Equivariant stable homotopy theory

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We will study equivariant homotopy theory for G a finite group (although this often easily generalizes to compact Lie groups). The general idea is that if we have two G-spaces X and Y, we'd like to study homotopy classes of equivariant maps between them:

$$[X,Y]^G = \mathrm{Map}^G(X,Y)/\mathrm{htpy},$$

where $\mathrm{Map}^G(X,Y) = \{f : X \to Y | f(gx) = gf(x) \text{ for all } g \in G\}$. In classical homotopy theory (i.e. when G is the trivial), we study CW complexes. The first question we should ask ourselves, then, is: What should a "G CW complex" be?

Classically, a CW complex is a space X equipped with *characteristic maps* $f_\alpha : D^n_\alpha \to X$ for $\alpha \in S_n$ (an indexing set) for $n = 0, 1, 2, \ldots$, such that:

- $\coprod \mathrm{int}\, D^n_\alpha \to X$ is a bijection;
- $f(D^n_\alpha)$ is contained in a finite union of $f(D^m_\beta)$ for $m < n$;
- X has the *weak topology*, i.e. whenever $X \to Y$ is continuous, then $D^n_\alpha \to X \to Y$ is continuous.

To generalize this appropriately, there are a few choices to be made. For starters, we need to decide where G acts. Certainly it should act on X. Then, it might also act on the D^n_α, and moreover it might even act on the S_n. (The main innovation for the Kervaire invariant problem exploits the interplay between the various choices.) The standard definition will be that G acts only on S_n and not on the D^n_α.

Example 1. Let $G = \mathbb{Z}/2$ and $X = S^1 \subset \mathbb{C}$, and G acts on X by conjugation. Then the 0-cells are given by $\{\pm 1\} \times D^0$, and the 1-cells are given by $G \times D^1$ (with the appropriate attaching maps).

Example 2. Suppose V is a finite-dimensional (real) representation of G. If G is finite, then V necessarily has an invariant metric, and we can define $S(V)$, the unit sphere in V, and S^V, the one-point compactification of V. In particular, when G is trivial and $V = \mathbb{R}^n$, then $S(V) = S^{n-1}$ and $S^V = S^n$.

In particular, we can take $G = \mathbb{Z}/2$ and V to be the sign representation (so $V = \mathbb{R}$ and G acts by negation). Then $S(V) = \{\pm 1\}$ and S^V is our circle with the conjugation action from before.

We have the following small fact.

Proposition 1. *In the given situation, $S(V)$ and S^V both admit the structure of a G CW complex.*

A theme we'd like to thread through these talks is the difference between an explicitly given G CW complex and a space that comes with a G action that we simply assert has such a structure. This will be a central idea in the third lecture, and we're all very excited about it.

Here are a few classical theorems in homotopy theory.

Theorem 1 (obstruction theory)**.** *Suppose X is a CW complex and Y is a space. Suppose that whenever X has an n-cell, $\pi_n(Y) = 0$. Then any map $X \to Y$ is homotopic to a constant map.*

Theorem 2 (Freudenthal suspension)**.** *If X is a CW complex of dimension m and Y is $(n-1)$-connected and $m < 2n$ (give or take), then $[X,Y] \xrightarrow{\sim} [S^1 \wedge X, S^1 \wedge Y]$.*

We need equivariant analogues of these notions. Specifically, we need equivariant notions of homotopy groups, dimension, and connectivity. The easiest way to figure out the right definitions is simply to follow through the proofs of these theorems and see what we need to say.

In proving the fundamental theorem of obstruction theory, one comes across things like $[X^{(n)}/X^{(n-1)}, Y]$, and observes that $X^{(n)}/X^{(n-1)} = \bigvee_{\alpha \in S_n} D^n_\alpha / \partial D^n_\alpha = \bigvee_{\alpha \in S_n} S^n_\alpha$. Thus, $[X^{(n)}/X^{(n-1)}, Y] = \prod_{\alpha \in S_n} \pi_n Y$.

When we equivariantify this, we see that we want to regard $\pi_n Y$ as a functor $G\text{-}\mathtt{sets} \to \mathtt{AbGrps}$ (for $n > 1$, or to $\mathtt{Grps}$ for $n = 1$, or $\mathtt{Sets}$ for $n = 0$). Namely, given $T \in G\text{-}\mathtt{sets}$, we have $T \mapsto [\bigvee_{t \in T} S^n, Y]^G = [T_+ \wedge S^n, Y]^G$ (where T_+ denotes T with a disjoint basepoint). We can write this as $\underline{\pi}_n Y(T)$; alternatively, we have the notation $\pi_n^G(Y) = \underline{\pi}_n Y(*) = [S^n, Y]^G$.

Of course, this functor is quite overspecified. Namely, $\underline{\pi}_n Y$ is determined by its value on transitive G-sets (i.e. those of the form $T = G/H$ for $H \leq G$). This is just because

$$\underline{\pi}_n Y(T_1 \coprod T_2) = \underline{\pi}_n Y(T_1) \times \underline{\pi}_n Y(T_2).$$

Moreover, it is not hard to see that

$$\underline{\pi}_n Y(G/H) = [(G/H)_+ \wedge S^n, Y]^G = [S^n, Y]^H = [S^n, Y^H] = \pi_n Y^H$$

(where in the second set H acts trivially, and $Y^H = \{y \in Y | hy = y \text{ for all } h \in H\}$ are the H fixed points). So, one can think of equivariant homotopy groups more pragmatically as remembering the fixed point sets. Of course, G/H is more than just a transitive G-set; it has a chosen basepoint H/H. This is how we obtained the above equality in the first place, but on the other hand one might prefer to be agnostic about one's G-sets. In fact, this small point can make many formulas much more complicated than they need to be.

Let us turn to the Freudenthal suspension theorem. Suppose X is a G CW complex, say $X = \coprod S_n \times D^n/\sim$, where $G \curvearrowright S_n$. Then $X^H = \coprod S_n^H \times D^n/\sim$, and in particular X^H is a CW complex. So, whatever sort of a gadget $\dim^G X$ is, it needs to remember $\dim X^H$ for all $H \leq G$. Thus, $\dim^G X$ is a function $\{H \leq G\} \to \mathbb{N}$ (which is of course conjugation-invariant). Similarly, for connectivity we define the *connectivity* of Y to be the function $C^G(Y) : \{H \leq G\} \to \mathbb{N}$ picking up the connectivity of Y^H. With these definitions in hand, many of the classical facts go through.

But actually, the equivariant analogue of Freudenthal is quite complicated – so complicated, in fact, that we're not even going to state it, at least not carefully.[1] Roughly, it says that if $\dim^G X < \kappa \cdot C^G$, then for all V, $[X, Y]^G \to S^V \wedge X, S^V \wedge Y]^G$ is a bijection. Here, κ is an equivariant generalization of the number 2.

The proofs of all these fundamental theorems run fairly elementarily, the way a classical homotopy theorist might expect. The proofs all tend to be reductions to classical homotopy theory rather than truly equivariant proofs.

The equivariant Freudental suspension theorem is the motivation for the category in which we do equivariant stable homotopy theory, the *equivariant Spanier-Whitehead category*, denoted $\mathscr{SW}^G$. The objects are the finite G CW complexes, and the morphisms are denoted and defined as

$$\{X, Y\}^G = \operatorname{colim}_V [S^V \wedge X, S^V \wedge Y]^G,$$

where V runs over all representations of G. Now, note that this colimit is also attained at a "finite stage", just as in the classical case. A nice way to see this that every representation occurs inside of the *regular representation* ρ_G, a $|G|$-dimensional representation given by

$$\bigoplus_G \mathbb{R} \cong \{f : G \to \mathbb{R}\}.$$

Then,

$$\{X, Y\}^G = \operatorname{colim}_{n \to \infty} [S^{n\rho_G} \wedge X, S^{n\rho_G} Y]^G,$$

and this is attained at a finite stage.

One can ask for an analogue of the *degree* of a map, and in particular one can ask for $\{S^0, S^0\}^G$. For this, we introduce the *Burnside category*, denoted $\mathtt{Burn}^G$, whose objects are finite G-sets and whose morphisms are *correspondences*. An example of a correspondence is a roof diagram $S \leftarrow S' \to T$ in $G\text{-}\mathtt{sets}$; these admit an obvious notion of equivalence. The set of equivalence classes is a commutative monoid under coproduct, and we define $\mathtt{Burn}^G(S, T)$ to be its group completion. This leads us to the following beautiful theorem.

[1] reference: Adams, *Prerequisites for Carlsson's work*

Theorem 3 (Segal, tom Dieck)**.** *The endofunctor $S \mapsto S_+$ on G-*`sets` *defines a functor* $\texttt{Burn}^G \to \mathscr{SW}^G$*, and this functor is fully faithful.*

This implies that $\texttt{Burn}^G(*,*) \xrightarrow{\sim} \{S^0, S^0\}^G$, where the source is the *Burnside ring* of G, i.e. the Grothendieck group of finite G-sets. This is kind of the answer one might expect. Namely, recall that classically, the degree can be defined by looking at the preimage of a regular value. This technique can indeed be proved in this method, although it's tougher because the idea of "transversality" is touchier. However, this theorem should also be surprising, because it exactly plays off these two points of view. The target is totally built up out of things like S^V, whereas the source is totally built up out of the category of finite G-sets.

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Today we'll explain the proof of the above theorem. The reason is that this will allow us to explore many new aspects of equivariant stable homotopy theory, and in particular it will bring up perhaps the one truly surprising thing in the field.

2.1 Duality

Suppose $\mathcal{C}$ is a symmetric monoidal category (for example, $\mathcal{C} = \texttt{Vect}$ with the usual tensor product, or $\mathcal{C} = \mathscr{SW}^G$ with smash product, or $\mathcal{C} = \texttt{Burn}^G$ with cartesian product). We say that an object $X \in \mathcal{C}$ is *dualizable* if there is some $Y \in \mathcal{C}$ and maps $X \otimes Y \to 1$ and $1 \to Y \otimes X$ such that in the diagram

$$\begin{array}{ccc} X \otimes 1 & \longrightarrow & X \otimes Y \otimes X \\ & \searrow^{\sim} & \downarrow \\ & & 1 \otimes X, \end{array}$$

the diagonal map is the composite functorial isomorphism $X \otimes 1 \cong X \cong 1 \otimes X$, and similarly for the diagram

$$\begin{array}{ccc} 1 \otimes Y & \longrightarrow & Y \otimes X \otimes Y \\ & \searrow^{\sim} & \downarrow \\ & & Y \otimes 1. \end{array}$$

Now, if X is dualizable, then $\mathcal{C}(X \otimes W, Z) \cong \mathcal{C}(W, Y \otimes Z)$ and $\mathcal{C}(Y \otimes W, Z) \cong \mathcal{C}(W, X \otimes Z)$. So for instance, if $\mathcal{C} = \texttt{Vect}$ then dualizable is equivalent to finite-dimensional.

Remark 1. We remark that this formulation is slightly misleading. One might have instead said that the functor $W \mapsto \mathcal{C}(X \otimes W, 1)$ is representable (by Y). The point is that being dualizable is a *condition*, not a collection of extra data.

Proposition 2. *A symmetric monoidal functor preserves dualizability.*

Now, let's return to our supposed functor $\texttt{Burn}^G \to \mathscr{SW}^G$. We know we've got $S \mapsto S_+$, but what do we do with a morphism $S \xleftarrow{f} U \xrightarrow{g} T$ in $\texttt{Burn}^G$? We need to send this to some $S^V \wedge S_+ \to S^V \wedge T_+$ for some $V >> 0$. This comes from a Pontryagin-Thom style argument. Namely, we choose an equivariant embedding $i : U \hookrightarrow V$ (e.g. $V = \bigoplus_{u \in U} \mathbb{R}$). This gives an embedding $(i, f) : U \hookrightarrow V \times S$, and now we apply the Pontryagin-Thom collapse: we take a small disk around each point of U in $V \times S$ and collapse everything else, and this gives us $S^V \wedge S_+ \to S^V \wedge U_+$. We then compose this with $1 \wedge g : S^V \wedge U_+ \to S^V \wedge T_+$.

Exercise 1. Check that this is compatible with composition.

Observe that in $\texttt{Burn}^G$, every object is dualizable; in fact, every object is self-dual. This is defined by the morphisms $T \times T \xleftarrow{\Delta} T \to *$ and $* \leftarrow T \xrightarrow{\Delta} T \times T$. This implies that the object $T_+ \in \mathscr{SW}^G$ is self-dual as well.

Remark 2. The Spanier-Whitehead category, as we've constructed it, has two properties: it satisfies *stability* with respect to representation spheres (i.e. $\{X, Y\} \xrightarrow{\sim} \{S^V \wedge X, S^V \wedge Y\}$), and finite G-sets are self-dual.

Now, suppose we wanted to do "equivariant homotopy theory" in an arbitrary category. It seems that the stability property is specially suited to topology, whereas the self-duality property is easy to generalize: $\{S_+, X\}^G = \{S^0, S_+ \wedge X\}$. This is essentially saying that $\bigvee_{s\in S} X \to \prod_{s\in S} X$ is a weak equivalence. To do this, we need to be working in an *additive category*, in which the map from a finite coproduct to a finite product is an isomorphism. So, we should think of this self-duality as an equivariant version of *additivity* in which G acts on the indexing set.

In fact, the stability and the self-duality are roughly equivalent. We give the following "theorem":

$$\text{stability w/r/t } S^n + \text{self-duality of finite } G\text{-sets} \;\Leftrightarrow\; \text{stability w/r/t } S^V.$$

If we had chosen to stabilize only with respect to ordinary spheres, we would have arrived at a different, less correct notion.

Exercise 2. In $\mathscr{SW}^G$, we have duality maps $T_+ \wedge T_+ \to S^0$ and $S^0 \to T_+ \wedge T_+$. One of these exists as a map of G CW complexes, while the other only exists stably. Which is which, and what is the map? Relate this to the previous remark.

Sketch of proof of the Segal-tom Dieck theorem. The first obvious thing we could do is use duality to rewrite both sides of the map, as $\texttt{Burn}^G(S \times T, *) \to \{S_+ \wedge T_+, S^0\}^G$. Thus, we may assume that $T = *$. Next, we can decompose S into orbits; both sides take disjoint unions (in the S-variable) to direct sums, so we can reduce to the case where S is a transitive G-set, i.e. $\texttt{Burn}^G(G/H, *) \to \{G/H_+, S^0\}^G$.

Now, the right side is
$$\{G/H_+, S^0\}^G = \operatorname{colim}[S^V \wedge G/H_+, S^V]^G = [S^V, S^V]^H,$$
and it is easy to check that the same is true on the other side: $\texttt{Burn}^G(G/H, Y) \cong \texttt{Burn}^H(*, Y)$. (More generally, this participates in a pair of adjoint functors relating $\texttt{Burn}^G$ and $\texttt{Burn}^H$.)

This reduces us to the case where $S = T = *$. That is, we only must prove that $\texttt{Burn}^G(*, *) \to \{S^0, S^0\}^G$ is an isomorphism. This we will prove by induction on G. When G is trivial, this is just the fact that $\pi_n S^n = \mathbb{Z}$ for $n \geq 1$. Next, it's easy to check that this map is a monomorphism. Indeed, the source is the free abelian group on the set of transitive G-sets, or equivalently a direct sum of copies of $\mathbb{Z}$ indexed by the conjugacy classes of subgroups $H \leq G$. Then, given $H \leq G$, we can restrict to H-fixed points to get $[S^V, S^V]^G \to [S^{V^H}, S^{V^H}] = \mathbb{Z}$ (by induction). This shows that the lower map in

$$\begin{array}{ccc}
\texttt{Burn}^G(*,*) & \longrightarrow & \{S^0, S^0\}^G \\
\Big\| & & \Big\downarrow \\
\bigoplus\limits_{H \text{ up to conjugacy}} \mathbb{Z} & \longrightarrow & \prod\limits_{H \text{ up to conjugacy}} \mathbb{Z}
\end{array}$$

is a monomorphism.

Lastly, we show that our map is an epimorphism. But since we're almost out of time, we'll only do it for $G = \mathbb{Z}/2$. (This is the surprising fact/tool that we alluded to earlier.) To do this, we examine an interesting cofiber sequence. Write σ for the sign representation of $\mathbb{Z}/2$ on $\mathbb{R}$, so that S^σ is the circle with the complex conjugation involution. Thus we have the cofiber sequence $\mathbb{Z}/2_+ \to S^0 \to S^\sigma$. This gives us the exact sequence

$$\begin{array}{ccccc}
\{S^0, \mathbb{Z}/2_+\}^G & \longrightarrow & \{S^0, S^0\}^G & \longrightarrow & \{S^0, S^\sigma\}^G \\
 & & \Big\uparrow & & \\
 & & \texttt{Burn}^G(*,*) & &
\end{array}$$

and we want to show that the vertical map is an epimorphism. By induction, we know that $\{S^0, \mathbb{Z}/2_+\}^G \cong$

$\{\mathbb{Z}/2_+, S^0\}^G \cong \{S^0, S^0\}$. By this isomorphism, we get

$$\begin{array}{ccccc} \{S^0, \mathbb{Z}/2_+\}^G & \longrightarrow & \{S^0, S^0\}^G & \longrightarrow & \{S^0, S^\sigma\}^G \\ \uparrow \wr & & \uparrow & & \\ \mathtt{Burn}^G(*, \mathbb{Z}/2) & \longrightarrow & \mathtt{Burn}^G(*, *), & & \end{array}$$

which a diagram chase allows us to reduce the problem to showing that the diagonal map of

$$\begin{array}{ccc} \{S^0, S^0\}^G & \longrightarrow & \{S^0, S^\sigma\}^G \\ \uparrow & \nearrow & \\ \mathtt{Burn}^G(*, *) & & \end{array}$$

is an epimorphism. Now, its target is $\{S^0, S^\sigma\}^G = \operatorname{colim}[S^V, S^V \wedge S^\sigma]^G$. Let us write $V_0 \subset V$ for the invariants. Restricting then gives us

$$[S^V, S^V \wedge S^\sigma]^G \to [S^{V_0}, S^V \wedge S^\sigma]^G = [S^{V_0}, S^{V_0} \wedge S^0] = [S^{V_0}, S^{V_0}] = \mathbb{Z},$$

and WE WILL SEE THAT THIS MAP IS AN ISOMORPHISM...... except that we won't, because it would take too much time away from the third and final talk. □

3

There is a problem with the Spanier-Whitehead category $\mathscr{SW}^G$ – it doesn't have enough objects! For instance, mapping cones don't exist; e.g. a morphism $X \to Y$ might only actually come from a function $S^V \wedge X \to S^V \wedge Y$. However, we can add these in. We do this by forming the category of *G-spectra*, denoted $\mathscr{S}^G$, where instead of adding in negative-dimensional spheres, we add in virtual-representation spheres. We will denote homotopy classes of maps again by square brackets; then $[X, S^{-V} \wedge Y]^G = \{S^V \wedge X, Y\}^G$. Every object of $\mathscr{S}^G$ can be (functorially) written as a filtered colimit of the form $\operatorname{colim}_V S^{-V} \wedge X_V$.

In the equivariant homotopy theory of spaces, we decided to think of π_n as a functor from finite G-sets to abelian groups; this is *additive* in the sense that it takes disjoint unions to direct sums (i.e. it preserves coproducts). In $\mathscr{S}^G$, we now have, for a finite G-set T, the group $\pi_n X = [S^n, X]^G$ and more generally $\pi_n X(T) = [S^n \wedge T_+, X]^G$. By the above theorem that we didn't quite finish proving, $\mathtt{Burn}^G(S,T) \xrightarrow{\sim} \{S_+, T_+\}^G = [S_+, T_+]^G$, then $\pi_n X$ is a contravariant additive functor $\mathtt{Burn}^G \to \mathtt{AbGrps}$. Such a functor is called a *Mackey functor*.

A natural question is: Which Mackey functors occur as homotopy groups? For instance, $\pi_0 S^0(T) = [T_+, S^0]^G = \mathtt{Burn}^G(T, *)$. More generally, if S is another finite G-set, then $(\pi_0 S_+)(T) = \mathtt{Burn}^G(T, S)$. That is: every representable Mackey functor occurs as homotopy groups. Instead, let's look at $\pi_{-i} S_+$ for $i > 0$. Then

$$\pi_{-i} S_+ = \lim [S^V, S^i \wedge S^V \wedge S_+]^G.$$

If you imagine you've decomposed S^V via an equivariant cell decomposition, then you can compute this on the skeleta, and these computations look like $[G \times_H D^m/S^{m-1}, S^i \wedge S^V \wedge S_+]^G$ (for some $m \le \dim V^H$). But this is just

$$[D^m/S^{m-1}, S^i \wedge S^V \wedge S_+]^H = [D^m/S^{m-1}, (S^i \wedge S^V \wedge S_+)^H] = [D^m/S^{m-1}, S^i \wedge S^{V^H} \wedge S_+] = 0$$

by the connectivity of spheres. This means that any Mackey functor occurs as a homotopy group. In fact, for M a Mackey functor we can construct an Eilenberg-MacLane spectrum HM, which is characterized by the property that $\pi_* HM = M$ concentrated in degree 0.

Remark 3. Suppose we have an abelian group A, and we want to build a space X with $\pi_n X = A$. To do this, one usually writes down a free resolution $F_1 \to F_2 \to A \to 0$ and then we get $\bigvee_{F_1} S^n \to \bigvee_{F_2} S^n \to X$. But this depends on knowing that $[S^n, S^n]$ is a free abelian group and that $[S^i, S^n] = 0$ for $i < n$. In the above situation, exactly the same computations arise: in $\mathscr{S}^G$, we have that $[S^n, S^n \wedge T_+]^G$ is a representable (free) Mackey functor and $[S^i, S^n \wedge T_+]^G = 0$ for $i < n$. This feels a bit more significant in equivariant homotopy theory, but it's not until one reaches motivic homotopy theory that this becomes a truly deep result (due to Morel).

Now, given our Eilenberg-MacLane spectra, we can talk about homology and cohomology. Suppose X is a pointed G CW complex. We will also write X for its suspension spectrum. Now if M is a Mackey functor, we can define $H^n_G(X;M) = [X, S^n \wedge HM]^G$ and $H_n^G(X;M) = [S^n, HM \wedge X]^G$. Just as in ordinary homotopy theory, these have a rather simple description. Namely, $H^n_G(X;M)$ can be computed as the cohomology groups of the cellular cochain complex $C^*_{cell}(X;M)$. Explicitly, $C^n(X;M) = M(T_n)$, where T_n is the G-set of n-cells of X. (A priori, this doesn't make sense if T_n is infinite, but one can always take an inverse limit over finite G-subsets.)

Exercise 3. Let ρ_G be the regular representation of G; this contains a trivial representation, and we let $\overline{\rho_G}$ its orthogonal complement. Let $X = S(\rho_G)_+$ be its unit sphere (with a disjoint basepoint). This actually comes with a G CW decomposition. Namely, $S(\overline{\rho_G}) = \partial\Delta[G]$, where $\Delta[G]$ is the standard simplex with vectors coming from G. Think through this.

Another thing we do with Eilenberg-MacLane spaces is construct Postnikov towers. Namely, there is a map from X to the tower $\cdots \to \widetilde{P}^n X \to \widetilde{P}^{n-1}X \to \cdots$, such that the maps $\pi_i X \to \pi_i \widetilde{P}^n X$ are isomorphisms from $i \leq n$ and sucht hat $\pi_i \widetilde{P}^n X = 0$ for $i > n$. The fibers of the tower sit as $\widetilde{P}^n_n X \to \widetilde{P}^n X \to \widetilde{P}^{n-1} X$, and $\widetilde{P}^n_n X = S^n \wedge HM$ for $M = \pi_n X$.

Now, in classical homotopy theory, an ideal situation occurs when $\widetilde{P}^n_n X = S^n \wedge HF$, where F is a free abelian group. (For example, this happens for MU and KU.) We often try to understand other cohomology theories in terms of these. You might think that in equivariant homotopy theory, the ideal situation would be when $\widetilde{P}^n_n X = S^n HM$ for M a free Mackey functor (i.e. $M(S) = \bigoplus_j \mathtt{Burn}^G(S, T_j)$, so that $M = \pi_0 T_+$ for some G-set T). However, this isn't really what comes out for equivariant versions of MU and KU. Luckily, this also isn't such a natural thing to want after all, anyways.

Let's consider the example of Atiyah's Real K-theory $K\mathbb{R}$. Recall that this is a $\mathbb{Z}/2$-equivariant cohomology theory which occurs from trying to look at the complex points of real vector bundles. Namely, $K\mathbb{R}^0(X)$ is the Grothendieck group of Real vector bundles over X, where a Real vector bundle is a complex vector bundle equipped with a $\mathbb{Z}/2$-action covering that on X which is *conjugate linear*, i.e. $\tau(\lambda v) = \overline{\lambda} \cdot \tau(v)$.

This enjoys the following nice properties.

1. If $X = Y \times \mathbb{Z}/2$, then $K\mathbb{R}(X) = KU(Y)$.
2. If X has trivial $\mathbb{Z}/2$-action, then $K\mathbb{R}(X) = KO(X)$.
3. There is the periodicity formula $K\mathbb{R}(X) \cong K\mathbb{R}(S^{\mathbb{C}} \wedge X)$, where $S^{\mathbb{C}}$ is the 1-point compactification of $\mathbb{C}$ with the conjugation action.

Now, when Atiyah proved the periodicity theorem, he actually just observed that the classical Bott periodicity theorems still held when stated as facts in algebraic geometry.

Note that $\mathbb{C}$ is the sum of a trivial representation and a sign representation, but by changing the basis we can also view it as the permutation representation; that is, $\mathbb{C} = \rho_{\mathbb{Z}/2}$. Thus, we can rewrite the periodicity theorem as $K\mathbb{R}(X) \cong K\mathbb{R}(S^{\rho_{\mathbb{Z}/2}} \wedge X)$. Beautifully, Atiyah showed that this actually implies both of Bott's original periodicity theorems. Luckily for us, it also admits an obvious generalization.

Now, recall that $K = KU$ has $K^0(S^0) = \mathbb{Z}$ and $K^i(X) = K^i(S^2 \wedge X)$. In other words, $\pi_0 K = \mathbb{Z}$ and $S^2 \wedge K \simeq K$. This tells us that $\widetilde{P}^{2n}_{2n} K = S^{2n} \wedge H\mathbb{Z}$ and $\widetilde{P}^{2n+1}_{2n+1} K = 0$. In other words, K has a filtration whose associated graded is $\bigvee_{n \in \mathbb{Z}} S^{2n} \wedge H\mathbb{Z}$. If we want to do the same thing for Real K-theory we run into $\pi_0 K\mathbb{R}$, which turns out to be the constand Mackey functor $\underline{\mathbb{Z}}$, i.e. $\underline{\mathbb{Z}}(S) = \mathrm{Map}^G(S, \mathbb{Z}) = \mathrm{Map}(S/G, \mathbb{Z})$. Given a correspondence $S \leftarrow U \to T$, we get $\underline{\mathbb{Z}}(S) \leftarrow \underline{\mathbb{Z}}(U) \leftarrow \underline{\mathbb{Z}}(T)$: the first map is given by adding up along the fibers, and the second map is just composition with $U \to T$. That is, if we have $f : U \to \mathbb{Z}$ then we get $f' : S \to \mathbb{Z}$ by $f'(s) = \sum_{x \in p^{-1}(s)} f(x)$ for $p : U \to S$ the projection. So, we write this concisely as $\widetilde{P}^0_0 K\mathbb{R} = H\underline{\mathbb{Z}}$. But this tells us nothing about the other levels; Atiyah's theorem tells us that $S^{\rho_{\mathbb{Z}/2}} \wedge K\mathbb{R} \simeq K\mathbb{R}$, but this is a shift by a representation sphere rather than an ordinary sphere. This suggests looking for a filtration whose associated graded is $\bigvee_{n \in \mathbb{Z}} S^{n\rho_{\mathbb{Z}/2}} \wedge H\underline{\mathbb{Z}}$.

In fact, this tower exists, and is called the *slice tower*. We write this as $\{P^n X\}$. We won't quite say what property characterizes it, but we can at least say that the fibers $P^n_n K\mathbb{R}$ are contractible for n odd and $S^{m\rho_{\mathbb{Z}/2}} \wedge H\underline{\mathbb{Z}}$ for $n = 2m$.

The case $G = \mathbb{Z}/2$ was done by Dan Dugger in his thesis. The general case was done by HHR in the course of their work on the Kervaire invariant. The ideal case of the Postnikov tower, recall, was $\widetilde{P}^n_n X = S^n \wedge HF$ for F free abelian. Relatedly, the ideal case of the equivariant Postnikov tower is $\widetilde{P}^n_n X = S^n \wedge HF$ for $F = \pi_0 T_+$ a free

Mackey functor. But the ideal case of the slice tower is $P_n^n X = \bigvee G_+ \wedge_H S^{m\rho_H} \wedge H\underline{\mathbb{Z}}$, a bouquet (indexed by G/H) of certain special representation spheres smashed with $H\underline{\mathbb{Z}}$. (Here, it must be that $n = m \cdot |H|$.) In this case, we say X is *pure*.

Theorem 4 (HHR)**.** *Many geometrically occurring equivariant cohomology theories are pure. (For instance, $K\mathbb{R}$ and the analog of MU are both pure.)*

Theorem 5 (HHR)**.** *If X is pure (and $G \neq \mathbb{Z}/3$) then $\pi_i X(*) = 0$ when $i = -3$ or $i = -1$ and $\pi_i X(*)$ is torsion-free when $i = -2$. In fact, usually $\pi_{-2} X(*)$ is usually 0.*

The first theorem is quite hard, but the second one is actually easy enough that we could've proved it here. It is called the *gap theorem.* Observe that it occurs for $K\mathbb{R}$, since $\pi_i K\mathbb{R}(*) = \pi_i KO$, and (beginning at degree -4) the Bott periodicity clock is

$$\mathbb{Z} \quad 0 \quad 0 \quad 0 \quad \mathbb{Z} \quad \mathbb{Z}/2 \quad \mathbb{Z}/2 \quad 0 \quad \mathbb{Z}.$$

So we could've expected those zeros!