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KERVAIRE INVARIANT ONE
[after M. A. Hill, M. J. Hopkins, and D. C. Ravenel]

by **Haynes MILLER**

INTRODUCTION

The *Kervaire sphere* K^{4k+1} (as described by Hirzebruch [13]) is the intersection of the complex hypersurface $z_0^3 + z_1^2 + \cdots + z_{2k+1}^2 = 0$ in $\mathbb{C}^{2k+2}$ with a sphere centered at the singularity at the origin. It is a smooth manifold homeomorphic to a sphere, and it is known to bound a smooth manifold W^{4k+2}. Attach a disk to W^{4k+2} by a homeomorphism $S^{4k+1} \to K^{4k+1}$, to obtain a topological manifold M^{4k+2}. Kervaire [16] showed that M^{10} (the case $k = 2$) is not of the homotopy type of a smooth manifold; and consequently that K^9 is not diffeomorphic to S^9.

The import of this example was brought out in subsequent joint work of Kervaire and Milnor [17] on "homotopy spheres." A homotopy n-sphere is a closed smooth manifold of the homotopy type of S^n (and hence, by Smale, of the homeomorphism type of S^n if $n > 4$). The set of oriented diffeomorphism classes of homotopy n-spheres forms a group Θ_n under connected sum. It turns out that any embedding of a homotopy sphere in high codimension has a trivializable normal bundle. A choice of trivialization gives a "framed manifold," and hence a class in the cobordism group Ω_n^{fr} of closed framed n-manifolds. The coset modulo the subgroup J_n of framings of S^n is well defined, and with $C_n = \Omega_n^{\mathrm{fr}}/J_n$ we get a homomorphism

$$\tau_n : \Theta_n \to C_n$$

which Kervaire and Milnor almost completely describe. They show that τ_{4k-1} is surjective with cyclic kernel of order explicitly described in terms of Bernoulli numbers; that τ_{4k} is an isomorphism; and that there is an exact sequence

$$0 \longrightarrow \Theta_{4k+2} \xrightarrow{\tau_{4k+2}} C_{4k+2} \longrightarrow \mathbb{Z}/2 \longrightarrow \Theta_{4k+1} \xrightarrow{\tau_{4k+1}} C_{4k+1} \longrightarrow 0$$

So in these dimensions there is an alternative: either there is a framed bordism class in $\Omega_{4n+2}^{\mathrm{fr}}$ that is not represented by a homotopy sphere, or there is a homotopy $(4k+1)$-sphere that bounds as a framed manifold. In [16], Kervaire had shown that the second alternative obtains when $k = 2$.

The composite $\kappa : \Omega_{4k+2}^{\mathrm{fr}} \to C_{4k+2} \to \mathbb{Z}/2$ is the *Kervaire invariant.* In small dimensions, framed manifolds of Kervaire invariant one are not uncommon. The parallelizations of S^1, S^3, and S^7 determine framings of their normal bundles, and with the resulting product framings $S^1 \times S^1$, $S^3 \times S^3$, and $S^7 \times S^7$ all have Kervaire invariant one (in dimensions 2, 6, and 14). In [17] Kervaire and Milnor speculated that these may be the only examples.

The Pontryagin-Thom construction establishes an isomorphism between Ω_n^{fr} and the stable homotopy group $\pi_n^s(S^0) = \pi_n(\mathbb{S})$ (for which see below), and in 1969 Browder [5] gave a homotopy theoretic interpretation of the Kervaire invariant which implied that $\kappa = 0$ unless $4k+2$ is of the form $2(2^j - 1)$. Homotopy theoretic calculations [22, 4, 2] soon muddied the waters by providing examples in dimensions 30 and 62. Much effort in the 1970's was focused on understanding the role of the Kervaire invariant and attempting inductive constructions of elements (always written θ_j) of Kervaire invariant one in dimension $2(2^j - 1)$ for any j. For example, Barratt and Mahowald proved that if θ_j exists with order 2 and square zero, then θ_{j+1} exists. Cohen, Jones, and Mahowald [6] defined a Kervaire invariant for oriented n-manifolds immersed in $\mathbb{R}^{n+2}$, and showed that this invariant does take on the value 1 in dimensions $2(2^j - 1)$.

The focus of this report is the following result.

Theorem 0.1 (Hill, Hopkins, Ravenel, 2009, [11]). — *The Kervaire invariant* $\kappa : \Omega_{4k+2}^{\mathrm{fr}} \to \mathbb{Z}/2$ *is trivial unless* $4k+2 = 2, 6, 14, 30, 62$, *or (possibly)* 126.

The case $4k+2 = 126$ remains open.

In his proof that κ is trivial in dimension 10, Kervaire availed himself of the current state of the art in homotopy theory (mainly work of Serre). Kervaire and Milnor relied on contemporaneous work of Adams. Over the intervening fifty years, further developments in homotopy theory have been brought to bear on the Kervaire invariant problem. In 1964 Brown and Peterson brought spin bordism into play to show that κ is trivial in dimensions $8k+2$, and Browder's work used the Adams spectral sequence. Over the past quarter century, however, essentially no progress has been made on this problem till the present work.

Hill, Hopkins, and Ravenel (hereafter HHR) marshall three major developments in stable homotopy theory in their attack on the Kervaire invariant problem:

1. The chromatic perspective based on work of Novikov and Quillen and pioneered by Landweber, Morava, Miller, Ravenel, Wilson, and many more recent workers;
2. The theory of structured ring spectra, implemented by May and many others; and
3. Equivariant stable homotopy theory, as developed by May and collaborators.

The specific application of equivariant stable homotopy theory was inspired by analogy with a fourth development, the motivic theory initiated by Voevodsky and Morel, and uses as a starting point the theory of "Real bordism" investigated by Landweber, Araki, Hu and Kriz. The introduction of equivariant homotopy theory into the study of the Kervaire invariant problem is reminiscent of Gunnar Carlsson's use of it in his resolution, a generation ago, of the Segal Conjecture. In their application of these ideas, HHR require significant extensions of the existing state of knowledge of this subject, and their paper provides an excellent account.

1. THE KERVAIRE INVARIANT

Any compact smooth n-manifold M embeds into a Euclidean space, and in high codimension any two embeddings are isotopic. The normal bundle ν is thus well defined up to addition of a trivial bundle. A *framing* of M is a bundle isomorphism $t : \nu \to M \times \mathbb{R}^q$. A parallelization of M—a trivialization of its tangent bundle—determines a framing, but not conversely: any codimension one embedding into a Euclidean space determines a framing, but most spheres, for example, are not parallelizable. The Pontryagin-Thom construction is the induced contravariant map on one-point compactifications, $S^{n+q} = \mathbb{R}^{n+q}_+ \to M \times \mathbb{R}^q_+ = M_+ \wedge S^q$, giving an elment of $\pi_{n+q}(M_+ \wedge S^q)$.

This group becomes independent of q for $q > n$, and is termed the nth *stable homotopy group* $\pi^s_n(M_+) = \lim_{q\to\infty} \pi_{n+q}(M_+ \wedge S^q)$ of M. The framing t determines a "stable homotopy theory fundamental class" $[M, t] \in \pi^s_n(M_+)$. Composing with the map collapsing M to a point gives an element of $\pi^s_n(S^0)$. A bordism between framed manifolds determines a homotopy between their Pontryagin-Thom collapse maps, and this construction gives an isomorphism from the bordism group to $\pi^s_n(S^0)$.

A framing of a manifold M of dimension $4k + 2$ determines additional structure in the cohomology of M. A cohomology class $x \in H^{2k+1}(M; \mathbb{F}_2)$ is represented by a well-defined homotopy class of maps $M_+ \to K(\mathbb{F}_2, 2k + 1)$. When we apply the stable homotopy functor we get a map $\pi^s_n(M_+) \to \pi^s_n(K(\mathbb{F}_2, 2k + 1))$. A calculation shows that the target group is $\mathbb{Z}/2$, so the class $[M, t]$ determines an element $\varphi_t(x) \in \mathbb{Z}/2$.

This *Kervaire form* $\varphi_t : H^{2k+1}(M; \mathbb{F}_2) \to \mathbb{Z}/2$ turns out to be a *quadratic refinement* of the intersection pairing $x \cdot y = \langle x \cup y, [M]\rangle$—that is to say,

$$\varphi_t(x + y) = \varphi_t(x) + \varphi_t(y) + x \cdot y$$

In the group (under direct sum) of isomorphism classes of finite dimensional $\mathbb{F}_2$ vector spaces with nondegenerate quadratic form, the Kervaire forms of cobordant framed manifolds are congruent modulo the subgroup generated by the *hyperbolic* quadratic space (H, q) with $H = \langle a, b\rangle$, $q(a) = q(b) = q(0) = 0$, $q(a + b) = 1$. This quotient is the *Witt group* of $\mathbb{F}_2$, and is of order 2. The element of $\mathbb{Z}/2$ corresponding to a quadratic space is given by the *Arf invariant*, given by the more popular of 0, 1, as values of q. The *Kervaire invariant* of (M, t) is then the Arf invariant of the quadratic space $(H^{2k+1}(M; \mathbb{F}_2), q_t)$. This defines the homomorphism

$$\kappa : \pi^s_{4k+2}(S^0) = \Omega^{\mathrm{fr}}_{4k+2} \to \mathbb{Z}/2$$

Regarding κ as defined on $\pi^s_{4k+2}(S^0)$ invites the question: What is a homotopy-theoretic interpretation of the Kervaire invariant? This was answered by Browder in a landmark paper [5], in terms of the Adams spectral sequence.

Discussion of these matters was streamlined in the 1960's by use of the *stable homotopy category* $h\mathcal{S}$. Its objects are designed to represent cohomology theories. There are many choices of underlying categories of spectra, but they all lead the same homotopy category $h\mathcal{S}$, which is additive, indeed triangulated, and symmetric monoidal (with tensor product given by the "smash product" $\wedge$). It is an analog of the derived category of a commutative ring. There is a "stabilization" functor Σ^∞ from the homotopy category of pointed CW complexes to $h\mathcal{S}$. It sends the two point space S^0 to the "sphere spectrum" $\Sigma^\infty S^0 = \mathbb{S}$, which serves as the unit for the smash product. The suspension functor Σ is achieved by smashing with $\Sigma^\infty S^1$. The *homotopy* of a spectrum E is $\pi_n(E) = [\Sigma^n\mathbb{S}, E]$, so that, for a space X, $\pi_n^s(X) = \pi_n(\Sigma^\infty X_+)$. Ordinary mod 2 cohomology of a space X (which we abbreviate to $H^*(X)$) is represented by the *Eilenberg-Mac Lane spectrum* $\mathbb{H}$—$H^n(X) = [\Sigma^\infty X_+, \Sigma^n\mathbb{H}]$—and, as explained by G. Whitehead, homology is obtained as $H_n(X) = [\Sigma^n\mathbb{S}, \Sigma^\infty X_+ \wedge \mathbb{H}]$. The cup product is represented by a structure map $\mathbb{H}\wedge\mathbb{H} \to \mathbb{H}$, making $\mathbb{H}$ into a "ring spectrum." The unit map for this ring struture, $\mathbb{S} \to \mathbb{H}$, represents a generator of $\pi_0(\mathbb{H}) = H^0(\mathbb{S})$. The graded endomorphism algebra of the object $\mathbb{H}$ is the well-known *Steenrod algebra* $\mathcal{A}$ of stable operations on mod 2 cohomology.

Evaluation of homology gives a natural transformation

$$d_X : \pi_n(X) = [\Sigma^n\mathbb{S}, X] \to \mathrm{Hom}_{\mathcal{A}}(H^*(X), H^*(\Sigma^n\mathbb{S}))$$

which is an isomorphism if $X = \mathbb{H}$. This leads to the *Adams spectral sequence*, which takes the form

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(X), \mathbb{F}_2) = \mathrm{Ext}_{\mathcal{A}}^{s}(H^*(X), H^*(\Sigma^t\mathbb{S})) \Longrightarrow \pi_{t-s}(X)$$

for X a spectrum such that $\pi_n(X) = 0$ for $n << 0$ and $H_n(X)$ is finite dimensional for all n. It converges to an appropriate 2-adic completion of the homotopy groups of X. In particular,

$$E_2^{s.t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_{t-s}(\mathbb{S})\,.$$

To this day these Ext groups remain quite mysterious overall, but Adams immediately computed them for $s \leq 2$. $E_2^{0,t}$ is of course just $\mathbb{F}_2$ in $t = 0$. The edge homomorphism

$$e : \pi_n(\mathbb{S}) \to E_2^{1,n+1}$$

is the "mod 2 Hopf invariant." One interpretation of this invariant is that $e(\alpha) \neq 0$ if and only if the mapping cone $S^0 \cup_\alpha e^{n+1}$ supports a nonzero Steenrod operation of positive degree. $E_2^{1,*}$ is the dual of the module of indecomposables in the Steenrod algbra, known since Adem to be given by the Steenrod squares $\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4, \mathrm{Sq}^8, \ldots$. The element dual to Sq^{2^j} is denoted by $h_j \in E_2^{1,2^j}$. There is an element of Hopf invariant one in $\pi_n(\mathbb{S})$ if and only if there is a nonzero permanent cycle in $E_2^{1,n+1}$. Adams proved that $d_2 h_j = h_0 h_{j-1}^2$, and that $h_0 h_{j-1}^2 \neq 0$ for $j > 3$ in $E_2^{3,2^j+1}$ (where the product is the usual one occuring in Ext, corresponding to the composition product in $\pi_*(\mathbb{S})$), so elements of Hopf invariant one occur only in dimensions 0, 1, 3, and 7—a celebrated theorem of Adams [1] from 1960.

Next, there is an edge homomorphism

$$f : \pi_{2k}(\mathbb{S}) \to E_2^{2,2k+2}$$

Adams computed these vector spaces and found that a basis is given by $\{h_i h_j : i = j \text{ or } i < j-1\}$. The product structure of the spectral sequence, together with computations in $E_2^{4,*}$, imply [20] that the only survivors to E_3 are h_0h_2, h_0h_3, h_2h_4, h_2h_5, h_3h_6, and the infinite families h_j^2 and h_1h_j. This is the context of the following key result.

THEOREM 1.1 (Browder, 1969, [5]). — *The Kervaire invariant vanishes except in dimensions of the form $2(2^j-1)$, where it is detected by h_j^2.*

Among subsequent simplifications of Browder's argument, let me point to Lannes's [18], in which consideration of manifolds with boundary identifies the Kervaire functional with a certain Hopf invariant. In more recent work this argument as been further simplified, and in joint work Lannes and the author have given a proof employing characteristic numbers of manifolds with corners. Later work [15] also revealed that it is not difficult to see, without invoking the Adams spectral sequence, that these are the only dimensions in which the Kervaire invariant can be nontrivial.

Thus the Kervaire invariant problem amounts to the question of whether h_j^2 is a permanent cycle. Since h_j is permanent for $j \leq 3$, the multiplicative structure implies that h_j^2 is too—and the squares of the framed manifolds representing the Hopf invariant one classes are manifolds of Kervaire invariant one. Difficult calculations in the Adams spectral sequence [22, 4, 2] verify that h_4^2 and h_5^2 are also permanent. The theorem of HHR amounts to the assertion that h_j^2 supports a differential in the Adams spectral sequence for all $j > 6$. One of the reasons homotopy theorists were hoping that the Kervaire classes did survive was that they had no idea what the targets of differentials on them might be. Unfortunately, the work of HHR sheds no light on this question—though proving that these classes die without saying how is the genius of the paper.

Incidentally, it is known that h_0h_2, h_0h_3, and h_2h_4 survive to homotopy classes and that h_2h_5 does not. The fates of h_3h_6 and of h_6^2 are at present unkown. All members of the remaining infinite family in $E_2^{2,*}$, the h_1h_j, do survive: this is a famous theorem of Mark Mahowald [19]. In some respects their behavior is the opposite of what was hoped for of the Kervaire classes. Cohen, Jones, and Mahowald [6] employed Mahowald's method to show that h_j^2 survives in the Adams spectral sequence for $\pi_*^s(\mathbb{C}P^\infty)$.

By the Hopf invariant one theorem, the operation $Sq^{2^{j+1}}$ must be trivial on the mapping cone of any map $S^{2^{j+1}-1} \to S^0$ if $j \geq 3$. What if we consider the mapping cone of a map $S^{2^{j+1}-1} \to S^0 \cup_2 e^1$ instead? The form of Adams's differential $d_2h_{j+1} = h_0h_j^2$ implies that $\mathrm{Sq}^{2^{j+1}}$ is nonzero in the mapping cone of such a map if and only if the composite $S^{2^{j+1}-1} \to S^0 \cup_2 e^1 \to S^1$ is a class of Kervaire invariant one. HHR leaves open the identity of the simplest space in which $\mathrm{Sq}^{2^{j+1}}$ can be nonzero.

The Kervaire invariant question is also important unstably. The 2-primary components of the homotopy groups of spheres are related to each other via the *EHP sequence*

$$\cdots \longrightarrow \pi_q(S^n) \xrightarrow{H} \pi_q(S^{2n-1}) \xrightarrow{P} \pi_{q-2}(S^{n-1}) \xrightarrow{E} \pi_{q-1}(S^n) \longrightarrow \cdots$$

Since $\pi_q(S^{2n-1}) = 0$ for $q < 2n-1$, the EHP sequence implies that the suspension maps starting with $E : \pi_{2n-4}(S^{n-1}) \to \pi_{2n-3}(S^n)$ are all isomorphisms, and all these groups are isomorphic to the stable homotopy group $\pi_{n-3}(\mathbb{S})$. Next, we have the exact sequence

$$\pi_{2n-1}(S^n) \xrightarrow{H} \pi_{2n-1}(S^{2n-1}) \xrightarrow{P} \pi_{2n-3}(S^{n-1}) \xrightarrow{E} \pi_{2n-2}(S^n) \longrightarrow 0$$

The degree homomorphism $\deg : \pi_{2n-1}(S^{2n-1}) \to \mathbb{Z}$ is an isomorphism, $\deg(\mathrm{id}) = 1$, and $\deg \circ H : \pi_{2n-1}(S^n) \to \mathbb{Z}$ is the traditional Hopf invariant. As described, $\pi_{2n-1}(S^n)$ contains an element of Hopf invariant one if and only if $n = 1, 2, 4$, or 8. If n is odd and greater than 1 then $\pi_{2n-1}(S^n)$ is a finite group and the image of the Hopf invariant is zero. In the remaining cases—n even and not 2, 4, or 8—the image of the Hopf invariant has index 2 in $\pi_{2n-1}(S^{2n-1}) = \mathbb{Z}$. Thus we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{P} \pi_{2n-3}(S^{n-1}) \xrightarrow{E} \pi_{2n-2}(S^n) \longrightarrow 0$$

The suspension here is the last one failing to be injective. Its kernel is generated by $P(1)$, which turns out to be the *Whitehead sqaure* $w_n \in \pi_{2n-3}(S^{n-1})$, i.e. the composite $S^{2n-3} \to S^{n-1} \vee S^{n-1} \to S^{n-1}$ of the pinch map with the attaching map of the top cell in the torus $S^{n-1} \times S^{n-1}$.

The next question is: does this short exact sequence split? That is to say, is w_{n-1} divisible by 2? A theorem of Mahowald asserts that if $2\theta = w_{n-1}$ in $\pi_{2n-3}(S^{n-1})$ then $E\theta \in \pi_{2n-2}(S^n) \xrightarrow{\cong} \pi_{n-2}(\mathbb{S})$ is an element of order two and Kervaire invariant one. So Browder's theorem implies that w_{n-1} is divisible by 2 only if n is a power of 2; computations show that it is divisible if $n = 16$ or 32; and HHR prove that it is not divisible if $n > 64$.

See [3] for more about the roles elements of Kervaire invariant one were supposed to play in unstable homotopy theory.

2. CHROMATIC HOMOTOPY THEORY

Much work in Algebraic Topology in the 1960s and early 1970s centered on developing the machinery of generalized homology theories, satisfying the Eilenberg-Steenrod axioms with the exception of the axiom specifying the homology of a point. These were exemplified by topological K-theory (created by Atiyah as an interpretation of Bott periodicity) and cobordism theory (Atiyah's interpretation of the Pontryagin-Thom theory of cobordism). Novikov [25] and Quillen [27] highlighted the centrality and convenience of complex cobordism, whose coefficient ring had been shown by Milnor to be a polynomial algebra: $MU^* = MU^*(*) = \mathbb{Z}[x_1, x_2, \ldots]$, $|x_i| = -2i$. They called attention to the class of "complex oriented" multiplicative homology theories $E^*(-)$: those such that the cohomology of $\mathbb{C}P^\infty$ is a power series ring over the coefficients on a single 2-dimensional generator—an Euler class for complex line bundles. Many of the standard calculations of ordinary cohomology carry over to this much wider setting.

The Euler class of a tensor product is no longer the sum of the Euler classes of the factors, but is rather given by a power series

$$e(\lambda_1 \otimes \lambda_2) = F(e(\lambda_1), e(\lambda_2))$$

Standard properties of the tensor product imply that this power series obeys identities making it a *formal group*:

$$F(x,y) = x + y + \cdots \quad , \quad F(x, F(y,z)) = F(F(x,y),z) \quad , \quad F(x,y) = F(y,x)$$

More precisely it is a graded formal group (always of degee 2) over the graded ring $E^* = E^*(*)$. The formal group associated with complex K-theory, for example, is the *multiplicative* formal group $G_m(x,y) = x + y - uxy$ where $u \in K^{-2}(*)$ is the Bott class, a unit by Bott periodicity. Any multipicative cohomology theory whose coefficient ring $E^*(*)$ is commutative and evenly graded is complex oriented. The spectrum $\mathbb{MU}$ representing complex cobordism is initial among such, and Quillen observed that its formal group enjoys the corresponding universal property: it is the *Lazard ring*, so that there is a bijection between graded formal groups over a graded ring R_* and graded ring homomorphisms $MU_* = \pi_*(\mathbb{MU}) \to R_*$.

The integral homology $H_*(\mathbb{MU};\mathbb{Z})$ also admits a modular interpretation: graded ring homomorphisms from it are bijective with formal groups $F(x,y)$ equipped with a strict isomorphism $\log(t)$ (a "logarithm") to the additive group $G_a(x,y) = x + y$: $\log(x) \equiv x \mod \deg 2$ and $\log G(x,y) = \log(x) + \log(y)$. Thus $H_*(\mathbb{MU};\mathbb{Z}) = \mathbb{Z}[m_1, m_2, \ldots]$ where $\log(t) = \sum_j m_{j-1} t^i$, and the Hurewicz map $\pi_*(\mathbb{MU}) \to H_*(\mathbb{MU};\mathbb{Z})$ classifies the formal group $\log^{-1}(\log(x) + \log(y))$.

In some respects complex bordism is just as useful as ordinary mod p homology. For instance, we have the Adams-Novikov spectral sequence,

$$E_2^{s,t}(X;\mathbb{MU}) \Longrightarrow \pi_{t-s}(X)$$

The E_2 term is computable by homological algebra over the Hopf algebra S_* that co-represents the functor sending a commutative ring to the group (under composition) of formal powers series $f(t) = t + b_1 t^2 + b_2 t^3 + \cdots$. As an algebra, $S_* = \mathbb{Z}[b_1, b_2, \ldots]$. Landweber and Novikov observed that $MU_*(X)$ is naturally a graded comodule for this Hopf algebra (in which we declare $|b_n| = 2n$). For example, the coaction $\psi : MU_* \to S_* \otimes MU_*$ (with $X = *$) is the map co-representing the action of power series on the set of formal groups by conjugation, ${}^f F(x,y) = f(F(f^{-1}(x), f^{-1}(y)))$. Then

$$E_2^{s,t}(X;\mathbb{MU}) = \mathrm{Ext}^{s,t}_{S_*}(\mathbb{Z}, MU_*(X))$$

The universal mod 2 Thom class for complex vector bundles is represented by a map $\mu : \mathbb{MU} \to \mathbb{H}$, and this map induces a map of spectral sequences

$$\mu : E_r^{s,t}(X;\mathbb{MU}) \to E_r^{s,t}(X;\mathbb{H})$$

The mod 2 Hopf invariant factors as

$$\pi_{n-1}(\mathbb{S}) \to E_2^{1,n}(\mathbb{S};\mathbb{MU}) \xrightarrow{\mu} E_2^{1,n}(\mathbb{S},\mathbb{H})$$

Novikov [25] observed (see also [24]) that for $j > 3$ the class h_j is not in the image of μ. This solves the Hopf invariant one problem, but does not tell us the differential in the Adams spectral sequence which is nonzero on h_j (though Novikov [25] (see also [23]) went on to use complex bordism to recover the Adams differential as well).

One may hope to address the Kervaire invariant one problem by an analogous strategy. Unfortunately, h_j^2 is always in the image of μ. In 1978 Ravenel [28] turned this defect to an advantage to prove that a certain analogue of h_j^2 in the classical mod p Adams spectral sequence (for p a prime larger than 3) does not survive. The HHR gambit is a variation on Ravenel's, which we therefore describe.

The elements in question are denoted b_j and they lie in $\mathrm{Ext}_{\mathcal{A}}^{2,2(p-1)p^{j+1}}(\mathbb{F}_p, \mathbb{F}_p)$. When $p = 2$, $b_j = h_{j+1}^2$. The first one, b_0, survives to an element $\beta_1 \in \pi_{2(p-1)p-2}(\mathbb{S})$. Toda proved in 1967 that for p odd, $d_{2p-1}b_1 = uh_0b_0^p \neq 0$ in the Adams spectral sequence, where $u \in \mathbb{F}_p^\times$ and $h_0 \in \mathrm{Ext}_{\mathcal{A}}^{1,2p-2}(\mathbb{F}_p, \mathbb{F}_p)$ represents $\alpha_1 \in \pi_{2p-3}(\mathbb{S})$, the first element of order p in $\pi_*(\mathbb{S})$.

Theorem 2.1 (Ravenel, 1978, [28]). — *For $p \geq 5$ and $j \geq 1$, the element b_j dies in the Adams spectral sequence.*

The elements h_0 and b_j are images of similarly defined elements in $E_2^{2,*}(\mathbb{S}; \mathbb{MU})$, and Toda's differential forces an analogous differential there (destroying Novikov's conjecture [25] that for odd primes it collapsed at E_2). Ravenel used Toda's calculation to ground an induction proving that in the Adams-Novikov spectral sequence

$$d_{2p-1}b_{j+1} \equiv ch_0b_j^p \mod \mathrm{Ann}\, b_0^{a_j} \quad , \quad a_j = p\,\frac{p^j - 1}{p-1} \quad , \quad c \in \mathbb{Z}_{(p)}^\times$$

Because of the spacing of nonzero groups at $E_2^{*,*}$, this is the first possibly nontrivial differential, so b_{j+1} will die as long as $h_0b_j^p \neq 0$ in $E_2^{2p+1,*}$. This is well beyond the computable range in general. However, Ravenel proved the following theorem, guaranteeing that the class b_{j+1} dies:

Proposition 2.2 (Ravenel, 1978, [28]). — *For any sequence $i_0, i_1, \ldots, i_k \geq 0$, the element $h_0b_0^{i_0}b_1^{i_1} \cdots b_k^{i_k} \in E_2^{*,*}(\mathbb{S}; \mathbb{MU})$ is nonzero.*

Corresponding elements occur in $E_2^{*,*}(\mathbb{S}; \mathbb{H})$, but no such theorem holds: this is one of the advantages of $\mathbb{MU}$ relative to $\mathbb{H}$.

Ravenel detected these classes under a map $(f, \varphi) : (MU_*, S_*) \to (R_*, B_*)$. The Hopf algebra B_* arises from a finite group G as the ring R_*^G of functions from G to R_* with Hopf algebra structure given by $\Delta(f)(g_1, g_2) = f(g_1g_2)$, under the identification $R_*^G \otimes_{R_*} R_*^G \xrightarrow{\cong} R_*^{G\times G}$ sending $f_1 \otimes f_1$ to $(g_1, g_2) \mapsto f_1(g_1)f_2(g_2)$. A graded ring homomorphism $f : MU_* \to R_*$ classifies a graded formal group F over R_*. The map φ arises from an action of G on F as a group of strict automorphisms: formal power series $g(t) \equiv t \mod \deg 2$ which conjugate F to itself. The map $\varphi : S_* \to R_*^G$ is determined by its composites with the evaluation maps $\mathrm{ev}_g : R_*^G \to R_*$, and $\mathrm{ev}_g \circ \varphi$ is to classify the formal power series $g(t)$. With these definitions,

$$\mathrm{Ext}^{s,t}_{R_*^G}(R_*, R_*) \cong H^s(G; R_t)$$

One must then find a graded formal group admitting a finite group of automorphisms with rich enough cohomology. It suffices to study graded formal groups F over graded rings of the form $R_* = A[u^{\pm 1}]$, $|u| = -2$. Such an F must be of the form $F(x, y) = u^{-1}F_0(ux, uy)$ for a formal group F_0 over A.

Formal groups over fields are well understood. Over an algebraically closed field of characteristic p, the "height" of a formal group determines it up to isomorphism. The height is defined in terms of the self-map given by $[p](x)$, where $[n](x)$ is defined inductively by $[0](x) = 0$ and $[n](x) = F([n-1](x), x)$. A key elementary fact is that either $[p](t) = 0$ (as is the case for the additive group $G_a(x, y) = x+y$), or $[p](t) = f(t^{p^h})$ with $f'(0) \neq 0$. The *height* is the integer h (declared to be infinite if $[p](t) = 0$). There is a formal group of any height h defined over $\mathbb{F}_p$ with the property that all endomorphisms over $\overline{\mathbb{F}}_p$ are already defined over $\mathbb{F}_{p^h}$. This endomorphism ring is well-studied (see e.g. [29]), and known to contain elments of multiplicative order p exactly when $(p-1)|h$.

Ravenel took the first possible case: F_0 is formal group of height $n = p - 1$ over $A = \mathbb{F}_q$, $q = p^n$, such that $\mathrm{Aut}_{\mathbb{F}_q}(F_0)$ contains an element of order p. Let C_p be the subgroup generated by any such element. Then the map

$$\varphi_* : \mathrm{Ext}^{*,*}_{S_*}(\mathbb{Z}, MU_*) \to H^*(C_p; R_*) = E[h] \otimes R_*[b] \quad , \quad |h| = 1, |b| = 2$$

sends $h_0 \mapsto u^{-n}h$ and $b_j \mapsto u^{-np^{j+1}}b$ for all j, proving the proposition.

We have not yet proved the theorem, however, since, for large j, $b_{j+1} \in E_2^{2,*}(\mathbb{S}; \mathbb{MU})$ is not the only class mapping to $b_{j+1} \in E_2^{2,*}(\mathbb{S}; \mathbb{H})$. The claim is that d_{2p-1} takes on the same value on all of them, modulo $\ker \varphi_*$; that is, $d_{2p-1}(\ker \mu) \subseteq \ker \varphi_*$. Consequently no class in $E_2^{2,*}(\mathbb{S}; \mathbb{MU})$ mapping to $b_{j+1} \in E_2^{2,*}(\mathbb{S}; \mathbb{H})$ survives, and so b_{j+1} does not survive either.

To see that $d_{2p-1}(\ker \mu) \subseteq \ker \varphi_*$, Ravenel invoked L. Smith's construction of a certain 8-cell complex $\mathbb{V}(2)$, with bottom cell $i : \mathbb{S} \to \mathbb{V}(2)$, such that the map $\varphi : MU_* \to R_*$ factors as $MU_*(\mathbb{S}) \to MU_*(\mathbb{V}(2)) \to R_*$. Thus $\ker i_* \subseteq \ker \varphi_*$. He checks that $\ker \mu \subseteq \ker i_*$. So if $a \in \ker \mu$ then $i_* d_{2p-1} a = d_{2p-1} i_* a = 0$, so $d_{2p-1} a \in \ker \varphi_*$.

Smith's complex $\mathbb{V}(2)$ exists only for $p > 3$, and this is where that assumption is required. In fact, at $p = 3$, b_1 dies but b_2 survives in the Adams spectral sequence, and the fate of the others is at present unknown. It is hoped that a modification of the HHR technique will resolve that case as well.

The chromatic approach to homotopy theory rests on the insight we owe to Morava, that the automorphism groups of formal groups over fields should control the structure of the Adams-Novikov E_2-term. Ravenel's result was the first one to depend upon this insight in an essential way.

3. THE DETECTION THEOREM

The attack at $p = 2$ by HHR is similar to Ravenel's odd primary approach, except that they circumvent the need to compute any Adams differentials at all.

THEOREM 3.1 (Hill, Hopkins, and Ravenel, 2009, [11]). — *There is a spectrum* $\mathbb{L}$ *with a map* $i : \mathbb{S} \to \mathbb{L}$ *such that, for* $j \geq 8$,
(1) (The detection property) $E_2^{0,2^j-1}(\mathbb{L}; \mathbb{MU}) = 0$ *and* $\ker i_* \subseteq \ker \mu$ *in*

$$\begin{tikzcd} E_2^{2,2^j}(\mathbb{S}; \mathbb{MU}) \arrow[r, "i_*"] \arrow[d, "\mu"'] & E_2^{2,2^j}(\mathbb{L}; \mathbb{MU}) \\ E_2^{2,2^j}(\mathbb{S}; \mathbb{H}) & \end{tikzcd}$$

(2) (The vanishing property) $\pi_{2^j-2}(\mathbb{L}) = 0$.

Proof of Theorem 0.1. An elementary argument (not invoking convergence) shows that any permanent cycle in $E_2^{2,2^j}(\mathbb{S}; \mathbb{MU})$ reduces to zero in $E_2^{2,2^j}(\mathbb{S}; \mathbb{H})$. An element in $\pi_{2^j-2}(\mathbb{S})$ represented by h_j^2 must be the image of a permanent cycle in $E_2^{2,2^j}(\mathbb{S}; \mathbb{MU})$, so θ_j does not exist. $\square$

In the rest of this section, we will sketch a line of thought which leads to a candidate for the spectrum $\mathbb{L}$. Completing the construction of this candidate spectrum and verifying the Vanishing Property requires a large infusion of equivariant homotopy theory, which we take up in later sections.

The starting point is finding a map $\bar{i}_* : E_2^{*,*}(\mathbb{S}, \mathbb{MU}) \to H^*(G; R_*)$ such that $\ker \bar{i}_* \subseteq \ker \mu$, for a certain finite group G acting (this time nontrivially) on a certain graded ring R_*.

The construction of $\bar{i}_*$ requires a slight extension of the notion of a Hopf algebra. A *Hopf algebroid* over a commutative ring K is a co-groupoid object in the category of commutative (perhaps graded) algebras over K. A Hopf algebroid co-represents a functor from commutative K-algebras to groupoids. For example, the functor sending a commutative graded ring R_* to the groupoid of formal groups over it is corepresentable by the Hopf algebroid $(MU_*, MU_* \otimes S_*)$, with structure morphisms arising from the Hopf algebra structure of S_* and its action on MU_*. Comodules for Hopf algebroids and the corresponding Ext groups are defined in the evident way. A comodule for $MU_* \otimes S_*$ is the same thing as an S_* module over MU_* and $\mathrm{Ext}^{*,*}_{MU_* \otimes S_*}(MU_*, M) = \mathrm{Ext}^{*,*}_S(\mathbb{Z}, M)$.

An action of a finite group on a graded ring determines a Hopf algebroid with object ring R_*, morphism ring R_*^G, and structure maps given as follows. The maps $\eta_L, \eta_R : R_* \to R_*^G$ are characterized by $\mathrm{ev}_g \circ \eta_L = \mathrm{id}$, $\mathrm{ev}_g \circ \eta_R = g$, $\epsilon = \mathrm{ev}_1 : R_*^G \to R_*$, and the diagonal map is defined by $\Delta(f)(g_1, g_2) = f(g_1 g_2)$ under the identification $\alpha : R_*^G \otimes_{R_*} R_*^G \to R_*^{G \times G}$ such that

$$\mathrm{ev}_{g_1,g_2} \alpha(f_1 \otimes f_2) = f_1(g_1) \cdot g_1 f(g_2)$$

A subtlety here is that (as usual in the Hopf algebroid setup) R_* must act via η_R on the left factor and via η_L on the right factor in the tensor product. Again, $\mathrm{Ext}_{R_*^G}(R_*, R_*) = H^*(G; R_*)$.

A map $(MU_*, MU_* \otimes S_*) \to (R_*, R_*^G)$ of Hopf algebroids classifies a graded formal group F over R_* along with a strict isomorphism $\theta_g : F \to gF$, for every $g \in G$, such that

$$\theta_1 = \mathrm{id} \quad \text{and} \quad \theta_{g_1 g_2} = g_1 \theta_{g_2} \circ \theta_{g_1} \tag{1}$$

Construction of the relevant action begins with the involution on the ring spectrum $\mathbb{MU}$ arising from complex conjugation. In homology the action is given by $\overline{m}_j = (-1)^j m_j$; so as graded commmutative algebras with C_2 action

$$H_*(\mathbb{MU}; \mathbb{Z}) = \mathrm{Sym}\left(\bigoplus_{j>0} \sigma^{\otimes j}\right) \tag{2}$$

where σ is the sign representation on $\mathbb{Z}$ and Sym denotes the symmetric algebra. In homotopy this involution classifies the formal group

$$\overline{F}(x, y) = -F(-x, -y)$$

Note that $-[-1]_F(t)$ is a strict isomorphism $F \to \overline{F}$. If R_* is a graded ring with involution $x \mapsto \gamma x$, then the map $\pi_*(\mathbb{MU}) \to R_*$ classifying a formal group F over R_* is equivariant if and only if $\overline{F} = \gamma F$.

This construction may be promoted to a finite group containing C_2 via the *norm* construction. This is a construction applicable in any cocomplete closed symmetric monoidal category $\mathcal{C}$. Given a group G and subgroup H of finite index, the restriction functor from G-objects in $\mathcal{C}$ to H-objects in $\mathcal{C}$ has a left adjoint Ind_H^G. When this is applied to the categories of commutative monoids in $\mathcal{C}$, with the tensor symmetric monoidal structure, the left adjoint is denoted N_H^G; $N_H^G R = R^{\otimes G/H}$ as a commutative monoid in $\mathcal{C}$. This functor lifts to a functor also denoted N_H^G from H-objects in $\mathcal{C}$ to G-objects in $\mathcal{C}$, satisfying many interesting properties. For example, $N_H^G(X \otimes Y) \cong N_H^G X \otimes N_H^G Y$. If H acts on a spectrum X such that $H_*(X; \mathbb{Z})$ is free then

$$H_*(N_H^G X; \mathbb{Z}) \cong N_H^G H_*(X; \mathbb{Z}) \tag{3}$$

HHR use the following distributivity of the norm over coproducts: Let I be a set and for each $i \in I$ let $M_i \in \mathcal{S}^H$. Pick a set F of orbit representatives for the right action of G on $\mathrm{map}(G/H, I)$. For $f \in F$ let $\mathrm{Stab}(f) = \{g \in G : fg = f\}$. Then

$$N_H^G\left(\coprod_{i \in I} M_i\right) \cong \coprod_{f \in F} \mathrm{Ind}_{\mathrm{Stab}(f)}^G N_H^{\mathrm{Stab}(f)}\left(\coprod_{g \in G/\mathrm{Stab}(f)} M_{f(g)}\right) \tag{4}$$

So we can form the G homotopy type

$$\mathbb{MU}^{(n)} = N_2^{2n} \mathbb{MU}$$

where $2n$ is shorthand for the cyclic group C_{2n} of order $2n$. This is a fundameantal object for HHR. Neglecting the group action, $\mathbb{MU}^{(n)} = \mathbb{MU}^{\wedge n}$. Its homology is easily described using (3), (2), and the fact that if V is a free ZH-module then $N_H^G \mathrm{Sym} V \cong \mathrm{Sym}(\mathrm{Ind}_H^G V)$: As a $\mathbb{C}_{2n}$-algebra,

$$H_*(\mathbb{MU}^{(n)}; \mathbb{Z}) = \mathrm{Sym}\left(\bigoplus_{j>0} \mathrm{Ind}_2^{2n}\left(\sigma^{\otimes j}\right)\right) \tag{5}$$

The homotopy of $\mathbb{MU}^{(n)}$ is a graded ring with C_{2n} action, and an equivariant map from it to a graded C_{2n}-ring R_* classifies a graded formal group F over R_* together with a strict isomorphism $\theta : F \to \gamma F$ (where γ is a generator of C_{2n}) such that

$$\gamma^n F = \overline{F} \quad \text{and} \quad \gamma^{n-1}\theta \circ \cdots \circ \gamma\theta \circ \theta = -[-1]_F \tag{6}$$

Given such data, the formula $\theta_{\gamma^k} = \gamma^{k-1}\theta \circ \cdots \circ \gamma\theta \circ \theta$ satisfies (1) and so defines a map $(MU_*, MU_* \otimes S_*) \to (R_*, R_*^{C_{2n}})$. In the universal case, the ring is $MU_*^{(n)} = \pi_*(\mathbb{MU}^{(n)})$; we get maps of Hopf algebroids

$$\bar{i} : (MU_*, MU_* \otimes S_*) \to (MU_*^{(n)}, (MU_*^{(n)})^{C_{2n}}) \to (R_*, R_*^{C_{2n}})$$

In cohomology we get

$$\bar{i}_* : E_2^{*,*}(\mathbb{S}; \mathbb{MU}) \to H^*(C_{2n}; MU_*^{(n)}) \to H^*(C_{2n}; R_*) \tag{7}$$

Triples (R_*, F, θ) satisfiying (6) arise from the theory of formal A-modules. Let A be a discrete valuation ring of characteristic zero, with quotient field K, uniformizer π, and residue field $A/(\pi) = \mathbb{F}_q$. Over K, any formal group has the form $F(x, y) = l^{-1}(l(x) + l(y))$ for a unique $l(t) \in tK[[t]]$ with $l'(0) = 1$ (the "logarithm" of F). The formal group is defined over A provided the Hazewinkel ([10], 8.3) functional equation

$$l(t) = t + \pi^{-1} l(t^q)$$

is satisfied. If this condition holds then for any $a \in A$ the power series $[a](t) = l^{-1}(al(t))$ has coefficients in A, and is an endomorphism of F. This construction defines a ring homomorphism $[-]_F : A \to \mathrm{End}_A(F)$ that splits the natural map $\mathrm{End}_A(F) \to A$ given by $f(t) \mapsto f'(0)$: F is a "formal A-module." For computations, HHR use the example ([10] 25.3.16)

$$l(t) = \sum_{i=0}^{\infty} \frac{t^{q^i}}{\pi^i}$$

Let $A = \mathbb{Z}_2[\zeta]$, where ζ is a primitive 2^kth root of unity. The element $\pi = \zeta - 1$ serves as a uniformizer. The discrete valuation ring A is totally ramified over $\mathbb{Z}_2$, so $A/(\pi) = \mathbb{F}_2$ and $q = 2$. Let F_0 be a formal A-module, $R_* = A[u^{\pm 1}]$ with $|u| = -2$, and $F(x, y) = u^{-1} F_0(ux, uy)$. Define an action of C_{2^k} on R_* by letting a generator γ act trivially on A and by $\gamma u = \zeta u$. Define

$$\theta(t) = \zeta^{-1} u^{-1} [\zeta](ut)$$

Then $\gamma^{2^{k-1}}F = \overline{F}$, $\theta : F \to \gamma F$ is a strict isomorphism satisfying (6), and we get an equivariant ring homomorphism $MU_*^{(2^{k-1})} \to R_*$.

HHR take $k = 3$ here, and check that for $j > 6$ the map $\bar{i}_*$ of (7) (with $n = 4$) satisfies the Detection Property $\ker \bar{i}_* \subseteq \ker \mu$.

The spectrum $\mathbb{L}$ is arranged so that the second map in (7) factors through $E_2^{*,*}(\mathbb{L}; \mathbb{MU})$. The middle term in (7) is the E_2 term of the a spectral sequence converging to the homotopy of the *homotopy fixed point spectrum* (for which see below) $(\mathbb{MU}^{(4)})^{hC_8}$ of the action of C_8 on $\mathbb{MU}^{(4)}$; so one might want to take $\mathbb{L}$ to be $(\mathbb{MU}^{(4)})^{hC_8}$. Unfortunately the Vanishing Property fails for this spectrum. Instead, HHR define $\mathbb{L}$ to be the homotopy fixed point spectrum of a certain localization of $\mathbb{MU}^{(4)}$ whose homotopy is given by $\Delta^{-1}\pi_*(\mathbb{MU}^{(4)})$ for an even-dimensional element Δ which maps to to a unit in R_*. The spectrum $\mathbb{L}$ is an $\mathbb{MU}$-module spectrum, from which it follows that there is a factorization of (7) as

$$E_2^{*,*}(\mathbb{S}; \mathbb{MU}) \to E_2^{*,*}(\mathbb{L}; \mathbb{MU}) \to H^*(C_{2n}; \Delta^{-1}\pi_*(\mathbb{MU}^{(4)})) \to H^*(C_{2n}; R_*)$$

The third term here is the E_2 term of the "homotopy fixed point spectral sequence" converging to $\pi_*(\mathbb{L})$, and, since $\Delta^{-1}\pi_*(\mathbb{MU}^{(4)})$ is evenly graded, the first clause of the Detection Property is clear for this spectral sequence and by the factorization the second clause continues to hold; so this is actually the spectral sequence employed by HHR. In fact the two spectral sequences coincide.

This completes the proof of the Detection Theorem 3.1, modulo (1) justification of the formation of the homotopy fixed point spectrum; (2) construction of the appropriate localization; and (3) verification of the Vanishing Property. The first point requires an action of C_{2n} on the spectrum $\mathbb{MU}^{(4)}$, not just on its homotopy type. The second and third points require a full-bore application of equivariant stable homotopy theory. We take up these topics next.

4. RING SPECTRA

Working in a homotopy theory, such as the derived category of a ring or the stable homotopy category $h\mathcal{S}$, offers a valuable simplification, and the axiomatic framework of triangulated categories is very attractive. But it imposes severe restrictions on the types of construction one can make, and at least since Quillen's fundamental document [26] homotopy theory has been recognized as something more than the study of a homotopy category. Quillen wanted to say what a homotopy *theory* was, and did so by means of the theory of *model categories*. Just as a homotopy type may be represented by many non-homeomorphic spaces, so a homotopy theory may be modeled by many categorically non-equivalent (but "Quillen equivalent") model categories. A fundamental example is given by the categories of spaces and simplicial sets. Quillen equivalent model categories have equivalent homotopy categories.

There are by now many models of the homotopy theory of spectra—many Quillen equivalent model categories of "spectra," all having $h\mathcal{S}$ as homotopy category. Some are even endowed with the structure of a symmetric monoidal category, in which the moniodal product descends to the smash product in $h\mathcal{S}$. This convenience allows one to define a "(commutative) ring spectrum" as a (commutative) monoid with respect to that operation. These form model categories in their own right.

Thom spectra, such as $\mathbb{MU}$, provide basic examples in which the structure of a commutative ring spectrum arises explictly as part of the construction. When this does not happen, one can resort to an obstruction theory. Using this approach, one [30, 9] can build a commutative ring spectrum $E(F/k)$ associated to any formal group F of finite height over a perfect field k. The homotopy ring is $\Lambda[u^{\pm 1}]$, where Λ denotes the Lubin-Tate ring supporting the universal deformation of F. The naturality of this construction implies that any finite group G of automorphisms of F acts on the spectrum $E(F/k)$.

Basic to homotopy theory is the construction of *homotopy limits*, in particular of the homotopy fixed point object of the action of a finite group G. In the category of spaces, the homotopy fixed point space of an action of G on X is provided by the space of equivariant maps $X^{hG} = \mathrm{map}^G(EG, X)$ from a contractible CW complex EG upon which G acts freely. The map $EG \to *$ induces a map from the actual fixed point set, $X^G \to X^{hG}$. The homotopy fixed point spectrum is the "derived" fixed point object, in the sense that it is the functor best approximating $X \mapsto X^G$ that respects suitable weak equivalences. Similar constructions work in greater generality, and allow one to form, for example, the homotopy fixed point ring spectrum $E(F/k)^{hG}$. These constructions come with a spectral sequence of the form

$$E_2^{s,t} = H^s(G; \pi_t(X)) \Longrightarrow \pi_{t-s}(X^{hG})$$

Early work by Hopkins and the author explored these objects in the simplest nontrivial case, when F has height $p-1$ and G contains an element of order p. A motivating case occurs when $p = 2$ and $F(x, y)$ is the multiplicative formal group $G_m(x, y) = x + y - xy$ over $\mathbb{F}_2$. One then finds that $E(G_m/\mathbb{F}_2)^{hC_2} = KO_2^\wedge$, the 2-adic completion of orthogonal K-theory. Use was made in this work, as well, of the equivariant homotopy theory of representation spheres, to compute differentials in the homotopy fixed point spectral sequence.

Given this work it was natural for HHR to hope to define $\mathbb{L}$ as $E(F/k)^{hG}$, for suitable choices of F, k, and G. This hope foundered on the difficulty of the computations, and was replaced by the consideration of $\mathbb{MU}^{(4)}$ and the far more elaborate appeal to equivariant topology required to prove the Vanishing Property. The norm construction may be carried out in $\mathcal{S}$ to produce a commutative ring spectrum $N_2^{2n}\mathbb{MU} = \mathbb{MU}^{(n)}$ with an action of C_{2n}, so the homotopy fixed point set may be formed. Proving the Vanishing Property for its localization $\mathbb{L}$ requires still finer structure.

5. EQUIVARIANT STABLE HOMOTOPY THEORY

In order to form the spectrum $\mathbb{L}$ and study its properties, HHR invoke a large collection of tools from equivariant stable homotopy theory. Diverse variants of this theory have been extensively developed by May and his collaborators, and HHR elect to work in one of them, namely orthogonal spectra [21]. At many points they require details which were not fully articulated in existing references, and have provided in their paper an exhaustive account of the fundamentals of this subject.

5.1. G-CW complexes

The starting point is G-equivariant *unstable* homotopy theory, where G is a finite group. A *G-CW complex* is a G-space X together with a filtration by "skelata" such that $\mathrm{Sk}_0 X$ is a G-set and for each $n \geq 0$ there is a G-set P_n and a pushout square

$$\begin{tikzcd} P_n \times S^{n-1} \arrow[r] \arrow[d] & P_n \times D^n \arrow[d] \\ \mathrm{Sk}_{n-1}X \arrow[r] & \mathrm{Sk}_n X \end{tikzcd}$$

(where D^n is the n-disk, S^{n-1} is its boundary sphere, and $S^{-1} = \varnothing$), and $X = \bigcup \mathrm{Sk}_n X$ in the weak topology. For example, the unit sphere $S(V)$ of an orthogonal representation V is a compact smooth G-manifold and hence (by a theorem of Verona) admits a finite G-CW structure. The one-point compactification S^V of V is the suspension of $S(V)$ and hence also admits a G-CW structure (with cells increased by one in dimension from those of $S(V)$).

When $G = C_{2^k}$ we can be more explicit. Decompose V into irreducibles—

$$V = a\epsilon \oplus b\sigma \oplus \lambda_1 \oplus \cdots \oplus \lambda_c$$

—where ϵ denotes the trivial one-dimensional representation, σ denotes the sign representation pulled back under the unique surjection $C_{2n} \to C_2$, and each λ_j is 2-dimensional. For each j, λ_j is a copy of $\mathbb{C}$ with a generator $\gamma \in C_{2^k}$ acting by a root of unity, ζ_j (which is well defined up to complex conjugation). Order the λ_j's so that if $j \leq k$ then $\langle \zeta_j \rangle \subseteq \langle \zeta_k \rangle$ as subgroups of the unit circle. Pick bases $v_1, \ldots v_b$ of $b\sigma$, and for each j pick a nonzero vector $w_j \in \lambda_j$. Define cones in V as follows:

$$H_a = a\epsilon \quad , \quad H_{a+i} = H_{a+i-1} \oplus \mathbb{R}_+\langle v_i \rangle \quad \text{for} \quad 1 \leq i \leq b \, ,$$

and for j such that $2 \leq 2j \leq 2^k - a - b$

$$H_{a+b+2j-1} = H_{a+b+2j-2} \oplus \mathbb{R}_+\langle w_j \rangle \quad , \quad H_{a+b+2j} = H_{a+b+2j-1} \oplus \mathbb{R}_+\langle \gamma w_j \rangle$$

Let $\mathrm{Sk}_0 S^V = \varnothing_+ = *$, and for $i > 0$ let

$$\mathrm{Sk}_i S^V = (G \cdot H_i)_+$$

This defines a C_{2^k}-CW filtration of S^V with cells in dimensions 0 and i with $a \le i \le \dim V$, each indexed by a transitive C_{2^k}-set (namely C_{2^k}/C_{2^k} in dimensions 0 and a, $C_{2^k}/C_{2^{k-1}}$ for $a+1 \le i \le a+b$, and $\langle \zeta_j \rangle$ in dimensions $a+b+2j-1$ and $a+b+2j$).

5.2. Orthogonal spectra

While we will suppress discussion of technicalities regarding spectra, it may be useful to spell out the definition of the variant adopted by HHR.

Let G be a finite group. Orthogonal G-spectra are defined in terms of a certain category $\mathcal{I}_G$ enriched in the category $\mathcal{T}^G$ of pointed G-spaces and equivariant maps. The objects of $\mathcal{I}_G$ are the finite-dimensional orthogonal representations of G. For $V, W \in \mathcal{I}_G$, let $O(V,W)$ be the Stiefel manifold of linear isometric embeddings from V into W. Let N be the vector bundle over $O(V,W)$ whose fiber at $i : V \hookrightarrow W$ is the orthogonal complement of the image of i in W. Then $\mathcal{I}_G(V,W)$ is defined to be the Thom space of this bundle. G acts on $O(V,W)$ by conjugation and N is an equivariant vector bundle, so $\mathcal{I}_G(V,W)$ receives a G-action.

Let $\mathcal{T}_G$ be the category of pointed G-spaces and all continuous pointed maps; this is enriched over $\mathcal{T}^G$. An *(orthogonal) G-spectrum* is an enriched functor $X : \mathcal{I}_G \to \mathcal{T}_G$. It assigns to every $V \in \mathcal{I}_G$ a G-space X_V, and to every pair $V, W \in \mathcal{I}_G$ a continuous equivariant map $\mathcal{I}_G(V,W) \to \mathrm{map}(X_V, X_W)$. Thus for any $i \in O(V,W)$ we receive a map $N(i)_+ \wedge X_V \to X_W$; they vary continuously and form an equivariant family. They are bonding maps for a spectrum. Morphisms in the enriched category $\mathcal{S}_G$ of G-spectra are simply the spaces of natural transformations with G acting by conjugation. Write $\mathcal{S}^G$ for the category enriched over $\mathcal{T}$ with $\mathcal{S}^G(X,Y) = \mathcal{S}_G(X,Y)^G$, and $\mathcal{S}_1 = \mathcal{S}$.

For example, the *sphere G-spectrum* $\mathbb{S}_G = S^0$ sends W to its one-point compactification S^W. The equivariant stabilization functor $\Sigma^\infty : \mathcal{T}_G \to \mathcal{S}_G$ is given on $K \in \mathcal{T}_G$ by $W \mapsto S^W \wedge K$. Any $V \in \mathcal{I}_G$ co-represents a G-spectrum, denoted by S^{-V}. By Yoneda, $\mathcal{S}_G(S^{-V}, X) = X_V$. This leads to a canonical expression of any G-spectrum as

$$X = \operatorname*{colim}_V S^{-V} \wedge X_V$$

Smash products are also handled gracefully in this context. The smash product of two G-spectra X and Y is the left Kan extension of $(V,W) \mapsto X_V \wedge Y_W$ along the functor $\mathcal{I}_G \times \mathcal{I}_G \to \mathcal{I}_G$ given by orthogonal direct sum. This makes $\mathcal{S}^G$ a closed symmetric monoidal category, and equivariant associative and commutative ring spectra are defined using this monoidal structure.

Let H be a subgroup of G. There is a restriction functor $\mathrm{res}^G_H : \mathcal{S}^G \to \mathcal{S}^H$, which has an adjoint

$$\mathrm{Ind}^G_H : \mathcal{S}^H \to \mathcal{S}^G$$

Restriction doesn't change the underlying homotopy type, and one may write $\mathrm{Ind}^G_H X = G_+ \wedge_H X$.

The following observation is attributed to Hesselholt and Hovey. Let $\mathcal{I}$ be the full subcategory of $\mathcal{I}_G$ consisting of just the trivial representations of G. Then the restriction functor $\mathcal{S}_G = \mathrm{Fun}(\mathcal{I}_G, \mathcal{T}_G) \to \mathrm{Fun}(\mathcal{I}, \mathcal{T}_G) = \mathrm{Fun}(G, \mathcal{S})$ participates in an adjoint equivalence of symmetric monoidal categories enriched over $\mathcal{T}_G$: the category of G-spectra is equivalent to the category of G-objects in spectra. Homotopy theoretic aspects are not preserved by this equivalence, but the observation does allow one to easily make constructions. For example, given $H \subseteq G$, HHR use this device to construct a *norm* functor

$$N_H^G : \mathcal{S}^H \to \mathcal{S}^G$$

The norm is a multiplicative induction: it lifts to a left adjoint of the restriction functor from commutative G ring spectra to commutative H ring spectra. Thus if R is a commutative G ring spectrum, there is a natural map of commutative G ring spectra

$$N_H^G \mathrm{res}_H^G R \to R \tag{8}$$

Using this norm, HHR get to view $\mathbb{MU}^{(n)}$ as a commutative C_{2n} ring spectrum.

5.3. Mackey functors and Bredon cohomology

Given finite G-sets P and Q, form the set $M_G^+(P,Q)$ of isomorphism classes of of finite G-sets X equipped with equivariant maps $X \to P$, $X \to Q$. These form the morphisms in a category, in which the composition is given by pull-back. Each $M_G^+(P,Q)$ is a commutative monoid under disjoint union. Formally adjoin inverses to get an abelian group $M_G(P,Q)$. The composition is bi-additive, so it extends to the group completions and we receive a pre-additive category M_G.

A *Mackey functor* is an additive functor from M_G to the category of abelian groups. For example, the "Burnside Mackey functor" is given by $\tilde{A} : P \mapsto M_G(P, *)$, so that $\tilde{A}(G/K)$ is the Burnside group $A(K)$ of isomorphism classes of virtual finite K-sets.

A G-module N determines a Mackey functor $\underline{N}$ with

$$\underline{N}(P) = \mathrm{map}^G(P, N)$$

The morphism $P \xleftarrow{p} X \xrightarrow{q} Q$ induces $\underline{N}(Q) \to \underline{N}(P)$ given by sending $f : Q \to L$ to

$$a \mapsto \sum_{px=a} f(qx)$$

The "constant" Mackey functor $\underline{\mathbb{Z}}$, for example, arises from $\mathbb{Z}$ as a trivial G-module.

The Mackey category M_G arises in equariant stable homotopy theory as the full subcategory of $h\mathcal{S}_G$ generated by the "discrete" spectra $\Sigma^\infty P_+$. Therefore for any cohomology theory E_G^* representable by a G-spectrum the functor $P \mapsto E_G^n(\Sigma^\infty P_+)$ defines a Mackey functor. Similarly, a G-spectrum E represents a homology theory with $E_n(X) = \pi_n(E \wedge X)$, which extends to a Mackey functor valued theory with $\underline{E}_n(X)(P) = E_n(\Sigma^\infty P_+ \wedge X)$. We have for example Mackey functor valued homotopy groups $\underline{\pi}_n$. If we regard G-spectra of the form $P_+ \wedge S^n$ (where A is a finite G-set) as spheres, then it is natural to extend the appellation "homotopy group" to $\underline{\pi}_n(X)(P) = [P_+ \wedge S^n, X]^G$.

Bredon cohomology with coefficients in a Mackey functor M is characterized by isomorphisms $H^0_G(\Sigma^\infty_+ A; M) = M(A)$ and $H^n_G(\Sigma^\infty_+ A; M) = 0$ for $n \neq 0$, natural in $A \in M_G$, and is represented by a G-spectrum HM. It may be computed from a G-CW structure by means of a cellular cochain complex constructed in the usual way. It follows for example that $H^*_G(X; \underline{\mathbb{Z}}) = H^*(X/G; \mathbb{Z})$.

From the C_{2^k}-CW structure described above, it is easy to deduce that if ρ is the regular representation of C_{2^k} then for any $m \geq 0$

$$H^j_{C_{2^k}}(S^{m\rho}; \underline{\mathbb{Z}}) = 0 \quad \text{for} \quad 1 \leq j \leq 3$$

This simple computation lies behind the Vanishing Property of Theorem 3.1.

5.4. Slices

The key to proving the Vanishing Property is the identification of an appropriate equivariant analogue of the Postnikov system. We being by recalling this notion, in an appropriate form. See [8, 12] for details.

Let $\mathcal{T}^c$ be the category of path-connected pointed spaces. For $n \geq 0$ let $\mathcal{T}^c_{\leq n}$ be the class of spaces A such that $\mathrm{map}_*(S^q, A)$ is contractible for all $q > n$. (It's equivalent to require that $\pi_q(A) = 0$ for all $q > n$.) Given any $X \in \mathcal{T}^c$, there is a map $p : X \to P^n X$ such that (1) $P^n X \in \mathcal{T}^c_{\leq n}$ and (2) p is a "$\mathcal{T}^c_{\leq n}$-equivalence": $p^* : \mathrm{map}_*(P^n X, A) \xrightarrow{\simeq} \mathrm{map}_*(X, A)$ for any $A \in \mathcal{T}^c_{\leq n}$. The map $X \to P^n X$ is the nth *Postnikov section* of X, and is well defined up to a contractible choice. Since $\mathcal{T}^c_{\leq n-1} \subseteq \mathcal{T}^c_{\leq n}$, there is a natural factorization $X \to P^n X \to P^{n-1} X$, in which the second map may be assumed to be a fibration. The fiber $P^n_n X$ of $P^n X \to P^{n-1} X$ is a $K(\pi_n(X), n)$. The inverse limit of this *Postnikov tower* is weakly equivalent to X.

The equivariant analogue employed by HHR uses a carefully chosen set of G-spectra in place of spheres, rejecting the "spheres" $G/H_+ \wedge S^n$ used in the construction of G-CW complexes in favor of objects induced from a restricted class of representation spheres.

Definition 5.1. — *A* slice cell *is a G-spectrum weakly equivalent to either* $\mathrm{Ind}_K^G S^{m\rho_K}$ *(the "regular" case) or* $S^{-1} \wedge \mathrm{Ind}_K^G S^{m\rho_K}$ *(the "irregular" case) for some subgroup K of G and some $m \in \mathbb{Z}$. The slice cell is* isotropic *if $K \neq 1$. The underlying homotopy type of a slice cell is a wedge of spheres of dimension $m|K|$ or $m|K| - 1$, and this number is declared to be its* dimension*.*

HHR use "$\widehat{S}$" to indicate a slice cell and "$\widehat{W}$" for a wedge of slice cells. The slice cells available of dimension n vary with n, but the equivalence

$$S^{\rho_G} \wedge \mathrm{Ind}_K^G S^{m\rho_K} \simeq \mathrm{Ind}_K^G S^{(m+[G:K])\rho_K} \tag{9}$$

implies that smashing with $S^{m\rho_G}$ induces a bijection between homotopy classes of slice cells in dimension n and dimension $n + m|G|$.

Let $(\mathcal{S}_G)_{\leq n}$ be the class of objects A such that $\mathcal{S}_G(\widehat{S}, A)$ is equivariantly contractible for all slice cells $\widehat{S}$ with $\dim \widehat{S} > n$. (It's equivalent to require that $[\widehat{S}, A]^G = 0$ for all slice cells $\widehat{S}$ with $\dim \widehat{S} > n$.) Given any $X \in \mathcal{S}_G$, there is an equivariant map

$p : X \to P^n X$ such that (1) $P^n X \in (\mathcal{S}_G)_{\leq n}$ and (2) p is an "$(\mathcal{S}_G)_{\leq n}$-equivalence": $p_* : \mathcal{S}_G(P^n X, A) \to \mathcal{S}_G(X, A)$ is an equivariant weak equivalence for any $A \in (\mathcal{S}_G)_{\leq n}$. The map $p : X \to P^n X$ is the nth *slice section* of X. These functors assemble into a natural tower of fibrations, the *slice tower*, whose inverse limit is weakly equivalent to X. The fiber of $P^n X \to P^{n-1} X$ is written $P^n_n X$ and is called the *n-slice* of X.

Unfortunately, slices can be quite complicated and may not be "Eilenberg Mac Lane objects" in any reasonable sense. This is because S^n (with trivial action) is usually not a slice cell. But S^0 and S^{-1} are (take $m = 0$ and $K = 1$), and this implies that $P^{-1}_{-1} X = H\underline{\pi}_{-1}(X)$. Zero slices are also of the form HM for a Mackey functor. The n-slice of an n-dimensional slice cell $\widehat{S}$ can be computed: $P^n_n(\widehat{S}) = H\underline{\mathbb{Z}} \wedge \widehat{S}$ if $\widehat{S}$ is regular, and $P^n_n(\widehat{S}) = H\tilde{A} \wedge \widehat{S}$ if $\widehat{S}$ is irregular.

The relationship between slice cells and equivariant spheres begins with the elementary observation that a slice cell $\widehat{S}$ of dimension n admits the structure of a finite G-CW complex with cells in dimensions k with

$$\begin{aligned} \lfloor n/|G| \rfloor \leq k \leq n \quad &\text{if} \quad n \geq 0 \\ n \leq k \leq \lfloor n/|G| \rfloor \quad &\text{if} \quad n < 0 \end{aligned}$$

This leads to the conclusion that if M is an n-slice then $\underline{\pi}_k(M) = 0$ unless

$$\begin{aligned} \lfloor n/|G| \rfloor \leq k \leq n \quad &\text{if} \quad n \geq 0 \\ n \leq k \leq \lfloor (n+1)/|G| \rfloor \quad &\text{if} \quad n < 0 \end{aligned} \tag{10}$$

The *slice spectral sequence* is obtained by applying equivariant homotopy to the slice tower. HHR index it as

$$E_2^{s,t} = \pi^G_{t-s}(P^t_t X) \Longrightarrow \pi^G_{t-s}(X)$$

This indexing is in accord with that of the Atiyah-Hirzebruch spectral sequence, but HHR prefer to display it as one does the Adams spectral sequence, drawing $t - s$ horizontally and s vertically. In this display, (10) implies that the spectral sequence is nonzero only in two wedges, in the northeast and southwest quadrants, bounded by lines of slope 0 and $|G| - 1$.

Since slices are not generally Eilenberg Mac Lane objects, the E_2 term of the slice spectral sequence is not generally expressed in terms of homology. However, the key observation of HHR is that the slices of the crucial C_{2n}-spectrum $\mathbb{MU}^{(n)}$ are extremely well behaved. Here is the definition (in slightly different language than HHR's).

Definition 5.2. — *A G-spectrum X is* pure *if for every n there is a wedge $\widehat{W}_n$ of regular n-dimensional slice cells and an equivariant weak equivalence $P^n_n X \simeq H\underline{\mathbb{Z}} \wedge \widehat{W}_n$, It is* isotropic *if all these slice cells are isotropic.*

If X is pure, then in the slice spectral sequence

$$E_2^{s,t} = \pi^G_{t-s}(P^t_t X) = H_{t-s}(\widehat{W}_t; \underline{\mathbb{Z}})$$

Theorem 5.3 (Slice Theorem). — *The C_{2n}-spectrum $\mathbb{MU}^{(n)}$ is pure and isotropic.*

The proof of the slice theorem is the heart of this work. Here is a sketch.

A *refinement of the homotopy* of a G-spectrum X is a wedge $\widehat{W}$ of slice cells together with an equivariant map $\alpha : \widehat{W} \to X$ for which there exists a map of graded abelian groups $H_*(\widehat{W}) \to \pi_*(\widehat{W})$ such that

$$\begin{tikzcd} & H_*(\widehat{W}) \arrow[dl, "\cong"'] \arrow[d] \arrow[dr, "\mathrm{id}"] & \\ \pi_*(X) & \pi_*(\widehat{W}) \arrow[l, "\alpha_*"] \arrow[r, "h"'] & H_*(\widehat{W}) \end{tikzcd}$$

commutes, where h denotes the Hurewicz map. If X is a G ring spectrum, the refinement is *multiplicative* if $\widehat{W}$ is equipped with a G ring structure for which i is multiplicative.

For example, algebra generators of $\pi_*(\mathbb{MU}_{\mathbb{R}})$ can be chosen to be restrictions to the trivial group of C_2-equivariant maps $S^{j\rho_2} \to \mathbb{MU}_{\mathbb{R}}$. It follows that

$$\alpha_1 : \widehat{W}(1) = \bigwedge_{j>0} \bigvee_{i\geq 0} S^{ij\rho_2} \to \mathbb{MU}_{\mathbb{R}}$$

(where the infinite smash product is the "weak smash," the homotopy colimit over finite sub smash products) is a multiplicative refinement of homotopy. Note also that $\widehat{W}(1)$ is regular and isotropic.

HHR use the theory of formal groups to define certain equivariant homotopy classes

$$\overline{r}_j^{(n)} : S^{j\rho_2} \to \mathrm{res}_2^{2n} \mathbb{MU}^{(n)}$$

By using the ring-spectrum structure of $\mathrm{res}_2^{2n}\mathbb{MU}^{(n)}$, these classes give an equivariant associative ring spectrum map

$$\bigwedge_{j>0} \bigvee_{i\geq 0} S^{ij\rho_2} \to \mathrm{res}_2^{2n} \mathbb{MU}^{(n)}$$

Apply the norm to this map, and compose with the adjunction (8) to get

$$\alpha_n : \widehat{W}(n) = N_2^{2n} \left(\bigwedge_{j>0} \bigvee_{i\geq 0} S^{ij\rho_2} \right) \to \mathbb{MU}^{(n)}$$

The C_{2n}-spectrum $\widehat{W}(n)$ is a wedge of slice cells in view of the multiplicativity of the norm and the distributivity formula (4): Take $M_i = S^{iV}$ for V a representation of H and $i \in I = \mathbb{N}$, and let $|f| = \sum_g f(g) \in \mathbb{N}$. Then (4) implies that

$$N_H^G \left(\bigvee_{i\geq 0} S^{iV} \right) \cong \bigvee_{f\in F} \mathrm{Ind}_{\mathrm{Stab}(f)}^G S^{|f| \mathrm{Ind}_H^{\mathrm{Stab}(f)} V} \tag{11}$$

When $V = \rho_H$, this is a wedge of slice cells. Since C_2 is normal in C_{2n}, $C_2 \subseteq \mathrm{Stab}(f)$ for any $f \in F$, so $\widehat{W}(n)$ is isotropic. HHR prove, starting with (5), that α_n is a multiplicative refinement of homotopy.

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