THE Ext^0-TERM OF THE REAL-ORIENTED ADAMS-NOVIKOV SPECTRAL SEQUENCE

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1. Introduction

The purpose of this note is to describe the Ext^0 elements of the spectral sequence

$$E_2 = Ext^*_{BP\mathbb{R}_\star BP\mathbb{R}}(BP\mathbb{R}_\star, BP\mathbb{R}_\star) \Rightarrow (\pi_\star^{\mathbb{Z}/2} S^0)^\wedge_2. \tag{1}$$

The spectral sequence (1) was introduced in [9] and [8]. Here, $BP\mathbb{R}$ is the Real-oriented Brown-Peterson spectrum, which was constructed from Landweber's Real cobordism spectrum $M\mathbb{R}$ [10] by Araki [2]. These are $\mathbb{Z}/2$-equivariant spectra, indexed on $RO(\mathbb{Z}/2)$. The subscript $\star$ refers to the $RO(\mathbb{Z}/2)$-indexing, i. e. all (bi)degrees $k + l\alpha$, $k, l \in \mathbb{Z}$, where α is the sign representation of $\mathbb{Z}/2$. Thus, the spectral sequence converges to the 2-primary components of the groups $\pi^{\mathbb{Z}/2}_{k+l\alpha} S^0 = \pi^{\mathbb{Z}/2}_k S^{-l\alpha}$.

In the coefficient ring $BP\mathbb{R}_\star = BP\mathbb{R}_\star^{\mathbb{Z}/2}$ (we will drop the group from the superscript to simplify the notation, see [9]), there are elements v_n, which are analogues of the usual generators of BP_*. We also have an element $a \in \pi_\star S^0_{\mathbb{Z}/2}$ defined by the cofiber sequence

$$\mathbb{Z}/2_+ \to S^0 \xrightarrow{a} S^\alpha \tag{2}$$

where the first map collapses $\mathbb{Z}/2$ to a single point. In addition to these, there are periodicity operators on monomials in the generators $v_n \in BP\mathbb{R}_\star$. We usually express these operators as powers of a certain symbol σ, which, however, is not itself an element. The degrees of v_n, a, and σ are as follows.

$$dim(v_n) = (2^n - 1)(1 + \alpha) \tag{3}$$

$$dim(a) = -\alpha \tag{4}$$

$$dim(\sigma) = \alpha - 1. \tag{5}$$

For further discussion of $BP\mathbb{R}_\star$, see Section 2 below.

For a set $\{x_i\}$, let $\mathbb{Z}\{x_i\}$ denote the free abelian group on the generators x_i. For an abelian group M, let $M\{x_i\}$ denote $M \otimes \mathbb{Z}\{x_i\}$. The following theorem describes $Ext^0_{BP\mathbb{R}_\star BP\mathbb{R}}(BP\mathbb{R}_\star, BP\mathbb{R}_\star)$.

Theorem 6. *As a $\mathbb{Z}_{(2)}$-module,*

$$\begin{aligned} &Ext^0_{BP\mathbb{R}_\star}(BP\mathbb{R}_\star, BP\mathbb{R}_\star) \\ &\quad = \mathbb{Z}_{(2)}\{v_0\sigma^{2l} \mid l \in \mathbb{Z}\} \\ &\quad \oplus a \cdot \mathbb{Z}/2[a] \\ &\quad \oplus \mathbb{Z}/2\{v_n^r \sigma^{l2^{n+1}} a^t \mid n, r \geq 1,\ l \in \mathbb{Z},\ 2^n - 1 \leq t \leq 2^{n+1} - 2\}. \end{aligned}$$

The degrees of these elements are determined by (3), (4), and (5).

In degrees $k + 0\alpha$, the spectral sequence (1) converges to

$$\pi_k^{\mathbb{Z}/2} S^0 \cong \pi_k \Sigma^\infty B\mathbb{Z}/2_+ \oplus \pi_k S^0.$$

By a simple computation of degrees, if $v_n^r \sigma^{l2^{n+1}} a^t$ has degree $k + 0\alpha$, then $l \leq 0$. The following table lists the first elements of Ext^0-summand of the E_2-term of (1) in degrees $k + 0\alpha$, of the form $v_n^r \sigma^{l2^{n+1}} a^t$, for $n \leq 4$, $0 \geq l \geq -7$. Following each element, the number in the parenthesis is the degree of the element.

	$n = 1$	2	3	4
$l = 0$	$v_1 a(1)$ $v_1^2 a^2(2)$	$v_2 a^3(3)$ $v_2^2 a^6(6)$	$v_3 a^7(7)$ $v_3^2 a^{14}(14)$	$v_4 a^{15}(15)$ $v_4^2 a^{30}(30)$
-1	$v_1^5 \sigma^{-4} a(9)$ $v_1^6 \sigma^{-4} a^2(10)$	$v_2^4 \sigma^{-8} a^4(20)$	$v_3^4 \sigma^{-16} a^{12}(44)$	$v_4^4 \sigma^{-32} a^{28}(92)$
-2	$v_1^9 \sigma^{-8} a(17)$ $v_1^{10} \sigma^{-8} a^2(18)$	$v_2^7 \sigma^{-16} a^5(37)$	$v_3^6 \sigma^{-32} a^{10}(74)$	$v_4^6 \sigma^{-64} a^{26}(154)$
-3	$v_1^{13} \sigma^{-12} a(25)$ $v_1^{14} \sigma^{-12} a^2(26)$	$v_2^9 \sigma^{-24} a^3(51)$ $v_2^{10} \sigma^{-24} a^6(54)$	$v_3^8 \sigma^{-48} a^8(104)$	$v_4^8 \sigma^{-96} a^{24}(216)$
-4	$v_1^{17} \sigma^{-16} a(33)$ $v_1^{18} \sigma^{-16} a^2(34)$	$v_2^{12} \sigma^{-32} a^4(68)$	$v_3^{11} \sigma^{-64} a^{13}(141)$	$v_4^{10} \sigma^{-128} a^{22}(278)$
-5	$v_1^{21} \sigma^{-20} a(41)$ $v_1^{22} \sigma^{-20} a^2(42)$	$v_2^{15} \sigma^{-40} a^5(85)$	$v_3^{13} \sigma^{-80} a^{11}(171)$	$v_4^{12} \sigma^{-160} a^{20}(340)$
-6	$v_1^{25} \sigma^{-24} a(49)$ $v_1^{26} \sigma^{-24} a^2(50)$	$v_2^{17} \sigma^{-48} a^3(99)$ $v_2^{18} \sigma^{-48} a^6(102)$	$v_3^{15} \sigma^{-96} a^9(201)$	$v_4^{14} \sigma^{-192} a^{18}(402)$
-7	$v_1^{29} \sigma^{-28} a(57)$ $v_1^{30} \sigma^{-28} a^2(58)$	$v_2^{20} \sigma^{-56} a^4(116)$	$v_3^{17} \sigma^{-112} a^7(231)$ $v_3^{18} \sigma^{-112} a^{14}(238)$	$v_4^{16} \sigma^{-224} a^{16}(464)$

The elements in the first row of the table $v_n a^{2^n - 1}$ and $v_n^2 a^{2^{n+1} - 1}$ are $\mathbb{Z}/2$-equivariant analogues of the elements h_n and h_n^2 in the classical Adams spectral sequence. The elements in the second row of the table $v_n^4 \sigma^{-2^{n+1}} a^{2^{n+1} - 4}$, $n \geq 2$, are analogues of the Adams spectral sequence elements g_{n-1}. And in the first column of the table, the elements $v_1^{4l+1}\sigma^{4l}a$ and $v_1^{4l+2}\sigma^{4l}a^2$ are analogues of the Adams spectral sequence elements $P^l h_1$ and $h_1 P^l h_1$, respectively [8].

A proof of Theorem 6 was given in [8], but we substantially simplify the argument here. We also give an interpretation of elements of the type $v_n\sigma^{l2^{n+1}}a^{2^n-1} \in Ext^0$ as Hopf invariant one elements in a certain sense. In [8], I also calculated an upper bound for the 1-line $Ext^1_{BP\mathbb{R}_\star BP\mathbb{R}}(BP\mathbb{R}_\star, BP\mathbb{R}_\star)$.

In Section 2 of the note, we recall some facts of Real-oriented homotopy theory used in constructing the Real Adams-Novikov spectral sequence. Section 3 is devoted to the proof of Theorem 6. In Section 4, we give the interpretation of the Ext^0 elements $v_n a^{2^n-1}$ as Hopf invariant one type elements.

2. The Real-oriented Adams-Novikov Spectral Sequence

In this section, we give a brief overview of the construction of the Real-oriented Adams-Novikov spectral sequence [9]. Only a small portion of the results from [9] are needed. We will recall it here in a form as self-contained as possible.

The term Real (with capitalized "R") was first introduced for K-theory by Atiyah, who defined a Real bundle ξ to be a complex bundle over an $\mathbb{Z}/2$-equivariant space, together with an action of $\mathbb{Z}/2$, which is complex antilinear fiberwise [3]. The Real cobordism spectrum $M\mathbb{R}$, introduced by Landweber and Araki [2, 10], is the Real analogue of the complex cobordism spectrum MU, and is defined as the Thom spectrum of canonical Real bundles. Specifically, the infinite Grassmannian $BU(n)$ has a $\mathbb{Z}/2$-action by complex conjugation. There is a canonical Real bundle γ_n of dimension n over $BU(n)$, giving the map on Thom spaces

$$\Sigma^{1+\alpha}BU(n)^{\gamma_n} \to BU(n+1)^{\gamma_{n+1}}.$$

This is a $\mathbb{Z}/2$-equivariant prespectrum, whose associated spectrum is $M\mathbb{R}$. Thus, $M\mathbb{R}$ is a $\mathbb{Z}/2$-equivariant spectrum indexed on the complete $RO(\mathbb{Z}/2)$-graded universe, i. e. all degrees $k + l\alpha$, $k, l \in \mathbb{Z}$. We will write the coefficient ring $M\mathbb{R}_\star$, the $\star$ indicating the $RO(\mathbb{Z}/2)$-grading. Unlike the complex-oriented case, $M\mathbb{R}$ does not represent cobordism classes of Real manifolds, i. e. manifolds whose stable normal bundles admit Real structure, in the sense that $M\mathbb{R}_{k+l\alpha}$ is not isomorphic to the cobordism group of Real manifolds of dimension $k + l\alpha$. However, there is still a map from the cobordism ring of Real manifolds to $M\mathbb{R}_\star$ given by the Pontrjagin-Thom construction. This map is not an isomorphism due to the lack of transversality (for further discussion, see [8]).

There is a notion of Real orientation, analogous to the notion of complex orientation. In particular, a Real orientation on a $\mathbb{Z}/2$-equivariant ring spectrum E is equivalent to ring spectrum map from $M\mathbb{R}$ to E.

The following proposition was shown in [9].

Proposition 7. *There is a ring isomorphism $MU_* \cong M\mathbb{R}_{*(1+\alpha)}$, where $M\mathbb{R}_{*(1+\alpha)}$ is the subring of $M\mathbb{R}_\star$ consisting of elements in degrees $k(1+\alpha)$, $k \in \mathbb{Z}$. The isomorphism takes MU_{2k} onto $M\mathbb{R}_{k(1+\alpha)}$.*

Also, $M\mathbb{R}$ is an E_∞-ring spectrum. So we can define $BP\mathbb{R}$, the Real-oriented version of the Brown-Peterson spectrum BP, in the manner of [5] as follows. Consider $M\mathbb{R}_{*(1+\alpha)} \cong MU_{2*} \cong \mathbb{Z}[x_i \mid i \geq 1]$, where x_i is in degree $2i(1+\alpha)$. It can be show that the x_i for $i \neq 2^n - 1$, ordered in any way, form a regular sequence in $M\mathbb{R}$. Killing this sequence in $M\mathbb{R}$ in the category of $M\mathbb{R}$-modules and localizing at the prime 2 gives $BP\mathbb{R}$. In fact, there is also a more elementary construction using the Quillen idempotent [2], but that requires a treatment of formal group laws.

We will use the Borel cohomology and Tate spectral sequences [7] to compute the coefficient ring $BP\mathbb{R}_\star$. Recall the standard cofiber sequence

$$E\mathbb{Z}/2_+ \to S^0 \to \widetilde{E\mathbb{Z}/2}.$$

Smashing with $BP\mathbb{R}$ and mapping to $F(E\mathbb{Z}/2_+, BP\mathbb{R})$ gives the Tate diagram

$$\begin{array}{ccccc} E\mathbb{Z}/2_+ \wedge BP\mathbb{R} & \longrightarrow & BP\mathbb{R} & \longrightarrow & \widetilde{E\mathbb{Z}/2} \wedge BP\mathbb{R} \\ \downarrow \simeq & & \downarrow & & \downarrow \\ E\mathbb{Z}/2_+ \wedge F(E\mathbb{Z}/2_+, BP\mathbb{R}) & \longrightarrow & F(E\mathbb{Z}/2_+, BP\mathbb{R}) & \longrightarrow & \widetilde{E\mathbb{Z}/2} \wedge F(E\mathbb{Z}/2_+, BP\mathbb{R}). \end{array}$$

The Borel cohomology of $BP\mathbb{R}$ is $F(E\mathbb{Z}/2_+, BP\mathbb{R})_\star$, and the Tate cohomology of $BP\mathbb{R}$ is $\widehat{BP\mathbb{R}}_\star = \widetilde{E\mathbb{Z}/2} \wedge F(E\mathbb{Z}/2_+, BP\mathbb{R})_\star$. For the $RO(\mathbb{Z}/2)$-graded coefficients, the Borel cohomology spectral sequence is

$$H^*(\mathbb{Z}/2, BP_*[\sigma, \sigma^{-1}]) \Rightarrow F(E\mathbb{Z}/2_+, BP\mathbb{R})_\star \tag{8}$$

where σ is a periodicity operator of degree $\alpha - 1$ (compare with the Introduction). This operator represents the $(\alpha - 1)$-periodicity in the homotopy groups of the spectrum $F(\mathbb{Z}/2_+, BP\mathbb{R})$: we have

$$F(\mathbb{Z}/2_+, BP\mathbb{R})_\star = BP_*[\sigma, \sigma^{-1}].$$

The Tate spectral sequence is

$$\widehat{H}^*(\mathbb{Z}/2, BP_*[\sigma, \sigma^{-1}]) \Rightarrow \widehat{BP\mathbb{R}}_\star \tag{9}$$

where $\widehat{H}^*$ denotes the Tate cohomology of $\mathbb{Z}/2$.

We can also look at the fixed-point version of the Tate spectral sequence

$$\widehat{H}^*(\mathbb{Z}/2, BP_*) \Rightarrow (\widehat{BP\mathbb{R}})_*^{\mathbb{Z}/2}.$$

As we shall see, this converges to the homotopy groups of the geometric fixed point spectrum $(\widetilde{E\mathbb{Z}/2} \wedge BP\mathbb{R})^{\mathbb{Z}/2}$ of $BP\mathbb{R}$ (see [9]). Thus, the spectrum $BP\mathbb{R}$ satisfies a "strong completion theorem" in the sense that

$$BP\mathbb{R} \simeq F(E\mathbb{Z}/2_+, BP\mathbb{R}).$$

Hence, the Borel cohomology spectral sequence (8) converges to $BP\mathbb{R}_\star$. The E_∞-term of (8) is the associated graded abelian group to $BP\mathbb{R}_\star$ with respect to the filtration by powers of the ideal (a). It is the following.

Proposition 10. *The E_∞-term of the Borel cohomology spectral sequence (8) is*

$$E_0 BP\mathbb{R}_\star = \mathbb{Z}_{(2)}[v_n\sigma^{l2^{n+1}}, a \mid l \in \mathbb{Z}, n \geq 0]/\sim \tag{11}$$

where the relations are

$$\begin{aligned} v_0 &= 2 \\ (v_n\sigma^{l2^{n+1}})a^{2^{n+1}-1} &= 0 \\ (v_m\sigma^{k2^{m+1}})(v_n\sigma^{l2^{m-n}2^{n+1}}) &= v_nv_m\sigma^{(k+l)2^{m+1}} \text{ for } n \leq m. \end{aligned}$$

The elements $v_n\sigma^{l2^{n+1}}$ has degree $(2^n-1)(1+\alpha)+l2^{n+1}(\alpha-1)$, and a has degree $-\alpha$.

Remark: It is shown in [9] that the ring on the right hand side of (11) is actually isomorphic to $BP\mathbb{R}_\star$. However, we do not need to use this fact in the present note.

The proof of Proposition 10 is given in [9], we paraphrase it here. We have

$$BP\mathbb{R}_{*(1+\alpha)} \cong BP_* \cong \mathbb{Z}_{(2)}[v_0, v_1, \ldots]$$

where $v_0 = 2$, and v_n has degree $(2^n-1)(1+\alpha)$. As remarked above, the element a is given by the cofiber sequence

$$\mathbb{Z}/2_+ \to S^0 \xrightarrow{a} S^\alpha.$$

Consider the Tate spectral sequence (9). Its E_1-term is

$$BP_*[a, a^{-1}, \sigma, \sigma^{-1}]$$

where the filtration degree of a monomial is its degree with respect to a. We have

$$d_1(\sigma^{-1}) = v_0 a = 2a$$

from the computation of $H^*(\mathbb{Z}/2, BP_*)$. We use this notation since it conforms with the pattern of the higher differentials. One must be careful, however, because the E_1-term is not a graded-commutative ring in any reasonable sense (it has nontorsion elements in all degrees). Alternatively, the E_2-term can be calculated as

$$E_2 = \widehat{H}^*(\mathbb{Z}/2, BP_*[\sigma, \sigma^{-1}]).$$

The action of $\mathbb{Z}/2$ on $BP_*[\sigma, \sigma^{-1}]$ is as follows. For reasons that will become clear shortly, we write the generators of BP_* as $v_n^{\mathbb{C}}$. For a sequence of nonnegative integers

$R = (r_0, r_1, \ldots)$, with only finitely many $r_i > 0$, we write the monomial

$$v_R^{\mathbb{C}} = \prod_{i\geq 0}(v_i^{\mathbb{C}})^{r_i}.$$

The degree of $v_R^{\mathbb{C}}$ is $|v_R^{\mathbb{C}}| = \Sigma_{i\geq 0}2r_i(2^i-1)$. Then the generator of $\mathbb{Z}/2$ acts on $v_R^{\mathbb{C}}\sigma^l$ by $(-1)^{\frac{|v_R^{\mathbb{C}}|}{2}+l}$. This gives

$$E_2 = BP_\star[\sigma^2, \sigma^{-2}, a, a^{-1}]/(2a) = BP_\star[\sigma^2, \sigma^{-2}, a, a^{-1}]/(2) \tag{12}$$

where $BP_\star$ is defined to be $\mathbb{Z}_{(2)}[v_n]$,

$$v_n = v_n^{\mathbb{R}} = v_n^{\mathbb{C}}\sigma^{2^n-1}. \tag{13}$$

We have $dim(v_n) = (2^n-1)(1+\alpha)$, $dim(\sigma) = \alpha - 1$, and $dim(a) = -\alpha$. To explain this notation, note that the generator of $\mathbb{Z}/2$ acts by 1 on v_n. Now for fixed $l \in \mathbb{Z}$, we have

$$\begin{aligned}\widehat{H}^i(\mathbb{Z}/2, \mathbb{Z}_{(2)}\{v_n\sigma^{2l}\}) =& \mathbb{Z}/2 \text{ for } i \text{ even}\\ & 0 \text{ for } i \text{ odd}\end{aligned} \tag{14}$$

and

$$\begin{aligned}\widehat{H}^i(\mathbb{Z}/2, \mathbb{Z}_{(2)}\{v_n\sigma^{2l+1}\}) =& \mathbb{Z}/2 \text{ for } i \text{ odd}\\ & 0 \text{ for } i \text{ even.}\end{aligned} \tag{15}$$

If we consider the action of the class $a : S^0 \to S^\alpha$ on $F(\mathbb{Z}/2_+, BP\mathbb{R})$, then (14) and (15), over all $l \in \mathbb{Z}$, combine into

$$\widehat{H}^*(\mathbb{Z}/2, \mathbb{Z}_{(2)}\{v_n\}[\sigma, \sigma^{-1}]) = \mathbb{Z}/2\{v_n\}[\sigma^2, \sigma^{-2}, a, a^{-1}].$$

We get a similar formula for monomials in the variables v_n. Putting together all the monomials gives (12). Thus, every $x \in E_2$ has an $RO(\mathbb{Z}/2)$-degree $k + l\alpha$. We will call the number $k + l$ the *total degree* of x.

Similarly, for the Borel cohomology spectral sequence, we have

$$\begin{aligned}H^i(\mathbb{Z}/2, \mathbb{Z}_{(2)}\{v_n\sigma^{2l}\}) =& \mathbb{Z}_{(2)} \text{ for } i = 0\\ & \mathbb{Z}/2 \text{ for } i > 0 \text{ even}\\ & 0 \text{ else}\end{aligned}$$

and

$$\begin{aligned}H^i(\mathbb{Z}/2, \mathbb{Z}_{(2)}\{v_n\sigma^{2l+1}\}) =& \mathbb{Z}/2 \text{ for } i > 0 \text{ odd}\\ & 0 \text{ else.}\end{aligned}$$

These combine into

$$H^*(\mathbb{Z}/2, \mathbb{Z}_{(2)}\{v_n\}[\sigma, \sigma^{-1}]) = \mathbb{Z}_{(2)}\{v_n\}[\sigma^2, \sigma^{-2}, a]/(2a).$$

Hence, the E_2-term of the Borel cohomology spectral sequence is

$$BP_\star[\sigma^2, \sigma^{-2}, a]/(2a).$$

Now from E_2 on, (9) is a spectral sequence of graded commutative rings, where the grading is by total degree. By sparsity, σ^{-2^n} survives to $E_{2^{n+1}-1}$. There is the differential

$$d_{2^{n+1}-1}(\sigma^{-2^n}) = v_n a^{2^{n+1}-1}. \tag{16}$$

These are primary differentials in the sense that they arise from the $\mathbb{Z}/2$-equivariant Steenrod operations (see [9]). These differentials determine the entire pattern of differentials in (9), as follows. For a monomial v_R, let $s_R = min\{i \mid r_i > 0\}$. For a monomial $v_R\sigma^{2^s l}a^k$, $k, l \in \mathbb{Z}$, l odd, suppose that $s \leq s_R$. Then $v_R\sigma^{2^s l}a^k$ survives to $E_{2^{s+1}-1}$. This is because $\sigma^{2^s l}$ survives to $E_{2^{s+1}-1}$, and $BP\mathbb{R}$ is a ring spectrum, so there is a multiplication map by $v_R a^k$

$$\Sigma^{|v_R a^k|} BP\mathbb{R} \to BP\mathbb{R}$$

which induces a map of Tate spectral sequences.

Now by (16)

$$\begin{aligned} d_{2^{s+1}-1}(v_R\sigma^{2^s l}a^k) &= v_R d_{2^{s+1}-1}((\sigma^{-2^s})^{-l})a^k \\ &= -l v_s v_R(\sigma^{-2^s})^{-l-1}a^{2^{s+1}-1+k} \\ &= v_s v_R \sigma^{2^s(l+1)}a^{2^{s+1}-1}. \end{aligned}$$

This is not 0 in $E_{2^{s+1}-1}$ by the previous paragraph, with R replaced by $R+\Delta_s$, where $\Delta_s = (0, \ldots, 0, 1, 0, \ldots)$ with the 1 in the s-th position.

By the same argument, if $s \geq s_R + 1$, the monomial $v_R\sigma^{2^{s_R}m}a^k$, with m even, is the target of a differential $d_{2^{s_R+1}-1}$. Hence, in the Tate spectral sequence (9), all elements except $\mathbb{Z}/2[a, a^{-1}]$ are wiped out. In particular, in degrees $k + 0\alpha$, the only surviving term is $\mathbb{Z}/2$ in degree 0. Thus, the fixed point spectrum of $\widehat{BP\mathbb{R}}$ is $H\mathbb{Z}/2$, which is the geometric fixed point spectrum of $BP\mathbb{R}$. Recall that $\mathbb{Z}/2$-equivariant spectra are equivalent if they are equivalent nonequivariantly and on fixed points. So $\widehat{BP\mathbb{R}} \simeq \widetilde{E\mathbb{Z}/2} \wedge BP\mathbb{R}$. Therefore, $BP\mathbb{R} \simeq F(E\mathbb{Z}/2_+, BP\mathbb{R})$, and we have the strong completion theorem for $BP\mathbb{R}$.

Now we turn to the Borel cohomology spectral sequence (8). This is the half of the Tate spectral sequence consisting of elements of filtration degree ≥ 0. By [4], the differentials in the Borel cohomology spectral sequence are exactly the differentials in the Tate spectral sequence whose sources and targets both have filtration degree ≥ 0. Hence, the only elements that survive in (8) are the targets of Tate differentials that originate from negative filtration degrees. The filtration degree of a monomial $v_R\sigma^{2^s l}a^k$ is k, and a differential d_t increases filtration degree by t. So these elements must be of the form $v_R\sigma^{2^{s_R+1}m}a^k$, where $k < 2^{s_R+1} - 1$, and m is even. This is the target of the differential

$$d_{2^{s_R+1}-1}(v_{R'}\sigma^{2^{s_R}(m-1)}a^{k-2^{s_R+1}+1}) = v_R\sigma^{2^{s_R}m}a^k$$

originating in filtration degree $k-2^{s_R+1}-1<0$ in the Tate spectral sequence. Here, R' denotes the sequence of nonnegative integers $(r'_0, r'_1, \ldots)$, where $r'_{s_R} = R_{s_R}-1$, and $r'_i = r_i$ for $i \neq s_R$. Thus, $v_R\sigma^{2^{s_R}m}a^k$ survives as a permanent cycle in the Borel cohomology spectral sequence. Therefore, the E_∞-term of the Borel cohomology spectral sequence consists of elements of the form $v_R\sigma^{2^{s_R+1}l}a^k$, $0 \leq k < 2^{s_R+1}-1$.

We remarked that we will not need to use the exact ring structure of $BP\mathbb{R}_\star$ (as opposed to $E_0BP\mathbb{R}_\star$). However, we will need the following basic fact.

Lemma 17. *Suppose $x \in BP\mathbb{R}_\star$ has total degree ≥ 0, and x is not a unit in $BP\mathbb{R}_{0+0\alpha}$. If xa^k has total degree <0 for some $k \geq 0$, then $xa^k = 0$.*

Proof. By the Borel cohomology spectral sequence, the only nontrivial elements in $BP\mathbb{R}_\star$ with total degree <0 are a^r, $r \geq 0$. For $k+l<0$, multiplication by a is an isomorphism from $BP\mathbb{R}_{k+l\alpha}$ to $BP\mathbb{R}_{k+(l-1)\alpha}$. If $k+l=0$, $2=v_0$ in $BP\mathbb{R}_\star$, so the isomorphism holds only modulo 2. Also, $2a=0$. Suppose that xa^j has total degree 0, i. e. $dim(xa^j) = k-k\alpha$, and that $xa^{j+1} \neq 0$. Then $xa^{j+1} = a$. In particular, $k=0$. Further, xa^j is not divisible by 2, or else xa^{j+1} would be divisible by $2a=0$. Therefore, xa^j is an odd multiple of unity in $BP\mathbb{R}_{0+0\alpha} = \mathbb{Z}_{(2)}$. If $j=0$, then x is a unit in $BP\mathbb{R}_{0+0\alpha}$. If $j>0$, this implies a is invertible in $BP\mathbb{R}_\star$. This is a contradiction, since if 1 is a multiple of a, then it would vanish nonequivariantly. □

By the theory of Real orientations, we also have

$$BP\mathbb{R}_\star BP\mathbb{R} = BP\mathbb{R}_\star[t_i \mid i \geq 1].$$

The elements t_i are in degrees $(2^i-1)(1+\alpha)$, and are the Real analogues of the generators of BP_*BP. Then $(BP\mathbb{R}_\star, BP\mathbb{R}_\star BP\mathbb{R})$ is a Hopf algebroid, where

$$\eta_R(a) = a \tag{18}$$

$$\eta_R(v_n\sigma^{l2^{n+1}}) = \eta_R(v_n)\sigma^{l2^{n+1}}. \tag{19}$$

The second formula follows from reasons of degree (see [9], Theorem 4.11). The formulas for the structure maps on v_n are the same as the formulas for the Hopf algebroid (BP_*, BP_*BP). This is because by formal group law theory, the Hopf algebroid (BP_*, BP_*BP) maps to $(BP\mathbb{R}_\star, BP\mathbb{R}_\star BP\mathbb{R})$ [9].

The Hopf algebroid $(BP\mathbb{R}_\star, BP\mathbb{R}_\star BP\mathbb{R})$ is flat. So by a construction similar to that for the classical Adams-Novikov spectral sequence, we get the Real-oriented Adams-Novikov spectral sequence (1).

3. Elements in Ext^0

In this section, we prove Theorem 6. First, we get an upper bound on

$$Ext^0_{BP\mathbb{R}_\star BP\mathbb{R}}(BP\mathbb{R}_\star, BP\mathbb{R}_\star).$$

Let η_L and η_R be the left and right unit maps of the Hopf algebroid

$$(BP\mathbb{R}_\star, BP\mathbb{R}_\star BP\mathbb{R}).$$

We can think of $Ext^*_{BP\mathbb{R}_\star BP\mathbb{R}}(BP\mathbb{R}_\star, BP\mathbb{R}_\star)$ as the cohomology of the cobar complex

$$Cobar_{BP\mathbb{R}_\star}(BP\mathbb{R}_\star, BP\mathbb{R}_\star BP\mathbb{R}, BP\mathbb{R}_\star),$$

whose n-th term is

$$BP\mathbb{R}_\star BP\mathbb{R} \otimes_{BP\mathbb{R}_\star} \cdots \otimes_{BP\mathbb{R}_\star} BP\mathbb{R}_\star BP\mathbb{R}$$

with n factors. The cobar differentials are the alternating sums of the left unit, the coproducts, and the right unit. So $Ext^0 \subseteq BP\mathbb{R}_\star$ is the kernel of the first cobar differential

$$d_1 = \eta_L - \eta_R = 1 - \eta_R : BP\mathbb{R}_\star \to BP\mathbb{R}_\star BP\mathbb{R}.$$

We have the filtration of $(BP\mathbb{R}_\star, BP\mathbb{R}_\star BP\mathbb{R})$ by powers of the ideal (a). Since $\eta_R(a) = \eta_L(a) = a$, this is indeed a filtration on the Hopf algebroid, and induces a filtration on $Ext^0_{BP\mathbb{R}_\star BP\mathbb{R}}(BP\mathbb{R}_\star, BP\mathbb{R}_\star)$. This filtration results from the Borel cohomology spectral sequence (8), which we used to compute $BP\mathbb{R}_\star$.

Define

$$BPA_\star = BP_\star[a]/(v_n a^{2^{n+1}-1} \mid n \geq 0)$$

(see 13). Then

$$E_0 BP\mathbb{R}_\star = \mathbb{Z}_{(2)}[v_n \sigma^{l2^{n+1}}, a \mid n \geq 0,\ l \in \mathbb{Z}]/(v_0 = 2, v_n a^{2^{n+1}-1} = 0) \subseteq BPA_\star[\sigma, \sigma^{-1}].$$

Also, let

$$BPA_\star BPA = BPA_\star[t_i \mid i \geq 1].$$

Then

$$E_0 BP\mathbb{R}_\star BP\mathbb{R} \subseteq BPA_\star BPA[\sigma, \sigma^{-1}].$$

Thus we can define a flat Hopf algebroid structure on

$$(BPA_\star[\sigma, \sigma^{-1}], BPA_\star BPA[\sigma, \sigma^{-1}])$$

by setting

$$\eta_R(\sigma) = \eta_L(\sigma) = \sigma \tag{20}$$

and $\eta_R(a) = \eta_L(a) = a$. The coproduct structure formulas on $v_i, i \geq 0$ are the same as in $BP_* BP$. By (20), $(BPA_\star, BPA_\star BPA)$ is a flat sub-Hopf algebroid of $(BPA_\star[\sigma, \sigma^{-1}], BPA_\star BPA[\sigma, \sigma^{-1}])$, and

$$\begin{aligned} &Ext^0_{BPA_\star BPA[\sigma,\sigma^{-1}]}(BPA_\star[\sigma, \sigma^{-1}], BPA_\star[\sigma, \sigma^{-1}]) \\ &= Ext^0_{BPA_\star BPA}(BPA_\star, BPA_\star)[\sigma, \sigma^{-1}] \\ &\subseteq BPA_\star[\sigma, \sigma^{-1}]. \end{aligned}$$

From the map of Hopf algebroids, we also get a map

$$Ext^0_{E_0 BP\mathbb{R}_\star BP\mathbb{R}}(E_0 BP\mathbb{R}_\star, E_0 BP\mathbb{R}_\star) \xrightarrow{f} Ext^0_{BPA_\star BPA}(BPA_\star, BPA_\star)[\sigma, \sigma^{-1}].$$

We have the following commutative diagram

$$\begin{array}{ccc} Ext^0_{E_0BP\mathbb{R}_\star BP\mathbb{R}}(E_0BP\mathbb{R}_\star, E_0BP\mathbb{R}_\star) & \xrightarrow{i} & E_0BP\mathbb{R}_\star \\ \downarrow f & & \downarrow j \\ Ext^0_{BPA_\star BPA}(BPA_\star, BPA_\star)[\sigma, \sigma^{-1}] & \xrightarrow[k]{} & BPA_\star[\sigma, \sigma^{-1}]. \end{array}$$

Since all the three maps i, j and k are inclusions, f is also an inclusion. This gives

$$Ext^0_{BPA_\star BPA}(BPA_\star, BPA_\star)[\sigma, \sigma^{-1}] \cap E_0BPR_\star$$

as an upper bound for $Ext^0_{BP\mathbb{R}_\star BP\mathbb{R}}(BP\mathbb{R}_\star, BP\mathbb{R}_\star)$.

To calculate $Ext^0_{BPA_\star BPA}(BPA_\star, BPA_\star)$, consider the cobar complex

$$Cobar_{BPA_\star}(BPA_\star, BPA_\star BPA, BPA_\star).$$

Since $a = \eta_L(a) = \eta_R(a)$, the coproduct formula on a is

$$\psi(a) = a \otimes 1 = 1 \otimes a.$$

Thus, the cobar complex is graded by powers of a. For $n \geq 0$, let $I_n \subset BP_*$ be the ideal $(v_0, \ldots, v_{n-1})$. In degrees t where $2^n - 1 \leq t < 2^{n+1} - 1$, $v_m a^t = 0$ for all $m < n$, so

$$\begin{aligned} \oplus_{t=2^n-1}^{2^{n+1}-2} Cobar_{BPA_\star}(BPA_\star, &BPA_\star BPA, BPA_\star)_t \\ &\cong \oplus_{t=2^n-1}^{2^{n+1}-1} Cobar_{BP_*}(BP_*, BP_*BP, BP_*/I_n)\{a^t\}. \end{aligned} \tag{21}$$

Thus,

$$Ext^0_{BPA_\star BPA}(BPA_\star, BPA_\star) = \oplus_{n\geq 0}(\oplus_{t=2^n-1}^{2^{n+1}-1} Ext^0_{BP_*BP}(BP_*, BP_*/I_n)\{a^t\}).$$

By the Morava-Landweber theorem [10, 11],

$$Ext^0_{BP_*BP}(BP_*, BP_*) = \mathbb{Z}_{(2)}$$

and

$$Ext^0_{BP_*BP}(BP_*, BP_*/I_n) = \mathbb{Z}/2[v_n]$$

for $n \geq 1$. So in degree 0, $Ext^0_{BPA_\star BPA}(BPA_\star, BPA_\star)$ is $\mathbb{Z}_{(2)}$, generated over $\mathbb{Z}/2$ by $v_0 = 2$. In degrees t where $2^n - 1 \leq t < 2^{n+1} - 1$, $n \geq 1$, it is

$$\oplus_{t=2^n-1}^{2^{n+1}-1}(\mathbb{Z}/2[v_n])\{a^t\}.$$

This gives that the upper bound on $Ext^0_{BP\mathbb{R}_\star BP\mathbb{R}}(BP\mathbb{R}_\star, BP\mathbb{R}_\star)$ is generated as a $\mathbb{Z}_{(2)}$-module by elements of the form

$$a^t,\ t \geq 0 \text{ and } v_n^r \sigma^{l2^{n+1}} a^t \tag{22}$$

where $r \geq 0$, $l \in \mathbb{Z}$ and $2^n - 1 \leq t \leq 2^{n+1} - 2$.

To finish the proof of Theorem 6, we need to show that elements of the above form are in fact in $Ext^0_{BP\mathbb{R}_\star BP\mathbb{R}}(BP\mathbb{R}_\star, BP\mathbb{R}_\star)$. Regardless of the exact multiplicative structure of $BP\mathbb{R}_\star$, we can choose a set of generators for the $\mathbb{Z}_{(2)}$-module spanned by (22), consisting of elements of the form

$$x = (v_n \sigma^{l_1 2^{n+1}}) \cdots (v_n \sigma^{l_t 2^{n+1}}) a^t, \;\; 2^n - 1 \leq t \leq 2^{n+1} - 1. \tag{23}$$

By (19), when we apply η_R to each of the factors $v_n \sigma^{l_i 2^{n+1}}$, we obtain $v_n \sigma^{l_i 2^{n+1}}$ plus multiples of elements of the form $v_m \sigma^{l_i 2^{n+1}}$, with $m < n$. However, by Lemma 17, these extra terms are annihilated by $a^{2^{m+1}-1}$, which divides a^j. Thus, $\eta_R(x) = x$, as claimed.

This shows that $Ext^0_{BP\mathbb{R}_\star BP\mathbb{R}}(BP\mathbb{R}_\star, BP\mathbb{R}_\star)$ is generated by elements of the form (22). For $n = 0$, the elements $v_0 \sigma^{2l}$ generate copies of $\mathbb{Z}_{(2)}$ since in the classical case, v_0 generates a copy $\mathbb{Z}_{(2)}$. For $n \geq 1$, the elements generate copies of $\mathbb{Z}/2$ since they contain nontrivial powers of a, and $2a = 0$. Likewise, a^t, $t > 1$ generate copies of $\mathbb{Z}/2$. This proves Theorem 6.

4. Hopf Invariant One Type Elements

In this section, we will consider the class of Ext^0 elements $v_n \sigma^{l 2^{n+1}} a^{2^n - 1}$. For $l = 0$, the element $v_n a^{2^n-1}$ is in degree $(2^n - 1) + 0\alpha$. In Propositions 7.13 and 7.14 of [9], it was shown that there is a filtration on $BP\mathbb{R}_\star$, such that there is an algebraic Novikov spectral sequence with E_2-term $Ext_{P_\star[a]}(\mathbb{Z}/2[a], E_0 BP\mathbb{R}_\star)$, converging to the E_2-term of the Real Adams-Novikov spectral sequence, where $P_\star[a]$ is a certain Hopf algebra over $\mathbb{Z}/2[a]$. There is also a Cartan-Eilenberg spectral sequence with the same E_2-term, and converging to the E_2-term of the $\mathbb{Z}/2$-equivariant Adams spectral sequence of Greenlees [6]. (This is the Adams spectral sequence based on the Borel cohomology Steenrod algebra $(H^c_\star, A^{cc}_\star)$. Here, H is the equivariant Eilenberg-MacLane spectrum indexed on the complete $\mathbb{Z}/2$-universe, obtained by applying the universe change functor to nonequivariant $H\mathbb{Z}/2$, considered as a fixed spectrum over the trivial $\mathbb{Z}/2$-universe. Then

$$H^c_\star = F(E\mathbb{Z}/2_+, H)_\star$$
$$A^{cc}_\star = F(E\mathbb{Z}/2_+, H \wedge H)_\star. \;)$$

Also, in degrees $k + 0\alpha$, the nonequivariant Adams spectral sequence E_2-term is a summand of the $\mathbb{Z}/2$-equivariant Adams spectral sequence E_2-term (see [8], Proposition 6.12). In this sense, the Real Adams-Novikov E_2-elements $v_n a^{2^n-1}$ correspond to the Hopf invariant one element h_n in the classical (nonequivariant) Adams spectral sequence [8, 9]. Recall from [1] that for $n \leq 3$, h_n is a permanent cycle, and represents the Hopf invariant one maps $S^{2^{n+1}-1} \to S^n$. Also by [1], one can say that nonequivariantly, the Hopf invariant one property holds for n if S^{2^n-1} is parallelizable.

The Hopf invariant one property in the $\mathbb{Z}/2$-equivariant category can be interpreted as follows. For any n, consider the free unit sphere $S(2^n\alpha)$ in the representation $2^n\alpha$.

The tangent bundle of $S(2^n\alpha)$ has the property that

$$\tau_{S(2^n\alpha)} \oplus 1 = 2^n\alpha.$$

The $\mathbb{Z}/2$-equivariant Hopf invariant one property can be formulated to say that $S(2^n\alpha)$ is parallelizable, i. e. $\tau_{S(2^n\alpha)} \cong 2^n - 1$, which is true if and only if $n \leq 3$. So in this case, we have

$$2^n|_{S(2^n\alpha)} \cong 2^n\alpha|_{S(2^n\alpha)}.$$

Stably, this gives

$$S(2^n\alpha)_+ \simeq \Sigma^{2^n(\alpha-1)}S(2^n\alpha)_+.$$

Consider the usual cofiber sequence

$$S(2^n\alpha)_+ \to S^0 \stackrel{a^{2^n}}{\to} S^{2^n\alpha}.$$

The Hopf invariant one map, as an element of the stable homotopy groups of spheres, is the composition

$$S^{2^n\alpha-1} \to S(2^n\alpha)_+ \stackrel{\simeq}{\to} \Sigma^{2^n(\alpha-1)}S(2^n\alpha)_+ \to S^{2^n(\alpha-1)}$$

where the last map collapses $S(2^n\alpha)$. This is an element of degree $(2^n-1)+0\alpha$, and by the comparison with the Adams spectral sequence, it is represented by $v_n a^{2^n-1}$ for $n \leq 3$ (see [8], Section 6.2).

By the previous section, we also have the elements

$$v_n\sigma^{l2^{n+1}}a^{2^n-1} \in Ext^0_{BP\mathbb{R}_\star BP\mathbb{R}}(BP\mathbb{R}_\star, BP\mathbb{R}).$$

To see these elements, note that for $n \leq 3$, we can iterate the periodicity of $S(2^n\alpha)_+$ to get families of Hopf invariant one maps

$$S^{2^n\alpha-1} \to S(2^n\alpha)_+ \stackrel{\simeq}{\to} \Sigma^{l2^n(\alpha-1)}S(2^n\alpha)_+ \to S^{l2^n(\alpha-1)}. \tag{24}$$

Proposition 25. *For l even, the map (24) is represented by 0 in*

$$Ext^0_{BP\mathbb{R}_\star BP\mathbb{R}}(BP\mathbb{R}_\star, BP\mathbb{R}_\star).$$

For l odd, (24) is represented by $v_n\sigma^{l2^{n+1}}a^{2^n-1}$ in $Ext^0_{BP\mathbb{R}_\star BP\mathbb{R}_\star}(BP\mathbb{R}_\star, BP\mathbb{R}_\star)$.

Proof. For $l=0$, the map (24) is 0 since it is just the composition of the two maps of the cofiber sequence

$$S^{2^n\alpha-1} \to S(2^n\alpha)_+ \to S^0.$$

For general l, recall the construction of the element $v_n\sigma^{2^{n+1}}$ ([9], Comment after Theorem 4.11). Namely, consider the cofiber sequence

$$S((2^{n+1}-1)\alpha)_+ \to S^0 \stackrel{a^{2^{n+1}-1}}{\longrightarrow} S^{2^{n+1}-1}. \tag{26}$$

Applying $BP\mathbb{R}^\star$ gives the connecting map

$$\delta : BP\mathbb{R}^\star S((2^{n+1}-1)\alpha)_+ \to BP\mathbb{R}^{\star+1-(2^{n+1}-1)\alpha}.$$

Since $v_n a^{2^{n+1}-1} = 0$, there is an element

$$s \in BP\mathbb{R}^{2^n(\alpha-1)} S((2^{n+1}-1)\alpha)_+$$

such that $\delta(s) = v_n$. Consider the analogue of the Borel cohomology spectral sequence ${}_{(2^{n+1}-1)}E$, converging to

$$F(S((2^{n+1}-1)\alpha)_+, BP\mathbb{R})_\star$$

obtained by replacing $E\mathbb{Z}/2_+$ by $S((2^{n+1}-1)\alpha)_+$ in the construction of the spectral sequence (8). This has the same E_2-term as the E_2-term of the Borel cohomology spectral sequence (8) for $BP\mathbb{R}$, but restricted to filtration degrees t, with $0 \leq t \leq 2^{n+1}-2$. The differentials are exactly the differentials of (8) whose sources and targets are both in filtration degrees t, $0 \leq t \leq 2^{n+1}-2$. We compare the Borel cohomology spectral sequences for $F(S((2^{n+1}-1)\alpha)_+, BP\mathbb{R})$ and for $BP\mathbb{R}$. Recall the differential

$$d_{2^{n+1}-1}\sigma^{-2^n} = v_n a^{2^{n+1}-1}$$

in the Borel cohomology spectral sequence (8) for $BP\mathbb{R}$ (see 16). But in the spectral sequence ${}_{(2^{n+1}-1)}E$ discussed above, the target does not exist, and the differential turns into the connecting map δ. Thus, the invertible element σ^{-2^n} in the Borel cohomology spectral sequence for $F(S((2^{n+1}-1)\alpha)_+, BP\mathbb{R})$ is a permanent cycle, and is realized by the element $s \in BP\mathbb{R}^\star S((2^{n+1}-1)\alpha)_+$. In particular, s is an invertible element of $BP\mathbb{R}^* S((2^{n+1}-1)\alpha)_+$, whose inverse is represented by σ^{2^n}. Comparing Tate spectral sequences for $S((2^{n+1}-1)\alpha)_+ \wedge BP\mathbb{R}$ and $BP\mathbb{R}$, one sees that $BP\mathbb{R}_\star(S((2^{n+1}-1)\alpha)_+)$ is in fact $2^n(\alpha-1)$-periodic, and the periodicity operator is realized by cap product with the cohomology class s. Also, s^2 corresponds to the periodicity operator $\sigma^{2^{n+1}}$ in $BP\mathbb{R}_\star$ on monomials containing $v_i\sigma^{l2^{i+1}}$, $i \leq n$.

Now compare the cofiber sequences (26) for $S(2^n\alpha)_+$ and for $S((2^{n+1}-1)\alpha)_+$ via the inclusion

$$S(2^n\alpha)_+ \to S((2^{n+1}-1)\alpha)_+.$$

We have the commutative diagram

$$\begin{array}{ccc} BP\mathbb{R}^\star S((2^{n+1}-1)\alpha)_+ & \xrightarrow{\delta} & BP\mathbb{R}^{\star+1-(2^{n+1}-1)\alpha} \\ \downarrow & & \downarrow {\scriptstyle a^{2^n-1}} \\ BP\mathbb{R}^\star S(2^n\alpha)_+ & \xrightarrow{\delta} & BP\mathbb{R}^{\star+1-2^n\alpha}. \end{array}$$

Let s' denote the image of the class $s \in BP\mathbb{R}^\star S((2^{n+1}-1)\alpha)_+$ in $BP\mathbb{R}^\star S(2^n\alpha)_+$. If Hopf invariant one holds (i. e. for $n \leq 3$), then we compare the spectral sequences ${}_{(2^n)}E$ and ${}_{(2^{n+1}-1)}E$ for $F(S(2^n\alpha)_+, BP\mathbb{R})$ and $F(S((2^{n+1}-1)\alpha)_+, BP\mathbb{R})$. In particular, by arguments similar to that for $F(S((2^{n+1}-1)\alpha)_+, BP\mathbb{R})$, we find that σ^{2n} is a permanent cycle in the Borel cohomology spectral sequence for $F(S(2^n\alpha)_+, BP\mathbb{R})$, and is realized by s'. So s' is an invertible element. It is the only element in $BP\mathbb{R}^{2^n(\alpha-1)} S(2^n\alpha)_+$, so it realizes the $2^n(1-\alpha)$-periodicity of $S(2^n\alpha)_+$. This identifies Real Adams-Novikov spectral sequence representatives of all the individual maps in (24). Namely, for l odd, we see that the map (24) is represented by the element $\sigma^{(l-1)2^n}v_n a^{2^n-1} \in Ext^0_{BP\mathbb{R}_\star BP\mathbb{R}}(BP\mathbb{R}_\star, BP\mathbb{R}_\star)$. For l even, the Hopf invariant one map (24) is represented by 0 in $Ext^0_{BP\mathbb{R}_\star BP\mathbb{R}}(BP\mathbb{R}_\star, BP\mathbb{R}_\star)$. This is because (24) is 0 when smashed with $BP\mathbb{R}_\star$, since as shown in [9], the elements $v_n\sigma^{l2^n}a^{2^n-1} \in BP\mathbb{R}_\star S(2^n\alpha)_+$ map to 0 in $BP\mathbb{R}_\star$ for l odd. □

References

[1] J. F. Adams. On the non-existence of elements of Hopf invariant one. *Ann. of Math.* 72 (1960), 20-104.

[2] S. Araki. τ-cohomology theories. *Japan J. Math.* (N. S.) 4 (1978), no. 2, 363-416.

[3] M. F. Atiyah. K-theory and Reality. *Quarterly J. Math. Oxford* Sec. (2), 17 (1966) 367-386.

[4] A. K. Bockstedt, I. Madsen. Topological cyclic homology of the integers, K-theory. *Astérisque* No. 226 (1994) 7-8, 57-143.

[5] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules and algebras in stable homotopy theory, with an appendix by M. Cole.* Mathematical Surveys and Monographs 47, AMS, 1997.

[6] J. P. C. Greenlees. *Adams spectral sequences in equivariant topology.* Ph. D. thesis, Cambridge University, 1985.

[7] J. P. C. Greenlees and J. P. May. Generalized Tate cohomology. *Mem. Amer. Math. Soc.* 113 (1995) no. 543.

[8] P. Hu. Cobordism of Real manifolds and calculations with the Real-oriented Adams-Novikov spectral sequence. Ph. D. thesis, 1998.

[9] P. Hu and I. Kriz. Real-oriented homotopy theory and an analogue of the Adams-Novikov spectral sequence. To appear in *Topology.*

[10] P. S. Landweber. Annihilator ideals and primitive elements in complex cobordism. *Illinois J. Math.* 17 (1973) 273-284.

[11] J. Morava. Noetherian localisations categories of cobordism comodules. *Annals of Math.* 121 (1985) 1-39.

[12] D. Ravenel. *complex cobordism and stable homotopy groups of spheres.* Academic Press, 1986.