

SPECTRA AND SYMMETRIC SPECTRA IN GENERAL MODEL CATEGORIES

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ABSTRACT. We give two general constructions for the passage from unstable to stable homotopy that apply to the known example of topological spaces, but also to new situations, such as the \mathbb{A}^1 -homotopy theory of Morel-Voevodsky [16, 23]. One is based on the standard notion of spectra originated by Boardman [24]. Its input is a well-behaved model category \mathcal{C} and an endofunctor G , generalizing the suspension. Its output is a model category $Sp^{\mathbb{N}}(\mathcal{C}, G)$ on which G is a Quillen equivalence. The second construction is based on symmetric spectra [11], and applies to model categories \mathcal{C} with a compatible monoidal structure. In this case, the functor G must be given by tensoring with a cofibrant object K . The output is again a model category $Sp^{\Sigma}(\mathcal{D}, K)$ where tensoring with K is a Quillen equivalence, but now $Sp^{\Sigma}(\mathcal{D}, K)$ is again a monoidal model category. We study general properties of these stabilizations; most importantly, we give a sufficient condition for these two stabilizations to be equivalent that applies both in the known case of topological spaces and in the case of \mathbb{A}^1 -homotopy theory.

INTRODUCTION

The object of this paper is to give two very general constructions of the passage from unstable homotopy theory to stable homotopy theory. Since homotopy theory in some form appears in many different areas of mathematics, this construction is useful beyond algebraic topology, where these methods originated. In particular, the two constructions we give apply not only to the usual passage from unstable homotopy theory of pointed topological spaces (or simplicial sets) to the stable homotopy theory of spectra, but also to the passage from the unstable \mathbb{A}^1 -homotopy theory of Morel-Voevodsky [23, 16] to the stable \mathbb{A}^1 -homotopy theory. This example is obviously important, and the fact that it is an example of a widely applicable theory of stabilization may come as a surprise to readers of [12], where specific properties of sheaves are used.

Suppose, then, that we are given a (Quillen) model category \mathcal{C} and a functor $G: \mathcal{C} \rightarrow \mathcal{C}$ that we would like to invert, analogous to the suspension. We will clearly need to require that \mathcal{C} be compatible with the model structure; specifically, we require G to be a left Quillen functor. We will also need some technical hypotheses on the model category \mathcal{C} , which are complicated to state and to check, but which are satisfied in almost all interesting examples, including \mathbb{A}^1 -homotopy theory. It is well-known what one should do to form the category $Sp^{\mathbb{N}}(\mathcal{C}, G)$ of spectra, as first written down for topological spaces in [1]. An object of $Sp^{\mathbb{N}}(\mathcal{C}, G)$

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is a sequence X_n of objects of \mathcal{C} together with maps $GX_n \rightarrow X_{n+1}$, and a map $f: X \rightarrow Y$ is a sequence of maps $f_n: X_n \rightarrow Y_n$ compatible with the structure maps. There is an obvious model structure, called the *projective model structure*, where the weak equivalences are the maps $f: X \rightarrow Y$ such that f_n is a weak equivalence for all n . It is not difficult to show that this is a model structure and that there is a left Quillen functor $G: Sp^{\mathbb{N}}(\mathcal{C}, G) \rightarrow Sp^{\mathbb{N}}(\mathcal{C}, G)$ extending G on \mathcal{C} . But, just as in the topological case, G will not be a Quillen equivalence. So we must localize the projective model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$ to produce the *stable* model structure, with respect to which G will be a Quillen equivalence. A new feature of this paper is that we are able to construct the stable model structure with minimal hypotheses on \mathcal{C} , using the localization results of Hirschhorn [9] (based on work of Dror-Farjoun [5]). We must pay a price for this generality, of course. That price is that stable equivalences are not stable homotopy isomorphisms, but instead are cohomology isomorphisms on all cohomology theories, just as for symmetric spectra [11]. If enough hypotheses are put on \mathcal{C} and G , then we show that stable equivalences are stable homotopy isomorphisms. Jardine [12] proves that stable equivalences are stable homotopy isomorphisms in the stable \mathbb{A}^1 -homotopy theory, using the Nisnevitch descent theorem. His result does not follow from our general theorem for the Morel-Voevodsky motivic model category, because the hypotheses we need do not hold there, but Voevodsky (personal communication) has constructed a simpler model category equivalent to the Morel-Voevodsky one that does satisfy our hypotheses.

As is well-known in algebraic topology, the category $Sp^{\mathbb{N}}(\mathcal{C}, G)$ is not sufficient to understand the smash product. That is, if \mathcal{C} is a symmetric monoidal model category, and G is the functor $X \mapsto X \otimes K$ for some cofibrant object K of \mathcal{C} , it almost never happens that $Sp^{\mathbb{N}}(\mathcal{C}, G)$ is symmetric monoidal. We therefore need a different construction in this case. We define a category $Sp^{\Sigma}(\mathcal{C}, K)$ just as in symmetric spectra [11]. That is, an object of $Sp^{\Sigma}(\mathcal{C}, K)$ is a sequence X_n of objects of \mathcal{C} with an action of the symmetric group Σ_n on X_n . In addition, we have Σ_n -equivariant structure maps $X_n \otimes K \rightarrow X_{n+1}$, but we must further require that the iterated structure maps $X_n \otimes K^{\otimes p} \rightarrow X_{n+p}$ are $\Sigma_n \times \Sigma_p$ -equivariant, where Σ_p acts on $K^{\otimes p}$ by permuting the tensor factors. It is once again straightforward to construct the projective model structure on $Sp^{\Sigma}(\mathcal{C}, K)$. The same localization methods developed for $Sp^{\mathbb{N}}(\mathcal{C}, G)$ apply again here to give a stable model structure on which tensoring with K is a Quillen equivalence. Once again, stable equivalences are cohomology isomorphisms on all possible cohomology theories, but this time it is very difficult to give a better description of stable equivalences even in the case of simplicial symmetric spectra (but see [22] for the best such result I know). We point out that our construction gives a different construction of the stable model category of simplicial symmetric spectra than the one appearing in [11].

We now have competing stabilizations of \mathcal{C} under the tensoring with K functor when \mathcal{C} is symmetric monoidal. Naturally, we need to prove they are the same in an appropriate sense. This was done in the topological (actually, simplicial) case in [11] by constructing a functor $Sp^{\mathbb{N}}(\mathcal{C}, G) \rightarrow Sp^{\Sigma}(\mathcal{C}, K)$, where $K = S^1$ and G is the tensor with S^1 functor, and proving it is a Quillen equivalence. We are unable to generalize this argument. Instead, following an idea of Hopkins, we construct a zigzag of Quillen equivalences $Sp^{\mathbb{N}}(\mathcal{C}, G) \rightarrow \mathcal{C} \leftarrow Sp^{\Sigma}(\mathcal{C}, K)$. However, we need to require that the cyclic permutation map on $K \otimes K \otimes K$ be homotopic to the identity by an explicit homotopy for our construction to work. This hypothesis holds in the

topological case with $K = S^1$ and in the \mathbb{A}^1 -local case with K equal to either the simplicial circle or the algebraic circle $\mathbb{A}^1 - \{0\}$. This section of the paper is by far the most delicate, and it is likely that we do not have the best possible result.

We also investigate the properties of these two stabilization constructions. There are some obvious properties one would hope for of a stabilization construction such as $Sp^{\mathbb{N}}(\mathcal{C}, G)$ or $Sp^{\Sigma}(\mathcal{C}, K)$. First of all, it should be functorial in the pair (\mathcal{C}, G) . We prove this for both stabilization constructions; the most difficult point is defining what one should mean by a map from (\mathcal{C}, G) to (\mathcal{D}, H) . Furthermore, it should be homotopy invariant. That is, if the map $(\mathcal{C}, G) \rightarrow (\mathcal{D}, H)$ is a Quillen equivalence, the induced map of stabilizations should also be a Quillen equivalence. We also prove this; one corollary is that the Quillen equivalence class of $Sp^{\Sigma}(\mathcal{C}, K)$ depends only on the homotopy type of K . Finally, the stabilization map $\mathcal{C} \rightarrow Sp^{\mathbb{N}}(\mathcal{C}, G)$ should be the initial map to a model category \mathcal{D} with an extension of G to a Quillen equivalence. However, this last statement seems to be asking for too much, because the category of model categories is itself something like a model category. This statement is analogous to asking for an initial map in a model category from X to a fibrant object, and such things do not usually exist. The best we can do is to say that if G is already a Quillen equivalence, then the map from $\mathcal{C} \rightarrow Sp^{\mathbb{N}}(\mathcal{C}, G)$ is a Quillen equivalence. This gives a weak form of uniqueness, and is the basis for the comparison between $Sp^{\mathbb{N}}(\mathcal{C}, G)$ and $Sp^{\Sigma}(\mathcal{C}, K)$. See also see [20] and [19] for uniqueness results for the usual stable homotopy category.

We point out that this paper leaves some obvious questions open. We do not have a good characterization of stable equivalences or stable fibrations in either spectra or symmetric spectra, in general, and we are unable to prove that spectra or symmetric spectra are right proper. We do have such characterizations for spectra when the original model category \mathcal{C} is sufficiently well-behaved, and the adjoint U of G preserves sequential colimits. These hypotheses include the cases of ordinary simplicial spectra and spectra in a new motivic model category of Voevodsky (but not the original Morel-Voevodsky motivic model category). We also prove that spectra are right proper in this situation. But we do not have a characterization of stable equivalences of symmetric spectra even with these strong assumptions. Also, we have been unable to prove that symmetric spectra satisfy the monoid axiom. Without the monoid axiom, we do not get model categories of monoids or of modules over an arbitrary monoid, though we do get a model category of modules over a cofibrant monoid. The question of whether commutative monoids form a model category is even more subtle and is not addressed in this paper. See [15] for commutative monoids in symmetric spectra of topological spaces.

There is a long history of work on stabilization, much of it not using model categories. As far as this author knows, Boardman was the first to attempt to construct a good point-set version of spectra; his work was never published (but see [24]), but it was the standard for many years. Generalizations of Boardman's construction were given by Heller in several papers, including [6] and [7]. Heller has continued work on these lines, most recently in [8]. The review of this paper in Mathematical Reviews by Tony Elmendorf (MR98g:55021) captures the response of many algebraic topologists to Heller's approach. I believe the central idea of Heller's approach is that the homotopy theory associated to a model category \mathcal{C} is the collection of all possible homotopy categories of diagram categories $ho \mathcal{C}^I$ and all functors between them. With this definition, one can then forget one had

the model category in the first place, as Heller does. Unfortunately, the resulting complexity of definition is overwhelming at present.

Of course, there has also been very successful work on stabilization by May and coauthors, the two major milestones being [14] and [4]. At first glance, May's approach seems wedded to the topological situation, relying as it does on homeomorphisms $X_n \rightarrow \Omega X_{n+1}$. This is the reason we have not tried to use it in this paper. However, there has been considerable recent work showing that this approach may be more flexible than one might have expected. I have mentioned [15] above, but perhaps the most ambitious attempt to generalize S -modules has been initiated by Mark Johnson [13].

Finally, we point out that Schwede [18] has shown that the methods of Bousfield and Friedlander [1] apply to certain more general model categories. His model categories are always simplicial and proper, and he is always inverting the ordinary suspension functor. Nevertheless, the paper [18] is the first serious attempt to define a general stabilization functor of which the author is aware.

This paper is organized as follows. We begin by defining the category $Sp^{\mathbb{N}}(\mathcal{C}, G)$ and the associated strict model structure in Section 1. Then there is the brief Section 2 recalling Hirschhorn's approach to localization of model categories. We then construct the stable model structure modulo certain technical lemmas in Section 3. The technical lemmas we need assert that if a model category \mathcal{C} is left proper cellular, then so is the strict model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$, and therefore we can apply the localization technology of Hirschhorn. We prove these technical lemmas, and the analogous lemmas for the strict model structure on $Sp^{\Sigma}(\mathcal{D}, K)$, in an Appendix. In Section 4, we study the simplifications that arise when the adjoint U of G preserves sequential colimits and \mathcal{C} is sufficiently well-behaved. We characterize stable equivalences as the appropriate generalization of stable homotopy isomorphisms in this case, and we show the stable model structure is right proper, giving a description of the stable fibrations as well. In Section 5, we prove the functoriality, homotopy invariance, and homotopy idempotence of the construction $(\mathcal{C}, G) \mapsto Sp^{\mathbb{N}}(\mathcal{C}, G)$. We also investigate monoidal structure.

Section 6 begins the second part of the paper, about symmetric spectra. Since we have developed all the necessary techniques in the first part, the proofs in this part are more concise. In Section 6 we discuss the category of symmetric spectra. In Section 7 we construct the projective and stable model structures on symmetric spectra, and in Section 8, we discuss some properties of symmetric spectra. This includes functoriality, homotopy invariance, and homotopy idempotence of the stable model structure. We conclude the paper in Section 9 by constructing the chain of Quillen equivalences between $Sp^{\mathbb{N}}(\mathcal{C}, G)$ and $Sp^{\Sigma}(\mathcal{C}, K)$, under the cyclic permutation hypothesis mentioned above. Finally, as stated previously, there is an Appendix verifying that the techniques of Hirschhorn can be applied to the projective model structures on $Sp^{\mathbb{N}}(\mathcal{C}, G)$ and $Sp^{\Sigma}(\mathcal{C}, K)$.

Obviously, considerable familiarity with model categories will be necessary to understand this paper. The original reference is [17], but a better introductory reference is [3]. More in depth references include [9], [10], and [2]. In particular, we rely heavily on the localization technology in [9].

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an arbitrary model category, a vision that could not be carried out without Phil Hirschhorn's devotion to getting the localization theory of model categories right. The idea of using almost finitely generated model categories in Section 4 is due to Voevodsky, and the idea of using bispectra to compare symmetric spectra with ordinary spectra (see Section 9) is due to Hopkins.

1. SPECTRA

In this section and throughout the paper, \mathcal{C} will be a model category and $G: \mathcal{C} \rightarrow \mathcal{C}$ will be a left Quillen endofunctor of \mathcal{C} with right adjoint U . In this section, we define the category $Sp^{\mathbb{N}}(\mathcal{C}, G)$ of spectra and construct its strict model structure.

The following definition is a straightforward generalization of the usual notion of spectra [1].

Definition 1.1. Suppose G is a left Quillen endofunctor of a model category \mathcal{C} . Define $Sp^{\mathbb{N}}(\mathcal{C}, G)$, the category of *spectra*, as follows. A *spectrum* X is a sequence $X_0, X_1, \dots, X_n, \dots$ of objects of \mathcal{C} together with structure maps $\sigma: GX_n \rightarrow X_{n+1}$ for all n . A *map of spectra* from X to Y is a collection of maps $f_n: X_n \rightarrow Y_n$ commuting with the structure maps, so that the diagram

$$\begin{array}{ccc} GX_n & \xrightarrow{\sigma_X} & X_{n+1} \\ Gf_n \downarrow & & \downarrow f_{n+1} \\ GY_n & \xrightarrow{\sigma_Y} & Y_{n+1} \end{array}$$

is commutative for all n .

Note that if \mathcal{C} is either the model category of pointed simplicial sets or the model category of pointed topological spaces, and G is the suspension functor given by smashing with the circle S^1 , then $Sp^{\mathbb{N}}(\mathcal{C}, G)$ is the Bousfield-Friedlander category of spectra [1].

Definition 1.2. Given $n \geq 0$, the *evaluation functor* $Ev_n: Sp^{\mathbb{N}}(\mathcal{C}, G) \rightarrow \mathcal{C}$ takes X to X_n . The evaluation functor has a left adjoint $F_n: \mathcal{C} \rightarrow Sp^{\mathbb{N}}(\mathcal{C}, G)$ defined by $(F_n X)_m = G^{m-n} X$ if $m \geq n$ and $(F_n X)_m = 0$ otherwise, where 0 is the initial object of \mathcal{C} . The structure maps are the obvious ones.

Note that F_0 is a full and faithful embedding of the category \mathcal{C} into $Sp^{\mathbb{N}}(\mathcal{C}, G)$.

Limits and colimits in the category of spectra are taken objectwise.

Lemma 1.3. *The category of spectra is bicomplete.*

Proof. Given a functor $X: \mathcal{J} \rightarrow Sp^{\mathbb{N}}(\mathcal{C}, G)$, we define $(\text{colim } X)_n = \text{colim } Ev_n \circ X$ and $(\text{lim } X)_n = \text{lim } Ev_n \circ X$. Since G is a left adjoint, it preserves colimits. The structure maps of the colimit are then the composites

$$G(\text{colim } Ev_n \circ X) \cong \text{colim}(G \circ Ev_n \circ X) \xrightarrow{\text{colim } \sigma \circ X} \text{colim } Ev_{n+1} \circ X.$$

Although G does not preserve limits, there is still a natural map $G(\text{lim } Y) \rightarrow \text{lim } GY$ for any functor $Y: \mathcal{J} \rightarrow \mathcal{C}$. Then the structure maps of the limit are the composites

$$G(\text{lim } Ev_n \circ X) \rightarrow \text{lim}(G \circ Ev_n \circ X) \xrightarrow{\text{lim } \sigma \circ X} \text{lim } Ev_{n+1} \circ X. \quad \square$$

Remark 1.4. Note that the evaluation functor $\text{Ev}_n: Sp^{\mathbb{N}}(\mathcal{C}, G) \rightarrow \mathcal{C}$ preserves colimits, so should have a right adjoint $R_n: \mathcal{C} \rightarrow Sp^{\mathbb{N}}(\mathcal{C}, G)$. We define $(R_n X)_m = U^{n-m} X$ if $m \leq n$, and $(R_n X)_m = 1$ if $m > n$. The structure map $GU^{n-m} X \rightarrow U^{n-m-1} X$ is adjoint to the identity map of $U^{n-m} X$ when $m < n$. We leave it to the reader to verify that R_n is the right adjoint of Ev_n .

We now show that the functors G and U extend to functors on $Sp^{\mathbb{N}}(\mathcal{C}, G)$.

Lemma 1.5. *Suppose $F: \mathcal{C} \rightarrow \mathcal{C}$ is a functor and $\tau: GF \rightarrow FG$ is a natural transformation. Then there is an induced functor $F: Sp^{\mathbb{N}}(\mathcal{C}, G) \rightarrow Sp^{\mathbb{N}}(\mathcal{C}, G)$, called the prolongation of F .*

Proof. Given $X \in Sp^{\mathbb{N}}(\mathcal{C}, G)$, we define $(FX)_n = FX_n$, with structure maps $GF X_n \xrightarrow{\tau} FG X_n \xrightarrow{F\sigma} FX_{n+1}$. Given a map $f: X \rightarrow Y$, we define Ff by $(Ff)_n = Ff_n$. \square

Note that the prolongation of F depends on the choice of natural transformation τ . Usually there is an obvious choice of τ .

Corollary 1.6. *The functors G and U prolong to adjoint functors G and U on $Sp^{\mathbb{N}}(\mathcal{C}, G)$.*

Proof. To prolong G , take τ to be the identity in Lemma 1.5. To prolong its right adjoint U , take τ to be the composite $GUX \rightarrow X \rightarrow UGX$ of the counit and unit of the adjunction. It is a somewhat involved exercise in adjoint functors to verify that the resulting prolongations G' and U' are still adjoint to each other. Like all the exercises in adjoint functors in this paper, the simplest way to proceed is to write the adjoint $Y \rightarrow UZ$ of a map $f: GY \rightarrow Z$ as the composition

$$Y \xrightarrow{\eta} UGY \xrightarrow{Uf} UZ$$

and to use the standard properties of the unit and counit of an adjunction. \square

The following remark is critically important to the understanding of our approach to spectra.

Remark 1.7. The definition we have just given of the prolongation of G to an endofunctor of $Sp^{\mathbb{N}}(\mathcal{C}, G)$ is the only possible definition under our very general hypotheses. However, this definition **does not generalize the definition of the suspension** when \mathcal{C} is the category of pointed topological spaces and $GX = X \wedge S^1$. Indeed, recall from [1] that the suspension of a spectrum X in this case is defined by $(X \wedge S^1)_n = X_n \wedge S^1$, with structure map given by

$$X_n \wedge S^1 \wedge S^1 \xrightarrow{1 \wedge T} X_n \wedge S^1 \wedge S^1 \xrightarrow{\sigma \wedge 1} X_{n+1} \wedge S^1,$$

where T is the twist isomorphism. On the other hand, if we apply our definition of the prolongation of G above, we get a functor $X \mapsto X \otimes S^1$ defined by $(X \otimes S^1)_n = X_n \otimes S^1$ with structure map

$$X_n \wedge S^1 \wedge S^1 \xrightarrow{\sigma \wedge 1} X_{n+1} \wedge S^1.$$

Said another way, Bousfield and Friedlander choose the natural transformation $\tau: X \wedge S^1 \wedge S^1 \rightarrow X \wedge S^1 \wedge S^1$ to be $1 \wedge T$, while we are taking it to be the identity. This is a crucial and subtle difference whose ramifications we will study in Section 9.

We now show that $Sp^{\mathbb{N}}(\mathcal{C}, G)$ inherits a model structure from \mathcal{C} , called the *projective model structure*. The functor $G: Sp^{\mathbb{N}}(\mathcal{C}, G) \rightarrow Sp^{\mathbb{N}}(\mathcal{C}, G)$ will be a left Quillen functor with respect to the projective model structure, but it will not be a Quillen equivalence. Our approach to the projective model structure owes much to [1] and [11, Section 5.1]. At this point, we will slip into the standard model category terminology and notation, all of which can be found in [10], mostly in Section 2.1.

Definition 1.8. A map $f \in Sp^{\mathbb{N}}(\mathcal{C}, G)$ is a *level equivalence* if each map f_n is a weak equivalence in \mathcal{C} . Similarly, f is a *level fibration* (resp. *level cofibration*, *level trivial fibration*, *level trivial cofibration*) if each map f_n is a fibration (resp. cofibration, trivial fibration, trivial cofibration) in \mathcal{C} . The map f is a *projective cofibration* if f has the left lifting property with respect to every level trivial fibration.

Note that level equivalences satisfy the two out of three property, and each of the classes defined above is closed under retracts. Thus we should be able to construct a model structure using these classes. To do so, we need the small object argument, and hence we assume that \mathcal{C} is cofibrantly generated (see [10, Section 2.1] for a discussion of cofibrantly generated model categories).

Definition 1.9. Suppose \mathcal{C} is a cofibrantly generated model category with generating cofibrations I and generating trivial cofibrations J . Suppose G is a left Quillen endofunctor of \mathcal{C} , and form the category of spectra $Sp^{\mathbb{N}}(\mathcal{C}, G)$. Define sets of maps in $Sp^{\mathbb{N}}(\mathcal{C}, G)$ by $I_G = \bigcup_n F_n I$ and $J_G = \bigcup_n F_n J$.

The sets I_G and J_G will be the generating cofibrations and trivial cofibrations for a model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$. There is a standard method for proving this, based on the small object argument [10, Theorem 2.1.14]. The first step is to show that the domains of I_G and J_G are small, in the sense of [10, Definition 2.1.3].

Proposition 1.10. *Suppose A is small relative to the cofibrations (resp. trivial cofibrations) in \mathcal{C} , and $n \geq 0$. Then $F_n A$ is small relative to the level cofibrations (resp. level trivial cofibrations) in $Sp^{\mathbb{N}}(\mathcal{C}, G)$.*

Proof. The main point is that Ev_n commutes with colimits. We leave the remainder of the proof to the reader. \square

To apply this to the domains of I_G , we need to know that the maps of I_G -cof are level cofibrations. Recall the right adjoint R_n of Ev_n constructed in Remark 1.4.

Lemma 1.11. *A map in $Sp^{\mathbb{N}}(\mathcal{C}, G)$ is a level cofibration (resp. level trivial cofibration) if and only if it has the left lifting property with respect to $R_n g$ for all $n \geq 0$ and all trivial fibrations (resp. fibrations) g in \mathcal{C} .*

Proof. By adjunction, a map f has the left lifting property with respect to $R_n g$ if and only if $Ev_n f$ has the left lifting property with respect to g . Since a map is a cofibration (resp. trivial cofibration) in \mathcal{C} if and only if it has the left lifting property with respect to all trivial fibrations (resp. fibrations), the lemma follows. \square

Proposition 1.12. *Every map in I_G -cof is a level cofibration. Every map in J_G -cof is a level trivial cofibration.*

Proof. Since G is a left Quillen functor, every map in I_G is a level cofibration. By Lemma 1.11, this means that $R_n g \in I_G$ -inj for all $n \geq 0$ and all trivial fibrations g . Since a map in I_G -cof has the left lifting property with respect to every map in I_G -inj, in particular it has the left lifting property with respect to $R_n g$. Another

application of Lemma 1.11 completes the proof for I_G -cof. The proof for J_G -cof is similar. \square

Proposition 1.10 and Proposition 1.12 immediately imply the following corollary.

Corollary 1.13. *The domains of I_G are small relative to I_G -cof. The domains of J_G are small relative to J_G -cof.*

Theorem 1.14. *Suppose \mathcal{C} is cofibrantly generated. Then the projective cofibrations, the level fibrations, and the level equivalences define a cofibrantly generated model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$, with generating cofibrations I_G and generating trivial cofibrations J_G . We call this the projective model structure. The projective model structure is left proper (resp. right proper, proper) if \mathcal{C} is left proper (resp. right proper, proper).*

Note that if \mathcal{C} is either the model category of pointed simplicial sets or pointed topological spaces, and G is the suspension functor, the projective model structure on \mathcal{C}_G is the strict model structure on the Bousfield-Friedlander category of spectra [1].

Proof. The retract and two out of three axioms are immediate, as is the lifting axiom for a projective cofibration and a level trivial fibration. By adjointness, a map is a level trivial fibration if and only if it is in I_G -inj. Hence a map is a projective cofibration if and only if it is in I_G -cof. The small object argument [10, Theorem 2.1.14] applied to I_G then produces a functorial factorization into a projective cofibration followed by a level trivial fibration.

Adjointness implies that a map is a level fibration if and only if it is in J_G -inj. We have already seen in Proposition 1.12 that the maps in J_G -cof are level equivalences, and they are projective cofibrations since they have the left lifting property with respect to all level fibrations, and in particular level trivial fibrations. Hence the small object argument applied to J_G produces a functorial factorization into a projective cofibration and level equivalence followed by a level fibration.

Conversely, we claim that any projective cofibration and level equivalence f is in J_G -cof, and hence has the left lifting property with respect to level fibrations. To see this, write $f = pi$ where i is in J_G -cof and p is in J_G -inj. Then p is a level fibration. Since f and i are both level equivalences, so is p . Thus f has the left lifting property with respect to p , and so f is a retract of i by the retract argument [10, Lemma 1.1.9]. In particular $f \in J_G$ -cof.

Since colimits and limits in $Sp^{\mathbb{N}}(\mathcal{C}, G)$ are taken levelwise, and since every projective cofibration is in particular a level cofibration, the statements about properness are immediate. \square

We also characterize the projective cofibrations. We denote the pushout of two maps $A \rightarrow B$ and $A \rightarrow C$ by $B \amalg_A C$.

Proposition 1.15. *A map $i: A \rightarrow B$ is a projective (trivial) cofibration if and only if the induced maps $A_0 \rightarrow B_0$ and $A_n \amalg_{GA_{n-1}} GB_{n-1} \rightarrow B_n$ for $n \geq 1$ are (trivial) cofibrations.*

Proof. We only prove the cofibration case, leaving the similar trivial cofibration case to the reader. First suppose $i: A \rightarrow B$ is a projective cofibration. We have

already seen in Proposition 1.12 that $A_0 \rightarrow B_0$ is a cofibration. Suppose $p: X \rightarrow Y$ is a trivial fibration in \mathcal{C} , and suppose we have a commutative diagram

$$\begin{array}{ccc} A_n \amalg_{GA_{n-1}} GB_{n-1} & \longrightarrow & X \\ \downarrow & & \downarrow p \\ B_n & \longrightarrow & Y \end{array}$$

We must construct a lift in this diagram. By adjointness, it suffices to construct a lift in the induced diagram

$$\begin{array}{ccc} A & \longrightarrow & R_n X \\ i \downarrow & & \downarrow \\ B & \longrightarrow & R_n Y \times_{R_{n-1} U Y} R_{n-1} U X \end{array}$$

where R_n is the right adjoint of Ev_n . Using the description of R_n given in Remark 1.4, one can easily check that the map $R_n X \rightarrow R_n Y \times_{R_{n-1} U Y} R_{n-1} U X$ is a level trivial fibration, so a lift exists.

Conversely, suppose that the map i satisfies the conditions in the statement of the proposition. Suppose $p: X \rightarrow Y$ is a level trivial fibration in $Sp^{\mathbb{N}}(\mathcal{C}, G)$, and suppose the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ i \downarrow & & \downarrow p \\ B & \xrightarrow{g} & Y \end{array}$$

commutes. We construct a lift $h_n: B_n \rightarrow X_n$, compatible with the structure maps, by induction on n . There is no difficulty defining h_0 , since i_0 has the left lifting property with respect to the trivial fibration p_0 . Suppose we have defined h_j for $j < n$. Then by lifting in the induced diagram

$$\begin{array}{ccc} A_n \amalg_{GA_{n-1}} GB_{n-1} & \xrightarrow{(f_n, \sigma \circ G h_{n-1})} & X_n \\ \downarrow & & \downarrow p_n \\ B_n & \xrightarrow{g_n} & Y_n \end{array}$$

we find the required map $h_n: B_n \rightarrow X_n$. □

Finally, we point out that the prolongation of G is still a Quillen functor.

Proposition 1.16. *Give $Sp^{\mathbb{N}}(\mathcal{C}, G)$ the projective model structure. Then the prolongation $G: Sp^{\mathbb{N}}(\mathcal{C}, G) \rightarrow Sp^{\mathbb{N}}(\mathcal{C}, G)$ of G is a Quillen functor. Furthermore, the functor $F_n: \mathcal{C} \rightarrow Sp^{\mathbb{N}}(\mathcal{C}, G)$ is a Quillen functor.*

Proof. The functor Ev_n obviously takes level fibrations to fibrations and level trivial fibrations to trivial fibrations. Hence Ev_n is a right Quillen functor, and so its left adjoint F_n is a left Quillen functor. Similarly, the prolongation of U to a functor $U: Sp^{\mathbb{N}}(\mathcal{C}, G) \rightarrow Sp^{\mathbb{N}}(\mathcal{C}, G)$ preserves level fibrations and level trivial fibrations, so its left adjoint G is a Quillen functor. □

2. BOUSFIELD LOCALIZATION

We will define the stable model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$ in Section 3 as a Bousfield localization of the projective model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$. In this section we recall the theory of Bousfield localization of model categories from [9].

To do so, we need some preliminary remarks related to function complexes. Details can be found in [10, Chapter 5], [2], and [9, Chapter 19]. Given an object A in a model category \mathcal{C} , there is a functorial cosimplicial resolution of A induced by the functorial factorizations of \mathcal{C} . By mapping out of this cosimplicial resolution we get a simplicial set $\text{Map}_\ell(A, X)$. Similarly, there is a functorial simplicial resolution of X , and by mapping into it we get a simplicial set $\text{Map}_r(A, X)$. One should think of these as replacements for the simplicial structure present in a simplicial model category. Let us define the homotopy function complex $\text{map}(A, X) = \text{Map}_r(QA, RX)$. Then $\text{map}(A, X)$ is canonically isomorphic in the homotopy category of simplicial sets to $\text{Map}_\ell(QA, RX)$, and makes $\text{Ho } \mathcal{C}$ an enriched category over the homotopy category $\text{Ho } \mathbf{SSet}$ of simplicial sets. In fact, $\text{Ho } \mathcal{C}$ is naturally tensored and cotensored over $\text{Ho } \mathbf{SSet}$, as well as enriched over it. In particular, if G is an arbitrary left Quillen functor between model categories with right adjoint U , we have $\text{map}((LG)X, Y) \cong \text{map}(X, (RU)Y)$ in $\text{Ho } \mathbf{SSet}$, where $(LG)X = GQX$ is the total left derived functor of G and $(RU)Y = URY$ is the total right derived functor of U .

Definition 2.1. Suppose we have a set \mathcal{S} of maps in a model category \mathcal{C} .

1. A \mathcal{S} -local object of \mathcal{C} is a fibrant object W such that, for every $f: A \rightarrow B$ in \mathcal{S} , the induced map $\text{map}(B, W) \rightarrow \text{map}(A, W)$ is a weak equivalence of simplicial sets.
2. A \mathcal{S} -local equivalence is a map $g: A \rightarrow B$ in \mathcal{C} such that the induced map $\text{map}(B, W) \rightarrow \text{map}(A, W)$ is a weak equivalence of simplicial sets for all \mathcal{S} -local objects W .

Note that \mathcal{S} -local equivalences between \mathcal{S} -local objects are in fact weak equivalences.

Then the main theorem of [9] is the following. We will define cellular model categories, a special class of cofibrantly generated model categories, in the Appendix.

Theorem 2.2. *Suppose \mathcal{S} is a set of maps in a left proper cellular model category \mathcal{C} . Then there is a left proper cellular model structure on \mathcal{C} where the weak equivalences are the \mathcal{S} -local equivalences and the cofibrations remain unchanged. The \mathcal{S} -local objects are the fibrant objects in this model structure. We denote this new model category by $L_{\mathcal{S}}\mathcal{C}$ and refer to it as the Bousfield localization of \mathcal{C} with respect to \mathcal{S} . Left Quillen functors from $L_{\mathcal{S}}\mathcal{C}$ to \mathcal{D} are in one to one correspondence with left Quillen functors $F: \mathcal{C} \rightarrow \mathcal{D}$ such that $F(Qf)$ is a weak equivalence for all $f \in \mathcal{S}$.*

We will also need the following fact about localizations, which is implicit in [9, Chapter 4].

Proposition 2.3. *Suppose \mathcal{C} and \mathcal{D} are left proper cellular model categories, \mathcal{S} is a set of maps in \mathcal{C} , and \mathcal{T} is a set of maps in \mathcal{D} . Suppose $F: \mathcal{C} \rightarrow \mathcal{D}$ is a Quillen equivalence with right adjoint U , and suppose $F(Qf)$ is a \mathcal{T} -local equivalence for all $f \in \mathcal{S}$. Then F induces a Quillen equivalence $F: L_{\mathcal{S}}\mathcal{C} \rightarrow L_{\mathcal{T}}\mathcal{D}$ if and only if, for every \mathcal{S} -local $X \in \mathcal{C}$, there is a \mathcal{T} -local Y in \mathcal{D} such that X is weakly equivalent in \mathcal{C}*

to UY . This condition will hold if, for all fibrant Y in \mathcal{D} such that UY is \mathcal{S} -local, Y is \mathcal{T} -local.

Proof. Suppose first that F does induce a Quillen equivalence on the localizations, and suppose that X is \mathcal{S} -local. Then QX is also \mathcal{S} -local. Let $L_{\mathcal{T}}$ denote a fibrant replacement functor in $L_{\mathcal{T}}\mathcal{D}$. Then, because F is a Quillen equivalence on the localizations, the map $QX \rightarrow UL_{\mathcal{T}}FQX$ is a weak equivalence in $L_{\mathcal{S}}\mathcal{C}$ (see [10, Section 1.3.3]). But both QX and $UL_{\mathcal{T}}FQX$ are \mathcal{S} -local, so $QX \rightarrow UL_{\mathcal{T}}FQX$ is a weak equivalence in \mathcal{C} . Hence X is weakly equivalent in \mathcal{C} to UY , where Y is the \mathcal{T} -local object $L_{\mathcal{T}}FQX$.

Before proving the converse, note that, since F is a Quillen equivalence before localizing, the map $FQUX \rightarrow X$ is a weak equivalence for all fibrant X . Since the functor Q does not change upon localization, and every \mathcal{T} -local object of \mathcal{D} is in particular fibrant, this condition still holds after localization. Thus F is a Quillen equivalence after localization if and only if F reflects local equivalences between cofibrant objects, by [10, Corollary 1.3.16].

Suppose $f: A \rightarrow B$ is a map between cofibrant objects such that Ff is a \mathcal{T} -local equivalence. We must show that $\text{map}(f, X)$ is a weak equivalence for all \mathcal{S} -local X . Adjointness implies that $\text{map}(f, UY)$ is a weak equivalence for all \mathcal{T} -local Y , and our condition then guarantees that this is enough to conclude that $\text{map}(f, X)$ is a weak equivalence for all \mathcal{S} -local X .

We still need to prove the last statement of the proposition. So suppose X is \mathcal{S} -local. Then QX is also \mathcal{S} -local, and, in \mathcal{C} , we have a weak equivalence $QX \rightarrow UR FQX$. Our assumption then guarantees that $Y = RFQX$ is \mathcal{T} -local, and X is in indeed weakly equivalent to UY . \square

The fibrations in $L_{\mathcal{S}}\mathcal{C}$ are not completely understood [9, Section 3.6]. The \mathcal{S} -local fibrations between \mathcal{S} -local fibrant objects are just the usual fibrations. In case both \mathcal{C} and $L_{\mathcal{S}}\mathcal{C}$ are right proper, there is a characterization of the \mathcal{S} -local fibrations in terms of homotopy pullbacks analogous to the characterization of stable fibrations of spectra in [1]. However, $L_{\mathcal{S}}\mathcal{C}$ need not be right proper even if \mathcal{C} is, as is shown by the example of Γ -spaces in [1], where it is also shown that the expected characterization of \mathcal{S} -local fibrations does not hold.

3. THE STABLE MODEL STRUCTURE

Our plan now is to apply Bousfield localization to the projective model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$ to obtain a model structure with respect to which G is a Quillen equivalence. In order to do this, we will have to prove that the projective model structure makes $Sp^{\mathbb{N}}(\mathcal{C}, G)$ into a cellular model category when \mathcal{C} is left proper cellular. We will prove this technical result in the appendix. In this section, we will assume that $Sp^{\mathbb{N}}(\mathcal{C}, G)$ is cellular, find a good set \mathcal{S} of maps to form the stable model structure as the \mathcal{S} -localization of the projective model structure, and prove that G is a Quillen equivalence with respect to the stable model structure.

Just as in symmetric spectra [11], we want the stable equivalences to be maps which induce isomorphisms on all cohomology theories. Cohomology theories will be represented by the appropriate analogue of Ω -spectra.

Definition 3.1. A spectrum X is a *U-spectrum* if X is level fibrant and the adjoint $X_n \xrightarrow{\tilde{\sigma}} UX_{n+1}$ of the structure map is a weak equivalence for all n .

Of course, if \mathcal{C} is the category of pointed simplicial sets or pointed topological spaces, and G is the suspension functor, U -spectra are just Ω -spectra. We will find a set \mathcal{S} of maps of $S\mathcal{P}^{\mathbb{N}}(\mathcal{C}, G)$ such that the \mathcal{S} -local objects are the U -spectra. To do so, note that if $\text{map}(A, X_n) \rightarrow \text{map}(A, UX_{n+1})$ is a weak equivalence of simplicial sets for all cofibrant A in \mathcal{C} , then $X_n \rightarrow UX_{n+1}$ will be a weak equivalence as required. Since \mathcal{C} is cofibrantly generated, we should not need all cofibrant A , but only those A related to the generating cofibrations. This is true, but the proof is somewhat technical; the reader might be well-advised to skip the following proof.

Proposition 3.2. *Suppose \mathcal{C} is a left proper cofibrantly generated model category with generating cofibrations I , and $f: X \rightarrow Y$ is a map. Then f is a weak equivalence if and only if $\text{map}(C, X) \rightarrow \text{map}(C, Y)$ is a weak equivalence for all domains and codomains C of maps of I .*

Proof. The only if half is clear. Since every cofibrant object is a retract of a cell complex (i.e. an object A such that the map $0 \rightarrow A$ is a transfinite composition of pushouts of maps of I), it suffices to show that $\text{map}(A, f)$, or, equivalently, $\text{Map}_r(A, Rf)$, is a weak equivalence for all cell complexes A . Given a cell complex A , there is an ordinal λ and a λ -sequence

$$0 = A_0 \rightarrow A_1 \rightarrow \dots \rightarrow A_\beta \rightarrow \dots$$

with colimit $A_\lambda = A$. We will show by transfinite induction on β that $\text{Map}_r(A_\beta, Rf)$ is a weak equivalence for all $\beta \leq \lambda$. Since $A_0 = 0$, getting started is easy. For the successor ordinal case of the induction, suppose $\text{map}(A_\beta, f)$ is a weak equivalence. We have a pushout square

$$\begin{array}{ccc} C & \longrightarrow & A_\beta \\ g \downarrow & & \downarrow i_\beta \\ D & \longrightarrow & A_{\beta+1} \end{array}$$

where g is a map of I . Factor the composite $QC \rightarrow C \rightarrow D$ into a cofibration $\tilde{g}: QC \rightarrow \tilde{D}$ followed by a trivial fibration $\tilde{D} \rightarrow D$. In the terminology of [9], \tilde{g} is a cofibrant approximation to g . By [9, Proposition 12.3.2], there is a cofibrant approximation $\tilde{i}_\beta: \tilde{A}_\beta \rightarrow \tilde{A}_{\beta+1}$ to i_β which is a pushout of \tilde{g} . This is where we need our model category to be left proper. For any fibrant object Z , the functor $\text{Map}_r(-, Z)$ converts colimits to limits and cofibrations to fibrations [10, Corollary 5.4.4]. Hence we have two pullback squares of fibrant simplicial sets

$$\begin{array}{ccc} \text{Map}_r(\tilde{A}_{\beta+1}, RZ) & \longrightarrow & \text{Map}_r(\tilde{D}, RZ) \\ \downarrow & & \downarrow \\ \text{Map}_r(\tilde{A}_\beta, RZ) & \longrightarrow & \text{Map}_r(QC, RZ) \end{array}$$

where $Z = X$ and $Z = Y$, respectively. Here the vertical maps are fibrations. There is a map from the square with $Z = X$ to the square with $Z = Y$ induced by f . By hypothesis, this map is a weak equivalence on every corner except possibly the upper left. But then Dan Kan's cube lemma (see [10, Lemma 5.2.6], where the dual of the version we need is proved, or [2]) implies that the map on the upper left corner is also a weak equivalence, and hence that $\text{Map}_r(A_{\beta+1}, Rf)$ is a weak equivalence.

Now suppose β is a limit ordinal and $\text{Map}_r(A_\gamma, Rf)$ is a weak equivalence for all $\gamma < \beta$. Then, for $Z = X$ or $Z = Y$, the simplicial sets $\text{Map}_r(A_\gamma, RZ)$ define a limit-preserving functor $\beta^{\text{op}} \rightarrow \mathbf{SSet}$ such that each map $\text{Map}_r(A_{\gamma+1}, RZ) \rightarrow \text{Map}_r(A_\gamma, RZ)$ is a fibration of fibrant simplicial sets. There is a natural transformation from the functor with $Z = X$ to the functor with $Z = Y$, and by hypothesis this map is a weak equivalence at every stage. It follows that it is a weak equivalence on the inverse limits, as one can see in different ways. The simplest is probably to note that the inverse limit is a right Quillen functor [10, Corollary 5.1.6]. Thus $\text{Map}_r(A_\beta, Rf)$ is a weak equivalence, as required. This completes the transfinite induction and the proof. \square

Note that the left properness assumption in Proposition 3.2 is unnecessary when the domains of the generating cofibrations are themselves cofibrant, since there is then no need to apply cofibrant approximation.

In view of Proposition 3.2, we need to choose our set \mathcal{S} so as to make

$$\text{map}(C, X_n) \rightarrow \text{map}(C, UX_{n+1})$$

a weak equivalence for all \mathcal{S} -local objects X and all domains and codomains C of the generating cofibrations I . Adjointness implies that, if X is level fibrant, $\text{map}(C, X_n) \cong \text{map}(F_n Q C, X)$ in $\text{Ho } \mathbf{SSet}$, since $F_n Q C = (L F_n) C$, where $L F_n$ is the total left derived functor of F_n . Also, $\text{map}(C, UX_{n+1}) \cong \text{map}(F_{n+1} G Q C, X)$. In view of this, we make the following definition.

Definition 3.3. Suppose \mathcal{C} is a left proper cellular model category with generating cofibrations I , and G is a left Quillen endofunctor of \mathcal{C} . Define the set \mathcal{S} of maps in $Sp^{\mathbb{N}}(\mathcal{C}, G)$ as $\{F_{n+1} G Q C \xrightarrow{s_n^{QC}} F_n Q C\}$, as C runs through the set of domains and codomains of the maps of I and n runs through the non-negative integers. Here the map s_n^{QC} is adjoint to the identity map of $G Q C$. Define the *stable model structure* on $Sp^{\mathbb{N}}(\mathcal{C}, G)$ to be the localization of the projective model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$ with respect to this set \mathcal{S} . We refer to the \mathcal{S} -local weak equivalences as *stable equivalences*, and to the \mathcal{S} -local fibrations as *stable fibrations*.

Theorem 3.4. *Suppose \mathcal{C} is a left proper cellular model category and G is a left Quillen endofunctor of \mathcal{C} . Then the stably fibrant objects in $Sp^{\mathbb{N}}(\mathcal{C}, G)$ are the U -spectra. Furthermore, for all cofibrant $A \in \mathcal{C}$ and for all $n \geq 0$, the map $F_{n+1} G A \xrightarrow{s_n^A} F_n A$ is a stable equivalence.*

Proof. By definition, X is \mathcal{S} -local if and only if X is level fibrant and

$$\text{map}(F_n Q C, X) \rightarrow \text{map}(F_{n+1} G Q C, X)$$

is a weak equivalence for all $n \geq 0$ and all domains and codomains C of maps of I . By the comments preceding Definition 3.3, this is equivalent to requiring that X be level fibrant and that the map $\text{map}(C, X_n) \rightarrow \text{map}(C, UX_{n+1})$ be a weak equivalence for all $n \geq 0$ and all domains and codomains C of maps of I . By Proposition 3.2, this is equivalent to requiring that X be a U -spectrum.

Now, by definition, s_n^A is a stable equivalence if and only if $\text{map}(s_n^A, X)$ is a weak equivalence for all U -spectra X . But by adjointness, $\text{map}(s_n^A, X)$ can be identified with $\text{map}(A, X_n) \rightarrow \text{map}(A, UX_{n+1})$. Since $X_n \rightarrow UX_{n+1}$ is a weak equivalence between fibrant objects, so is $\text{map}(s_n^A, X)$. \square

We would now like to claim that the stable model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$ that we have just defined is a generalization of the stable model structure on spectra of topological spaces or simplicial sets defined in [1]. This cannot be a trivial observation, however, both because our approach is totally different and because of Remark 1.7.

Corollary 3.5. *If \mathcal{C} is either the category of pointed simplicial sets or pointed topological spaces, and G is the suspension functor given by smashing with S^1 , then the stable model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$ coincides with the stable model structure on the category of Bousfield-Friedlander spectra [1].*

Proof. We know already that the cofibrations are the same in the stable model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$ and the stable model structure of [1]. We will show that the weak equivalences are the same. In any model category at all, a map f is a weak equivalence if and only if $\text{map}(f, X)$ is a weak equivalence of simplicial sets for all fibrant X . Construction of $\text{map}(f, X)$ requires replacing f by a cofibrant approximation f' and building cosimplicial resolutions of the domain and codomain of f' . In the case at hand, we can do the cofibrant replacement and build the cosimplicial resolutions in the strict model category of spectra, since the cofibrations do not change under localization. Thus $\text{map}(f, X)$ is the same in both the stable model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$ and in the stable model category of Bousfield and Friedlander. Since the stably fibrant objects are also the same, the corollary holds. \square

The purpose of the stable model structure is to make the prolongation of G into a Quillen equivalence. We begin the process of proving this with the following corollary.

Corollary 3.6. *Suppose \mathcal{C} is a left proper cellular model category and G is a left Quillen endofunctor of \mathcal{C} . Then the prolongation of G to a functor $G: Sp^{\mathbb{N}}(\mathcal{C}, G) \rightarrow Sp^{\mathbb{N}}(\mathcal{C}, G)$ is a Quillen functor with respect to the stable model structure.*

Proof. In view of Hirschhorn's localization theorem 2.2, we must show that $G(Qf)$ is a stable equivalence for all $f \in \mathcal{S}$. Since the domains and codomains of the maps of \mathcal{S} are already cofibrant, it is equivalent to show that Gf is a stable equivalence for all $f \in \mathcal{S}$. Since $GF_n = F_nG$, we have $Gs_n^A = s_n^{GA}$. In view of Theorem 3.4, this map is a weak equivalence whenever A , and hence GA , is cofibrant. Taking $A = QC$, where C is a domain or codomain of a map of I , completes the proof. \square

We will now show that G is in fact a Quillen *equivalence* with respect to the stable model structure. To do so, we introduce the shift functors.

Definition 3.7. Suppose \mathcal{C} is a model category and G is a left Quillen endofunctor of \mathcal{C} . Define the *shift functors* $t: Sp^{\mathbb{N}}(\mathcal{C}, G) \rightarrow Sp^{\mathbb{N}}(\mathcal{C}, G)$ and $s: Sp^{\mathbb{N}}(\mathcal{C}, G) \rightarrow Sp^{\mathbb{N}}(\mathcal{C}, G)$ by $(sX)_n = X_{n+1}$ and $(tX)_n = X_{n-1}$, $(tX)_0 = 0$, with the evident structure maps. Note that t is left adjoint to s .

It is clear that s preserves level equivalences and level fibrations, so t is a left Quillen functor with respect to the strict model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$, and s is a right Quillen functor. Also, $sU = Us$, so $tG = Gt$. Similarly, $\text{Ev}_n s = \text{Ev}_{n+1}$, so $tF_n = F_{n+1}$. It follows that $ts_n^A = s_{n+1}^A$, so that t is a Quillen functor with respect to the stable model structure as well.

We have now come to the main advantage of our approach to spectra. We can test whether a spectrum X is a U -spectrum by checking that X is level fibrant and by checking that the map $X \rightarrow sUX$, adjoint to the structure map of X , is a level equivalence. The analogous statement is **false** in the Bousfield-Friedlander category [1], because the extra twist map they use (see Remark 1.7) means that there is **no map of spectra** $X \rightarrow sUX$ adjoint to the structure map of X ! Our interpretation of this is that the Bousfield-Friedlander approach, while excellent at what it does, is probably not the right general construction. Further evidence for this is provided by the extreme simplicity of the following proof.

Theorem 3.8. *Suppose \mathcal{C} is a left proper cellular model category and G is a left Quillen endofunctor of \mathcal{C} . Then the functors $G: Sp^{\mathbb{N}}(\mathcal{C}, G) \rightarrow Sp^{\mathbb{N}}(\mathcal{C}, G)$ and $t: Sp^{\mathbb{N}}(\mathcal{C}, G) \rightarrow Sp^{\mathbb{N}}(\mathcal{C}, G)$ are Quillen equivalences with respect to the stable model structures. Furthermore, Rs is naturally isomorphic to LG , and RU is naturally isomorphic to Lt .*

Proof. There is a natural map $X \rightarrow sUX$ which is a weak equivalence when X is a stably fibrant object of $Sp^{\mathbb{N}}(\mathcal{C}, G)$. This means that the total right derived functor $R(sU)$ is naturally isomorphic to the identity functor on $\text{Ho } Sp^{\mathbb{N}}(\mathcal{C}, G)$ (where we use the stable model structure). On the other hand, $R(sU)$ is naturally isomorphic to $Rs \circ RU$ and also to $RU \circ Rs$, since s and U commute with each other. Thus the natural isomorphism from the identity to $R(sU)$ gives rise to a natural isomorphism $1 \rightarrow Rs \circ RU$ and a natural isomorphism $RU \circ Rs \rightarrow 1$, displaying RU and Rs as adjoint equivalences of categories. It follows that U and s are both Quillen equivalences, as required. Since RU is adjoint to LG and Rs is adjoint to Lt , we must also have RU naturally isomorphic to Lt and Rs naturally isomorphic to LG . \square

4. THE ALMOST FINITELY GENERATED CASE

The reader may well object at this point that we have defined the stable model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$ without ever defining stable homotopy groups. This is because stable homotopy groups do not detect stable equivalences in general. The usual simplicial and topological situation is very special. The goal of this section is to put some hypotheses on \mathcal{C} and G so that the stable model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$ behaves similarly to the stable model structure on ordinary simplicial spectra. In particular, we show that, if \mathcal{C} is almost finitely generated (defined below), the usual Q construction gives a stable fibrant replacement functor; thus, a map f is a stable equivalence if and only if Qf is a level equivalence. This allows us to characterize $\text{Ho } Sp^{\mathbb{N}}(\mathcal{C}, G)(F_0 A, X)$ for well-behaved A as the usual sort of colimit $\text{colim } \text{Ho } \mathcal{C}(G^n A, X_n)$. It also allows us to prove that the stable model structure is right proper, so we get the expected characterization of stable fibrations.

Most of the results in this section do not depend on the existence of the stable model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$, so we do not usually need to assume \mathcal{C} is left proper cellular.

We now define almost finitely generated model categories, as suggested to the author by Voevodsky.

Definition 4.1. An object A of a category \mathcal{C} is called *finitely presented* if the functor $\mathcal{C}(A, -)$ preserves direct limits of sequences $X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_n \rightarrow \dots$. A cofibrantly generated model category \mathcal{C} is said to be *finitely generated* if the

domains and codomains of the generating cofibrations and the generating trivial cofibrations are finitely presented. A cofibrantly generated model category is said to be *almost finitely generated* if the domains and codomains of the generating cofibrations are finitely presented, and if there is a set of trivial cofibrations J' with finitely presented domains and codomains such that a map f whose codomain is fibrant is a fibration if and only if f has the right lifting property with respect to J' .

This definition differs slightly from other definitions. In particular, an object A is usually said to be finitely presented if $\mathcal{C}(A, -)$ preserves all directed (or, equivalently, filtered) colimits. We are trying to assume the minimum necessary. Finitely generated model categories were introduced in [10, Section 7.4], but in that definition we assumed only that $\mathcal{C}(A, -)$ preserves (transfinitely long) direct limits of sequences of *cofibrations*. The author would now prefer to call such model categories *compactly generated*. Thus, the model category of simplicial sets is finitely generated, but the model category on topological spaces is only compactly generated. Since we will only be working with (almost) finitely generated model categories in this section, our results will not apply to topological spaces. We will indicate where our results fail for compactly generated model categories, and a possible way to amend them in the compactly generated case.

The definition of an almost finitely generated model category was suggested by Voevodsky. The problem with finitely generated, or, indeed, compactly generated, model categories is that they are not preserved by localization. That is, if \mathcal{C} is a finitely generated left proper cellular model category, and S is a set of maps, then the Bousfield localization $L_S\mathcal{C}$ will not be finitely generated, because we lose all control over the generating trivial cofibrations in $L_S\mathcal{C}$. However, if S is a set of cofibrations such that $X \otimes K$ is finitely presented for every domain or codomain X of a map of S and every finite simplicial set K , then $L_S\mathcal{C}$ will still be almost finitely generated. (We use a framing on \mathcal{C} to construct $X \otimes K$). Indeed, the horns

$$(A \otimes \Delta[n]) \amalg_{A \otimes \Lambda^k[n]} (B \otimes \Lambda^k[n]) \rightarrow B \otimes \Delta[n]$$

on the maps $A \rightarrow B$ of S are used to detect S -local fibrant objects, and an S -local fibration between S -local fibrant objects is just an ordinary fibration. We can therefore take the set J' to consist of the horns on the maps of S together with the old set of generating trivial cofibrations.

In particular, Voevodsky has informed the author that he can make an unstable motivic model category that is almost finitely generated, using this approach. For the reader's benefit, we summarize his construction. The category \mathcal{C} is the category of simplicial presheaves (of sets) on the category of smooth schemes over some base scheme k . There is a projective model structure on this category, where weak equivalences and fibrations are defined objectwise from weak equivalences and fibrations of simplicial sets. The projective model structure is finitely generated (using the fact that smooth schemes over k is an essentially small category). There is an embedding of smooth schemes into \mathcal{C} as representable functors. We need to localize this model structure to take into account both the Nisnevich topology and the fact that the functor $X \mapsto X \times \mathbb{A}^1$ should be a Quillen equivalence. To do so, we define a set S' to consist of the maps $X \times \mathbb{A}^1 \rightarrow X$ for every smooth scheme X

and maps $P \rightarrow X$ for every pullback square of smooth schemes

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ A & \xrightarrow{j} & X \end{array}$$

where p is étale, j is an open embedding, and $p^{-1}(X - A) \rightarrow X - A$ is an isomorphism. Here P is the mapping cylinder $(B \amalg Y) \amalg_{B \amalg B} (A \times \Delta[1])$. We then define S to consist of mapping cylinders on the maps of S' . The maps of S are then cofibrations whose domains and codomains are finitely presented (and remain so after tensoring with any finite simplicial set), so the Bousfield localization will be almost finitely generated.

There is then some work involving properties of the Nisnevich topology to show that this model category is equivalent to the Morel-Voevodsky motivic model category of [16], and to the model category used by Jardine [12].

The reason we need almost finitely generated model categories is because, in an almost finitely generated model category \mathcal{C} , sequential colimits preserve trivial fibrations, fibrant objects, and fibrations between fibrant objects. Indeed, suppose we have a map of sequences $p_n: X_n \rightarrow Y_n$ that is a fibration between fibrant objects for all n . We show $\text{colim } X_n$ is fibrant by testing that $X \rightarrow 0$ has the right lifting property with respect to J' . We then test that $\text{colim } p_n$ is a fibration by testing that it has the right lifting property with respect to J' . The proof that sequential colimits preserve trivial fibrations is similar.

Now, given a spectrum X , there is an obvious candidate for a stable fibrant replacement of X .

Definition 4.2. Suppose G is a left Quillen endofunctor of a model category \mathcal{C} with right adjoint U . Define $R: Sp^{\mathbb{N}}(\mathcal{C}, G) \rightarrow Sp^{\mathbb{N}}(\mathcal{C}, G)$ to be the functor sU , where s is the shift functor. Then we have a natural map $\iota_X: X \rightarrow RX$, and we define

$$R^\infty X = \text{colim}(X \xrightarrow{\iota_X} RX \xrightarrow{R\iota_X} R^2X \xrightarrow{R^2\iota_X} \dots \xrightarrow{R^{n-1}\iota_X} R^n X \xrightarrow{R^n\iota_X} \dots).$$

Let $j_X: X \rightarrow R^\infty X$ denote the obvious natural transformation.

The following lemma, though elementary, is crucial.

Lemma 4.3. *The maps $\iota_{RX}, R\iota_X: RX \rightarrow R^2X$ coincide.*

Proof. The map ι_{RX} is the adjoint of the structure map

$$GUX_{n+1} \xrightarrow{\varepsilon} X_{n+1} \xrightarrow{\eta} UGX_{n+1} \xrightarrow{U\sigma} UX_{n+2}$$

of RX , where ε denotes the counit of the adjunction, η denotes the unit, and σ denotes the structure map of X . Thus ι_{RX} is the composite

$$UX_{n+1} \xrightarrow{\eta} UGX_{n+1} \xrightarrow{U\varepsilon} UX_{n+1} \xrightarrow{U\eta} U^2GX_{n+1} \xrightarrow{U^2\sigma} U^2X_{n+2}.$$

Since $U\varepsilon \circ \eta$ is the identity, it follows that $\iota_{RX} = R\iota_X$. \square

We stress that Lemma 4.3 fails for symmetric spectra, and it is the major reason we must work with finitely generated model categories rather than compactly generated model categories. Indeed, in the compactly generated case, R^∞ is not a good functor, since maps out of one of the domains of the generating cofibrations

will not preserve the colimit that defines R^∞ . The obvious thing to try is to replace the functor R by a functor W , obtained by factoring $X \rightarrow RX$ into a projective cofibration $X \rightarrow WX$ followed by a level trivial fibration $WX \rightarrow RX$. The difficulty with this plan is that we do not see how to prove Lemma 4.3 for W . An alternative plan would be to use the mapping cylinder $X \rightarrow W'X$ on $X \rightarrow RX$; this might make Lemma 4.3 easier to prove, but the map $X \rightarrow W'X$ will not be a cofibration. The map $X \rightarrow W'X$ may, however, be good enough for the required smallness properties to hold. It is a closed inclusion if \mathcal{C} is topological spaces, for example. The author knows of no good general theorem in the compactly generated case.

This lemma leads immediately to the following proposition.

Proposition 4.4. *Suppose G is a left Quillen endofunctor of a model category \mathcal{C} , and suppose that its right adjoint U preserves sequential colimits. Then the map $\iota_{R^\infty X}: R^\infty X \rightarrow R(R^\infty X)$ is an isomorphism. In particular, if X is level fibrant, \mathcal{C} is almost finitely generated, and U preserves sequential colimits, then $R^\infty X$ is a U -spectrum.*

Proof. The map $\iota_{R^\infty X}$ is the colimit of the vertical maps in the diagram below.

$$\begin{array}{ccccccccccc}
 X & \xrightarrow{\iota_X} & RX & \xrightarrow{R\iota_X} & R^2X & \xrightarrow{R^2\iota_X} & \dots & \xrightarrow{R^{n-1}\iota_X} & R^nX & \xrightarrow{R^n\iota_X} & \dots \\
 \iota_X \downarrow & & \iota_{RX} \downarrow & & \iota_{R^2X} \downarrow & & & & \iota_{R^nX} \downarrow & & \\
 RX & \xrightarrow{R\iota_X} & R^2X & \xrightarrow{R^2\iota_X} & R^3X & \xrightarrow{R^3\iota_X} & \dots & \xrightarrow{R^n\iota_X} & R^{n+1}X & \xrightarrow{R^{n+1}\iota_X} & \dots
 \end{array}$$

Since the vertical and horizontal maps coincide, the result follows. For the second statement, we note that if X is level fibrant, each $R^n X$ is level fibrant since R is a right Quillen functor (with respect to the projective model structure). Since sequential colimits in \mathcal{C} preserve fibrant objects, $R^\infty X$ is level fibrant, and hence a U -spectrum. \square

Proposition 4.5. *Suppose G is a left Quillen endofunctor of a model category \mathcal{C} with right adjoint U . If \mathcal{C} is almost finitely generated, and X is a U -spectrum, then the map $j_X: X \rightarrow R^\infty X$ is a level equivalence.*

Proof. By assumption, the map $\iota_X: X \rightarrow RX$ is a level equivalence between level fibrant objects. Since R is a right Quillen functor, $R^n\iota_X$ is a level equivalence as well. Then the method of [10, Corollary 7.4.2] completes the proof. Recall that this method is to use factorization to construct a sequence of projective trivial cofibrations $Y_n \rightarrow Y_{n+1}$ with $Y_0 = X$ and a level trivial fibration of sequences $Y_n \rightarrow R^n X$. Then the map $X \rightarrow \text{colim } Y_n$ will be a projective trivial cofibration. Since sequential colimits in \mathcal{C} preserve trivial fibrations, the map $\text{colim } Y_n \rightarrow R^\infty X$ will still be a level trivial fibration. \square

Proposition 4.5 gives us a slightly better method of detecting stable equivalences.

Corollary 4.6. *Suppose G is a left Quillen endofunctor of a model category \mathcal{C} with right adjoint U . Suppose \mathcal{C} is almost finitely generated and U preserves sequential colimits. Then a map $f: A \rightarrow B$ is a stable equivalence in $\text{Sp}^{\mathbb{N}}(\mathcal{C}, G)$ if and only if $\text{map}(f, X)$ is a weak equivalence for all level fibrant spectra X such that $\iota_X: X \rightarrow RX$ is an isomorphism.*

Proof. By definition, f is a stable equivalence if and only if $\text{map}(f, Y)$ is a weak equivalence for all U -spectra Y . But we have a level equivalence $Y \rightarrow R^\infty Y$ by Proposition 4.5, and so it suffices to know that $\text{map}(f, R^\infty Y)$ is a weak equivalence for all U -spectra Y . But, by Proposition 4.4, $\iota_{R^\infty Y}$ is an isomorphism. \square

This corollary, in turn, allows us to prove that R^∞ detects stable equivalences. The following theorem is similar to [11, Theorem 3.1.11].

Theorem 4.7. *Suppose G is a left Quillen endofunctor of a model category \mathcal{C} with right adjoint U . Suppose that \mathcal{C} is almost finitely generated and sequential colimits in \mathcal{C} preserve finite products. Suppose also that U preserves sequential colimits. If $f: A \rightarrow B$ is a map in $Sp^{\mathbb{N}}(\mathcal{C}, G)$ such that $R^\infty f$ is a level equivalence, then f is a stable equivalence.*

Proof. Suppose X is a U -spectrum such that the map $\iota_X: X \rightarrow RX$ is an isomorphism. We will show that $\text{map}(f, X)$ is a retract of $\text{map}(R^\infty f, R^\infty X)$; this will obviously complete the proof. We first note that there is a natural map $\text{map}(R^\infty C, R^\infty X) \rightarrow \text{map}(C, X)$ obtained by precomposition with $C \rightarrow R^\infty C$ and postcomposition with $k: R^\infty X \rightarrow X$. Here k is the inverse of the map $j_X: X \rightarrow R^\infty X$, which is an isomorphism since ι_X is so. On the other hand, we claim that there is also a natural map $\text{map}(C, X) \rightarrow \text{map}(R^\infty C, R^\infty X)$ obtained by applying the total right derived functor of R^∞ . This is not obvious, since we are asserting that the total right derived functor of R^∞ preserves the enrichment of the projective homotopy category of $Sp^{\mathbb{N}}(\mathcal{C}, G)$ over the homotopy category of simplicial sets, even though R^∞ is not a right Quillen functor. Nevertheless, if we assume that this natural map exists, it follows easily that the composite $\text{map}(C, X) \rightarrow \text{map}(R^\infty C, R^\infty X) \rightarrow \text{map}(C, X)$ is the identity (in the homotopy category of simplicial sets), and therefore that $\text{map}(f, X)$ is a retract of $\text{map}(R^\infty f, R^\infty X)$.

It remains to show that the total right derived functor of R^∞ preserves the enriched structure. We first point out that R^∞ preserves level fibrations between level fibrant objects and all level trivial fibrations, because \mathcal{C} is almost finitely generated. It follows from Ken Brown's lemma [10, Lemma 1.1.12] that R^∞ preserves level equivalences between level fibrant objects. Because sequential colimits in \mathcal{C} preserve finite products, R^∞ also preserves finite products.

We claim that, for any functor H on a model category \mathcal{D} that preserves fibrations between fibrant objects, weak equivalences between fibrant objects, and finite products, the total right derived functor of H preserves the enriched structure over HoSSet . Indeed, analysis of the definition of this enrichment [10, Chapter 5] shows that it suffices to check that such a functor H preserves simplicial frames on a fibrant object Y . A simplicial frame on Y is a factorization $\ell_\bullet Y \xrightarrow{\alpha} Y_* \xrightarrow{\beta} r_\bullet Y$ in the diagram category of simplicial objects in \mathcal{D} , where $\ell_\bullet Y$ is the constant simplicial diagram on Y , the n th space of $r_\bullet Y$ is the product Y^{n+1} , α is a level equivalence, and β is a level fibration. We further require that β is an isomorphism in degree 0. Since H preserves products, weak equivalences between fibrant objects, and fibrations between fibrant objects, it follows that HY_* is a simplicial frame on HY . \square

Corollary 4.8. *Suppose G is a left Quillen endofunctor of a model category \mathcal{C} with right adjoint U . Suppose that \mathcal{C} is almost finitely generated and sequential colimits*

in \mathcal{C} preserve finite products. Suppose also that U preserves sequential colimits. Then $j_A: A \rightarrow R^\infty A$ is a stable equivalence for all $A \in Sp^{\mathbb{N}}(\mathcal{C}, G)$.

Proof. One can easily check that $R^\infty j_A$ is an isomorphism, using Proposition 4.4. \square

Finally, we get the desired characterization of stable equivalences.

Theorem 4.9. *Suppose G is a left Quillen endofunctor of a model category \mathcal{C} with right adjoint U . Suppose that \mathcal{C} is almost finitely generated and sequential colimits in \mathcal{C} preserve finite products. Suppose as well that U preserves sequential colimits. Let L' denote a fibrant replacement functor in the projective model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$. Then, for all $A \in Sp^{\mathbb{N}}(\mathcal{C}, G)$, the map $A \rightarrow R^\infty L'A$ is a stable equivalence into a U -spectrum. Also, a map $f: A \rightarrow B$ is a stable equivalence if and only if $R^\infty L'f$ is a level equivalence.*

Proof. The first statement follows immediately from Proposition 4.4 and Corollary 4.8. By the first statement, if f is a stable equivalence, so is $R^\infty L'f$. Since $R^\infty L'f$ is a map between U -spectra, it is a stable equivalence if and only if it is a level equivalence. The converse follows from Theorem 4.7. \square

Since we did not need the existence of the stable model structure to prove Theorem 4.9, one can imagine attempting to construct it from the functor $R^\infty L'$. This is, of course, the original approach of Bousfield-Friedlander [1], and this approach has been generalized by Schwede [18]. Also, if one has some way to detect level equivalences in \mathcal{C} , say using appropriate generalizations of homotopy groups, Theorem 4.9 implies that stable equivalences in $Sp^{\mathbb{N}}(\mathcal{C}, G)$ are detected by the appropriate generalizations of stable homotopy groups. One can see these generalizations in the following corollary as well.

Corollary 4.10. *Suppose \mathcal{C} is a pointed, left proper, cellular, almost finitely generated model category where sequential colimits preserve finite products. Suppose $G: \mathcal{C} \rightarrow \mathcal{C}$ is a left Quillen functor whose right adjoint U commutes with sequential colimits. Finally, suppose A is a finitely presented cofibrant object of \mathcal{C} that has a finitely presented cylinder object $A \times I$. Then*

$$\mathrm{Ho} Sp^{\mathbb{N}}(\mathcal{C}, G)(F_k A, Y) = \mathrm{colim}_m \mathrm{Ho} \mathcal{C}(A, U^m Y_{k+m}).$$

for all level fibrant $Y \in Sp^{\mathbb{N}}(\mathcal{C}, G)$.

Here we are using the stable model structure to form $\mathrm{Ho} Sp^{\mathbb{N}}(\mathcal{C}, G)$, of course.

Proof. We have $\mathrm{Ho} Sp^{\mathbb{N}}(\mathcal{C}, G)(F_k A, Y) = Sp^{\mathbb{N}}(\mathcal{C}, G)(F_k A, R^\infty Y) / \sim$, by Theorem 4.9, where \sim denotes the left homotopy relation. We can use the cylinder object $F_k(A \times I)$ as the source for our left homotopies. Then adjointness implies that $Sp^{\mathbb{N}}(\mathcal{C}, G)(F_k A, R^\infty Y) / \sim = \mathcal{C}(A, \mathrm{Ev}_k R^\infty Y) / \sim$. Since A and $A \times I$ are finitely presented, we get the required result. \square

By assuming slightly more about \mathcal{C} , we can also characterize the stable fibrations.

Corollary 4.11. *Suppose \mathcal{C} is a pointed, proper, cellular, almost finitely generated model category such that sequential colimits preserve pullbacks. Suppose $G: \mathcal{C} \rightarrow \mathcal{C}$ is a left Quillen functor whose right adjoint U commutes with sequential colimits. Then the stable model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$ is proper. In particular, a map*

$f: X \rightarrow Y$ is a stable fibration if and only if f is a level fibration and the diagram

$$\begin{array}{ccc} X & \longrightarrow & R^\infty L'X \\ f \downarrow & & \downarrow Lf \\ Y & \longrightarrow & R^\infty L'Y \end{array}$$

is a homotopy pullback square in the projective model structure, where L' is a fibrant replacement functor in the projective model structure.

Proof. We will actually show that, if $p: X \rightarrow Y$ is a level fibration and $f: B \rightarrow X$ is a stable equivalence, the pullback $B \times_A Y \rightarrow X$ is a stable equivalence. The first step is to use the right properness of the projective model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$ to reduce to the case where B and Y are level fibrant. Indeed, let $Y' = L'Y$, $B' = L'B$, and $f' = L'f$. Then factor the composite $X \rightarrow Y \rightarrow Y'$ into a projective trivial cofibration $X' \rightarrow Y'$ followed by a level fibration $p': X' \rightarrow Y'$. Then we have the commutative diagram below,

$$\begin{array}{ccccc} B & \xrightarrow{f} & Y & \xleftarrow{p} & X \\ \downarrow & & \downarrow & & \downarrow \\ B' & \xrightarrow{f'} & Y' & \xleftarrow{p'} & X' \end{array}$$

where the vertical maps are level equivalences. Then Proposition 12.2.4 and Corollary 12.2.8 of [9], which depend on the projective model structure being right proper, imply that the induced map $B \times_Y X \rightarrow B' \times_{Y'} X'$ is a level equivalence. Hence $B \times_Y X \rightarrow X$ is a stable equivalence if and only if $B' \times_{Y'} X' \rightarrow X'$ is a stable equivalence, and so we can assume B and X are level fibrant.

Now let S denote the pullback square below.

$$\begin{array}{ccc} B \times_Y X & \longrightarrow & X \\ \downarrow & & \downarrow p \\ B & \xrightarrow{f} & Y \end{array}$$

Then $R^n S$ is a pullback square for all n , and there are maps $R^n S \xrightarrow{R^n \iota_S} R^{n+1} S$. Since pullbacks commute with sequential colimits, $R^\infty S$ is a pullback square. Furthermore, $R^\infty p$ is a level fibration, since sequential colimits in \mathcal{C} preserve fibrations between level fibrant objects. Since f is a stable equivalence between level fibrant spectra, $R^\infty f$ is a level equivalence by Theorem 4.9. So, since the projective model structure is right proper, the map $R^\infty(B \times_Y X \rightarrow X)$ is a level equivalence, and thus $B \times_Y X \rightarrow X$ is a stable equivalence.

The characterization of stable fibrations then follows from [9, Proposition 3.6.8]. \square

5. PROPERTIES OF THE STABILIZATION FUNCTOR

In this section we explore some of the properties of the correspondence $(\mathcal{C}, G) \mapsto Sp^{\mathbb{N}}(\mathcal{C}, G)$, where, throughout this section, we mean the stable model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$. We begin by showing that, if G is already a Quillen equivalence, then the embedding $\mathcal{C} \rightarrow Sp^{\mathbb{N}}(\mathcal{C}, G)$ is a Quillen equivalence. This important fact is as close as we can get to proving that $Sp^{\mathbb{N}}(\mathcal{C}, G)$ is the initial, up to homotopy, stabilization

of \mathcal{C} with respect to G . We also show that $Sp^{\mathbb{N}}(\mathcal{C}, G)$ is functorial in the pair (\mathcal{C}, G) , with a suitable definition of maps of pairs. Under mild hypotheses, we show that $Sp^{\mathbb{N}}(\mathcal{C}, G)$ preserves Quillen equivalences in the pair (\mathcal{C}, G) . In particular, this means, for example, that the Bousfield-Friedlander category of spectra of simplicial sets does not depend, up to Quillen equivalence, on which model of the circle S^1 one chooses. We conclude the section by pointing out that our stabilization condition preserves some monoidal structure. For example, if \mathcal{C} is a simplicial model category, and G is a simplicial functor, then $Sp^{\mathbb{N}}(\mathcal{C}, G)$ is again a simplicial model category, and the extension of G is again a simplicial functor. However, if \mathcal{C} is monoidal, and G is a monoidal functor, $Sp^{\mathbb{N}}(\mathcal{C}, G)$ will almost never be a monoidal category; this is the reason we need the symmetric spectra introduced in the next section.

Theorem 5.1. *Suppose \mathcal{C} is a left proper cellular model category, and suppose G is a left Quillen endofunctor of \mathcal{C} that is a Quillen equivalence. Then $F_0: \mathcal{C} \rightarrow Sp^{\mathbb{N}}(\mathcal{C}, G)$ is a Quillen equivalence, where $Sp^{\mathbb{N}}(\mathcal{C}, G)$ has the stable model structure.*

Proof. We first point out that the right adjoint $Ev_0: Sp^{\mathbb{N}}(\mathcal{C}, G) \rightarrow \mathcal{C}$ reflects weak equivalences between fibrant objects. Indeed, suppose X and Y are U -spectra, and $f: X \rightarrow Y$ is a map such that $Ev_0 f = f_0$ is a weak equivalence. Then, because X and Y are U -spectra, $U^n f_n$ is a weak equivalence for all n . Since G is a Quillen equivalence, U reflects weak equivalences between fibrant objects by [10, Corollary 1.3.16]. Thus f_n is a weak equivalence for all n , and so f is a level equivalence and hence a stable equivalence, as required.

In view of [10, Corollary 1.3.16], to complete the proof it suffices to show that $X \rightarrow (L_S F_0 X)_0$ is a weak equivalence for all cofibrant $X \in \mathcal{C}$, where L_S denotes a stably fibrant replacement functor in $Sp^{\mathbb{N}}(\mathcal{C}, G)$. Let R' denote a fibrant replacement functor in the strict model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$. Then $X \rightarrow (R' F_0 X)_0$ is certainly a weak equivalence. We claim that $R' F_0 X$ is already a U -spectrum. Suppose for the moment that this is true; then by lifting we can construct a stable equivalence $R' F_0 X \rightarrow L_S F_0 X$, and since $R' F_0 X$ is a U -spectrum, this map is in fact a level equivalence. Hence the map $X \rightarrow (L_S F_0 X)_0$ is a weak equivalence, as required.

It remains to prove that $R' F_0 X$ is a U -spectrum. Since G is a Quillen equivalence, the map $(F_0 X)_n = G^n X \rightarrow URG^{n+1} X = UR(F_0 X)_{n+1}$ is a weak equivalence. By lifting, we can factor the weak equivalence $(F_0 X)_{n+1} \rightarrow (R' F_0 X)_{n+1}$ through the trivial cofibration $(F_0 X)_{n+1} \rightarrow R(F_0 X)_{n+1}$. This implies that the map $(F_0 X)_n \rightarrow U(R' F_0 X)_{n+1}$ is a weak equivalence, and hence that $R' F_0 X$ is a U -spectrum, as required. \square

In particular, this theorem means that the passage $(\mathcal{C}, G) \mapsto (Sp^{\mathbb{N}}(\mathcal{C}, G), G)$ is idempotent, up to Quillen equivalence. This suggests that we are doing some kind of fibrant replacement of (\mathcal{C}, G) , but the author knows no way of making this precise.

We now examine the functoriality of the stable model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$.

Definition 5.2. Suppose \mathcal{C} and \mathcal{D} are left proper cellular model categories, G is a left Quillen endofunctor of \mathcal{C} , and H is a left Quillen endofunctor of \mathcal{D} . A *map of pairs* $(\Phi, \tau): (\mathcal{C}, G) \rightarrow (\mathcal{D}, H)$ is a left Quillen functor $\Phi: \mathcal{C} \rightarrow \mathcal{D}$ and a natural transformation $\tau: \Phi G \rightarrow H\Phi$ such that τ_A is a weak equivalence for all cofibrant $A \in \mathcal{C}$.

Note that there is an obvious associative and unital composition of maps of pairs.

Proposition 5.3. *Suppose $(\Phi, \tau): (\mathcal{C}, G) \rightarrow (\mathcal{D}, H)$ is a map of pairs. Then there is an induced map of pairs $(Sp^{\mathbb{N}(\Phi)}, Sp^{\mathbb{N}(\tau)}): (Sp^{\mathbb{N}}(\mathcal{C}, G), G) \rightarrow (Sp^{\mathbb{N}}(\mathcal{D}, H), H)$ such that $Sp^{\mathbb{N}(\Phi)} \circ F_n = F_n \Phi$. This induced map of pairs is compatible with composition and identities.*

Proof. Suppose G has right adjoint U , H has right adjoint V , and Φ has right adjoint Γ . The natural transformation τ induces a dual natural transformation $D\tau: \Gamma V \rightarrow U\Gamma$. Define $Sp^{\mathbb{N}(\Gamma)}: Sp^{\mathbb{N}}(\mathcal{D}, H) \rightarrow Sp^{\mathbb{N}}(\mathcal{C}, G)$ by $(Sp^{\mathbb{N}(\Gamma)}Y)_n = \Gamma Y_n$, with structure maps adjoint to the composite

$$\Gamma Y_n \xrightarrow{\Gamma \tilde{\sigma}} \Gamma V Y_{n+1} \xrightarrow{D\tau} U \Gamma Y_{n+1}$$

where $\tilde{\sigma}$ is adjoint to the structure map of Y . The functor $Sp^{\mathbb{N}(\Gamma)}$ is analogous to restriction in the theory of group representations, and we must now define the analog to induction $Sp^{\mathbb{N}(\Phi)}$. To do so, first note that τ defines natural transformations $\tau^q: \Phi G^q \rightarrow H^q \Phi$ for all q , by iteration. Define $(Sp^{\mathbb{N}(\Phi)}X)_n$ to be the coequalizer of the two maps

$$\coprod_{p+q+r=n} H^p \Phi G^q X_r \rightrightarrows \coprod_{p+q=n} H^p \Phi X_q$$

where the top map is induced by $H^p \Phi G^q X_r \rightarrow H^p \Phi X_{q+r}$ and the bottom map is induced by $H^p \Phi G^q X_r \xrightarrow{H^p \tau^q} H^{p+q} \Phi X_r$. To define the structure map of $Sp^{\mathbb{N}(\Phi)}X$, note that the coequalizer diagram for $H(Sp^{\mathbb{N}(\Phi)}X)_n$ is just the subdiagram of the coequalizer diagram for $(Sp^{\mathbb{N}(\Phi)}X)_{n+1}$ consisting of all terms with a positive power of H . The inclusion of diagrams induces the desired structure map $H(Sp^{\mathbb{N}(\Phi)}X)_n \rightarrow (Sp^{\mathbb{N}(\Phi)}X)_{n+1}$. We leave to the reader the exercise in adjointness required to prove that $Sp^{\mathbb{N}(\Phi)}$ is left adjoint to $Sp^{\mathbb{N}(\Gamma)}$.

The functor $Sp^{\mathbb{N}(\Gamma)}$ clearly preserves level fibrations and level trivial fibrations, so $Sp^{\mathbb{N}(\Phi)}$ is a left Quillen functor with respect to the projective model structures. Also, since $\text{Ev}_n Sp^{\mathbb{N}(\Gamma)} = \Gamma \text{Ev}_n$, we have $F_n \Phi = Sp^{\mathbb{N}(\Phi)} F_n$. To show that $Sp^{\mathbb{N}(\Phi)}$ is a left Quillen functor with respect to the stable model structures, we must show that $Sp^{\mathbb{N}(\Phi)} s_n^A$ is a stable equivalence for all cofibrant A , by Theorem 2.2. Using the fact that $Sp^{\mathbb{N}(\Phi)} F_n = F_n \Phi$ and the fact that τ_A is a weak equivalence, we reduce to showing that $s_n \Phi A$ is a stable equivalence in $Sp^{\mathbb{N}}(\mathcal{D}, H)$. This follows from Theorem 3.4, so $Sp^{\mathbb{N}(\Phi)}$ is a left Quillen functor with respect to the stable model structures. the map Φs_n^A is a stable equivalence. Thus $\Phi: Sp^{\mathbb{N}}(\mathcal{C}, G) \rightarrow Sp^{\mathbb{N}}(\mathcal{D}, H)$ is a left Quillen functor with respect to the stable model structures.

We define $Sp^{\mathbb{N}(\tau)}$ by defining its adjoint $DSp^{\mathbb{N}(\tau)}: Sp^{\mathbb{N}(\Gamma)}V \rightarrow USp^{\mathbb{N}(\Gamma)}$. Indeed, $D\tau$ is just the prolongation of the adjoint $D\tau$ of τ . Since τ is a weak equivalence on all cofibrant objects of \mathcal{C} , $D\tau$ is a weak equivalence on all fibrant objects of \mathcal{D} . To see this, note that τ induces a natural isomorphism in the homotopy category. Adjointness implies that $D\tau$ also induces a natural isomorphism in the homotopy category, and it follows that $D\tau$ is a weak equivalence on all fibrant objects of \mathcal{D} . Thus $DSp^{\mathbb{N}(\tau)}$ will be a level equivalence on all level fibrant objects of $Sp^{\mathbb{N}}(\mathcal{D}, H)$, so $Sp^{\mathbb{N}(\tau)}$ is a level equivalence on all cofibrant objects of $Sp^{\mathbb{N}}(\mathcal{C}, G)$. We leave it to the reader to check compatibility of $(Sp^{\mathbb{N}(\Phi)}, Sp^{\mathbb{N}(\tau)})$ with compositions and identities. \square

Proposition 5.3 and Theorem 5.1 give us a weak universal property of $Sp^{\mathbb{N}}(\mathcal{C}, G)$.

Corollary 5.4. *Suppose $(\Phi, \tau): (\mathcal{C}, G) \rightarrow (\mathcal{D}, H)$ is a map of pairs such that H is a Quillen equivalence. Then there is a functor $\tilde{\Phi}: \text{Ho } Sp^{\mathbb{N}}(\mathcal{C}, G) \rightarrow \mathcal{D}$ such that $\tilde{\Phi} \circ LF_0 = L\tilde{\Phi}: \text{Ho } \mathcal{C} \rightarrow \text{Ho } \mathcal{D}$.*

This corollary is trying to say that $(Sp^{\mathbb{N}}(\mathcal{C}, G), G)$ is homotopy initial among maps of pairs $(\mathcal{C}, G) \rightarrow (\mathcal{D}, H)$ where H is a Quillen equivalence. Though the statement of the corollary is the best statement of this concept we have been able to find, we suspect there is a better one.

Proof. By Proposition 5.3 there is a map of pairs

$$(Sp^{\mathbb{N}(\Phi)}, Sp^{\mathbb{N}(\tau)}): (Sp^{\mathbb{N}}(\mathcal{C}, G), G) \rightarrow (Sp^{\mathbb{N}}(\mathcal{D}, H), H)$$

induced by (Φ, τ) . By Theorem 5.1, $F_0: \mathcal{D} \rightarrow Sp^{\mathbb{N}}(\mathcal{D}, H)$ is a Quillen equivalence. Define $\tilde{\Phi}$ to be the composite $REv_0 \circ LSp^{\mathbb{N}(\Phi)}$. \square

We have now shown that the correspondence $(\mathcal{C}, G) \mapsto (Sp^{\mathbb{N}}(\mathcal{C}, G), G)$ is functorial. We would like to know that it is homotopy invariant. In particular, we would like to know that $(Sp^{\mathbb{N}(\Phi)}, Sp^{\mathbb{N}(\tau)})$ is a Quillen equivalence of pairs when Φ is a Quillen equivalence. Our proof of this seems to require some hypotheses.

Theorem 5.5. *Suppose $(\Phi, \tau): (\mathcal{C}, G) \rightarrow (\mathcal{D}, H)$ is a map of pairs such that Φ is a Quillen equivalence. Suppose as well that either the domains of the generating cofibrations for \mathcal{C} can be taken to be cofibrant, or that τ_X is a weak equivalence for all X . Then, in the induced map of pairs $(Sp^{\mathbb{N}(\Phi)}, Sp^{\mathbb{N}(\tau)}): (Sp^{\mathbb{N}}(\mathcal{C}, G), G) \rightarrow (Sp^{\mathbb{N}}(\mathcal{D}, H), H)$, the Quillen functor $Sp^{\mathbb{N}(\Phi)}$ is a Quillen equivalence.*

Proof. We will first show that $Sp^{\mathbb{N}(\Phi)}$ is a Quillen equivalence on the projective model structures. Use the same notation as in the proof of Proposition 5.3, so that Γ denotes the right adjoint of Φ . Then, since Φ is a Quillen equivalence, Γ reflects weak equivalences between fibrant objects, by [10, Corollary 1.3.16]. It follows that $Sp^{\mathbb{N}(\Gamma)}$ reflects level equivalences between level fibrant objects. Hence to show that $Sp^{\mathbb{N}(\Phi)}$ is a Quillen equivalence of the projective model structures, it suffices to show that $X \rightarrow Sp^{\mathbb{N}(\Gamma)}RSp^{\mathbb{N}(\Phi)}X$ is a level equivalence for all cofibrant X , where R is a fibrant replacement functor in the projective model structure on $Sp^{\mathbb{N}}(\mathcal{D}, H)$. Thus, we need only show that $X_n \rightarrow \Gamma R(Sp^{\mathbb{N}(\Phi)}X)_n$ is a weak equivalence for all n and all cofibrant X , where now R is a fibrant replacement functor in \mathcal{D} . Since X_n is cofibrant and Φ is a Quillen equivalence, it suffices to show that $\Phi X_n \rightarrow (Sp^{\mathbb{N}(\Phi)}X)_n$ is a weak equivalence for all n and all cofibrant X . In fact, we can assume that X is an I_G -cell complex.

Write X as the colimit of a λ -sequence

$$0 = X^0 \rightarrow X^1 \rightarrow X^2 \rightarrow \dots \rightarrow X^\beta \rightarrow \dots \rightarrow X^\lambda = X$$

where each map $X^\beta \rightarrow X^{\beta+1}$ is a pushout of a map of I_G . We will prove that, for all $\beta \leq \lambda$, $\Phi X_n^\beta \rightarrow (Sp^{\mathbb{N}(\Phi)}X^\beta)_n$ is a weak equivalence for all n , by transfinite induction on β . Getting started is easy. The limit ordinal part of the induction follows from [9, Proposition 17.9.12], since each of the maps $\Phi X_n^\beta \rightarrow \Phi X_n^{\beta+1}$ and each of the maps $(Sp^{\mathbb{N}(\Phi)}X^\beta)_n \rightarrow (Sp^{\mathbb{N}(\Phi)}X^{\beta+1})_n$ is a cofibration of cofibrant objects.

For the successor ordinal part of the induction, suppose $X^\beta \rightarrow X^{\beta+1}$ is a pushout of the map $F_m C \xrightarrow{F_m f} F_m D$ of I_G . Then we have a pushout diagram

$$\begin{array}{ccc} \Phi(F_m C)_n & \longrightarrow & \Phi(F_m D)_n \\ \downarrow & & \downarrow \\ \Phi X_n^\beta & \longrightarrow & \Phi X_n^{\beta+1} \end{array}$$

and another pushout diagram

$$\begin{array}{ccc} (Sp^{\mathbb{N}(\Phi)} F_m C)_n & \longrightarrow & (Sp^{\mathbb{N}(\Phi)} F_m D)_n \\ \downarrow & & \downarrow \\ (Sp^{\mathbb{N}(\Phi)} X^\beta)_n & \longrightarrow & (Sp^{\mathbb{N}(\Phi)} X^{\beta+1})_n \end{array}$$

in \mathcal{D} . Note that $\Phi(F_m C)_n = \Phi G^{n-m} C$, where we interpret $G^{n-m} C$ to be the initial object if $n < m$. Similarly, $(Sp^{\mathbb{N}(\Phi)} F_m C)_n = (F_m \Phi C)_n = H^{n-m} \Phi C$. Thus the natural transformation τ induces a map from the first of these pushout squares to the second. If C (and hence also D) is cofibrant, then this map of pushout squares is a weak equivalence at both the upper left and upper right corners. Or, if τ is a weak equivalence for all X , then again this map is a weak equivalence at both the upper left and upper right corners. It is also a weak equivalence at the lower left corner, by the induction hypothesis. Since the top horizontal map is a cofibration in \mathcal{D} , Dan Kan's cube lemma [10, Lemma 5.2.6] implies that the map is a weak equivalence on the lower right corner. This completes the induction.

We have now proved that $Sp^{\mathbb{N}(\Phi)}$ is a Quillen equivalence with respect to the projective model structures. In view of Proposition 2.3, to show that $Sp^{\mathbb{N}(\Phi)}$ is a Quillen equivalence with respect to the stable model structures, we need to show that if Y is level fibrant in $Sp^{\mathbb{N}(\mathcal{D}, H)}$ and $Sp^{\mathbb{N}(\Gamma)} Y$ is a U -spectrum, then Y is a V -spectrum. To see this, note that, since $Sp^{\mathbb{N}(\Gamma)} Y$ is a U -spectrum, the natural map $\Gamma Y_n \rightarrow U \Gamma Y_{n+1}$ is a weak equivalence for all n . There is a natural transformation $D\tau: \Gamma V \rightarrow U \Gamma$ dual to τ . Furthermore, $(D\tau)_X$ is a weak equivalence for all fibrant X , as we have seen in the proof of Proposition 5.3. Thus, the natural map $\Gamma Y_n \rightarrow \Gamma V Y_{n+1}$ is a weak equivalence for all n . Since Γ reflects weak equivalences between fibrant objects, it follows that Y is a V -spectrum, as required. \square

As an example of Theorem 5.5, suppose we take a pointed simplicial set K weakly equivalent to S^1 . Then there is a weak equivalence $K \rightarrow RS^1$, where R is the fibrant replacement functor. This induces a natural transformation of left Quillen functors $\tau: K \wedge - \rightarrow RS^1 \wedge -$. In Theorem 5.5, take $\mathcal{D} = \mathcal{C}$ equal to the model category of pointed simplicial sets, take Φ to be the identity, and take τ to be this natural transformation. Then we get a Quillen equivalence between the stable model categories of spectra obtained by inverting K and inverting RS^1 . Therefore, the choice of simplicial circle does not matter, up to Quillen equivalence, for Bousfield-Friedlander spectra.

We now investigate to what extent the correspondence that takes (\mathcal{C}, G) to the stable model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$ preserves monoidal structure. We begin by assuming that \mathcal{C} is a \mathcal{D} -model category, for some symmetric monoidal model category \mathcal{D} . This means that \mathcal{D} is a symmetric monoidal category with a compatible model structure. Since we will need to work with this compatibility, we remind the reader of the precise definition (see also [10, Chapter 4]).

Definition 5.6. Suppose \mathcal{D} is a monoidal category. Given maps $f: A \rightarrow B$ and $g: C \rightarrow D$, we define the *pushout product* $f \square g$ of f and g to be the map $f \square g: (A \otimes D) \amalg_{A \otimes C} (B \otimes C) \rightarrow B \otimes D$ induced by the commutative square

$$\begin{array}{ccc} A \otimes C & \xrightarrow{f \otimes 1} & B \otimes C \\ 1 \otimes g \downarrow & & \downarrow 1 \otimes g \\ A \otimes D & \xrightarrow{f \otimes 1} & B \otimes D \end{array}$$

The compatibility condition we require is then that, if f and g are cofibrations, then so is $f \square g$, and furthermore, if one of f and g is a trivial cofibration, so is $f \square g$. We must also require that, if S is the unit of \otimes and $QS \rightarrow S$ is a cofibrant approximation, then $QS \otimes X \rightarrow X$ is still a weak equivalence.

Then, by saying that \mathcal{C} is a \mathcal{D} -model category, we mean that \mathcal{C} is tensored, cotensored, and enriched over \mathcal{D} , compatibly with the model structure. This compatibility is precisely analogous to the compatibility above.

We then have the following theorem.

Theorem 5.7. *Let \mathcal{D} be a cofibrantly generated monoidal model category, and suppose the domains of the generating cofibrations are cofibrant. Suppose \mathcal{C} is a left proper cellular \mathcal{D} -model category, and that G is a left \mathcal{D} -Quillen endofunctor of \mathcal{D} . This means that $G(X \otimes K) \cong GX \otimes K$, coherently, for $X \in \mathcal{C}$ and $K \in \mathcal{D}$. Then $Sp^{\mathbb{N}}(\mathcal{C}, G)$, with the stable model structure, is again a \mathcal{D} -model category, and the extension of G is a \mathcal{C} -Quillen self-equivalence of $Sp^{\mathbb{N}}(\mathcal{C}, G)$.*

Of course, the Quillen functors $F_n: \mathcal{C} \rightarrow Sp^{\mathbb{N}}(\mathcal{C}, G)$ will be \mathcal{D} -Quillen functors as well.

Proof. We define the action of \mathcal{D} on $Sp^{\mathbb{N}}(\mathcal{C}, G)$ levelwise. That is, given $X \in Sp^{\mathbb{N}}(\mathcal{C}, G)$ and $K \in \mathcal{D}$, we define $(X \otimes K)_n = X_n \otimes K$. The structure map is given by

$$G(X_n \wedge K) \cong GX_n \wedge K \rightarrow X_{n+1} \wedge K.$$

One can easily verify that this makes $Sp^{\mathbb{N}}(\mathcal{C}, G)$ tensored over \mathcal{D} . Similarly, define $(X^K)_n = X_n^K$, with structure maps $G(X_n^K) \rightarrow X_{n+1}^K$ adjoint to the composite

$$G(X_n^K) \wedge K \cong G(X_n^K \wedge K) \xrightarrow{G(\text{ev})} GX_n \rightarrow X_{n+1}$$

where $\text{ev}: X_n^K \wedge K$ is the evaluation map, adjoint to the identity of X_n^K . This makes $Sp^{\mathbb{N}}(\mathcal{C}, G)$ cotensored over \mathcal{D} . Finally, given X and Y in $Sp^{\mathbb{N}}(\mathcal{C}, G)$, define $\text{Map}(X, Y) \in \mathcal{D}$ to be the equalizer of the two maps

$$\alpha, \beta: \prod_n \text{Map}(X_n, Y_n) \rightarrow \prod_n \text{Map}(X_n, UY_{n+1})$$

where α is the product of the maps $\text{Map}(X_n, Y_n) \rightarrow \text{Map}(X_n, UY_{n+1})$ induced by the adjoint of the structure map of Y , and β is the product of the maps

$$\text{Map}(X_{n+1}, Y_{n+1}) \rightarrow \text{Map}(UX_{n+1}, UY_{n+1}) \rightarrow \text{Map}(X_n, UY_{n+1}).$$

Here the first map exists since G preserves the \mathcal{D} action, and the second map is induced by the structure map of X . This functor makes $Sp^{\mathbb{N}}(\mathcal{C}, G)$ enriched over \mathcal{D} .

We must now check that these structures are compatible with the model structure. We begin with the projective model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$. One can easily check that $F_n f \square g = F_n(f \square g)$. Thus, if f is one of the generating cofibrations of the projective model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$, and g is a cofibration in \mathcal{D} , then $f \square g$ is a cofibration in $Sp^{\mathbb{N}}(\mathcal{C}, G)$. It follows that $f \square g$ is a cofibration for f an arbitrary cofibration of $Sp^{\mathbb{N}}(\mathcal{C}, G)$ (see [21, Lemma 2.3] and [11, Corollary 5.3.5]). A similar argument shows that $f \square g$ is a projective trivial cofibration in $Sp^{\mathbb{N}}(\mathcal{C}, G)$ if either f is a projective cofibration in $Sp^{\mathbb{N}}(\mathcal{C}, G)$ or g is a trivial cofibration in \mathcal{D} . Finally, if $QS \rightarrow S$ is a cofibrant approximation to the unit S in \mathcal{D} , and X is cofibrant in $Sp^{\mathbb{N}}(\mathcal{C}, G)$, then each X_n is cofibrant in \mathcal{C} , so the map $X \otimes QS \rightarrow X$ is a level equivalence as required. Thus $Sp^{\mathbb{N}}(\mathcal{C}, G)$ with its projective model structure is a \mathcal{D} -model category.

To show that $Sp^{\mathbb{N}}(\mathcal{C}, G)$ with its stable model structure is also a \mathcal{D} -model category, we need to show that, if f is a stable trivial cofibration and g is a cofibration in \mathcal{D} , then $f \square g$ is a stable equivalence. It suffices to check this for $g: K \rightarrow L$ one of the generating trivial cofibrations of \mathcal{D} . In this case, by hypothesis, K and L are cofibrant in \mathcal{D} . Thus the functor $-\otimes K$ is a Quillen functor with respect to the projective model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$, and similarly for L . Furthermore, if $s_n^{QC}: F_{n+1}GQC \rightarrow F_nQC$ is an element of the set \mathcal{S} , then $s_n^{QC} \otimes K \cong s_n^{QC \otimes K}$, since G preserves the \mathcal{D} -action. In view of Theorem 3.4, the map $s_n^{QC \otimes K}$ is a stable equivalence. Theorem 2.2 then implies that $-\otimes K$ is a Quillen functor with respect to the stable model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$, and similarly for $-\otimes L$. Thus, if f is a stable trivial cofibration, so are $f \otimes K$ and $f \otimes L$. It follows from the two out of three property that $f \square g$ is a stable equivalence, as required. \square

Remark 5.8. Suppose that the functor G is actually given by $GX = X \otimes K$ for some cofibrant object K of \mathcal{D} . We then have **two different ways** of tensoring with K on $Sp^{\mathbb{N}}(\mathcal{C}, G)$. The first way is the extension of G to a Quillen equivalence of $Sp^{\mathbb{N}}(\mathcal{C}, G)$. Recall from Remark 1.7 that this functor, which we denote by $X \mapsto X \overline{\otimes} K$, does not use the twist map. On the other hand, we also have the functor $X \mapsto X \otimes K$ that is part of the \mathcal{D} -action on $Sp^{\mathbb{N}}(\mathcal{C}, G)$ constructed in Theorem 5.7. This functor *does* use the twist map as part of its structure map; indeed, in order to construct the isomorphism $G(X \otimes K) \cong GX \otimes K$ we need to permute the two different copies of K . Therefore, we **do not know** that $X \mapsto X \otimes K$ is a Quillen equivalence, even though $X \mapsto X \overline{\otimes} K$ is. We will have to deal with this point more thoroughly in Section 9, when we compare $Sp^{\mathbb{N}}(\mathcal{C}, G)$ with symmetric spectra.

Theorem 5.7 gives us a functorial stabilization. We first simplify the notation. Suppose K is a cofibrant object of a symmetric monoidal model category \mathcal{D} . Then $G = - \otimes K$ is a left Quillen functor on any \mathcal{D} -model category \mathcal{C} . In this case, we denote $Sp^{\mathbb{N}}(\mathcal{C}, G)$ by $Sp^{\mathbb{N}}(\mathcal{C}, K)$.

Corollary 5.9. *Suppose K is a cofibrant object of a cofibrantly generated symmetric monoidal model category \mathcal{D} where the domains of the generating cofibrations can be taken to be cofibrant. Then the correspondence $\mathcal{C} \mapsto Sp^{\mathbb{N}}(\mathcal{C}, K)$ defines an endofunctor of the category of left proper cellular \mathcal{D} -model categories.*

Note that the “category” of left proper cellular \mathcal{D} -model categories is not really a category, because the Hom-sets need not be sets. It is really a 2-category, and the correspondence $\mathcal{C} \mapsto Sp^{\mathbb{N}}(\mathcal{C}, K)$ is actually a 2-functor. See [10] for a description of this point of view on model categories.

Proof. Given a left proper cellular \mathcal{D} -model category \mathcal{C} , we have seen in Theorem 5.7 that $Sp^{\mathbb{N}}(\mathcal{C}, K)$ is a \mathcal{D} -model category. Just as in Lemma 1.5, a \mathcal{D} -Quillen functor $H: \mathcal{C} \rightarrow \mathcal{C}'$ induces a functor $H: Sp^{\mathbb{N}}(\mathcal{C}, K) \rightarrow Sp^{\mathbb{N}}(\mathcal{C}', K)$, as does its right adjoint V . Since H is defined levelwise, it preserves the action of \mathcal{D} . It is easy to check that V preserves level fibrations and level trivial fibrations, so that H is a \mathcal{C} -Quillen functor with respect to the projective model structures. Furthermore, we have $Hs_n^{QC} = s_n^{HQC}$, so Theorem 3.4 and Theorem 2.2 imply that H is a \mathcal{C} -Quillen functor with respect to the stable model structures as well. \square

We now point out that, if \mathcal{D} is a symmetric monoidal model category, and $G = - \otimes K$ for some cofibrant object K , the category $Sp^{\mathbb{N}}(\mathcal{D}, G)$ is almost never itself monoidal, though, as we have seen, it has an action of \mathcal{D} . To see this, consider the category $\mathcal{D}^{\mathbb{N}}$ of sequences from \mathcal{D} . An object of $\mathcal{D}^{\mathbb{N}}$ is a sequence X_n of objects of \mathcal{D} , and a map $f: X \rightarrow Y$ is a sequence of maps $f_n: X_n \rightarrow Y_n$. Then $\mathcal{D}^{\mathbb{N}}$ is a symmetric monoidal category, where we define $(X \wedge Y)_n = \coprod_{p+q=n} X_p \wedge Y_q$. The functor G defines a monoid $T = (S^0, GS^0, G^2S^0, \dots, G^n S^0, \dots)$ in this category, using the fact that G preserves the \mathcal{D} -action.

Lemma 5.10. *Suppose \mathcal{D} is a symmetric monoidal model category and G is a left \mathcal{D} -Quillen functor. Then $Sp^{\mathbb{N}}(\mathcal{D}, G)$ is the category of left modules over the monoid $T = (S^0, GS^0, \dots, G^n S^0, \dots)$.*

We leave the proof of this lemma to the reader. The important point is that the monoid T is almost never commutative, and therefore $Sp^{\mathbb{N}}(\mathcal{D}, G)$ can not be a symmetric monoidal category with unit T . Indeed, let $K = GS \in \mathcal{D}$, so that $G = - \otimes K$. Then T is commutative if and only if the commutativity isomorphism on $K \wedge K$ is the identity. This happens only very rarely.

6. SYMMETRIC SPECTRA

We have just seen that the stabilization functor $Sp^{\mathbb{N}}(\mathcal{C}, G)$ is not good enough in case \mathcal{D} is a symmetric monoidal model category and G is a \mathcal{D} -Quillen functor, because $Sp^{\mathbb{N}}(\mathcal{C}, G)$ is not usually itself a symmetric monoidal model category. In this section, we begin the construction of a better stabilization functor $Sp^{\Sigma}(\mathcal{D}, K)$ for this case. We will concentrate on the category theory in this section, leaving the model structures for the next section. The terms used for the algebra of symmetric monoidal categories and modules over them are all defined in [10, Section 4.1].

Through most of this section, then, \mathcal{D} will be a bicomplete closed symmetric monoidal category with unit S , and K will be an object of \mathcal{D} . The category \mathcal{C} will be a bicomplete category enriched, tensored, and cotensored over \mathcal{D} . Note that any \mathcal{D} -functor G on \mathcal{D} is of the form $G(X) = X \wedge K$ for $K = GS$, so we will only consider such functors. Because of this, we will drop the letter G from our notations and replace it with K .

This section is based on the symmetric spectra and sequences of [11]. The main idea of symmetric spectra is that the commutativity isomorphism of \mathcal{D} makes $K^{\wedge n}$ into a Σ_n -object of \mathcal{D} , where Σ_n is the symmetric group on n letters. We must keep track of this action if we expect to get a symmetric monoidal category of K -spectra.

The following definition is [11, Definition 2.1.1].

Definition 6.1. Let $\Sigma = \coprod_{n \geq 0} \Sigma_n$ be the category whose objects are the sets $\bar{n} = \{1, 2, \dots, n\}$ for $n \geq 0$, where $\bar{0} = \emptyset$. The morphisms of Σ are the isomorphisms

of \bar{n} . Given a category \mathcal{C} , a *symmetric sequence* in \mathcal{C} is a functor $\Sigma \rightarrow \mathcal{C}$. The category of symmetric sequences is the functor category \mathcal{C}^Σ .

A symmetric sequence in \mathcal{C} is a sequence $X_0, X_1, \dots, X_n, \dots$ of objects of \mathcal{C} with an action of Σ_n on X_n . It is sometimes more useful to consider a symmetric sequence as a functor from the category of finite sets and isomorphisms to \mathcal{C} ; since the category Σ is a skeleton of the category of finite sets and isomorphisms, there is no difficulty in doing so.

As a functor category, the category of symmetric sequences in \mathcal{C} is bicomplete if \mathcal{C} is so; limits and colimits are taken objectwise. Furthermore, if \mathcal{D} is a closed symmetric monoidal category, so is \mathcal{D}^Σ , as explained in [11, Section 2.1]. Recall that the monoidal structure is given by

$$(X \otimes Y)(C) = \prod_{A \cup B = C, A \cap B = \emptyset} X(A) \otimes Y(B)$$

where we think of X , Y , and $X \otimes Y$ as functors from finite sets to \mathcal{D} . Equivalently, though less canonically, we have

$$(X \otimes Y)_n = \prod_{p+q=n} \Sigma_n \times_{\Sigma_p \times \Sigma_q} (X_p \otimes Y_q).$$

The unit of the monoidal structure is the symmetric sequence $(S, 0, \dots, 0, \dots)$, where 0 is the initial object of \mathcal{C} . To define the closed structure, we first define $\text{Hom}_{\Sigma_n}(X, Y)$ for $X, Y \in \mathcal{D}^{\Sigma_n}$ in the usual way, as an equalizer of the two obvious maps $\text{Hom}(X, Y) \rightarrow \text{Hom}(X \times \Sigma_n, Y)$. The closed structure is then given by

$$\text{Hom}(X, Y)_k = \prod_n \text{Hom}_{\Sigma_n}(X_n, Y_{n+k}).$$

If \mathcal{C} is enriched, tensored, and cotensored over \mathcal{D} , then \mathcal{C}^Σ is enriched, tensored, and cotensored over \mathcal{D}^Σ . Indeed, the same definition as above works to define the tensor structure. The cotensor structure is defined as follows. First we define $\text{Hom}_{\Sigma_n}(K, X)$ for $X \in \mathcal{C}^{\Sigma_n}$ and $K \in \mathcal{D}^{\Sigma_n}$ as an appropriate equalizer. Then, for $X \in \mathcal{C}^\Sigma$ and $K \in \mathcal{D}^\Sigma$, we define $X_k^K = \prod_n \text{Hom}_{\Sigma_n}(K_n, X_{n+k})$. The enrichment $\text{Map}(X, Y)$ is defined similarly. In the same way, if \mathcal{C} is an enriched monoidal category over \mathcal{D} , then \mathcal{C}^Σ is an enriched monoidal category over \mathcal{D}^Σ .

Consider the free commutative monoid $\text{Sym}(K)$ on the object $(0, K, 0, \dots, 0, \dots)$ of \mathcal{D}^Σ . One can easily check that $\text{Sym}(K)$ is the symmetric sequence $(S^0, K, K \otimes K, \dots, K^{\otimes n}, \dots)$ where Σ_n acts on $K^{\otimes n}$ by the commutativity isomorphism, as in [11, Section 4.4].

Definition 6.2. Suppose \mathcal{D} is a symmetric monoidal model category, \mathcal{C} is a \mathcal{D} -model category, and K is an object of \mathcal{D} . The category of *symmetric spectra* $Sp^\Sigma(\mathcal{C}, K)$ is the category of modules in \mathcal{C}^Σ over the commutative monoid $\text{Sym}(K)$ in \mathcal{D}^Σ . That is, a symmetric spectrum is a sequence of objects $X_n \in \mathcal{C}^{\Sigma_n}$ and Σ_n -equivariant maps $X_n \otimes K \rightarrow X_{n+1}$, such that the composite

$$X_n \otimes K^{\otimes p} \rightarrow X_{n+1} \otimes K^{\otimes p-1} \rightarrow \dots \rightarrow X_{n+p}$$

is $\Sigma_n \times \Sigma_p$ -equivariant for all $n, p \geq 0$. A map of symmetric spectra is a collection of Σ_n -equivariant maps $X_n \rightarrow Y_n$ compatible with the structure maps of X and Y .

Because $\text{Sym}(K)$ is a commutative monoid, the category $Sp^\Sigma(\mathcal{D}, K)$ is a bi-complete closed symmetric monoidal category, with $\text{Sym}(K)$ itself as the unit (see

Lemma 2.2.2 and Lemma 2.2.8 of [11]). We denote the monoidal structure by $X \wedge Y = X \otimes_{\text{Sym}(K)} Y$, and the closed structure by $\text{Hom}_{\text{Sym}(K)}(X, Y)$. Similarly, $Sp^\Sigma(\mathcal{C}, K)$ is bicomplete, enriched, tensored, and cotensored over $Sp^\Sigma(\mathcal{D}, K)$ with the tensor structure denoted $X \wedge Y$ again, and, if \mathcal{C} is a \mathcal{D} -monoidal model category, then $Sp^\Sigma(\mathcal{C}, K)$ will be a monoidal category enriched over $Sp^\Sigma(\mathcal{D}, K)$.

Of course, if we take $\mathcal{C} = \mathbf{SSet}_*$ and $K = S^1$, we recover the definition of symmetric spectra given in [11], except that we are using right modules instead of left modules.

Definition 6.3. Given $n \geq 0$, the *evaluation functor* $\text{Ev}_n: Sp^\Sigma(\mathcal{C}, K) \rightarrow \mathcal{C}$ takes X to X_n . the evaluation functor has a left adjoint $F_n: \mathcal{C} \rightarrow Sp^\Sigma(\mathcal{C}, K)$, defined by $F_n X = \tilde{F}_n X \otimes \text{Sym}(K)$, where $\tilde{F}_n X$ is the symmetric sequence $(0, \dots, 0, \Sigma_n \times X, 0, \dots)$.

Note that $F_0 X = (X, X \otimes K, \dots, X \otimes K^{\otimes n}, \dots)$, and in particular $F_0 S = \text{Sym}(K)$. Also, if $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, there is a natural isomorphism $F_n X \wedge F_m Y \cong F_{n+m}(X \otimes Y)$, just as in [11, Proposition 2.2.6]. In particular, $F_0: \mathcal{D} \rightarrow Sp^\Sigma(\mathcal{D}, K)$ is a (symmetric) monoidal functor, and so $Sp^\Sigma(\mathcal{C}, K)$ is naturally enriched, tensored, and cotensored over \mathcal{D} . In fact, this structure is very simple. Indeed, if $X \in Sp^\Sigma(\mathcal{C}, K)$ and $L \in \mathcal{D}$, $X \otimes L = X \otimes_{\text{Sym}(K)} F_0 L$ is just the symmetric sequence whose n th term is $X_n \otimes L$. The structure map is the composite

$$X_n \otimes L \otimes K \xrightarrow{1 \otimes \tau} X_n \otimes K \otimes L \rightarrow X_{n+1} \otimes L.$$

Note the presence of the twist map; this is required even when $L = K$ to get a symmetric spectrum, unlike the case of ordinary spectra. Similarly, $X^L = \text{Hom}_{\text{Sym}(K)}(F_0 L, X)$ is the symmetric sequence whose n th term is X_n^L , with the twist map again appearing as part of the structure map.

Remark 6.4. Just as in the spectrum case, the functor Ev_n has a right adjoint $R_n: \mathcal{C} \rightarrow Sp^\Sigma(\mathcal{C}, K)$. Indeed, $R_n X = \text{Hom}(\text{Sym}(K), \tilde{R}_n X)$, where $\tilde{R}_n X$ is the symmetric sequence concentrated in degree n whose n th term is X^{Σ_n} .

7. MODEL STRUCTURES ON SYMMETRIC SPECTRA

Throughout this section, \mathcal{D} will denote a left proper cellular symmetric monoidal model category, \mathcal{C} will denote a left proper cellular \mathcal{D} -model category, and K will denote a cofibrant object of \mathcal{D} . In this section, we discuss the projective and stable model structures on the category $Sp^\Sigma(\mathcal{C}, K)$ of symmetric spectra. The results in this section are very similar to the corresponding results for spectra, so we will leave most of the proofs to the reader.

Definition 7.1. A map $f \in Sp^\Sigma(\mathcal{C}, K)$ is a *level equivalence* if each map f_n is a weak equivalence in \mathcal{C} . Similarly, f is a *level fibration* (resp. *level cofibration*, *level trivial fibration*, *level trivial cofibration*) if each map f_n is a fibration (resp. cofibration, trivial fibration, trivial cofibration) in \mathcal{C} . The map f is a *projective cofibration* if f has the left lifting property with respect to every level trivial fibration.

Then, just as in Definition 1.9, if we denote the generating cofibrations of \mathcal{C} by I and the generating trivial cofibrations by J , we define $I_K = \bigcup_n F_n I$ and $J_K = \bigcup_n F_n J$.

We have analogues of 1.10–1.13 with the same proofs. This gives us the projective model structure.

Theorem 7.2. *The projective cofibrations, the level fibrations, and the level equivalences define a left proper cellular model structure on $Sp^\Sigma(\mathcal{C}, K)$.*

The set I_K is the set of generating cofibrations of the projective model structure, and J_K is the set of generating trivial cofibrations. The cellularity of the projective model structure is proved in the Appendix.

Note that Ev_n takes level (trivial) fibrations to (trivial) fibrations, so Ev_n is a right Quillen functor and F_n is a left Quillen functor.

Theorem 7.3. *The category $Sp^\Sigma(\mathcal{D}, K)$, with the projective model structure, is a symmetric monoidal model category. The category $Sp^\Sigma(\mathcal{C}, K)$, with its projective model structure, is a $Sp^\Sigma(\mathcal{D}, K)$ -model category.*

Proof. We first show that the pushout product $f \square g$ is a (trivial) cofibration when f is a cofibration in $Sp^\Sigma(\mathcal{C}, K)$, and g is a cofibration in $Sp^\Sigma(\mathcal{D}, K)$ (and one of them is a level equivalence). As explained in [10, Corollary 2.5], we may as well assume that f and g belong to the sets of generating cofibrations or generating trivial cofibrations. In either case, we have $f = F_m f'$ and $g = F_n g'$. But then $f \square g = F_{m+n}(f' \square g')$. Since F_{m+n} is a Quillen functor, the result follows.

Now let QS denote a cofibrant replacement for the unit S in \mathcal{D} . Then $F_0 QS$ is a cofibrant replacement for $F_0 S = \text{Sym}(K)$ in $Sp^\Sigma(\mathcal{D}, K)$. Indeed, $F_0 QS$ is cofibrant, and $Ev_n F_0 QS$ is just $QS \otimes K^{\otimes n}$. Since K is cofibrant and \mathcal{D} is a monoidal model category, the map $F_0 QS \rightarrow F_0 S$ is a level equivalence. Now, if X is cofibrant in $Sp^\Sigma(\mathcal{C}, K)$, then each X_n is cofibrant. Hence the map $X_n \otimes QS \rightarrow X_n$ is a weak equivalence for all n , and so the map $X \wedge F_0 QS \rightarrow X$ is a level equivalence, as required. \square

We point out here that one can show that the projective model structure on $Sp^\Sigma(\mathcal{D}, K)$ satisfies the monoid axiom of [21], assuming that \mathcal{D} itself does so. This means there is a projective model structure on the category of monoids in $Sp^\Sigma(\mathcal{D}, K)$ and on the category of modules over any monoid. We do not include the proofs of these statements since we have not been able to prove the analogous statements for the stable model structure.

The projective cofibrations of symmetric spectra are more complicated than they are in the case of ordinary spectra.

Definition 7.4. Define the symmetric spectrum $\overline{\text{Sym}(K)}$ in $Sp^\Sigma(\mathcal{D}, K)$ to be 0 in degree 0 and $K^{\otimes n}$ in degree n , for $n > 0$, with the obvious structure maps. Define the n th latching space $L_n X$ of $X \in Sp^\Sigma(\mathcal{C}, K)$ by $L_n X = Ev_n(X \wedge \overline{\text{Sym}(K)})$.

The obvious map $i: \overline{\text{Sym}(K)} \rightarrow \text{Sym}(K)$ induces a Σ_n -equivariant natural transformation $L_n X \rightarrow X$.

Note that the latching space is a Σ_n -object of \mathcal{C} . There is a model structure on Σ_n -objects of \mathcal{C} where the fibrations and weak equivalences are the underlying ones. This model structure is cofibrantly generated: if I is the set of generating cofibrations of \mathcal{C} , then the set of generating cofibrations of \mathcal{C}^{Σ_n} is the set $\Sigma_n \times I$. Here, for an object A , $\Sigma_n \times A$ is the coproduct of $n!$ copies of A , given the obvious Σ_n -structure.

Proposition 7.5. *A map $f: X \rightarrow Y$ in $Sp^\Sigma(\mathcal{C}, K)$ is a projective (trivial) cofibration if and only if the induced map $Ev_n(f \square i): X_n \amalg_{L_n X} L_n Y \rightarrow Y_n$ is a (trivial) cofibration in \mathcal{C}^{Σ_n} for all n .*

Proof. We only prove the cofibration case, as the trivial cofibration case is analogous. If each map $X_n \amalg_{L_n X} L_n Y \rightarrow Y_n$ is a cofibration, then we can show that f is a projective cofibration by showing f has the left lifting property with respect to level trivial fibrations. Indeed, we construct a lift by induction, just as in the proof of Proposition 1.15. To prove the converse, it suffices to show that $\text{Ev}_n(f \square i)$ is a Σ_n -cofibration for $f \in I_K$, since Ev_n is itself a left Quillen functor. Then we can write $f = F_m g$, and we find that $\text{Ev}_n(f \square i)$ is an isomorphism when $n \neq m$, and is the map $\Sigma_m \times g$ when $n = m$. This is a Σ_m -cofibration, as required. \square

We must now localize the projective model structure to obtain the stable model structure.

Definition 7.6. A symmetric spectrum $X \in Sp^\Sigma(\mathcal{C}, K)$ is an Ω -spectrum if X is level fibrant and the adjoint $X_n \rightarrow X_{n+1}^K$ of the structure map is a weak equivalence for all n .

Just as with Bousfield-Friedlander spectra, we would like the Ω -spectra to be the fibrant objects in the stable model structure. We invert the same maps we did in that case.

Definition 7.7. Define the set of maps \mathcal{S} in $Sp^\Sigma(\mathcal{C}, K)$ to be $\{F_{n+1}(QC \otimes K) \xrightarrow{s_n^{QC}} F_n QC\}$ as C runs through the domains and codomains of the generating cofibrations of \mathcal{C} , and $n \geq 0$. The map s_n^{QC} is adjoint to the map $QC \otimes K \rightarrow \text{Ev}_{n+1} F_n QC = \Sigma_{n+1} \times (QC \otimes K)$ corresponding to the identity of Σ_{n+1} . Define the *stable model structure* on $Sp^\Sigma(\mathcal{C}, K)$ to be the Bousfield localization with respect to \mathcal{S} of the projective model structure on $Sp^\Sigma(\mathcal{C}, K)$. The \mathcal{S} -local weak equivalences are called the *stable equivalences*, and the \mathcal{S} -local fibrations are called the *stable fibrations*.

The following theorem is then analogous to Theorem 3.4, and has the same proof.

Theorem 7.8. *The stably fibrant symmetric spectra are the Ω -spectra. Furthermore, for all cofibrant $A \in \mathcal{C}$ and for all $n \geq 0$, the map $F_{n+1}(A \otimes K) \xrightarrow{s_n^A} F_n A$ is a stable equivalence.*

Just as in Corollary 3.5, this theorem implies that, when $\mathcal{C} = \mathbf{SSet}_*$ or \mathbf{Top}_* , $Sp^\Sigma(\mathcal{D}, K)$ is the same as the stable model category on (simplicial or topological) symmetric spectra discussed in [11].

The analog of Corollary 3.6 also holds, with the same proof, so that tensoring with K is a Quillen endofunctor of $Sp^\Sigma(\mathcal{C}, K)$. Of course, we want this functor to be a Quillen equivalence. As in Definition 3.7, we prove this by introducing the shift functors.

Definition 7.9. Define the *right shift functor* $s: Sp^\Sigma(\mathcal{C}, K) \rightarrow Sp^\Sigma(\mathcal{C}, K)$ by $sX = \text{Hom}_{\text{Sym}(K)}(F_1 S, X)$. Thus $(sX)_n = X_{n+1}$, where the Σ_n -action on X_{n+1} is induced by the usual inclusion $\Sigma_n \rightarrow \Sigma_{n+1}$. The structure maps of sX are the same as the structure maps of X . Define the *left shift functor* $t: Sp^\Sigma(\mathcal{C}, K) \rightarrow Sp^\Sigma(\mathcal{C}, K)$ by $tX = X \otimes_{\text{Sym}(K)} F_1 S$, so that t is left adjoint of s . We have $(tX)_n = \Sigma_n \times_{\Sigma_{n-1}} X_{n-1}$, with the induced structure maps.

Note that adjointness gives natural isomorphisms

$$(sX)^K \cong \text{Hom}_{\text{Sym}(K)}(F_1 K, X) \cong s(X^K).$$

There is a map $F_1K \rightarrow F_0S$ which is the identity in degree 1. By adjointness, this map induces a map

$$X = \mathrm{Hom}_{\mathrm{Sym}(K)}(F_0S, X) \rightarrow \mathrm{Hom}_{\mathrm{Sym}(K)}(F_1K, X) = (sX)^K.$$

X is an Ω -spectrum if and only if this map is a level equivalence and X is level fibrant. Therefore, the same method used to prove Theorem 3.8 also proves the following theorem.

Theorem 7.10. *The functors $X \mapsto X \otimes K$ and t are Quillen equivalences on $Sp^\Sigma(\mathcal{C}, K)$. Furthermore, Rs is naturally isomorphic to $L(- \otimes K)$ and $R((-)^K)$ is naturally isomorphic to Lt .*

We have now shown that $Sp^\Sigma(\mathcal{C}, K)$ is a K -stabilization of \mathcal{C} . However, for this construction to be better than ordinary spectra, we must show that $Sp^\Sigma(\mathcal{D}, K)$ is a symmetric monoidal model category.

Theorem 7.11. *Suppose that the domains of the generating cofibrations of both \mathcal{C} and \mathcal{D} are cofibrant. Then the category $Sp^\Sigma(\mathcal{D}, K)$ is a symmetric monoidal model category, and the category $Sp^\Sigma(\mathcal{C}, K)$ is a $Sp^\Sigma(\mathcal{D}, K)$ -model category.*

Proof. We prove this theorem in the same way as Theorem 5.7. Since the cofibrations in the stable model structure are the same as the cofibrations in the projective model structure, the only thing to check is that $f \square g$ is a stable equivalence when f and g are cofibrations and one of them is a stable equivalence. We may as well assume that $f: F_nA \rightarrow F_nB$ is a generating cofibration in $Sp^\Sigma(\mathcal{C}, K)$ and g is a stable trivial cofibration in $Sp^\Sigma(\mathcal{D}, K)$; the argument for f a stable trivial cofibration and g a generating cofibration in $Sp^\Sigma(\mathcal{D}, K)$ is the same. Then, by hypothesis, A and B are cofibrant in \mathcal{C} . We claim that $F_nA \wedge -$ is a Quillen functor $Sp^\Sigma(\mathcal{D}, K)$ to $Sp^\Sigma(\mathcal{C}, K)$ with their stable model structures, and similarly for $F_nB \wedge -$. Indeed, in view of Theorem 2.2, it suffices to show that $F_nA \wedge s_m^{QC}$ is a stable equivalence for all $m \geq 0$ and all domains or codomains C of the generating cofibrations of \mathcal{C} . But one can easily check that $F_nA \wedge s_m^{QC} = s_{n+m}^{A \otimes QC}$. Then Theorem 7.8 implies that this map is a stable equivalence, as required.

Thus, both functors $F_nA \wedge -$ and $F_nB \wedge -$ are Quillen functors in the stable model structures. A two out of three argument, as in Theorem 5.7, then shows that $f \square g$ is a stable equivalence, as required. \square

Note that the functor $F_0: \mathcal{D} \rightarrow Sp^\Sigma(\mathcal{D}, K)$ is a symmetric monoidal Quillen functor, so of course $Sp^\Sigma(\mathcal{C}, K)$ is a \mathcal{D} -model category as well, under the hypotheses of Theorem 7.11. In fact, we only need the domains of the generating cofibrations of \mathcal{D} to be cofibrant to conclude that $Sp^\Sigma(\mathcal{C}, K)$ is a \mathcal{D} -model category, using the argument of Theorem 7.11.

As we mentioned above, we do not know if the stable model structure satisfies the monoid axiom. Given a particular monoid R , one could attempt to localize the projective model structure on R -modules to obtain a stable model structure. However, for this to work one would need to know that the projective model structure is cellular, and the author does not see how to prove this. This plan will certainly fail for the category of monoids, since the projective model structure on monoids will not be left proper in general.

We also point out that it may be possible to prove some of the results of Section 4 for symmetric spectra. All of those results cannot hold, since stable homotopy isomorphisms do not coincide with stable equivalences even for symmetric spectra

of simplicial sets. Nevertheless, in that case, every stable homotopy isomorphism is a stable equivalence [11, Theorem 3.1.11], and there is a replacement for the functor R^∞ constructed in [22]. We do not know if these results hold for symmetric spectra over a general well-behaved finitely generated model category.

8. PROPERTIES OF SYMMETRIC SPECTRA

In this section, we point out that the arguments of Section 5 also apply to symmetric spectra. In particular, if smashing with K is already a Quillen equivalence on \mathcal{C} , then $F_0: \mathcal{C} \rightarrow Sp^\Sigma(\mathcal{C}, K)$ is a Quillen equivalence. This means that, under mild hypotheses, the homotopy category of \mathcal{C} is enriched, tensored, and cotensored over $\text{Ho } Sp^\Sigma(\mathcal{D}, K)$. We also show symmetric spectra are functorial in an appropriate sense. In particular, we show that the Quillen equivalence class of $Sp^\Sigma(\mathcal{C}, K)$ is an invariant of the homotopy type of K .

Throughout this section, \mathcal{D} will denote a left proper cellular symmetric monoidal model category, \mathcal{C} will denote a left proper cellular \mathcal{C} -model category, and K will denote a cofibrant object of \mathcal{D} .

The proof of the following theorem is the same as the proof of Theorem 5.1.

Theorem 8.1. *Suppose smashing with K is a Quillen equivalence on \mathcal{C} . Then $F_0: \mathcal{C} \rightarrow Sp^\Sigma(\mathcal{C}, K)$ is a Quillen equivalence.*

Corollary 8.2. *Suppose that the domains of the generating cofibrations of both \mathcal{C} and \mathcal{D} are cofibrant, and suppose that smashing with K is a Quillen equivalence on \mathcal{C} . Then $\text{Ho } \mathcal{D}$ is enriched, tensored, and cotensored over $\text{Ho } Sp^\Sigma(\mathcal{D}, K)$.*

Proof. Note that $\text{Ho } Sp^\Sigma(\mathcal{C}, K)$ is certainly enriched, tensored, and cotensored over $\text{Ho } Sp^\Sigma(\mathcal{D}, K)$. Now use the equivalence of categories $LF_0: \text{Ho } \mathcal{C} \rightarrow \text{Ho } Sp^\Sigma(\mathcal{C}, K)$ to transport this structure back to \mathcal{C} . \square

Recall that the homotopy category of any model category is naturally enriched, tensored, and cotensored over $\text{Ho } \mathbf{SSet}$ [10, Chapter 5]. This corollary is the first step to the assertion that the homotopy category of any stable (with respect to the suspension) model category is naturally enriched, tensored, and cotensored over the homotopy category of (simplicial) symmetric spectra. See [20] for further results along these lines.

Symmetric spectra are functorial in a natural way.

Theorem 8.3. *Suppose the domains of the generating cofibrations of \mathcal{D} , \mathcal{C} , and the left proper cellular \mathcal{C} -model category \mathcal{C}' are cofibrant. Then any \mathcal{D} -Quillen functor $\Phi: \mathcal{C} \rightarrow \mathcal{C}'$ extends naturally to a $Sp^\Sigma(\mathcal{D}, K)$ -Quillen functor*

$$Sp^\Sigma(\Phi): Sp^\Sigma(\mathcal{C}, K) \rightarrow Sp^\Sigma(\mathcal{C}', K).$$

Furthermore, if Φ is a Quillen equivalence, so is $Sp^\Sigma(\Phi)$.

Proof. The functor Φ induces a \mathcal{D}^Σ -functor $\mathcal{C}^\Sigma \rightarrow (\mathcal{C}')^\Sigma$, which takes the symmetric sequence (X_n) to the symmetric sequence (ΦX_n) . It follows that Φ induces a $Sp^\Sigma(\mathcal{D}, K)$ -functor $Sp^\Sigma(\Phi): Sp^\Sigma(\mathcal{C}, K) \rightarrow Sp^\Sigma(\mathcal{C}', K)$, that takes the symmetric spectrum (X_n) to the symmetric spectrum (ΦX_n) , with structure maps

$$\Phi X_n \otimes K \cong \Phi(X_n \otimes K) \rightarrow \Phi X_{n+1}.$$

Let Γ denote the right adjoint of Φ . Then the right adjoint of $Sp^\Sigma(\Phi)$ is $Sp^\Sigma(\Gamma)$, which takes the symmetric spectrum (Y_n) to the symmetric spectrum (ΓY_n) , with structure maps adjoint to the composite

$$\Phi(\Gamma Y_n \otimes K) \cong \Phi \Gamma Y_n \otimes K \rightarrow Y_n \otimes K \rightarrow Y_{n+1}.$$

Since $Ev_n Sp^\Sigma(\Gamma) = \Gamma Ev_n$, it follows that $Sp^\Sigma(\Phi)F_n = F_n \Phi$.

It is clear that $Sp^\Sigma(\Gamma)$ preserves level fibrations and level equivalences, so $Sp^\Sigma(\Phi)$ is a Quillen functor with respect to the projective model structure. In view of Theorem 2.2, to see that $Sp^\Sigma(\Phi)$ defines a Quillen functor with respect to the stable model structures, it suffices to show that $Sp^\Sigma(\Phi)(s_n^{QC})$ is a stable equivalence for all domains and codomains of C of the generating cofibrations of \mathcal{C} . But one can readily verify that $Sp^\Sigma(\Phi)(s_n^{QC}) = s_n^{\Phi QC}$, which is a stable equivalence as required. Thus $Sp^\Sigma(\Phi)$ is a Quillen functor with respect to the stable model structures.

If Φ is a Quillen equivalence, one can easily check that $Sp^\Sigma(\Phi)$ is a Quillen equivalence with respect to the projective model structure. To see that it is still a Quillen equivalence with respect to the stable model structures, we need only show that $Sp^\Sigma(\Gamma)$ reflects stably fibrant objects, in view of Proposition 2.3. But, if X is level fibrant and $Sp^\Sigma(\Gamma)(X)$ is an Ω -spectrum, this means that the map $\Gamma X_n \rightarrow (\Gamma X_{n+1})^K \cong \Gamma(X_{n+1}^K)$ is a weak equivalence for all n . Since Γ reflects weak equivalences between fibrant objects, this means that X is an Ω -spectrum, as required. \square

Symmetric spectra are also functorial, in a limited sense, in the cofibrant object K .

Theorem 8.4. *Suppose $f: K \rightarrow K'$ is a weak equivalence of cofibrant objects of \mathcal{D} , and suppose the domains of the generating cofibrations of \mathcal{D} and \mathcal{C} are cofibrant. Then f induces a Quillen equivalence $Sp^\Sigma(\mathcal{C}, f): Sp^\Sigma(\mathcal{C}, K) \rightarrow Sp^\Sigma(\mathcal{C}, K')$ which is natural with respect to \mathcal{D} -Quillen functors of \mathcal{C} .*

Proof. The map f induces a map of commutative monoids $\text{Sym}(K) \rightarrow \text{Sym}(K')$. This induces the usual induction and restriction adjunction

$$Sp^\Sigma(\mathcal{C}, f): Sp^\Sigma(\mathcal{C}, K) \rightarrow Sp^\Sigma(\mathcal{C}, K').$$

That is, if X is in $Sp^\Sigma(\mathcal{C}, K)$, then $Sp^\Sigma(\mathcal{C}, f)(X) = X \otimes_{\text{Sym}(K)} \text{Sym}(K')$. Restriction obviously preserves level fibrations and level equivalences, so is a Quillen functor with respect to the projective model structure. One can easily check that $Sp^\Sigma(\mathcal{C}, f) \circ F_n = F_n$. It follows that $Sp^\Sigma(\mathcal{C}, f)(s_n^{QC})$ is the map

$$F_{n+1}(QC \otimes K) \rightarrow F_n QC$$

in $Sp^\Sigma(\mathcal{C}, K')$. The weak equivalence $QC \otimes K \rightarrow QC \otimes K'$ induces a level equivalence $F_{n+1}(QC \otimes K) \rightarrow F_{n+1}(QC \otimes K')$. Since the map $F_{n+1}(QC \otimes K') \rightarrow F_n QC$ is a stable equivalence, so is the given map. Thus induction is a Quillen functor with respect to the stable model structures.

We now prove that induction is a Quillen equivalence between the projective model structures. The proof of this is similar to the proof of Theorem 5.5. That is, restriction certainly reflects level equivalences between level fibrant objects. It therefore suffices to show that the map $X \rightarrow X \otimes_{\text{Sym}(K)} \text{Sym}(K')$ is a level equivalence for all cofibrant X . The argument of Theorem 5.5 will prove this without difficulty.

To show that induction is a Quillen equivalence between the stable model structures, we need only check that restriction reflects stably fibrant objects. This follows from the fact that the map $Z^{K'} \rightarrow Z^K$ is a weak equivalence for all fibrant Z . \square

In particular, it does not matter, up to Quillen equivalence, what model of the simplicial circle one takes in forming the symmetric spectra of [11].

9. COMPARISON OF SPECTRA AND SYMMETRIC SPECTRA

In this section, suppose \mathcal{D} is a left proper cellular symmetric monoidal model category, K is a cofibrant object of \mathcal{D} , and \mathcal{C} is a left proper cellular \mathcal{D} -model category. Let G denote the left Quillen endofunctor $GX = X \otimes K$ of \mathcal{C} . Then we have two different stabilizations of \mathcal{C} , namely $Sp^{\mathbb{N}}(\mathcal{C}, G)$ and $Sp^{\Sigma}(\mathcal{C}, K)$. The object of this section is to compare them. We show that $Sp^{\mathbb{N}}(\mathcal{C}, G)$ and $Sp^{\Sigma}(\mathcal{C}, K)$ are related by a chain of Quillen equivalences whenever the cyclic permutation self-map of $K \otimes K \otimes K$ is homotopic to the identity.

This is not the ideal theorem; one might hope for a direct Quillen equivalence rather than a zigzag of Quillen equivalences, and one might hope for weaker hypotheses, or even no hypotheses. However, some hypotheses are necessary, as pointed out to the author by Jeff Smith. Indeed, the category $\text{Ho } Sp^{\Sigma}(\mathcal{D}, K)$ is symmetric monoidal, and therefore $\text{Ho } Sp^{\Sigma}(\mathcal{D}, K)(F_0S, F_0S)$, the self-maps of the unit, form a commutative monoid. If we have a chain of Quillen equivalences between $Sp^{\mathbb{N}}(\mathcal{D}, K)$ and $Sp^{\Sigma}(\mathcal{D}, K)$ that preserves the functors F_0 , then $\text{Ho } Sp^{\mathbb{N}}(\mathcal{D}, K)(F_0S, F_0S)$ would also have to be a commutative monoid. With sufficiently many hypotheses on \mathcal{D} and K , for example if \mathcal{D} is the category of simplicial sets and K is a finite simplicial set, we have seen in Section 4 that this mapping set is the colimit $\text{colim } \text{Ho } \mathcal{D}(K^{\otimes n}, K^{\otimes n})$. There are certainly examples where this monoid is not commutative; for example K could be the mod p Moore space, and then homology calculations show this colimit is not commutative. In fact, this monoid will be commutative if and only if the cyclic permutation map of $K \otimes K \otimes K$ becomes the identity in $\text{Ho } \mathcal{D}$ after tensoring with sufficiently many copies of K . Hence we need some hypothesis on the cyclic permutation map.

The heart of our argument is the following theorem. The argument of this theorem can be summarized by saying that commuting stabilization functors are equivalent, and as such, was suggested to the author by Mike Hopkins in a different context. Recall that there are two different ways to tensor with K on $Sp^{\mathbb{N}}(\mathcal{C}, G)$; the functor $X \mapsto X \overline{\otimes} K$ that is a Quillen equivalence but does not involve the twist map, and the functor $X \mapsto X \otimes K$ that may not be a Quillen equivalence but does involve the twist map.

Theorem 9.1. *Suppose that the functor $X \mapsto X \otimes K$ is a Quillen equivalence of $Sp^{\mathbb{N}}(\mathcal{C}, G)$, and also that the domains of the generating cofibrations of \mathcal{D} are cofibrant. Then there is a \mathcal{D} -model category \mathcal{E} together with \mathcal{D} -Quillen equivalences $Sp^{\Sigma}(\mathcal{C}, K) \rightarrow \mathcal{E} \leftarrow Sp^{\mathbb{N}}(\mathcal{C}, G)$. Furthermore, we have a natural isomorphism $[\text{Ho } Sp^{\Sigma}(\mathcal{C}, K)](F_0A, F_0B) \cong [\text{Ho } Sp^{\mathbb{N}}(\mathcal{C}, G)](F_0A, F_0B)$ for $A, B \in \mathcal{C}$.*

Proof. We take $\mathcal{E} = Sp^{\mathbb{N}}(Sp^{\Sigma}(\mathcal{C}, K), K)$, where the functor $G: Sp^{\Sigma}(\mathcal{C}, K) \rightarrow Sp^{\Sigma}(\mathcal{C}, K)$ used to form \mathcal{E} is defined by $GX = X \otimes K$ and is part of the \mathcal{D} -model structure of $Sp^{\Sigma}(\mathcal{C}, K)$ (see the comment following Theorem 7.11). This means that the structure map of G involves the twist map. By Theorem 8.1, $F_0: Sp^{\Sigma}(\mathcal{C}, K) \rightarrow \mathcal{E}$ is a \mathcal{D} -Quillen equivalence. On the other hand, consider

$Sp^\Sigma(Sp^\mathbb{N}(\mathcal{C}, G), K)$, where now we use the action of \mathcal{D} on $Sp^\mathbb{N}(\mathcal{C}, G)$ that comes from Theorem 5.7. As pointed out in Remark 5.8, this means that we are using the functor $X \mapsto X \otimes K$ to form $Sp^\Sigma(Sp^\mathbb{N}(\mathcal{C}, G), K)$, not the functor $X \mapsto X \overline{\otimes} K$. By hypothesis, this functor is already a Quillen equivalence, so Theorem 5.1 implies that $F_0: Sp^\mathbb{N}(\mathcal{C}, G) \rightarrow Sp^\Sigma(Sp^\mathbb{N}(\mathcal{C}, G), K)$ is a \mathcal{D} -Quillen equivalence.

It remains to prove that \mathcal{E} and $Sp^\Sigma(Sp^\mathbb{N}(\mathcal{C}, G), K)$ are isomorphic as model categories. This is mostly a matter of unwinding definitions. An object of \mathcal{E} is a set $\{Y_{m,n}\}$ of objects of \mathcal{C} , where $m, n \geq 0$. There is an action of Σ_n on $Y_{m,n}$, and there are Σ_n -equivariant maps $Y_{m,n} \otimes K \xrightarrow{\nu} Y_{m+1,n}$ and $Y_{m,n} \otimes K \xrightarrow{\rho} Y_{m,n+1}$. In addition, the composite $Y_{m,n} \otimes K^{\otimes p} \rightarrow Y_{m,n+p}$ is $\Sigma_n \times \Sigma_p$ -equivariant, and there is a compatibility between ν and ρ , expressed in the commutativity of the following diagram.

$$\begin{array}{ccccc} Y_{m,n} \otimes K \otimes K & \xrightarrow{1 \otimes T} & Y_{m,n} \otimes K \otimes K & \xrightarrow{\rho \otimes 1} & Y_{m,n+1} \otimes K \\ \nu \otimes 1 \downarrow & & & & \downarrow \nu \\ Y_{m+1,n} \otimes K & \xlongequal{\quad} & Y_{m+1,n} \otimes K & \xrightarrow{\rho} & Y_{m+1,n+1} \end{array}$$

An object $\{Z_{m,n}\}$ of the category $Sp^\Sigma(Sp^\mathbb{N}(\mathcal{C}, G), K)$ has the same description if we switch m and n . There is then an isomorphism of categories between \mathcal{E} and $Sp^\Sigma(Sp^\mathbb{N}(\mathcal{C}, G), K)$ which simply switches m and n . The model structures are also the same. Indeed, they are both the localization of the evident bigraded projective model structure with respect to the maps $F_{m,n+1}(QC \otimes K) \rightarrow F_{m,n}QC$ and $F_{m+1,n}(QC \otimes K) \rightarrow F_{m,n}QC$, where $F_{m,n}$ is left adjoint to the evaluation functor $\text{Ev}_{m,n}$.

The natural isomorphism $[\text{Ho } Sp^\Sigma(\mathcal{C}, K)](F_0, F_0B) \cong [\text{Ho } Sp^\mathbb{N}(\mathcal{C}, G)](F_0A, F_0B)$ follows from the fact that the composites

$$\mathcal{C} \xrightarrow{F_0} Sp^\Sigma(\mathcal{C}, K) \xrightarrow{F_0} \mathcal{E}$$

and

$$\mathcal{C} \xrightarrow{F_0} Sp^\mathbb{N}(\mathcal{C}, G) \xrightarrow{F_0} Sp^\Sigma(Sp^\mathbb{N}(\mathcal{C}, G), K) \cong \mathcal{E}$$

are equal. \square

In particular, we have calculated $[\text{Ho } Sp^\Sigma(\mathcal{C}, G)](F_0A, F_0B)$ in Corollary 4.10; when both the hypotheses of that corollary and the hypotheses of Theorem 9.1 hold, we get the expected result

$$[\text{Ho } Sp^\Sigma(\mathcal{C}, K)](F_0A, F_0B) = \text{colim } \text{Ho } \mathcal{C}(A \wedge K^{\wedge n}, B \wedge K^{\wedge n})$$

for cofibrant A and B .

Theorem 9.1 indicates that we should try to prove that $- \otimes K$ is a Quillen equivalence of $Sp^\mathbb{N}(\mathcal{C}, G)$. The only reasonable way to do this is by comparing this functor to $-\overline{\otimes} K$, which we know is a Quillen equivalence. The basic idea of the proof is to compare $X \otimes K \otimes K$ to $X \overline{\otimes} K \overline{\otimes} K$. Both of these spectra have the same spaces, and their structure maps differ precisely by the cyclic permutation self-map of $K \otimes K \otimes K$. So if we knew that this map were the identity, they would be the same spectra. The hope is then that, if we know that the cyclic permutation map is homotopic to the identity, these two spectra are equivalent in $\text{Ho } Sp^\mathbb{N}(\mathcal{C}, G)$. One can in fact construct a map of spectra $X \overline{\otimes} K \overline{\otimes} K \rightarrow R(X \otimes K \otimes K)$, where R is a level fibrant replacement functor and X is cofibrant, by inductively modifying the

identity map. Unfortunately, the author does not know how to do this modification in a natural way, so is unable to prove that the derived functors $L(X \otimes K \otimes K)$ and $L(X \overline{\otimes} K \overline{\otimes} K)$ are equivalent in this way.

Instead, we will follow a suggestion of Dan Dugger and construct a new functor F on cofibrant objects of $Sp^{\mathbb{N}}(\mathcal{C}, G)$ and natural level equivalences $FX \rightarrow X \overline{\otimes} K \overline{\otimes} K$ and $FX \rightarrow X \otimes K \otimes K$, for cofibrant X . It will follow immediately that $L(X \overline{\otimes} K \overline{\otimes} K)$ is naturally equivalent to $L(X \otimes K \otimes K)$, and so that $X \mapsto X \otimes K$ is a Quillen equivalence on $Sp^{\mathbb{N}}(\mathcal{C}, G)$. Unfortunately, to make this construction we will need to make some unpleasant assumptions that ought to be unnecessary.

Definition 9.2. Given a symmetric monoidal model category \mathcal{D} whose unit S is cofibrant, a *unit interval* in \mathcal{D} is a cylinder object I for S such that there exists a map $H_I: I \otimes I \rightarrow I$ satisfying $H_I \circ (1 \otimes i_0) = H_I \circ (i_0 \otimes 1) = i_0 \pi$ and $H_I \circ (1 \otimes i_1)$ is the identity. Here $i_0, i_1: S \rightarrow I$ and $\pi: I \rightarrow S$ are the structure maps of I . Given a cofibrant object K of \mathcal{D} , we say that K is *symmetric* if there is a unit interval I and a homotopy

$$H: K \otimes K \otimes K \otimes I \rightarrow K \otimes K \otimes K$$

from the cyclic permutation to the identity.

Note that $[0, 1]$ is a unit interval in the usual model structure on compactly generated topological spaces, and $\Delta[1]$ is a unit interval in the category of topological spaces. Indeed, the required map $H_1: \Delta[1] \times \Delta[1] \rightarrow \Delta[1]$ takes both of the nondegenerate 2-simplices 011×001 and 001×011 to 001 . Similarly, the standard unit interval chain complex of abelian groups is a unit interval in the projective model structure on chain complexes. Also, any symmetric monoidal left Quillen functor preserves unit intervals. It follows, for example, that the unstable A^1 -model category of Morel-Voevodsky has a unit interval.

Our goal, then, is to prove the following theorem.

Theorem 9.3. *Suppose \mathcal{D} is a symmetric monoidal model category with cofibrant unit S , and \mathcal{C} is a left proper cellular \mathcal{D} -model category. Suppose that K is a cofibrant object of \mathcal{D} , and that either K is itself symmetric or the domains of the generating cofibrations of \mathcal{C} are cofibrant and K is weakly equivalent to a symmetric object of \mathcal{D} . Then the functor $X \mapsto X \otimes K$ is a Quillen equivalence of $Sp^{\mathbb{N}}(\mathcal{C}, G)$.*

This theorem is certainly not the best one can do. For example, by considering the analogous functors with more than three tensor factors of K , it should be possible to show that the same theorem holds if there is a left homotopy between some even permutation of $K^{\otimes n}$ and the identity. Also, it seems clear that one should only need the cyclic permutation, or more generally some even permutation, to be equal to the identity in $\text{Ho } \mathcal{D}$. That is, we should not need a specific left homotopy. But the author does not know how to remove this hypothesis.

In any case, we have the following corollary.

Corollary 9.4. *Suppose \mathcal{D} is a left proper cellular symmetric monoidal model category whose unit S is cofibrant, and whose generating cofibrations can be taken to have cofibrant domains. Suppose \mathcal{C} is a left proper \mathcal{D} -model category. Suppose K is a cofibrant object of \mathcal{D} , and either that K is itself symmetric, or else that the domains of the generating cofibrations of \mathcal{C} are cofibrant and K is weakly equivalent to a symmetric object of \mathcal{D} . Then there is a \mathcal{D} -model category \mathcal{E} and \mathcal{D} -Quillen*

equivalences

$$Sp^\Sigma(\mathcal{C}, K) \rightarrow \mathcal{E} \leftarrow Sp^\mathbb{N}(\mathcal{C}, G).$$

We will prove Theorem 9.3 in a series of lemmas.

Lemma 9.5. *Suppose \mathcal{D} is a symmetric monoidal model category whose unit S is cofibrant. Suppose we have a square*

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ r \downarrow & & \downarrow s \\ B & \xrightarrow{g} & Y \end{array}$$

in a \mathcal{D} -model category \mathcal{C} , where A and B are cofibrant, and a left homotopy $H: A \otimes I \rightarrow Y$ from gr to sf , for some unit interval I . Then there is an object B' of \mathcal{C} , a weak equivalence $B' \xrightarrow{q} B$, a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ r' \downarrow & & \downarrow s \\ B' & \xrightarrow{g'} & Y \end{array}$$

such that $qr' = r$, and a left homotopy $H': B' \otimes I \rightarrow Y$ between gq and g' . Furthermore, this construction is natural in an appropriate sense.

Naturality means that, if we have a map of such homotopy commutative squares that preserves the homotopies, then we get a map of the resulting commutative squares that preserves the maps q and H' . The precise statement is complicated, and we leave it to the reader.

Proof. We let B' be the mapping cylinder of r . That is, we take B' to be the pushout in the diagram below.

$$\begin{array}{ccc} A & \xrightarrow{i_0} & A \otimes I \\ r \downarrow & & \downarrow h \\ B & \xrightarrow{j} & B' \end{array}$$

The map r' is then the composite hi_1 , and the map g' is the map that is g on B and H on $A \otimes I$. It follows that $g'r' = Hi_1 = sf$, as required. The map $q: B' \rightarrow B$ is defined to be the identity on B and the composite $A \otimes I \xrightarrow{\pi} A \xrightarrow{r} B$ on $A \otimes I$. Since j is a trivial cofibration (as a pushout of i_0), it follows that q is a weak equivalence, and it is clear that $qr' = r$. We must now construct the homotopy H' . First note that B' is cofibrant, since B is so and j is a trivial cofibration, and so $B' \otimes I$ is a cylinder object for B' . In fact, $B' \otimes I$ is the pushout of $A \otimes I \otimes I$ and $B \otimes I$ over $A \otimes I$. Define H' to be the constant homotopy $B \otimes I \xrightarrow{\pi} B \xrightarrow{g} Y$ on $B \otimes I$ and the homotopy

$$A \otimes I \otimes I \xrightarrow{1 \otimes H_I} A \otimes I \xrightarrow{H} Y$$

on $A \otimes I \otimes I$. The fact that $H_I \circ (i_0 \otimes 1) = i_0 \pi$ guarantees that H' is well-defined, and the other conditions on H_I guarantee that H' is a left homotopy from qg to g' . We leave the naturality of this construction to the reader. \square

We also need the following lemma about the behavior of unit intervals.

Lemma 9.6. *Suppose \mathcal{D} is a symmetric monoidal model category whose unit S is cofibrant. Let I and I' be unit intervals, and define J by the pushout diagram below.*

$$\begin{array}{ccc} S & \xrightarrow{i_0} & I' \\ i_1 \downarrow & & \downarrow j_1 \\ I & \xrightarrow{j_0} & J \end{array}$$

Then J is a unit interval.

Proof. The reader is well-advised to draw a picture in the topological or simplicial case, from which the proof should be clear. We think of J as the interval whose left half is I and whose right half is I' . In particular, J is a cylinder object for S , where $i'_0 = j_0 i_0$ and $i'_1 = j_1 i_1$. Then, because the tensor product preserves pushouts, we can think of $J \otimes J$ as a square consisting of a copy of $I \otimes I$ in the lower left, a copy of $I \otimes I'$ in the upper left, a copy of $I' \otimes I$ in the lower right, and a copy of $I' \otimes I'$ in the upper right. We define the necessary map $G: J \otimes J \rightarrow J$ by defining G on each subsquare. On the lower left, we use the composite $I \otimes I \xrightarrow{H} I \xrightarrow{j_0} J$, where H is the homotopy making I into a unit interval. Similarly, on the upper right square, we use the composite $j_1 H'$. On the upper left square we use the constant homotopy $j_0(1 \otimes \pi)$, and on the lower right square we use the constant homotopy $j_0(\pi \otimes 1)$. We leave it to the reader to check that this makes J into a unit interval. \square

The importance of these two lemmas for spectra is indicated in the following consequence.

Lemma 9.7. *Suppose \mathcal{D} is a left proper cellular symmetric monoidal model category with a unit interval I , whose unit S is cofibrant. Let K be a cofibrant object of \mathcal{D} , and let \mathcal{C} be a left proper cellular \mathcal{D} -model category. Suppose $A, B \in Sp^{\mathbb{N}}(\mathcal{C}, G)$, where A is cofibrant, and we have maps $f_n: A_n \rightarrow B_n$ for all n and a homotopy $H_n: A_n \otimes K \otimes I \rightarrow B_{n+1}$ from $f_{n+1} \sigma_A$ to $\sigma_B(f_n \otimes 1)$, where $\sigma_{(-)}$ is the structure map of the spectrum $(-)$. Then there is a spectrum C , a level equivalence $C \xrightarrow{h} A$, and a map of spectra $C \xrightarrow{g} B$ such that g_n is homotopic to $f_n h_n$. Furthermore, this construction is natural in an appropriate sense.*

Once again, the naturality involves the homotopies H_n as well as the maps f_n . We leave the precise statement to the reader.

Proof. We define C_n, h_n, g_n and a homotopy $H'_n: C_n \otimes I_n \rightarrow B_n$ from g_n to $f_n h_n$, where I_n is a unit interval, inductively on n , using Lemma 9.5. To get started, we take $C_0 = A_0$, h_0 to be the identity, g_0 to be f_0 , and H'_0 to be the constant homotopy (with $I_0 = I$). For the inductive step, we apply Lemma 9.5 to the

diagram

$$\begin{array}{ccc} C_n \otimes K & \xrightarrow{g_n \otimes 1} & B_n \otimes K \\ \sigma(h_n \otimes 1) \downarrow & & \downarrow \sigma \\ A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} \end{array}$$

and the homotopy obtained as follows. We have a homotopy

$$C_n \otimes K \otimes I_n \xrightarrow{1 \otimes T} C_n \otimes I_n \otimes K \xrightarrow{H'_n \otimes 1} B_n \otimes K \xrightarrow{\sigma} B_{n+1}$$

from $\sigma(f_n \otimes 1)(h_n \otimes 1)$ to $\sigma(g_n \otimes 1)$. On the other hand, we also have the homotopy $H_n(h_n \otimes 1)$ from $f_{n+1}\sigma(h_n \otimes 1)$ to $\sigma(f_n \otimes 1)(h_n \otimes 1)$. We can amalgamate these to get a homotopy $G_n: C_n \otimes K \otimes I_{n+1} \rightarrow B_{n+1}$ from $f_{n+1}\sigma(h_n \otimes 1)$ to $\sigma(g_n \otimes 1)$, and I_{n+1} is still a unit interval by Lemma 9.6. Hence Lemma 9.5 gives us an object C_{n+1} , a map $\sigma: C_n \otimes K \rightarrow C_{n+1}$, and a map $g_{n+1}: C_{n+1} \rightarrow B_{n+1}$ such that $g_{n+1}\sigma = \sigma(g_n \otimes 1)$. It also gives us a map $h_{n+1}: C_{n+1} \rightarrow A_{n+1}$ such that $h_{n+1}\sigma = \sigma(h_n \otimes 1)$ and a homotopy $H'_{n+1}: C_{n+1} \otimes I_{n+1} \rightarrow B_{n+1}$ from $f_{n+1}h_{n+1}$ to g_{n+1} . This completes the induction step and the proof (we leave naturality to the reader). \square

With this lemma in hand we can now give the proof of Theorem 9.3.

Proof of Theorem 9.3. We first reduce to the case where K is itself symmetric. So suppose the generating cofibrations of \mathcal{C} have cofibrant domains, and suppose K' is symmetric and weakly equivalent to K ; this means there are weak equivalences $K \rightarrow RK \rightarrow RK' \leftarrow K'$, where R denotes a fibrant replacement functor. This means the total left derived functors $X \mapsto X \otimes^L K$ and $X \mapsto X \otimes^L K'$ are naturally isomorphic on the homotopy category of any \mathcal{D} -model category. In particular, it suffices to show that $X \mapsto X \otimes K'$ is a Quillen equivalence on $Sp^{\mathbb{N}}(\mathcal{C}, K)$. On the other hand, by Theorem 5.5, there are \mathcal{D} -Quillen equivalences

$$Sp^{\mathbb{N}}(\mathcal{C}, K) \rightarrow Sp^{\mathbb{N}}(\mathcal{C}, RK) \rightarrow Sp^{\mathbb{N}}(\mathcal{C}, RK') \leftarrow Sp^{\mathbb{N}}(\mathcal{C}, K').$$

It therefore suffices to show that $X \mapsto X \otimes K'$ is a Quillen equivalence of $Sp^{\mathbb{N}}(\mathcal{C}, K')$; that is, we can assume that K itself is symmetric.

Let H denote the given homotopy from the cyclic permutation to the identity of $K \otimes K \otimes K$. Let X be a cofibrant spectrum, let $\tilde{\sigma}$ denote the structure map of $X \overline{\otimes} K \overline{\otimes} K$, and let σ denote the structure map of $X \otimes K \otimes K$. These two structure maps differ by the cyclic permutation, and therefore we are in the situation of Lemma 9.7, with $A = X \overline{\otimes} K \overline{\otimes} K$, $B = X \otimes K \otimes K$, f_n equal to the identity map, and $H_n = (\sigma_X \otimes 1 \otimes 1)(X_{n+1} \otimes H)$. It follows that we get a functor F defined on cofibrant objects of $Sp^{\mathbb{N}}(\mathcal{C}, G)$ and natural level equivalences $FX \xrightarrow{h} X \overline{\otimes} K \overline{\otimes} K$ and $FX \xrightarrow{g} X \otimes K \otimes K$, where the latter map is a level equivalence since g_n is homotopic to h_n . Thus the total left derived functors of $(-)\overline{\otimes} K \overline{\otimes} K$ and $(-)\otimes K \otimes K$ are naturally isomorphic. Since we know already that $(-)\overline{\otimes} K \overline{\otimes} K$ is a Quillen equivalence, so is $(-)\otimes K \otimes K$, and hence so is $(-)\otimes K$. \square

APPENDIX A. CELLULAR MODEL CATEGORIES

In this section we define cellular model categories and show that the projective model structures on $Sp^{\mathbb{N}}(\mathcal{C}, G)$ and $Sp^{\Sigma}(\mathcal{C}, K)$ are cellular when \mathcal{C} is so. This is

necessary to be sure that the Bousfield localizations used in the paper do in fact exist. The definitions in this section are taken from [9]. Throughout this section, then, G will be a left Quillen endofunctor of \mathcal{C} ; when we refer to $Sp^\Sigma(\mathcal{C}, K)$, we will be thinking of \mathcal{C} as a \mathcal{D} -model category, where \mathcal{D} is some symmetric monoidal model category, and of K as a cofibrant object of \mathcal{D} .

A cellular model category is a special kind of cofibrantly generated model category. Three additional hypotheses are needed.

Definition A.1. A model category \mathcal{C} is *cellular* if there is a set of cofibrations I and a set of trivial cofibrations J making \mathcal{C} into a cofibrantly generated model category and also satisfying the following conditions.

1. The domains and codomains of I are compact relative to I .
2. The domains of J are small relative to the cofibrations.
3. Cofibrations are effective monomorphisms.

The first hypothesis above requires considerable explanation, which we will provide below. We first point out that the second hypothesis will hold in the projective model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$ or $Sp^\Sigma(\mathcal{C}, K)$ when it holds in \mathcal{C} .

Lemma A.2. *Suppose \mathcal{C} is a cofibrantly generated model category with generating cofibrations I and generating trivial cofibrations J , and G is a left Quillen endofunctor of \mathcal{C} . If the domains of J are small relative to the cofibrations in \mathcal{C} , then the domains of the generating trivial cofibrations J_G of the projective model structure on $Sp^{\mathbb{N}}(\mathcal{C}, G)$ ($Sp^\Sigma(\mathcal{C}, K)$) are small relative to the cofibrations in $Sp^{\mathbb{N}}(\mathcal{C}, G)$ ($Sp^\Sigma(\mathcal{C}, K)$).*

Proof. For $Sp^{\mathbb{N}}(\mathcal{C}, G)$, this follows immediately from the definition of J_G , Proposition 1.10, and Proposition 1.12. The proof for $Sp^\Sigma(\mathcal{C}, K)$ is similar. \square

We now discuss the third hypothesis.

Definition A.3. Suppose \mathcal{C} is a category. A map $f: X \rightarrow Y$ is an *effective monomorphism* if f is the equalizer of the two obvious maps $Y \rightrightarrows Y \amalg_X Y$.

Proposition A.4. *Suppose \mathcal{C} is a cofibrantly generated model category and G is a left Quillen endofunctor of \mathcal{C} . If cofibrations are effective monomorphisms in \mathcal{C} , then level cofibrations, and in particular projective cofibrations, are effective monomorphisms in $Sp^{\mathbb{N}}(\mathcal{C}, G)$ and in $Sp^\Sigma(\mathcal{C}, K)$.*

Proof. This is immediate, since limits in $Sp^{\mathbb{N}}(\mathcal{C}, G)$ and $Sp^\Sigma(\mathcal{C}, K)$ are taken level-wise. \square

We must now define compactness. This will involve some preliminary definitions. These definitions are extremely technical; the reader is advised to keep the example of CW-complexes in mind.

Definition A.5. Suppose I is a set of maps in a cocomplete category. A *relative I -cell complex* is a map which can be written as the transfinite composition of pushouts of coproducts of maps of I . That is, given a relative I -cell complex f , there is an ordinal λ and a λ -sequence $X: \lambda \rightarrow \mathcal{C}$ and a collection $\{(T^\beta, e^\beta, h^\beta)_{\beta < \lambda}\}$ satisfying the following properties.

1. f is isomorphic to the transfinite composition of X .
2. Each T^β is a set.
3. Each e^β is a function $e^\beta: T^\beta \rightarrow I$.

4. Given $\beta < \lambda$ and $i \in T^\beta$, if $e_i^\beta: C_i \rightarrow D_i$ is the image of i under e^β , then h_i^β is a map $h_i^\beta: C_i \rightarrow X_\beta$.
5. Each $X_{\beta+1}$ is the pushout in the diagram

$$\begin{array}{ccc} \coprod_{T^\beta} C_i & \longrightarrow & \coprod_{T^\beta} D_i \\ \Pi h_i^\beta \downarrow & & \downarrow \\ X_\beta & \longrightarrow & X_{\beta+1} \end{array}$$

The ordinal λ together with the λ -sequence X and the collection $\{(T^\beta, e^\beta, h^\beta)_{\beta < \lambda}\}$ is called a *presentation* of f . The set $\coprod_{\beta} T_\beta$ is the *set of cells* of f , and given a cell e , its *presentation ordinal* is the ordinal β such that $e \in T^\beta$. The *presentation ordinal* of f is λ .

We also need to define subcomplexes of relative I -cell complexes.

Definition A.6. Suppose \mathcal{C} is a cocomplete category and I is a set of maps in \mathcal{C} . Given a presentation $\lambda, X: \lambda \rightarrow \mathcal{C}$, and $\{(T^\beta, e^\beta, h^\beta)_{\beta < \lambda}\}$ of a map f as a relative I -cell complex, a *subcomplex* of f (or really of the presentation of f), is a collection $\{(\tilde{T}^\beta, \tilde{e}^\beta, \tilde{h}^\beta)_{\beta < \lambda}\}$ such that the following properties hold.

1. Every \tilde{T}^β is a subset of T^β , and \tilde{e}^β is the restriction of e^β to \tilde{T}^β .
2. There is a λ -sequence $\tilde{X}: \lambda \rightarrow \mathcal{C}$ such that $\tilde{X}_0 = X_0$ and a map of λ -sequences $\tilde{X} \rightarrow X$ such that, for every $\beta < \lambda$ and $i \in \tilde{T}^\beta$, the map $\tilde{h}_i^\beta: C_i \rightarrow \tilde{X}_\beta$ is a factorization of $h_i^\beta: C_i \rightarrow X_\beta$ through the map $\tilde{X}_\beta \rightarrow X_\beta$.
3. Every $X_{\beta+1}$ is the pushout in the diagram

$$\begin{array}{ccc} \coprod_{\tilde{T}^\beta} C_i & \longrightarrow & \coprod_{\tilde{T}^\beta} D_i \\ \Pi \tilde{h}_i^\beta \downarrow & & \downarrow \\ \tilde{X}_\beta & \longrightarrow & \tilde{X}_{\beta+1} \end{array}$$

Given a subcomplex of f , the *size* of that subcomplex is the cardinality of its set of cells $\coprod_{\beta < \lambda} \tilde{T}^\beta$. Usually, I will be a set of cofibrations in a model category where the cofibrations are essential monomorphisms. This condition guarantees that a subcomplex is uniquely determined by its set of cells [9, Proposition 12.5.9]. Of course, every set of cells does not give rise to a subcomplex.

We can now define compactness.

Definition A.7. Suppose \mathcal{C} is a cocomplete category and I is a set of maps in \mathcal{C} .

1. Given a cardinal κ , an object X is κ -*compact relative to I* if, for every relative I -cell complex $f: Y \rightarrow Z$ and for every presentation of f , every map $X \rightarrow Z$ factors through a subcomplex of size at most κ .
2. An object X is *compact relative to I* if X is κ -compact relative to I for some cardinal κ .

The following proposition is adapted from an argument of Phil Hirschhorn's.

Proposition A.8. *Suppose \mathcal{C} is a cellular model category with generating cofibrations I , and G is a left Quillen endofunctor of \mathcal{C} . Let A be a domain or codomain of I . Then $F_n A$ is compact relative to I_G in $Sp^{\mathbb{N}}(\mathcal{C}, G)$ or $Sp^{\Sigma}(\mathcal{C}, K)$.*

Proof. We will prove the proposition only for $Sp^{\mathbb{N}}(\mathcal{C}, G)$, as the $Sp^{\Sigma}(\mathcal{C}, K)$ case is similar. Throughout this proof we will use Proposition A.4, which guarantees that

subcomplexes in \mathcal{C}_G are determined by their cells. Choose an infinite cardinal γ such that the domains and codomains of I are all γ -compact relative to I . When dealing with relative I -cell complexes, we can assume that we have a presentation as a transfinite composition of pushouts of maps of I , rather than as a transfinite composition of pushouts of coproducts of maps of I , using [10, Lemma 2.1.13] or [9, Section 12.2]. A similar comment holds for relative I_G -cell complexes. We will proceed by transfinite induction on β , where the induction hypothesis is that for every presented relative I_G -cell complex $f: X \rightarrow Y$ whose presentation ordinal is $\leq \beta$, and for every map $F_n A \xrightarrow{f} Y$ where n is an integer and A is a domain or codomain of I , f factors through a subcomplex with at most γ I_G -cells. Getting the induction started is easy. For the induction step, suppose the induction hypothesis holds for all ordinals $\alpha < \beta$, and suppose we have a presentation

$$X = X^0 \rightarrow X^1 \rightarrow \dots \rightarrow X^\alpha \rightarrow \dots X^\beta = Y$$

of $f: X \rightarrow Y$ as a transfinite composition of pushouts of maps of I_G . Then the boundary of each I_G -cell of this presentation is represented by a map $F_m C \rightarrow X^\alpha$, for some $\alpha < \beta$, some $m \geq 0$, and some domain C of a map of I . This map factors through a subcomplex with at most γ I_G -cells, by induction. It follows that the I_G -cell itself is contained in a subcomplex of at most γ I_G -cells, since we can just attach the interior of the I_G -cell to the given subcomplex.

Now suppose we have an arbitrary map $F_n A \rightarrow Y$, where A is a domain or codomain of a map of I . Such a map is determined by a map $A \rightarrow Y_n$ in \mathcal{C} . The map $f_n: X_n \rightarrow Y_n$ is the transfinite composition of the cofibrations $X_n^\alpha \rightarrow X_n^{\alpha+1}$. For each α , there is an m and a map h of I such that $X_n^\alpha \rightarrow X_n^{\alpha+1}$ is the pushout of $G^m h$, where we interpret $G^m h$ as the identity map if m is negative. Apply [9, Lemma 12.4.19] to write the β -sequence X_n^α as a retract of a β -sequence

$$X_n = Z^0 \rightarrow Z^1 \rightarrow \dots \rightarrow Z^\alpha \rightarrow \dots Z^\beta = Z$$

where each map $Z^\alpha \rightarrow Z^{\alpha+1}$ is a relative I -cell complex. We denote the retraction by $r: Z \rightarrow Y_n$, noting that the restriction of r to Z^α factors (uniquely) through X_n^α . We can think of the entire map $X_n \rightarrow Z$ as a relative I -cell complex, each cell e of which appears in the relative I -cell complex $Z^{t(e)} \rightarrow Z^{t(e)+1}$ for some unique ordinal $t(e)$, and so has associated to it the I_G -cell $c(e)$ of f used to form $X^{t(e)} \rightarrow X^{t(e)+1}$. The composite $A \rightarrow Y_n \rightarrow Z$ then factors through a subcomplex V with at most γ I -cells. The proof will be completed if we can find a subcomplex W of Y with at most γ I_G -cells such that the restriction of r to V factors through W_n .

Take W to be a subcomplex of Y containing the I_G -cells $c(e)$ as e runs through the cells of V . Then W_n contains $r(e)$ for every cell of V , so W_n contains rV , as required. Furthermore, since each I_G -cell $c(e)$ lies in a subcomplex with no more than γ I_G -cells, and V has no more than γ cells, there is a choice for W which has no more than $\gamma^2 = \gamma$ cells. This completes the induction step and the proof. \square

Altogether then, we have the following theorem.

Theorem A.9. *Suppose \mathcal{C} is a left proper cellular model category, and G is a left Quillen endofunctor on \mathcal{C} . Then the category $Sp^{\mathbb{N}}(\mathcal{C}, G)$ of G -spectra and the category $Sp^{\Sigma}(\mathcal{C}, K)$ of symmetric spectra, with the projective model structures, are left proper cellular model categories.*

REFERENCES

- [1] A. K. Bousfield and E. M. Friedlander, *Homotopy theory of Γ -spaces, spectra, and bisimplicial sets*, Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II (M. G. Barratt and M. E. Mahowald, eds.), Lecture Notes in Math., vol. 658, Springer, Berlin, 1978, pp. 80–130.
- [2] W. G. Dwyer, P. S. Hirschhorn, and D. M. Kan, *Model categories and general abstract homotopy theory*, in preparation.
- [3] W. G. Dwyer and J. Spalinski, *Homotopy theories and model categories*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 73–126.
- [4] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory. With an appendix by M. Cole*, Mathematical Surveys and Monographs, vol. 47, American Mathematical Society, Providence, RI, 1997, xii+249 pp.
- [5] Emmanuel Dror Farjoun, *Cellular spaces, null spaces and homotopy localization*, Springer-Verlag, Berlin, 1996.
- [6] Alex Heller, *Stable homotopy categories*, Bull. Amer. Math. Soc. **74** (1968), 28–63.
- [7] ———, *Completions in abstract homotopy theory*, Trans. Amer. Math. Soc. **147** (1970), 573–602.
- [8] ———, *Stable homotopy theories and stabilization*, J. Pure Appl. Algebra **115** (1997), no. 2, 113–130.
- [9] Philip S. Hirschhorn, *Localization of model categories*, preprint, available at <http://www-math.mit.edu/~psh>, 2/22/2000.
- [10] Mark Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1998.
- [11] Mark Hovey, Brooke Shipley, and Jeff Smith, *Symmetric spectra*, J. Amer. Math. Soc. **13** (2000), no. 1, 149–208.
- [12] J. F. Jardine, *Motivic symmetric spectra*, preprint, 2000.
- [13] Mark W. Johnson, *Enriched sheaf theory as a framework for stable homotopy theory*, Ph.D. thesis, University of Washington, 1999.
- [14] L. G. Lewis, Jr., J. P. May, and M. Steinberger, *Equivariant stable homotopy theory*, Lecture Notes in Mathematics, vol. 1213, Springer-Verlag, 1986.
- [15] M. Mandell, J. P. May, B. Shipley, and S. Schwede, *Model categories of diagram spectra*, to appear in Proc. London Math. Soc., 1999.
- [16] F. Morel and V. Voevodsky, *\mathbf{A}^1 -homotopy theory of schemes*, preprint, 1998.
- [17] Daniel G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, vol. 43, Springer-Verlag, 1967.
- [18] Stefan Schwede, *Spectra in model categories and applications to the algebraic cotangent complex*, J. Pure Appl. Algebra **120** (1997), no. 1, 77–104.
- [19] Stefan Schwede, *The stable homotopy category has a unique model at the prime 2*, preprint, 2000.
- [20] Stefan Schwede and Brooke Shipley, *A uniqueness theorem for stable homotopy theory*, preprint, 16pp., 2000.
- [21] Stefan Schwede and Brooke E. Shipley, *Algebras and modules in monoidal model categories*, Proc. London Math. Soc. (3) **80** (2000), no. 2, 491–511.
- [22] Brooke Shipley, *Symmetric spectra and topological Hochschild homology*, K-Theory **19** (2000), 155–183.
- [23] Vladimir Voevodsky, *\mathbf{A}^1 -homotopy theory*, Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998), vol. 1998, pp. 579–604 (electronic).
- [24] R. Vogt, *Boardman’s stable homotopy category*, Aarhus Univ. Lect. Notes, vol. 21, Aarhus University, 1970.

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