AN ALTERNATIVE APPROACH TO EQUIVARIANT STABLE HOMOTOPY THEORY

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Abstract. Building on the work of Martin Stolz [Sto11], we develop the basics of equivariant stable homotopy theory starting from the simple idea that a G-spectrum should just be a spectrum with an action of G on it, in contrast to the usual approach in which the definition of a G-spectrum depends on a choice of universe.

1. Introduction

In the standard approach to equivariant stable homotopy theory, pioneered by Lewis, May, and Steinberger in [LMSM86] and reaching the current state of the art in the work of Mandell and May [MM02], the definition of a G-spectrum for a compact Lie group G depends on a choice of G-universe $\mathcal{U}$; that is, on a set of orthogonal G-representations closed under finite direct sums and summands, containing the trivial one-dimensional representation. We convert the universe into a category enriched over pointed G-spaces by defining $\mathcal{U}(V,W)$ to be the space of orthogonal isomorphisms $O(V,W)_+$ with G acting by conjugation. A $\mathcal{U}$-space is then an enriched functor from $X\colon \mathcal{U} \to G\,\mathrm{Top}_*$ to the category of based G-spaces. The sphere S is a $\mathcal{U}$-space, where $S(V) = S^V$, the one-point compactification of V. A $G\mathcal{U}$-spectrum X is then an external module over S, in the sense that we have an associative and unital natural transformation of functors $\mathcal{U}\times\mathcal{U} \to G\,\mathrm{Top}_*$ from $S(V)\wedge X(W)$ to $X(V\oplus W)$. Then one defines homotopy groups $\pi_q^H(X)$ for each closed subgroup H of G and for all integers q, and declares the homotopy isomorphisms to be the weak equivalences.

Historically, the universe has been thought to be central to the definition of a G-spectrum. For example, a $G\mathcal{U}$-spectrum where $\mathcal{U}$ consists of the trivial G-representations has been called a "naive" G-spectrum, whereas a $G\mathcal{U}$-spectrum based on a complete G-universe $\mathcal{U}$, in which every finite-dimensional orthogonal G-representation occurs, has been called a "genuine" G-spectrum.

In this paper we show that we can think of a universe as a Quillen model structure on the category of naive G-spectra. That is:

(1) For us, a G-spectrum is always an orthogonal spectrum X with a continuous action of G on it, so a set of based $G\times O(n)$-spaces X_n with associative unital $G\times O(p)\times O(q)$-structure maps
$$S^p \wedge X_q \to X_{p+q}.$$
(2) Given a G-spectrum X and an orthogonal G-representation V of dimension n, there is a G-space $\mathrm{Ev}_V(X) = X(V)$ defined by $X(V) = X_n$ as an $O(n) = O(V)$-space, with G-action
$$g\cdot x = \rho(g)(gx) = g(\rho(g)x))$$

where $\rho\colon G \to O(n)$ corresponds to V. The functor Ev_V from G-spectra to based G-spaces has a left adjoint F_V. The structure maps of X then make X into a $G\mathcal{U}$-spectrum for any universe $\mathcal{U}$. Thus the category of naive G-spectra is equivalent to the category of $G\mathcal{U}$-spectra for any G-universe $\mathcal{U}$.

(3) Given a G-universe $\mathcal{U}$, there is a symmetric monoidal $\mathcal{U}$-level model structure on G-spectra, in which a map f is a weak equivalence or fibration if and only if $f(V)$ is a weak equivalence or fibration of G-spaces for all $V \in \mathcal{U}$.

(4) One can then form the smallest symmetric monoidal Bousfield localization of the $\mathcal{U}$-level model structure with respect to the maps

$$S^V \wedge F_V S^0 = F_V S^V \to S = F_0 S^0$$

for $V \in \mathcal{U}$, where this map is adjoint to the identity map $S^V \to \mathrm{Ev}_V F_0 S^0 = S^V$. In the homotopy category of this Bousfield localization, smashing with S^V for $V \in \mathcal{U}$ is an equivalence with inverse given by smashing with $F_V S^0$. The resulting model category structure coincides with the stable model structure on $G\mathcal{U}$-spectra of [MM02] under the equivalence between G-spectra and $G\mathcal{U}$-spectra.

We stress that the general idea of this paper has been known to the experts for a long time. In Elmendorf and May's 1997 paper [EM97], they showed that equivariant S-modules are independent of the universe up to equivalence and that different universes correspond to different model category structures. In Section V.I of [MM02], Mandell and May show if $\mathcal{U}'$ is a subuniverse of $\mathcal{U}$, then the category of $G\mathcal{U}$-spectra is equivalent to the category of $G\mathcal{U}'$-spectra, and they describe the model structure on $G\mathcal{U}$-spectra for a complete universe $\mathcal{U}$ that corresponds to a subuniverse $\mathcal{U}'$. Our approach is the reverse to theirs, as we start with the trivial universe; we acknowledge a debt to Neil Strickland, who told the first author such an approach was possible sometime around the year 2000. Stolz's 2011 thesis [Sto11] contains a very similar approach to ours, the primary difference being that we use Bousfield localization instead of stable homotopy isomorphisms, and the secondary difference being that Stolz prefers to use the more abstruse definition of a G-orthogonal spectrum as a certain kind of G-functor. It would be fair to think of this paper as a popularization and simplification of Stolz's work.

2. G-spaces and G-spectra

We first fix notation. A topological space is a compactly generated, weak Hausdorff space, and all constructions, such as limits and colimits, are carried out in this bicomplete closed symmetric monoidal category Top or its pointed analogue Top_*. The symbol G will always denote a compact Lie group. A G-space is a space with a continuous left action of G, and a based G-space is a G-space with a distinguished basepoint that is fixed by the action of G. The category Top_G of G-spaces and nonequivariant maps is closed symmetric monoidal, where we use the diagonal action of G on $X \times Y$, and the conjugation action of G on the (non-equivariant) mapping space $\mathrm{Map}(X,Y)$. That is, $(g \cdot f)(x) = g \cdot f(g^{-1} \cdot x)$. The category $G\,\mathrm{Top}$ of G-spaces and equivariant maps is also closed symmetric monoidal, as a subcategory of G-spaces and nonequivariant maps. The category $G\,\mathrm{Top}$ is enriched, tensored, and cotensored over Top via the symmetric monoidal left adjoint that takes $X \in \mathrm{Top}$ to X with the trivial G-action. The enrichment over Top is then given by the subspace $\mathrm{Map}_G(X, Y)$ of equivariant maps.

Given a G-space X and a closed subgroup H of G, we can consider the fixed points X^H. This is a functor from $G\,\mathrm{Top}$ to Top with left adjoint the functor that takes Y to $G/H \times Y$. (Note that the reason to assume that H is closed is so that the usual definition of G/H as cosets of H with the quotient topology is weak Hausdorff—if H were not closed then we would have to take the closure anyway to stay in weak Hausdorff spaces).

The category $G\,\mathrm{Top}$ and its pointed analogue $G\,\mathrm{Top}_*$ are proper, cellular, topological, symmetric monoidal model categories. For the standard notions of model category theory, see [Hov99] or [Hir03]. A map f in $G\,\mathrm{Top}_*$ is a weak equivalence (resp. fibration) if and only if f^H is a weak equivalence (resp. fibration) in Top_* for every closed subgroup H. The generating cofibrations are the maps

$$(G/H)_+ \wedge S^{n-1}_+ \to (G/H)_+ \wedge D^n_+$$

for all $n \geq 0$ (where S^{-1} is the empty set) and for all closed subgroups H of G. The generating trivial cofibrations are the maps

$$(G/H)_+ \wedge D^n_+ \to (G/H)_+ \wedge D^n_+ \wedge D^1_+$$

for $n \geq 0$ and for all closed subgroups H of G.

We note that there are in fact many different model structures on $G\,\mathrm{Top}_*$, one for each collection of closed subgroups of G, and also that G need not be compact Lie for this to work. However, there are some subtleties with the symmetric monoidal structure when one works with a general collection of closed subgroups. Fausk [Fau08] has an excellent treatment of these issues.

We can now define a G-spectrum. Throughout this paper, $O(n)$ denotes the orthogonal group of $n \times n$ orthogonal real matrices.

Definition 2.1. For a compact Lie group G, a G**-spectrum** X is a sequence of pointed $G \times O(n)$-spaces X_n for $n \geq 0$, together with $G \times O(p) \times O(q)$-equivariant structure maps

$$\nu_{p,q} \colon S^p \wedge X_q \to X_{p+q}$$

that are associative and unital. Here S^p is the one-point compactification of $\mathbb{R}^p$, so inherits an $G \times O(p)$-action where G acts trivially and the point at infinity is the fixed basepoint. The unital condition is simply that $\nu_{0,q}$ is the identity. The associative condition is that the composite

$$S^p \wedge S^q \wedge X_r \xrightarrow{1\wedge\nu_{q,r}} S^p \wedge X_{q+r} \xrightarrow{\nu_{p,q+r}} X_{p+q+r}$$

is equal to the composite

$$S^p \wedge S^q \wedge X_r \xrightarrow{\mu_{p,q}\wedge 1} S^{p+q} \wedge X_r \xrightarrow{\nu_{p+q,r}} X_{p+q+r},$$

where $\mu_{p,q}$ is the isomorphism induced by the standard isomorphism $\mathbb{R}^p \oplus \mathbb{R}^q \cong \mathbb{R}^{p+q}$. We will denote the category of G-spectra by $G\text{-}\mathrm{Sp}^O$, where a map of G-spectra $f \colon X \to Y$ is a collection of $G \times O(n)$-equivariant maps $f_n \colon X_n \to Y_n$ that are compatible with the structure maps.

If $G = *$, a G-spectrum is just an orthogonal spectrum [MMSS01], and a G-spectrum for general G is just an orthogonal spectrum with an action of G on it. As such, the category of G-spectra is closed symmetric monoidal. The easiest way to see this is to note that a G-spectrum is an S-module in the category of

G-orthogonal sequences, and S is a commutative monoid in this closed symmetric monoidal category. Here a **G-orthogonal sequence** X is a collection of pointed $G \times O(n)$-spaces X_n for $n \geq 0$. The category of G-orthogonal sequences is closed symmetric monoidal where

$$(X \otimes Y)_n = \bigvee_{p+q=n} O(n)_+ \wedge_{O(p)\times O(q)} (X_p \wedge Y_q)$$

with diagonal G-action. The closed structure is given by

$$\mathrm{Hom}(X, Y)_n = \prod_{m \geq n} \mathrm{Map}_{O(m-n)}(X_{m-n}, Y_m)$$

with $O(n)$ acting on a map by acting on the target Y_m using the inclusion

$$O(n) \subseteq O(m-n) \times O(n) \to O(m).$$

Note that the maps in $\mathrm{Hom}(X, Y)_n$ are not G-equivariant, so G can act by conjugation as usual. However, just as with G-spaces, G-orthogonal sequences are enriched over (pointed) topological spaces, and the enrichment is given by

$$\mathrm{Map}(X, Y) = \prod_n \mathrm{Map}_{G\times O(n)}(X_n, Y_n).$$

Now, S is the G-orthogonal sequence whose nth space is S^n, the one-point compactification of $\mathbb{R}^n$ with induced pointed orthogonal action and trivial G-action. This is a monoid using the $G \times O(p) \times O(q)$-equivariant isomorphisms

$$S^p \wedge S^q \to S^{p+q}.$$

It is a commutative monoid because the commutativity isomorphism of the symmetric monoidal structure on G-orthogonal spectra involves a (p, q)-shuffle, just as with symmetric spectra [HSS00]. The only thing this requires is that the (p, q)-shuffle be an element of $O(p + q)$, which it of course is.

It is then clear that G-spectra are S-modules, and so inherit a closed symmetric monoidal structure. The category of G-spectra is also enriched over topological spaces, where $\mathrm{Map}_{G\text{-}\mathrm{Sp}^{\mathcal{O}}}(X, Y)$ is the subspace of $\mathrm{Map}(X, Y)$ consisting of maps of orthogonal spectra.

3. Level structures

A G-spectrum has levels X_n for each integer n, and X_n is a $G \times O(n)$-space. However, if V is an orthogonal representation of G corresponding to $\rho\colon G \to O(n)$, we can twist the G-action on X_n by ρ to obtain a new G-space $X(V)$ with $X(V) = X_n$ as an $O(n)$-space, but where

$$g \cdot x = \rho(g)(gx) = g(\rho(g)x).$$

We can also think of

$$X(V) = O(\mathbb{R}^n, V)_+ \wedge_{O(n)} X_n$$

with diagonal G-action, where G acts on the set $O(\mathbb{R}^n, V)$ of orthogonal maps from $\mathbb{R}^n$ to V by sending τ to $g\tau g^{-1}$, which in this case is just $g\tau$ since G acts trivially on $\mathbb{R}^n$.

Note that $X(V)$ is not a $G \times O(n)$-space; it is instead an $O(n) \rtimes_\rho G$-space, where the semi-direct product is taken with respect to the action of G on $O(n)$ where g acts by conjugating by $\rho(g)$.

Let us denote by Ev_V the evaluation functor $\mathrm{Ev}_V \colon G\text{-}\mathrm{Sp}^O \to G\,\mathrm{Top}_*$ that takes X to $X(V)$. This functor should have a left adjoint F_V whose Vth space $(F_V K)(V)$ is $O(V)_+ \wedge K$; this means that if $n = \dim V$, we should have

$$(F_V K)_n = O(V, \mathbb{R}^n)_+ \wedge_{O(V)} (O(V)_+ \wedge K) = O(V, \mathbb{R}^n)_+ \wedge K.$$

In terms of the representation $\rho\colon G \to O(n)$ corresponding to V, we have

$$(F_V K)_n = O(n)_+ \wedge K,$$

with G-action $g(\tau, x) = (\tau\rho(g^{-1}), gx)$. This obviously commutes with $O(n)$-action, so gives us a $G \times O(n)$-space.

Proposition 3.1. *If V is an orthogonal n-dimensional G-representation, the functor $\mathrm{Ev}_V \colon G\text{-}Sp^O \to G\,\mathrm{Top}_*$ has a left adjoint F_V defined by*

$$(F_V K)_{n+k} = O(n+k)_+ \wedge_{O(k)\times O(n)} (S^k \wedge (O(V, \mathbb{R}^n)_+ \wedge K)),$$

and there is a natural isomorphism

$$F_V(K) \wedge F_W(L) \cong F_{V\oplus W}(K \wedge L).$$

Proof. Let us first note that $F_V K$ is in the fact the free S-module on the G-orthogonal sequence that is $O(V, \mathbb{R}^n)_+ \wedge K$ in degree n and the basepoint elsewhere. Thus a map from $F_V K$ to a spectrum X is the same thing as a map of $G \times O(n)$-spaces

$$\alpha\colon O(V, \mathbb{R}^n)_+ \wedge K \to X_n$$

The left-hand side is a free $O(n)$-space on K, but the G-action is twisted. Working this out gives that f is equivalent to a map

$$\beta\colon K \to X_n$$

such that $\beta(gk) = \rho(g)(g\beta(k))$, where $\rho\colon G \to O(n)$ corresponds to the representation V. This is then the same thing as an equivariant map $K \to \mathrm{Ev}_V X$.

For the last part of the proposition, because $F_V(K)$ is a free S-module, it is enough to check that

$$\begin{aligned} O(n+m)_+ \wedge_{O(n)\times O(m)} ((O(V, \mathbb{R}^n)_+ \wedge K) \wedge (O(W, \mathbb{R}^m)_+ \wedge L)) \\ \cong O(V \oplus W, \mathbb{R}^{n+m})_+ \wedge (K \wedge L) \end{aligned}$$

as $G \times O(n+m)$-spaces. We leave this to the reader. □

Now by choosing a set of representations V, we can use the functors F_V and Ev_V to construct a level model structure. Note that the functor F_0, where 0 is the only 0-dimensional representation, plays a special role as it is symmetric monoidal.

Definition 3.2. Given a set $\mathcal{U}$ of finite-dimensional orthogonal G-representations, define a map f of G-spectra to be a $\mathcal{U}$-level equivalence (resp., $\mathcal{U}$-level fibration) if $\mathrm{Ev}_V f$ is a weak equivalence (resp. fibration) of based G-spaces for all $V \in \mathcal{U}$. Define f to be a $\mathcal{U}$-cofibration if it has the left lifting property with respect to all maps that are both $\mathcal{U}$-level equivalences and $\mathcal{U}$-level fibrations.

Theorem 3.3. *The $\mathcal{U}$-cofibrations, $\mathcal{U}$-level fibrations, and $\mathcal{U}$-level equivalences define a proper cellular topological model structure on G-spectra. This $\mathcal{U}$-level model structure is symmetric monoidal when $\mathcal{U}$ is closed under finite direct sums.*

Of course, it is usual to take $\mathcal{U}$ to be a G-**universe**; that is, a set of representations closed under direct sums and summands that contains the one-dimensional trivial representation. At this point, it is unnecessary to put such a restriction on $\mathcal{U}$.

The proof of this theorem is standard, and so we only give a sketch below.

Proof. The generating cofibrations are the maps $F_V i$ for $V \in \mathcal{U}$ and for i a generating cofibration

$$(G/H \times S^{n-1})_+ \to (G/H \times D^n)_+$$

of based G-spaces. The generating trivial cofibrations are the maps $F_V j$ for j a generating trivial cofibration

$$(G/H \times D^n)_+ \to (G/H \times D^n \times D^1)_+$$

of based G-spaces. The heart of the proof that this does define a model structure is the fact that transfinite compositions of pushouts of maps of the form $F_V j$ are $\mathcal{U}$-level equivalences. The basic point is that the maps $\mathrm{Ev}_W F_V j$ are in fact inclusions of G-deformation retracts, so that transfinite compositions of pushouts of them are still G-homotopy equivalences and so weak equivalences. In addition, one must also ensure that the set colimit is the same as the space colimit, which can of course go wrong for weak Hausdorff spaces. This is dealt with just as in [MM02, Theorem 2.4].

The key to proving that the $\mathcal{U}$-level model structure is symmetric monoidal is the isomorphism $F_V i \,\square\, F_W j \cong F_{V \oplus W}(i \,\square\, j)$, where $f \,\square\, g$ is the map

$$(\mathrm{dom}\, f \wedge \mathrm{codom}\, g) \amalg_{\mathrm{dom}\, f \wedge \mathrm{dom}\, g} (\mathrm{codom}\, f \wedge \mathrm{dom}\, g) \to \mathrm{codom}\, f \wedge \mathrm{codom}\, g.$$

This isomorphism follows from the last part of Proposition 3.1, and makes it clear that the $\mathcal{U}$-level model structure is symmetric monoidal because $G\,\mathrm{Top}_*$ is so.

The topological structure is similar but easier, since it is given by the symmetric monoidal functor

$$\mathrm{Top}_* \to G\,\mathrm{Top}_* \xrightarrow{F_0} G\text{-}\mathrm{Sp}^O$$

where the first map takes X to X with trivial G-action. □

4. The stable model structure

We now want to localize the $\mathcal{U}$-level model structure to produce a stable model structure. For any finite-dimensional orthogonal representation V of G, S^V denotes the one-point compactification of V with fixed basepoint the point at infinity. The point of the stable model structure is to make the G-spectra $F_0 S^V$ invertible under the smash product, for $V \in \mathcal{U}$, so that we can desuspend by representation spheres in $\mathcal{U}$. If we want to get a symmetric monoidal result, we should assume that $\mathcal{U}$ is closed under finite direct sums. If we also want to get a result that is stable in the usual sense of being able to desuspend by the circle, we should assume that $\mathcal{U}$ contains the one-dimensional trivial representation. We will such a $\mathcal{U}$ a G-**preuniverse**. It is usual to assume that $\mathcal{U}$ is closed under summands as well, so is a G-universe, but this is not necessary. We discuss this a bit more later.

Now, the obvious candidate for an inverse of S^V is $F_V S^0$, because this is S "shifted by V." However, $F_V S^0 \wedge S^V$ is $F_V S^V$, when we want it to be S. Fortunately, there is a canonical map

$$\lambda_V \colon F_V S^V \to S$$

adjoint to the identity map

$$S^V \to \mathrm{Ev}_V S = O(\mathbb{R}^n, V)_+ \wedge_{O(n)} S^n \cong S^V$$

where $n = \dim V$.

Bousfield localization [Hir03] is a general theory that starts with a (nice) model category and a map and produces a new model category in which that map is now a weak equivalence, while introducing as few other new weak equivalences as possible. So we'd like to define the $\mathcal{U}$-stable model structure as the (left) symmetric monoidal Bousfield localization of the $\mathcal{U}$-level model structure with respect to the maps λ_V for V an irreducible representation in $\mathcal{U}$. The second author has such a theory of symmetric monoidal Bousfield localizations, but as it has not appeared, we just carry it out in this special case, which is simplified by the fact that λ_V is a map between cofibrant objects. If λ_V is to be a weak equivalence in a symmetric monoidal model structure, we will need $\lambda_V \wedge A$ to be a weak equivalence as well for all cofibrant A. The cofibrant objects in the level model structure are all built out of the domains and codomains of the generating cofibrations of the level model structure. In our case, the codomains of the generating cofibrations are contractible, so we don't need them.

We therefore make the following definition.

Definition 4.1. For a compact Lie group G and a G-preuniverse $\mathcal{U}$, we define the **$\mathcal{U}$-stable model structure** on G-spectra to be the left Bousfield localization of the $\mathcal{U}$-level model structure with respect to the maps $\lambda_V \wedge F_W((G/H)_+ \wedge S^{n-1}_+)$, where $V, W \in \mathcal{U}$, H is a closed subgroup of G, and $n \geq 0$.

Let us recall that Bousfield localization produces a new model structure on the same category with the same cofibrations. It works by first constructing the locally fibrant objects and then using them to construct the local equivalences. In our case, then, a G-spectrum X will be **$\mathcal{U}$-stably fibrant** if it is $\mathcal{U}$-level fibrant and the maps

$$\mathrm{Map}(\lambda_V \wedge F_W((G/H)_+ \wedge S^{n-1}_+), X)$$

are weak equivalences of topological spaces.

Note that in Hirschhorn's book [Hir03] these mapping spaces are in fact built from framings on the model category, and do not refer to topological mapping spaces. However, if the model category is simplicial, the source is cofibrant, and the target is fibrant, the mapping spaces created by framings are weakly equivalent to the simplicial mapping spaces. For simplicial model categories, then, we can use simplicial mapping spaces instead of framings to form the Bousfield localization with respect to f if f is a map of cofibrant objects. Every topological model category is also simplicial through the geometric realization functor. The simplicial mapping spaces in a topological model category are just $\mathrm{Sing}\,\mathrm{Map}(X, Y)$, where Sing denotes the singular complex functor. But Sing preserves and reflects weak equivalences. Thus, for topological model categories, we can use topological mapping spaces to form the Bousfield localization with respect to f as long as f is a map of cofibrant objects.

The process of Bousfield localization then continues by defining a map f to be a **$\mathcal{U}$-stable equivalence** if $\mathrm{Map}(Qf, X)$ is a weak equivalence of topological spaces for all $\mathcal{U}$-stably fibrant X. Here Qf denotes any cofibrant approximation to f in the $\mathcal{U}$-level model structure. That is, if $f\colon\ A \to B$, then we would have a commutative square

$$\begin{array}{ccc} A' & \xrightarrow{Qf} & B' \\ \downarrow & & \downarrow \\ A & \xrightarrow[f]{} & B \end{array}$$

where A' and B' are cofibrant and the vertical maps are $\mathcal{U}$-level equivalences. Such an approximation is easily obtained by taking a cofibrant approximation QA to A and then factoring the composite $QA \to A \to B$ into a cofibration followed by a trivial fibration in the $\mathcal{U}$-level model structure. The point of making this cofibrant approximation is so we can use topological mapping spaces, as explained in the preceding paragraph.

The process of Bousfield localization concludes by defining f to be a **$\mathcal{U}$-stable fibration** if f has the right lifting property with respect to all maps that are both $\mathcal{U}$-level cofibrations and $\mathcal{U}$-stable equivalences.

Of course we expect the $\mathcal{U}$-stably fibrant objects to be Ω-spectra in an appropriate sense. For this to make sense, we need to note that any G-spectrum X possesses natural maps

$$S^V \wedge X(W) \to X(V \oplus W)$$

that are both G and $O(m) \times O(n)$-equivariant, where $m = \dim V$ and $n = \dim W$. Indeed, remember that $S^V = S^m$ as an $O(m)$-space, and $X(W) = X_n$ as an $O(n)$-space, so these maps are just the structure maps $\nu_{m,n}$ of X. We just have to check that $\nu_{m,n}$ is G-equivariant with respect to the twisted G-actions. So let $\rho_1 \colon G \to O(m)$ and $\rho_2 \colon G \to O(n)$ denote the homomorphisms corresponding to V and W, so that the composite

$$\rho_1 \times \rho_2 \colon G \xrightarrow{(\rho_1,\rho_2)} O(m) \times O(n) \to O(m+n)$$

corresponds to $V \oplus W$. We compute:

$$\nu_{m,n}(g \cdot (x,y)) = \nu_{m,n}(\rho_1(g)gx, \rho_2(g)gy)$$
$$= (\rho_1(g) \times \rho_2(g))\nu_{m,n}(gx, gy) = (\rho_1 \times \rho_2)(g)g\nu_{m,n}(x,y),$$

as required.

By taking adjoints, this means that any G-spectrum X has maps

$$X(W) \to \Omega^V X(V \oplus W) = \mathrm{Map}(S^V, X(V \oplus W))$$

of G-spaces for all V and W. Note that G acts by conjugation on $\Omega^V X(V \oplus W)$, as usual with mapping spaces.

Definition 4.2. Given a G-preuniverse $\mathcal{U}$, a **$\mathcal{U} - \Omega$-spectrum** is a G-spectrum X such that the map

$$X(W) \to \Omega^V X(V \oplus W)$$

is a weak equivalence in $G\,\mathrm{Top}_*$ for all $V, W \in \mathcal{U}$.

Theorem 4.3. *The $\mathcal{U}$-stably fibrant G-spectra are the $\mathcal{U} - \Omega$-spectra.*

Proof. We have a series of isomorphisms

$$\begin{aligned}
&\mathrm{Map}_{G\text{-}\mathrm{Sp}^{\mathcal{O}}}(F_V S^V \wedge F_W((G/H)_+ \wedge S^{n-1}_+), X)\\
&\cong \mathrm{Map}_{G\text{-}\mathrm{Sp}^{\mathcal{O}}}(F_{V\oplus W}(S^V \wedge (G/H)_+ \wedge S^{n-1}_+), X)\\
&\cong \mathrm{Map}_{G\,\mathrm{Top}_*}(S^V \wedge (G/H)_+ \wedge S^{n-1}_+, X(V\oplus W))\\
&\cong \mathrm{Map}_{G\,\mathrm{Top}_*}((G/H)_+ \wedge S^{n-1}_+, \Omega^V X(V\oplus W))\\
&\cong \mathrm{Map}_{\mathrm{Top}_*}(S^{n-1}_+, (\Omega^V X(V\oplus W))^H).
\end{aligned}$$

and a similar isomorphism

$$\mathrm{Map}_{G\text{-}\mathrm{Sp}^{\mathcal{O}}}(S \wedge F_W((G/H)_+ \wedge S^{n-1}_+), X) \cong \mathrm{Map}_{\mathrm{Top}_*}(S^{n-1}_+, (X(V))^H).$$

Tracing the maps through this series of isomorphisms shows that X is $\mathcal{U}$-stably fibrant if and only if the map

$$X(V) \to \Omega^V X(V\oplus W)$$

is a weak equivalence of G-spaces for all V and W in $\mathcal{U}$. □

Corollary 4.4. *For $V \in \mathcal{U}$, the map $\lambda_V\colon F_V S^V \to S$ is a $\mathcal{U}$-stable equivalence. In fact $\lambda_V \wedge F_W K$ is a $\mathcal{U}$-stable equivalence for all $W \in \mathcal{V}$ and all cofibrant pointed G-spaces K.*

Proof. We get the first statement by taking $W = 0$ in Therorem 4.3. To get the general statement, we repeat the argument of Theorem 4.3 to see that

$$\mathrm{Map}_{G\text{-}\mathrm{Sp}^{\mathcal{O}}}(\lambda_V \wedge F_W K, X)$$

is the map

$$\mathrm{Map}_{G\,\mathrm{Top}_*}(K, X(W)) \to \mathrm{Map}_{G\,\mathrm{Top}_*}(K, \Omega^V X(V\oplus W)).$$

If X is a $\mathcal{U}$-stably fibrant G-spectrum, this map is a weak equivalence since K is cofibrant. □

We then have the following theorem.

Theorem 4.5. *Fix a G-preuniverse $\mathcal{U}$. The category of G-spectra equipped with the $\mathcal{U}$-stable model structure is a left proper cellular topological stable symmetric monoidal model category in which the G-spectra $F_0 S^V$ for $V \in \mathcal{U}$ are invertible under the smash product in the homotopy category.*

Proof. Bousfield localizations preserve left proper cellular model categories. To see that the $\mathcal{U}$-stable model structure is symmetric monoidal, we start by showing that if $W \in \mathcal{U}$ and K is a cofibrant pointed G-space, then $F_W K \wedge (-)$ is a left Quillen functor with respect to the $\mathcal{U}$-stable model structure. It is of course a left Quillen functor with respect to the $\mathcal{U}$-level model structure. The general theory of Bousfield localization then tells us that it is a left Quillen functor with respect to the $\mathcal{U}$-stable model structure if and only if $F_W K \wedge f$ is a $\mathcal{U}$-stable equivalence for all the maps $f = \lambda_V \wedge F_{W'}((G/H)_+ \wedge S^{n-1}_+)$ with respect to which we are localizing. But $F_W K \wedge f$ is of the form $\lambda_V \wedge F_T(L)$ for some $T \in \mathcal{U}$ and a cofibrant L, so this follows from Corollary 4.4.

Now, since the cofibrations don't change in passing to the stable model structure, to prove that the $\mathcal{U}$-stable model structure is symmetric monoidal, it suffices to check that $f \,\square\, g$ is a $\mathcal{U}$-stable equivalence when $f\colon F_W K \to F_W L$ is one of the generating cofibrations of the $\mathcal{U}$-level model structure and $g\colon C \to D$ is a cofibration and a $\mathcal{U}$-stable equivalence. But we have just seen that $F_W K \wedge (-)$ and $F_W L \wedge (-)$ are left Quillen functors on the stable model category. Therefore the map

$$F_W L \wedge C \to (F_W L \wedge C) \amalg_{F_W K \wedge C} (F_W K \wedge D)$$

is a $\mathcal{U}$-stable trivial cofibration, as a pushout of $F_W K \wedge g$. But the composite

$$F_W L \wedge C \to (F_W L \wedge C) \amalg_{F_W K \wedge C} (F_W K \wedge D) \xrightarrow{f \square g} F_W L \wedge D$$

is $F_W L \wedge g$, so it too is a $\mathcal{U}$-stable trivial cofibration. Thus $f \square g$ must be a $\mathcal{U}$-stable equivalence.

The $\mathcal{U}$-stable model structure is topological through the same left Quillen symmetric monoidal functor $\mathrm{Top}_* \to G\text{-}\mathrm{Sp}^O$ that takes X to $F_0 X$ with the trivial G-action. The fact that

$$\lambda_V \colon F_V S^V \cong F_V S^0 \wedge F_0 S^V \to S$$

is a $\mathcal{U}$-stable equivalence shows that $F_V S^0$ is a smash inverse to $F_0 S^V$ (for $V \in \mathcal{U}$) in the homotopy category of the $\mathcal{U}$-stable model structure. In particular, we can take V to to be the one-dimensional trivial representation to see that the suspension is invertible in the homotopy category, so the $\mathcal{U}$-stable model structure is in fact stable in the usual sense. □

It is natural to think that if $\mathcal{U}$ is a G-preuniverse and $\mathcal{U}'$ is the G-universe generated by $\mathcal{U}$, so just the collection of summands of $\mathcal{U}$, then the $\mathcal{U}$-stable model structure should be equivalent to the $\mathcal{U}'$-stable model structure. The argument for this would be that if $V \oplus W$ is in $\mathcal{U}$, then the map

$$S^V \wedge (S^W \wedge F_{V\oplus W} S^0) = S^{V\oplus W} \wedge F_{V\oplus W} S^0 \to S$$

is a weak equivalence in $\mathcal{U}$-stable model structure, and so $S^W \wedge F_{V\oplus W} S^0$ is a smash inverse of S^V. This is wrong, though, because the left-hand side is not the derived smash product since neither factor is cofibrant in the $\mathcal{U}$-model structure. So we cannot say that S^V is invertible under smash product in the homotopy category of the $\mathcal{U}$-stable model structure.

5. Comparison to Mandell-May method

In this section, we compare our approach to equivariant stable homotopy theory to the approach of Mandell and May [MM02]. If we fix a universe $\mathcal{U}$, a $G\mathcal{U}$-spectrum is in particular a G-functor from the universe, thought of as a G-category via $\mathcal{U}(V, W) = O(V, W)_+$ with diagonal G-action when $\dim V = \dim W$ and $*$ otherwise, to the category of pointed G-spaces. That is, such a functor has natural G-equivariant maps

$$O(V, W)_+ \wedge X(V) \to X(W)$$

that are associative and unital. A $G\mathcal{U}$-spectrum is such a functor equipped with an associative and unital natural transformation

$$S^V \wedge X(W) \to X(V \oplus W)$$

of G-functors on $\mathcal{U} \times \mathcal{U}$.

There is then an obvious forgetful functor β from $G\mathcal{U}$-spectra to G-spectra with $(\beta X)_n = X(\mathbb{R}^n)$. Note that $(\beta X)_n$ is an $G \times O(n)$-space through the G-map

$$O(\mathbb{R}^n, \mathbb{R}^n)_+ \wedge X(\mathbb{R}^n) \to X(\mathbb{R}^n).$$

We get G-maps

$$S^p \wedge (\beta X)_q \to (\beta X)_{p+q}$$

by restricting the structure map of X to $V = \mathbb{R}^p$. These maps are $G \times O(p) \times O(q)$-equivariant because the structure map of X is a natural transformation of G-functors on $\mathcal{U} \times \mathcal{U}$. They are associative and unital because the structure map of X is so.

Conversely, we define a functor α from G-spectra to $G\mathcal{U}$-spectra by defining

$$(\alpha X)(V) = O(\mathbb{R}^n, V)_+ \wedge_{O(n)} X_n$$

where $n = \dim V$. Equivalently, we define $(\alpha X)(V) = X_n$ with group action

$$g \cdot x = \rho(g)(gx) = g(\rho(g)x)$$

where $\rho\colon G \to O(n)$ is the representation corresponding to V. Then αX becomes a G-functor from $\mathcal{U}$ to pointed G-spaces, because the map

$$j_{V,W}\colon O(V,W)_+ \wedge X(V) \to X(W)$$

defined by $j(\tau, x) = \tau x$ for $\tau \in O(V,W) = O(n)$ and $x \in X(V) = X_n = X(W)$ is G-equivariant. Indeed, let $\rho_1, \rho_2\colon G \to O(n)$ correspond to V and W, respectively. Then we compute:

$$\begin{aligned} j_{V,W}(g(\tau,x)) &= j_{V,W}(\rho_2(g)\tau\rho_1(g)^{-1}, \rho_1(g)(gx)) \\ &= \rho_2(g)\tau\rho_1(g)^{-1}\rho_1(g)(gx) \\ &= \rho_2(g)\tau(gx) = \rho_2(g)g(\tau x) = g \cdot j_{V,W}(\tau, x). \end{aligned}$$

A similar computation shows that the structure maps

$$\nu_{p.q}\colon S^p \wedge X_q \to X_{p+q}$$

are G-equivariant maps

$$S^V \wedge X(W) \to X(V \oplus W).$$

In fact, we have already done this, just before Definition 4.2. We leave the proof that the structure maps are compatible with the $O(V,V')$ and $O(W,W')$-actions, so define a natural transformation of G-functors on $\mathcal{U} \times \mathcal{U}$, to the reader. The associativity and unit axioms for αX follow immediately from the ones for X, since the structure maps are the same.

Note that the composite functor $\beta\alpha$ is the identity functor. On the other hand, if X is a $G\mathcal{U}$-spectrum and $\dim V = n$, then the G-map

$$X(\mathbb{R}^n, V)_+ \wedge X(\mathbb{R}^n) \to X(V),$$

coming from the fact that X is a G-functor, descends to an isomorphism

$$X(\mathbb{R}^n, V)_+ \wedge_{O(n)} X(\mathbb{R}^n) \to X(V)$$

and so an isomorphism $\alpha\beta X \to X$.

We have therefore proved the following proposition, also proved in [MM02, Theorem V.1.5].

Proposition 5.1. *The functors α and β are adjoint equivalences of categories.*

Mandell and May also show that both α and β are symmetric monoidal, and of course they commute with the functors F_V and Ev_V for $V \in \mathcal{U}$.

The following proposition is then clear.

Proposition 5.2. *Let $\mathcal{U}$ be a G-universe. With respect to the adjoint equivalences α and β, the $\mathcal{U}$-level model structure on G-spectra and the level model structure on $G\mathcal{U}$-spectra coincide. That is, α and β preserve and reflect cofibrations, fibrations, and weak equivalences.*

Of course, we want the stable model structures to coincide as well, and they do.

Theorem 5.3. *Let $\mathcal{U}$ be a G-universe. With respect to the adjoint equivalences α and β, the $\mathcal{U}$-stable model structure on G-spectra and the stable model structure on $G\mathcal{U}$-spectra coincide.*

Proof. We first prove that α is a left Quillen functor. Since α is already a left Quillen functor on the $\mathcal{U}$-level model structure, the general theory of Bousfield localization tells us that α is a left Quillen functor on the $\mathcal{U}$-stable model structure if and only if $\alpha(\lambda_V \wedge F_W((G/H)_+ \wedge S^{n-1}_+))$ is a stable equivalence for all $V, W \in \mathcal{U}$, closed subgroups H and $n \geq 0$. But α is symmetric monoidal, so this is

$$\alpha(\lambda_V) \wedge \alpha(F_W((G/H)_+ \wedge S^{n-1}_+)).$$

The map $\alpha(\lambda_V)$ is proved to be a a stable equivalence (that is, an isomorphism on stable homotopy groups) in [MM02, Lemma III.4.5]. Since it is a stable equivalence of cofibrant objects in a symmetric monoidal model category, it remains so after smashing with any cofibrant object, such as $F_W((G/H)_+ \wedge S^{n-1}_+)$. Thus α is a left Quillen functor.

It now follows that α preserves all stable equivalences. Indeed, if $f\colon A \to B$ is a stable equivalence, we can take a cofibrant approximation Qf to f that is level equivalent to f. More precisely, we take a level equivalence $p\colon QA \to A$ where QA is cofibrant, and then factor $f \circ p$ into a cofibration $Qf\colon QA \to QB$ followed by a level equivalence $q\colon QB \to B$. Then Qf is a stable equivalence between cofibrant objects. Thus $\alpha(Qf)$ is a stable equivalence. Since $\alpha(p)$ and $\alpha(q)$ are level equivalences, it follows that $\alpha(f)$ is a stable equivalence.

Of course, α and β also preserve and reflect stably fibrant objects, since these are Ω-spectra with respect to $\mathcal{U}$ in both cases. (These are called $\Omega - G$-spectra by Mandell and May, and Corollary III.4.10 of [MM02] identifies them as the stably fibrant objects). We now use this to show that β preserves stable equivalences whose target is stably fibrant. Indeed, suppose $f\colon X \to Y$ is a stable equivalence of $G\mathcal{U}$-spectra and Y is stably fibrant. Let $j\colon \beta X \to Z$ be a stable trivial cofibration to a stably fibrant G-spectrum Z. Then

$$\alpha j\colon X \cong \alpha(\beta X) \to \alpha Z$$

is a stable trivial cofibration to the stably fibrant $G\mathcal{U}$-spectrum αZ. Since Y is stably fibrant, we can find a lift $g\colon \alpha(Z) \to Y$ in the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha j"'] & Y \arrow[d] \\ \alpha Z \arrow[r] & * \end{tikzcd}$$

such that $g \circ (\alpha j) = f$. Thus g is a stable equivalence between stably fibrant $G\mathcal{U}$-spectra. By Theorem V.3.4 of [MM02], g is a level equivalence. Thus βg is a $\mathcal{U}$-level equivalence, and so

$$\beta f = (\beta g) \circ (\beta \alpha j) = (\beta g) \circ j$$

is a $\mathcal{U}$-stable equivalence.

We can now use this to prove that β preserves arbitrary stable equivalences. Indeed, if $f\colon X \to Y$ is a stable equivalence, let $j\colon Y \to RY$ be a stable trivial cofibration to a stably fibrant $G\mathcal{U}$-spectrum RY. Then factor $j \circ f = (Rf) \circ i$, where $i\colon X \to RX$ is a stable trivial cofibration and $Rf\colon RX \to RY$ is a stable fibration. Note that Rf is necessarily a stable equivalence. Applying β, and using the fact that β preserves stable equivalences whose target is stably fibrant, we see that βi, βj, and $\beta(Rf)$ are all stable equivalences. It follows that βf is also a stable equivalence.

Since β preserves cofibrations and stable equivalences and has right adjoint α, β is a left Quillen functor with respect to the stable model structures. It follows that α preserves fibrations. Of course, α is also a left Quillen functor, so β preserves fibrations as well, completing the proof. □

Of course, since the $\mathcal{U}$-stable model structure coincides with the Mandell-May model structure under the equivalences α and β, all of the properties that Mandell and May prove hold for the $\mathcal{U}$-stable model structure as well.

We therefore have the following corollary.

Corollary 5.4. *Suppose $\mathcal{U}$ is a G-universe.*

(1) *The weak equivalences in the $\mathcal{U}$-stable model structure are the maps that induce isomorphisms on all stable homotopy groups $\pi_q^H(-)$ for q an integer and H a closed subgroup of G, where*
$$\pi_q^H(X) = \operatorname{colim}_{V\in\mathcal{U}} \pi_q(\Omega^V X(V)^H)$$
for $q \geq 0$ and
$$\pi_q^H(X) = \operatorname{colim}_{V\in\mathcal{U}} \pi_0^H(\Omega^V X(V\oplus\mathbb{R}^q)^H)$$
for $q < 0$.

(2) *The stable fibrations in the $\mathcal{U}$-stable model structure are the level fibrations $p\colon X \to Y$ such that the diagram of G-spaces*
$$\begin{array}{ccc} X(V) & \longrightarrow & \Omega^W X(V\oplus W) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \Omega^W p} \\ Y(V) & \longrightarrow & \Omega^W Y(V\oplus W) \end{array}$$
is an homotopy pullback for all $V, W \in \mathcal{U}$.

(3) *The $\mathcal{U}$-stable model structure is right proper.*

(4) *The $\mathcal{U}$-stable model structure satisfies the monoid axiom.*

(5) *Cofibrant objects are flat in the $\mathcal{U}$-stable model structure, in the sense that if X is cofibrant then $X \wedge (-)$ preserves stable equivalences.*

The last two properties are the essential properties needed for a good theory of monoids and modules over them in a model category. For a good theory of commutative monoids, one typically needs a positive model structure, and this too is provided in Mandell and May, and so holds also for the $\mathcal{U}$-stable model structure. We mention that Stolz [Sto11] also has a different positive model structure, analogous to the convenient positive model structure of Shipley [Shi04], where a cofibration of commutative monoids is in particular a cofibration in the underlying category with its positive model structure.

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