Example. To illustrate these notions, we give at least a couple of the coherence diagrams which the above formalism implies. Suppose, for example, we have three rigged surfaces x, y, z where x has one inbound and one outbound boundary component. Let $\check{x}$ be obtained by gluing those two boundary components. Analogously, we obtain rigged surfaces $(x \amalg y)^\vee$, $(x \amalg z)^\vee$, etc. by gluing the same two boundary components. (*Note*: we use $X^\vee$ in the same meaning as $\check{X}$; the former notation is preferable when X is a longer expression.) Then we have, for example, the following two commutative coherence diagrams (the isos are coherence isos supplied by the lax structure):

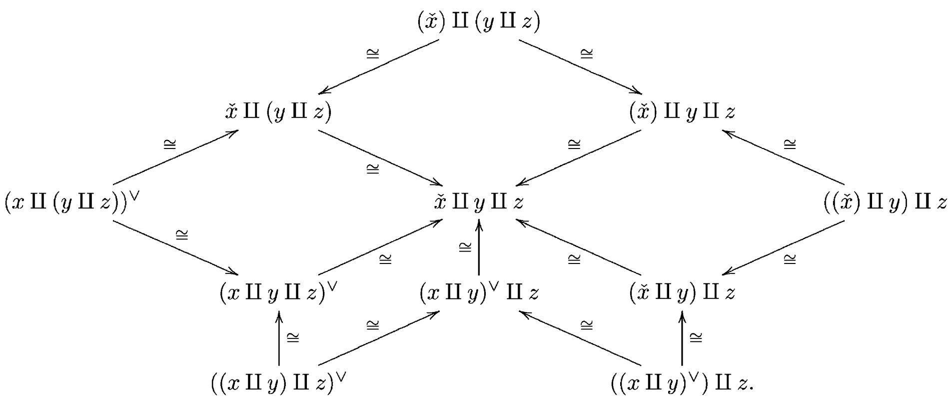
$$\begin{array}{ccccc} & & (\check{x}) \amalg (y \amalg z) & & \\ & \overset{\cong}{\swarrow} & & \overset{\cong}{\searrow} & \\ (x \amalg (y \amalg z))^\vee & & & & ((\check{x}) \amalg y) \amalg z \\ \downarrow \cong & & & & \downarrow \cong \\ ((x \amalg y) \amalg z)^\vee & & \xleftarrow{\quad\cong\quad} & & ((x \amalg y)^\vee) \amalg z), \end{array} \tag{11}$$

$$\begin{array}{ccccc} & & (\check{x}) \amalg y & & \\ & \overset{\cong}{\swarrow} & & \overset{\cong}{\searrow} & \\ y \amalg (\check{x}) & & & & (x \amalg y)^\vee \\ & \underset{\cong}{\searrow} & & \underset{\cong}{\swarrow} & \\ & & (y \amalg x)^\vee. & & \end{array} \tag{12}$$

If the reader wishes to consider stacks, we may replace x, y, z by holomorphic families of rigged surfaces.

While coherences (11) and (12) are certainly obvious geometrically, note that it is by no means obvious how one would write down *all* such coherences naively. It is also impossible to develop a model for $\mathscr{C}$ where all the arrows of diagrams such as (11), (12) would be identities: it is well known that even for the category of finite sets, there is no consistent set theory where the operation $\amalg$ would be *strictly* commutative associative unital. Thus, lax structures must be considered.

In the theory and 2-theory formalism, the trick of reducing all coherence diagrams such as (11) and (12) to a uniform shape is to consider *composite* operations in the theory or 2-theory. For example, in (11), we may consider composite operations $x\amalg y\amalg z$, $\check{x}\amalg y$, $\check{x}\amalg y\amalg z$. We omit parentheses to distinguish these from compositions of operations, as one must in a lax algebra. Thus, for example, the composite operation $\check{x}\amalg y\amalg z$ is to be distinguished from $(\check{x})\amalg y\amalg z$ which is the composition of gluing followed by $?\amalg ?\amalg ?$. Now from this point of view, diagram (11) is broken up into 2-theory coherence diagrams



(13)

All squares in (13) are coherence diagrams corresponding to the associativity of composition in 2-theories. Diagram (12) can be broken up analogously, but this time we would also need to use 2-theory coherence diagrams coming from the Σ_k-equivariance of 2-theories.

Finally, we would like to point out that while it is possible to follow these examples, it is also apparent that in diagram (13), our notation was already becoming awkward. This is due to the fact that $\check{?}$ was just an abbreviation for a whole system of operations, indexed by the incoming and outgoing boundary components to be glued. The notion of 2-theory is precisely designed to capture such indexing of operations. Therefore, these examples should help explain why what may have seemed as esoteric definitions above are in fact abstractions forced by the structure present: any rigorous axiomatization of the structure of $\mathscr{C}$ must include these, or equivalent, abstractions.

There is one generalization of $\mathscr{C}$ which will be also useful. Let K be some finite set. Then instead of the groupoid S of sets, we can consider the groupoid S_K of finite sets over K, i.e. whose objects are maps

$$a \to K$$

where a is a finite set, and maps are maps $a \to b$ which commute with the maps to K. We think of K as a set of *labels*. Then we can consider, in the same sense as above, the stack over $\mathscr{B}$

$$Q_K = lax\, i_{\#} S_K, \tag{14}$$

or, explicitly, the stack of covering spaces with locally finitely many sheets, labelled locally constantly by elements of K. Then we denote by $\mathscr{C}_K$ the fiber product of stacks

$$\mathscr{C} \times_{Q \times Q} (Q_K \times Q_K),$$

i.e. stack of LCMCs, consisting of (families of) rigged surfaces with boundary components labelled by elements of K.

It is tempting to go even further and consider the case when K would be a *groupoid*; such structures would be useful in encoding conformal field-like theories corresponding to vertex intertwining algebras [23]. However, note that then sets over K form a 2-category. The appropriate fully lax analogue of (14) gives, as sections over an object B of $\mathscr{B}$, the set of all *gerbes* on a covering space of B. Gerbes, however, do not form a stack, but rather a 2-*stack*, which is the right analogue of stack over a 3-category (whose objects are 2-categories). While this is an interesting direction, we shall not pursue it any further here.

We now proceed to use the notion of LCMC to define *conformal field theory*. Let $\mathscr{H}_1, \dots, \mathscr{H}_n$ be complex (separable) Hilbert spaces. Then on $\mathscr{H}_1 \otimes \cdots \otimes \mathscr{H}_n$, there is a natural inner product

$$\langle a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n \rangle = \langle a_1, b_1 \rangle \langle a_2, b_2 \rangle \cdots \langle a_n, b_n \rangle.$$

The Hilbert completion of this inner product space is called the *Hilbert tensor product*

$$\mathscr{H}_1 \hat{\otimes} \cdots \hat{\otimes} \mathscr{H}_n. \tag{15}$$

Now an element of (15) is called *trace class* if there exist unit vectors $e_{ij} \in \mathscr{H}$ where $j = 1, \dots, n$ and i runs through some countable indexing set I such that

$$x = \sum_{i \in I} \mu_i (e_{i1} \otimes \cdots \otimes e_{in})$$

and

$$\sum_{i\in I} |\mu_i| < \infty.$$

The vector subspace of (15) of vectors of trace class will be denoted by

$$\mathscr{H}_1 \boxtimes \cdots \boxtimes \mathscr{H}_n. \tag{16}$$

Note that (16) is not a Hilbert space. We have, however, canonical maps

$$\boxtimes : (\mathscr{H}_1 \boxtimes \cdots \boxtimes \mathscr{H}_n) \otimes (\mathscr{H}_{n+1} \boxtimes \cdots \boxtimes \mathscr{H}_{m+n}) \to \mathscr{H}_1 \boxtimes \cdots \boxtimes \mathscr{H}_{m+n}$$

and, if $\mathscr{H}^*$ denotes the dual Hilbert space to a complex Hilbert space $\mathscr{H}$,

$$tr : \mathscr{H} \boxtimes \mathscr{H}^* \boxtimes \mathscr{H}_1 \boxtimes \cdots \boxtimes \mathscr{H}_n \to \mathscr{H}_1 \boxtimes \cdots \boxtimes \mathscr{H}_n.$$

This allows us to define a particular example of stack of LCMCs based on $\mathscr{H}$, which we will call $\underline{\mathscr{H}}$. The underlying stack of lax commutative monoids (T-algebras) is Q. Now let $B \in \mathscr{B}$. Let s, t be sections of the stack Q over B, i.e. covering spaces of B with finitely many sheets. Then we have an infinite-dimensional holomorphic bundle over B

$$(\mathscr{H}^*)^{\boxtimes s} \boxtimes \mathscr{H}^{\boxtimes t}. \tag{17}$$

What we mean by that is that there is a well defined sheaf of holomorphic sections of (17) (note that it suffices to understand the case when s, t are constant covering spaces, which is obvious). Now a section of $\underline{\mathscr{H}}$ over a pair of sections s, t of Q is a *global* section of (17) over b; the only automorphisms of these sections covering $Id_s \times Id_t$ are identities. The operation $+$, $\check{?}$ are given by the operations $\boxtimes$, tr (see above).

We can also define a variation of this LCMC for the case of labels indexed over a finite set K. We need a collection of Hilbert spaces

$$\mathscr{H}_K = \{\mathscr{H}_k \mid k \in K\}.$$

Then we shall define a stack of LCMCs $\underline{\mathscr{H}_K}$. The underlying stack of T-algebras (commutative monoids) is Q_K. Let s, t be sections of Q_K over $B \in \mathscr{B}$. The place of (17) is taken by

$$(\mathscr{H}_K^*)^{\boxtimes s} \boxtimes \mathscr{H}_K^{\boxtimes t}. \tag{18}$$

By the sheaf of holomorphic section of (18) when B is a point we mean that $\boxtimes$-powers of $\mathscr{H}_k$ (or $\mathscr{H}_k^*$) for each label $k \in K$ are taken according to the number of points of $\Gamma(t)$ (resp. $\Gamma(s)$); when s and t are constant covering spaces B, the space of sections of (18) is simply the set of holomorphically varied elements of the spaces of sections over points of B (which are identified). This is generalized to the case of general s, t in the obvious way (using functoriality with respect to permutations of coordinates). As above, the only automorphisms of these sections covering $Id_s \times Id_t$ are identities.

Now by an *abstract chiral conformal field theory* (CFT) on a stack of LCMCs $\mathscr{D}$ with underlying stack of commutative monoids Q we mean a Hilbert space $\mathscr{H}$ together with a map of stacks of LCMCs

$$\phi : \mathscr{D} \to \underline{\mathscr{H}} \tag{19}$$

over the map of underlying stacks of commutative monoids

$$Id : Q \to Q. \tag{20}$$

More generally, by an *abstract CFT* with set of labels K we mean a stack of LCMCs with underlying stack of commutative monoids Q_K, a collection of Hilbert spaces $\mathscr{H}_K$ and a map of stacks of LCMCs

$$\phi : \mathscr{D} \to \underline{\mathscr{H}_K} \tag{21}$$

over the map of underlying stacks of commutative monoids

$$Id : Q_K \to Q_K. \tag{22}$$

Since we did not specify above explicitly what we mean by a morphism of stacks of LCMCs, we should say that here we are referring to *strict* morphisms. Note, however, that in the present cases, there is no ambiguity, since the target has only one morphism over the identity on each section of the underlying stack over an object $B \in \mathscr{B}$.

For our purposes, however, we should like to be much more specific about the stack of LCMCs $\mathscr{D}$ which is the source of maps (19) and (21). We shall start with the notion of $\mathbb{C}^\times$*-central extension* (or, equivalently, *one-dimensional modular functor*) on an LCMC $\mathscr{D}$. This is a strict morphism of stacks of LCMCs

$$\psi : \tilde{\mathscr{D}} \to \mathscr{D} \tag{23}$$

over (20) with the following additional structure: For each object B of $\mathscr{B}$, and each pair of sections s, t of Q over B, and each section α of $\mathscr{D}$ over $s, t, B, B' \to B$,

$$\psi^{-1}(\alpha|_{B'}) \tag{24}$$

with varying B' is the space of sections of a complex holomorphic line bundle over B. Furthermore, functoriality maps supplied by the structure of stack of LCMCs on $\tilde{D}$ are linear maps on these holomorphic line bundles. Regarding the operation $+$, we require that the map induced by $+$

$$\psi^{-1}(\alpha|_{B'}) \times \psi^{-1}(\beta|_{B'}) \to \psi^{-1}((\alpha + \beta)|_{B'}) \tag{25}$$

be a bilinear map, which induces an isomorphism of holomorphic line bundles

$$\psi^{-1}(\alpha|_{B'})\otimes_{\mathcal{O}_{B'}}\psi^{-1}(\beta|_{B'})\to\psi^{-1}((\alpha+\beta)|_{B'}) \tag{26}$$

($\mathcal{O}_B$ is the holomorphic structure sheaf on B).

Regarding the operation $\check{?}$, we simply require that if α is a section of $\mathcal{D}$ over $s+u$, $t+u$, B where u is another section of Q over B, and $\check{\alpha}$ is the section over s, t, B which is obtained by applying the operation $\check{?}$ to α, then the map of holomorphic line bundles coming from LCMC structure

$$\psi^{-1}(\alpha|_{B'})\to\psi^{-1}(\check{\alpha}|_{B'}) \tag{27}$$

($B'\to B$) be an isomorphism of holomorphic line bundles.

By a *chiral CFT with one-dimensional modular functor over* $\mathcal{D}$ we shall mean a CFT

$$\phi:\tilde{D}\to\underline{\mathscr{H}} \tag{28}$$

where $\tilde{\mathcal{D}}$ is a $\mathbb{C}^\times$-central extension of $\mathcal{D}$ which has the property that ϕ is a linear map on the spaces of sections (24).

This concept is easily generalized to CFTs with general modular functors: For a finite set of labels K, and a stack of LCMCs $\mathcal{D}$ with underlying stack of commutative monoids Q, let

$$\mathcal{D}_K=\mathcal{D}\times_{Q\times Q}(Q_K\times Q_K)$$

be the corresponding stack of LCMCs with underlying stack of commutative monoids Q_K. Then a *modular functor* $\tilde{\mathcal{D}}_K$ over $\mathcal{D}_K$ is a strict morphisms of LCMCs

$$\tilde{\mathcal{D}}_K\to\mathcal{D}_K$$

over (22) with the following additional structure: For each object B of $\mathcal{B}$, and each pair of sections s, t of Q_K over B, and each section α of $\mathcal{D}_K$ over $s, t, B, B'\to B$, (24) with varying B' is the space of sections of a complex holomorphic (finite-dimensional) bundle over B. Furthermore, functoriality maps supplied by the structure of stack of LCMCs on $\tilde{D}_K$ are linear maps on these holomorphic bundles. Regarding the operation $+$, we require that map (25) induced by $+$ be a bilinear map, which induces an isomorphism of holomorphic bundles (26), as before.

The operation $\check{?}$ is slightly more complicated in the present general modular functor case: suppose u is a section of Q which is a constant covering space, β is a section over s, t, B, and the image of β, s, t in $\mathcal{D}$ (resp. Q) is γ, resp. p, r. Now assume that α is a section of $\mathcal{D}$ over $p+u$, $r+u$, B such that γ is obtained from α by the gluing operation $\check{?}$. Assume further that for each lift v of u to Q_K, $\alpha(v)$ is the section of $\mathcal{D}_K$ over $s+v$, $t+v$, B such that β is obtained from $\alpha(v)$ by the gluing operation $\check{?}$ (note that because of our definition of $\mathcal{D}_K$, $\alpha(v)$ is necessarily uniquely determined). Then we require that the map of holomorphic line bundles coming from

LCMC structure

$$\bigoplus_{v} \psi^{-1}(\alpha(v)|_{B'}) \to \psi^{-1}(\beta|_{B'}) \tag{29}$$

$(B' \to B)$ is an isomorphism of holomorphic bundles, where the sum runs over all lifts v of u to Q_K.

By a *chiral CFT with modular functor over* $\mathscr{D}_K$ we mean a CFT

$$\phi : \tilde{D}_K \to \underline{\mathscr{H}}_K \tag{30}$$

which has the property that ϕ is a linear map on the spaces of sections (24). If we set

$$\mathscr{D} = \mathscr{C},$$

this is the rigorous version of CFT following the outline in [36]. With the caveat that this notion does not capture super-CFTs or twisted CFTs (field-theoretic notions corresponding to intertwining vertex algebras), this definition is from many points of view the correct one.

The authors have been asked what is the improvement of the present definition over the definition of Segal [36]. The answer is that this is the wrong question: the concepts we describe are (up to some possible variations some of which will be discussed below) precisely what one gets when including all the desired features outlined in [36] and all the details not given in [36]. It seems meaningless to consider notions where some of these axioms would be omitted (for example, a notion of modular functor M where a coherence isomorphism $M_{x \amalg y} \cong M_x \otimes M_y$ would be required, but the corresponding coherence diagrams would not). Therefore, rather than improving on the definition [36], what we claim to have done is just writing the whole definition down in detail.

We remark that while it is important for foundational reasons to have the full force of maps (28) and (30), the stack notation is awkward, and it is usually sufficient to refer to fibers over a point. Thus, e.g. in (28), for a rigged surface X, we usually speak simply of the element of

$$\hat{\bigotimes}_{in} \mathscr{H}^* \hat{\otimes} \hat{\bigotimes}_{out} \mathscr{H}$$

given by (28) (the products are over inbound resp. outbound boundary components of X). In this notation, the element is usually denoted by U_X and call the *vacuum vector* or *field operator*, depending on context.

There are several reasons why we do not precisely follow this definition here, and rather introduce several modifications. The main reason is that there are at the present time still very few examples of CFTs in which the full structure of CFT with modular functor, as defined above, can be proven rigorously. Still unknown cases include for example the completed Moonshine module CFT and the higher level chiral WZW models (not discussed here). A CFT associated with an even lattice L is now known. However, the general construction of lattice CFTs requires a detailed discussion of the labels which is beyond the scope of the present paper and will be described elsewhere. In this paper, we will only describe the lattice CFT in the case of even unimodular lattice L (see the next section) and use a weaker notion for a general lattice. A similar weaker structure for the completed Moonshine module will be discussed in Section 4.

However, there is another, less pragmatic reason why we do not consider the full CFT structures with modular functor: in our construction of elliptic cohomology theory (see Section 5 below), it is apparent that only modularity with respect to the genus 1 mapping class group (= modular group) is relevant to the modular invariance of our theory. Thus, it makes perfect sense to restrict our consideration to rigged surfaces of genus $\leqslant 1$. This gives field-theoretic concepts corresponding to *rational vertex operator algebras* (see next section for more details). Also, although in this paper we focused, following [36], on a "stringy" approach to CFT, in physics however there is an alternate quantum field theory approach using Schwinger functions, which can be considered entirely one worldsheet at a time.

Even at genus $\leqslant 1$, however, modular functor considerations involve labels, which present an additional complication; since the main purpose of this paper is to present *some examples* of rigorously defined elliptic cohomology theories based on CFT, we seek to define alternative concepts of CFT on rigged surfaces of genus $\leqslant 1$, which do not require labels. These considerations lead to the following definitions:

Let $n \in \mathbb{N}$. Then by $\mathscr{C}_n$ we mean the substack of $\mathscr{C}$ whose section over $B \in \mathscr{B}$, s, t sections of Q over B, consist of all those families of rigged surfaces over B whose fiber over each point has connected components of genus $\leqslant n$ (the genus of a rigged surface x with boundary is the genus of the closed surface obtained by gluing disks to the boundary components of x). Let, also, $\mathscr{C}_0^+$ denote the substack of $\mathscr{C}_0$ consisting of families of rigged surfaces whose each connected component has exactly one outbound boundary component, and let $\mathscr{C}_1^+$ denote the substack of $\mathscr{C}_1$ consisting of families of rigged surfaces whose each connected component of genus i has exactly $1-i$ outbound boundary components, $i = 0, 1$. Note that, curiously, $\mathscr{C}_1^+$ is actually an LCMC. On the other hand,

$$\mathscr{C}_n, \mathscr{C}_0^+ \tag{31}$$

are not LCMCs, but the operation $+$ (disjoint union) is still well defined on (31), and the operation $\check{?}$ is partially defined by the operation $\mathscr{C}$, defined precisely when it produces a section of the respective stack (31).

By a *stack of partial LCMCs* over (31) we shall mean a map ϕ of stacks over Q from a $\mathscr{D}$ to (31) where $\mathscr{D}$ has an operation $+$ and a partial operation $\check{?}$ defined if and only it is defined in the target of ϕ, and satisfies all the axioms of LCMC we defined. Now a *partial CFT* over a stack of partial LCMCs $\mathscr{D}$ over (31) is a map $\mathscr{D} \to \underline{\mathscr{H}}$ which satisfies the axioms of map of LCMCs, whenever operations in the source are defined. Similarly, one defines the notions of *one-dimensional modular functor* on a stack of partial LCMCs over (31) and *partial CFT with one-dimensional modular functor* on a stack of partial LCMCs over (31).

We shall be only concerned with $n = 0,1$ here. By a 0-*CFT* (resp. *directed* 0-*CFT*) we shall mean a partial CFT with one-dimensional modular functor over $\mathscr{C}_0$ (resp. $\mathscr{C}_0^+$). It may seem natural to define a 1-CFT as a partial CFT with one-dimensional modular functor over $\mathscr{C}_1$, but we want to be more general than that, to allow for modular groups Γ which are subgroups of $PSL_2(\mathbb{Z})$ of finite index.

To that end, consider the 2-categories

$$C_1, C_0, C_1^+, C_0^+ \tag{32}$$

whose objects are stacks of partial LCMCs over

$$\mathscr{C}_1, \mathscr{C}_0, \mathscr{C}_1^+, \mathscr{C}_0^+ \tag{33}$$

(the word 'partial' does not apply in the penultimate case). The morphisms of (32) are lax morphisms of stacks of partial LCMCs. Now consider the forgetful functors

$$U: C_1 \to C_0, \quad U^+: C_1^+ \to C_0^+. \tag{34}$$

These functors have lax left adjoints

$$L: C_0 \to C_1, \quad L^+: C_0^+ \to C_1^+, \tag{35}$$

and we are going to be interested in the stack of partial LCMCs over $\mathscr{C}_1$

$$\mathscr{D}_\infty = L(\mathscr{C}_0), \tag{36}$$

and the stack of LCMCs over $\mathscr{C}_1^+$

$$\mathscr{D}_\infty^+ = L^+(\mathscr{C}_0^+). \tag{37}$$

Stacks (36) and (37) are not difficult to describe. Note that it suffices to describe sections over s, t, B where B is an object of $\mathscr{B}$ and s, t are constant sections of Q over B. Now let x be a section of $\mathscr{C}_1$ over s, t, B. Clearly, it suffices to consider the case when the fibers of x over each point of B is connected and of genus 1. Then the sections of $\mathscr{D}_\infty$ over x are equivalence classes of sections y of $\mathscr{C}_0$ over $s+e$, $t+e$ where e is a 1-element set, and $\check{y} = x$; two such choices y, y' are considered equivalent if over each point c of B, the images of the glued boundary component from y, y' in the elliptic curve E obtained from gluing disks to the boundary components of the restriction of x to c are homotopic when parametrized suitably. Eq. (37) can be constructed similarly. We see therefore that the fibers of the forgetful maps

$$\mathscr{D}_\infty \to \mathscr{C}_1, \tag{38}$$

$$\mathscr{D}_\infty^+ \to \mathscr{C}_1^+ \tag{39}$$

over sections x of the target over constant sections s, t of Q over B are in canonical bijective correspondence to the set of cosets

$$PSL_2(\mathbb{Z})/\mathbb{Z}. \tag{40}$$

Here the subgroup $\mathbb{Z} \subset PSL_2(\mathbb{Z})$ consists of modular transformations

$$\tau \mapsto \tau + n, \quad n \in \mathbb{Z}.$$

Now let Γ be a subgroup of $PSL_2(\mathbb{Z})$ of finite index, containing this $\mathbb{Z}$. Then there is a well defined stack of partial LCMCs over $\mathscr{C}_1$ (resp. stack of LCMCs over $\mathscr{C}_1^+$) which we could denote as

$$\mathscr{D}_\Gamma = \mathscr{D}_\infty/\Gamma,$$

$$\mathscr{D}_\Gamma^+ = \mathscr{D}_\infty^+/\Gamma.$$

In the above situation, i.e. over a section x of $\mathscr{C}_1$ (resp. $\mathscr{C}_1^+$) over a pair of constant sections s, t of Q over B, the sections are the set of orbits of (40) by the action of $\Gamma \subset \mathbb{Z}$ (i.e. these are single, not double cosets).

Now we are ready for our main definition: By a 1-*CFT with modularity group* Γ (resp. a *directed* 1-*CFT with modularity group* Γ) we shall mean a partial CFT with one-dimensional modular functor on $\mathscr{D}_\Gamma$ (resp. CFT with one-dimensional modular functor on $\mathscr{D}_\Gamma^+$).

Remark 1. From a physical point of view, Moore and Seiberg [29] proved a result which, in the present language, says roughly that every 1-CFT gives rise to a CFT. However, this result has not yet been checked in the rigorous mathematical framework, and so, at the moment, from a mathematical point of view is still conjectural.

Remark 2. We have so far mentioned variants which are weaker than the original full concept of CFT defined above. However, one can also impose additional conditions on a CFT. Notably, these include reality conditions (i.e. requiring that the Hilbert space $\mathscr{H}$, or the system $\mathscr{H}_K$, have a suitable real form) and boundary convergence conditions for limit worldsheets. In this paper, we shall find one such (very mild) condition useful: Considering the standard annuli $A_q = \{z \in \mathbb{C} \mid ||q|| \leqslant ||z|| \leqslant 1\}$ with the standard boundary parametrizations z and qz, these annuli specify an action of the semigroup $\mathbb{C}^*_{<1} = \{z \in \mathbb{C} \mid 0 < ||z|| < 1\}$ (under multiplication). One can assume that this extends to a continuous action of $\mathbb{C}^*_{\leqslant 1}$ by bounded operators.

This technical condition clearly can be formulated for a 0-CFT (hence also a 1- CFT), and we shall assume it for the rest of this paper.

At least for motivational reasons, it is beneficial to clarify the relationship between the concepts we introduced and vertex operator algebras (VOAs), which is the same as conformal vertex algebras, see [11,12]. First of all, the nature of the relationship is that both concepts are different mathematical models of related physical structures; neither structure is in general known to imply presence of the other. Thus, it is appropriate to speak in terms of analogy, although sometimes, a more direct connection exists. A *vertex algebra* is a non-negatively graded complex vector space

$$V = \bigoplus_{n \in \mathbb{N}} V_n, \tag{41}$$

where the degree is called *weight*, together with series called *vertex operators*

$$Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \in Hom(V, V)[[z, z^{-1}]]. \tag{42}$$

The operator $v_n : V \to V$ is of weight $m - n - 1$ if $v \in V_m$. This conforms with the older (but still used) indexing [11], the newer indexing [12] is shifted. Operators (42) are, of course, required to obey certain axioms (see [11,12]). This concept is analogous to our concept of directed 0-CFT. More concretely, let $\lambda_* : V \to V$ be the linear operator given by multiplication by λ^n on V_n. Then given a $z \in \mathbb{C}$ with

$$0 < ||z|| < 1,$$

we can choose $\lambda \in \mathbb{C}^\times$, $\mu \in \mathbb{C}^\times$ such that the disks λD, $\mu D + z$ are disjoint and contained in the interior of D. Then let $A_{\lambda,\mu,z}$ be the rigged surface in $\mathbb{C}$ whose parametrized boundary components are S^1, λS^1, $\mu S^1 + z$. Then we may hope that there exists a directed 0-CFT

$$\mathscr{H} \subset \hat{V} = \prod_n V_n$$

such that, if we rewrite (42), for a given z, (assuming the series converges), as a map

$$Y : V \otimes V \to \hat{V}, \tag{43}$$

then

$$Y(\lambda_* u, \mu_* v) = U_{A_{\lambda,\mu,z}}(u \otimes v). \tag{44}$$

In that case, the term vertex *operator* algebra refers to the presence of a certain element $\omega \in V_2$ such that the vertex operator associated with ω encodes (infinitesimal) boundary reparametrizations. We refer the reader to [2,11,22] for details. Anyway, in this case, we may refer to the directed 0-CFT $\mathscr{H}$ as a *Hilbert completion* of the V(O)A V; however, note that the Hilbert structure on $\mathscr{H}$ plays a marginal role, and, in fact, can often be varied (for example in the case of *bc*-systems Segal [26,36]). Analogously, the axioms of V(O)A call for no inner product on V. Thus, such "Hilbert completion" is not a canonical operation, since it is not always defined, and if it is defined, is not unique. On the other hand, however, note that if we have an directed 0-CFT, assuming that for standard annuli A_q (i.e. submanifolds of $\mathbb{C}$ where the parametrized boundary components are S^1 and qS^1) we have

$$\lim_{q\to 1} U_{A_q} = Id_{\mathscr{H}}, \tag{45}$$

we can always set V_n to be the subspace of $\mathscr{H}$ on which U_{A_q} acts by q^n. We may then use (41) to define V. Assume each V_n is finite dimensional. We have $\mathscr{H} \in \hat{V}$, so we may use (44) to define (43). Further, the assumption we made about holomorphy shows that (43) can be expanded to the form (42). Thus, a directed 0-CFT with some mild assumptions does always give rise to a VOA. Under such assumptions, similarly a directed 1-CFT gives rise to a rational VOA in the sense of [9], [47].

One important point in this discussion is the 'central charge'. A VOA is, in particular, a representation of the universal central extension of the Lie algebra $Vect(S^1)_{\mathbb{C}}$ of polynomial complex vector fields on S^1: such representations have an important invariant, called *central charge*, and denoted by c [10]. Accordingly, assuming (45), a directed 0-CFT is also, in particular, a representation of the group $Diff^+(S^1)$ of analytic orientation-preserving diffeomorphisms of S^1, which also has an invariant called central charge. We shall use this invariant below. See [26,33,36], for more details and further relevant discussion.

We may ask what property of vertex algebra corresponds to a 0-CFT (not directed). Note that in such notion, we must have operators corresponding to reversing the orientations of boundary components of $A_{\lambda,\mu,z}$. We shall consider the "infinitesimally thick" annulus A_- whose both boundary components are S^1, the "outside one" parametrized by z, while the "inside one" parametrized by $1/z$ (It is appropriate to think of A_- as the limit at $q\to 1$ of the annuli $A_{q,-}$ with boundary components parametrized by z and q/z, $z\in S^1$.). Then we assume there is an operator in $\mathscr{H}\to\mathscr{H}^*$ associated to A_-. This corresponds to a *bilinear form B* on the complex vector space V. The annulus $A_{q,-}$ has an involution automorphism reversing its boundary components. Taking the limit $q\to 1$ leads to the assumption that

$$B \text{ is a non-degenerate symmetric bilinear form.} \tag{46}$$

To find the appropriate condition on the symmetric bilinear form V which correspond to a directed 0-CFT extending to a 0-CPT (not directed), we consider the fact that if, in a genus 0 connected rigged surface X with 2 inbound and 1 outbound boundary component we reverse the parametrization of one inbound and one outbound boundary component (by composing with $1/z : S^1 \to S^1$), we obtain a rigged surface of the same kind (i.e. an element of the same connected component of the moduli space).

The appropriate model case of X to consider in the case of VOAs is $A_{\lambda,\mu,z}$. However, we imagine that $\lambda,\mu\to 1$ (so we, of course, no longer have an actual surface, although we could consider this as a surface with tubes in the sense of [22]). The two "boundary components" with identical images are parametrized by Id_{S^1}, the remaining boundary component is parametrized by

$$t \mapsto z + t. \tag{47}$$

If we apply the transformation

$$z \mapsto 1/z,$$

(47) becomes

$$t \mapsto \frac{1}{z+t}. \tag{48}$$

Our condition is that the operator

$$Y'(v, z) \tag{49}$$

obtained by composing with $v \in V_n$ via the "surface" with two boundary components Id_{S^1} and a third boundary component (48) (where we insert v) be B-adjoint to

$$Y(z, v). \tag{50}$$

To calculate operator (49) in vertex operator algebra terms, we note that

$$\frac{1}{z+t} = \frac{1}{z} + \frac{(-t/z^2)}{1 + (-t/z^2) \cdot (-z)}. \tag{51}$$

This gives

$$Y'(v, z) = (-z^2)^n Y\left(e^{zL_1} v, \frac{1}{z}\right). \tag{52}$$

Thus, our condition requires that (52) be B-adjoint to (50). Vertex operator algebras satisfying this condition will be called *reflexive*. Note that in (52), the right-hand side converges (is a sum of finitely many factors, since L_1 has weight -1). Note that this condition cannot quite be phrased in the language of VAs, since, at least, one need an L_1 which satisfies the usual Lie relations (of $sl_2\mathbb{C}$) with L_0, L_{-1}. (Note that this makes every VOA, or even more generally any Virasoro algebra representation, an $sl_2\mathbb{C}$-representation, but that this representation on a VOA can *never* be exponentiated to a Lie algebra representation, since if it did, multiplication by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

would reverse weights.)

Note also that if

$$v = \omega = L_{-2}(1)$$

is the conformal vector, then

$$L_1\omega = 0,$$

so reflexivity implies that L_n is B-adjoint to L_{-n}.

Note, finally, that the notion of reflexivity is implicitly introduced in [2, p. 417], where it is also remarked that VOAs associated with lattices and the Moonshine module are reflexive. In fact, the condition discussed in [2] is somewhat stronger: suppose in addition to reflexivity that V, ω, B, $Y(v,z)$ are all defined over $\mathbb{R}$. Assuming V has this property, and, in addition, B is positive-definite, we call the VOA V *reflection-positive*.

The corresponding condition on 0-CFT is also called *reflection-positivity*. It can be phrased on sections over a point, so the stack language is not needed. The condition simply says that if $\bar{X}$ is the opposite rigged surface to a rigged surface X (i.e. has opposite complex structure and the same boundary parametrizations), then

$$U_{\bar{X}}, \quad U_X$$

be adjoint. This means that

$$\phi(U_{\bar{X}}) = U_X$$

where, for a Hilbert space K (in our case a Hilbert tensor product of copies of H and H^*),

$$\phi : K \to K^*$$

is the antiisomorphism given by the inner product.

Note that in the case of a reflection-positive VOA V, V has a positive-definite inner product

$$\langle u, v \rangle = B(u, \bar{v}),$$

so we may discuss the *canonical* Hilbert completion $\mathscr{H}$ of V. Further, there is a canonical candidate for field operators U_X on $\mathscr{H}$ coming from the VOA structure, and the question of whether $\mathscr{H}$ is a 0-CFT is thus a question of convergence.

A subtle point [2, p. 417] is that VOAs associated with even lattices are not reflection-positive, because the form B with respect to which they are reflexive is not positive-definite; it is, in particular, not the "obvious" form. See next section for a more detailed discussion. On the other hand, the Moonshine module is reflection-positive.

3. Examples of 1-CFTs: lattice theories

Let L be an even lattice, i.e. a free abelian group with a $\mathbb{Z}$-valued quadratic form such that $\langle x, x \rangle$ is even for every $x \in L$. We shall denote

$$T = L_{\mathbb{C}}/L.$$

By T_{S^1}, we denote the group of analytic maps $S^1 \to T$ with the topology of uniform convergence of all derivatives. For a rigged surface X, we set

$$T_{\partial X} = \prod_{\partial X} T_{S^1}$$

(the product is over all boundary components of X). We also denote by

$$T_X$$

the group of holomorphic maps $X \to T$. We have an map

$$T_X \to T_{\partial X}$$

(by restriction), which is an embedding if X has no closed connected components. We consider the topology on T_X given by restriction.

We will start by constructing the following data:

1. A central extension $\tilde{T}_{S^1}$ of T_{S^1} by $\mathbb{C}^\times$ together with a specific lift of the canonical action of the group $Diff^+ S^1$ of analytic oriented diffeomorphisms of S^1 on T_{S^1} to an action on $\tilde{T}_{S^1}$ such that, moreover, the central extension induced by an orientation-reversing diffeomorphism of S^1 is opposite.
2. If we denote by $\tilde{T}_{\partial X}$ the induced (product) $\mathbb{C}^\times$-central extension of $T_{\partial X}$, a canonical splitting s_X of the induced $\mathbb{C}^\times$-central extension $\tilde{T}_X$ of T_X compatible with gluing in the sense that when $\check{X}$ is obtained from X by gluing and $\check{f}: \check{X} \to T$ is a holomorphic function which pulls back to a holomorphic function $f: X \to T$, then $s_{\check{X}}(\check{f})$ is a restriction of $s_X(f)$.

Extracting data (1) and (2) from an even lattice L involves some subtle points, and is not completely canonical. We follow [36], although the truth is that the authors of the present paper could not locate a version of [36] complete enough to treat all the details. Nevertheless, one must assume that all of the information presented in this section is known to the author of [36].

One begins by choosing a bilinear form

$$b: L \times L \to \mathbb{Z}/2$$

which satisfies

$$b(x,x) \equiv \tfrac{1}{2}\langle x,x\rangle \bmod 2.$$

Note that this implied (by considering $b(x+y)$)

$$b(x,y) + b(y,x) \equiv \langle x,y\rangle \bmod 2. \tag{53}$$

In the sequel, let

$$exp(z) = e^{2\pi iz}.$$

Let, for a function $f: S^1 \to T$, $\tilde{f}$ be a lift of f to $L_{\mathbb{C}}$, i.e. a function

$$\tilde{f}: [0,1] \to L_{\mathbb{C}}$$

such that the following diagram commutes:

$$\begin{array}{ccc} [0,1] & \xrightarrow{\tilde{f}} & L_{\mathbb{C}} \\ {\scriptstyle exp}\downarrow & & \downarrow{\scriptstyle proj.} \\ S^1 & \xrightarrow[f]{} & T. \end{array}$$

We set $\Delta_{\tilde{f}} = \tilde{f}(1) - \tilde{f}(0)$. Then the group T'_{S^1} of all such pairs $(\tilde{f}, f)$ is a universal covering of T_{S^1}. Then a $\mathbb{C}^\times$-valued 2-cocycle on T'_{S^1} is given by

$$c(\tilde{f}, \tilde{g}) = exp\frac{1}{2}\left(\oint_{S^1} \tilde{f}\, d\tilde{g} - \Delta_{\tilde{f}}\tilde{g}(0) + b(\Delta_{\tilde{f}}, \Delta_{\tilde{g}})\right). \tag{54}$$

To prove (1), we note that the restriction of c to the subgroup $L \subset T'_{S^1}$ of constant functions with values in L is 0, so the corresponding $\mathbb{C}^\times$-central extension $\tilde{T}'_{S^1}$ of T'_{S^1} specified by (54) splits canonically when restricted to L. Thus, we get a canonical homomorphism of groups

$$L \subset \tilde{T}'_{S^1}. \tag{55}$$

Note carefully that we have

$$c(\tilde{f}, k) = c(k, \tilde{f}) \quad \text{for } k \in L,$$

so (55) is a normal subgroup. We then set

$$\tilde{T}_{S^1} := \tilde{T}'_{S^1}/L.$$

The $Diff(S^1)$-action stated in (1) is then induced by using the obvious invariance of the cocycle c under the universal cover of $Diff^+(S^1)$, and projecting the resulting action down to $\tilde{T}_{S^1}$ (using the fact that L is an even lattice).

So far, the summand $\frac{1}{2}b(\Delta_{\tilde{f}}, \Delta_{\tilde{g}})$ in (54) played no role: the discussion of (1) would be equally valid without it. The situation is, however, different in the discussion of (2).

To begin this discussion, select a universal cover

$$\pi: \tilde{X} \to X$$

with a fundamental domain X'. X' can be chosen by selecting a simple analytic curve: $[0,1] \to X$ such that $Im(c) \cap \partial X$ consists of a collection of points which are images of 1 under the boundary component parametrizations. We can then demand

$$\pi : interior(X') \stackrel{\cong}{\to} X - (\partial X \cup Im(c)).$$

Note carefully two details: First, orientation of $\partial X'$ determines orientation of the parametrizations of the connected components of ∂X. Reversal of orientation of $\partial X'$ will reverse the orientations of all the parametrizations of connected components of ∂X, but any subset of these orientations may be separately reversed by making a different selection of c.

The other point is that a selection of c determines an *order* of the connected components of ∂X. Call these boundary components, in this order,

$$c_1, \ldots, c_n.$$

Now let $f \in T_X$, let $\tilde{f}$ be a lift to the group $T_{X'}$ of holomorphic maps $X' \to \mathbb{C}$. We would like to compare

$$c(\tilde{f}, \tilde{g})$$

to

$$exp\frac{1}{2}\left(\oint_{\partial X'} \tilde{f} d\tilde{g}\right) \tag{56}$$

which vanishes by Stokes' theorem (as $d(\tilde{f} d\tilde{g}) = 0$). Let $\tilde{f}_i = \tilde{f}_{c_i}$. One then has

$$exp\frac{1}{2}\left(\oint_{\partial X'} \tilde{f} d\tilde{g}\right) = exp\frac{1}{2}\left(\sum_{i=1}^{n}\left(\oint_{c_i} \tilde{f}_i\, d\tilde{g}_i - \Delta_{\tilde{f}_i}\tilde{g}_i(0)\right) + \sum_{i<j} \Delta_{\tilde{f}_i}\Delta_{\tilde{g}_i}\right). \tag{57}$$

Since the left-hand side vanishes, comparing (57) and (54) gives that

$$c(\tilde{f}, \tilde{g}) = exp\frac{1}{2}\left(\sum_{i<j} \Delta_{\tilde{f}_i}\Delta_{g_j} + \sum_{i=1}^{n} b(\Delta_{\tilde{f}_i}, \Delta_{\tilde{g}_i})\right). \tag{58}$$

Notice two points. First,

$$c(\tilde{f}, \tilde{g}) \in \{-1, 1\}.$$

Second,

$$c(\tilde{f}, \tilde{g}) = c(\tilde{g}, \tilde{f}), \tag{59}$$

as, by (53),

$$\frac{c(\tilde{f},\tilde{g})}{c(\tilde{g},\tilde{f})} = exp\frac{1}{2}\left(\sum_{i=1}^{n} \Delta_{\tilde{f}_i}\right)^2 = 1.$$

We know that a symmetric 2-cocycle with 2-divisible kernel is a coboundary (of $a = \frac{1}{2}c(x,x)$), however, for our purposes, we need a *canonical* choice of a (subject to the choices we made thus far).

To this end, choose a lift

$$\tilde{b}: L \times L \to \mathbb{Z}$$

of b. We see that

$$a_{\tilde{f}} = exp\left(\sum_{i=1}^{n} \frac{1}{4}\tilde{b}(\Delta_{\tilde{f}_i}, \Delta_{\tilde{f}_i}) - \frac{1}{8}\langle \Delta_{\tilde{f}_i}, \Delta_{\tilde{f}_i}\rangle\right), \tag{60}$$

which makes the correct choice of splitting

$$T_X \to \tilde{T}_{\partial X}$$

$$f \mapsto (f, a_{\tilde{f}}). \tag{61}$$

Obviously, this is independent of the choice of $\tilde{f}$. Note, however, that we can choose $\tilde{b}$ so that $a = 0$: For any choice of $\tilde{b}$,

$$q(x,y) := \tilde{b}(x,y) + \tilde{b}(y,x) - \langle x,y\rangle \in 2\mathbb{Z},$$

so we may replace $\tilde{b}$ by

$$\tilde{b}(x,y) - \tfrac{1}{2}q(x,y)$$

to make the difference 0. The choice of the lift $\tilde{b}$ of b is still not canonical (it can be altered by adding any antisymmetric form $S(x,y)$ on $\mathbb{Z} \times \mathbb{Z}$), but at this point, no data of our theory depend on it.

Now the recipe for constructing a 0-CFT from data (1), (2) is essentially formal (as pointed out in [36]). However, certain details must still be handled with care. First, the Hilbert space $\mathscr{H}$ is the basic representation of $\tilde{T}_{S^1}$ which is reflection positive, or, in other words, is a complexification of the basic unitary representation of $\widetilde{Map}(S^1, L_{\mathbb{R}}/L)$. (We speak here of reflection-positivity of the action of $\tilde{T}_{S^1}$; the reader must distinguish it carefully from any reflection-positivity of CFT structure.) Note also the usual caveat that only the *real* subgroup of $\tilde{T}_{S^1}$ acts by unitary (hence bounded) operators on $\mathscr{H}$. General elements of $\tilde{T}_{S^1}$ send only a dense subspace of $\mathscr{H}$ to $\mathscr{H}$. The whole story is told in substantial detail in [33], but we can summarize it as follows: first, the central extension $\tilde{T}_{S^1}$ splits canonically on T_D (D is the unit disk), since that is a special case of (2). Thus, we obtain a canonical representation

$$\tilde{T}_D = \mathbb{C}^\times \times T_D \to \mathbb{C}^\times \tag{62}$$

which is Id on the first factor and trivial on the second. Roughly speaking, one can think of $\mathscr{H}$ as the induced representation from (62) via the embedding $\tilde{T}_D \subset \tilde{T}_{S^1}$. However, one must be more precise about the Hilbert space structure.

This is best done as follows: Consider the subgroup $T_{S^1,0}$ of functions of degree 0, and the restricted central extension $\tilde{T}_{S^1,0}$. Note that there is a canonical short exact sequence

$$1 \to \mathbb{C}^\times \to T_{S^1,0} \to V \to 1,$$

where the kernel consists of constant functions, and V is a vector space. Further, on V, the cocycle takes the simple and completely canonical form

$$c(f,g) = \oint \tilde{f}\, d\tilde{g},$$

thus giving the pulled back central extension $\tilde{V}$, which is a Heisenberg group.

Now there is a completely rigorous and developed theory of Heisenberg representations ([33, 9.5], which is an infinite-dimensional analogue of [31]). The essential point is that V is identified with the space of analytic functions

$$f : S^1 \to L_{\mathbb{C}} \tag{63}$$

with

$$\oint_{S^1} f = 0. \tag{64}$$

We may therefore consider the real subspace $V_{\mathbb{R}}$ of analytic functions

$$f : S^1 \to L_{\mathbb{R}}$$

with (64). This real structure gives a complex conjugation, and the choice of isotropic subspace $A \subset V$ consisting of all functions holomorphically extending to D uniquely determines a Heisenberg representation $\mathscr{H}_0$ which is a Hilbert space (see [33, 9.5]).

The Hilbert space $\mathscr{H}$ then can be defined as

$$\mathscr{H}_0 \hat{\otimes} \ell^2(L),$$

and as a representation is

$$ind_{\tilde{T}_{S^1,0}}^{\tilde{T}_{S^1}}(\mathscr{H}_0).$$

(This has a clear meaning, since the kernel of the extension over which we are inducing is the discrete group L.)

To define the 0-CFT structure, consider a rigged surface X with no closed connected components. Assume first that all n boundary components are oriented outbound. (We shall specify later what to do in case of reversal of orientation.) Then we have an identification

$$T_{\partial X} = \prod_{i=1}^{n} T_{S^1},$$

and therefore a representation of $\tilde{T}_{\partial X}$ on

$$\hat{\bigotimes_{i=1}^{n}} \mathscr{H}. \tag{65}$$

Note that (65) is L-graded by sum of degrees of factors. Let $\mathscr{H}_X$ be the subspace of (65) consisting of all elements of degree 0 invariant under the action of T_X (see property (2)). (We will see that the condition limiting the degree to 0, is, in fact, not needed, since no elements other than in degree 0 can be invariant; this is, however, not important.)

Proposition 1. *If X is a connected rigged surface of genus* 0, *then $\mathscr{H}_X$ is a one-dimensional complex vector space*, *and moreover consists of trace-class elements.*

Granting this for the moment, $\mathscr{H}_X$ defines the $\mathbb{C}^{\times}$-central extension $\tilde{\mathscr{C}}_0$ and the embedding of $\mathscr{H}_X$ into (65) defines the 0-CFT supported on $\tilde{\mathscr{C}}_0$. To make that complete, we must discuss the case when some boundary components of X are oriented inbound. To this end, define an involution $a : T_{S^1} \to T_{S^1}$ by

$$a(f)(z) = f(1/z). \tag{66}$$

Then one notices that (on universal covers) the resulting cocycle is the exact reciprocal of cocycle (54). This means that the two central extensions are opposite (their product, pushed to the same kernel, canonically splits). This gives an isomorphism

$$\mathscr{H} \cong \mathscr{H}^*, \tag{67}$$

as the basic representations with opposite cocycles are dual Hilbert spaces. Now simply reverse the parametrization of any inbound boundary components of X by (66), find the ray $\mathscr{H}_X$, and apply iso (67) on factors (65) which correspond to boundary components with reversed parametrizations. The desired behaviour of $\mathscr{H}_X$ under gluing now follows from Properties 1 and 2 (see the beginning of this section). It remains to be shown that the 0-CFT we have defined extends to a 1-CFT, but in view of [47], this follows from the results of [8]; some discussion of modularity of the lattice 0-CFTs will be carried out below in Section 6 in connection with θ-series.

To relate this to the discussion in [2, p. 417] regarding reflection-positivity, it is appropriate to discuss, in more detail, the symmetric bilinear form B associated with isomorphism (67), and specifically compare it with the ‘‘standard’’ bilinear form B_0.

First of all, to define B_0, we point out that $\mathscr{H}$ has a real structure: it suffices to specify that on $\mathscr{H}_0$. Recall that we have

$$\mathscr{H}_0 = \widehat{Sym}(A), \tag{68}$$

where A is the space of holomorphic functions $f : D \to L_{\mathbb{C}}$ with average 0. Then let $A_{\mathbb{R}}$ be the real vector subspace of A consisting of functions which map real numbers to $L_{\mathbb{R}}$. This induces the desired real structure on (68). As usual, we obtain an associated symmetric bilinear form

$$B_0(x, y) = \langle x, \bar{y} \rangle.$$

(We make the convention that inner products on complex inner product spaces are linear in first coordinate and antilinear in second.) Note that B_0 coincides with the inner product when restricted to $\mathscr{H}_{\mathbb{R}}$, and hence is positive-definite.

Now recall that the map

$$\theta : f \mapsto -f$$

on T'_{S^1} preserves our cocycle, and thus defines an involution automorphism

$$\tilde{\theta} : \tilde{T}_{S^1} \to \tilde{T}_{S^1}.$$

Consequently, we obtain an automorphism

$$\theta_0 : \mathscr{H} \to \mathscr{H}$$

defined up to scalar multiple; we normalize it by requiring

$$\theta_0(1) = 1,$$

where 1 is the vacuum vector. Then θ_0 is an involution. Now we have

Lemma 2. *Let B be the symmetric bilinear form on $\mathscr{H}$ associated with isomorphism* (67). *Then*

$$B(x, y) = B_0(x, \theta_0 y).$$

Proof. We shall confine ourselves to $\mathscr{H}_0$; the discussion of the general degrees is then standard and we omit it.

On $\mathscr{H}_0$, the advantage is that our discussion becomes restricted to Heisenberg groups, and the choice of lattice L does not matter. We may restrict attention to $L \coloneqq \mathbb{Z}$. In fact, let us study iso (67). First, recall the iso

$$a : \tilde{T}_{S^1} \to \tilde{T}_{S^1}$$

by (66). This reverses the cocycle, and therefore acts by $1/z$ on the kernel of the central extension $\tilde{T}_{S^1}$. Composing the representation $\mathscr{H}$ with a gives a representation W of $\tilde{T}_{S^1}$ of level -1; another representation of the same group of level -1 is the dual $\mathscr{H}^*$ of the representation $\mathscr{H}$. Then classification of representations of $\tilde{T}_{S^1}$ shows that we must have an iso of representations

$$W \cong \mathscr{H}^*.$$

Up to scalar multiple, this is (67), which we use to define B. We can normalize the iso by

$$B(1,1) = 1.$$

Now we study the dual representation $\mathscr{H}^*$: for $f \in \mathscr{H}^*$, a dual representation in general acts by

$$g(f)(x) = f(g^{-1}(x)),$$

so in our case, for $g \in A$,

$$g(f)(x) = f(-g(x)). \tag{69}$$

We can, however, take

$$g(?) = B_0(?, y)$$

for some $y \in \mathscr{H}_0$. Then using (69), our statement reduces to

$$B_0(g(x), y) = B_0(x, a(g)(y)). \tag{70}$$

In turn, it suffices to show this for $g \in A_{\mathbb{R}}$, where (70) coincides with

$$\langle g(x), y \rangle = \langle x, \bar{g}(y) \rangle,$$

($\bar{?} : A \to \bar{A}$), which is the reflection-positivity (unitarity) formula for the Heisenberg representation. □

Proof of Proposition 1. The argument that

$$dim(\mathscr{H}_X) \leqslant 1 \tag{71}$$

is carried out in [33, Section 8.11]: If X has n boundary components, the double coset space

$$T_X \backslash T_{\partial X} \Big/ \prod_{i=1}^{n} T_D \tag{72}$$

is the moduli space of holomorphic T-bundles on $\mathbb{P}^1$, which is discrete and isomorphic to L (by degree). Hence, T_X acts transitively on

$$\left(T_{\partial X} \bigg/ \prod_{i=1}^{n} T_D\right)_0, \tag{73}$$

the part of degree 0. Now (65) can be interpreted as a certain space of functions on the ∞-dimensional complex manifold (73) (a suitable subspace of the space of holomorphic functions). In any case, a T_X-invariant holomorphic function on (73) is determined by its value on a single point, hence (71). (In fact, we see that on the bottom weight vector in non-zero degrees the action of $T \subset T_X$ is non-trivial, which is why the degree 0 condition is unnecessary.)

To show that

$$dim(\mathscr{H}_X) \geqslant 1, \tag{74}$$

we grade (65) by L^n, considering degrees of all individual factors. Let

$$L_0^n = \left\{(\lambda_1, \dots, \lambda_n) \,\middle|\, \sum \lambda_i = 0\right\}.$$

We shall construct a non-zero element $x \in \mathscr{H}_X$ as

$$x = \sum_{k \in L_0^n} x_k, \tag{75}$$

where $deg(x_k) = k$. From this point of view, it suffices to construct x_0, as we may set

$$x_k = f_k(x_0),$$

where $f_k \in T_X$ is any holomorphic function of degree k (although we then must show that sum (75) converges and is trace class).

But now x_0 does not depend on the lattice L, and can be obtained by a method called *boson-fermion correspondence*. This means that the 0-CFT $\mathscr{H}_0$ is isomorphic to the Hilbert tensor product of $dim(L)$ copies of the degree 0 part $\mathscr{F}_0$ of the bc-system $\mathscr{F}$ associated with the space of 1/2-forms $\Omega^{1/2}(S^1, \mathbb{C})$. The CFT $\mathscr{F}$ is defined on a $\mathbb{C}^\times$-central extension of the stack of LCMCs $\mathscr{C}_{\text{spin}}$ of rigged surfaces with spin, and is treated in detail in [26]. At any rate, what is important for us is only that its degree 0 part $\mathscr{F}_0$ is a 0-CFT, and the tensor product of $dim(L)$ copies of its vacuum vector over X is the element x_0 we seek. The required properties of x_0 easily follow from properties of the boson-fermion correspondence (cf. [33]).

To be a little more specific, recall [33] that the *projective* representation of T_{S^1} on $\mathscr{H}_0 \cong \mathscr{F}_0$ is induced by the action of T_{S^1} on $Gr_{\text{res}}\Omega^{1/2}(S^1, \mathbb{C})$. Thus, it is obvious that the vacuum vector x_0 is invariant under that projective action. However, that amounts only to saying that for $f \in T_X$, $(deg(f) = 0)$,

$$f(x_0) = \lambda_f x_0 \tag{76}$$

for some constant λ_f: we must show that

$$\lambda_f = 1. \tag{77}$$

To this end, we note that obviously

$$\lambda_{f \amalg g} = \lambda_f \lambda_g, \tag{78}$$

but also

$$\lambda_{\check{f}} = \lambda_f. \tag{79}$$

(*Note*: To prove (79) in the case boundary parametrizations of arbitrary orientation, we need the fact that the action of $\tilde{T}_{S^1}$ on $\mathscr{F}$ is B-adjoint to the action obtained by reversing parametrization where B is the symmetric bilinear form involved in reflexivity. But restricting to real forms of both group and representation, this follows from the fact that the real form of the representation is orthogonal, see [33, Chapters 10, 12, 13].)

Anyway, in view of (78), (79), it suffices to prove (77) for annuli and the standard disk. The statement for the standard disk is tautological (on any Heisenberg representation, the action of $\bar{A}$ on the vacuum is the identity). For annuli, the statement essentially amounts to saying that $\mathscr{H}_0$, $\mathscr{F}$ are the same representations of $\widetilde{Diff}^+(S^1)$. That follows from the fact that these representations have the same invariants: central charge $c = 1$ and rotation number 0 ([26,36]).

Thus, we have reduced the proof of Proposition 1 to the following statement. □

Lemma 3. *The element $x \in \prod_L \mathscr{H}_0$ constructed above is an element of $\mathscr{H}$, and is trace class.*

We first prove that x_0 is of trace class. To this end, we make a brief excursion to the theory $\mathscr{F}$. The state space of this theory is

$$K = \hat{\bigwedge}(\Omega^{1/2}S^1_+ \oplus \overline{\Omega^{1/2}S^1_-}), \tag{80}$$

where $\Omega^{1/2}S^1_+$ is the closed subspace of $\Omega^{1/2}S^1$ spanned by $z^n dz^{1/2}$ for $n \geqslant 0$, and $\Omega^{1/2}S^1_-$ is its orthogonal complement. Let X be as above, and let

$$u_X \in \hat{\bigotimes_{in}} K^* \hat{\otimes} \hat{\bigotimes_{out}} K$$

be the vacuum vector where the tensor products are over inbound and outbound boundary components of X, as above. Assume, without loss of generality, that X

only has outbound boundary components, so

$$u_X \in \hat{\bigotimes} K.$$

In admissible basis notation ([26,33]), u_X is of the form

$$\bigwedge p_i, \tag{81}$$

where p_i ranges over boundary values of a basis of the space of 1/2-forms on X which are $z^n dz^{1/2}$ on one boundary component c_i, $n \geqslant 0$. However, if we choose a number $0 < \lambda < 1$ such that for each boundary component c of X, an annulus $A_q, ||q|| = \lambda$ can be mapped conformally to X so that S^1 maps to c by the parametrization, and the other boundary component of A_q maps into the interior of X, then

$$p_i = (z^{n_i} dz^{1/2})_{c_i} + f_i,$$

where $||f_i|| \leqslant \lambda^{n_i + 1/2}$. This implies that (81) is trace class, and hence so is x_0.

We now need the following

Lemma 4. *Let X be a genus 0 rigged surface with all boundary components $c_1, \ldots, c_n$ oriented outbound. Let $d_1 \ldots, d_n \in \mathbb{Z}$, $\sum d_i = 0$. Then there exists a holomorphic function $f : X \to \mathbb{C}^\times$ such that*

$$deg(f|c_i) = d_i, \tag{82}$$

$$||f(z)|| \text{ is constant on } z \in c_i. \tag{83}$$

Proof. Recall that there exists a harmonic function

$$h_i : X \to [0, 1]$$

where 0,1 are regular points,

$$(h_i)^{-1}(0) = c_i, \tag{84}$$

$$(h_i)^{-1}(1) = \coprod_{j \neq i} c_j. \tag{85}$$

Such function is a solution to the Dirichlet problem (see [24, 8.7]).

Now if we denote by $\underline{Hol}$, $\underline{Harm}$ the sheaves of holomorphic and real harmonic functions, and by $\underline{\mathbb{R}}$, $\underline{\mathbb{Z}}$ the constant sheaves, we have a short exact sequence

$$0 \to \underline{\mathbb{R}} \to \underline{Hol} \to \underline{Harm} \to 0 \tag{86}$$

(The second map is $f \mapsto f + \bar{f}$, the first map is $\lambda \mapsto i\lambda$.) We have, in effect, a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{\mathbb{Z}} & \longrightarrow & \underline{Hol} & \longrightarrow & \underline{Hol^\times} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow = & & \downarrow \phi & & \\ 0 & \longrightarrow & \underline{\mathbb{R}} & \longrightarrow & \underline{Hol} & \longrightarrow & \underline{Harm} & \longrightarrow & 0. \end{array} \tag{87}$$

The top right map is

$$f \mapsto exp(f),$$

the right column ϕ is

$$f \mapsto \ln(f) + \ln(\bar{f}) = \ln(f\bar{f}) = 2\ln||f||. \tag{88}$$

Let $Hol_X^\times = \Gamma_X \underline{Hol}^\times$, $Hol_X = \Gamma_X \underline{Hol}$, $Harm_X = \Gamma_X \underline{Harm}$. Then (87) gives a diagram with exact rows

$$\begin{array}{ccccccc} Hol_X & \longrightarrow & Hol_X^\times & \xrightarrow{\beta} & H^1(X, \mathbb{Z}) & \longrightarrow & H^1(X, \underline{Hol}) \\ \downarrow = & & \subseteq \downarrow \phi & & \downarrow & & \downarrow = \\ Hol_X & \longrightarrow & Harm_X & \xrightarrow{\delta} & H^1(X, \mathbb{R}) & \longrightarrow & H^1(X, \underline{Hol}). \end{array} \tag{89}$$

Note in (89) that any function f such that

$$\phi(f) \in \langle h_1, \dots, h_n \rangle$$

has, by (88), constant modulus on each c_i.

Now we shall study the map δ for the standard annulus $A_{1-\varepsilon}$ where $0 < \varepsilon < 1$. Then

$$H^1(A_{1-\varepsilon}, \mathbb{R}) = \mathbb{R},$$

and it is easy to see that, with suitable normalization, $\delta(f)$ has the same sign as

$$\oint_{S^1} \frac{f(z)}{z} dz - \oint_{(1-\varepsilon)S^1} \frac{f(z)}{z} dz.$$

Consequently, returning back to X, if δ_i is the composition

$$Harm_X \xrightarrow{\delta} H^1(X, \mathbb{R}) \to H^1(X, c_i),$$

then by (84) and (85),

$$sign(\delta_i h_i)$$

is opposite to

$$sign(\delta_j h_i), \quad j \neq i.$$

This implies that

$$rank(\langle \delta h_1, \ldots, \delta h_n \rangle) \geqslant n-1,$$

and hence

$$\langle \delta h_1, \ldots, \delta h_n \rangle = H^1(X, \mathbb{R}).$$

Now choose $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that

$$\lambda_1 \delta h_1 + \cdots + \lambda_n \delta h_n \tag{90}$$

is any element of $H^1(X, \mathbb{Z})$. Then, by (89), (90) is in the image of β, say, equal to $\beta(f)$. Then f is the desired function. □

We now return to the proof of Lemma 3. We first prove a weaker statement. Let $||x_k||_1$ be the infimum of $\sum ||\gamma_i||$ where $x_k = \sum \gamma_i x_{k1} \otimes \cdots \otimes x_{kn}$.

Lemma 5. *$||x_k||_1$ grows at most exponentially in $||k||$.*

Proof. Use finitely many functions $f_1, \ldots, f_d$ of Lemma 4, $d = dim(L)$, whose degrees generate L_0^n. Then $f_i(x_0)$ can be calculated by first calculating by the constant loop (equal to the modulus), which acts trivially on each factor of x_0, then by the map of modulus 1 (which preserves norm), and then by a central term, which, by definition of the cocycle, grows at most exponentially as specified. □

But now let Y be such that X is obtained from Y by gluing on a standard annulus A_q to each boundary component. Then if y_k are the analogues of x_k with X replaced by Y, the action of these annuli on y_k multiplies norm by $\leqslant ||q||^{\frac{1}{2}||k||^2}$ (since the k-graded summand of (65) has energy $\geqslant \frac{1}{2}||k||^2$). Replacing x by y in Lemma 5 gives the statement of Lemma 3. □

The reader may ask what is missing in the above discussion that would allow us to define a full CFT associated with an even lattice L, instead of just a 1-CFT. The answer is that we would need a discussion of the labels. The situation is actually quite simple: The labels correspond to elements of L'/L where L' is the dual lattice, and the "Verlinde algebra" (cf. [45], also [36]) is just the group algebra $\mathbb{C}[L'/L]$. However, since the rest of this paper avoids labels, we do not want to introduce them here, and will therefore discuss the CFT associated with a general lattice elsewhere. On the other hand, there is one case which does not need labels, namely the case of L even unimodular. We therefore state the following

Proposition 6. *If L is an even unimodular lattice, then the* 1-*CFT associated with L constructed above in this section extends to a CFT with one-dimensional modular functor, as defined in the previous section.*

Proof. We shall make use of the fact that every rigged surface can be obtained from a genus 0 rigged surface by gluing. We can use this to construct a canonical splitting of the restriction of the central extension $\tilde{T}_{\partial X}$ to T_X for rigged surfaces X of genus >0 by cutting X via non-separating curves into a surface of genus 0, and using the fact that on boundary components with opposite orientation, the cocycle has opposite signs.

There is, however, a subtle point. Suppose X has genus $g>0$ and we cut X into a genus 0 rigged surface X_1 using a complete system of non-separating curves $c_1, \dots, c_g$. Now take holomorphic functions $f, g : X \to T$, which then give rise to functions $f_1, g_1 : X_1 \to T$ and functions $\tilde{f}_1, \tilde{g}_1$ into $L_{\mathbb{C}}$ on the universal cover of ∂X_1, as above. We have

$$c(\tilde{f}_1, \tilde{g}_1) = 0$$

so we would like to say the same thing for their restrictions $\tilde{f}, \tilde{g}$ to the universal cover of ∂X. It may indeed seem that the cocycle summands corresponding to the two copies $c_1, \dots, c_g, c'_1, \dots, c'_g$ of the cutting curves of opposite orientations relative to X will cancel. This is almost true, but not exactly. The problem is that the values of the function $\tilde{f}_1$ (resp. $\tilde{g}_1$) on c_i, c'_i may differ by a constant. Keeping track of these constants, we find that

$$c(\tilde{f}, \tilde{g}) = exp \sum_{i=1}^{g} \frac{1}{2} (\Delta_{\tilde{f}}^{a_i} \Delta_{\tilde{g}}^{b_i} - \Delta_{\tilde{g}}^{a_i} \Delta_{\tilde{f}}^{b_i}), \tag{91}$$

where $a_1, b_1, \dots, a_g, b_g$ form a hyperbolic basis of $H_1(X, \delta X; \mathbb{Z})$ and $\Delta_?^a$ denotes the degree of ? along a. (For example, a_i may come from c_i and b_i from a dual cutting system $d_1, \dots, d_g$.) Note that this implies $c(\tilde{f}, \tilde{g}) \in \{1, -1\}$ and does not depend on the choice of lifts.

Therefore, we have found that while a splitting

$$T_X \to \tilde{T}_{\partial X} \tag{92}$$

cannot be defined by the same formula as in the genus 0 case (which was $f \mapsto \tilde{f}$), (92) can be defined by the corrected formula

$$f \mapsto (\tilde{f}, c(\tilde{f}, \tilde{f})), \tag{93}$$

where the second coordinate is in the center. One then sees from (91) that (93) is a homomorphism of groups $T_X \to \tilde{T}_{\partial X}$. Then (91) also shows that this is the correct splitting needed for compatibility with gluing, which in addition does not depend on the choice of cutting curves $c_1, \dots, c_g$.

Thus, we may speak of the space $\mathscr{H}_X = \mathscr{H}^{T_X}$ as above for every rigged surface X without closed connected components. Indeed, using Proposition 1, we can construct a non-zero trace class element

$$x \in \mathscr{H}_X$$

for any such X.

What is then left is showing that (71) generalizes to surfaces with higher genus. Let, to this end, Y be the closed surface obtained by gluing standard disks to the boundary components of X. As before, we may form space (72) which is isomorphic to

$$H^1(Y, \underline{Hol_T}), \tag{94}$$

where $\underline{Hol_T}$ is the sheaf of T-valued holomorphic functions on Y. There is a degree map from (94) to L, and constants act by the character equal to degree, so it suffices to consider the subgroup

$$H^1_0(Y, \underline{Hol_T}) \tag{95}$$

of (94) of elements of degree $0 \in L$. Now (95) is the set of $\mathbb{C}$-points of an abelian variety E and the statement we must prove is that the space of sections of the line bundle $\mathscr{L}$ on E corresponding to the $\mathbb{C}^\times$-principal bundle

$$T_X \backslash \tilde{T}_{\partial X} / \prod_{i=1}^{n} T_D$$

is one-dimensional.

But the line bundle $\mathscr{L}$ can be identified. Indeed, as above, by compatibility with gluing, the line bundle remains canonically isomorphic when we cut along additional curves, so we may cut Y along a complete system of non-separating curves $c_1, \dots c_g$ instead. Then we know an element of (95) can be represented by a system of *constant* transition functions

$$const_{z_i} : c_i \to T, \quad i = 1, \dots, g. \tag{96}$$

Now z_i and $z_i + \omega_{ij}$ represent the same point in (95) where $(\omega_{1j}, \dots, \omega_{gj})$ is the period of Y corresponding to degrees $h_j = (0, \dots, 0, h, 0, \dots, 0), h \in L$, ($h$ is the jth entry) along $c_1, \dots, c_g$.

How do sections of $\mathscr{L}$ transform under this shift? It will be by multiplication by a certain $\lambda_j \in \mathbb{C}^\times$. To find λ_j, let Y_1 be, again, the genus 0 rigged surface obtained by cutting Y along $c_1, \dots, c_g$. We must consider a holomorphic function $f : Y_1 \to T$ of degree h_j along $c_1, \dots, c_g$. The number λ_j is the central term obtained by commuting the constant z_j past the function of degree h_j! But it is well know that if $z = e^{2\pi i \tau}$,

$\tau \in L_{\mathbb{C}}$, the commutation number is

$$\lambda_j = e^{2\pi i \langle \tau, h_j \rangle}$$

(see e.g. [33, Proposition 4.7.1] for an equivalent statement). We have therefore identified $\mathscr{L}$ as a theta-bundle on (95).

However, to identify the sections, we need to be a bit more explicit. To this end, first consider the case $L = \mathbb{Z}$ (this is not an even lattice, but this part of the discussion is not affected). Then the Jacobian of Y can be identified with

$$J(Y) = \mathbb{C}^g / (\mathbb{Z}^g \oplus \Omega \mathbb{Z}^g),$$

where Ω is a symmetric matrix (the period matrix) and we have that $Im(\Omega)$ is positive definite. Then the space of sections of the theta-bundle on $J(Y)$ is one-dimensional and generated by the theta-function

$$\theta(z) = \sum_{x \in \mathbb{Z}^g} e^{\pi i (x^T \Omega x + 2 z^T x)}. \tag{97}$$

However, (95) is isomorphic (as an abelian group) to $J(Y) \otimes L$. In this situation, the space of sections of the theta-bundle $\mathscr{L}$ described above is $|L'/L|^g$-dimensional, and generated freely by the theta-functions

$$\theta_\alpha(z) = \sum_{x \in \mathbb{Z}^g \otimes L + \alpha} e^{\pi i (x^T \Omega x + 2 z^T x)} \tag{98}$$

for $\alpha \in \mathbb{Z}^g \otimes (L'/L)$. (The exponents in (98) are calculated by contracting, for $z, x \in \mathbb{C}^g \otimes L$, by the matrix product in the $\mathbb{C}^g$-coordinate, and the L-inner product in the L-coordinate.) For more discussion of theta-functions, see Section 6 below.

For an even unimodular lattice L, we have $L' = L$, so the space of sections of $\mathscr{L}$ is 1-dimensional, thus proving

$$dim(\mathscr{H}_X) \leqslant 1$$

for genus $g > 0$, as desired. We have not explicitly discussed Y closed, but in that case the Hilbert space is just $\mathbb{C}$, so the discussion reduces to defining the modular functor section $\mathscr{H}_Y$. We see that the above discussion forces

$$\mathscr{H}_Y = \Gamma(\mathscr{L})$$

which is one-dimensional for L unimodular, as needed. Our proof of Proposition 6 is complete. □

Comment. The CFTs associated with the even unimodular lattices $E(8) \times E(8)$ and $E(16)$ of dimension 16 are used in the construction of the heterotic string theories; thus, Proposition 6 contributes to mathematical foundations of those theories. It is worth mentioning that implicit in string theory is a conjecture stating that these are the only chiral CFTs with one-dimensional modular functor and central charge 16.

4. An example of 1-CFT: the completed Moonshine module

We will now give an outline of how to extend the techniques of the previous section to constructing a (reflection-positive) 1-CFT which is a Hilbert completion of the Moonshine module. This is needed because we wish to discuss the Moonshine module 1-CFT (and, in fact, make conjectures about it). On the other hand, for the main definition of this paper, which will be presented in the next section, this example is not necessary: it is merely important that *some examples* exist, which we have already shown. Because of this, we will proceed in somewhat less detail than above. We shall use some ideas implicit in [7]. First of all, we will restrict ourselves to the 0-CFT structure on $\mathscr{H}^\natural$. Second, we will only construct the Hilbert version of the twisted module $\mathscr{H}_T$ where $\mathscr{H}$ is the lattice 1-CFT associated with an even lattice L—we are thinking of the Leech lattice, but that does not matter. The point is that, if we think of the construction of $\mathscr{H}^\natural$ as a convergence question of vertex operators in $V^\natural$ (as discussed at the end of Section 2), then these are the only convergence results we need; the remaining 0-CFT operators of $\mathscr{H}^\natural$ arise by means of averaging with respect to θ_0, composition and reversals of orientation of boundary components. For a good discussion of this from the VOA point of view, see [21].

To describe $\mathscr{H}_T$, one must discuss the twisted loop group $T^{1/2}_{S^1}$: This is the group of (analytic) maps

$$f : [0,1] \to T = L_{\mathbb{C}}/L$$

such that $f(0) = \frac{1}{f(1)}$. Of course, we need a central extension of $T^{1/2}_{S^1}$. To this end, we consider the group $T^{1/2}_{S^1}$ of maps

$$f : [0,1] \to L_{\mathbb{C}}, \quad \Delta_f := f(0) + f(1) \in L.$$

We have a short exact sequence

$$0 \to L \to T^{1/2}_{S^1} \to T^{1/2}_{S^1} \to 0.$$

(Note that unlike the untwisted case, where the kernel consisted of functions f with $\Delta_f = 0$, here we know only $\Delta_f \in 2L$ when f is in the kernel.)

As before, on starts by defining a $\mathbb{C}^\times$-values 2-cocycle on $T^{1/2}_{S^1}$, which can, in fact, be defined by using (with a changed meaning) formula (54). Similarly as before, the cocycle is 0 when restricted to L, and if $\tilde{T}^{1/2}_{S^1}$ is the corresponding $\mathbb{C}^\times$-central extension, we obtain a short exact sequence

$$0 \to L \to \tilde{T}^{1/2}_{S^1} \to \tilde{T}^{1/2}_{S^1} \to 0.$$

Now $\mathscr{H}_T$ is the basic reflection-positive Hilbert representation of the twisted loop group $\tilde{T}^{1/2}_{S^1}$. As before, care is needed to describe this accurately, but we note that

$T^{1/2}_{S^1}$ contains the subgroup

$$V^{1/2} = \{f : [0,1] \to L_{\mathbb{C}} \,|\, f(0) = -f(1)\}. \tag{99}$$

In fact, we have a short exact sequence

$$0 \to L/2L \to T^{1/2}_{S^1} \to V^{1/2} \to 0. \tag{100}$$

Now (99) is a vector space and the induced central extension is a Heisenberg group. Since we also have a choice of isotropic space (at least up to a choice of Gr_{res}, see [33, Chapter 7]), we therefore have a canonical associated Heisenberg representation [33, 9.5], which we will denote by $\mathscr{H}_{T,0}$. We let $\mathscr{H}_T$ be the induced representation via (100). Note that this is somewhat simpler than in the untwisted case, since the kernel is finite.

Note at this point also that, similarly to the twisted case, there is an automorphism

$$\theta^{1/2} : T^{1/2}_{S^1} \to T^{1/2}_{S^1}$$

given by $\theta^{1/2}(f)(z) = 1/f(z)$. This preserves the cocycle and kernel, and hence lifts to an automorphism

$$\tilde{\theta}^{1/2} : \tilde{T}^{1/2}_{S^1} \to \tilde{T}^{1/2}_{S^1}.$$

Thus, we obtain an isomorphism of the $\mathscr{H}_T$ with the representation obtained by composition with $\tilde{\theta}^{1/2}$:

$$\theta^{1/2}_0 : \mathscr{H}_T \to \mathscr{H}_T. \tag{101}$$

If $dim(L) = 24$, the usual normalization is

$$\theta^{1/2}_0(1) = -1 \tag{102}$$

whereby $\theta^{1/2}_0$ becomes an involution. Reasons for this choice will become apparent later.

Now consider a genus 0 connected rigged surface X whose boundary components are decorated 0 or T, where the number of components decorated T is even. Assume (just for simplicity of notation) that all boundary components of X are oriented outbound. Consider the covering space

$$X^{1/2} \to X$$

corresponding to the map

$$\rho : \pi_1 X \to \mathbb{Z}/2$$

where $\rho(\alpha) = 1$ (resp. 0) on elements α conjugate to the parametrizations of boundary components decorated T (resp. 0). We denote by $T_X^{1/2}$ the space of all holomorphic functions $f : X^{1/2} \to T$ where

$$f(\imath(x)) = \frac{1}{f(x)},$$

$\imath : X^{1/2} \to X^{1/2}$ being the non-trivial deck transformation. Analogously as above, we can consider

$$\phi : T_X^{1/2} \to \prod_0 T_{S^1} \times \prod_T T_{S^1}^{1/2}, \tag{103}$$

where on the right-hand side the products are over boundary components of X decorated 0 resp. T.

Analogously as above, the pullback of the cocycle via ϕ is 0, so we can consider the $T_X^{1/2}$-fixed subspace

$$\mathscr{H}_X^{1/2} \tag{104}$$

of the basic representation

$$\hat{\bigotimes_0} \mathscr{H} \hat{\otimes} \hat{\bigotimes_T} \mathscr{H}_T \tag{105}$$

of the right-hand side of (103). We will restrict ourselves here to showing the following

Proposition 7. *If X is a genus 0 connected rigged surface with two boundary components decorated T and the other boundary components decorated 0, then $\mathscr{H}_X^{1/2}$ is 1-dimensional, and its elements are trace class.*

Remark. This amounts to saying that $\mathscr{H}_T$ is a "reflexive version" of twisted module over 0-CFT $\mathscr{H}$. The reader can fill in the details of that definition. A "non-reflexive" version of module over a directed 0-CFT would be obtained if we require that all boundary components of X be oriented inbound except one which is oriented outbound and labelled T (of course, the obvious gluing properties are required in both cases). The word "twisted" comes from the fact that we must choose sheets of $X^{1/2}$ when considering the $\tilde{T}_{S^1}^{1/2}$-action on the copies of $\mathscr{H}_T$ corresponding to the T-labelled boundary components of X, but loops in the moduli space of the surfaces X may permute the sheets of $X^{1/2}$.

The Hilbert-completed Moonshine module is

$$\mathscr{H}^{\natural} := \mathscr{H}^{\theta_0} \oplus \mathscr{H}^{\theta_0^{1/2}}.$$

The operators U_X are obtained from averaging elements of $\mathscr{H}_X^{1/2}$ over the direct product of $\mathbb{Z}/2$'s generated by θ_0's and $\theta_0^{1/2}$'s over boundary components of X decorated 0 and T, respectively. If X has more than 2 boundary components decorated T, we use gluing. Obviously, a consistency discussion with respect to gluing is needed, but this follows by noting that Proposition 7 is just a convergence theorem for (twisted) vertex operators, where the discussion has been thorough in the literature (see e.g. [13,21]).

Finally, note that this point of view elucidates choice (102) of the normalization of $\theta_0^{1/2}$: when X is an annulus with one inbound and one outbound boundary component decorated T, then Proposition 7 gives, in particular, a representation of a $\mathbb{C}^\times$-central extension of $Diff^+(S^1)$. Such central extensions are characterized by central charge and rotation number; the rotation number characterizes the pullback of the central extension to the subgroup S^1 of rigid rotations. This determines the weight of the vacuum vector. The point is that in the case of the twisted module $\mathscr{H}_T$, that weight comes out to be

$$dim(L)/16,$$

which, in the case $dim(L) = 24$ becomes 3/2. Hence, (102).

Proof of Proposition 7. With a slick moduli argument analogous to the untwisted case not readily available, we use a more pedestrian argument for uniqueness: if X is an annulus, the statement follows from the irreducibility of the representation $\mathscr{H}_T$ of $\tilde{T}_{S^1}^{1/2}$ (for example, by turning X into a standard annulus with reparametrized boundary components). In the general case, one shows that

$$closure\left(\prod_0 T_{D^-}\right)\Big\backslash \prod_0 T_{S^1} \times \prod_T T_{S^1}^{1/2} \Big/ closure\left(\prod_T T_X^{1/2}\right)$$

is equal to $T_{S^1}^{1/2}$ (where, as usual, the products are over boundary components decorated 0 and T as indicated, and D^- is the outside of the unit circle in $\mathbb{C}\cup\{\infty\}$). This reduces the proof to the annulus case.

We shall now turn to the extension theorem, which is the more interesting part. As in the previous section, the key is showing an analogous statement of the Proposition with $\mathscr{H}$ replaced by $\mathscr{H}_0$ and $\mathscr{H}_T$ replaced by $\mathscr{H}_{T,0}$ (the parts of degree 0), i.e. constructing an element

$$x_0 \in \hat{\prod_0} \mathscr{H}_0 \hat{\otimes} \hat{\bigotimes_T} \mathscr{H}_{T,0} \tag{106}$$

invariant under the vector subspace of holomorphic functions $X^{1/2} \to T$ intersected with the corresponding Heisenberg groups. The desired element $x \in \mathscr{H}_X^{1/2}$ can then be obtained again by summing the images of x_0 under a suitable series of holomorphic functions; same arguments as in the previous section can be used to establish convergence (note the finiteness of the twisted part).

Thus, we shall focus on the right-hand side of (106), i.e. on the pair of Hilbert spaces

$$\mathscr{H}_0, \ \mathscr{H}_{T,0}. \tag{107}$$

The striking property of these spaces is that they are Heisenberg representations, and hence, once again, are independent of the particular lattice L. Thus, for the purposes of constructing x_0, one may replace L by $\mathbb{Z}$. In that case, it is provocative to ask whether we may, again, use some type of boson-fermion correspondence which would explain the (super)-CFT structure on (107).

Such boson-fermion correspondence does, indeed, exist. We begin by giving a "fermionic" description of $\mathscr{H}_T$ (for $L = \mathbb{Z}$). Consider the Hilbert space K spanned by

$$\langle z^n dz^{1/2} \mid n \in \tfrac{1}{2}\mathbb{Z} \rangle. \tag{108}$$

Here the elements (108) shall form an orthonormal basis. We shall call the subspace of K spanned by

$$\langle z^n dz^{1/2} \mid n \in \mathbb{Z} \rangle$$

K^{even} and the subspace spanned by

$$\langle z^n dz^{1/2} \mid n \in \mathbb{Z} + \tfrac{1}{2} \rangle$$

K^{odd}. We then have a symmetric bilinear form B on K given by

$$B(\eta, \xi) = \begin{cases} \oint \eta\xi & \text{if } \eta, \xi \in K^{\text{even}}, \\ -\oint \eta\xi & \text{if } \eta, \xi \in K^{\text{odd}}, \\ 0 & \text{else.} \end{cases} \tag{109}$$

The purpose of this is to consider the subgroup

$$V_+^{1/2} \subset V^{1/2}$$

consisting of loops of the form

$$1 + h_{1/2} z^{1/2} + h_1 z^1 + h_{3/2} z^{3/2} + \cdots, \quad h_n \in \mathbb{C} \tag{110}$$

where

$$(1 + h_{1/2} z^{1/2} + h_1 z^1 + h_{3/2} z^{3/2} + \cdots)(1 - h_{1/2} z^{1/2} + h_1 z^1 - h_{3/2} z^{3/2} + \cdots) = 1. \tag{111}$$

(Note that condition (111) is the right condition to assure that (110) belongs to the twisted loop group.) The group $V^{1/2}$ embeds to GL_{res} (with the usual polarization) by sending (110) to the matrix

$$\begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 1 & h_{1/2} & h_1 & h_{3/2} & h_2 & \cdots \\ \cdots & 0 & 1 & h_{1/2} & h_1 & h_{3/2} & \cdots \\ \cdots & 0 & 0 & 1 & h_{1/2} & h_1 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & h_{1/2} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

(the rows corresponding to $z^n dz^{1/2}$, $n \in \frac{1}{2}\mathbb{Z}$ in order), which, by (111), is inverse to

$$\begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 1 & -h_{1/2} & h_1 & -h_{3/2} & h_2 & \cdots \\ \cdots & 0 & 1 & -h_{1/2} & h_1 & -h_{3/2} & \cdots \\ \cdots & 0 & 0 & 1 & -h_{1/2} & h_1 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & -h_{1/2} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

which, however, by (109) is B-adjoint. Thus, we obtain a map

$$V_+^{1/2} \to SO_{\mathrm{res}}K.$$

The cocycle splits on $V_+^{1/2}$, so the map lifts to $Spin_{\mathrm{res}}K$ (see [33, Chapter 12]) so $V_+^{1/2}$ acts naturally on the *Spin*-representation of $Spin_{\mathrm{res}}K$, which is

$$\hat{\bigwedge}\left\langle z^n dz^{1/2} \mid n<0, n\in\frac{1}{2}\mathbb{Z}\right\rangle. \tag{112}$$

Defining the group $V_-^{1/2}$ in the same way as $V_+^{1/2}$ with z^n replaced by z^{-n}, we obtain an analogous action, with the appropriate commutation relation, so we get a map

$$\tilde{V}^{1/2} \to Spin_{\mathrm{res}}K,$$

and a representation of $\tilde{V}^{1/2}$ on (112).

Now because the Heisenberg representation $\mathscr{H}_{T,0}$ of $\tilde{V}^{1/2}$ is irreducible, we obtain an isometry ι from $\mathscr{H}_{T,0}$ to (112) which hence must be an iso, since both sides have equal partition functions

$$\prod_{n\in\mathbb{N}} (1-q^{n+1/2})^{-1} = \prod_{n\in\mathbb{N}} (1+q^{n/2}).$$

Thus, (112) is the desired fermionic description of $\mathscr{H}_{T,0}$. To construct the field theory operator x_0, recall from the previous section that $\mathscr{H}_0$ can be described as the degree 0 part of the spinor of the space

$$H_{\mathbb{C}} = H_{\mathbb{R}} \otimes \mathbb{C} = H \oplus H^*. \tag{113}$$

where H is Hilbert completion of the space of antiperiodic analytic 1/2-forms on S^1. Then $H \cong H^*$ (using the symmetric bilinear form $\oint \eta\xi$). Denote the right-hand side of (113) as $H^{\mathrm{even}} \oplus H^{\mathrm{odd}}$.

Thus, the field operator of the fermionic model of (107) ($L = \mathbb{Z}$) can be characterized as the Pfaffian line of the maximal isotropic space

$$H_X^{\mathrm{even}} \oplus H_X^{\mathrm{odd}} \subset \bigoplus_0 H^{\mathrm{even}} \oplus H^{\mathrm{odd}} \oplus \bigoplus_T K^{\mathrm{even}} \oplus K^{\mathrm{odd}}$$

where H_X^{even} is the closure of the subspace of all 1/2-forms on X antiperiodic on all boundary components and H_X^{odd} is the closure of the subspace of all 1/2-forms on X antiperiodic on all boundary components labelled 0 and periodic on all boundary components labelled T. Note (cf. [26]) that to get isotropy, the formula for B on H is

$$B(\xi,\eta) = \begin{cases} \oint \eta\xi & \text{if } \eta, \xi \in H^{\mathrm{even}}, \\ -\oint \eta\xi & \text{if } \eta, \xi \in H^{\mathrm{odd}}. \end{cases}$$

Remark. Actually, to have this theory behave exactly right, it appropriate to take (112) for one half of the T-labelled boundary components and

$$\hat{\bigwedge}\left\langle z^n dz^{1/2} \mid n \leqslant 0, n \in \frac{1}{2}\mathbb{Z} \right\rangle$$

for the other half. As remarked by Deligne (cf. [26]), the resulting structure is not a CFT even on any $\mathbb{C}^{\times}$-central extension of the stack of LCMCs $\mathscr{C}_{\mathrm{spin}}$, but rather a still more complicated object, which we may call a "CFT twisted by the super-Brauer group of $\mathbb{C}$". However, this is not relevant here, since we are considering only the degree 0 (Heisenberg) part of the theory, where the distinction is not visible.

5. Stringy bundles

Comment. The referee pointed out that some material of this paper overlaps with previous work of Brylinski, Segal and McLaughlin.

Let $\mathscr{H}$ be a 1-CFT. Let $\mathscr{V} := (\mathscr{H} - \{0\})/\mathbb{C}^{\times}$. We call the 1-CFT $\mathscr{H}$ *regular* if $\mathscr{H}$ satisfies (45) (so central charge is defined) and

$$U_A : \mathscr{H}^{\otimes k} \to \mathscr{H},$$

for a connected genus 0 rigged surface A with k inbound and 1 outbound boundary component induces a map

$$\mathscr{V}^{\times k} \to \mathscr{V}.$$

This means that $U_A(x_1 \otimes \cdots \otimes x_k) \neq 0$ if $x_1, \ldots, x_k \neq 0$. Note that it follows from the discussion in Sections 3 and 4 that the 1-CFTs associated with even lattices, as well as the Hilbert-completed Moonshine module, are regular. In the rest of this paper, we shall assume that $\mathscr{H}$ is a regular 1-CFT.

Let X be a closed complex curve (= conformal surface). A *stringy bundle* B on X consists of the following data:

1. A (finite) discrete set S of points on X, called punctures.
2. For every holomorphic embedding $h : A \to X$ where A is a genus 0 rigged surface with k outbound and one inbound boundary component with $h(\partial A) \cap S = \emptyset$, a map

$$U_h : \mathscr{V}^{\times k} \to \mathscr{V}$$

   coming from a projective operator $\mathscr{H}^{\otimes k} \to \mathscr{H}$. These maps are compatible under gluing of A, and continuous with respect to the analytic topology on the space of embeddings. In more detail, by compatibility under gluing we mean that if $h_1, \ldots, h_k : A_1, \ldots, A_k \to X$, $h_i(\partial A_i) \cap S = \emptyset$, are holomorphic embeddings which glue with h to an embedding $j : B \to X$ (B is obtained by gluing $A_1, \ldots, A_k, A$), then

$$U_j = U_h \circ (U_{h_1}, \ldots, U_{h_k}).$$

3. If $h(A) \cap S = \emptyset$, then $U_h = U_A$, the vacuum operator coming from the conformal field theory structure on $\mathscr{H}$.

Clearly, it suffices to specify elements

$$U_h \in \mathscr{V}$$

for holomorphic embeddings $h : D \to X$ where D is the unit disk and $h(D)$ contains exactly one puncture.

Two $\mathscr{H}$-stringy bundles on X are considered equal if they coincide upon enlarging the finite set of punctures S. A topology on the space $\tilde{B}_X\mathscr{H}$ of $\mathscr{H}$-stringy bundles on X is given as follows:

Let, for a conformal surface X (not necessarily compact), $L^0(X)$ denote the space of analytic Jordan curves in X (analytic injective maps $S^1 \to X$). On $L^0(X)$, we consider the analytic topology.

Now choose a compact set $K \subset L^0(X)$ and an open set $\mathscr{U} \subset \mathscr{V}$. Let $I(K, \mathscr{U})$ denote the set of stringy bundles on X which can be written as (S, U_h) for a finite set of punctures S such that

$$K \subset L^0(X - S) \quad \text{and} \quad U_h \in \mathscr{U} \text{ for } h \in K.$$

Then we let all the sets $I(K, \mathscr{U})$ form a subbasis of the space $\tilde{B}_X \mathscr{H}$.

Proposition 8. *There is a canonical homotopy class of maps* $p_1 : \tilde{B}_X \mathscr{H} \to K(\mathbb{Z}, 4)$ (*which we will call the first Pontrjagin class*).

The idea is that $\tilde{B}_X \mathscr{H}$ can be thought of as a space of particles decorated by $\mathscr{V} \simeq \mathbb{C}P^\infty$; the particles collide according to the 1-CFT structure. To get the Pontrjagin class, we look at the "relative" space of particles, suppressing particles which are outside a fixed disk in X. This relative space is $B^2\mathbb{C}P^\infty$. The details of this proof will be given in the appendix.

Now a *map of stringy bundles* (or *stringy homomorphism*) $(S, U_h) \to (S', U'_h)$ consists of the following data:

1. A set of punctures T in X which contains both S and S', and for each Jordan curve c in $X - T$, a projective map

   $$\phi_c : \mathscr{V} \to \mathscr{V}$$

   (i.e. such that ϕ_c is induced by a bounded linear map $\mathscr{H} \to \mathscr{H}$); ϕ_c is required to depend on c continuously.
2. If $h : A \to X$ is a holomorphic embedding of a genus 0 rigged surface A with inbound boundary components $c_1, \dots, c_k$ and outbound boundary component d, $h(\partial A) \cap T = \emptyset$, then

   $$U'_h \circ (\phi_{c_1}, \dots, \phi_{c_k}) = \phi_d \circ U_h,$$

   where U_A is the vacuum vector interpreted as a map $\mathscr{H}^{\otimes k} \to \mathscr{H}^{\otimes \ell}$ defined up to scalar multiple.

Two maps of $\mathscr{H}$-stringy bundles are considered the same if they coincide for some choice of punctures T (possibly larger than the original choices). We have a topology on the set of all maps of stringy bundles, which is similar to the topology on the set of stringy bundles:

Let $L(X)$ be the space of all analytic Jordan curves in X, with analytic topology. Choose a compact set $Q \subset L(X)$ and an open set $\mathscr{W} \subset Map(\mathscr{V}, \mathscr{V})$. Let $J(Q, \mathscr{W})$ consist of all maps of string bundles which can be written as (ϕ, T), $\phi : (S, B) \to (S', B')$, $T \supset S \coprod S'$ such that $K \subset L(X - T)$ and $\phi_c \in \mathscr{W}$ for $c \in Q$.

We let the sets $J(Q,\mathscr{W})$, along with the subbasis sets $I(K,\mathscr{U})$ for the spaces of source and target stringy bundles, be the subbasis of topology on the space of all maps of $\mathscr{H}$-stringy bundles on X. Note that the maps, as well as their sources and targets, are allowed to vary.

Let $E_\tau = \mathbb{C}/\langle 1,\tau\rangle$ be an elliptic curve. Then E_τ acts on itself holomorphically by translation. An *elliptic bundle* on E_τ is a stringy bundle B on E_τ together with maps

$$\phi_e : e^*B \to B, \quad e \in E_\tau. \tag{114}$$

Here e^* denotes the stringy bundle induced by composition with the given holomorphic self-map of E_τ in the obvious sense. We require that (114) be a continuous map from E_τ to the space of maps of stringy bundles, and that

$$\phi_{e_2} e_2^*(\phi_{e_1}) = \phi_{e_2 e_1} \tag{115}$$

(compatibility under composition). Furthermore, we shall require a 'positive energy condition' on an elliptic bundle (S,B) on E_τ. To formulate this condition, consider a compact 1-parametric subgroup $S^1 \cong T \subset E_\tau$. This 1-parametric subgroup can also be considered as a Jordan curve, which we will denote by c. Then

$$(\phi_e)_c \circ U_e : \mathscr{V} \to \mathscr{V}, \quad e \in T \tag{116}$$

specifies a projective action of S^1 on $\mathscr{H}$. Here U_e is the limit of U_A as A tends to an infinitesimally thin annulus with one inbound and one outbound boundary component, where the boundary parametrizations are linear and differ by a rigid rotation by e (using the identification isomorphism $S^1 \cong T$). Recall (Remark 2 of Section 2) that we assume as a part of the definition of 1-CFT that such operators exist and are bounded. To see that (116) indeed specifies a projective action, let $e,f \in T$. Then the fact that ϕ_e is a stringy isomorphism gives

$$(\phi_e)_c U_f = U_f(\phi_e)_{c-f}.$$

(By $c-f$ we mean the Jordan curve c shifted by $-f$.) Now compute

$$(\phi_e)_c U_e (\phi_f)_c U_f = (\phi_e)_c (\phi_f)_{c+e} U_e U_f = (\phi_{e+f})_c U_{e+f}.$$

The last equality follows from (115).

Now the positive energy condition states that this action is induced by a genuine action of T on $\mathscr{H}$, whose weight spaces are finite-dimensional, and the weights are non-negative, with bottom weight space one-dimensional.

A morphism of $\mathscr{H}$-elliptic bundles on E_τ is a map of stringy bundles compatible with the translation maps. Also, note that by methods analogous with above we get a canonical topology on the space of all $\mathscr{H}$-elliptic bundles on E_τ. This space will be denoted by

$$B_{E_\tau}\mathscr{H}.$$

In a variant, we can also take a union of those spaces over $\tau \in \mathbb{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$ with the obvious topology. We shall call the resulting space

$$B_{\text{ell}}\mathscr{H}.$$

Then Proposition 8 gives a canonical map

$$p_1 : B_{\text{ell}}\mathscr{H} \to K(\mathbb{Z}, 4)$$

(the first Pontrjagin class).

Now let Φ be a set of representatives of elements of a set of isomorphism classes of 1-CFTs. We assume that $\otimes$ is defined and is strictly associative on Φ. This is not a big assumption, as in all the examples considered in this paper in Sections 6 and 7 below (see in particular the Remark at the end of Section 6), Φ is just a set of the form

$$\{\mathscr{H}_1^{\hat{\otimes} n_1} \hat{\otimes} \cdots \hat{\otimes} \mathscr{H}_k^{\hat{\otimes} n_k} \mid n_1, \ldots, n_k \in \mathbb{N}\} \tag{117}$$

for some fixed 1-CFTs $\mathscr{H}_1, \ldots, \mathscr{H}_k$, we can just use given representatives of elements (117), and redefine tensor products of elements of (117) using the coherences of $\hat{\otimes}$ to achieve strict associativity. Then we have canonical maps

$$\otimes : B_{\text{ell}}\mathscr{H}_1 \times B_{\text{ell}}\mathscr{H}_2 \to B_{\text{ell}}\mathscr{H}_1 \hat{\otimes} \mathscr{H}_2.$$

Thus, if we put

$$B_{\text{ell}}\Phi = \coprod_{\mathscr{H} \in \Phi} B_{\text{ell}}\mathscr{H},$$

then $B_{\text{ell}}\Phi$ is a strictly associative unital H-space with respect to $\otimes$. We let

$$\tilde{\mathscr{E}} = \Omega B(B_{\text{ell}}\Phi).$$

There will be two distinguished cohomology classes on $\tilde{\mathscr{E}}$. First, each $\mathscr{H}$ comes with a central charge $c_{\mathscr{H}} \in \mathbb{Q}$. Letting $A \subset \mathbb{Q}$ be the additive subgroup generated by all $c_{\mathscr{H}}$ with $\mathscr{H} \in \Phi$, we get a map of (1-fold) loop spaces

$$c : \tilde{\mathscr{E}} \to K(A, 0) = A. \tag{118}$$

We denote by

$$\bar{\mathscr{E}}$$

the homotopy fiber of (118). Next, the Pontrjagin class is additive with respect to $\otimes$, and hence we get a map

$$p_1 : \bar{\mathscr{E}} \to K(\mathbb{Z}, 4). \tag{119}$$

Let

$$\mathscr{E}$$

be the fiber of (119). We shall pick certain classes

$$\omega \in \pi_{2k} \Sigma^{\infty} \mathscr{E}_{+} \tag{120}$$

(for some k) and define

$$E = \omega^{-1} \Sigma^{\infty} \mathscr{E}_{+}. \tag{121}$$

To explain this notation, note that $\Sigma^{\infty}\mathscr{E}_{+}$ is an A_{∞} ring spectrum. Thus, multiplication by (120) defines a self-map

$$\Sigma^{2k} \Sigma^{\infty} \mathscr{E}_{+} \to \Sigma^{\infty} \mathscr{E}_{+}. \tag{122}$$

Then (121) is defined as the telescope of the map (122).

(A reader in need of a quick introduction to homotopy theory is referred to [28].)

For motivation, we look at the following classical example. Consider the inclusion

$$h : S^2 = \mathbb{C}P^1 \subset \mathbb{C}P^{\infty}.$$

Then h specifies a homotopy class

$$\beta \in \pi_2 \Sigma^{\infty} \mathbb{C}P^{\infty}_{+}.$$

Since $\Sigma^{\infty}\mathbb{C}P^{\infty}_{+}$ is an E_{∞} ring spectrum, $\beta^{-1}\Sigma^{\infty}\mathbb{C}P^{\infty}_{+}$ is well defined. Furthermore, however, if we take the map

$$\mathbb{C}P^{\infty} \to K \tag{123}$$

given by the identical representation of S^1, then (123) maps β to the Bott class, and hence induces a map

$$\phi : \beta^{-1} \Sigma^{\infty} \mathbb{C}P^{\infty}_{+} \to K. \tag{124}$$

Then we have the following well known result [39,40].

Proposition 9. *The map ϕ of* (124) *is an equivalence.*

We proceed to define a map

$$\tilde{\mathscr{E}} \to K[[q]] \tag{125}$$

which induces a map

$$\gamma : E \to K[[q]][q^{-1}],$$

as desired.

Let (S, B) be an elliptic bundle on E_τ. Let, for $\lambda \in [0, 1]$, c_λ be a Jordan curve in E_τ of the form

$$e^{2\pi i t} \mapsto \lambda\tau + t.$$

Now the action of $S^1 \cong T = [0,1] \subset E_\tau$ by $(\phi_e)_{c_\lambda} \circ U_e$, $e \in T$, specifies an S^1-action on $\mathscr{H}$. (Recall (114) for the definition of ϕ_e.) Moreover, if

$$0 < \lambda_0 < \cdots < \lambda_n < 1, \tag{126}$$

we have, by (115)

$$(\phi_{\lambda_1 - \lambda_0})_{c_{\lambda_0}} \dots (\phi_{\lambda_n - \lambda_{n-1}})_{c_{\lambda_{n-1}}} (\phi_{\lambda_0 - \lambda_n})_{c_{\lambda_n}} = Id.$$

Thus, choosing (126), B specifies an element

$$((\phi_{\lambda_1 - \lambda_0})_{c_{\lambda_0}} | \dots | (\phi_{\lambda_n - \lambda_{n-1}})_{c_{\lambda_{n-1}}}) \in B_n \prod_{m \geqslant 0} \widetilde{GL}(dim(\mathscr{H}(m)))$$

where $\mathscr{H}(n)$ is the weight decomposition of $\mathscr{H}$ with respect to the T-action (in particular, we have assumed $dim(\mathscr{H}(n)) < \infty$), and $\widetilde{GL}(k)$ is the category of $\mathbb{C}$-vector spaces of dimension k and their isomorphisms.

Now the choice of (126) is arbitrary (including the number n), subject to the condition that no puncture lie on any $\lambda_i\tau + [0, 1]$. Thus, we see that the set of possible choices of (126) is directed under the ordering by inclusion, and hence its simplicial realization is contractible. Roughly speaking, this gives a map

$$B^B_{\mathrm{ell}}\mathscr{H} \to B\left(\left(\prod_{n \geqslant 0} \widetilde{GL}(dim(\mathscr{H}(n)))\right) \Big/ \mathbb{C}^\times\right) \tag{127}$$

where $B^B_{\mathrm{ell}}\mathscr{H} \subset B_{\mathrm{ell}}\mathscr{H}$ is the connected component containing the stringy bundle B. Taking the quotient by $\mathbb{C}^\times$ comes from the fact that the maps ϕ_e are only determined up to scalar multiple in our setup.

More precisely, to construct map (127), we must consider an intermediate object which contains information on both punctures and the points (126) as parts of its data. The correct object is a simplicial space whose nth stage consists of an $(n+1)$-tuple (126) along with an elliptic bundle on E_τ which has a representative with no punctures on the curves $c_{\lambda_0}, \dots, c_{\lambda_n}$. Let $\bar{B}^B_{\mathrm{ell}}(\mathscr{H})$ be the simplicial realization of this simplicial space. Then our construction gives a variant of the map (127) with source $\bar{B}^B_{\mathrm{ell}}(\mathscr{H})$. Along with this, we obtain a projection

$$\bar{B}^B_{\mathrm{ell}}(\mathscr{H}) \to B_{\mathrm{ell}}(\mathscr{H}). \tag{128}$$

We need to show that (128) is an equivalence. Clearly the fibers of (128) are contractible. In addition to this, one shows that (128) is a quasifibration. This can be done using the Dold–Thom criterion; the kth stratum is the closed subset of $\bar{B}_{\mathrm{ell}}(\mathscr{H})$ consisting of bundles which can be written as (S, B) with $|S| \leqslant k$.

Now put

$$\widetilde{GL}(\infty) = \varinjlim \widetilde{GL}(k).$$

Then stabilizing, we clearly get a multiplicative map

$$B_{\mathrm{ell}}\Phi \to B\left(\left(\prod_{n\geqslant 0} \widetilde{GL}(\infty)\right)/\mathbb{C}^\times\right) \simeq B(U[[q]]/S^1). \tag{129}$$

Here $U[[q]] = \prod_{n=0}^\infty U$ where the tuples $(u_0, u_1, \ldots) \in \prod_{n=0}^\infty U$ are written as $u_0 + u_1 q + u_2 q^2 + \cdots$ Since we assumed $dim\, \mathscr{H}(0) = 1$ for $\mathscr{H} \in \Phi$, $\pi_0 B_{\mathrm{ell}}\Phi$ is $\otimes$-invertible in the target of (129), and thus we get a map

$$\tilde{\mathscr{E}} \to B(U[[q]]/S^1).$$

When inverting the Bott class, the projective factor disappears, and we get the map (125).

In more detail, the Bott element

$$\beta \in \pi_2 B(U[[q]]/S^1)$$

(induced from the usual element of $\pi_2 BU[[q]]$ which is a product of the classical Bott elements of $\pi_2 BU$) gives a map

$$B(U[[q]]/S^1) \to \Omega^2 B(U[[q]]/S^1). \tag{130}$$

This gives rise to a generalized cohomology theory, but note that this theory is just $K[[q]]$, since

$$\Omega^4 B(U[[q]]) \to \Omega^4 B(U[[q]]/S^1)$$

is an equivalence.

Comment. This construction of map (125) was at the root of the motivation of our definition of $B_{\mathrm{ell}}(\mathscr{H})$. Consider the group $\mathscr{G}$ of invertible maps $\mathscr{H} \to \mathscr{H}$. Then $\mathscr{G}$ is S^1-equivariant (by conjugation), and to get $\prod Aut(\mathscr{H}(n))$ from $\mathscr{G}$, we apply S^1-fixed points. Then we take the bar construction on the resulting group to get, roughly, a model of (a part of) K-theory. One can attempt to model this construction by first forming the cyclic bar construction (Hochschild homology) of $\mathcal{G}$, and then applying $S^1 \times S^1$- (or E-) fixed points (recall that the cyclic bar construction is a model of $LB\mathcal{G}$). The definition of $B_{\text{ell}}(\mathcal{H})$ came from searching for a modular-invariant construction which would map into $LB\mathcal{G}$. Note, however, that the $LB\mathcal{G}$ approach can be taken only metaphorically, and not literally, since Hochschild homology does not provide a correct model for the S^1-fixed points of $LB\mathcal{G}$.

Note also that the construction $B_{\text{ell}}\mathcal{H}$ makes sense for a 0-CFT $\mathcal{H}$, in fact even for a directed 0-CFT. However, for a 1-CFT with modular group $\Gamma \subset PSL_2\mathbb{Z}$, the entire structure of $\mathcal{H}$ is invariant under the action of Γ, which is what we mean by saying that the construction of $B_{\text{ell}}\mathcal{H}$ is *manifestly modular*. We could ask if this automatically implies that the image of the map γ constructed above is contained in the ring of Γ-modular forms. This, however, is not so simple. The difficulty is that the target of the map γ is itself not Γ-invariant, so we cannot use a simple transport of structure argument. We do not, in fact, have a theorem of this nature in general, although in the examples we shall compute in the subsequent sections, the conclusion holds (see also the Comment under the Conjecture in the next section).

The attentive reader has noticed that in order to give a concrete example of our elliptic cohomology theory E, we must make a choice of Φ, and of the element ω. Such choice depends on calculations, which are the content of the remainder of this paper (excluding the appendix).

6. Theta elements

In this section, we shall construct first examples of elements of $\pi_* B_{\text{ell}}\mathcal{H}$, giving rise to first examples of E. Let $\mathcal{H}$ be a 1-CFT constructed from an even lattice L. Assume there is an element $\alpha \in L$ be such that $\langle \alpha, \alpha \rangle = 2$. Then we have a map

$$\psi_\alpha : \mathcal{H}_1 \to \mathcal{H},$$

where $\mathcal{H}_1$ is the 1-CFT which is the basic representation of $\widetilde{LSU(2)}$. (In the language of Section 3, we can consider $\mathcal{H}_1$ as the 1-CFT associated with the root lattice of $SU(2)$. While not using the language of CFT, a considerable amount of information about $\mathcal{H}_1$ can be found in [33].) The main point is that the loop group $LSU(2)$ acts projectively on $\mathcal{H}_1$ and the action extends, in fact, to an action on $\mathcal{H}$ (see Chapter 13 of [33]). We shall now construct a map

$$\Xi : \mathbb{C}P^\infty \to B_{\text{ell}}\mathcal{H}. \tag{131}$$

Identify $\mathbb{C}P^\infty$ with the space C of principal divisors on E_τ. (This is a standard fact, which follows for example from the fact that the space of all divisors is the free abelian group on E_τ—see the appendix for more on this.) Then for a divisor $D \in C$, we obtain a stringy bundle B^D by choosing a function f with divisor D, letting the set of punctures be $|D|$, and letting for a Jordan curve c in $E_\tau - |D|$ bounding a rigged disk in E_τ, B_c^D be the image of the vacuum under $f|_c \in LS^1$, which is embedded in $LSU(2)$ by a fixed embedding of a maximal torus

$$S^1 \subset SU(2). \tag{132}$$

Note that the function f is only determined up to scalar multiple, but the choice does not matter, since multiplication by a constant loop preserves the vacuum.

Note, further, that the restriction of Ξ to $\mathbb{C}P^2$ canonically factors through S^4:

$$\begin{array}{ccccc} \mathbb{C}P^2 & \longrightarrow & \mathbb{C}P^\infty & \xrightarrow{\Xi} & B_{ell}\mathscr{H} \\ & \searrow & & \nearrow_{\Lambda} & \\ & & S^4. & & \end{array} \tag{133}$$

Here the map $\mathbb{C}P^2 \to S^4$ is of degree 1, i.e. collapse to the top cell. To see this, consider $S^2 \subset \mathbb{C}P^\infty$ represented by

$$[0] - [\lambda] - [\mu\tau] + [\lambda + \mu\tau], \quad \lambda, \mu \in [0, 1].$$

Then the restriction of Ξ to S^2, by varying the maximal torus (132), extends to a map

$$\Lambda : S^4 \to B_{\text{ell}}\mathscr{H}$$

which is easily seen to satisfy (133) (by considering the space of $SU(2)$-bundles on E_τ).

It is natural to conjecture that Ξ factors through a map $BSU(2) \to B_{\text{ell}}\mathscr{H}$. This is indeed true, as we shall prove in the next section. In fact, more generally, we have

Conjecture 1. *For every regular* 1-*CFT* $\mathscr{H}$, *there is a canonical map*

$$\alpha : B\,Aut(\mathscr{H}) \to B_{\text{ell}}\mathscr{H}$$

such that the diagram

$$\begin{array}{ccc} BAut(\mathscr{H}) & \xrightarrow{\alpha} & B_{ell}\mathscr{H} \\ & {}_{\tau}\searrow \quad \swarrow_{\gamma\beta} & \\ & K[[q]] & \end{array}$$

commutes, where τ *is the character map* (*note that an automorphism of* $\mathscr{H}$ *preserves grading*).

In Section 7, we prove this conjecture in the case when $\mathscr{H} = \mathscr{H}_G$ is the 1-conformal field theory associated with the root lattice of a simply connected simply laced compact Lie group G.

One can go somewhat further with this conjecture. Denote by $Str(1)$ the space of stringy isomorphisms $1 \to \eta$ (for some η). Clearly, any such stringy iso gives rise to an elliptic bundle, an automorphisms of $\mathscr{H}$ give rise to stringy isos $1 \to 1$, and trivial elliptic bundles. We have, therefore, a map

$$\alpha' : Str(1)/Aut(\mathscr{H}) \to B_{\mathrm{ell}}\mathscr{H},$$

and one could ask if $Str(1)$ is always contractible. That seems to be a difficult question in general, but the results of next section will be obtained by finding suitable contractible subspaces of $Str(1)$.

Comment. The above discussion points out the need for a method of constructing stringy isos $1 \to \eta$, i.e. elements of $Str(1)$. We point out that there exists a general approach to obtaining maps which have the properties of stringy isos at least locally (although we merely outline the construction and shall not discuss convergence). To this end, we need to recall the language of vertex operator algebras. Let

$$V = \bigoplus_{n \in \mathbb{N}} \mathscr{H}(n),$$

where $\mathscr{H}(n)$ is the summand of $\mathscr{H}$ of vectors of energy n (i.e. on which rigid rotations act by $q \mapsto q^n$). We then assume that the 0-CFT structure on $\mathscr{H}$ is, indeed obtained from a vertex operator algebra structure on V (see comments at the conclusion of Section 2). This means that for $v \in V$, there is assigned a vertex operator

$$Y(v,z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \in Hom(V,V)[[z,z^{-1}]]$$

with certain properties (see [13]). Set

$$Y_+(v,z) = \sum_{n \in \mathbb{N}}^{v} z^{-n-1}, \quad Y_-(v,z) = Y(v,z) - Y_+(v,z).$$

Recall our setup from Section 2: $0 < ||z|| < 1$, and $0 < ||\lambda||, ||\mu|| < 1$ such that $B_\lambda(0) \cap B_\mu(z) = \emptyset$, $B_\lambda(0) \cup B_\mu(z) \subset interior(D)$ (we set $B_\varepsilon(p) = \{x \in \mathbb{C} \mid ||x-p|| < \varepsilon\}$). Recall that $A = A_{\lambda,\mu,z}$ is the closure of $D - (B_\lambda(0) \cup B_\mu(z))$ with boundary components parametrized by $1 : S^1 \to S^1, \lambda \cdot 1, \mu \cdot 1$. Let also $\lambda_* : V \to V$ be defined as λ^n on $\mathscr{H}(n)$. We denote by $(?)^\lambda$ conjugation by $\lambda_* : \phi^\lambda = \lambda_*^{-1} \phi \lambda_*$. Then we have the formula

$$U_A((1 + Y_+(v,z)^{\lambda^{-1}}\, dt)u, 1 + \mu_* v\, dt) = (1 + Y_-(v,z)\, dt)(\lambda_*^{-1} u). \tag{134}$$

This means that we expand both sides of (134) in dt, and both sides of (134) are required to agree up to linear terms in dt. Formula (134) can be interpreted as saying that vertex operators incorporate renormalization of CFT structure.

Anyway, (134) has a generalization: we have

$$U_A((1+Y_+(v,r)^{\lambda^{-1}}\,dt)u,(1+Y_-(v,r-z)^{\mu}\,dt)w)=(1+Y_-(v,z)\,dt)U_A(u,w) \quad (135)$$

(in (134), $w=1$). Analogously, if $||z||>1$,

$$U_A((1+Y_+(v,r)^{\lambda^{-1}}\,dt)u,(1+Y_+(v,r-z)^{\mu}\,dt)w)=(1+Y_+(v,z)\,dt)U_A(u,w), \quad (136)$$

and (135), (136) can be readily generalized, at least up to scalar multiple, to any pair of pants A with general boundary parametrizations, by conjugating by elements of $Diff^+S^1$.

We see from (136) that if $||z||>1$, then granted appropriate convergence, a stringy iso $1\to\eta$ on D (without puncture) can be defined on a Jordan curve c in $interior(D)$ as

$$e^{Y_+(v,z)^{A_c}}, \quad (137)$$

where A_c is the rigged annulus with boundary S^1, c. (We set, as usual, $e^f=\sum_{n\in\mathbb{N}}(1/n!)f^n$, where the exponent denotes composition.) Similarly, for $||z||<1$, we can define (granted convergence) a stringy iso $1\to\eta$ with puncture at z by (137) when $ind_z(c)=0$ and

$$e^{Y_-(v,z)^{A_c}} \quad (138)$$

when $ind_z(c)=1$.

It is notable that the actions of simply laced groups on the VOAs associated with their lattices, as well as the action the Monster on the Moonshine module, are constructed (essentially) by this method.

We would need to control convergence of formulas (137) and (138), and find a sufficiently large contractible space of compositions of operators (137), (138) which are meromorphic on E_τ in order to use this to develop an approach to Conjecture 1.

Remark. The previous comment points us in yet a new direction: It suggests the possibility of a completely *internal* (i.e. quantized) version of the concept of B_{ell}. From that point of view, in our present setting, the base of a stringy bundle is not quantized, while the fiber is: one could try to quantize both. At the present time, we can formulate these ideas only vaguely, and in the language of physics. We drop, therefore, at least for the purpose of this Remark, the rigorous standards applied elsewhere in this paper, and use the language of physics with liberty. Recall (cf. Zoumolodchikov–Fateev [46]) that there is, at least in physics, a notion of moduli space of CFTs; one can start trying to understand such moduli spaces by means of local deformation of conformal field theories, using truly marginal operators, or (1,1)-fields [16]. Note that the type of conformal field theories $\mathscr{H}$ we are considering is chiral (holomorphic), and hence their moduli space would be discrete.

However, this is not the kind of moduli space we see. Instead, the fully quantized version of $B_{\mathrm{ell}}\mathscr{H}$ should be a moduli space of *bundles of CFTs* with fiber isomorphic to $\mathscr{H}$ over σ-models of elliptic curves. Recall that the moduli space of σ-models of elliptic curves fully reflects the geometry of the moduli space of elliptic curves, (it has the same tangent space), and in fact gives its compactification.

What is, however, a bundle of CFTs over another CFT? Roughly speaking, a bundle of CFTs over a base CFT V should be given by a Hilbert basis e_i, $i \in I$, of V, and Hilbert spaces $\mathscr{H}_i$, $i \in I$. For a rigged surface X with k inbound and ℓ outbound boundary components, we would be given a matrix

$$(u_{i_1,\dots,i_k,j_1,\dots,j_\ell})_{i_1,\dots,i_k,j_1,\dots,j_\ell \in I}$$

where

$$u_{i_1,\cdots,i_k,j_1,\dots,j_\ell} \in \mathscr{H}_{i_1} \otimes \cdots \otimes \mathscr{H}_{i_k} \otimes \overline{\mathscr{H}_{j_1}} \otimes \cdots \otimes \overline{\mathscr{H}_{j_\ell}}.$$

Of course, suitable axioms are needed.

But one can see how, in this (at present unrigorous) language, a tensor product of CFTs is a special case of bundle of CFTs, and, moreover, one can see that an $\mathscr{H}$-stringy isomorphism meromorphic on an elliptic curve E should give a bundle of CFTs over the σ-model V of E which is a deformation of $V \hat{\otimes} \mathscr{H}$.

Knowing at least what the trivial bundle is, we could try a "perturbative approach", i.e. look for local data which would allow an infinitesimal deformation of the trivial CFT bundle $V \hat{\otimes} \mathscr{H}$ (we mean, again, a deformation in the direction of the bundle structure, not a deformation of the base). In the case of the σ-model V of E, the above comments suggest that before quantizing E, the data we need is essentially a map

$$LE \to \mathscr{H} \tag{139}$$

which would be equivariant under the semigroup $\mathscr{E}$ of holomorphic embeddings $f : D \to D$, $f(0) = 0$ (D is the unit disk). Thus, the quantized version of (139), and therefore an infinitesimal deformation of the trivial quantized stringy bundle, should be a map of $\mathscr{E}$-representations

$$V \to \mathscr{H}.$$

Note that this is potentially a much more far reaching approach, since instead of σ-models of elliptic curves, we could consider moduli spaces of σ-models of Calabi–Yau varieties [16], and hence we can speak of

$$B_{k-CY}\mathscr{H},$$

which is the (coarse) moduli space of 2-CFTs with fiber $\mathscr{H}$ over σ-models of k-dimensional Calabi–Yau varieties.

However, let us return to the theta elements, and to the rigorous discussion prior to the Remark and the Comment. Let $\Xi_\alpha = \Psi_\alpha \Xi$, $\Lambda_\alpha = \Psi_\alpha \Lambda$ (recall (131)). Now recall the theta series

$$\theta_L(\tau, u) = \sum_{x \in L} q^{\frac{1}{2}||x||^2} e^{2\pi i \langle u, x \rangle}$$

for $u \in L_{\mathbb{C}}$. As usual, $q = e^{2\pi i \tau}$, and for future reference also $z = e^{2\pi i u}$. Then the partition function of $\mathscr{H}$ satisfies

$$Z_{\mathscr{H}}(\tau) = tr U_{\mathscr{H}}(A_\tau) = q^{c/24} \eta(\tau)^{-c} \theta(\tau, 0),$$

where $c = rank(L)$ (cf. [2,13]).

Now consider the map we constructed

$$\Gamma : B_{\text{ell}} \mathscr{H} \to K[[q]].$$

Recall that

$$K^* \mathbb{C}P^\infty = k^*[[z-1]], \tag{140}$$

where z stands for the identical representation of S^1.

Proposition 10. *Under correspondence* (140), *the element*

$$\Gamma \Xi_\alpha \in K[[q]]^* \mathbb{C}P^\infty$$

corresponds to

$$q^{c/24} \eta(\tau)^{-c} \theta_L(\tau, u\alpha). \tag{141}$$

Proof. For this purpose, we start with a slightly different model of trivial bundles ξ on E_τ. Consider a sequence (126). For $\lambda_1 < \lambda_2$, $0 < \lambda_2 - \lambda_1 < 1$, consider the annulus

$$A_{\lambda_1, \lambda_2} = \{\lambda \tau + [0,1] \mid \lambda_1 \leqslant \lambda \leqslant \lambda_2\} \subset E_\tau.$$

We will assume that the bundle is trivial on each $A_{\lambda_{i-1}, \lambda_i}$ $(i = 0, \ldots, n,\ \lambda_{-1} = \lambda_n)$, and that *constant* transition functions z_i are given from sections on $A_{\lambda_{i-1}, \lambda_i}$ to $A_{\lambda_i, \lambda_{i+1}}$. This means that for an analytic function on ξ given by a function on $A_{\lambda_{i-1}, \lambda_i}$, to be analytically continued to $A_{\lambda_i, \lambda_{i+1}}$, it must be multiplied by z_i on $A_{\lambda_i, \lambda_{i+1}}$. The triviality of ξ requires that

$$\prod_{i=0}^{n} z_i = 1.$$

Then the space of all possible choices of the z_i's for all possible sequences (126) is the bar construction

$$B\mathbb{C}^\times \simeq \mathbb{C}P^\infty.$$

Further, for these bundles, we get *canonical* identification of ($\mathscr{H}$, with canonical rotation action) with ($\mathscr{H}$, with action induced by the equivariance of ξ): the identity. Now the function $(\phi_{\lambda_{i-1},\lambda_i})_{c_{\lambda_i}}$ in this setting is simply multiplication by constant loops, the character of which are well known to be the theta functions (see [33]).

Now to link the space $\mathscr{B}$ of trivial bundles on E_τ in this sense with the space $\mathbb{Z}_0[E_\tau]$ of principal divisors, introduce a space $\tilde{\mathbb{Z}}_0[E_\tau]$ which maps into both. The space

$$\tilde{\mathbb{Z}}_0[E_\tau]$$

consists of choices (126), and (finite) degree 0 divisors D_i on $Int(A_{\lambda_{i-1},\lambda_i})$ for each $i = 0, \dots, n$ such that

$$\sum \varepsilon(D_i) = 0,$$

where ε is augmentation to the covering group of E_τ corresponding to $\pi_1(T)$ where T is the image of [0, 1] in E_τ. Then we have an obvious forgetful map

$$\tilde{\mathbb{Z}}_0[E_\tau] \to \mathbb{Z}_0[E_\tau],$$

which is an equivalence (by comparison of homotopy types). On the other hand, there is a map

$$\tilde{\mathbb{Z}}_0[E_\tau] \to \mathscr{B}$$

by making the transition function at $\lambda_i\tau + [0,1]\varepsilon(D_i)$. Now we see that on $\tilde{\mathbb{Z}}_0[E_\tau]$, the map to $B((\prod \widetilde{GL}(dim(\mathscr{H}(n))))/\mathbb{C}^\times)$, defined via projections to $\mathbb{Z}_0[E_\tau]$, $\mathscr{B}$ coincide.

In more detail, to identify the S^1-equivariant Hilbert spaces involved in the two bar constructions, normalize the elliptic function f with divisor $\sum D_i$ so that it is 1 at the point $\lambda_n\tau$. At $\lambda_i\tau + [0,1]$, multiply this by the actions of the constant loop $z_0 \cdot \dots \cdot z_i$. To see that the constant loop actions correspond to the actions given by the elliptic function f, note that there is a holomorphic function h on $A_{\lambda_{i-1},\lambda_i}$ such that

$$\frac{f(u + \lambda_i - \lambda_{i-1})h(u + \lambda_i - \lambda_{i-1})}{f(u)h(u)} = z_i$$

for $u \in \lambda_{i-1}\tau + [0,1]$. $\square$

Rationalizing, we get

$$\mathbb{C}P^\infty_{\mathbb{Q}} = \bigvee_{n \geqslant 0} S^{2n}_{\mathbb{Q}}.$$

The images of the cells can be detected by taking the Chern character. This corresponds to taking the logarithm of z in (141) (i.e. considering (141) as a function of u). Thus, we have proven

Proposition 11. *The image of* $S_{\mathbb{Q}}^{2n}$ *in* $K^{2n}[[q]]$ *under* $(\Gamma\Theta_\alpha)_{\mathbb{Q}}$ *is the coefficient of* (141) *at* u^n. $\square$

Now suppose we have lattices $L_1, \ldots, L_k, M_1, \ldots, M_k$ where

$$\sum_{i=1}^{k} rank(L_i) = \sum_{i=1}^{k} rank(M_i), \tag{142}$$

and suppose $\alpha_i \in L_i, \beta_i \in M_i$ satisfy

$$||\alpha_i||^2 = ||\beta_i||^2 = 2.$$

We want to consider the element

$$\Lambda_{\beta_1,\ldots,\beta_k}^{\alpha_1,\ldots,\alpha_k} = \frac{\Lambda_{\alpha_1} \cdot \ldots \cdot \Lambda_{\alpha_k}}{\Lambda_{\beta_1} \cdot \ldots \cdot \Lambda_{\beta_k}} \in \pi_4(\mathscr{E}). \tag{143}$$

Note that by (142), element (143) has central charge 0. Note also that the α_i's and β_i's obviously all have the same Pontrjagin class, and hence element (143) has Pontrjagin class 0. The product in (143) is the loop product in $\tilde{\mathscr{E}}$.

Now by Proposition 11, the image of (143) in $K^4[[q]]$ is the coefficient at u^2 in

$$\phi_{\beta_1,\ldots,\beta_k}^{\alpha_1,\ldots,\alpha_k} = \frac{\prod_{i=1}^{k} \theta_{L_i}(\tau, u\alpha_i)}{\prod_{i=1}^{k} \theta_{M_i}(\tau, u\beta_i)}. \tag{144}$$

Now let $\Gamma_m \subset PSL_2\mathbb{Z}$ be the subgroup generated by the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and the subgroup $\Gamma(m)SL_2\mathbb{Z}$ of matrices congruent to $Id \bmod m$, for $m \in \mathbb{N}$.

To recall the modularity properties of function (144) we review some basic facts about θ-functions of lattices [30,31,38]. This is a very special case of the theory of Siegel modular forms [30], in fact, in some sense, the "trivial" case, i.e. one which reduces to the classical context. By a theta series associated with L we mean a holomorphic function f: $H \times L_{\mathbb{C}} \to \mathbb{C}$ $(H = \{z \in \mathbb{C} \mid Im(z) > 0\})$ such that

$$f(\tau, z+\alpha) = f(\tau, z) \quad \alpha \in L,$$

$$f(\tau, z+\tau\alpha) = f(\tau, z) exp\left(-\tfrac{1}{2}\langle \alpha, \alpha \rangle \tau - \langle z, \alpha \rangle\right). \tag{145}$$

Then f determines a section of a certain holomorphic line bundle $\mathscr{L} = \mathscr{L}_\tau$ on the abelian variety

$$L_{\mathbb{C}}/L \oplus \tau L, \tag{146}$$

which is thereby defined. Note that while, of course, (146) is isomorphic to a product of copies of the elliptic curve E_τ, the line bundle $\mathscr{L}$ does depend on L. Let L' be the dual lattice of L. The space of sections $V = \Gamma(\mathscr{L})$, which is the $\mathbb{C}$-vector space of functions (145), has dimension $|L'/L|$ ($|?|$ denotes cardinality), and basis

$$\theta_\alpha(\tau, z) = \sum_{x \in L+\alpha} exp\left(\tfrac{1}{2}||x||^2\tau + \langle z, x\rangle\right), \quad \alpha \in L'/L. \tag{147}$$

The basic point when investigating the modularity of these theta series is that when we substitute τ by

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1,$$

then we obtain an isomorphic abelian variety

$$L_{\mathbb{C}}/L \oplus \tau' L.$$

The isomorphism ϕ: $z \in L_{\mathbb{C}}/L \oplus \tau L \mapsto z' \in L_{\mathbb{C}}/L \oplus \tau' L$ is given by

$$z' = \frac{z}{c\tau + d}. \tag{148}$$

The key point is that the iso ϕ of abelian varieties is also covered by an iso of line bundles

$$\mathscr{L}_\tau \to \mathscr{L}_{\tau'}.$$

To see this, one applies the "τ-scaled Fourier transformation"

$$f(\tau, z) \mapsto f(-1/\tau, z') \tag{149}$$

(see [38]). Let d be the discriminant of L, $n = rank(L)$. Then, for example,

$$\theta(\tau, z) = \frac{i^{n/2}}{\sqrt{d}\tau^{n/2}} \sum_{\alpha \in L'/L} \theta_\alpha(-1/\tau, z') exp(-\langle z, z\rangle/2\tau). \tag{150}$$

This shows that the function

$$exp(-\langle z, z\rangle/2\tau) \tag{151}$$

defines an iso $\iota: \mathscr{L}_\tau \to \phi^* \mathscr{L}_{\tau'}$ (covering the identity). Note that the desired iso is furthermore unique up to $\mathbb{C}^\times$-multiplication, and can be normalized, say, by saying that it be 1 at $z = 0$ (which (149) satisfies). Further, note that $\mathscr{L}_\tau$ is clearly identified with $\mathscr{L}_{\tau+1}$ (note that L is even), so we have, indeed, specified an action of the modular group $SL_2\mathbb{Z}$ on V, considered as a bundle over H (covering the standard action of $SL_2\mathbb{Z}$ on H).

There is another point of view which is also beneficial. The space V can be viewed as the Heisenberg representation of a finite Heisenberg group. Let A, $\bar{A}$ be two copies of L'/L. We let A act on V by

$$\alpha(f)(\tau, z) = f(\tau, z - \alpha), \quad \alpha \in A, \tag{152}$$

and $\bar{A}$ by

$$\iota\alpha(\iota^{-1}f). \tag{153}$$

Then the actions of A and $\bar{A}$ do not commute, but rather generate a central extension of $A \oplus \bar{A}$ by a finite cyclic group. Extending the kernel to $\mathbb{C}^\times$, we obtain a Heisenberg group G:

$$1 \to \mathbb{C}^\times \to G \to A \oplus \bar{A} \to 1, \tag{154}$$

where the kernel acts on V by multiplication. The subgroup A is maximal isotropic, and V is the Heisenberg representation. In fact, one may replace A by $L_{\mathbb{C}}$ and define a Heisenberg group

$$1 \to \mathbb{C}^\times \to \mathscr{G} \to L_{\mathbb{C}} \oplus \bar{L}_{\mathbb{C}} \to 1,$$

where the Heisenberg representation is the Hilbert space of rapidly decreasing holomorphic functions on $L_{\mathbb{C}}$ (see [31, Chapter 1]). Then $L \subset L_{\mathbb{C}}$ is an isotropic subgroup and $L^\perp = L'$; then V is the space of functions invariant under L and $\bar{L}$, which, by Mumford [31, Proposition 1.4], is the Heisenberg representation of G. However, being invariant under L, $\bar{L}$ turns out to be equivalent to conditions (145), thus proving $V = \Gamma(\mathscr{L}_\tau)$.

Returning to the question of modularity, now note that we have a canonical action of $SL_2\mathbb{Z}$ on $A \oplus \bar{A} \cong (\mathbb{Z} \oplus \mathbb{Z}) \otimes_{\mathbb{Z}} A$ by acting on the first factor. We need to understand what kind of action this induces on G. In fact, we shall specify an $\alpha \in A$, and look at $\mathbb{Z}/m \cong \langle \alpha \rangle \subset A$; we may study the pullback of (154) to

$$1 \to \mathbb{C}^\times \to M \to \mathbb{Z}/m \oplus \overline{\mathbb{Z}/m} \to 1. \tag{155}$$

Note that the group of automorphisms of M over $Id_{\mathbb{Z}/m \oplus \overline{\mathbb{Z}/m}}$ is

$$H := Hom(\mathbb{Z}/m \oplus \overline{\mathbb{Z}/m}, \mathbb{C}^\times). \tag{156}$$

Consequently, we have an extension

$$1 \to H \to \widetilde{SL_2}\mathbb{Z} \to SL_2\mathbb{Z} \to 1, \tag{157}$$

determined by the map $SL_2(\mathbb{Z}) \to Aut(\mathbb{Z}/m \oplus \overline{\mathbb{Z}/m})$. This extension is not central. Rather, $SL_2\mathbb{Z}$ acts on H in the standard way (by substitution). We may ask if extension (143) is the trivial element of $H^2(SL_2\mathbb{Z}, H)$ (i.e., a semidirect product). In our case, this is true: recall that formula (148) gives a lift of the action of $SL_2\mathbb{Z}$ to an action on M.

On the other hand, note that in that case, the set of all lifts to an action of $SL_2\mathbb{Z}$ on M forms a torsor T over

$$H^1(SL_2\mathbb{Z}, H),$$

i.e. the set of crossed homomorphisms $SL_2\mathbb{Z} \to H$.

Which particular element of T we have is important to us. We can, for example, ask if $\Gamma(m) \subset SL_2\mathbb{Z}$ acts trivially on M (we have a well-defined homomorphism $\Gamma(m) \to H$ which we would like to be 0).

To this end, note that we have not yet specified a cocycle of the Heisenberg group G; if α is the generator of A, then a choice of cocycle is specified by a choice of lift of

$$\alpha + \bar{\alpha} \tag{158}$$

to G. Note that a convenient choice of (158) in G would be

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \alpha \tag{159}$$

(the action of $SL_2\mathbb{Z}$ coming from the projective action on V determining the moduarity of θ-series; note that we have

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \theta_a(\tau, z) = exp\left(\tfrac{1}{2}\langle a, a \rangle\right)\theta_a(\tau, z)).$$

With choice (159) as representative for (158), it is easy to see that the cocycle is the antisymmetric $\mathbb{C}^{\times}$-valued bilinear form

$$\tfrac{1}{2} S \tag{160}$$

where S is the commutator. At the same time, we also see that such choice is only possible if m is odd. If m is even, we can choose cocycle (160) only when we pull back the central extension (155) to $\mathbb{Z}/2m \oplus \overline{\mathbb{Z}/2m}$. (Note that the pullback of G is not a Heisenberg group.) Nevertheless, the action of $SL_2\mathbb{Z} = Sp_1\mathbb{Z}$ preserves the antisymmetric cocycle, so we proved

Lemma 12. *$\Gamma(m)$ acts trivially on G when m is odd and $\Gamma(2m)$ acts trivially on G when m is even.* □

Now let m_L be the maximum possible order of an element of L'/L. Let, in (144), m be the least common multiple of m_{L_i}, m_{M_i}. We have

Proposition 13. *The function* (144) *is even in u. The coefficient at u^{2n} is Γ_m-automorphic of weight 2n if m is odd, and Γ_{2m}-automorphic of weight 2n if m is even.*

Proof. When passing to $\phi = \phi^{\alpha_1,\dots,\alpha_k}_{\beta_1,\dots,\beta_k}$, the factors (151) cancel out, and so do weight factors, so

$$\phi(\tau', z') = u(g)\phi(\tau, z),$$

where

$$z' = \frac{z}{c\tau + d}, \quad \tau' = \frac{a\tau + b}{c\tau + d},$$

$ad - bc = 1$ for some character $u: \widetilde{SL_2\mathbb{Z}} \to \mathbb{C}^\times$. Consequently, a coefficient at z^{2m} is automorphic of weight $2m$. The modular group contains $\Gamma(m)$ if m is odd and $\Gamma(2m)$ if m is even by Lemma 12, and also obviously $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. $\square$

Example. If L_n is the root lattice of $SU(n)$, then L_n is generated by roots $e_1, \dots, e_{n-1}$ which, together with 0, form vertices of a regular simplex (these are not simple roots). Then the dual lattice L_n' is generated by

$$e_1, \dots, e_{n-1}, \frac{1}{n}(e_1 + \cdots + e_{n-1}),$$

so $L_n'/L_n \cong \mathbb{Z}/n$. We see that $m = n$, so the modular group is Γ_n if n is odd, and Γ_{2n} if n is even.

Example. We shall calculate explicitly the image

$$\omega \in K^4[[q]]$$

of

$$\Lambda^{\alpha_1\alpha_3}_{\alpha_2^2},$$

where $\alpha_n \in L_{n+1}$ is a root of $SU(n+1)$, and L_{n+1} is as above. Then

$$L_2 = \mathbb{Z} \cdot \sqrt{2},$$

so its theta function is

$$\theta_1 = \theta_{L_2}(\tau, u\alpha_1) = \sum_{n \in \mathbb{Z}} q^{n^2} z^{-2n}$$

$(z = exp(u))$. We shall also consider the other basis element

$$\theta_2 = \theta_{L_1,\sqrt{2}/2}(\tau, u\alpha_1) = \sum_{n \in \mathbb{Z}} q^{\left(n+\frac{1}{2}\right)^2} z^{-2n+1}.$$

Note that

$$\theta_1|_{u=0} = 1, \quad \theta_2|_{u=0} = 0 \tag{161}$$

(because of sign cancellations). Now L_3 is the honeycomb lattice which can be written as

$$L_3 = \mathbb{Z} \cdot \sqrt{6} \otimes L_2 \cup \left(\mathbb{Z} + \frac{1}{2}\right) \cdot \sqrt{6} \otimes \left(L_2 + \frac{\sqrt{2}}{2}\right).$$

Consequently, if we put

$$a = \theta_{\mathbb{Z} \cdot \sqrt{6}}(\tau, 0) = \sum_{n \in \mathbb{Z}} q^{3n^2},$$

$$b = \theta_{\left(\mathbb{Z}+\frac{1}{2}\right)\sqrt{6}}(\tau, 0) = \sum_{n \in \mathbb{Z}} q^{3\left(n+\frac{1}{2}\right)^2},$$

then we have

$$\theta_{L_3}(\tau, u\alpha_2) = a\theta_1 + b\theta_2.$$

Now L_4 is the three-dimensional sphere packing lattice

$$L_4 = L_2 \otimes \mathbb{Z} \cdot \sqrt{2} \otimes \mathbb{Z} \cdot 2 + \left(L_2 + \tfrac{\sqrt{2}}{2}\right) \otimes \left(\mathbb{Z} + \tfrac{1}{2}\right) \cdot \sqrt{2} \otimes \left(\mathbb{Z} + \tfrac{1}{2}\right) \cdot 2.$$

Thus, if we put

$$c = \theta_{\mathbb{Z}\sqrt{2} \otimes \mathbb{Z} \cdot 2}(\tau, 0) = \left(\sum_{n \in \mathbb{Z}} q^{n^2}\right)\left(\sum_{n \in \mathbb{Z}} q^{2n^2}\right),$$

$$d = \theta_{\left(\mathbb{Z}+\frac{1}{2}\right)\sqrt{2} \otimes \left(\mathbb{Z}+\frac{1}{2}\right) \cdot 2}(\tau, 0) = \left(\sum_{n \in \mathbb{Z}} q^{\left(n+\frac{1}{2}\right)^2}\right)\left(\sum_{n \in \mathbb{Z}} q^{2\left(n+\frac{1}{2}\right)^2}\right),$$

then

$$\theta_{L_4}(\tau, u\alpha_3) = c\theta_1 + d\theta_2.$$

Thus,

$$\omega = Coeff_{u^2}\left(\frac{\theta_1(c\theta_1 + d\theta_2)}{(a\theta_1 + b\theta_2)^2}\right). \tag{162}$$

Now, as expected, θ_1, θ_2 are reciprocal functions in z, and hence even functions of u, and hence the argument A of $Coeff_{u^2}$ in (162) is a function of $U = u^2$.

Moreover, such coefficient is obviously determined by applying

$$D = \frac{d}{dU}\bigg|_{U=0}.$$

We have (since $z^k = e^{2\pi iku} = 1 + iku - \frac{k^2u^2}{2} + \cdots$), up to a constant,

$$t_1 := D\theta_1 = \sum_{n\in\mathbb{Z}} n^2 q^{n^2},$$

$$t_2 := D\theta_2 = \sum_{n\in\mathbb{Z}} \left(n+\frac{1}{2}\right)^2 q^{\left(n+\frac{1}{2}\right)^2}.$$

By (161),

$$A(0) = \frac{c}{a^2}. \tag{163}$$

Thus, we have

$$\begin{aligned} const\cdot\omega = DA &= A(0)\cdot\frac{DA}{A(0)} \\ &= \frac{c}{a^2}\cdot\left(\frac{D\theta_1}{\theta_1(0)} + \frac{cD\theta_1 + dD\theta_2}{c\theta_1(0)+d\theta_2(0)} - 2\frac{aD\theta_1 + bD\theta_2}{a\theta_1 + b\theta_2}\right) \\ &= \frac{c}{a^2}\left(t_1 + \frac{ct_1 + dt_2}{c} - \frac{2(at_1 + bt_2)}{a}\right) \\ &= \frac{c}{a^2}\left(\frac{d}{c} - 2\frac{b}{a}\right)t_2. \end{aligned}$$

We need to verify that $\omega\neq 0$. To this end, it suffices to verify that

$$\frac{d}{c} - 2\frac{b}{a}\neq 0. \tag{164}$$

The left-hand side is

$$\begin{aligned} &\frac{(2q^{1/4} + 2q^{9/4} + \cdots)(2q^{1/2} + 2q^{9/2} + \cdots)}{(1 + 2q + 2q^4 + \cdots)(1 + 2q^2 + 2q^8 + \cdots)} \\ &\quad - 2\frac{2q^{3/4} + 2q^{27/4} + \cdots}{1 + 2q^3 + 2q^{12} + \cdots}. \end{aligned}$$

The lowest coefficient, at $q^{3/4}$ is, indeed, 0. However, the next coefficient, at $q^{7/4}$, is $-8\neq 0$.

Remark. We can use ω in (121) if we choose as Φ, say, the set of all isomorphism classes of tensor products of the VOAs of the lattices L_2, L_3, L_4, thus obtaining a first concrete example of an elliptic cohomology theory E in the sense of the last section. Note that we must invert the prime 2 in order for our class ω to be invertible, so we will obtain a map

$$E\to K[[q]][\tfrac{1}{2}].$$

7. The Coxeter elements, lattices and Moonshine

We shall begin with one more interpretation of map (131). Consider the space of meromorphic functions $f : E_\tau \to \mathbb{C}^\times$ (i.e. algebraic functions defined on a Zariski open subset of E_τ). We consider the topology on spaces of meromorphic functions given as follows: Choose a compact set L in the source space (in this case E_τ) and an open set $\mathscr{U}$ in the target space (in this case $\mathbb{C}^\times$). Let $\mathscr{K}(L, \mathscr{U})$ be the set of all meromorphic functions f which have no singularities on L and satisfy $f(L) \subset \mathscr{U}$. We let the sets $\mathscr{K}(L, \mathscr{U})$ be the subbasis of topology on the given space of meromorphic functions.

The topological vector space K of meromorphic functions on E_τ is ∞-dimensional, and hence $K^\times = K - \{0\}$ is contractible. Now any $f \in K^\times$ determines a well-defined stringy isomorphism

$$\phi^f : 1 \to B^f, \tag{165}$$

where ϕ_c^f is the action of the loop $f|_c \in L\mathbb{C}^\times$ on $\mathscr{H}$ (we fix a torus $S^1 \subset SU(2)$; recall that $\mathscr{H}$ is an even lattice which contains an element of square length 2), and 1 is the constant stringy bundle.

Now, however, note that B^f depends only on the divisor D^f of f, so the target of (165) is invariant under the action of $\mathbb{C}^\times$ on $K^\times$. Thus, we get a map

$$K^\times/\mathbb{C}^\times \to \tilde{B}_{E_\tau}\mathscr{H}. \tag{166}$$

However, note that the action of $\mathbb{C}^\times$ on $K^\times$ is given as follows: for $\lambda \in \mathbb{C}^\times$ (a constant function) we have

$$\phi^\lambda : 1 \to 1$$

and

$$\phi^{\lambda \cdot f} = \phi^f \circ \phi^\lambda.$$

This means that, while for $D \in K^\times/\mathbb{C}^\times$ the stringy isomorphism (165) does depend on the choice of representative f of D, the isomorphism

$$\phi^{e^* f} \circ (\phi^f)^{-1}, \quad e \in E_\tau$$

does not. We conclude that (166) lifts to a map

$$K^\times/\mathbb{C}^\times \to B_{\mathrm{ell}}\mathscr{H},$$

which is the map Ξ. Note, however, that an obvious generalization of this method now leads to the following

Proposition 14. *Suppose $\mathscr{H}$ is a 1-conformal field theory, and $\mathscr{K}$ is an E_τ-equivariant space such that to every $x \in \mathscr{K}$, there is, continuously and E_τ-equivariantly assigned a stringy isomorphism of $\mathscr{H}$-bundles on E_τ*

$$f^x : 1 \to B^x. \tag{167}$$

Suppose, further, that an E_τ-fixed subset $G \subset \mathscr{K}$ has a group structure and $\mathscr{K}$ has a structure of a right E_τ-equivariant G-space such that

$$f^g : 1 \to 1, \tag{168}$$

and, for $g \in G$, $x \in \mathscr{K}$,

$$f^{xg} = f^x \circ f^g, \quad x \in \mathscr{K}, \ g \in G. \tag{169}$$

Then (167) *induces a natural map*

$$\mathscr{K}/G \to B_{\mathrm{ell}}\mathscr{H}. \tag{170}$$

In particular, if $\mathscr{K}$ is contractible, then we have a canonical map $BG \simeq \mathscr{K} \times_G EG \to \mathscr{K}/G$, so we get a natural map

$$\Xi : BG \to B_{\mathrm{ell}}\mathscr{H}.$$

Proof. By (168), B^x is invariant under changing x to xg, $g \in G$, so we get a map

$$\mathscr{K}/G \to \{\mathscr{H}\text{-stringy bundles on } E_\tau\},$$

where $\mathscr{K}/G$ denotes the orbits of $\mathscr{K}$ under the right G-action. Now the function f^x depends on the choice of representative x of a class $\alpha \in \mathscr{K}/G$, but for $e \in E_\tau$, we have

$$\begin{aligned} &f^{e^*(gx)}(f^{xg})^{-1} = && \text{by (169).} \\ &f^{e^*x} f^{e^*g} f^{g^{-1}} (f^x)^{-1} = && \text{since } G \text{ is } E_\tau\text{-fixed} \\ &f^{e^*x} f^g f^{g^{-1}} (f^x)^{-1} = f^{e^*x}(f^x)^{-1}, && \end{aligned}$$

so the function $f^{e^*x}(f^x)^{-1}$ depends only on α. Thus, we get map (170). $\square$

Example. Let $\mathscr{H} = \mathscr{H}_1$, or more generally, any lattice with a specified point of square length 2. Then let $\mathscr{K}$ be the set of all 'meromorphic maps'

$$E_\tau \to SL_2\mathbb{C},$$

i.e. algebraic maps

$$U \to SL_2\mathbb{C}$$

for a Zariski open set $U \subset E_\tau$. Note that we have a fibration

$$Mer(E_\tau, \mathbb{C}) \to \mathscr{K} \to Mer(E_\tau, \mathbb{C}^2 - \{0\}),$$

thus proving that $\mathscr{K}$ is contractible. The condition of Proposition 14 are clearly satisfied, so we get a map

$$BSU(2) \simeq BSL_2\mathbb{C} \to B_{\mathrm{ell}}\mathscr{H},$$

as promised in Section 6.

This method generalizes to other algebraic groups, but not all of them. For example, the smallest non-trivial irreducible representation of $E(8)$ is the adjoint representation, so there clearly are no representations V where $V - \{0\}$ would be transitive (cf. [35]). However, all canonical complexifications of compact Lie groups are rational varieties (by BN-decomposition). We claim that for any rational smooth variety Z over $\mathbb{C}$, and any complex algebraic curve X, $Mer(X,Z)$ is contractible: if Z is a Zariski-open set in $\mathbb{A}^k$, then $Mer(X,Z)$ is a complement in the infinite-dimensional vector space $Mer(X,\mathbb{A}^k)$ of a set of infinite codimension, so it is contractible. In more detail, given a continuous map

$$f : S^m \to Mer(X,Z),$$

f can be approximated by a map

$$f' : S^m \to Mer(X,\mathbb{A}^k)$$

whose target is contained in an affine subspace V of finite dimension; if f, f' are close enough, $tf + (1-t)f'$ land in $Mer(X,Z)$ for all $t \in [0,1]$. But now a generic affine subspace of finite dimension in $Mer(X,\mathbb{A}^k)$ is contained in $Mer(X,Z)$, so V can be chosen so that f' is homotopic to a constant through a linear homotopy.

For Z arbitrary, we have $Mer(X,Z)$ covered by a directed system of contractible open subsets

$$\{Mer(X,U) \mid U \subset Z \text{ open affine}\},$$

so $Mer(X,Z)$ is contractible. Thus, we have

Proposition 15. *If $\mathscr{H}_G$ is the root lattice of a simply laced compact Lie group G, then we have a canonical map*

$$\Xi : BG \to B_{\mathrm{ell}}\mathscr{H}_G.$$

Example. The most interesting case of Proposition 15 is $G = E_8$. Denote the root lattice of E_8 by Γ_8. The degrees [4, Chapter V, Section 6.2] of E_8 are

$$d_i = 2, 8, 12, 14, 18, 20, 24, 30, \quad i = 1, \dots, 8$$

[4, Chapter VI, Section 4.10], so

$$H^*(BE_8, \mathbb{Q}) \cong \mathbb{Q}[\alpha_1, \dots, \alpha_8], \tag{171}$$

where $dim(\alpha_i) = 2d_i$, $i = 1, \dots, 8$. By rational homotopy theory, we get classes $\tilde{\omega}_i \in \pi_{2i}(BE_8)$ representing certain integral multiples of α_i. Put

$$\omega_i = \phi_i(\tilde{\omega}_i) \in \pi_{2d_i} B_{\mathrm{ell}}\mathscr{H}_{E_8}. \tag{172}$$

Concretely,

$$\omega_1 \in \pi_4 B_{\text{ell}} \mathscr{H}_{E_8},$$

$$\omega_2 \in \pi_{16} B_{\text{ell}} \mathscr{H}_{E_8},$$

$$\omega_3 \in \pi_{24} B_{\text{ell}} \mathscr{H}_{E_8}.$$

Note that

$$p_1(\omega_i) = 0 \quad \text{for } i > 0$$

(by connectivity), we expect the image

$$\bar{\omega}_i \in K_{2d_i}[[q]]$$

of ω_i to be modular. We shall calculate these images explicitly for $i = 2, 3$.

This can be done by Proposition 10. If T is the maximal torus of E_8, the image of H_*BT in $K_*[[q]]$ is linearly spanned by coefficients at monomials in $Sym[\Gamma_8 \otimes \mathbb{C}]$ of the theta function

$$\theta_{\Gamma_8}(\tau, u) \in Sym[\Gamma_8 \otimes \mathbb{C}][[q]].$$

It is quite remarkable that in each dimension, most of these coefficients must be linearly dependent by (171).

Now concretely, recall from [14] that for a suitable basis $x_1, \ldots, x_8$ of $\Gamma_8 \otimes \mathbb{C}$,

$$\theta_{\Gamma_8}(\tau, u) = \frac{1}{2}\left(\prod_{i=1}^{8} \theta_1(\tau, x_i) + \prod_{i=1}^{8} \theta_2(\tau, x_i) + \prod_{i=1}^{8} \theta_3(\tau, x_i) + \prod_{i=1}^{8} \theta_4(\tau, x_i)\right),$$

where θ_i, $i = 1, \ldots, 4$ are the Jacobi theta functions

$$\theta_3 = \sum_{m \in \mathbb{Z}} q^{\frac{1}{2}m^2} e^{2\pi i m u},$$

$$\theta_1 = \sum_{m \in \mathbb{Z}} q^{\frac{1}{2}\left(m+\frac{1}{2}\right)^2} e^{2\pi i \left(m+\frac{1}{2}\right)\left(u+\frac{1}{2}\right)},$$

$$\theta_2 = \sum_{m \in \mathbb{Z}} q^{\frac{1}{2}\left(m+\frac{1}{2}\right)^2} e^{2\pi i \left(m+\frac{1}{2}\right) u},$$

$$\theta_4 = \sum_{m \in \mathbb{Z}} q^{\frac{1}{2}m^2 e^{2\pi i m}\left(u+\frac{1}{2}\right)}.$$

To get rid of some constants, change variables to $s = 2\pi i u$. Then we have

$$\theta_3 = \sum_{k \geqslant 0} \frac{s^{2k}}{(2k)!} \left(\sum_m m^{2k} q^{\frac{1}{2}m^2} \right),$$

$$\theta_2 = \sum_{k \geqslant 0} \frac{s^{2k}}{(2k)!} \left(\sum_{\left(m+\frac{1}{2}\right)} m^{2k} q^{\frac{1}{2}\left(m+\frac{1}{2}\right)^2} \right),$$

$$\theta_4 = \sum_{k \geqslant 0} \frac{s^{2k}}{(2k)!} \left(\sum_m (-1)^m m^{2k} q^{\frac{1}{2}m^2} \right),$$

$$\theta_1 = i \sum_{k \geqslant 0} \frac{s^{2k+1}}{(2k+1)!} \left(\sum_m (-1)^m (m + \tfrac{1}{2})^{2k+1} q^{\frac{1}{2}\left(m+\frac{1}{2}\right)^2} \right).$$

Notice that the functions θ_2, θ_3, θ_4 are even, while θ_1 is odd. As it turns out, θ_1 supplies the coefficients we are interested in. Put

$$a_k = \sum_m (-1)^m \left(m + \frac{1}{2} \right)^{2k+1} q^{\frac{1}{2}\left(m+\frac{1}{2}\right)^2}.$$

Then we have the following theta function coefficients in degrees 8,12 (i.e. dimensions 16,24).

$$a_1^8 = \Delta \cdot const,$$

$$-\frac{3}{2} a_5 a_1^7 + \frac{5}{2} a_3^2 a_1^6 = const \cdot g_2 \cdot \Delta. \tag{173}$$

It follows that the first element (173) is $\bar{\omega}_2$, the second is $\bar{\omega}_3$. Thus, if we have $\mathscr{H}_{E_8} \in \Phi$, we can take

$$\omega := \omega_3 \cdot (1_{E_8})^{-1} \in \pi_{24} \mathscr{E},$$

where 1_{E_8} is the constant map from S^{24} to the trivial $\mathscr{H}_{E_8}$-elliptic bundle on E_τ. Then the image of ω in $K_*[[q]]$ is the discriminant form, as desired.

However, note that the coefficient ring of the corresponding spectrum E still won't be exactly the 'right' one: if we look at the element

$$\rho := \omega_2 \cdot (1_{E_8})^{-1} \in \pi_{16} \mathscr{E},$$

then its image in $K_*[[q]]$ is (up to constant)

$$\Delta/g_2,$$

which is an automorphic function, but not a modular form, because it has a singularity. This means that if we consider the coefficients E_* as a moduli space, then it will exclude the elliptic curve which lives at this singularity. One could speculate that to remove this defect, one must work with the Moonshine module instead of $\mathscr{H}_{E_8}$. We shall say more about this below.

First, however, we give another application of Proposition 14.

Proposition 16. *For a CFT $\mathscr{H}_L$ associated with an even lattice L, there is a natural map*

$$B(T) \to B_{\text{ell}}\mathscr{H}_L.$$

Proof. We shall use Proposition 14. We let $\mathscr{K} = Mer(E_\tau, T)$ (recall $T = L_{\mathbb{C}}/L$). Then to each $g \in \mathscr{K}$, there is, equivariantly, assigned a stringy iso

$$f^g : 1 \to B^g$$

given on an analytic smooth Jordan curve c in $E_\tau - \{\text{singularities of } g\}$, by

$$f^g_c = \text{multiplication by } f^g|_c$$

(recall from Section 3 that $\mathscr{H}_L$ is the basic representation of T_{S^1}). It follows from the discussion of Section 3 that f^g is a stringy iso. The conditions of Proposition 14 are obviously satisfied (see above comments for contractibility of $\mathscr{K}$). This concludes our proof. □

Example. If Conjecture 1 of Section 5 holds, we would get a map

$$B(Co_0 \ltimes \Lambda_{\mathbb{C}}/\Lambda) \to B_{\text{ell}}\mathscr{H}_\Lambda,$$

where Λ is the Leech lattice. One can detect images of elements in $\pi_i B(Co_0 \ltimes \Lambda_{\mathbb{C}}/\Lambda')_{\mathbb{Q}}$ in $K_*[[q]]$ using the θ-function of Λ similarly as we did above for E_8.

For example, using the standard coordinate frame $u_1, \ldots, u_{24}$ ([6]), we can investigate the coefficient of

$$\theta_\Lambda(\tau, u) \quad \text{at} \quad \prod_{j=1}^{24} u_j. \tag{174}$$

If we put

$$g_2 = \frac{1}{12}\left(1 + 240\sum_{n\geqslant 0}\frac{n^3q^n}{1-q^n}\right),$$

$$g_3 = \frac{1}{6^3}\left(-1 + 504\sum_{n\geqslant 0}\frac{n^5q^n}{1-q^n}\right)$$

(omitting some 2π powers), then using MAPLE one can verify that

$$c = (-3\cdot 6^6\cdot g_3^2 - 5440\Delta)\cdot\Delta^2. \tag{175}$$

Thus, we have a modular form of weight 36, conjecturally an element of $\pi_{48}B_{\mathrm{ell}}\mathscr{H}_\Lambda$.

However, recalling that

$$\theta_\Lambda(\tau, 0) = 1728g_2^3 - 720\Delta,$$

$$1728\Delta = 1728g_2^3 - 6^3g_3^2,$$

we see that the corresponding element in elliptic cohomology would have character

$$\begin{aligned}\frac{c}{\theta_\Lambda(\tau,0)} &= \frac{-5184g_2^3 - 256\Delta}{1728g_2^3 - 720\Delta}\Delta^2\\ &= \frac{-3j - 256}{j - 720}\Delta^2.\end{aligned}$$

We see that this is, again, an automorphic function with some singularities, i.e. not an automorphic form.

This however suggests a general conjecture about the θ-functions of lattices, which reflects the homotopy situation we suggested, but can be phrased without the use of any homotopical concepts. Let L be an even lattice of dimension n, and let G be its automorphism group. We have conjectured that there is a map

$$B(G \rtimes (S^1)^n) \to B_{\mathrm{ell}}\mathscr{H}_L.$$

Now we have

$$H^*(B(G\rtimes(S^1)^n), \mathbb{Q}) = H^*((\mathbb{C}P^\infty)^n, \mathbb{Q})^G = \mathbb{Q}[u_1, \ldots, u_n]^G.$$

Call the dual A of $\mathbb{Q}[u_1, \ldots, u_n]$ the $\mathbb{Q}$-coalgebra of *coefficients*. Let PA be the module of primitives of a $\mathbb{Q}$-coalgebra A. Recall from rational homotopy theory that if X is a formal generalized nilpotent space, then the Hurewicz map gives an onto map

$$\pi_*X_{\mathbb{Q}} \twoheadrightarrow PH_*X_{\mathbb{Q}}. \tag{176}$$

Thus, at least if we knew that $B_{\rm ell}\mathscr{H}_L$ is formal and rationally nilpotent, then PA would supply elements in the rational homotopy groups of $B_{\rm ell}\mathscr{H}_L$. Such elements would have $p_1 = 0$ provided that they are in dimension $2\ell > 4$, and hence their images in $K[[q]]$-theory (which are given by theta functions) should be automorphic of weight $\ell + \frac{n}{2}$. This leads to the following

Conjecture 2. *Let L be an even lattice. Note that the coefficients of the theta series*

$$\theta_L(\tau, u)$$

give a map

$$\alpha : A/G = A \otimes_{\mathbb{Q}[G]} \mathbb{Q} \to \mathbb{Q}[[q]].$$

Then for a homogeneous element

$$x \in P(A/G)$$

of degree $\ell > 2$,

$$\alpha(x)$$

is an automorphic form of weight

$$\ell + \frac{n}{2}.$$

The modular group is Γ_{m_L} or Γ_{2m_L} depending on whether m_L (the maximal order of an element of L'/L) is odd or even.

Example. The most interesting case of our discussion is

$$B_{\rm ell}\mathscr{H}^{\natural}$$

where $\mathscr{H}^{\natural}$ is the Hilbert completion of the Moonshine module $V^{\natural}$ (see Section 4 above and [13]). We have

$$\mathscr{H}^{\natural} = (\mathscr{H}_{\Lambda} \oplus \mathscr{H}^T_{\Lambda})^{\theta_0}, \tag{177}$$

where Λ is the Leech lattice, $\mathscr{H}_{\Lambda}$ is, again, the basic representation of the loop group $L(G)$ with

$$G = \Lambda_{\mathbb{C}}/\Lambda' \tag{178}$$

and $\mathscr{H}^T_{\Lambda}$ is the basic representation of the twisted loop group $L_\alpha(G)$ where α is the automorphism of G given by $\alpha(g) = g^{-1}$ (and $L_\alpha(G) = \{f : [0,1] \to G \,|\, f(1) = \alpha f(0)\}$).

Recall that in (177), θ_0 is the appropriate lift of the involution $x \mapsto -x$ of Λ. One can put a 1-conformal field theory structure on $\mathscr{H}^\natural$; some of the main points of that construction were outlined in Section 4.

There is no analogue of Proposition 16 for $\mathscr{H}^\natural$, because the θ_0-fixed points destroy all of these automorphisms. On the other hand, the monster F_1 acts on $\mathscr{H}^\natural$ [13], so Conjecture 1 of Section 6 would imply the existence of a "moonshine map"

$$BF_1 \to B_{\mathrm{ell}} \mathscr{H}^\natural. \tag{179}$$

Composing with the map

$$B_{\mathrm{ell}} \mathscr{H}^\natural \to K[[q]][q^{-1}] \tag{180}$$

which is map (125) multiplied by q^{-1} (this differs from the normalization of Section 5 by a factor of Δ), we obtain an element

$$\alpha \in K[[q]][q^{-1}]^* BF_1 \tag{181}$$

which, interpreted as a series of F_1-representations by the Atiyah completion theorem, is the moonshine module $V^\natural$.

Now drawing a parallel with Conjecture 2 suggests looking at the "coefficients" for element (181). These are characters of the representation $V^\natural$, known as Thompson series. The Moonshine conjectures [6], proven by Borcherds [2], state that the Thompson series are Hauptmoduln, in particular they are *modular* functions (automorphic of weight 0) with respect to appropriate subgroups of $PSL_2\mathbb{Z}$, with a simple pole at the cusp $z = \infty$, and no other singularities. Although the singularities are as desired, we see however that the element of $\pi_* B_{\mathrm{ell}} V^\natural$ which we are seeking is *not* among the Thompson series, since it is a modular function of positive weight. We formulate therefore the following

Conjecture 3. (1) *There is a positive integer n and an element*

$$x \in \pi_{24n} B_{\mathrm{ell}} \mathscr{H}^\natural$$

whose image under map (181) (*on coefficients*) *is*

$$\Delta^n \cdot J$$

($J = j - 744$).

(2) *Every element of* $\pi_m B_{\mathrm{ell}} V^\natural$ *is of the form*

$$\omega \cdot J,$$

where ω *is an automorphic form of weight* $\frac{m}{2}$ (*with full modular group*, *no character and no singularities*).

We could refer to this as the 'higher homotopy counterpart of the Moonshine conjectures'. It also suggests the set of tensor powers of $V^\natural$ as a choice of Φ from Section 5.

In fact, one can go a bit further in this direction. The main ingredient of the proof [2] of the Moonshine conjectures is the construction of a "Monster Lie algebra" M. Neglecting, for the moment, questions of Hilbert completion, we take the VOA

$$\mathscr{H} := V^\natural \otimes V_{II_{1,1}}, \tag{182}$$

where $II_{1,1}$ is the unimodular Lorentzian lattice with matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and $V_{II_{1,1}}$ is the corresponding vertex algebra [2]. Notice that (182) comes with an indefinite inner product $\langle ?, ? \rangle$. Now (a completion of) M is obtained by taking the vector subspace P of fields of type (1,0) in the conformal field theory (182) (see [36, Section 9]), and factoring out the kernel of $\langle ?, ? \rangle$ on P. (In fact, it is desirable to consider all of this in real form, see [2] and Section 3 above.) The no ghost theorem [11,15] asserts that if we consider the bigrading of M inherited from $V_{II_{1,1}}$, then the (m,n)-graded piece is

$$\begin{array}{ll} \mathbb{R}^2 & \text{if } m = n = 0, \\ V_{mn} & \text{otherwise,} \end{array}$$

where V_m is the weight $m+1$ piece of $V^\natural$. Further, M is a Lie algebra, and in fact a (generalized) Kac-Moody algebra of indefinite type with root lattice $II_{1,1}$. Using this, Borcherds proved that

$$p^{-1}\Lambda\left(\sum_{\substack{m>0 \\ n\in\mathbb{Z}}} V_{mn}p^m q^n\right) = \sum_m V_m p^m - \sum_n V_n q^n, \tag{183}$$

where we can look at p, q as formal variables, and the coefficients V_m are considered as representations of the Monster. This works because the Monster's action on $V^\natural$ carries through the no ghost theorem. In [2], one then uses (183) to prove Norton's replication formulae [32], which constitute the main step in the Moonshine conjecture proof. More explicitly, one remarks that

$$\Lambda(U) = exp\left(-\sum_{i>0} \psi^i(U)/i\right), \tag{184}$$

where ψ^i is the Adams operations on a G-representation U (here $G = F_1$ is the Monster), and that further

$$Tr(g|\psi^i U) = Tr(g^i|U),$$

so we get

$$p^{-1}\, exp\left(-\sum_{i>0}\sum_{\substack{m>0\\ n\in\mathbb{Z}}} Tr(g^i|V_{mn})p^{mi}q^{ni}/i\right) = \sum_{m\in\mathbb{Z}} Tr(g|V_m)p^m - \sum_{n\in\mathbb{Z}} Tr(g|V_n)q^n,$$

which is the Norton formula.

When trying to obtain a higher homotopy analogue of formula (183), the first step is thinking of the G-representations V_m as maps

$$BG \to BU(|V_m|) \to BU \times \mathbb{Z}. \tag{185}$$

In the second map (185), we map into the $|V_m|$th component of $BU \times \mathbb{Z}$. Thus, taking products over $m \geqslant -1$, (185) gives an element of

$$K[[q]][q^{-1}]^0 BG. \tag{186}$$

Now, on the other hand, Section 5 gives a map

$$\alpha \in K[[q]][q^{-1}]^0 B_{\text{ell}} V^{\natural} \tag{187}$$

and Conjecture 1 states that there is a map $BG \to B_{\text{ell}} V^{\natural}$ such that (187) factors through (186). Further, for a homotopy class $\omega \in \pi_{2k} B_{\text{ell}} V^{\natural}$, $\alpha\omega$ is the character of ω, i.e., conjecturally some cusp form, e.g. $J \cdot \Delta^n$.

Now in [2], one points out that *any* graded endomorphism of the vector space $V^{\natural}$ acts naturally on the Monster Lie algebra M, although to get (183), we must use the fact that the no ghost theorem identifies the bigraded pieces of M together with their G-actions.

We do not know what exactly is the right analogue of the bigraded action for $B_{\text{ell}} V^{\natural}$, but the above discussion suggests that the right substitute for V_m in (183) may be the element

$$\rho = \alpha\omega \in K[[q]][q^{-1}]^0 S^{2k} \tag{188}$$

corresponding by (187) to a homotopy class of $B_{\text{ell}} V^{\natural}$. Note that

$$K^0 S^{2k} = \mathbb{Z} \oplus \mathbb{Z}u,$$

where $u = \beta^k$ is the kth power of the Bott element. Given the fact that the mth coordinate of $\alpha\omega$ is realized on

$$BU(|V_m|) \subset BU \times \{|V_m|\},$$

we have

$$\rho = \omega(\tau) \cdot u + J(\tau) \tag{189}$$

(expressed as a function of $q = e^{2\pi i\tau}$). We now write down the left-hand side of (183), using (184), and substituting

$$\rho \text{ for } \sum V_m q^m.$$

Define, when applicable, $c_n(\phi)$ by

$$\phi(\tau) = \sum c_n(\phi) q^n.$$

Then, noting that

$$\psi^i u = i^k u,$$

the left-hand side of (183) becomes (for $k>0$)

$$\begin{aligned} p^{-1}\, exp&\left(-\sum_{i>0}\sum_{m,n>0}(i^k c_{mn}(\omega)u + c_{mn}(J))p^{mi}q^{ni}/i\right) \\ &= exp\left(-\sum_{i>0}\sum_{m,n>0} i^{k-1}c_{mn}(\omega)up^{mi}q^{ni})(J(p)-J(q))\right) \\ &= \left(1-\sum_{i>0}\sum_{m,n>0} i^{k-1}c_{mn}(\omega)up^{mi}q^{ni})\right)(J(p)-J(q)). \end{aligned} \tag{190}$$

Comments. In the summation, we can put $n>0$ because ω is a cusp form. The first equality is by the denominator formula, which is obtained from (183) by replacing V_m by its dimension. The second equality is because higher powers of the element $u \in K^0 S^{2k}$ are 0.

We have no conceptual prediction of what the analogue of the right-hand side of (183) should be, but (190) can be evaluated. Thus, the proposed higher homotopy analogue of the Norton formula is given by the following statement. We specialize to cusp forms ω of weight $12s$. Every modular form of weight $12s$ is of the form

$$\bar{\Delta}^k \Delta^\ell, \quad k+\ell = s, \tag{191}$$

where $\bar{\Delta} = \Delta J$. Let $\omega_q = \omega(\tau)$, $q = e^{2\pi i r}$, for any modular form ω.

Proposition 17. *We have, for* $\ell > 0$,

$$(j(p) - j(q)) \cdot \sum_{i>0} \sum_{\substack{m>0 \\ n>0}} i^{12(k+\ell)-1} c_{mn}(\bar{\Delta}^k \Delta^\ell) p^{mi} q^{ni}$$
$$= \bar{\Delta}_p^k \bar{\Delta}_q^k (\bar{\Delta}_p^\ell \Delta_q^\ell - \Delta_p^\ell \bar{\Delta}_q^\ell) + \mu, \tag{192}$$

where μ *is an antisymmetric* (*with respect to* p, q) *polynomial in* Δ_p, $\bar{\Delta}_p$, Δ_q, $\bar{\Delta}_q$ *homogeneous of degree* $k + \ell$ *in the variables* Δ_p, $\bar{\Delta}_p$ *and the variables* Δ_q, $\bar{\Delta}_q$, *divisible by* $\Delta_p \Delta_q$.

Proof. We have, for any cusp form ω of weight $2s$,

$$(j(p) - j(q)) \cdot \sum_{i>0} \sum_{\substack{m>0 \\ n>0}} i^{2s-1} c_{mn}(\omega) p^{mi} q^{ni}$$
$$= (j(p) - j(q)) \cdot \sum_{i>0} \sum_{\substack{m,n>0 \\ i|m,n}} i^{2s-1} c_{mn/i^2}(\omega) p^m q^n$$
$$= (j(p) - j(q)) \sum_{m>0} T_m(\omega_q) p^m, \tag{193}$$

where T_m is the Hecke operator [38]. Hecke operators preserve the space of modular forms of a given weight, and also cusp forms, so (193) is of the form

$$\sum_{m \geqslant 0} (\alpha_m)_q p^m,$$

where α_m is a modular form of weight $2k$. Moreover, (193) is manifestly antisymmetric in p, q, and it is easy to check that the coefficients at powers of p give

$$\omega(p).$$

Specializing to $\omega = \bar{\Delta}^k \Delta^\ell$, we see that

$$\bar{\Delta}_p^k \bar{\Delta}_q^k (\bar{\Delta}_p^\ell \Delta_q^\ell - \Delta_p^\ell \bar{\Delta}_q^\ell) \tag{194}$$

satisfies this coefficient condition. Thus, if we subtract this form from (193), we get an antisymmetric series μ, a modular form of weight $12(k + \ell)$ in each variable (when fixing the other variable), with 0 coefficients at the powers of p. Now dividing μ by $\Delta_p \Delta_q$, we decrease the weight by 12. Then repeating the process of subtracting terms of the form (194) (with varying k, ℓ), and dividing by $\Delta_p \Delta_q$, we can bring the weight of the remainder term to 0, at which point it vanishes. Thus, μ is as stated. $\square$

Remarks. (1) For $k=0$, $\ell=1$, the only possible choice for μ is $\mu=0$. For $k=\ell=1$, we have

$$\begin{aligned}(j(p)-j(q))\sum_{i>0}\sum_{m,n>0} i^{23}c_{mn}(\bar{\Delta}\Delta)p^{mi}q^{ni}\\ =(\bar{\Delta}_p\bar{\Delta}_q+20414592\Delta_p\Delta_q)(\Delta_p\bar{\Delta}_q-\Delta_q\bar{\Delta}_p).\end{aligned}$$

(2) Note that in the proof [2] of the denominator formula (Lemma 7.1 of [2]), which we mimicked exactly, the reason one gets just $j(p)-j(q)$ on the right-hand side is that a modular form of weight 0 is necessarily constant. Note also that the absence of higher exponent terms in (192), which came from $u\cup u=0$, is calculationally necessary in the proof of Proposition 17, since otherwise we would be adding modular forms of different weights in (193).

(3) In discussing the motivation for Proposition 17, we used the cohomology theory $K[[q]][q^{-1}]$ and Adams operations. However, it would be nice to phrase these ideas is in terms of the power operations in elliptic cohomology considered by Ando [1]. We also refer the reader to [18] for more background and further considerations.

8. Appendix. Some homotopy theory

The main point of this section is to give a

Proof of Proposition 8. Let $D=\{z\in\mathbb{C}\mid ||z||\leqslant 1\}$. We shall define a space

$$\tilde{B}_{(D,\partial D)}\mathscr{H}$$

whose elements are stringy bundles on U for some open set $\mathbb{C}\supset U\supset D$ (the definition of stringy bundles extends to non-compact surfaces), with two stringy bundles B, B' identified if $B_c=B'_c$ for every Jordan curve whose image is in D.

Now choose a holomorphic embedding $D\subset X$. This clearly determines a restriction map

$$\tilde{B}_X\mathscr{H}\to\tilde{B}_{(D,\partial D)}\mathscr{H}.$$

The proof will be completed if we can construct a map

$$\tilde{B}_{(D,\partial D)}\mathscr{H}\to K(\mathbb{Z},4). \tag{195}$$

Now consider the set P whose elements are pairs

$$((B,S),Q), \tag{196}$$

where $(B,S)\in\tilde{B}_{(D,\partial D)}\mathscr{H}$ and Q is a collection of disjoint Jordan curves in $\mathbb{C}$ homothetic to $S^1=\{z\in\mathbb{C}\mid ||z||=1\}$; two collections are considered equal if they only differ in Jordan curves whose interiors are disjoint from D. We require that for some choice of S, each element $s \in S$ be contained in the interior of a Jordan curve $c \in Q$ which is *minimal* in the sense that the interior of c contains the image of no other element of Q.

Then P is a topological poset with ordering

$$((B,S),Q) \leqslant ((B,S),Q')$$

if $Q \subseteq Q'$. Let W be the classifying space of P. There is an obvious forgetful map

$$v : W \to \tilde{B}_{(D,\partial D)}\mathscr{H}. \tag{197}$$

However, we claim that (197) is a quasifibration with contractible fiber, hence an equivalence. For example, to show that

$$v^{-1}(B,S) \text{ is contractible,} \tag{198}$$

look at the poset $P_{(B,S)}$ of all possible choices of Q for (B,S) in (196). Then we need to show that the classifying space of $P_{(B,S)}$ is contractible. This can be done in two steps. First consider the subposet $P'_{(B,S)}$ in which each $s \in S$ is contained in the interior of a *different* minimal curve. Then

$$P'_{(B,S)} \subset P_{(B,S)}$$

induces an equivalence of classifying spaces by Quillen's theorem A (since we can always, uniquely up to homotopy, add to Q small minimal curves containing the individual points of S). On the other hand, the classifying space of $P'_{(B,S)}$ is contractible, because we may omit from $q \in P'_{(B,S)}$ all curves except the minimal curves containing points of S.

To finish the proof of (198), one then uses the Dold–Thom criterion, with k-stratum consisting of all B for which there is a choice of S with $||S|| \leqslant k$. We omit the details.

Now since (197) is an equivalence, it therefore suffices to give a map

$$W \to K(\mathbb{Z}, 4). \tag{199}$$

To this end, let P_0 be the topological poset consisting of pairs

$$(Q, \alpha),$$

where $Q \in P$, and α is a map assigning to each minimal curve with interior contained in D an element of $\mathscr{V}$; the ordering is

$$(Q, \alpha) \leqslant (Q', \alpha')$$

if $Q \subseteq Q'$ and for each minimal curve c' in Q' whose interior is contained in D, if $c_1, \dots, c_n$ are the minimal curves of Q contained in the interior of c', and A is the rigged surface with boundary components $c_1, \ldots, c_n, c'$, we have

$$U_A \circ (\alpha(c_1), \ldots, \alpha(c_n)) = \alpha'(c').$$

Then the classifying space W_0 of P_0 is a model of the double loop space $B^2\mathscr{V}$ with respect to the double-loop space structure on $\mathscr{V}$ given by the action of the operad $\mathscr{C}$ of n-tuples of Jordan curves $c_1, \ldots, c_n$ with disjoint interiors in D homothetic to the boundary (with identity parametrization). The operad acts by the CFT structure. (Note that the operad is equivalent to the little 2-cube operad.) In any case, we have $\mathscr{V} \simeq K(\mathbb{Z}, 2)$, so

$$B^2\mathscr{V} \simeq K(\mathbb{Z}, 4),$$

regardless of the choice of double loop space structure. Thus, it remains to construct a map

$$W \to W_1.$$

This map is induced by a map of posets

$$P \to P_0$$

which sends $((B, S), Q)$ to (Q, α) where

$$\alpha(c) = B_c. \qquad \square$$

Remark. Recall that the space of divisors on X, or the free abelian group on X, is homotopically equivalent to

$$Map(X, K(\mathbb{Z}, 2)) \simeq JX \times K(\mathbb{Z}, 2) \times \mathbb{Z},$$

where JX is the Jacobian on X (here we are just considering the category of topological spaces, *Map* denotes the space of continuous maps). It therefore seems reasonable to ask if

$$\tilde{B}_X \mathscr{H} \simeq Map(X, K(\mathbb{Z}, 4)).$$

The method of the above proof seems applicable for X of genus 1, since then X is parallelizable. If X is not parallelizable, the difficulty is that there is no consistent notion of homothety, so one must consider rotations of the boundary parametrizations.

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