A PRIMER ON HOMOTOPY COLIMITS

DANIEL DUGGER

Contents

1. Introduction

This is an expository paper on homotopy colimits and homotopy limits. These are constructions which should arguably be in the toolkit of every modern algebraic topologist, yet there does not seem to be a place in the literature where a graduate student can easily read about them. Certainly there are many fine sources: [BK], [DS], [H], [HV], [V1], [V2], [CS], [S], among others. Of these my favorites are [DS] and [H], the first as a general introduction and the second as an excellent reference work. Yet [H] demands that the student absorb quite a bit before reaching homotopy colimits, and [DS] does not delve deeply into the topic. The remaining sources mentioned above present other difficulties to readers encountering these ideas for the first time.

What I found myself wanting was a relatively short paper that would start with the basic ideas and then proceed to give students a 'crash course' in homotopy colimits—a paper which would survey the basic techniques for working with them and show some examples, but not weigh the reader down with too many details. That is the aim of the present document. Like most such documents, it probably fails to truly meet its goals—as one example, it is not very short!

Many proofs are avoided, or perhaps just sketched, and the reader is encouraged to seek out the complete proofs in the above sources.

1.1. **Prerequisites.** The reader is assumed to be familiar with basic category theory, in particular with colimits and limits. [ML] is a fine reference. Some experience with simplicial sets will be helpful, as well as some experience with model categories. For the former we recommend [C], and for the latter [DS].

Almost no model category theory is used in the first eight sections, where we keep the focus on topological spaces for the most part. Readers will only have to know that a cellular inclusion is the main example of a cofibration, and that a CW-complex is the main example of a cofibrant object. "Weak equivalence" means weak homotopy equivalence—that is to say, a map inducing isomorphisms on all homotopy groups.

In Sections 7–10 model category theory is much more prevalent. Although one can state the basic properties of homotopy colimits and limits without using model categories, the most elegant proofs all use model category techniques. So it is very useful to become proficient in this way of thinking about things.

What we have just outlined is something like the 'minimum basic requirements' assumed in the paper. In reality we have assumed more, because we assume throughout that the reader has a certain amount of experience with many basic homotopy-theoretic constructions (classifying spaces, spectral sequences, etc.) Hopefully students with just one or two years experience past their first algebraic topology course will find the paper accessible, though.

1.2. **Organization.** Part 1 of the paper (Sections 2–6) develops the basic definition of homotopy colimits and limits, as well as some foundational properties. Everything is done in the context of topological spaces, although the entire discussion adapts more or less verbatim to other simplicial model categories.

Parts 2 and 3 of the paper (Sections 7–12) concern more advanced perspectives on homotopy colimits and limits. We develop spectral sequences for computing some of their invariants, explain how to adapt the constructions to arbitrary model categories, and in Part 2 we intensively discuss the connection with the theory of derived functors.

To conclude the paper we have Part 4, concerning examples. Most of the material here only depends on Part 1, but every once in a while we need to use something more advanced. Most readers will be able to understand the basic ideas without having read Parts 2 and 3 first, but will occasionally have to flip back for complete details.

1.3. **Notation.** If $\mathcal{C}$ is a category and X and Y are objects, then we will write $\mathcal{C}(X, Y)$ instead of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. The category $(\mathcal{C} \downarrow X)$ is the category whose objects are pairs $[A, A \to X]$ consisting of an object A in $\mathcal{C}$ and a map $A \to X$. A map $[A, A \to X] \to [B, B \to X]$ consists of a map $A \to B$ making the evident triangle commute. Occasionally we will denote an object of $(\mathcal{C} \downarrow X)$ as $[A, X \leftarrow A]$, depending on the circumstance.

Part 1. **Getting started**

2. First examples

The theory of homotopy colimits arises because of the following basic difficulty. Let I be a small category, and consider two diagrams $D, D' \colon I \to \mathcal{T}op$. If one has a natural transformation $f \colon D \to D'$, then there is an induced map $\operatorname{colim} D \to \operatorname{colim} D'$. If f is a natural weak equivalence—i.e., if $D(i) \to D'(i)$ is a weak equivalence for all $i \in I$—it unfortunately does not follow that $\operatorname{colim} D \to \operatorname{colim} D'$ is also a weak equivalence. Here is an example:

Example 2.1. Let I be the 'pushout category' with three objects and two non-identity maps, depicted as follows: $1 \longleftarrow 0 \longrightarrow 2$. Let D be the diagram

$$* \longleftarrow S^n \longrightarrow D^{n+1}$$

and let D' be the diagram

$$* \longleftarrow S^n \longrightarrow *.$$

Let $f \colon D \to D'$ be the natural weak equivalence which is the identity on S^n and collapses all of D^{n+1} to a point. Then $\operatorname{colim} D \cong S^{n+1}$ and $\operatorname{colim} D' = *$, so the induced map $\operatorname{colim} D \to \operatorname{colim} D'$ is certainly not a weak equivalence.

So the colimit functor does not preserve weak equivalences (one sometimes says that the colimit functor is not "homotopy invariant", and it means the same thing). The *homotopy* colimit functor may be thought of as a 'correction' to the colimit, modifying it so that it *is* homotopy invariant.

There is one simple example of a homotopy colimit which nearly everyone has seen: the mapping cone. We generalize this slightly in the following example, which concerns homotopy pushouts.

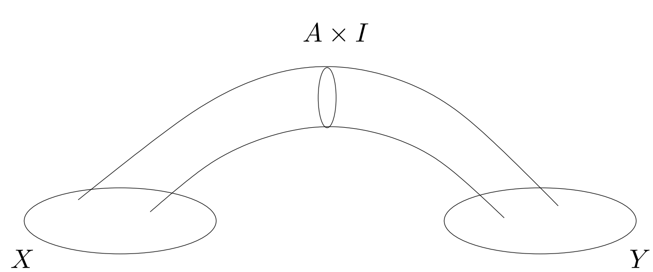
Example 2.2. Consider a pushout diagram of spaces $X \xleftarrow{f} A \xrightarrow{g} Y$. Call this diagram D. The pushout of D is obtained by gluing X and Y together along the images of the space A: that is, $f(a)$ is glued to $g(a)$ for every $a \in A$. The homotopy pushout, on the other hand, is constructed by gluing together X and Y 'up to homotopy'. Specifically, we form the following quotient space:

$$\operatorname{hocolim} D = \big[X \amalg (A \times I) \amalg Y\big] / \sim$$

where the equivalence relation has

$$(a, 0) \sim f(a) \qquad \text{and} \qquad (a, 1) \sim g(a), \qquad \text{for all } a \in A.$$

We can depict this space by the following picture:



Consider the open cover $\{U, V\}$ of hocolim D where U is the union of X with the image of $A \times [0, \frac{3}{4}]$, and V is the union of Y with the image of $A \times [\frac{1}{4}, 1]$. Note that U deformation retracts down to X, V deformation retracts down to Y, and that the map $A \to U \cap V$ given by $a \mapsto (a, \frac{1}{2})$ is a homotopy equivalence. The Mayer-Vietoris sequence then gives a long exact sequence relating the homology of hocolim D with $H_*(X)$, $H_*(Y)$, and $H_*(A)$. Similarly, the Van Kampen theorem shows (assuming X, Y, and A are path-connected, for simplicity) that $\pi_1(\text{hocolim}\, D)$ is the pushout of the diagram of groups $\pi_1(X) \longleftarrow \pi_1(A) \longrightarrow \pi_1(Y)$. The moral is that the space hocolim D is pretty easy to study using the standard tools of algebraic topology—in contrast to colim D, which is much harder.

It is now easy to prove that our construction of hocolim D preserves weak equivalences. Suppose D' is another pushout diagram $X' \longleftarrow A' \longrightarrow Y'$, and that $D \to D'$ is a natural weak equivalence. Let $\{U', V'\}$ be the cover of hocolim D' defined analogously to $\{U, V\}$. Note that the map hocolim $D \to$ hocolim D' restricts to maps $U \to U'$, $V \to V'$, and $U \cap V \to U' \cap V'$, and these restrictions are all weak equivalences (because both U and U' deformation retract down to X, and so forth). It then follows from the naturality of the Van Kampen theorem, and of the Mayer-Vietoris sequence, that hocolim $D \to$ hocolim D' induces isomorphisms on π_1 and on all homology groups. So it is a weak equivalence by the Whitehead theorem.

Before leaving this example we should relate it to mapping cones. If $f\colon A \to X$ is a map, then the quotient $X/f(A)$ is the pushout of $* \longleftarrow A \longrightarrow X$. The homotopy pushout of $* \longleftarrow A \longrightarrow X$, as defined above, is nothing other than the mapping cone of f.

There are several things to be learned from the above example, and we will return to it often as we develop the general theory. For now, here are four basic things to notice right away:

(1) Whereas the colimit of a diagram is obtained by taking the spaces in the diagram and gluing them together, the homotopy colimit will be constructed by gluing them 'up to homotopy'. Sometimes one says that the homotopy colimit is a 'fattened up' version of the colimit. The above example is perhaps misleadingly simple, because the indexing category I is so simple—for general categories quite a bit more will be involved in encoding the necessary homotopies. Still, this basic idea of 'gluing up to homotopy' is the important one.

(2) Note that in the above example one has a map hocolim $D \to$ colim D obtained by collapsing the homotopy. Specifically, one defines a map $X \amalg (A \times I) \amalg Y \to (X \amalg_A Y)$ by letting it be the natural inclusions on the X and Y factors, and on the $A \times I$ factor it is the projection $A \times I \to A$ followed by the natural inclusion. This map respects the identifications in the definition of hocolim D, so we get our map hocolim $D \to (X \amalg_A Y)$.

    This situation is typical. When we finally define hocolim D for general diagrams we will find that there is a natural map hocolim $D \to$ colim D obtained by 'collapsing homotopies'.

(3) Many algebraic-topological invariants of the space hocolim D should be computable in terms of the invariants for the D_i's. We will see, for instance, that this is true for any cohomology theory $E^*(-)$ and any homology theory $E_*(-)$. This is one of the main ways in which homotopy colimits are better

than colimits—they interact in predictable ways with the standard machinery of algebraic topology.

(4) It is not completely obvious, but it turns out that in our construction of $\operatorname{hocolim} D$ we could have replaced the interval I by any contractible space Z admitting a cofibration $\{0,1\} \rightarrowtail Z$. So we could have defined $\operatorname{hocolim} D$ as $[X \amalg (A \times Z) \amalg Y]/\sim$ where $(a,0) \sim f(a)$ and $(a,1) \sim g(a)$. This gives a space which is weakly equivalent to the definition we used above. (Even more, we could have replaced $A \times Z$ with any space B admitting a cofibration $A \amalg A \rightarrowtail B$ and a weak equivalence $B \to A$ coequalizing these two maps $A \to B$). What this is telling us is that there is not really a *single* homotopy colimit of a diagram; rather, there are lots of different 'models' for the homotopy colimit, all weakly equivalent to each other. The model where we used the interval I is in some sense more natural than the others, but we don't always want to be tied down to one model.

2.3. **The million-dollar question.** Why should one learn about homotopy colimits? How are they useful? These are the kind of questions every student should ask their professors before learning about something. It is often hard to give a simple answer, but here are my attempts:

(a) As remarked above, it is relatively easy to compute the homology or cohomology of a homotopy colimit ("easy" in the sense that there is a spectral sequence). So if one is studying a space X and can identify it as being a certain homotopy colimit (or more precisely, weakly equivalent to a certain homotopy colimit), then one has a good chance of computing the homology and cohomology groups of X.

(b) Many things that happen in algebraic topology come down, in the end, to showing that two spaces X and Y are weakly equivalent. As we will see, there are many techniques for showing that different homotopy colimits are weakly equivalent. So if one can first identify X and Y as certain homotopy colimits, there are suddenly a number of tools available for proving that $X \simeq Y$.

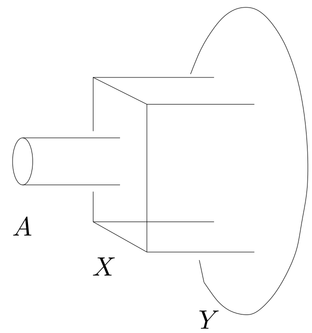
(c) Algebraic topology is full of *machinery*. This word can mean lots of things, but what I mean at the moment is a method for starting with some input data and producing a space or a sequence of spaces. For instance, one can start with a category and produce its classifying space; or start with a symmetric monoidal category and produce a Γ-space, and from the Γ-space get a spectrum. In algebraic K-theory one starts with a ring, considers the exact category of R-modules, and from this data constructs a K-theory space $K(R)$. These are only the most obvious examples—a complete list of such 'machines' would probably fill hundreds of pages.

Anyway, the point I want to make is that homotopy colimits (and limits) play an important role in the construction of the output spaces for many of these machines. If you are a student of homotopy theory and haven't yet encountered homotopy colimits, it is only a matter of time.

2.4. **One more example.** Before ending this section we examine another brief example. Consider a diagram of spaces

$$A \xrightarrow{f} X \xrightarrow{g} Y.$$

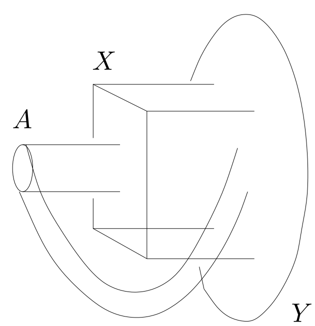
One way to construct the homotopy colimit in this case is as the double mapping cylinder shown below



This is the space $[(A \times I) \amalg (X \times I) \amalg Y]/\sim$ in which we have identified $(a,1) \sim (f(a),0)$ and $(x,1) \sim g(x)$, for all $a \in A$ and $x \in X$. Note that this space deformation retracts down to Y.

Now consider the following. For the colimit of a diagram D, every map $f\colon D_i \to D_j$ in the diagram tells us to glue $a \in D_i$ to $f(a) \in D_j$. In the homotopy colimit we are supposed to glue up to homotopy, and this is what we tried to do in the double mapping cylinder above. But note that we have only done this for f and g, whereas there is a third map in our diagram—namely, the composite gf! Maybe we should glue in a homotopy for that map, too.

This suggests that we should do the following. Start with $A \amalg X \amalg Y$ and glue in a cylinder for f, g, and gf. This gives us the following space, which we'll call W:



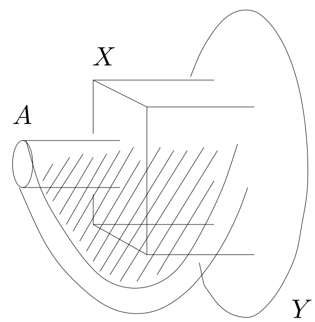
Unfortunately W is clearly not homotopy equivalent to Y, and therefore not homotopy equivalent to our double mapping cylinder above. But we can fix this as follows.

There is an evident map $A \times \partial\Delta^2$ into W: we have an $A \times I$ occuring in the mapping cylinders for f and gf, forming two of the 'sides' of $A \times \partial\Delta^2$. The third side comes the composite $A \times I \xrightarrow{g \times id} X \times I \to W$, where the second map is the mapping cylinder for g. What we will do is take W and attach a copy of $A \times \Delta^2$

along the image of $A \times \partial\Delta^2$; that is, we form the pushout

$$\begin{array}{ccc} A \times \partial\Delta^2 & \longrightarrow & W \\ \downarrow & & \vdots \\ A \times \Delta^2 & \cdots\cdots\rightarrow & W'. \end{array}$$

It is hard to draw a picture for W', but maybe we can try something like this:



This new space W' is homotopy equivalent to the double mapping cylinder we started with: the cylinder corresponding to gf can be squeezed down into the double mapping cylinder, via the $A \times \Delta^2$ piece we just attached. So W' is another model for the homotopy colimit of our diagram

2.5. **Summary.** The previous example suggests the following. Suppose given a small category I and a diagram $D\colon I \to \mathcal{T}op$. To construct $\operatorname{hocolim} D$ we should start with $\amalg_i D(i)$, and then for every map $f\colon i \to j$ in I we should glue in a cylinder $D(i) \times \Delta^1$ corresponding to f. Then for every pair of composable maps

$$i \xrightarrow{f} j \xrightarrow{g} k$$

in I we should glue in a copy of $D(i) \times \Delta^2$. Continuing the evident pattern, for every sequence of n composable maps

$$i_0 \to i_1 \to i_2 \to \cdots \to i_n$$

we should glue in a copy of $D(i_0) \times \Delta^n$. The problem is to figure out how to keep track of all this gluing in an efficent way! We'll begin developing the techniques for this in the next section.

3. Simplicial spaces

Before giving a general construction of homotopy colimits we need some preliminary machinery.

Let Δ be the cosimplicial indexing category: the objects are the finite ordered sets $[n] = \{0, 1, \ldots, n\}$ for $n \geq 0$, and the maps are the monotone increasing functions. Note that there is an inclusion $\Delta \hookrightarrow \mathcal{T}op$ which sends $[n]$ to Δ^n and sends a map $\sigma\colon [n] \to [k]$ to the corresponding linear map $\Delta^n \to \Delta^k$ which coincides with σ on the vertices of Δ^n. Sometimes we will blur the distinction between Δ and this subcategory of $\mathcal{T}op$ which is its image; in fact, historically the category Δ first arose *as this subcategory*—the description in terms of ordered sets is really just a modern convenience.

If $\mathcal{C}$ is any category, a simplicial object in $\mathcal{C}$ is a functor $X\colon \Delta^{op} \to \mathcal{C}$. This is commonly drawn as a diagram consisting of spaces $X_n = X([n])$ together with 'face' and 'degeneracy' maps between them:

$$\cdots \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; X_2 \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; X_1 \;\rightrightarrows\; X_0\ .$$

The face maps decrease dimension, and the degeneracies increase dimension; we will usually not draw the degeneracies, for typographical reasons. A cosimplicial object in $\mathcal{C}$ is a functor $Z\colon \Delta \to \mathcal{C}$, which is a similar diagram with all the arrows going in the other direction.

3.1. **Geometric realization.** Suppose $X\colon \Delta^{op} \to \mathcal{T}op$ is a simplicial space. The *geometric realization* of X is the space

$$|X| = \text{coeq}\left[\coprod_{[n]\to[k]} X_k \times \Delta^n \rightrightarrows \coprod_n X_n \times \Delta^n\right]. \tag{3.2}$$

This is a 'coequalizer', which is just another name for a colimit of a diagram consisting of two parallel arrows: so the coequalizer of two arrows $f, g\colon S \rightrightarrows T$ is the quotient space $T/\sim$ in which one identifies $f(s) \sim g(s)$ for all $s \in S$.

To finish explaining the formula in (3.2), we should mention that the first coproduct in the coequalizer is taken over all maps in Δ. If $\sigma\colon [n] \to [k]$ is a map in Δ then there are two evident maps from $X_k \times \Delta^n$ into $\coprod_i X_i \times \Delta^i$. The first sends $X_k \times \Delta^n$ to $X_n \times \Delta^n$ via the map $\sigma^*\colon X_k \to X_n$, and the second sends $X_k \times \Delta^n$ to $X_k \times \Delta^k$ via the map $s\colon \Delta^n \to \Delta^k$. This gives the two parallel maps in the coequalizer diagram.

A little thought shows that the above formula for $|X|$ can also be written as

$$|X| = \left(\coprod_n X_n \times \Delta^n\right)\Big/\sim$$

where the equivalence relation has

$$(d_i x, t) \sim (x, d^i t) \qquad \text{and} \qquad (s_i x, t) \sim (x, s^i t).$$

Here the d_i's and s_i's are the face and degeneracy maps in X, and the d^i's and the s^i's are the coface and codegeneracy maps in the cosimplicial object $\Delta \to \mathcal{T}op$ sending $[n] \mapsto \Delta^n$.

Remark 3.3. Note that if each X_n is a discrete space then we can regard X as a functor $\Delta^{op} \to \mathcal{S}et$ and the above construction is the same as the usual geometric realization of a simplicial set.

3.4. **Homotopy invariance of geometric realization.** By a map of simplicial spaces $X \to Y$ we mean a natural transformation of functors. Such a map is said to be an objectwise weak equivalence if $X_n \to Y_n$ is a weak equivalence of spaces, for all n. It is not quite true that if $X \to Y$ is an objectwise weak equivalence of simplicial spaces then $|X| \to |Y|$ is a weak equivalence of spaces. At about the same time, Segal [Se] and May [M] independently developed conditions under which this *is* true. We will describe a modern version of such conditions next.

If $s_i \colon X_{n-1} \to X_n$ is a degeneracy map, $0 \leq i \leq n-1$, then note that one of the simplicial identities is $d_i s_i = id$; so X_{n-1} is a retract of X_n. We then have that s_i is injective, and a point-set-topology argument shows that the topology on X_{n-1} coincides with the subspace topology on its image. So s_i is an inclusion. If X_n is Hausdorff (which is necessarily true if X_n is cofibrant), more point-set topology shows that s_i is in fact a closed inclusion.

Define the ***n*th latching object** of X to be the subspace

$$L_n X = \bigcup_{i=0}^{n-1} s_i(X_{n-1}) \subseteq X_n.$$

The inclusion $L_n X \hookrightarrow X_n$ is called the ***n*th latching map**.

The first few latching spaces are easy to picture: $L_0 X = \emptyset$, $L_1 X \cong X_0$, and $L_2 X \cong X_1 \amalg_{X_0} X_1$. These spaces get more complicated as n grows. For instance, $L_3 X$ consists of three copies of X_2 glued together along three copies of X_1, all containing a single copy of X_0.

A simplicial space X is called **Reedy cofibrant** if the latching maps $L_n X \to X_n$ are cofibrations, for all n. If X is Reedy cofibrant then each X_n is cofibrant, by an induction starting with the fact that the 0th latching map is $\emptyset \to X_0$.

Theorem 3.5. *Suppose $X \to Y$ is an objectwise weak equivalence between two simplicial spaces, both of which are Reedy cofibrant. Then $|X| \to |Y|$ is also a weak equivalence.*

Sketch of proof. Let $\operatorname{Sk}_n |X|$ denote the subspace of $|X|$ defined by

$$\operatorname{Sk}_n |X| = \operatorname{coeq}\left[\coprod_{\substack{[k]\to[l]\\ k,l\leq n}} X_l \times \Delta^k \rightrightarrows \coprod_{k\leq n} X_k \times \Delta^k \right].$$

Then there is a sequence of closed inclusions

$$\operatorname{Sk}_0 |X| \hookrightarrow \operatorname{Sk}_1 |X| \hookrightarrow \operatorname{Sk}_2 |X| \hookrightarrow \cdots$$

and the colimit is $|X|$. One shows that there are pushout squares

$$\begin{array}{ccc}
(L_n X \times \Delta^n) \amalg_{(L_n X \times \partial\Delta^n)} (X_n \times \partial\Delta^n) & \longrightarrow & \operatorname{Sk}_{n-1} |X| \\
\downarrow & & \downarrow \\
X_n \times \Delta^n & \longrightarrow & \operatorname{Sk}_n |X|
\end{array}$$

for each n, and our assumption that X is Reedy cofibrant implies that the left vertical map is a cofibration.

Using that $X \to Y$ is an objectwise weak equivalence, one shows inductively that each $L_nX \to L_nY$ is a weak equivalence, and then that each $\mathrm{Sk}_n |X| \to \mathrm{Sk}_n |Y|$ is a weak equivalence. It then follows that $|X| \to |Y|$ is also a weak equivalence. $\square$

Remark 3.6 (The fat realization)**.** Let X be a simplicial space. Define

$$||X|| = \mathrm{coeq}\left[\coprod_{[n]\hookrightarrow[k]} X_k \times \Delta^n \rightrightarrows \coprod_n X_n \times \Delta^n.\right]$$

where the left coproduct runs over all *injections* in Δ. Note that this definition completely ignores the degeneracy maps in the simplicial space X. The space $||X||$ is called the *fat realization* of X.

The disadvantage of $||X||$ over $|X|$ is that the former space is always much bigger and more complicated—in fact, it is always infinite-dimensional! For instance, suppose X is the simplicial space consisting of one point in every dimension. Then $|X|$ is just a point, but $||X||$ is a space consisting of one 0-cell, one 1-cell, one 2-cell, etc. This is because the degenerate stuff in X hasn't been collapsed, as it was in $|X|$.

The advantage of $||X||$ over $|X|$ is that this fat construction preserves weak equivalences under much weaker hypotheses. If $X \to Y$ is an objectwise weak equivalence between simplicial spaces which are cofibrant in each dimension, then $||X|| \to ||Y||$ is a weak equivalence. We'll see a proof of this in Example 8.13 below.

3.7. **Collapsing the geometric realization.** One often thinks of the $X_n \times \Delta^n$ pieces in $|X|$ as 'higher homotopies'. Consider the process of collapsing them, in which one shrinks every Δ^n to a point. Thus, we consider the diagram

$$\begin{array}{ccc} \coprod_{[n]\to[k]} X_k \times \Delta^n & \rightrightarrows & \coprod_{[n]} X_n \times \Delta^n \\ \downarrow & & \downarrow \\ \coprod_{[n]\to[k]} X_k & \rightrightarrows & \coprod_{[n]} X_n \end{array}$$

where the vertical maps come from the projections $X_k \times \Delta^n \to X_k$ and $X_n \times \Delta^n \to X_n$. The coequalizer of the bottom two arrows is precisely $\mathrm{colim}_{\Delta^{op}} X$. Thus, we have a natural map

$$|X| \to \mathrm{colim}\, X.$$

Now, $\mathrm{colim}\, X$ can be identified with the coequalizer of the first two arrows $d_0, d_1 \colon X_1 \to X_0$. This is an exercise for the reader; clearly there is a map $\mathrm{coeq}(X_1 \rightrightarrows X_0) \to \mathrm{colim}\, X$, and one can prove using the simplicial identities that any map $X_0 \to Z$ which coequalizes $d_0, d_1 \colon X_1 \to X_0$ actually induces a map $\mathrm{colim}\, X \to Z$. Thus, one gets a map $\mathrm{colim}\, X \to \mathrm{coeq}(X_1 \rightrightarrows X_0)$, and one readily sees that the two compositions are the identities.

Putting everything together, we have shown that there is a natural map

$$|X| \to \mathrm{coeq}\Big[X_1 \rightrightarrows X_0\Big].$$

Remark 3.8. Note that if X is a simplicial *set* then this coequalizer is just $\pi_0(X)$, the set of path components. In this case our map is just the usual one from $|X|$ to its set of path components (equipped with the discrete topology).

3.9. **Degenerate simplicial spaces.** A simplicial space X is **degenerate in dimension q and above** if the maps $L_kX \to X_k$ are homeomorphisms for all $k \geq q$. It follows that the spaces X_k, $k \geq q$, all get collapsed inside of $|X|$. The reason is that if $x \in X_k$ then $x = s_{i_1}s_{i_2}\dots s_{i_r}y$ for some $y \in X_{q-1}$. So for any $t \in \Delta^k$ we have

$$(x,t) = (s_{i_1}\dots s_{i_r}y, t) \sim (y, s^{i_1}\cdots s^{i_r}t)$$

in $|X|$. A little thought shows that in this case we can write

$$|X| = \mathrm{Sk}_q\,|X| = \mathrm{coeq}\left[\coprod_{\substack{[n]\to[k]\\ n,k\leq q}} X_k \times \Delta^n \rightrightarrows \coprod_{n\leq q} X_n \times \Delta^n\right].$$

This observation simplifies the process of computing $|X|$ in many cases, and we'll use it in the next sections when we're faced with some specific examples.

3.10. **Contracting homotopies.** Suppose X_* is a simplicial *set* and $*$ is a 0-simplex of X. A *contracting homotopy* for X is a collection of combinatorial data which will guarantee that $|X|$ deformation-retracts down to $*$. So we need to deform each n-simplex of X down to a point, and the deformations for different simplices need to be compatible. The easiest way to accomplish this is to specify the following data:

- For each 0-simplex a of X, a 1-simplex $S(a)$ connecting a to $*$;
- For each 1-simplex b of X, a 2-simplex $S(b)$ whose base is b, whose remaining vertex is $*$, and whose 'sides' are the 1-simplices previously specified;
- And so on—for each n-simplex c of X we will need an $(n+1)$-simplex whose base is c, whose remaining vertex is $*$, and whose sides coincide with previously specified data.

A contracting homotopy for X will therefore be a collection of maps $S\colon X_n \to X_{n+1}$ which are required to satisfy some identities. These identities will take a different form depending on whether we want the simplices $S(a)$ to point *towards* the simplex $*$ or *away from* the simplex $*$. We'll differentiate these cases by calling them "forward" and "backward" contracting homotopies, respectively.

Before giving the formal definition it will be useful to generalize somewhat. By an **augmented simplicial set** we mean a simplicial set X together with a set W and a map $X_0 \to W$ which coequalizes the two maps $X_1 \rightrightarrows X_0$. This is the same as having a map of simplicial sets $X \to cW$, where cW is the constant simplicial set having W in every dimension. A contracting homotopy for an augmented simplicial set $X_* \to W$ will be a map $W \to X_0$ such that $W \to X_0 \to W$ is the identity together with a way of deformation-retracting X_* down to the image of W in X_0.

Finally, we wish to generalize our discussion from simplicial *sets* to simplicial *spaces*. The basic formalism is the same, and in particular the definition of augmented simplicial *space* is the same.

Definition 3.11. *Let $X_* \to W$ be an augmented simplicial space. It will be convenient to define X_{-1} to be W, and to have the map $X_0 \to W$ be denoted by d_0. Then a* ***forward contracting homotopy*** *is a collection of maps $S\colon X_n \to X_{n+1}$ for $n \geq -1$ such that for each $a \in X_n$ one has*

$$d_i(Sa) = \begin{cases} S(d_ia) & \text{if } 0 \leq i < n \\ a & \text{if } i = n \end{cases} \qquad \text{and} \qquad S(s_ia) = s_i(Sa) \text{ for } 0 \leq i \leq n.$$

A ***backward contracting homotopy*** *for X is a collection of maps $S\colon X_n \to X_{n+1}$ for $n \geq -1$ such that for each $a \in X_n$ one has*

$$d_i(Sa) = \begin{cases} a & \text{if } i = 0 \\ S(d_{i-1}a) & \text{if } 0 < i \leq n \end{cases} \qquad \text{and} \qquad S(s_i a) = s_{i+1}(Sa) \text{ for } 0 \leq i \leq n.$$

Proposition 3.12. *Let $X_* \to W$ be an augmented simplicial space which admits either a forward or backward contracting homotopy. Then $|X| \to W$ is a homotopy equivalence.*

Proof. An easy exercise, or see ?????. □

Example 3.13. Let X be the simplicial set Δ^n. The k-simplices of X are all the monotone increasing sequences of length $k+1$ taking values in $\{0, 1, \ldots, n\}$. We regard X as augmented by the one-point space, so we set $X_{-1} = \{*\}$; it is useful to think of the element of X_{-1} as the "empty sequence".

One can define a backwards contracting homotopy for X by having $S\colon X_n \to X_{n+1}$ send a sequence $a_0 \ldots a_n$ to the sequence $0a_0 \ldots a_n$. In other words, the contracting homotopy inserts a 0 at the beginning of every sequence. One can also define a forwards contracting homotopy for X, by inserting an n at the *end* of every sequence.

Example 3.14. Let $f\colon X \to Y$ be a map of topological spaces, and consider the simplicial space $\check{C}(f)$ defined by

$$[n] \mapsto X \times_Y X \times_Y \cdots \times_Y X \qquad ((n+1) \text{ factors}).$$

If $(x_0, \ldots, x_n)$ is an element of $\check{C}(f)_n$, then the ith face map omits x_i and the jth degeneracy repeats x_j. This simplicial space is called the **Čech complex** of f. If we forget the topological structure then this is the nerve of a category, where there is one object for every element of X and a unique map between any two objects which have the same image under f.

We may regard $\check{C}(f)$ as being augmented by Y, via the map f. Suppose $s\colon Y \to X$ is a section of f. Define a backwards contracting homotopy for $\check{C}(X)$ by sending the point $(x_0, \ldots, x_n)$ to $(s(f(x_0)), x_0, \ldots, x_n)$. Note that one can also obtain a forwards contracting homotopy by appending $s(f(x_n))$ to the *end* of the tuple. So if f admits a section then $|\check{C}(f)| \to Y$ is a homotopy equivalence.

Example 3.15. This final example will not be needed until Part 2, but we include it here as a titillating exercise. Let $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ be adjoint functors between two categories. Recall that such a pair is equipped with natural transformations $LR(X) \to X$ and $Z \to RL(Z)$, which we'll refer to as 'contraction' and 'expansion'. These natural transformations have the property that the two composites $RX \to RLR(X) \to RX$ and $LZ \to LRLZ \to LZ$ (both obtained by first expanding and then contracting in the evident way) are the identities.

For each $X \in \mathcal{D}$ one can construct a simplicial object $B_{LR}(X)$ over $\mathcal{C}$ having the form

$$[n] \mapsto (LR)^{n+1}(X).$$

If the LR pairs in $B_{LR}(X)_n$ are labelled as 0 through n, then the face map d_i applies contraction to the ith LR pair; the jth degeneracy s_j applies an expansion between the L and R of the jth LR pair. Using only the facts stated in the previous paragraph, one may check that these face and degeneracy maps indeed satisfy the axioms for a simplcial object.

Note that the contraction map $LR(X) \to X$ provides an augmentation for $B_{LR}(X)$.

Now apply R levelwise to $B_{LR}(X)$ to obtain a simplicial object over $\mathcal{C}$. One can check that $RB_{LR}(X) \to RX$ admits a backwards contracting homotopy, where the map $S\colon R[B_{LR}(X)]_n \to R[B_{BL}(X)]_{n+1}$ is simply an expansion before the first R—that is, S is the map $Z \to RL(Z)$ where $Z = B_{LR}(X)_n$. It is routine to check that the necessary identities are satisfied.

Likewise, consider the case where $X = LA$. The augmented simplicial object $B_{LR}(LA) \to LA$ admits a forward contracting homotopy, where the map $B_{LR}(LA)_n \to B_{LR}(LA)_{n+1}$ inserts an expansion between the L and the A.

Exercise 3.16. Given a map of topological spaces $f\colon X \to Y$, there are adjoint functors

$$L\colon (\mathcal{T}op \downarrow X) \rightleftarrows (\mathcal{T}op \downarrow Y)\colon R$$

where L is composition with f and R is pullback along f. Check that the bar construction for LR, applied to the terminal object of $(\mathcal{T}op \downarrow Y)$, is $\check{C}(f)$. Which of the two contracting homotopies of Example 3.14 is related to the ones in Example 3.15?

4. Construction of homotopy colimits

Let I be a small category, and let $D\colon I \to \mathcal{T}op$ be a diagram. We will now explain how to construct the homotopy colimit of D (really we should say, "*a* homotopy colimit of D").

The **simplicial replacement** of D is the simplicial space

$$\coprod_{i_0} D(i_0) \leftleftarrows \coprod_{i_0 \leftarrow i_1} D(i_1) \;\Lleftarrow\; \coprod_{i_0 \leftarrow i_1 \leftarrow i_2} D(i_2) \;\Lleftarrow\; \dots$$

We will denote this $\mathrm{srep}(D)$. So we have

$$\mathrm{srep}(D)_n = \coprod_{i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n} D(i_n)$$

where the coproduct ranges over chains of composable maps in I. We must define the face and degeneracy maps. If $\sigma = [i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n]$ is a chain and $0 \le j \le n$, then we can 'cover up' i_j and obtain a chain of $n-1$ composable maps—call this new chain $\sigma(j)$. When $j < n$, the map $d_j\colon \mathrm{srep}(D)_n \to \mathrm{srep}(D)_{n-1}$ sends the summand $D(i_n)$ corresponding to σ to the identical copy of $D(i_n)$ in $\mathrm{srep}(D)_{n-1}$ indexed by $\sigma(j)$. When $j = n$ we must modify this slightly, as covering up i_n now yields a chain that ends with i_{n-1}. So $d_n\colon \mathrm{srep}(D)_n \to \mathrm{srep}(D)_{n-1}$ sends the summand $D(i_n)$ corresponding to the chain σ to the summand $D(i_{n-1})$ corresponding to $\sigma(n)$, and the map we use here is the map $D(i_n) \to D(i_{n-1})$ induced by the last map in σ.

The degeneracy maps $s_j\colon \mathrm{srep}(D)_n \to \mathrm{srep}(D)_{n+1}$, $0 \le j \le n$, are a bit easier to describe. Each s_j sends the summand $D(i_n)$ corresponding to the chain $\sigma = [i_0 \leftarrow i_1 \leftarrow \cdots i_n]$ to the identical summand $D(i_n)$ corresponding to the chain $\sigma[j]$ in which one has inserted the identity map $i_j \leftarrow i_j$.

Example 4.1. The **nerve** of a small category I is the simplicial set NI which in dimension n consists of all strings $[i_0 \to i_1 \to \cdots \to i_n]$ of n composable arrows. The face map d_j corresponds to 'covering up' the object i_j, as above. The **classifying space** of I is the geometric realization of the nerve; it will be denoted BI.

The nerve of the opposite category I^{op} may be identified with the simplicial set which in dimension n consists of all strings $[i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n]$ of n composable arrows, where the face map d_j again corresponds to covering up the object i_j. This is very similar to the nerve of I, but not identical—the order of the faces and degeneracies have been reversed. These simplicial sets are not isomorphic, but they *are* naturally weakly equivalent.

Suppose $D\colon I \to \mathcal{T}op$ is the diagram for which $D(i) = *$ for all $i \in I$. Then $\mathrm{srep}(D)$ is just the nerve of the category I^{op}.

Remark 4.2. Note that we have made a choice when defining the simplicial replacement. We could have defined the nth object to be

$$\coprod_{i_0 \to i_1 \to \cdots \to i_n} D(i_0) \tag{4.3}$$

and again defined the degeneracy d_j to be the map associated to 'covering up' i_j. This is related to the distinction between the nerve of a category I and the nerve of its opposite. The simplicial space from (4.3) is not isomorphic to $\mathrm{srep}(D)$, although their geometric realizations are homeomorphic.

So there are two natural definitions of the simplicial replacement (as well as for the nerve of a category), and one is forced to choose. Our choices were made to agree with the conventions in [H].

It turns out to be useful to have *both* definitions around at the same time. They are brought together in the *two-sided bar construction* which we will talk about in Section 10.

Remark 4.4. Note that if each $D(i)$ is a cofibrant space, then the simplicial replacement is automatically Reedy cofibrant. This is because the nth latching object of $\text{srep}(D)$ is just the subspace of $\text{srep}(D)_n$ consisting of all summands corresponding to chains which have identity maps in them. So the latching object is just a summand inside the whole space, and the complementary summand is cofibrant (being a disjoint union of cofibrant spaces). Thus, $L_n(\text{srep}(D)) \to \text{srep}(D)_n$ is a cofibration.

Definition 4.5. *The homotopy colimit of a diagram $D\colon I \to \mathcal{T}op$ is the geometric realization of its simplicial replacement. That is,*

$$\operatorname{hocolim} D = |\operatorname{srep}(D)|.$$

Sometimes we will write $\operatorname{hocolim}_I D$ *to remind us of the indexing category.*

4.6. **Homotopy invariance of the homotopy colimit.**

Proposition 4.7. *If $D, D'\colon I \to \mathcal{T}op$ are two diagrams consisting of cofibrant objects and $\alpha\colon D \to D'$ is a natural weak equivalence, then the induced map* $\operatorname{hocolim} D \to \operatorname{hocolim} D'$ *is a weak equivalence.*

Proof. We get a map of simplicial spaces $\text{srep}(D) \to \text{srep}(D')$, and this is an objectwise weak equivalence. Since $\text{srep}(D)$ and $\text{srep}(D')$ are both Reedy cofibrant, it follows that the induced map of realizations is also a weak equivalence. □

Remark 4.8. Note that we could have instead defined $\operatorname{hocolim} D$ to be $\|\operatorname{srep}(D)\|$. That is, we could have used the fat realization instead of the usual geometric realization. This would still give a homotopy invariant construction, and would be weakly equivalent to the definition of $\operatorname{hocolim} D$ adopted above. This is further demonstration that there is not really a *single* homotopy colimit construction; there are many such constructions, all weakly equivalent to each other.

Remark 4.9 (Cofibrancy assumptions)**.** Proposition 4.7 is perhaps weaker than one would hope for, because of the cofibrancy conditions on the objects of D and D'. There are two things to say about this. In a general model category, to get the 'correct' homotopy colimit of a diagram D one should first arrange things so that all the objects are cofibrant—for instance, by applying a cofibrant-replacement functor to all the objects of D. Then one can apply specific formulas for the hocolim, such as the one above.

In the category $\mathcal{T}op$, though, an 'accident' occurs, in that the cofibrancy conditions on the objects are not necessary at all! That is to say, Proposition 4.7 is true even without these conditions. A proof can be found in [DI, Appendix]. We will tend to ignore this, however, and continue to state results with the objectwise cofibration hypotheses in them. This is because we want to state the results so that they generalize to other model categories.

4.10. **The natural map from the homotopy colimit to the colimit.** Note that $\operatorname{colim} D$ is the coequalizer of d_0 and d_1 in $\operatorname{srep}(D)$: that is, it is the quotient space $[\amalg_i D(i)]/\sim$ where for every map $\sigma\colon i \to j$ in I we identify points $x \in D(i)$ with $\sigma_*(x) \in D(j)$. The canonical map

$$|\operatorname{srep}(D)| \to \operatorname{coeq}\Big[\operatorname{srep}(D)_1 \rightrightarrows \operatorname{srep}(D)_0\Big]$$

from Section 3.7 therefore can be written as a map $\operatorname{hocolim} D \to \operatorname{colim} D$.

Example 4.11. Let's return to our most basic example, where I is the pushout category and D is a diagram $X \xleftarrow{f} A \xrightarrow{g} Y$. The simplicial replacement has $X \amalg A \amalg Y$ in dimension 0, and $X \amalg A \amalg A \amalg Y$ in dimension 1; everything in dimensions 2 and higher is degenerate. So by the discussion in Section 3.9, when forming $|\operatorname{srep}(D)|$ we only have to pay attention to the spaces in dimensions 0 and 1.

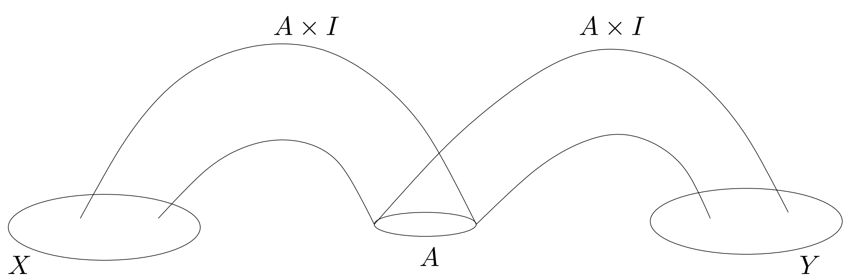
It's perhaps better to write $\operatorname{srep}(D)_1 = X_{id} \amalg A_f \amalg A_g \amalg Y_{id}$, where we are now keeping track of the maps in I indexing the summands (thus, "A_f" is the copy of A indexed by the map f). We see that the X and Y are degenerate, and a little thought shows that $|\operatorname{srep}(D)|$ is the quotient space

$$[X \amalg A \amalg Y \amalg (A_f \times \Delta^1) \amalg (A_g \times \Delta^1)]/\sim$$

in which the following identifications are made:

(1) $(a, 0) \in A_f \times \Delta^1$ is identified with $f(a) \in X$, whereas $(a, 1) \in A_f \times \Delta^1$ is identified with $a \in A$.
(2) $(a, 0) \in A_g \times \Delta^1$ is identified with $g(a) \in Y$, whereas $(a, 1) \in A_f \times \Delta^1$ is identified with $a \in A$.

We thus get something like the following picture (but where the two cylinders do not really intersect except at their ends):



Note that this is homeomorphic to the space from Example 2.2.

Exercise 4.12. Work through the definition of $\operatorname{hocolim} D$ when D is the diagram $A \to X \to Y$, and check that it is homeomorphic to the space W' from our example in Section 2.4.

4.13. **A different formula.** Here is another formula for the homotopy colimit. Although it looks quite different at first, the space it describes is homeomorphic to that of our previous definition (we will explain why below). The new formula is:

$$\operatorname*{hocolim}_I D = \operatorname{coeq}\left[\coprod_{i \to j} D_i \times B(j \downarrow I)^{op} \rightrightarrows \coprod_i D_i \times B(i \downarrow I)^{op}\right]. \tag{4.14}$$

There are a few things to say about this formula. If $\mathcal{C}$ is a category, then $B\mathcal{C}$ is its classifying space—the geometric realization of its nerve. And $\mathcal{C}^{op}$ denotes the opposite category. The *op*'s are needed in the above formula only to make it conform with the choices we made in defining the simplicial replacement. Finally, if $i \to j$ is a map in I then there is an evident induced map of categories $(j \downarrow I) \to (i \downarrow I)$, and this is being used in one of the maps from our coequalizer diagram.

The formula in (4.14) gives a more direct comparison between the homotopy colimit and the ordinary colimit. The colimit is, after all, the coequalizer

$$\operatorname{colim}_I D = \operatorname{coeq}\left[\coprod_{i \to j} X_i \rightrightarrows \coprod_i X_i\right].$$

One finds a map from the previous coequalizer diagram to this one simply by collapsing the spaces $B(i \downarrow I)^{op}$ to a point; thus, one gets the map $\operatorname{hocolim} D \to \operatorname{colim} D$.

Below we will prove rigorously that the space defined in (4.14) is homeomorphic to the space $|\operatorname{srep}(D)|$, but let's pause to explain the general idea. In constructing $|\operatorname{srep}(D)|$, for every chain $i_n \leftarrow i_{n-1} \leftarrow \cdots \leftarrow i_0$ we have added a copy of $D_{i_0} \times \Delta^n$. So if we fix a particular spot D_i of the diagram, this means that we are adding a copy of $D_i \times \Delta^n$ for every string $i_n \leftarrow i_{n-1} \leftarrow \cdots \leftarrow i_1 \leftarrow i$. Such a string gives an n-simplex in $B(i \downarrow I)^{op}$, corresponding to the chain

$$[i, i_n \leftarrow i] \leftarrow [i, i_{n-1} \leftarrow i] \leftarrow \cdots \leftarrow [i, i_1 \leftarrow i] \leftarrow [i, i \leftarrow i \colon id]$$

(which is a chain in $(i \downarrow I)$). In the formula (4.14) we are simply grouping all these $D_i \times \Delta^n$'s together—fixing i and letting n vary—into the space $D_i \times B(i \downarrow I)^{op}$. In other words, the space $B(i \downarrow I)^{op}$ is parameterizing all the 'D_i-homotopies' that are being added into the homotopy colimit.

Here is a simple example:

Example 4.15. Consider again the case where I is the pushout category $1 \leftarrow 0 \to 2$ and D is a diagram $X \leftarrow A \to Y$. Then $(1 \downarrow I)$ and $(2 \downarrow I)$ are both the trivial category, whereas $(0 \downarrow I)$ is the category $a \leftarrow b \to c$ (isomorphic to I again). So $B(0 \downarrow I)$ is the space consisting of two intervals joined at one endpoint:

$$\bullet \text{———} \bullet \text{———} \bullet.$$

The above formula says

$$\operatorname{hocolim}_I D = \left[X \amalg \left(A \times B(0 \downarrow I)^{op}\right) \amalg Y\right] \Big/ \sim$$

and one checks that the quotient relations give the same space we saw in Example 4.11.

If one is willing to learn some more machinery, there is a very slick proof that our two formulas for $\operatorname{hocolim} D$ are naturally homeomorphic. We give this in Section 10. For the moment we will be content with an argument which is more longwinded, but requires less background.

A couple of observations are needed. First, if K is a simplicial set than $X \times |K|$ can be identified with the geometric realization of the simplicial space

$$[n] \mapsto X \times K_n = \coprod_{K_n} X.$$

Consider the following big diagram:

$$\begin{array}{ccccc}
\cdots & & \cdots & & \cdots \\
\downdownarrows\!\downarrow & & \downdownarrows\!\downarrow & & \downdownarrows\!\downarrow \\
\coprod_{i,k_1 \leftarrow k_0 \leftarrow j \leftarrow i} X_i & \rightrightarrows & \coprod_{i,j_1 \leftarrow j_0 \leftarrow i} X_i & \longrightarrow & \coprod_{j_0 \leftarrow j_1} X_{j_1} \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
\coprod_{i,k_0 \leftarrow j \leftarrow i} X_i & \rightrightarrows & \coprod_{i,j_0 \leftarrow i} X_i & \longrightarrow & \coprod_{j_0} X_{j_0}
\end{array}$$

Each column is a simplicial space. The rightmost column is $\mathrm{srep}(X)$, the middle column is $\coprod_i (X_i \times N(I \downarrow i)^{op})$, and the leftmost column is $\coprod_{i \to j} (X_i \times N(I \downarrow j)^{op})$.

We have a map of simplicial spaces from the middle column to the right column. In dimension n this sends the summand X_i corresponding to the string $[j_0 \leftarrow j_1 \leftarrow \cdots j_n \leftarrow i]$ to the summand X_{j_n} corresponding to $[j_0 \leftarrow \cdots j_n]$ via the map $X_i \to X_{j_n}$ induced by $i \to j_n$. This is clearly compatible with face and degeneracies.

We have *two* maps of simplicial spaces from the left column to the middle column. In dimension n, one map sends the summand X_i corresponding to the index $[i, k_n \leftarrow k_{n-1} \leftarrow \cdots \leftarrow k_0 \leftarrow j \leftarrow i]$ to the summand X_i indexed by $[i, k_n \leftarrow \cdots \leftarrow k_0 \leftarrow i]$ (forget about j). The other map sends our summand X_i to the summand X_j indexed by $[j, k_n \leftarrow \cdots \leftarrow k_0 \leftarrow j]$ (forget about i).

Now, it's easy to check that each horizontal level of our diagram is a coequalizer diagram; that is to say, the objects in the right column are the coequalizers of the objects in the other two columns. Geometric realization is a left adjoint, and therefore will commute with coequalizers. So this identifies $|\,\mathrm{srep}(D)|$ with the coequalizer of

$$\coprod_{i \to j} |X_i \times N(I \downarrow j)^{op}| \rightrightarrows \coprod_i |X_i \times N(I \downarrow i)^{op}|.$$

This is the identification that we wanted.

5. Homotopy limits and some useful adjunctions

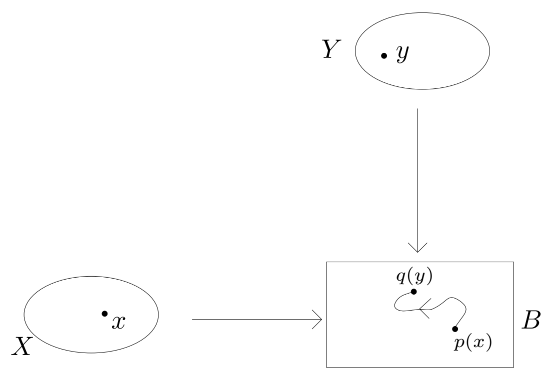
We haven't yet talked about homotopy *limits*. The story is completely dual to that for homotopy colimits, the main difference being that the pictures are not quite as easy to draw. We will just outline the basic constructions, accentuating the small differences.

Example 5.1. We again start with the most basic example, generalizing slightly the notion of a homotopy fiber. Let I be the pullback category $1 \to 0 \leftarrow 2$, and let $D\colon I \to \mathcal{T}op$ be a diagram $X \xrightarrow{p} B \xleftarrow{q} Y$. A point in the pullback $X \times_B Y$ consists of a point $x \in X$ and a point $y \in Y$ such that $p(x) = q(y)$. A point in the homotopy pullback will consist of a point $x \in X$, a point $y \in Y$, and a *path* from $p(x)$ to $q(y)$.

Formally, we define $\operatorname{holim} D$ to be the pullback of the diagram

$$\begin{array}{ccc} & & B^I \\ & & \downarrow \\ X \times Y & \xrightarrow{p \times q} & B \times B \end{array}$$

where B^I is the space of maps $\gamma\colon I \to B$ and the map $B^I \to B$ sends γ to $(\gamma(0), \gamma(1))$. It is sometimes useful to depict a point in $\operatorname{holim} D$ via a picture like the following:



Note that if $X \xrightarrow{p} B$ is a map and $* \in B$ is a basepoint, then the homotopy fiber of p, as classicaly defined, is just the homotopy pullback of the diagram $X \longrightarrow B \longleftarrow *$.

Generally speaking, if I is any indexing category and $D\colon I \to \mathcal{T}op$ is a diagram, then a point in $\lim D$ consists of points in each $D(i)$ which 'match up' as you move around the diagram. A point in $\operatorname{holim} D$ will consist of points in each $D(i)$, *together with* paths connecting their images as you move around the diagram, as well as 'higher homotopies' connecting the paths, and paths of paths, etc. It is a bit hard to describe, but here is one more example.

Example 5.2. Consider a diagram D of the form $A \xrightarrow{f} X \xrightarrow{g} Y$. A point in $\operatorname{holim} D$ will consist of points $a \in A$, $x \in X$, $y \in Y$, together with the following extra data. First, we need a path α from $f(a)$ to x, a path β from $g(x)$ to y, and a path γ from $g(f(a))$ to y. Applying g to α gives a path from $g(f(a))$ to $g(x)$, and so now we have a map $\partial\Delta^1 \to Y$ consisting of the three paths $g(\alpha)$, β, and γ.

Finally, we also require a map $\Delta^2 \to Y$ extending our map $\partial\Delta^1 \to Y$. This is a 'higher homotopy'.

5.3. **Tot of a cosimplicial space.** A cosimplicial space is a functor $X: \Delta \to \mathcal{T}op$, drawn as follows:

$$X_0 \rightrightarrows X_1 \overset{\to}{\rightrightarrows} X_2 \ \substack{\rightrightarrows\\ \rightrightarrows} \cdots$$

(and here we are omitting the codegeneracy maps for typographical reasons). Let Δ^* denote the cosimplicial space corresponding to the standard inclusion $\Delta \hookrightarrow \mathcal{T}op$. As a cosimplicial space, Δ^* is

$$\Delta^0 \rightrightarrows \Delta^1 \overset{\to}{\rightrightarrows} \Delta^2 \ \substack{\rightrightarrows\\ \rightrightarrows} \cdots$$

If X is any cosimplicial space we can talk about the space of maps from Δ^* to X: the points are the natural transformations $\Delta^* \to X$, and they are topologized as a subspace of $\prod_n X_n^{\Delta^n}$. This space of maps is sometimes denoted $\mathrm{Map}(\Delta^*, X)$, but is more commonly denoted $\mathrm{Tot}\, X$. It is called the **totalization** of X, or usually just "Tot of X", for short. We can also describe it as an equalizer:

$$\mathrm{Tot}\, X = \mathrm{eq}\left[\prod_n X_n^{\Delta^n} \rightrightarrows \prod_{[n]\to[k]} X_k^{\Delta^n}\right].$$

The two maps in the equalizer can be defined as follows, using that any map $\sigma: [n] \to [k]$ induces a corresponding map $\sigma_*: \Delta^n \to \Delta^k$. Given a sequence of elements $s_n \in X_n^{\Delta^n}$, one of our maps sends this sequence to the collection $\sigma \mapsto s_k \circ \sigma_* \in X_k^{\Delta^n}$. The other map sends the sequence s_n to the collection $\sigma \mapsto X(\sigma) \circ s_n \in X_k^{\Delta^n}$, where $X(\sigma)$ is the induced map $X_n \to X_k$.

In words, a point in $\mathrm{Tot}\, X$ consists of a point $x_0 \in X_0$, an edge x_1 in X_1, a 2-simplex x_2 in X_2, and so on, which are 'compatible' in the following two ways:

(1) The two images of x_0 under $X_0 \rightrightarrows X_1$ are the two endpoints of x_1; the three images of x_1 under the maps $d^0, d^1, d^2: X_1 \to X_2$ are the three faces of the 2-simplex x_2; and so on.

(2) The image of x_1 under the codegeneracy $X_1 \to X_0$ is the map $\Delta^1 \to X_0$ collapsing everything to x_0; the image of x_2 under the two codegeneracies $X_2 \rightrightarrows X_1$ are the two maps $\Delta^2 \rightrightarrows \Delta^1 \xrightarrow{x_1} X_1$, etc.

There doesn't seem to be a particularly simple way to think about all this! Usually I think of a point in $\mathrm{Tot}\, X$ as being a point $x_0 \in X_0$ plus an edge connecting its two images in X_1, plus a 2-simplex connecting the three images of this edge in X_2, and so on, with the proviso that all this data must be compatible under the codegeneracies.

Note that there is an evident map $\mathrm{eq}(X_0 \rightrightarrows X_1) \to \mathrm{Tot}\, X$ defined as follows. If $x_0 \in X_0$ is equalized by the two maps to X_1, then we can choose our 1-simplex x_1 in X_1 to be constant. Then we can also choose our 2-simplex in X_2 to be constant, and so on down the line. All of these choices are automatically compatible under codegeneracies, so we get a point in $\mathrm{Tot}\, X$.

5.4. **Reedy fibrancy.** It is not true that if $X \to Y$ is an objectwise weak equivalence between cosimplicial objects then $\mathrm{Tot}\, X \to TotY$ is a weak equivalence. It *is* true if X and Y satisfy some conditions, which we now explain.

Let X be a cosimplicial object and let $a \in X_n$. Applying the codegeneracy maps to a gives an n-tuple $(s^0 a, s^1 a, \ldots, s^{n-1} a) \in (X_{n-1})^n$. This is not an arbitrary

n-tuple, as the cosimplicial identities give us some relations among the coordinates. If we relabel this n-tuple as $(x_0, \ldots, x_{n-1})$, we find that $s^i x_i = s^i x_{i+1}$ for each i in the range $0 \leq i \leq n-2$. The ***n*th matching object** of X is the subspace of all n-tuples satisfying these relations; that is,

$$M_n X = \{(y_0, y_1, \ldots, y_{n-1}) \in (X_{n-1})^n \mid s^i y_i = s^i y_{i+1} \text{ for } 0 \leq i \leq n-2\}.$$

The map $X_n \to M_n X$ sending a to $(s^0 a, \ldots, s^{n-1} a)$ is called the ***n*th matching map**.

Definition 5.5. *A cosimplicial space is* ***Reedy fibrant*** *if the matching maps $X_n \to M_n X$ are fibrations, for all $n \geq 0$.*

Proposition 5.6. *Let $X \to Y$ be an objectwise weak equivalence between cosimplicial spaces, each of which is Reedy fibrant. Then $\operatorname{Tot} X \to \operatorname{Tot} Y$ is a weak equivalence of spaces.*

Proof. See [BK, ????]. □

5.7. **Construction of homotopy limits.** Let I be a small category and $D\colon I \to \mathcal{T}op$ a diagram. The **cosimplicial replacement** of D is the cosimplicial space $\operatorname{crep}(D)$ defined as

$$\operatorname{crep}(D)_n = \prod_{i_0 \to i_1 \to \cdots \to i_n} D(i_n).$$

The cofaces and codegeneracies are the evident ones, defined analogously to the case of simplicial replacements.

The cosimplicial replacement of a diagram is always Reedy fibrant, provided that the diagram was objectwise fibrant (which is always true in $\mathcal{T}op$, since all spaces are fibrant). So one defines the homotopy limit of D by

$$\operatorname{holim} D = \operatorname{Tot}[\operatorname{crep}(D)].$$

It readily follows from Proposition 5.6 that this construction is homotopy invariant.

The equalizer of $\operatorname{crep}(D)_0 \rightrightarrows \operatorname{crep}(D)_1$ is just $\lim D$; a point in this equalizer consists of a choice of point in each D_i which are compatible as one moves around the diagram. The natural map from this equalizer into $\operatorname{Tot}(\operatorname{crep}(D))$ gives us a natural map $\lim D \to \operatorname{holim} D$.

Just as for homotopy colimits, we can describe $\operatorname{holim} D$ via another formula—this time an equalizer formula:

$$\operatorname{holim} D \cong \operatorname{eq}\left[\prod_i X_i \times B(i \downarrow I) \rightrightarrows \prod_{i \to j} X_j \times B(i \downarrow I)\right].$$

5.8. **Adjunctions.** If $D\colon I \to \mathcal{T}op$ and $X \in \mathcal{T}op$, there is a useful adjunction formula

$$\mathcal{T}op(\operatorname*{colim}_I D, X) \cong \lim_I \mathcal{T}op(D(i), X).$$

Here $\mathcal{T}op(A, B)$ denotes the set of maps from A to B in the category $\mathcal{T}op$. The formula just says that giving a map $\operatorname{colim} D \to X$ is the same as giving a bunch of maps $D(i) \to X$ which are compatible as i changes. There is a similar formula

$$\mathcal{T}op(A, \lim_I D) \cong \lim_I \mathcal{T}op(A, D(i))$$

which has an analogous interpretation.

When generalizing to homotopy limits and colimits, the difference is that one replaces the set of maps in $\mathcal{T}op$ with the mapping space. For this we need to assume we are working in a 'good' category of spaces where the mapping space is a true right adjoint (like the category of compactly-generated spaces). We then have natural maps

$$\operatorname{Map}(\operatorname{hocolim}_I D, X) \to \operatorname{holim}_{I^{op}} \operatorname{Map}(D(i), X) \tag{5.9}$$

and

$$\operatorname{Map}(A, \operatorname{holim}_I D) \to \operatorname{holim}_I \operatorname{Map}(A, D(i)). \tag{5.10}$$

We'll explain the map in (5.9), as the other one is similar. Using the description of hocolim D from (4.14), we have maps

$$\begin{array}{c}
\operatorname{Map}(\operatorname{hocolim} D, Z) \\
\downarrow \cong \\
\operatorname{eq}\left[\operatorname{Map}\left(\coprod_i D_i \times B(i \downarrow I)^{op}, Z\right) \rightrightarrows \operatorname{Map}\left(\coprod_{i\to j} D_i \times B(j \downarrow I)^{op}, Z\right)\right] \\
\downarrow \cong \\
\operatorname{eq}\left[\prod_i \operatorname{Map}\left(D_i \times B(i \downarrow I)^{op}, Z\right) \rightrightarrows \prod_{i\to j} \operatorname{Map}\left(D_i \times B(j \downarrow I)^{op}, Z\right)\right] \\
\downarrow \cong \\
\operatorname{eq}\left[\prod_i \operatorname{Map}\left(D_i, Z\right)^{B(i\downarrow I)^{op}} \rightrightarrows \prod_{i\to j} \operatorname{Map}\left(D_i, Z\right)^{B(j\downarrow I)^{op}}\right] \\
\downarrow \cong \\
\operatorname{eq}\left[\prod_i \operatorname{Map}\left(D_i, Z\right)^{B(I^{op}\downarrow i)} \rightrightarrows \prod_{i\to j} \operatorname{Map}\left(D_i, Z\right)^{B(I^{op}\downarrow j)}\right] \\
\| \\
\operatorname{holim}_{I^{op}} \operatorname{Map}(D(i), Z).
\end{array}$$

In the first two maps we are using that $\operatorname{Map}(-, Z)$ takes colimits to limits, which follows from the adjointness properties. The third map just uses the adjunction, and in the fourth map we have used the identification $(i \downarrow I)^{op} = (I^{op} \downarrow i)$.

6. Changing the indexing category

As mentioned briefly in Section 2.3, one is often in the situation of wanting to prove that the homotopy colimits of two different diagrams are weakly equivalent. There are a variety of techniques for this, and we will describe a few in this section. Unfortunately, the proofs of these results require more technology than is yet at our disposal—so we will defer the proofs until Section 9.

Let $\alpha\colon I \to J$ be any functor between small categories. Then given any diagram $X : J \to \mathcal{T}op$, one obtains a new diagram $\alpha^* X : I \to \mathcal{T}op$ by $\alpha^* X = X \circ \alpha$. We wish to compare $\operatorname{hocolim}_J X$ with $\operatorname{hocolim}_I(\alpha^* X)$. In particular, under what conditions will they be weakly equivalent?

6.1. **The classical problem for colimits.** The corresponding problem in the case of ordinary colimits is probably familiar. There is a canonical map

$$\operatorname*{colim}_I(\alpha^* X) \to \operatorname*{colim}_J X$$

and one wants to know when this is an isomorphism. A common situation is that I is a subcategory of J, and one usual definition for I to be 'cofinal' in J is something like:

(1) For each $j \in J$, there is an $i \in I$ and a map $j \to i$.
(2) For any two parallel maps $j \rightrightarrows i$ where $i \in I$, there is a map $i \to i'$ in I such that the two composites $j \to i'$ are the same.

This is actually a special case of a much more general definition. Reall that for any $j \in J$, the overcategory $(j \downarrow \alpha)$ is the category whose objects are pairs (i, f) consisting of an object $i \in I$ and a map $f\colon j \to \alpha(i)$ in J. A map from (i, f) to (i', f') consists of a map $i \to i'$ in I making the diagram

$$\begin{array}{ccc} j & \xrightarrow{f} & \alpha(i) \\ & \searrow^{f'} & \downarrow \\ & & \alpha(i') \end{array}$$

commute.

Definition 6.2. *The functor $\alpha\colon I \to J$ is **terminal** (or **final**, or **left cofinal**) if for each $j \in J$ the overcategory $(j \downarrow \alpha)$ is non-empty and connected.*

Theorem 6.3. *If α is terminal then for every diagram $X\colon J \to \mathcal{T}op$, the map $\operatorname{colim}_I(\alpha^* X) \to \operatorname{colim}_J X$ is an isomorphism.*

Proof. See [ML, Thm. IX.3.1]. □

Remark 6.4. There is a nice way to remember the above definition and theorem. One particularly simple case is when J has a terminal object w, and $I = \{w\}$ is the subcategory consisting of this single object. In this case it's clear that $\operatorname{colim}_J X$ should just be $X(w)$, which is $\operatorname{colim}_I(\alpha^* X)$.

The condition for being a terminal object is that the overcategories $(j \downarrow \{w\})$ are trivial categories consisting of one object and an identity map. This is a very special case of the connectedness condition above.

6.5. **Extension to the case of homotopy colimits.** Let $\alpha\colon I \to J$ be a functor between small categories. We first note that for diagrams $X\colon J \to \mathcal{T}op$ there is a natural map of simplicial spaces

$$\operatorname{srep}(\alpha^* X) \to \operatorname{srep}(X).$$

In dimension n this is the map

$$\coprod_{i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n} (\alpha^* X)(i_n) \longrightarrow \coprod_{j_0 \leftarrow j_1 \leftarrow \cdots \leftarrow j_n} X(j_n)$$

which sends the summand $(\alpha^* X)(i_n)$ corresponding to the chain $[i_0 \leftarrow \cdots \leftarrow i_n]$ to the summand $X(i_0)$ corresponding to the chain $[\alpha(i_0) \leftarrow \cdots \leftarrow \alpha(i_n)]$. Note that $(\alpha^* X)(i_n) = X(\alpha(i_n))$, and the map is really just the identity on these summands. This is clearly compatible with the face and degeneracy maps, and so gives a map of simplicial spaces.

Taking realizations gives us a natural map

$$\operatorname*{hocolim}_I \alpha^* X \to \operatorname*{hocolim}_J X.$$

Definition 6.6. *The functor $\alpha\colon I \to J$ is* ***homotopy terminal*** *(or* ***homotopy final****, or* ***homotopy left cofinal****) if for each $j \in J$ the overcategory $(j \downarrow \alpha)$ is non-empty and contractible (meaning that its nerve is contractible).*

See Remark 6.13 for more about the above choices in terminology.

Theorem 6.7 (Cofinality Theorem)**.** *If α is homotopy terminal then for every diagram $X\colon J \to \mathcal{T}op$, the map $\operatorname{hocolim}_I(\alpha^* X) \to \operatorname{hocolim}_J X$ is a weak equivalence.*

Proof. See Sections 9.6 and 10. □

There is one special case of Theorem 6.7 which we *will* prove now, both because the proof is simple and because we will need it later.

Lemma 6.8. *Suppose that J has a terminal object z. Then for every diagram $X\colon J \to \mathcal{T}op$, the map $\operatorname{hocolim}_J X \to \operatorname{colim}_J X \to X(z)$ is a weak equivalence.*

Proof. Consider the simplicial space $\operatorname{srep}(X)$. There is an evident augmentation $\operatorname{srep}(X) \to X(z)$, and we claim that this augmented simplicial space admits a backwards contracting homotopy (see Definition 3.11). The contraction $S\colon \operatorname{srep}(X)_n \to \operatorname{srep}(X)_{n+1}$ will send the summand $X(i_n)$ labelled by $i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n$ to the summand $X(i_n)$ labelled by $z \leftarrow i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n$. It is routine to check that this satisfies the identities for a contracting homotopy, and therefore by Proposition 3.12 we find that $|\operatorname{srep}(X)| \to X(z)$ is a homotopy equivalence. □

It is often useful to know how the maps α^* behave under composition. Suppose now that $I_1 \xrightarrow{\alpha} I_2 \xrightarrow{\beta} I_3$ are two functors between categories, and that $X\colon I_3 \to \mathcal{T}op$ is a diagram. We have three natural maps of cosimplicial spaces, forming a triangle which is readily checked to commute:

$$\begin{array}{ccc} \operatorname{crep}[(\beta\alpha)^* X] & \longrightarrow & \operatorname{crep}(\beta^* X) \\ & \searrow & \downarrow \\ & & \operatorname{crep}(X). \end{array}$$

This yields a commutative triangle of homotopy colimits:

$$\begin{array}{ccc} \operatorname{hocolim}_{I_1}(\beta\alpha)^*X & \xrightarrow{\alpha_*} & \operatorname{hocolim}_{I_2}\alpha^*X \\ & \searrow^{(\beta\alpha)_*} & \downarrow{\beta_*} \\ & & \operatorname{hocolim}_{I_3} X. \end{array}$$

Here is another result about changing the indexing category. Suppose again that $\alpha\colon I \to J$ is a functor and $X\colon J \to \mathcal{T}op$. For each $j \in J$, let $u_j\colon (\alpha \downarrow j) \to J$ be the map sending $[i, \alpha(i) \to j] \mapsto \alpha(i)$. Notice that there is a canonical map

$$\operatorname*{colim}_{(\alpha\downarrow j)} u_j^* X \to X_j.$$

Theorem 6.9. *Let $\alpha\colon I \to J$ be a functor, and let $X\colon J \to \mathcal{T}op$. Suppose that for each $j \in J$ the composite map*

$$\operatorname*{hocolim}_{(\alpha\downarrow j)} u^*X \to \operatorname*{colim}_{(\alpha\downarrow j)} u_j^*X \to X_j$$

is a weak equivalence. Then the map $\operatorname{hocolim}_I \alpha^*X \to \operatorname{hocolim}_J X$ is a weak equivalence.

Proof. See Section 9.6. □

6.10. **Dual results for homotopy limits.** Suppose $\alpha\colon I \to J$ and $X\colon J \to \mathcal{T}op$. There is a natural map of cosimplicial spaces

$$\operatorname{crep}(X) \to \operatorname{crep}(\alpha^*X),$$

and after taking Tot this gives a map $\alpha^*\colon \operatorname{holim}_J X \to \operatorname{holim}_I(\alpha^*X)$.

Definition 6.11. *The functor $\alpha\colon I \to J$ is **homotopy initial** (or **homotopy cofinal**, or **homotopy right cofinal**) if for each $j \in J$ the overcategory $(\alpha \downarrow j)$ is non-empty and contractible (meaning that its nerve is contractible).*

Theorem 6.12. *If α is homotopy initial then for every diagram $X\colon J \to \mathcal{T}op$, the map $\operatorname{holim}_J X \to \operatorname{holim}_I(\alpha^*X)$ is a weak equivalence.*

Remark 6.13. The terms 'final/cofinal' and—even worse—'left/right cofinal' are easily mixed up, and it's also easy to mix up which one goes with colimits and which one goes with limits. The terms 'homotopy initial' and 'homotopy terminal' are better in this regard, as they fit naturally with the notions of initial and terminal object.

If a category has a terminal object, it is easy to compute the homotopy *colimit*. The condition that a category I has a terminal object ω says something about the overcategories $(i \downarrow \omega)$; likewise, the condition that a functor $\alpha\colon K \to I$ be homotopy terminal says something about the overcategories $(i \downarrow \alpha)$. So the adjective 'terminal' lets one remember how to connect all these concepts (and likewise for 'initial').

One also has the following analog of Theorem 6.9:

Theorem 6.14. *Let $\alpha\colon I \to J$ be a functor, and let $X\colon J \to \mathcal{T}op$. Suppose that for each $j \in J$ the composite map*

$$X_j \to \lim_{(j\downarrow\alpha)} u_j^*X \to \operatorname*{holim}_{(j\downarrow\alpha)} u^*X$$

is a weak equivalence. Then the map $\operatorname{holim}_J X \to \operatorname{holim}_I \alpha^*X$ is a weak equivalence.

6.15. **Further techniques.** We give one more result related to changing the indexing category. We'll only state the hocolim version; the holim version is entirely analogous.

Suppose that $\alpha, \alpha' \colon I \to J$ are two functors and $\eta \colon \alpha \to \alpha'$ is a natural transformation. If $X \colon J \to \mathcal{T}op$ then η induces a natural transformation $\eta_* \colon \alpha^* X \to (\alpha')^* X$. The following triangle commutes in the homotopy category $\mathrm{Ho}\,(\mathcal{T}op)$:

$$\begin{array}{ccc} \operatorname{hocolim}_I \alpha^* X & \xrightarrow{\alpha_*} & \operatorname{hocolim}_J X \\ {\scriptstyle \eta_*}\downarrow & \nearrow{\scriptstyle \alpha'_*} & \\ \operatorname{hocolim}_I (\alpha')^* X. & & \end{array}$$

Proposition 6.16. *Let $\alpha \colon I \to J$ be a functor between small categories, and let $X \colon J \to \mathcal{T}op$ be a diagram. Suppose that there is a functor $\beta \colon J \to I$ together with natural transformations $\eta \colon \alpha\beta \to id_J$ and $\theta \colon \beta\alpha \to id_I$ such that the following two conditions hold:*

(1) Applying X to the maps $\eta(j) \colon \alpha\beta(j) \to j$ yields weak equivalences, for all $j \in J$; and

(2) Applying $\alpha^* X$ to the maps $\theta(i) \colon \beta\alpha(i) \to i$ also yields weak equivalences, for all $i \in I$.

Then the induced map $\operatorname{hocolim}_I \alpha^* X \to \operatorname{hocolim}_J X$ *is a weak equivalence.*

Moreover, the same conclusion holds if there are zig-zags of natural transformations between $\beta\alpha$ and id_I, and between $\alpha\beta$ and id_J, provided each step in the zig-zags induces weak equivalences after applying X and $\alpha^* X$, respectively.

Proof. See [D, Proposition A.4]. □

Part 2. A closer look

So far we have understood the homotopy colimit as a 'fattened up' version of the colimit. Whereas taking a colimit can be thought of as gluing objects together, taking a homotopy colimit amounts to indirectly gluing them together via homotopies and higher homotopies. We saw that this process can be described by a certain formula (the geometric realization of the simplicial replacement), which is not hard to describe but perhaps not so easy to manipulate.

In the next few sections we will take a closer look at this 'formulaic' approach to homotopy colimits, and we'll encounter several variations of the main idea. The ostensible goal will be to learn some clever techniques for manipulating these formulas, but along the way we will make discoveries which slowly take us further and further away from the formulaic perspective. In Part 3 we will then take up those discoveries from a more abstract point-of-view.

There is a central theme which drives most of what follows. Given a diagram $D\colon I \to \mathcal{T}op$, there is a way of constructing the homotopy colimit by first replacing D with an 'equivalent' diagram $QD\colon I \to \mathcal{T}op$ (having $QD_i \simeq D_i$ for each i) and then taking the ordinary colimit of QD. The diagram QD is in some sense a *resolution* of D, and this leads us to view the homotopy colimit as a derived functor of the colimit. When we first encounter this idea in Section 8 it might seem like there's not much content to it—-we are just rewriting the old formula for the homotopy colimit in a different way. But the power of homological (or homotopical) algebra comes in realizing that one doesn't have to use the same resolution every time; *any* nice enough resolution will do the job. So in the end this new way of looking at things will prove very useful.

7. Brief review of model categories

Model categories will weave their way in and out of the next few sections. They have proven themselves to be a valuable ally when dealing with derived functors and homotopical algebra.

We will not recall the notion of a model category here. The reader may consult [DS], [H], or [Ho] for nice overviews. Suffice it to say that a model category $\mathcal{M}$ is a category equipped with three collections of maps—the cofibrations, fibrations, and weak equivalences—which are required to satisfy five basic axioms. A map is called a 'trivial cofibration' if it is both a cofibration and a weak equivalence, and similarly for 'trivial fibration'.

The basic examples are as follows:

(1) $\mathcal{T}op$, where the weak equivalences are weak homotopy equivalences, the fibrations are Serre fibrations, and the cofibrations are retracts of cellular inclusions.
(2) $s\mathcal{S}et$, where the weak equivalences are the maps which become weak homotopy equivalences after geometric realization. The fibrations are the Kan fibrations, and the cofibrations are the monomorphisms.
(3) $Ch_{\geq 0}(R)$, where R is a ring. This is the category of non-negatively graded chain complexes. The weak equivalences are the quasi-isomorphisms, the fibrations are maps which are surjective in positive dimensions, and the cofibrations are the monomorphisms which in each level are split with projective cokernel.
(4) $Ch_{\leq 0}(R)$, where R is a ring. This is the category of non-positively graded chain complexes (or cochain complexes, after re-indexing). The weak equivalences

are the quasi-isomorphisms, the cofibrations are the monomorphisms, and the fibrations are the surjections which in each level are split with injective kernel.

There are many other examples, for instance several different model categories of *spectra*.

7.1. **Quillen functors.** If $\mathcal{M}$ and $\mathcal{N}$ are two model categories, a **Quillen pair** is an adjoint pair

$$L\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon R$$

which satisfies the following two equivalent conditions:

(1) L preserves cofibrations and trivial cofibrations—that is so say, if f is a cofibration (resp. trivial cofibration) in $\mathcal{M}$ then $L(f)$ is a cofibration (resp. trivial cofibration) in $\mathcal{N}$.
(2) R preserves fibrations and trivial fibrations.

The most familiar example is the adjoint pair

$$|-|\colon s\mathcal{S}et \rightleftarrows \mathcal{T}op\colon \text{Sing}$$

where $|-|$ is geometric realization and Sing is the functor which sends a space X to the simplicial set $[n] \mapsto \mathcal{T}op(\Delta^n, X)$.

One can prove that when (L, R) is a Quillen pair, L preserves weak equivalences between cofibrant objects and R preserves weak equivalences between fibrant objects. The 'derived functor' of L applied to an object $A \in \mathcal{M}$ is obtained by choosing a weak equivalence $QA \to A$ in which QA is cofibrant, and then applying L to QA. If $Q'A \to A$ is another weak equivalence in which $Q'A$ is cofibrant, then the model category axioms show that there is a weak equivalence $QA \to Q'A$; thus, $L(QA) \to L(Q'A)$ is also a weak equivalence. This tells us that the derived functor of L gives a well-defined homotopy type.

Similarly, the derived functor of R applied to an object $Z \in \mathcal{N}$ is obtained by choosing a weak equivalence $Z \to FZ$ in which FZ is fibrant, and then applying R to FZ.

8. The derived functor perspective

In this section we explain a sense in which the homotopy colimit is the derived functor of the colimit functor. We also discuss a universal property (of sorts) enjoyed by the homotopy colimit.

Example 8.1. To motivate what follows, we return to our basic example of a pushout diagram $X \xleftarrow{f} A \xrightarrow{g} Y$. Recall that the homotopy pushout consists of a copy of X, a copy of Y, and a cylinder $A \times I$ in which the two ends of the cylinder have been glued to X and Y via the maps f and g. We can arrive at this construction in a different way, as follows.

Let $\mathrm{Cyl}(f)$ and $\mathrm{Cyl}(g)$ denote the mapping cylinders of f and g; for example, the former is the quotient space $[X \amalg (A \times I)]/\sim$ where $(a, 0) \sim f(a)$. Let $i\colon A \hookrightarrow \mathrm{Cyl}(f)$ denote the inclusion $a \mapsto (a, 1)$, and let $j\colon A \hookrightarrow \mathrm{Cyl}(g)$ be defined similarly. We have the new pushout diagram of the form $\mathrm{Cyl}(f) \xleftarrow{i} A \xrightarrow{j} \mathrm{Cyl}(g)$; let's call this new diagram QD. Note that there is a natural weak equivalence $QD \to D$ obtained by collapsing the cylinders, and that the colimit of QD is $\operatorname{hocolim} D$.

To summarize, we have found the following prescription for constructing the homotopy colimit. First replace the diagram D by a new one QD in which one adds homotopies to the objects in a certain way, without affecting their homotopy type. Sometimes this is called 'fattening up' the diagram D. The homotopy colimit of D is then just the colimit of the new diagram QD.

8.2. **Construction of QX.** We'll next explain how to adapt the above example to the general case. Let $X\colon I \to \mathcal{T}op$ be a diagram. Basically what we want to do is replace each object X_i with the homotopy colimit of all the objects in the diagram mapping *to* X_i. To say this precisely, for each $i \in I$ consider the classifying space $(I \downarrow i)$ and the forgetful functor $u_i\colon (I \downarrow i) \to I$ sending the map $[j \to i]$ to j. Write

$$(QX)_i = \operatorname*{hocolim}_{(I \downarrow i)} u_i^* X.$$

Note that $(I \downarrow i)$ has a terminal object, namely the identity map $[i \to i]$. It follows from the Cofinality Theorem (6.7) that the induced map $X_i \to (QX)_i$ is a weak equivalence.

Now suppose that we have a map $f\colon i \to j$, and let u_f denote the functor $(I \downarrow i) \to (I \downarrow j)$ sending $[k \to i]$ to $[k \to j]$ (obtained by composing with f). This functor induces a map

$$(u_f)_*\colon (QX)_i = \operatorname*{hocolim}_{(I \downarrow i)} u_i^* X \to \operatorname*{hocolim}_{(I \downarrow j)} u_j^* X = (QX)_j$$

If $i \xrightarrow{f} j \xrightarrow{g} k$ are two maps in I then we have a commutative diagram

$$\begin{array}{ccc} (I \downarrow i) & \xrightarrow{u_f} & (I \downarrow j) \\ & {\scriptstyle u_{gf}} \searrow & \downarrow {\scriptstyle u_g} \\ & & (I \downarrow k) \end{array}$$

and therefore get a commutative triangle

$$\begin{array}{ccc} (QX)_i & \xrightarrow{(u_f)_*} & (QX)_j \\ & \searrow^{(u_{gf})_*} & \downarrow{\scriptstyle (u_g)_*} \\ & & (QX)_k. \end{array}$$

So QX is a new diagram $I \to \mathcal{T}op$.

The natural maps $\operatorname{hocolim}_{(I\downarrow i)} u_i^* X \to \operatorname{colim}_{(I\downarrow i)} u_i^* X \cong X_i$ compile to give a map of diagrams $QX \to X$. We observed above that we have weak equivalences $X_i \to QX_i$ coming from the terminal object of $(I \downarrow i)$, but these are not compatible as i varies. But the composite $X_i \to (QX)_i \to X_i$ is the identity, and so it follows that $(QX)_i \to X_i$ is a weak equivalence as well. Thus, $QX \to X$ is an objectwise weak equivalence.

Our final claim is that $\operatorname{colim}_I(QX) \cong \operatorname{hocolim}_I X$. It is not so hard to just think about it and see that this must be true. We will be able to explain it better after a brief detour, though.

8.3. **Homotopy coherent maps and the universal property.** Let $X, Y \colon I \to \mathcal{T}op$ be two diagrams. A map of diagrams $X \to Y$ consists of a collection of maps $X_i \to Y_i$ which are compatible as i varies. A **homotopy coherent** map $X \to Y$ consists of a collection of maps $X_i \to Y_i$ (which might not be compatible as i varies), *together with* the following data:

(1) For every map $i \to j$ in I, a homotopy $X_i \times \Delta^1 \to Y_j$ between the composites $X_i \to X_j \to Y_j$ and $X_i \to Y_i \to Y_j$.
(2) For every composable pair $i \to j \to k$, a map $X_i \times \Delta^2 \to Y_k$ whose restriction to $X_i \times \partial\Delta^1$ gives the three homotopies corresponding to the maps $i \to j$, $i \to k$, and $j \to k$.
(3) For every chain of n morphisms $i_0 \to i_1 \to \cdots \to i_n$, a map $X_{i_0} \times \Delta^n \to Y_{i_n}$ which extends previous data on the subspace $X_{i_0} \times \partial\Delta^n$.

Of course we have been very sloppy in writing down the third condition. One way to be more rigorous is as follows. One can form a cosimplicial space $\operatorname{Map}(X, Y)$

$$\prod_i \operatorname{Map}(X_i, Y_i) \rightrightarrows \prod_{i_0 \to i_1} \operatorname{Map}(X_{i_0}, Y_{i_1}) \Rrightarrow \prod_{i_0 \to i_1 \to i_2} \operatorname{Map}(X_{i_0}, Y_{i_2}) \cdots \ldots$$

and a homotopy coherent map $X \to Y$ is precisely a point in Tot of this cosimplicial space.

Remark 8.4. Let Z be a space. To give a map $\operatorname{colim} X \to Z$ is equivalent to giving a map of diagrams $X \to cZ$, where cZ is the constant diagram containing Z at every spot (and all identity maps). The reader may check that to give a map $\operatorname{hocolim} X \to Z$ is the same as giving a *homotopy coherent* map of diagrams $X \to cZ$. This can be thought of as the 'universal property' for homotopy colimits.

Let $\operatorname{hc}(X, Y)$ denote the set of homotopy coherent maps from X to Y. Note that maps $X' \to X$ and $Y \to Y'$ give maps $\operatorname{hc}(X, Y) \to \operatorname{hc}(X', Y)$ and $\operatorname{hc}(X, Y) \to \operatorname{hc}(X, Y')$ in the evident way.

Let QX denote the diagram constructed in the previous section. We claim that to give a map of diagrams $QX \to Y$ is the same as giving a homotopy coherent map $X \to Y$:

Proposition 8.5. *There is a natural bijection between* $\mathcal{T}op^I(QX, Y)$ *and* $\mathrm{hc}(X, Y)$*.*

Proof. This is just a matter of chasing through the definitions. □

Remark 8.6. We can now explain why $\mathrm{colim}_I\, QX \cong \mathrm{hocolim}_I\, X$. To give a map $\mathrm{colim}_I\, QX \to Z$ is to give, for each $i \in I$, maps $\mathrm{hocolim}_{(I\downarrow i)}\, X \to Z$ which are compatible as i varies. This is the same as giving, for each $i \in I$, a homotopy coherent map $X|_{(I\downarrow i)} \to (cZ)|_{(I\downarrow i)}$, which are again compatible as i varies. But clearly this is the same thing as just giving a homotopy coherent map $X \to cZ$! Since this is in turn the same as giving a map $\mathrm{hocolim}_I\, X \to Z$, it follows that $\mathrm{colim}_I\, QX \cong \mathrm{hocolim}_I\, X$.

Said differently, for any space Z we have a sequence of natural bijections

$$\mathcal{T}op(\mathrm{colim}\, QX, Z) \cong \mathcal{T}op^I(QX, cZ) \cong \mathrm{hc}(X, cZ) \cong \mathcal{T}op(\mathrm{hocolim}\, X, Z).$$

This implies that $\mathrm{colim}\, QX \cong \mathrm{hocolim}\, X$.

For future reference we make the following observation.

Proposition 8.7. *Let* $E \to E'$ *be an objectwise trivial fibration of* I*-diagrams, and let* D *be an objectwise cofibrant* I*-diagram. Then* $\mathrm{hc}(D, E) \to \mathrm{hc}(D, E')$ *is surjective.*

Proof. The proof is a straightforward induction. Suppose $D \to E'$ is a homotopy coherent map. The maps $D_i \to E'_i$ lift to maps $D_i \to E_i$. Then for each map $i \to j$ in I we have a diagram

$$\begin{array}{ccc} D_i \times \partial\Delta^1 & \longrightarrow & E_j \\ \downarrow & & \downarrow \\ D_i \times \Delta^2 & \longrightarrow & E'_j \end{array}$$

and we get a lifting because the left vertical map is a cofibration and the right vertical map is a trivial fibration. And so on. □

8.8. **Model categories of diagrams.** Model categories provide a very useful way for understanding the derived functor perspective on homotopy colimits. Let $\mathcal{T}op^I$ denote the category of diagrams $I \to \mathcal{T}op$, where the maps are natural transformations. It turns out that $\mathcal{T}op^I$ has a model category structure in which a map $X \to Y$ is a

(1) weak equivalence if and only if each $X_i \to Y_i$ is a weak equivalence, and
(2) a fibration if and only if each $X_i \to Y_i$ is a fibration.

The cofibrations are a bit awkward to describe, but they are the maps with the left-lifting-property with respect to the trivial fibrations. We will talk more about the cofibrant objects in Section ?????.

There are adjoint functors

$$\mathrm{colim}\colon \mathcal{T}op^I \rightleftarrows \mathcal{T}op\colon c$$

where c is the constant diagram functor. Clearly c preserves fibrations and trivial fibrations, so this is a Quillen pair. To compute the derived functor of colim applied to a diagram X, one first chooses a weak equivalence $\hat{X} \to X$ where $\hat{X}$ is cofibrant, and then $\mathbf{L}\,\mathrm{colim}(X)$ is just $\mathrm{colim}\, \hat{X}$.

We claim that QX is precisely a cofibrant-replacement for X in $\mathcal{T}op^I$. Since $\mathrm{colim}\, QX \cong \mathrm{hocolim}\, X$, this identifies hocolim with the derived functor of colim.

We have already remarked that $QX \to X$ is an objectwise weak equivalence, so we just need to prove that QX is cofibrant. Let $W \to Z$ be an objectwise trivial fibration. Then the map $\mathcal{T}op^I(QX, W) \to \mathcal{T}op^I(QX, Z)$ is isomorphic to $\mathrm{hc}(X, W) \to \mathrm{hc}(X, Z)$. By Proposition 8.7, this is surjective; so we have a lift $QX \to W$ as desired.

Remark 8.9. Below it will be useful to have a name for a construction parallel to QX. Namely, if $D\colon I \to \mathcal{T}op$ is a diagram let

$$\operatorname{hocolim}_I\text{'}\, D = \operatorname{coeq}\left[\coprod_{i\to j} D_i \times B(j\downarrow I) \rightrightarrows \coprod_i D_i \times B(i\downarrow I)\right]$$

(this is the same formula as (4.14), but without the "op"'s on the overcategories). Then, if $X\colon I \to \mathcal{T}op$ is a diagram define $Q'X$ to be the diagram

$$i \mapsto \operatorname{hocolim}_{(I\downarrow i)}\text{'}(u_i^* X).$$

Repeating the arguments from above, one finds that $Q'X$ is also a cofibrant-replacement for X in $\mathcal{T}op^I$.

Note that if $*$ denotes the constant diagram $I \to \mathcal{T}op$ whose value is a single point, then $Q'(*)$ is the diagram $i \mapsto B(I \downarrow i)$.

8.10. **Tensor products of diagrams.** Suppose $X\colon I \to \mathcal{T}op$ and $\Omega\colon I^{op} \to \mathcal{T}op$. The tensor product $X \otimes \Omega$ is defined to be

$$X \otimes \Omega = \operatorname{coeq}\left[\coprod_{i\to j} X_i \times \Omega_j \rightrightarrows \coprod_i X_i \times \Omega_i\right].$$

This kind of construction is called a **coend**, and we have seen it several times already.

Example 8.11. A simplicial space is a functor $X\colon \Delta^{op} \to \mathcal{T}op$. If $j\colon \Delta \hookrightarrow \mathcal{T}op$ is the canonical functor, then $|X|$ is just $X \otimes j$.

If $X'\colon \Delta_f^{op} \to \mathcal{T}op$ denotes the restriction of X to Δ_f^{op}, and $j'\colon \Delta_f \to \mathcal{T}op$ is the restriction of j, then $||X||$ is $X' \otimes j'$.

Example 8.12. Let $X\colon I \to \mathcal{T}op$ be a diagram. Let $B(-\downarrow I)^{op}\colon I^{op} \to \mathcal{T}op$ denote the functor $i \mapsto B(i \downarrow I)^{op}$. Then $\operatorname{hocolim}_I D \cong X \otimes B(-\downarrow I)^{op}$.

If we fix X, the functor $X \otimes (-)$ has a nice adjointness property. Namely, it is the left adjoint to the functor $\mathcal{T}op \to \mathcal{T}op^{I^{op}}$ which sends Z to the diagram $i \mapsto \mathcal{T}op(X_i, Z)$. We'll call this functor $\operatorname{Hom}(X, -)$. Our adjoint pair is therefore

$$X \otimes (-)\colon \mathcal{T}op^{I^{op}} \rightleftarrows \mathcal{T}op\colon \operatorname{Hom}(X, -).$$

Assuming X is objectwise cofibrant, then $\operatorname{Hom}(X, -)$ takes fibrations to objectwise fibrations, and trivial fibrations to objectwise trivial fibrations. So the above is a Quillen pair. One useful consequence is that the left adjoint preserves weak equivalences between cofibrant objects.

Let B_I denote the diagram I^{op}-diagram $i \mapsto B(i \downarrow I)^{op}$. There is of course a map $B_I \to *$, and this is an objectwise weak equivalence because each category $(i \downarrow I)$ has an initial object and is therefore contractible. What's more, B_I is actually cofibrant in $\mathcal{T}op^{I^{op}}$. This is because B_I is none other than the diagram

$Q'(*)$, where $*$ denotes the constant I^{op}-diagram consisting of a point in every spot (see Remark 8.9 for Q'). That is to say, for each $i \in I$ one has

$$Q'(*)_i = B(I^{op} \downarrow i) \cong B(i \downarrow I)^{op}.$$

The second isomorphism is canonical, and so gives an isomorphism of diagrams.

So now we understand the formula for the homotopy colimit from another perspective: it came from taking a cofibrant approximation to $*$ in $\mathcal{T}op^{I^{op}}$. But model category theory now tells us that we could have used *any* cofibrant approximation to $*$, and we would have gotten something weakly equivalent (since any two cofibrant approximations are weakly equivalent, and $X \otimes (-)$ preserves weak equivalences between cofibrant objects). This is useful for obtaining other models for homotopy colimits.

Example 8.13. Recall that Δ_f denotes the subcategory of Δ consisting only of inclusions. Let $D\colon \Delta_f \to \mathcal{T}op$ denote the diagram $[n] \mapsto \Delta^n$, obtained by restricting the canonical diagram $\Delta \to \mathcal{T}op$. The map $D \to *$ is obviously an objectwise weak equivalence, and we claim additionally that D is cofibrant in $\mathcal{T}op^{\Delta_f}$. This is easy to see because if $X \xrightarrow{\sim} Y$ is an objectwise trivial fibration and $D \to Y$ is a map, then one can inductively product a lifting $D \to X$.

We conclude that if $X\colon \Delta_f^{op} \to \mathcal{T}op$ is any diagram, then $X \otimes D$ is weakly equivalent to $\operatorname{hocolim} X$ (naturally in X). But $X \otimes D$ is just the fat realization $||X||$.

This justifies our claim—from way back in Remark 3.6—that if $X \to Y$ is an objectwise weak equivalence between objectwise cofibrant simplicial spaces, that $||X|| \to ||Y||$ is necessarily a weak equivalence.

9. More on changing the indexing category

We discuss relative homotopy colimits (also called homotopy left Kan extensions), and use these to revisit the problem of changing the indexing category. Combining these new ideas with the techniques from the last section, we will be able to give proofs of two results we skipped in Part 1.

9.1. **Relative homotopy colimits.** Let $\alpha\colon I \to J$ be a functor. Let $\alpha^*\colon \mathcal{T}op^J \to \mathcal{T}op^I$ denote the functor sending a diagram $X\colon J \to \mathcal{T}op$ to the composition $I \to J \to \mathcal{T}op$. We call this functor 'restriction along α'. It has a left adjoint called the **relative colimit** or **left Kan extension**, denoted $\operatorname{colim}_{I\to J}$ or $\operatorname{colim}_\alpha$. In the case where $J = *$, the trivial category, this is the usual colimit functor.

There is a simple formula for $\operatorname{colim}_{I\to J}(A)$, where $A \in \mathcal{T}op^I$. Namely, it is the diagram in $\mathcal{T}op^J$ given by

$$j \mapsto \operatorname*{colim}_{(\alpha\downarrow j)}(u_j^* A)$$

where $u_j\colon (\alpha \downarrow j) \to I$ is the forgetful functor.

The adjoint pair

$$\operatorname*{colim}_{I\to J}\colon \mathcal{T}op^I \rightleftarrows \mathcal{T}op^J : \alpha^*$$

is a Quillen pair, as the right adjoint α^* clearly preserves fibrations and trivial fibrations. If $A\colon I \to \mathcal{T}op$ one defines the **relative homotopy colimit** (or homotopy left Kan extension) to be the J-diagram given by

$$\operatorname*{hocolim}_{I\to J} A = \operatorname*{colim}_{I\to J} QA.$$

Observe that this is the derived functor of $\operatorname{colim}_{I\to J}$.

We can also give a more explicit description of the relative homotopy colimit:

Proposition 9.2. *For $A \in \mathcal{T}op^I$, $\operatorname{hocolim}_{I\to J} A$ is the J-diagram*

$$j \mapsto \operatorname*{hocolim}_{(\alpha\downarrow j)}(u_j^* A).$$

Proof. Notice that

$$[\operatorname*{hocolim}_{I\to J} A]_j = [\operatorname*{colim}_{I\to J} QA]_j = \operatorname*{colim}_{(\alpha\downarrow j)} u_j^*(QA) \quad \text{and} \quad \operatorname*{hocolim}_{(\alpha\downarrow j)} u_j^* A = \operatorname*{colim}_{(\alpha\downarrow j)} Q(u_j^* A).$$

So it suffices to prove that $Q(u_j^*A) = u_j^*(QA)$. The former is the $(\alpha \downarrow j)$-diagram sending

$$[i, \alpha(i) \to j] \mapsto \operatorname*{hocolim}_{(\alpha\downarrow j)\downarrow[i,\alpha(i)\to j]} u_j^* A.$$

One readily checks that the category $\big((\alpha \downarrow j) \downarrow [i, \alpha(i) \to j]\big)$ may be identified with $(I \downarrow i)$, and so we are looking at the diagram

$$[i, \alpha(i) \to j] \mapsto \operatorname*{hocolim}_{(I\downarrow i)} u_i^* A.$$

But this is just $u_j^*(QA)$, so we are done. ☐

9.3. **Changing the indexing category.** Let $\alpha\colon I \to J$ be a functor. Our next goal will be to relate $\operatorname{hocolim}_I \alpha^* X$ to relative homotopy colimits. Let $\alpha^{op}\colon I^{op} \to J^{op}$ be the associated functor of opposite categories. Finally, let $B(-\downarrow \alpha)^{op}$ denote the diagram $J^{op} \to \mathcal{T}op$ sending $j \mapsto B(j \downarrow \alpha)^{op}$.

The following proposition uses the functor Q' discussed in Remark 8.9.

Proposition 9.4. *For any $\alpha\colon I \to J$ one has:*

(a) $B(-\downarrow \alpha)^{op} \cong \operatorname*{colim}_{I^{op}\to J^{op}} Q'(*)$, where $*$ is the constant I^{op}-diagram whose value is $*$.

(b) For any $X\colon I \to \mathcal{M}$, there is a natural isomorphism $\operatorname{hocolim}_I X \cong X \otimes_{I^{op}} Q'(*)$.

(c) $B(-\downarrow \alpha)^{op}$ is cofibrant in $\mathcal{T}op^{J^{op}}$.

(d) There is a natural isomorphism

$$X \otimes B(-\downarrow \alpha)^{op} \cong \operatorname*{hocolim}_I \alpha^* X.$$

Proof. For part (a) we begin by applying Proposition 9.2—or more precisely, the analogous result where every hocolim is relaced with hocolim'. This tells us that for every object j in J,

$$\Big[\operatorname*{colim}_{I^{op}\to J^{op}} Q'(*)\Big]_j \cong \operatorname*{hocolim'}_{(\alpha^{op}\downarrow j)} * \cong B(\alpha^{op}\downarrow j) = B(j\downarrow \alpha)^{op}.$$

Part (b) is an immediate consequence of (a) (using $\alpha = id$) and formula (4.14). Part (c) is also an immediate consequence of (a), since $Q'(*)$ is cofibrant in $\mathcal{T}op^{I^{op}}$ and $\operatorname{colim}_{I^{op}\to J^{op}}$ is a left Quillen functor.

Part (d) is an argument with adjunctions. For all spaces Z we have

$$\begin{aligned}
\mathcal{T}op\Big(X \otimes B(-\downarrow\alpha)^{op}, Z\Big) &\cong \mathcal{T}op\Big(X \otimes \operatorname*{colim}_{\alpha^{op}} Q'(*),\, Z\Big)\\
&\cong \mathcal{T}op^{J^{op}}\Big(\operatorname*{colim}_{\alpha^{op}} Q'(*),\, \operatorname{Hom}(X,Z)\Big)\\
&\cong \mathcal{T}op^{I^{op}}\Big(Q'(*),\, \alpha^*\operatorname{Hom}(X,Z)\Big)\\
&= \mathcal{T}op^{I^{op}}\Big(Q'(*),\, \operatorname{Hom}(\alpha^* X,Z)\Big)\\
&= \mathcal{T}op\big(\alpha^* X \otimes Q'(*),\, Z\big).
\end{aligned}$$

Since these isomorphisms are natural and hold for all spaces Z, it follows that

$$X \otimes B(-\downarrow\alpha)^{op} \cong \alpha^* X \otimes Q'(*).$$

But by (b) the object on the right is precisely $\operatorname{hocolim}_I \alpha^* X$. □

Remark 9.5. Note that in part (a) we could also have written

$$B(-\downarrow\alpha)^{op} \cong \operatorname*{hocolim}_{I^{op}\to J^{op}} *.$$

9.6. **Proof of the cofinality theorem.** We can now give two of the proofs we skipped over in Part 1.

Proof of Theorem 6.7. First note that if $\alpha\colon I \to J$ is a functor then there is a map of J^{op}-diagrams

$$B(-\downarrow\alpha)^{op} \to B(-\downarrow J)^{op}.$$

So for any diagram $X\colon J \to \mathcal{T}op$ there is an induced map

$$X \otimes B(-\downarrow\alpha)^{op} \to X \otimes B(-\downarrow J)^{op}.$$

The right object is $\operatorname{hocolim}_J X$, and by Proposition 9.4(c) the left object is $\operatorname{hocolim}_I \alpha^* X$. One checks that the above is the natural map $\operatorname{hocolim}_I \alpha^* X \to \operatorname{hocolim}_J X$.

Suppose now that $\alpha\colon I \to J$ is homotopy terminal. This means that for all $j \in J$, the space $B(j \downarrow \alpha)$ is contractible. So the map of J^{op}-diagrams

$$B(- \downarrow \alpha)^{op} \to B(- \downarrow J)^{op}$$

is an objectwise weak equivalence, since both diagrams are objectwise contractible. As $X \otimes (-)$ is a left Quillen functor, it necessarily preserves weak equivalences between cofibrant objects. So

$$X \otimes B(- \downarrow \alpha)^{op} \to X \otimes B(- \downarrow J)^{op}$$

is a weak equivalence of spaces, which is what we wanted. □

Proof of Theorem 6.9. Recall that $\alpha\colon I \to J$, $X\colon J \to \mathcal{T}op$, and we assume that for each $j \in J$ the composite

$$\operatorname*{hocolim}_{(\alpha\downarrow j)} u_j^* X \to \operatorname*{colim}_{(\alpha\downarrow j)} u_j^* X \to X_j \tag{9.7}$$

is a weak equivalence. Consider the two adjoint pairs

$$\mathcal{T}op^I \underset{\alpha^*}{\overset{\operatorname{colim}_\alpha}{\rightleftarrows}} \mathcal{T}op^J \underset{c}{\overset{\operatorname{colim}}{\rightleftarrows}} \mathcal{T}op.$$

The composite of the right adjoints is the constant diagram functor, so the composite of the left adjoints is the colimit functor.

We are starting with a diagram $X \in \mathcal{T}op^J$. Consider the composite

$$\operatorname*{colim}_\alpha(Q(\alpha^* X)) \to \operatorname*{colim}_\alpha(\alpha^* X) \to X.$$

This is a map of J-diagrams, and in spot j it is precisely the map from (9.7). So our assumption is that this map is an objectwise weak equivalence.

Now, the diagram $Q(\alpha^* X)$ is cofibrant in $\mathcal{T}op^I$. So $\operatorname{colim}_\alpha[Q(\alpha^*)]$ is cofibrant in $\mathcal{T}op^J$, as $\operatorname{colim}_\alpha$ is a left Quillen functor. So

$$\operatorname*{colim}_\alpha Q(\alpha^* X) \to QX$$

is an objectwise weak equivalence between cofibrant diagram. Applying colim to this, have the map

$$\operatorname*{hocolim}_I \alpha^* X \to \operatorname*{hocolim}_J X.$$

Since left Quillen functors preserve weak equivalences between cofibrant objects, this map is a weak equivalence and we are done. □

10. The two-sided bar construction

The material in this section is from the beautiful paper [HV]. We will see that there is a single construction which unifies almost everything we have talked about so far. Using this, one obtains very slick proofs of most of the main theorems.

10.1. **Basic definitions.** Let $\mathcal{M}$ be a simplicial model category (the reader is free to assume $\mathcal{M} = \mathcal{T}op$, but we have reason for the extra generality).

Let I be a small category and let $X\colon I \to \mathcal{M}$ and $W\colon I^{op} \to \mathcal{M}$. Define $B_\bullet(W, I, X)$ to be the simplicial object

$$[n] \mapsto \coprod_{i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n} W(i_0) \times X(i_n).$$

The face map d_j corresponds to 'covering up i_j', with two provisos. In $d_n\colon B_n(W, I, X) \to B_{n-1}(W, I, X)$ one must use the map $X(i_n) \to X(i_{n-1})$, whereas in d_0 one must use the map $W(i_0) \to W(i_1)$. The degeneracies correspond to insertion of identity maps, as we are used to. The simplicial object $B_\bullet(W, I, X)$ is called the **two-sided bar construction**.

Example 10.2. For the case $W = *$ (the constant diagram) one has $B_\bullet(*, I, X) = \text{srep}(X)$. We can also regard X as a functor $(I^{op})^{op} \to \mathcal{M}$ and thereby consider the object $B_\bullet(X, I^{op}, *)$. This is not $\text{srep}(X)$ but rather the 'other' simplicial replacement that was defined in Remark 4.2.

Let $B(W, I, X) = |B_\bullet(W, I, X)|$. Note that one has a natural map

$$B(W, I, X) \to \text{coeq}\Big[B_1(W, I, X) \rightrightarrows B_0(W, I, X)\Big] = W \otimes_I X.$$

One thinks of $B(W, I, X)$ as a fattened up version of the tensor product; or sometimes as the 'homotopy tensor product'. Note that if $X \to X'$ and $W \to W'$ are objectwise weak equivalences of objectwise-cofibrant diagrams, then the induced maps

$$B(W, I, X) \to B(W', I, X) \qquad \text{and} \qquad B(W, I, X) \to B(W, I, X')$$

are both weak equivalences. This uses the fact that $B_\bullet(W, I, X)$ is a Reedy cofibrant simplicial object, which is true for the same reason as for the simplicial replacement—the nth latching object sits inside $B_n(W, I, X)$ as a summand of the coproduct.

Note that if S is a set and $X \in \mathcal{M}$ then the notation $S \times X$ makes sense: it means the coproduct of copies of X, one for each element $s \in S$. If $S \to T$ and $X \to Y$ there are natural maps $S \times X \to T \times X$ and $S \times X \to S \times Y$. Using this observation, the construction $B_\bullet(Y, I, X)$ makes sense if $X\colon I \to \mathcal{M}$ and $Y\colon I^{op} \to \mathcal{S}et$, or if $X\colon I \to \mathcal{S}et$ and $Y\colon I^{op} \to \mathcal{M}$. It even makes sense if $X\colon I \to \mathcal{S}et$ and $Y\colon I^{op} \to \mathcal{S}et$, in which case it produces a simplicial set.

Example 10.3. $B_\bullet(*, I, *)$ is the nerve of I^{op}.

As explained in [HV], it is useful to think of the theory of diagrams as being a generalization of the theory of modules. One should think of a diagram $X\colon I \to \mathcal{M}$ as a 'left I-module', and a diagram $W\colon I^{op} \to \mathcal{M}$ as a right I-module. This is particularly satisfying if $\mathcal{M}$ is a subcategory of $\mathcal{S}et$: for an $x \in X(i)$ and a map

$f\colon i \to j$, write $f.x$ for the image of x under $X(i) \to X(j)$; for $w \in W(j)$ right $w.f$ for the image of w under $W(j) \to W(i)$.

If I and J are small categories, then an $\boldsymbol{I-J}$ **bimodule** is a diagram $I \times J^{op} \to \mathcal{M}$. If W is an $I-J$ bimodule and X is a $J-K$ bimodule, then by $B(W,J,X)$ we mean the $I-K$ bimodule defined as

$$(i,k) \mapsto B(W_i, J, X_k).$$

Here W_i is the J^{op}-diagram $j \mapsto W(i,j)$, and X_k is the J-diagram $j \mapsto X(j,k)$. Note that the construction of $B(W,J,X)$ makes sense even if the target of X is $\mathcal{S}et$, or if the target of W is $\mathcal{S}et$.

Example 10.4. If I is a category then for each $i \in I$ we obtain a left I-module $I(-,i)$ and a right I-module $I(-,i)$. These are free modules, in the following sense. For any object $Z \in \mathcal{M}$, consider the left I-module $I(-,i) \times W$ sending $j \mapsto I(j,i) \times W$. Then there is a bijection

$$\mathrm{Hom}_{I-\mathcal{M}od}(I(-,i) \otimes Z, X) \cong \mathrm{Hom}_{\mathcal{M}}(Z, X_i)$$

obtained by restricting to the canonical copy of Z in the ith spot of the diagram. Similarly, there are bijections

$$\mathrm{Hom}_{\mathcal{M}od-I}(I(i,-) \times Z, W) \cong \mathrm{Hom}_{\mathcal{M}}(Z, W_i)$$

for each right I-module W.

An easy adjointness argument now shows that $I(-,i) \otimes_I X \cong X_i$ and $W \otimes I(i,-) \cong W_i$.

Example 10.5. Putting the left and right modules $I(-,i)$ and $I(j,-)$ together, we have an $I-I$ bimodule given by the functor $I \times I^{op} \to \mathcal{S}et$ sending $(i,j) \mapsto I(j,i)$. We will call this functor I, by abuse. [The switching in the order of i and j is annoying, but seems unavoidable; the problem is that the notation in mathematics always wants to be right to left, so that to talk about maps from a to b we should really write "Hom(b,a)"; but we don't.]

By the above observations, for any left I-module X (that is to say, for any diagram $X\colon I \to \mathcal{M}$) we get a left I-module $B(I,I,X)$. Similarly, for any right I-module W we get another right I-module $B(W,I,I)$. We will see in a moment that these are precisely the diagrams QX and $Q'W$ defined in Section 8.

Exercise 10.6. If $X\colon I \to \mathcal{S}et$, then $B_\bullet(I,I,X)$ is an I-diagram of simplicial sets. Check that $B_\bullet(I,I,*)$ is the diagram $i \mapsto N(I \downarrow i)^{op}$. Similarly, check that $B_\bullet(*,I,I)$ is the diagram $i \mapsto N(i \downarrow I)^{op}$.

Exercise 10.7. Let $\alpha\colon I \to J$ be a functor. There there is a functor $J \times I^{op} \to \mathcal{S}et$ given by $(j,i) \mapsto J(\alpha(i),j)$. This is really obtained by starting with the $J-J$ bimodule J and restricting the right action along α. We will still call this bimodule J, but now regard it as a $J-I$ bimodule.

Check that $B_\bullet(J,I,*)$ is the left J-module given by $j \mapsto N(\alpha \downarrow j)^{op}$. Similarly, $B_\bullet(*,I,J)$ is the right J-module given by $j \mapsto N(j \downarrow \alpha)^{op}$.

10.8. **Main properties and applications.** The central result of [HV] is the following:

Theorem 10.9. *Let I, J, K, and L be small categories. Suppose given an $I-J$ bimodule X, a $J-K$ bimodule Y, and a $K-L$ bimodule Z. Then there is a canonical isomorphism*

$$B(X,J,Y)\otimes_K Z \cong B(X,J,Y\otimes_K Z)$$

of $I-L$ bimodules.

Similarly, if W is an $H-I$ bimodule then there is a canonical isomorphism

$$W\otimes_I B(X,J,Y)\cong B(W\otimes_I X,J,Y)$$

of $H-K$ bimodules.

Remark 10.10. The above theorem has an open-ended interpretation, as we have not specified the target categories for the bimodules X, Y, and Z. For instance, X and Y could take their values in $\mathcal{S}et$ and Z could take its values in $\mathcal{M}$; or X and Z could take their values in $\mathcal{S}et$ and Y could take its values in $\mathcal{M}$; or all three functors could take their values in $\mathcal{S}et$. The isomorphism of the theorem is valid in all these cases.

The proof of Theorem 10.9 is a simple exercise in adjoint functors. We will give it at the end of the section. What we will do now is point out that the theorem allows one to give very slick proofs of many of our results about homotopy colimits.

Example 10.11 (The two formulas for hocolim)**.** Recall that if $X\colon I\to \mathcal{T}op$ then $B(*,I,X)=|\operatorname{srep}(X)|$. By the theorem, we can also write

$$B(*,I,X)\cong B(*,I,I\otimes_I X)\cong B(*,I,I)\otimes_I X.$$

But $B(*,I,I)$ is the diagram $i\mapsto N(i\downarrow I)^{op}$, and so the right-most object is the formula from (4.14). This seems to be the slickest proof that the two formulas for $\operatorname{hocolim} X$ are isomorphic.

Example 10.12 (The diagrams QX)**.** Let $X\colon I\to\mathcal{M}$ and consider the left I-module $B(I,I,X)$. This is the diagram

$$i\mapsto B(I(-,i),I,X)=B(I(-,i),I,I)\otimes_I X.$$

But it's easy to check that $B_\bullet(I(-,i),I,I)=B_\bullet(*,I\downarrow i,I)$. So we are really looking at the diagram

$$i\mapsto B(*,I\downarrow i,I)\otimes_I X=B(*,I\downarrow i,X)=\operatorname*{hocolim}_{I\downarrow i} u_i^* X.$$

Therefore $B(I,I,X)$ is the I-diagram QX defined in Section 8.

Recall that we have a natural map of I-diagrams $B(I,I,X)\to I\otimes_I X=X$. This is our map $QX\to X$.

Finally, note that one has

$$\operatorname*{colim}_I QX=\operatorname*{colim}_I B(I,I,X)=*\otimes_I B(I,I,X)\cong B(*,I,X)\cong\operatorname*{hocolim}_I X.$$

Example 10.13 (The diagrams $Q'X$)**.** We again start with $X\colon I\to\mathcal{M}$, but now we regard X as a right I^{op}-module. It is easy to see that $B_\bullet(X,I^{op},*)$ is the 'other' simplicial replacement for X considered in Remark 4.2; and so $B(X,I^{op},*)$ is what we called $\operatorname{hocolim}' X$ in Remark 8.9.

The object $B(X,I^{op},I^{op})$ is a right I^{op}-module, or equivalently a left I-module; in other words, it is a diagram $I\to\mathcal{M}$. An analysis similar to the one in the previous example shows that this is precisely the diagram $Q'X$ defined in (8.9).

Just as in the previous example, we find that

$$\operatorname*{colim}_I Q'X = B(X, I^{op}, I^{op}) \otimes_I * \cong B(X, I^{op}, *) = \operatorname*{hocolim'}_I X.$$

Example 10.14 (Changing the indexing category)**.** Suppose $\alpha\colon I \to J$ is a functor and $X\colon J \to \mathcal{M}$. Then we can write

$$\operatorname*{hocolim}_I \alpha^* X = B(*, I, X) \cong B(*, I, J \otimes_J X) \cong B(*, I, J) \otimes_J X.$$

Note that the natural map $\operatorname{hocolim}_I \alpha^* X \to \operatorname{hocolim}_J X$ is the map

$$\alpha_*\colon B(*, I, J) \otimes_J X \to B(*, J, J) \otimes_J X.$$

Observe that $B(*, I, J)$ is the J^{op}-diagram given by $j \mapsto N(j \downarrow \alpha)^{op}$. So the above formula for $\operatorname{hocolim}_I \alpha^* X$ recovers Proposition 9.4(d).

We can also recover the other parts of Proposition 9.4. For instance, let us consider part (a). For any diagram $X\colon I \to \mathcal{M}$, we have already remarked that $Q'X = B(X, I^{op}, I^{op})$. So if we want to apply Q' to the constant I^{op}-diagram whose value is a point, then we have

$$Q'(*) = B(*, I, I).$$

It follows that

$$\operatorname*{colim}_{I^{op} \to J^{op}} Q'(*) = Q'(*) \otimes_I J = B(*, I, I) \otimes_I J = B(*, I, I \otimes_I J) = B(*, I, J).$$

But we have already identified $B(*, I, J)$ with the J^{op}-diagram $j \mapsto N(j \downarrow \alpha)^{op}$, in Example 10.7 above.

This completes our examples. Hopefully they demonstrate the power of learning to manipulate the two-sided bar construction. After proving just a few basic results, many significant corollaries come along almost for free.

For ease of future reference, we now summarize the relations between the two-sided bar construction and other objects considered in this paper. In the following, $I \to J$ is a map of small categories and X is a diagram $I \to \mathcal{M}$.

$$\begin{aligned}
B(*, I, *) &= BI^{op} \\
B(I, I, X) &= QX = B(I, I, I) \otimes_I X \\
B(X, I^{op}, I^{op}) &= Q'X = X \otimes_{I^{op}} B(I^{op}, I^{op}, I^{op}) \\
B(*, I, X) &= \operatorname*{hocolim}_I X = B(*, I, I) \otimes_I X \\
B(J, I, X) &= \operatorname*{hocolim}_{I \to J} X = B(J, I, I) \otimes_I X \\
B(X, I^{op}, *) &= \operatorname{hocolim}'_I X = X \otimes_{I^{op}} B(I^{op}, I^{op}, *) \\
B(X, I^{op}, J^{op}) &= \operatorname{hocolim}'_{I \to J} X = X \otimes_{I^{op}} B(I^{op}, I^{op}, J^{op})
\end{aligned}$$

11. Function spaces and the cobar construction

Let I be a small category, and let X and Y be left I-modules. One defines

$$F_I(X,Y) = \mathrm{eq}\left(\prod_i \mathrm{Map}(X(i),Y(i)) \rightrightarrows \prod_{i\to j} \mathrm{Map}(X(i),Y(j))\right).$$

Note that this is a simplicial set.

If X is an $I-K$ bimodule, then the natural extension of this definition gives a left K-module $F_I(X,Y)$. Likewise, if Y is an $I-K$ bimodule then $F_I(X,Y)$ is a right K-module.

Proposition 11.1. *Let Z be a left K-module, X an $I-K$ bimodule, and Y a left I-module. Then there are natural adjunction isomorphisms*

(a) $\mathrm{Hom}_{K-\mathcal{M}od}(Z, F_I(X,Y)) \cong \mathrm{Hom}_{I-\mathcal{M}od}(X \otimes_K Z, Y)$*, and*

(b) $F_K(Z, F_I(X,Y)) \cong F_I(X \otimes_K Z, Y)$.

Just as the tensor product $(-)\otimes_I(-)$ can be expanded to a homotopical version $B(-,I,-)$, its adjoint $F_I(-,-)$ also has a homotopical version which we denote $\Omega_I(X,Y)$. We define $\Omega_I(X,Y)$ to be the cosimplicial object

$$[n] \mapsto \prod_{i_0\to i_1\to\cdots\to i_n} \mathrm{Map}(X(i_0),Y(i_n))$$

and we define $\Omega_I^{tot}(X,Y) = \mathrm{Tot}\,\Omega_I(X,Y)$. Note that there is a natural map $F_I(X,Y) \to \Omega_I^{tot}(X,Y)$.

Theorem 11.2. *There are natural isomorphisms*

(a) $F_K(Z, \Omega_I^{tot}(X,Y)) \cong \Omega_I^{tot}(X \otimes_K Z, Y)$,

(b) $\Omega_K^{tot}(Z, F_I(X,Y)) \cong F_I(B(X,K,Z),Y)$*, and*

(c) $\Omega_K^{tot}(Z, \Omega_I^{tot}(X,Y)) \cong \Omega_I^{tot}(B(X,K,Z),Y)$.

Part 3. The homotopy theory of diagrams

12. ???

13. Cofibrant diagrams

There are often geometrically natural ways for expressing a certain topological space as a colimit—perhaps because the space comes to us with a stratification or polyhedral decomposition. If these colimit diagrams are also *homotopy colimit* diagrams, then we immediately have spectral sequences for computing the homology and cohomology, by the results of Section 15. So this leads us to the following problem. Given a diagram $D\colon I \to \mathcal{T}op$, we have seen that there is a natural map $\operatorname{hocolim}_I D \to \operatorname{colim}_I D$. Under what conditions will this map be a weak equivalence? In this section we develop some basic results addressing this issue.

Recall that the category of diagrams $\mathcal{T}op^I$ has a model category structure in which a map $X \to Y$ is a fibration (resp. weak equivalence) if each map $X_i \to Y_i$ is a fibration (resp. weak equivalence) of topological spaces. We have adjoint functors

$$\operatorname{colim}\colon \mathcal{T}op^I \rightleftarrows \mathcal{T}op\colon c$$

where c is the "constant diagram" functor, and these are a Quillen pair because c preserves fibrations and trivial fibrations.

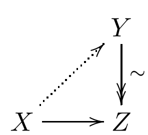
We have seen that $\operatorname{hocolim}_I X = \operatorname{colim}_I(QX)$, where $QX \to X$ is a certain cofibrant-replacement in $\mathcal{T}op^I$. This identifies hocolim as the derived functor of colim.

Recall that left Quillen functors preserve all weak equivalences between cofibrant objects. So if X was a cofibrant diagram, then $QX \to X$ would be such a weak equivalence, and it would follow that

$$\operatorname{hocolim} X = \operatorname{colim}(QX) \to \operatorname{colim} X$$

would be a weak equivalence. So this gives us a certain kind of answer to our original question: the map from the homotopy colimit to the colimit is a weak equivalence provided that the diagram X is cofibrant in $\mathcal{T}op^I$.

So how do we recognize when a diagram is cofibrant? In general, this is not an easy thing to do. One way is to test this directly in terms of the lifting property: a diagram X is cofibrant if every diagram



has a lifting, where $Y \to Z$ is a trivial fibration. This can be difficult to check, but it is manageable in some simple cases which we'll describe next.

Definition 13.1. *A category I is a* ***directed Reedy category*** *if there is a degree function assigning each object of I a non-negative integer such that all non-identity maps in I strictly raise the degree.*

Example 13.2.

(1) The pushout category $\mathbf{1} \leftarrow \mathbf{0} \to \mathbf{2}$ is a directed Reedy category. For instance, we can have $\deg(\mathbf{0}) = 0$ and $\deg(\mathbf{1}) = \deg(\mathbf{2}) = 1$.
(2) The coequalizer category $\mathbf{0} \rightrightarrows \mathbf{1}$ is a directed Reedy category.
(3) The subcategory $\Delta_f \hookrightarrow \Delta$ consisting of all the monomorphisms is a directed Reedy category.

(4) Let S be a set, and consider the poset of finite subsets of S, ordered by inclusion. Regard this poset as a category in the usual way, so that any two objects have at most one map between them. Then this poset, as well as any subcategory, is a directed Reedy category.

Let I be a directed Reedy category, and let $i \in I$. Let $(I \downarrow i)'$ be the full subcategory of $(I \downarrow i)$ consisting of all objects except the identity map $i \to i$. Write $u\colon (I \downarrow i)' \to I$ for the forgetful functor sending $[d, d \to i]$ to d. If $X\colon I \to \mathcal{T}op$ is a diagram, then the **latching object** of X at i is the space

$$L_i(X) = \operatorname*{colim}_{(I\downarrow i)'}(Xu).$$

Intuitively, $L_i(X)$ is the colimit of all the objects "below" X_i in the diagram. Note that there is a natural map $L_i(X) \to X_i$; this is called the **latching map** of X at i.

Example 13.3.

(1) Suppose $X_1 \leftarrow X_0 \to X_2$ is a pushout diagram. Then $L_0(X) = \emptyset$, $L_1(X) = X_0$, and $L_2(X) = X_0$.
(2) Suppose $X_0 \rightrightarrows X_1$ is a diagram. Then $L_0(X) = \emptyset$ and $L_1(X) = X_0 \amalg X_0$.
(3) Let I be the category $\mathbf{0} \rightrightarrows \mathbf{1} \to \mathbf{2}$ where the two maps $0 \to 2$ are the same. If $X\colon I \to \mathcal{T}op$ then $L_0(X) = \emptyset$, $L_1(X) = X_0 \amalg X_0$, and $L_2(X) = \operatorname{coeq}(X_0 \rightrightarrows X_1)$.
(4) Define P_n to be the poset of subsets of $\{1, 2 \ldots, n\}$ *excluding* the set $\{1, \ldots, n\}$ itself, ordered by inclusion. So P_2 is just the pushout category, for instance. Suppose $X\colon P_n \to \mathcal{T}op$ is a diagram. For any $S \subseteq \{1, 2, \ldots, n\}$, we can write

$$L_S(X) = \Big[\coprod_{T \subset S} X_T\Big]\Big/\!\sim$$

where the quotient relation says that for any two proper subsets $T, T' \subset S$ and any $x \in X_{T\cap T'}$, the images of x under $X_{T\cap T'} \to X_T$ and $X_{T\cap T'} \to X_{T'}$ are identified. Note that we could also describe $L_S(X)$ as a quotient space $[\coprod X_U]/\sim$ where U runs over the proper subsets of S with $|U| = |S| - 1$.

Proposition 13.4. *Let I be a directed Reedy category. Then a diagram $X\colon I \to \mathcal{T}op$ is cofibrant in $\mathcal{T}op^I$ if for every $i \in I$ the latching map $L_i(X) \to X_i$ is a cofibration.*

Proof. First, choose a degree function for the category I. Let $Y \to Z$ be a trivial fibration in $\mathcal{T}op^I$, and let $X \to Z$ be any map. We will inductively construct a lifting $X \to Y$.

Suppose that we have inductively produced a partial map of diagrams $X \to Y$, defined on all objects of I having degree less than n. (The base case is $n = 0$, which is trivial because there are no objects of I having degree less than 0). For each object i of degree n, consider the diagram

$$\begin{array}{ccc} L_i(X) & \longrightarrow & Y_i \\ \big\downarrow & & \big\downarrow{\scriptstyle\sim} \\ X_i & \longrightarrow & Z_i. \end{array}$$

Since $L_i(X) \to X_i$ is a cofibration, we can choose a lifting $X_i \to Y_i$. Doing this for all objects of degree n now gives us a partial map of diagrams $X \to Y$ defined on all objects of degree less than $n+1$. Continuing by induction, we produce the desired map $X \to Y$. (For more details, the reader may consult [H, Theorem 15.2.1]). □

Remark 13.5. It is worth mentioning that the above proposition can be set in a broader context. When I is a Reedy category, there is a so-called *Reedy model category structure* on $\mathcal{T}op^I$. The weak equivalences are the objectwise weak equivalences, but the cofibrations and fibrations are different from the 'projective' model structure we have used up until now. We refer the reader to [H, Chapter 15] for a thorough discussion. When I is a *directed* Reedy category, however, it is easy to see that the Reedy fibrations are precisely the objectwise fibrations. It follows that the Reedy and projective model category structures coincide in this case.

13.6. **An application.** Here is a simple application of what we have learned so far. Let X be a topological space, and let $\{A_1, \ldots, A_n\}$ be a collection of closed sets which cover X. Let Q_n denote the poset of all subsets of $\{1, 2, \ldots, n\}$ except the emptyset, ordered by reverse inclusion (note that this is basically the same as P_n). There is a diagram $A\colon Q_n \to X$ sending a subset $\sigma = \{i_1, \ldots, i_k\}$ to

$$A_\sigma = A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}.$$

The map $\operatorname{colim}_{Q_n} A \to X$ is clearly a bijection, and it is easy to check that it is actually a homeomorphism (but note that this uses that we have a *finite* cover). Since Q_n is a directed Reedy category, we know that $\operatorname{hocolim}_{Q_n} A \to \operatorname{colim}_{Q_n} A$ will be a weak equivalence provided that the latching maps in A are cofibrations. So we obtain the following corollary:

Corollary 13.7. *Assume that for every proper subset $S \subset \{1, \ldots, n\}$, the inclusion $\bigcup_{T \subset S} A_T \hookrightarrow A_S$ is a cofibration. Then* $\operatorname{hocolim}_{Q_n} A \to X$ *is a weak equivalence. As a result, for any cohomology theory $\mathcal{E}^*$ there exists a spectral sequence*

$$E_2^{p,q} = H^p(Q_n^{op}; \mathcal{E}^q(A)) \Rightarrow \mathcal{E}^{p+q}(X).$$

Proof. One simply checks that the latching maps are the maps $\bigcup_{T \subset S} A_T \to A_S$. The spectral sequence is then a result of Proposition 15.9. □

Exercise 13.8. For the above spectral sequence to be useful, one has to be able to compute the groups $H^p(Q_n^{op}; D)$ for diagrams $D\colon Q_n^{op} \to \mathcal{A}b$. Prove that this group is the same as the pth cohomology of the chain complex

$$\prod_{\#S=1} D(S) \longrightarrow \prod_{\#S=2} D(S) \longrightarrow \prod_{\#S=3} D(S) \longrightarrow \cdots$$

Each product ranges over all nonempty subsets $S \subseteq \{1, \ldots, n\}$ of the specified size, and the differential is described as follows. For any subset $T \subseteq \{1, \ldots, n\}$, let $\partial_i(T)$ denote the subset obtained by removing the ith element of T (where the elements are ordered in the usual way). Then if $x = (x_S)$ is a tuple in $\prod_{\#S=k} D(S)$, we define $d(x)$ to be the tuple $y = (y_T) \in \prod_{\#T=k+1} D(T)$ given by

$$y_T = \sum_i (-1)^i x_{(\partial_i T)}.$$

13.9. **Pushout diagrams.** Suppose that $X_1 \leftarrow X_0 \rightarrow X_2$ is a pushout diagram of spaces. Based on the work from the last section, we know that if X_0 is cofibrant and both $X_0 \rightarrow X_1$ and $X_0 \rightarrow X_2$ are cofibrations, then the map $\operatorname{hocolim} X \rightarrow \operatorname{colim} X$ is a weak equivalence. We can actually do somewhat better, however, and this improvement is useful enough that it is worth noting.

Proposition 13.10. *Let $X_1 \leftarrow X_0 \rightarrow X_2$ be a pushout diagram in which every object is cofibrant and $X_0 \rightarrow X_1$ is a cofibration. Then the map* $\operatorname{hocolim} X \rightarrow \operatorname{colim} X$ *is a weak equivalence.*

Proof. Factor $X_0 \rightarrow X_2$ as $X_0 \rightarrowtail Z \xrightarrow{\sim} X_2$. Then we have two pushout diagrams

$$\begin{array}{ccccc} X_1 & \leftarrowtail & X_0 & \longrightarrow & X_2 \\ \| & & \| & & \uparrow{\scriptstyle\sim} \\ X_1 & \leftarrowtail & X_0 & \rightarrowtail & Z \end{array}$$

and the resulting square

$$\begin{array}{ccc} \operatorname{hocolim}(X_1 \leftarrow X_0 \rightarrow X_2) & \longrightarrow & \operatorname{colim}(X_1 \leftarrow X_0 \rightarrow X_2) \\ {\scriptstyle\sim}\uparrow & & \uparrow \\ \operatorname{hocolim}(X_1 \leftarrow X_0 \rightarrow Z) & \xrightarrow{\sim} & \operatorname{colim}(X_1 \leftarrow X_0 \rightarrow Z). \end{array}$$

Hence, it will be enough to show that $X_1 \amalg_{X_0} Z \rightarrow X_1 \amalg_{X_0} X_2$ is a weak equivalence.

A map $A \rightarrow B$ between cofibrant objects is a weak equivalence if and only if $\operatorname{Map}(B, W) \rightarrow \operatorname{Map}(A, W)$ is a weak equivalence of simplicial sets for every space W. Applying this to $X_1 \amalg_{X_0} Z \rightarrow X_1 \amalg_{X_0} X_2$, we must show that

$$\operatorname{Map}(X_1, W) \times_{\operatorname{Map}(X_0,W)} \operatorname{Map}(X_2, W) \rightarrow \operatorname{Map}(X_1, W) \times_{\operatorname{Map}(X_0,W)} \operatorname{Map}(Z, W)$$

is a weak equivalence of simplicial sets. That is to say, we must show that the map of pullback diagrams

$$\begin{array}{ccccc} \operatorname{Map}(X_1, W) & \twoheadrightarrow & \operatorname{Map}(X_0, W) & \longleftarrow & \operatorname{Map}(X_2, W) \\ {\scriptstyle\sim}\uparrow & & {\scriptstyle\sim}\uparrow & & {\scriptstyle\sim}\uparrow \\ \operatorname{Map}(X_1, W) & \twoheadrightarrow & \operatorname{Map}(X_0, W) & \twoheadleftarrow & \operatorname{Map}(Z, W) \end{array}$$

induces a weak equivalence on the pullbacks. Using that all the objects are fibrant and the indicated maps fibrations, this follows from the right properness of $s\mathcal{S}et$ (and it's also a fairly easy exercise). □

Remark 13.11. Recall that the pushout category is equal to P_2, the poset of proper subsets of $\{1, 2\}$. It seems likely that there is an analog of Proposition 13.10 for P_n-diagrams—that is to say, a condition for the map $\operatorname{hocolim} X \rightarrow \operatorname{colim} X$ to be a weak equivalence that is weaker than the requirement that all latching maps are cofibrations. I don't know results along these lines, however.

14. Simplicial diagrams and homotopy coherence

Let $\mathcal{C}at$ denote the category of small categories. By a **simplicial category** we mean a simplicial object in $\mathcal{C}at$ where the categories in each level have the same object set. Alternatively, we may regard such a think as a category enriched over $s\mathcal{S}et$. To be concrete, a (small) simplicial category I consists of

(1) A set of objects (denoted I by abuse);
(2) For each $i, j \in I$, a simplicial set $I(i,j)$;
(3) For each $i \in I$, a distinguished 0-simplex $id_i \in I(i,i)$;
(4) For each $i, j, k \in I$, composition maps $I(j,k) \times I(i,j) \to I(i,k)$ which satisfy associativity and unital axioms.

One defines functors between simplicial categories in the evident manner.

If $\mathcal{C}$ is another simplicial category, then an I-diagram in $\mathcal{C}$ is just a functor $X\colon I \to \mathcal{C}$. Concretely, this consists of a collection of objects $X_i \in \mathcal{C}$ together with maps of simplicial sets $I(i,j) \to \mathcal{C}(X_i, X_j)$ for each $i, j \in I$ such that

(1) id_i maps to id_{X_i}, and
(2) for each $i, j, k \in I$, the diagram

$$\begin{array}{ccc} I(j,k) \times I(i,j) & \longrightarrow & I(i,k) \\ \downarrow & & \downarrow \\ \mathcal{C}(X_j, X_k) \times \mathcal{C}(X_i, X_j) & \longrightarrow & \mathcal{C}(X_i, X_k) \end{array}$$

commutes.

Now let $\mathcal{M}$ be a simplicial model category, and let $X\colon I \to \mathcal{M}$ be a diagram. Note that the maps $I(i,j) \to \underline{\mathcal{M}}(X_i, X_j)$ yield maps $I(i,j) \otimes X_i \to X_j$ via adjointness. So an I-diagram in $\mathcal{M}$ can be thought of as a collection of objects X_i and a collection of 'action' maps $I(i,j) \otimes X_i \to X_j$ satisfying the evident associativity and unital conditions. Just as we did for diagrams indexed by ordinary categories, we will think of diagrams $I \to \mathcal{M}$ as 'left I-modules'.

14.1. **Resolutions of categories.** Let I be an ordinary category. We can regard I as a simplicial category by regarding all its mappings sets as discrete simplicial sets. By a **resolution of** I we mean a simplicial category $\tilde{I}$ with the same set of objects as I, together with a map of simplicial categories $\tilde{I} \to I$ with the property that for every $i, j \in I$ the map $\tilde{I}(i,j) \to I(i,j)$ is a weak equivalence.

Part 4. Other useful tools

15. Spectral sequences for holims and hocolims

If $D\colon I \to \mathcal{T}op$ is a diagram, there is a spectral sequence for computing $\pi_*(\operatorname{holim} D)$ from knowledge of $\pi_*(D_i)$ for each i. Actually, this is not always a true spectral sequence due to the fact that π_0 may not be a group, and π_1 may not be an abelian group. So one has these problems on the 'fringe'. There are ways to deal with these problems, but very often one is in a situation where they actually aren't there. One way this can happen is if one is really dealing with spectra rather than spaces. Another way is if one is dealing with spaces which are all connected with abelian fundamental groups. We will develop things in these two special cases.

If E is a cohomology theory, then there is a related spectral sequence for computing $E^*(\operatorname{hocolim} D)$ from knowledge of $E^*(D_i)$, for all i. In fact this is a special case of the above, using the adjunctions in the category of spectra

$$E^n(\operatorname{hocolim} D) = \pi_{-n} \operatorname{Map}(\operatorname{hocolim} D, E) = \pi_{-n}\Big[\operatorname{holim}_I \operatorname{Map}(D(i), E)\Big].$$

Here we are writing E also for some spectrum representing our given cohomology theory.

15.1. Cohomology of a category with coefficients in a functor.

Fix an abelian category $\mathcal{A}$. In our applications below this will always be the category of abelian groups.

Let $\mathcal{C}$ be a small category, and let $F\colon \mathcal{C} \to \mathcal{A}$ be a functor. We will define objects $H^p(\mathcal{C}; F)$ in $\mathcal{A}$, for each $p \geq 0$. One approach starts by writing down the cosimplicial replacement for F:

$$\prod_c F(c) \rightrightarrows \prod_{c_0 \to c_1} F(c_1) \Rrightarrow \prod_{c_0 \to c_1 \to c_2} F(c_2) \ldots$$

This is a cosimplicial object over $\mathcal{A}$. Taking the alternating sum of the coface maps gives a cochain complex over $\mathcal{A}$, and we define $H^p(\mathcal{C}; F)$ to the the pth cohomology group of this complex.

Note that $H^0(\mathcal{C}; F)$ is just the equalizer of the first two arrows in our cosimplicial object, which is precisely $\lim F$. So in this sense the groups $H^p(\mathcal{C}; F)$ are 'higher limit functors'. One somtimes writes

$$H^p(\mathcal{C}; F) = \lim{}^p F.$$

Here is another description of the same groups, assuming that $\mathcal{A}$ has enough injectives. Note that the diagram category $\mathcal{A}^I$ is another abelian category, and also has enough injectives. The functor $\lim\colon \mathcal{A}^I \to \mathcal{A}$ is readily seen to be left exact, and $H^p(\mathcal{C}; F)$ is just the pth right derived functor. The specific complex for computing this which we gave above comes from a kind of 'bar resolution' for the object $F \in \mathcal{A}^I$.

Consider the category of cochain complexes over $\mathcal{A}$. By this we mean complexes of the form

$$X^0 \to X^1 \to X^2 \to \cdots$$

Denote the category of cochain complexes as $Ch^{\geq 0}(\mathcal{A})$. Under mild hypotheses this has a model category structure where the weak equivalences and cofibrations are determined levelwise; complexes of injectives are fibrant. There is an embedding

$\mathcal{A} \hookrightarrow Ch^{\geq 0}(\mathcal{A})$ obtained by regarding each $X \in \mathcal{A}$ as a cochain complex concentrated in degree 0. Then our functor $F\colon \mathcal{C} \to \mathcal{A}$ can be regarded as a diagram $\mathcal{C} \to Ch^{\geq 0}(A)$. We can therefore consider its homotopy limit, which will be another cochain complex. It's easy to see that $H^p(\mathcal{C}; F)$ is just $H^p(\operatorname{holim} F)$.

Finally, we give one more description—although this is more of a related fact. If X is any object in $\mathcal{A}$, let cX denote the constant diagram $I \to \mathcal{A}$ which has X at every spot, and where every map is the identity map. One can talk about Ext groups in the category $\mathcal{A}^I$, and there is an isomorphism

$$\operatorname{Ext}^p(cX, F) \cong \mathcal{A}(X, H^p(\mathcal{C}; F)).$$

Here is one way this is sometimes useful. If $\mathcal{A}$ has enough projectives, then so does $\mathcal{A}^I$. So one can compute $\operatorname{Ext}^p(cX, F)$ either via an injective resolution for F or via a projective resolution for cX.

Remark 15.2. One should note that the above constructions are interesting even when $\mathcal{A}$ is the category of vector spaces over a field. That is to say, even in this 'simple' case one can have nontrivial objects $H^p(\mathcal{C}; F)$ for $p > 0$.

In fact, let $\mathcal{A}$ be the category of vector spaces over a field k. Let G be a group, and let $\mathcal{C}$ be the category with one object whose endomorphism group is G. Then a functor $F\colon \mathcal{C} \to \mathcal{A}$ is just a representation of G over k, and $H^*(\mathcal{C}; F)$ is just classical group cohomology.

15.3. **Homology of a category with coefficients in a functor.** There are dual definitions for the *homology* of $\mathcal{C}$ with coefficients in $F\colon \mathcal{C} \to \mathcal{A}$. One writes down the simplicial replacement for F, and then $H_p(\mathcal{C}; F)$ is the pth homology group of the associated chain complex over $\mathcal{A}$ (obtained by taking the alternating sum of the face maps).

15.4. **The spectral sequence for a homotopy limit.** Let $D\colon I \to \mathcal{T}op$ be a diagram. Assume that each $D(i)$ is connected, with abelian fundamental group. Then for each n, one obtains an associated diagram $\pi_n D\colon I \to \mathcal{A}b$ sending $i \mapsto \pi_n D(i)$. Here is our theorem:

Theorem 15.5. *There is a spectral sequence $E_2^{p,q} = H^p(I; \pi_q D) \Rightarrow \pi_{q-p}(\operatorname{holim} D)$. The differentials have the form $d_r\colon E_r^{p,q} \to E_r^{p+r,q+r-1}$.*

This is an immediate consequence of the following result about cosimplicial spaces. If X is a cosimplicial pointed space, one may form a cosimplicial abelian group by applying $\pi_n(-,*)$ to each level (assuming that $n \geq 2$, or that the spaces X_i are connected with abelian fundamental group). After taking the alternating sum of the coface maps, the cosimplicial abelian group becomes a cochain complex. Let $H^p(\pi_q(X))$ denote the pth cohomology group.

Theorem 15.6. *Let X be a Reedy fibrant simplicial space, such that each X_n is connected with abelian fundamental group. Then there is a spectral sequence of the form $E_2^{p,q} = H^p(\pi_q(X)) \Rightarrow \pi_{q-p}(\operatorname{Tot} X)$, where the differentials have the form $d_r\colon E_r^{p,q} \to E_r^{p+r,q+r-1}$.*

Remark 15.7. There is an easy way to remember how the differentials work in the above spectral sequence, at least if one understands the two spectral sequences associated to a double chain complex. Suppose that, instead of X being a cosimplicial space, X were a cosimplicial chain complex. That is, suppose that instead of working in the model category $\mathcal{T}op$ we were working in the model category $Ch(\mathbb{Z})$.

Now each X_n is a chain complex, which we draw vertically with the differentials going down. We are now looking at a cosimplicial chain complex, which after taking the alternating sum of coface maps becomes a double complex. In this case $\operatorname{Tot} X$ "is" the totalization of this double complex, and the spectral sequence "is" the spectral sequence obtained by first taking homology groups in the vertical direction and then in the horizontal direction. So if one knows how the indexing works in the latter spectral sequence, one also knows how it works in the former.

15.8. **Spectral sequences for homotopy colimits.** Suppose $D\colon I \to \mathcal{T}op$, and that $\mathcal{E}^*$ is a cohomology theory represented by a spectrum $\mathcal{E}$. Note that for each n one obtains an I^{op}-diagram of abelian groups by $i \mapsto \mathcal{E}^n(D(i))$. We'll call this diagram $\mathcal{E}^n(D)$, for short.

Proposition 15.9. *There is a spectral sequence* $E_2^{p,q} = H^p(I^{op}; \mathcal{E}^q(D)) \Rightarrow \mathcal{E}^{p-q}(\operatorname{hocolim} D)$. *The differentials have the form* $d_r\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$.

Proof. This is obtained by dualizing the spectral sequence for a homotopy limit. □

One can also derive a spectral sequence for computing the E-homology of a homotopy colimit. This is based on the following spectral sequence for the homotopy groups of a geometric realization of spectra:

Proposition 15.10. *Let* $[n] \mapsto G_n$ *be a simplicial spectrum. Then there is a spectral sequence*

$$E^1_{p,q} = \pi_p G_q \Rightarrow \pi_{p+q}|G|$$

where the differentials have the form $d^r\colon E^r_{p,q} \to E^r_{p+r-1,q-r}$. *The differential* d_1 *is the alternating sum of the face maps in the cosimplicial abelian group* $[n] \mapsto \pi_* G_n$.

Proof. This is the homotopy spectral sequence associated to the tower of homotopy cofiber sequences

$$\begin{array}{ccccccccc}
* & \longrightarrow & |\operatorname{Sk}_0 G| & \longrightarrow & |\operatorname{Sk}_1 G| & \longrightarrow & |\operatorname{Sk}_2 G| & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & G_0 & & \Sigma G_1 & & \Sigma^2 G_2 & & \cdots
\end{array}$$

In spectra, homotopy cofiber sequences are also homotopy fiber sequences—so each layer in the tower gives a long exact sequence in homotopy groups, resulting in an exact couple. □

Remark 15.11. Again, there is a nice way to remember how the differentials go in the above spectral sequence. Imagine the parallel situation in which the G_i are chain complexes rather than spectra. Then what we really have is a double complex, and we are looking at the spectral sequence whose G_2-term is obtained by first taking the homology of the G_i's and then taking homology in the other direction. Provided one can remember how the differentials work in the spectral sequence of a double complex, one also knows how they work in the spectral sequence of Proposition 15.10.

Proposition 15.12. *Let* $\mathcal{E}$ *be a spectrum and let* $X\colon I \to \mathcal{T}op$ *be a diagram of spaces. Then for each* p *one gets a diagram of abelian group* $i \mapsto \mathcal{E}_p(X_i)$*; call this diagram* $\mathcal{E}_p X$.

There is a spectral sequence

$$E^2_{p,q} = H_q(I; \mathcal{E}_p X) \Rightarrow \mathcal{E}_{p+q}(\operatorname*{hocolim}_I X).$$

The differentials have the form $d^r : E^r_{p,q} \to E^r_{p+r-1,q-r}$.

Proof. Consider the simplicial spectrum $[n] \mapsto \mathcal{E} \wedge \Sigma^\infty(\mathrm{srep}(X)_n)$. The geometric realization of this simplicial spectrum is $\mathrm{hocolim}_I(\mathcal{E} \wedge \Sigma^\infty(X_i)_+)$, which is the same as

$$\mathcal{E} \wedge \Sigma^\infty(\operatorname*{hocolim}_I X_i).$$

The spectral sequence of Proposition 15.10 converges to the $(p+q)$th homotopy group of this geometric realization, which is therefore $\mathcal{E}_{p+q}(\mathrm{hocolim}_I X)$. □

16. Homotopy limits and colimits in other model categories

So far we have been almost exclusively working in the model category of topological spaces. In this section we will explain some of the ways in which our methods adapt to more general model categories. In many cases this takes the form, "If a model category satisfies P and Q then everything we did before works exactly the same. However, if the model category does *not* satisfy P or Q then one can still get the same basic results, but it requires harder work."

We also make some remarks particular to the case where the model category is chain complexes over an abelian category. Here, the study of homotopical algebra is really just ordinary homological algebra. So the theory of homotopy colimits can be phrased in somewhat more algebraic terms. We make some of this explicit.

16.1. **Simplicial model categories.** A model category $\mathcal{M}$ is called *simplicial* if for every $X, Y \in \mathcal{M}$ and $K \in s\mathcal{S}et$ one has functorial constructions

$$X \otimes K \in \mathcal{M}, \qquad F(K, X) \in \mathcal{M}, \qquad \text{and} \qquad \mathrm{Map}(X, Y) \in s\mathcal{S}et$$

together with adjunction isomorphisms

$$\mathrm{Map}(X \otimes K, Y) \cong \mathrm{Map}(X, F(K, Y)) \cong \underline{s\mathcal{S}et}(K, \mathrm{Map}(X, Y))$$

(note that these are isomorphisms of simplicial sets). One assumes there is a composition law $\mathrm{Map}(Y, Z) \times \mathrm{Map}(X, Y) \to \mathrm{Map}(X, Z)$ and identity maps $* \to \mathrm{Map}(X, X)$ satisfying the expected properties, and also an isomorphism $\mathrm{Map}(X, Y)_0 \cong \mathcal{M}(X, Y)$ that commutes with composition. Finally, one assumes the *pushout-product axiom* SM7; there are several equivalent versions, but we will use the one saying that if $i \colon A \rightarrowtail B$ is a cofibration in $\mathcal{M}$ and $j \colon K \hookrightarrow L$ is a cofibration in $s\mathcal{S}et$, then the map

$$i \Box j \colon (A \otimes L) \amalg_{(A \otimes K)} (B \otimes K) \to B \otimes L$$

is a cofibration which is a weak equivalence if either i or j is so. A detailed treatment of simplicial model categories can be found in [H, Chapter 9].

Example 16.2. The model category $\mathcal{T}op$ is a simplicial model category, where one defines

$$X \otimes K = X \times |K|, \qquad F(K, X) = X^{|K|}$$

and where $\mathrm{Map}(X, Y)$ is the simplicial set $[n] \mapsto \mathcal{T}op(X \times \Delta^n, Y)$.

Similarly, $s\mathcal{S}et$ is a simplicial model category where one defines

$$X \otimes K = X \times K, \quad F(K, X) = \underline{s\mathcal{S}et}(K, X), \quad \text{and} \quad \mathrm{Map}(X, Y) = \underline{s\mathcal{S}et}(X, Y).$$

In a simplicial model category, one can give formulas for homotopy limits and colimits exactly like what we have described for $\mathcal{T}op$. One uses exactly the same definitions, and all the same results hold.

16.3. **The homotopy theory of diagrams.** Let $\mathcal{M}$ be any model category, and let I be a small category. Let $\mathcal{M}^I$ denote the category of I-diagrams and natural transformations.

One would like there to be a model category structure on $\mathcal{M}^I$ where the weak equivalences are the objectwise weak equivalences. Unfortunately this probably doesn't exist in general. However, it does exist if I is a so-called Reedy category, and for all I if $\mathcal{M}$ is a *cofibrantly-generated* model category.

Theorem 16.4. *Assume $\mathcal{M}$ is a cofibrantly-generated model category. Then for any small category I there is a model category structure on $\mathcal{M}^I$ where the weak equivalences and fibrations are determined objectwise. This is commonly called the **projective** model category structure on $\mathcal{M}^I$.*

Proof. See [H, Section 11.6]. □

If $\mathcal{M}$ is cofibrantly-generated, we can again consider the adjoint functors

$$\operatorname{colim}\colon \mathcal{M}^I \rightleftarrows \colon \mathcal{M}\colon c$$

and these are again a Quillen pair. One can define the homotopy colimit of a diagram as the derived functor of the colimit, just as we did in $\mathcal{T}op$. Notice that this works even if $\mathcal{M}$ is not simplicial! Relative homotopy colimits can also be defined, and the whole theory is exactly the same as for $\mathcal{T}op$.

The dual story for homotopy limits is also a little different. Here one wants a model category structure on $\mathcal{M}^I$ where the weak equivalences and *cofibrations* are defined objectwise. For the following, recall that a model category is called combinatorial if it is cofibrantly-generated and the underlying category is locally presentable.

Theorem 16.5 (J. Smith, unpublished)**.** *Assume that $\mathcal{M}$ is a combinatorial model category. Then for any small category I there is a model category structure on $\mathcal{M}^I$ in which the weak equivalences and cofibrations are determined objectwise. This is commonly called the **injective** model category structure on $\mathcal{M}^I$.*

If $\mathcal{M}$ is a combinatorial model category one can then consider the adjoint functors

$$c\colon \mathcal{M} \rightleftarrows \mathcal{M}^I\colon \lim$$

(where c is the left adjoint), and observe that c preserves cofibrations and trivial cofibrations. To this is a Quillen pair, and one can define the homotopy limit of a diagram to be the derived functor of lim.

Remark 16.6. Even if the appropriate model category structure on $\mathcal{M}^I$ does not exist, there are other techniques for making the derived functor perspective work. One can still define a homotopy category of diagrams $\operatorname{Ho}(\mathcal{M}^I)$, even though an underlying model category structure may not exist. And one can still talk about the derived functors of colim and lim. See [DHKS] for this approach.

For yet another approach to homotopy limits and colimits in general model categories, see [CS].

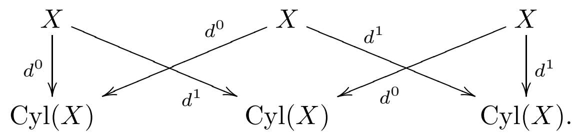
16.7. **Non-simplicial model categories.** Formulas for homotopy limits and colimits can also be given without assuming a simplicial structure on the model category; one just has to work a little harder. This is due to Dwyer-Kan, and it is described in detail in [H, Chapters 16, 19].

If X is a cofibrant object in a simplicial model category, then one can obtain a cylinder object for X by looking at $X \otimes \Delta^1$. One also has cylinder objects in non-simplicial model categories: they can be constructed by factoring the fold map $\nabla\colon X \amalg X \to X$ into a cofibration followed by a trivial fibration:

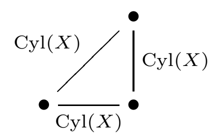
$$X \amalg X \rightarrowtail \operatorname{Cyl}(X) \xrightarrow{\sim} X.$$

These are even functorial, using that our factorizations are functorial.

In the same way, in any model category one can construct objects which "look like" $X \otimes \Delta^2$, $X \otimes \Delta^3$, etc. This is due to Dywer-Kan and is referred to as the theory of *framings*. For instance, to construct an object that looks like $X \otimes \Delta^2$ one does the following. Recall that our cylinder object $\mathrm{Cyl}(X)$ came with two maps $d^0, d^1 \colon X \to \mathrm{Cyl}(X)$. One can make an object that looks like $X \otimes \partial\Delta^1$ by forming the colimit of the diagram



This corresponds to gluing three copies of $\mathrm{Cyl}(X)$ to make the picture



corresponding to $\partial\Delta^1$. Let Z denote this colimit.

Our canonical map $\mathrm{Cyl}(X) \to X$ coequalizes d^0 and d^1, and therefore induces a map $Z \to X$. Factoring this again as a cofibration followed by trivial fibration gives

$$Z \rightarrowtail X[2] \xrightarrow{\sim} X$$

and this $X[2]$ is our object which "looks like" $X \otimes \Delta^2$. Note that it is also functorial in X, due to the functoriality of our factorizations.

For an object $X \in \mathcal{M}$, let cX denote the constant cosimplicial object which is X in every dimension. Briefly, a *cosimplicial frame* on X is a cosimplicial object $\hat{X}$ in $\mathcal{M}$, together with an objectwise weak equivalence $\hat{X} \to cX$ which is an isomorphism in level 0. When X is cofibrant, one also requires that $\hat{X}$ satisfy a certain Reedy cofibrancy condition having to do with latching maps being cofibrations—we will not write this down. The nth object of $\hat{X}$ is our object which "looks like" $X \otimes \Delta^n$. Dwyer and Kan showed that cosimplicial frames exist in any model category, essentially by inductively continuing the procedure we began above.

Let I be a small category. Given a diagram $D \colon I \to \mathcal{M}$, a cosimplicial frame on D is a diagram $\hat{D} \colon I \to c\mathcal{M}$ (a diagram of cosimplicial objects on $\mathcal{M}$) together with natural weak equivalences $\hat{D}(i) \to c[D(i)]$ which make each $\hat{D}(i)$ a cosimplicial frame on $D(i)$. Again, cosimplicial frames on diagrams always exist.

Once one has a cosimplicial frame on D, one can again write down explicit formulas for the homotopy colimit. (For the homotopy limit one needs a simplicial frame on D—we have not defined this but it is completely dual). The formulas are exactly what we wrote down in the simplicial case, one just has to develop enough machinery to realize that they really do make sense.

There is no point in us describing this theory in more detail because the reader should just go read [H]. The theory of frames and homotopy limits/colimits in general model categories is wonderfully presented there.

16.8. **Abelian categories.** Let $\mathcal{A}$ be an abelian category with enough projectives and injectives. Then there are model categories on $Ch_{\geq 0}(\mathcal{A})$ and $Ch_{\leq 0}(\mathcal{A})$ which exactly parallel the two model category structures described at the beginning of this section, when $\mathcal{A}$ is the category of modules over a ring. In these categories the theory of homotopy limits and colimits becomes somewhat simpler and more familiar.

Recall that if $\mathcal{B}$ is an additive category then there is an equivalence between the category of simplicial objects in $\mathcal{B}$ and the category $Ch_{\geq 0}(B)$. In one direction one replaces a simplicial object by its normalized chain complex; up to quasi-isomorphism, this is the same as the chain complex obtained by just taking the alternating sum of face maps.

Also, recall that if $D_{*,*}$ is a double chain complex then one may form a total chain complex in two ways. One way has $\mathrm{Tot}^{\oplus}(D)_n = \bigoplus_{p+q=n} D_{p,q}$ and the other has $\mathrm{Tot}^{\otimes}(D)_n = \bigotimes_{p+q=n} D_{p,q}$. We will have need for both of these.

Suppose given a simplicial object X_* of $Ch_{\geq 0}(\mathcal{A})$. Since the category of chain complexes is additive, we may take the alternating sum of face maps. . .and what we get is a double complex! Let X_*^{alt} denote this double complex. The result we are after is the following:

Proposition 16.9. *The two chain complexes* $\operatorname{hocolim} X_*$ *and* $\mathrm{Tot}^{\oplus}(X_*^{alt})$ *are quasi-isomorphic.*

Similarly, suppose Z^* is a cosimplicial object in $Ch_{\leq 0}(\mathcal{A})$. Let Z^*_{alt} denote the double complex obtained by taking the alternating sum of coface maps. Then

Proposition 16.10. *The complexes* $\operatorname{holim} Z^*$ *and* $\mathrm{Tot}^{\otimes}(Z^*_{alt})$ *are quasi-isomorphic.*

What these propositions say is that the theory of homotopy colimits in $Ch_{\geq 0}(\mathcal{A})$ (and of homotopy limits in $Ch_{\leq 0}(\mathcal{A})$) can be drastically simplified by using total complexes. For instance, if $D\colon I \to Ch_{\geq 0}(\mathcal{A})$ is a diagram then to construct $\operatorname{hocolim} D$ one can form the simplicial replacement, take the alternating sum of faces, and then apply $\mathrm{Tot}^{\oplus}$. No geometric realization (or Tot) is needed.

What about homotopy *limits* in $Ch_{\geq 0}(\mathcal{A})$? Here the story is a little more complicated, but only barely. The difficulty is as follows. Suppose Z^* is a cosimplicial object in $Ch_{\geq 0}(\mathcal{A})$. Taking alternating sums of coface maps gives a double complex Z^*_{alt}. But taking the total complex now gives a complex which has terms in negative degrees, so it does not lie in $Ch_{\geq 0}(\mathcal{A})$. How does one fix this? Well, for any $\mathbb{Z}$-graded chain complex C_* one can obtain a non-negatively graded chain complex by considering the truncation $\tau_{\geq 0}(C_*)$ given by

$$Z_0 \leftarrow C_1 \leftarrow C_2 \leftarrow \cdots$$

where Z_0 is the subobject of cycles in degree 0.

Proposition 16.11. *If Z^* is a cosimplicial object in $Ch_{\geq 0}(\mathcal{A})$, then* $\operatorname{holim} Z^*$ *is quasi-isomorphic to* $\tau_{\geq 0}\,\mathrm{Tot}^{\otimes}[Z^*_{alt}]$.

Similarly, we have

Proposition 16.12. *If X_* is a simplicial object in $Ch_{\leq 0}(\mathcal{A})$, then* $\operatorname{hocolim} X_*$ *is quasi-isomorphic to* $\tau_{\leq 0}\,\mathrm{Tot}^{\oplus}[X^*_{alt}]$. *Here, if C_* is a $\mathbb{Z}$-graded chain complex then $\tau_{\leq 0}(C_*)$ is the non-positively graded chain complex given by*

$$C_0/B_0 \to C_{-1} \to C_{-2} \to C_{-3} \to \cdots$$

where B_0 is the subobject of boundaries in degree 0.

Again, the above propositions show that the theory of homotopy limits and colimits in $Ch_{\geq 0}(\mathcal{A})$ and $Ch_{\leq 0}(\mathcal{A})$ can be drastically simplified by using total complexes in place of geometric realizations or Tot.

17. Various results concerning simplicial objects

This section is under construction!

Let I and J be two small categories, and let $X \to I \times J \to \mathcal{T}op$ be a diagram. Note that for any $i \in I$ we get a J-diagram by $j \mapsto X(i,j)$, and likewise for any $j \in J$ we get an I-diagram

Proposition 17.1. *There are canonical zig-zags of weak equivalences between the three objects*

$$\operatorname*{hocolim}_I [i \mapsto \operatorname*{hocolim}_J X(i,-)], \qquad \operatorname*{hocolim}_J [j \mapsto \operatorname*{hocolim}_I X(-,j)],$$

and $\operatorname{hocolim}_{I\times J} X$.

17.2. **Homotopy colimits and realizations.** Let $X\colon \Delta^{op} \to \mathcal{T}op$. We have already talked about the geometric realization $|X|$, but we can also form the homotopy colimit $\operatorname{hocolim} X$. These are both homotopy invariant constructions, but they are usually different. We can compare them, though:

Proposition 17.3. *There is a natural map* $\operatorname{hocolim} X \to |X|$ *called the* ***Bousfield-Kan map****. It is a weak equivalence when X is Reedy cofibrant.*

Similarly, if $Z\colon \Delta \to \mathcal{T}op$ is a cosimplicial space then there is a natural map $\operatorname{Tot} Z \to \operatorname{holim} Z$; this is a weak equivalence if Z is Reedy fibrant.

The proof of the above proposition requires more techniques than we have at the moment. However, we can at least describe the map. Recall that

$$\begin{aligned}\operatorname*{hocolim}_{\Delta^{op}} X &= \operatorname{coeq}\Biggl[\coprod_{[n]\to[k]} X_k \times B([n]\downarrow \Delta^{op})^{op} \rightrightarrows \coprod_n X_n \times B([n]\downarrow\Delta^{op})^{op}\Biggr]\\ &= \operatorname{coeq}\Biggl[\coprod_{[n]\to[k]} X_k \times B(\Delta\downarrow[n]) \rightrightarrows \coprod_n X_n \times B(\Delta\downarrow[n])\Biggr].\end{aligned}$$

Likewise, we have

$$|X| = \operatorname{coeq}\Biggl[\coprod_{[n]\to[k]} X_k \times \Delta^n \rightrightarrows \coprod_n X_n \times \Delta^n\Biggr].$$

We can produce a map $\operatorname{hocolim} X \to |X|$ by finding maps $\alpha_n\colon B(\Delta\downarrow[n]) \mapsto \Delta^n$ having the property that for every $\sigma\colon [n]\to[k]$ one gets a commutative square

$$\begin{array}{ccc} B(\Delta\downarrow[n]) & \xrightarrow{\sigma_*} & B(\Delta\downarrow[k]) \\ \downarrow & & \downarrow \\ \Delta^n & \xrightarrow{\sigma_*} & \Delta^k. \end{array}$$

We'll actually produce maps of simplicial sets $N(\Delta\downarrow[n]) \mapsto \Delta^n$. Recall that Δ^n is the simplicial set $[k] \mapsto \Delta([k],[n])$. A k-simplex in $N(\Delta\downarrow[n])$ is a string

$$[i_0] \to [i_1] \to \cdots \to [i_k] \to [n].$$

We can produce a map $[k]\to[n]$—that is, a k-simplex in Δ^n—by sending an element $j \in [k]$ to the image in $[n]$ of the last vertex of $[i_j]$ under the above composition of maps. Note that this gives a monotone increasing function $[k]\to[n]$, as desired. The resulting map $N(\Delta\downarrow[n]) \to \Delta^n$ is called the **last vertex map**. The reader may easily check that it gives the necessary commutative squares.

17.4. **Fat vs. non-fat.** Recall that $\Delta_f \subseteq \Delta$ is the subcategory consisting of all the co-face maps. A 'Δ-complex' is a functor $\Delta_f^{op} \to \mathcal{T}op$—it is a simplicial set without the degeneracy maps. If Z is a Δ-complex then the above Bousfield-Kan construction gives a natural map $\operatorname{hocolim}_{\Delta_f} Z \to ||Z||$.

So if X is a simplicial space one has the following square:

$$\begin{array}{ccc} \operatorname{hocolim}_{\Delta^{op}} X & \longrightarrow & |X| \\ \uparrow & & \uparrow \\ \operatorname{hocolim}_{\Delta_f^{op}} X & \longrightarrow & ||X|| \end{array}$$

Proposition 17.5. *If X is objectwise cofibrant, the two maps with domain* $\operatorname{hocolim}_{\Delta_f^{op}} X$ *are weak equivalences. If X is also Reedy cofibrant, the other two maps are weak equivalences as well.*

Part 5. Examples

18. Homotopy initial and terminal functors

In this section we present several specific examples of functors which are homotopy initial or terminal.

Our first example is a functor which is merely initial, not homotopy initial:

Example 18.1. Let $J \hookrightarrow \Delta$ denote the subcategory consisting of the objects [0], [1], and the two maps d^0, d^1 between them. We claim that $J \hookrightarrow \Delta$ is initial; this is equivalent to saying that $J^{op} \to \Delta^{op}$ is terminal. This will justify our claim from Section 3.7 that if X is a simplicial space then $\mathrm{colim}_{\Delta^{op}} X$ is homeomorphic to the coequalizer of $d_0, d_1 \colon X_1 \rightrightarrows X_0$.

To see that $i \colon J \hookrightarrow \Delta$ is initial, we must verify that for every $n \geq 0$ the category $(i \downarrow [n])$ is connected. The objects in this category consist of all maps $[0] \to [n]$ and all maps $[1] \to [n]$. Let $e_k \colon [0] \to [n]$ denote the map whose image is $\{k\}$. Let $f_k \colon [1] \to [n]$ denote the map whose image is $\{k\}$. Finally, if $k < l$ let $g_{k,l} \colon [1] \to [n]$ be the map sending $0 \mapsto k$ and $1 \mapsto l$. These are all the objects in $(i \downarrow [n])$.

One readily checks that there are maps in $(i \downarrow [n])$ from e_k to $g_{k,l}$, e_l to $g_{k,l}$, and from e_k to f_k. This proves that $(i \downarrow [n])$ is connected.

Example 18.2. Let $\Delta_f \hookrightarrow \Delta$ be the subcategory consisting of all maps which are monomorphisms (that is, all coface maps). We claim that the inclusion functor $i \colon \Delta_f \hookrightarrow \Delta$ is homotopy initial. As a consequence, i^{op} is homotopy terminal; so the homotopy colimit of a simplicial object can be obtained by instead taking the homotopy colimit of the object obtained by forgetting all degeneracies.

We must prove that for every $n \geq 0$, the overcategory $(i \downarrow [n])$ is contractible. To do this, consider the functor

$$F \colon (i \downarrow [n]) \longrightarrow (i \downarrow [n])$$

which sends a map $\sigma \colon [k] \to [n]$ to the map $F\sigma \colon [k+1] \to [n]$ given by

$$(F\sigma)(0) = 0, \qquad (F\sigma)(i) = \sigma(i-1) \text{ if } i \geq 1.$$

This becomes a functor in the evident way.

Let $e \colon [0] \to [n]$ denote the map whose image is 0, and let $E \colon (i \downarrow [n]) \to (i \downarrow [n])$ be the functor which sends every object to e and every map to the identity. We thus have three functors

$$F, id, e \colon (i \downarrow [n]) \longrightarrow (i \downarrow [n]).$$

The reader can check that there are natural transformations $id \to F$ and $e \to F$. This shows that upon taking classifying spaces the maps induced by F, id, and e are all homotopic. In particular, the identity map is null-homotopic—so $(i \downarrow [n])$ is contractible.

The argument from the above example actually shows the following. For each $\sigma \colon [k] \to [n]$ in Δ, let $sh(\sigma)$ denote the map $[k+1] \to [n+1]$ which sends $0 \mapsto 0$ and $i \mapsto \sigma(i-1) + 1$ for $i \geq 1$. So $sh(\sigma)$ is a 'shift' of the map σ.

Proposition 18.3. *Let $J \hookrightarrow \Delta$ be a subcategory satisfying the following:*

(1) For each map $\sigma \in J$, $sh(\sigma)$ is also in J;
(2) For each $n \geq 0$, the 'add 1 map' $[n] \to [n+1]$ given by $i \mapsto i+1$ belongs to J.
(3) For each $n \geq 0$, the map $[0] \to [n]$ whose image is $\{0\}$ belongs to J.

Then $J \hookrightarrow \Delta$ is homotopy initial.

The reader may check that Δ_f is the smallest subcategory of Δ satisfying the three conditions in the above proposition.

Exercise 18.4. Let Ω be the full subcategory of $\mathcal{S}et$ consisting of the objects $[0], [1], [2], \ldots$. Recall that $[n] = \{0, 1, \ldots, n\}$. Note that Δ is a subcategory of Ω, with the only maps in Δ being the monotone increasing functions.

Adapt the method used in Example 18.2 to prove that the inclusion $\Delta \hookrightarrow \Omega$ is homotopy initial.

Example 18.5. Consider the product category $\Delta \times \Delta$. Objects are pairs $([n_1], [n_2])$, and a map $([k_1], [k_2]) \to ([n_1], [n_2])$ simply consists of two maps $k_1 \to n_1$ and $k_2 \to n_2$.

Let $d\colon \Delta \to \Delta \times \Delta$ denote the diagonal functor. We claim that this is homotopy initial. As a consequence, d^{op} is homotopy terminal; so if $X_{*,*}$ is a bisimplicial space then its homotopy colimit is weakly equivalent to the homotopy colimit of the simplicial space $[n] \mapsto X_{n,n}$.

To justify the claim, we prove that $\big(d \downarrow ([p], [q])\big)$ is contractible for any p and q. The method is similar to that of the previous example. Recall that an object of $\big(d \downarrow ([p], [q])\big)$ consists of an object $[n]$ in Δ and a map $d([n]) \to ([p], [q])$. So we have an $[n]$ and two maps $[n] \to [p]$ and $[n] \to [q]$. Given an $[n']$ and two maps $[n'] \to [p]$ and $[n'] \to [q]$, a map from the first object to this one consists of a map $[n] \to [n']$ making the two evident triangles commute.

Let

$$F\colon \big(d \downarrow ([p], [q])\big) \longrightarrow \big(d \downarrow ([p], [q])\big)$$

be the functor which sends $([n], \sigma_1\colon [n] \to [p], \sigma_2\colon [n] \to [q])$ to the object $([n+1], [n+1] \to [p], [n+1] \to [q])$ where the first map sends $0 \mapsto 0$ and $i \mapsto \sigma_1(i-1)$ for $i \geq 1$, while the second map sends $0 \mapsto 0$ and $i \mapsto \sigma_2(i-1)$ for $i \geq 1$. The functor F has the evident behavior on maps.

Let $e\colon (d \downarrow ([p], [q])) \longrightarrow (d \downarrow ([p], [q]))$ denote the functor which sends all objects to $([0], e_0, e_0)$ where $e_0\colon [k] \to [n]$ always denotes the map whose image is $\{0\}$.

The reader can check that there are natural transformations $id \to F$ and $e \to F$. So after taking classifying spaces one finds that the identity is null-homotopic, and therefore $(d \downarrow ([p], [q]))$ is contractible.

18.6. **Truncated simplicial objects.** Let $\Delta_{\leq n}$ be the subcategory of Δ consisting of all objects $[k]$ where $k \leq n$. A functor $(\Delta_{\leq n})^{op} \to X$ is called an ***n*-truncated** simplicial space, or an ***n*-skeletal** simplicial space.

When taking homotopy colimits of an n-truncated simplicial space, one can no longer throw away the degeneracies and be guaranteed the same answer. That is, the subcategory of $(\Delta_{\leq n})^{op}$ consisting of the face maps is no longer homotopy final. One can see that the proof in Example 18.2 breaks down, as that proof used the infinite nature of the category Δ. Still, there is a nice reduction one can make.

Let Sub_n be the poset of subsets of $\{0, 1, \ldots, n\}$, ordered by inclusion, regarded as a category in the usual way. A picture of this category would look like an n-cube, hence the name. Note that Sub_n can also be thought of as the category of sub-simplices of Δ^n—so the sub-simplices of a simplex form a cube! Let iSub_n be the full subcategory consisting of all objects except $\{0, 1, \ldots, n\}$ (the 'i' is for 'initial').

Notice that there is a functor $\Gamma\colon \mathrm{iSub}_n \to \Delta_{\leq n}$, defined as follows. For any subset $S = \{i_0, \ldots, i_k\}$ of $[n]$, there is a unique order-preserving bijection between

S and $[k]$. Using this, an inclusion of subsets gives rise to an inclusion in $\Delta_{\leq n}$. The map Γ just sends the subset S to $[k]$, and has the evident behavior on maps. For instance, the inclusion $\{1\} \hookrightarrow \{0,1\}$ is sent to the map $[0] \mapsto [1]$ whose image is 1; the inclusion $\{1,3\} \hookrightarrow \{1,2,3\}$ is send to the map $[1] \to [2]$ whose image is $\{0,2\}$.

Proposition 18.7. *The functor* $\Gamma\colon \text{iSub}_n \to \Delta_{\leq n}$ *is homotopy initial. So* $\Gamma^{op}\colon (\text{iSub}_n)^{op} \to (\Delta_{\leq n})^{op}$ *is homotopy terminal.*

Remark 18.8. Let X be an n-truncated simplicial object. The above proposition shows that when computing $\operatorname{hocolim} X$ the degeneracies don't really matter—in the sense that one can write down a cubical diagram, using only face maps, whose homotopy colimit is $\operatorname{hocolim} X$. However, this does *not* say that if you look at the subdiagram of X consisting only of face maps that the homotopy colimit of *that* diagram is also the same as $\operatorname{hocolim} X$. The subcategory of $(\Delta_{\leq n})^{op}$ consisting of the face maps is not homotopy final!

The proof of the above proposition is more involved than what we have done so far. The classifying spaces of the overcategories are somewhat complicated, and their contractibility has to be proven by a combinatorial argument. A nice reference in the literature is [Si, Section 6].

Let $I_{n,k}$ denote the overcategory $(\text{iSub}_n \downarrow [k])$, where $k \leq n$. Note that an object of $I_{n,k}$ is a pair (σ, ϕ) where $\sigma \subseteq [n]$ and $\phi\colon \Gamma(\sigma) \to [k]$ is an order-preserving map. It is useful to drop the 'Γ', and regard ϕ just as an order-preserving map $\sigma \to [k]$. To have a map $(\sigma,\phi) \to (\sigma',\phi')$ means that $\sigma \subseteq \sigma'$ and ϕ is the restriction of ϕ'. From this it is easy to see that $I_{n,k}$ is a poset.

We wish to ultimately show that each $I_{n,k}$ is contractible, but we'll start by describing a certain stratification of $I_{n,k}$. For each order-preserving map $\alpha\colon [n] \to [k]$, let J_α denote the full subcategory of $I_{n,k}$ consisting of pairs (σ,ϕ) such that ϕ is the restriction of α. It's easy to check that J_α is isomorphic to the category iSub_n (in effect, the data in ϕ is redundant), and so the nerve of J_α is $\operatorname{sd} \Delta^n$.

If α and β are maps $[n] \to [k]$ in Δ, then $J_\alpha \cap J_\beta$ consists of pairs (σ,ϕ) such that ϕ is the restriction of both α and β. If we let S denote the maximal subset of $[n]$ on which α and β agree, then $J_\alpha \cap J_\beta$ is isomorphic to the category of subsets of S; hence $J_\alpha \cap J_\beta$ is $\operatorname{sd} \Delta^i$ for some i (or empty). This same reasoning applies to any iterated intersection $J_{\alpha_1} \cap J_{\alpha_2} \cap \ldots \cap J_{\alpha_l}$.

Order-preserving maps $[n] \to [k]$ are in bijective correspondence with monotone increasing sequences of length $n+1$, with values in $\{0,1,\ldots,k\}$. There are $\binom{n+k+1}{k}$ such sequences (they are in bijective correspondence with monomials of degree $n+1$ in the variables $X_0, X_1, \ldots, X_k$, where the exponent of X_i is the number of times i appears in the sequence). So we have seen how to decompose the nerve of $I_{n,k}$ into $\binom{n+k+1}{k}$ copies of $\operatorname{sd} \Delta^n$, and the intersection of any number of these copies is a copy of $\operatorname{sd} \Delta^i$ for some i (or else empty).

Exercise 18.9. Using the above description, work out explicit pictures of $I_{1,1}$, $I_{2,1}$, and $I_{2,2}$. The first, for instance, is the union of 3 copies of $\operatorname{sd} \Delta^1$, glued together in a certain way.

The above description tells us that the nerve of $I_{n,k}$ is the barycentric subdivision of a certain complex we'll call $L_{n,k}$. We can describe this complex as follows:

(1) The n-simplices correspond to monotone increasing sequences $a_0 \ldots a_n$ whose values are in $\{0,\ldots,k\}$ (i.e., to maps $[n] \to [k]$).

(2) The $(n-i)$-simplices correspond to sequences as in (1) except where i of the a_j's have been replaced by the symbol '?'.
(3) The face-map d_i corresponds to replacing the ith entry of the sequence with a '?'.

For instance, in $L_{1,1}$ there are three 1-simplices, indexed by the sequences 00, 01, and 11. We have that $d_1(00) = 0?$ and $d_1(01) = 0?$, etc. So $L_{1,1}$ consists of three 1-simplices which are glued together sequentially, with one pair head-to-head and the other pair tail-to-tail: $\cdot \to \cdot \leftarrow \cdot \to \cdot$

We need to show that $L_{n,k}$ is contractible.

Lemma 18.10. *Let X be a simplicial complex which is purely of dimension d (meaning that every simplex is contained in a d-simplex). Suppose the d-simplices can be ordered as $F_1, F_2, \ldots, F_M$ in such a way that for each $i \geq 1$*

(a) the subcomplex $F_{i+1} \cup F_{i+2} \cup \cdots \cup F_M$ intersects F_i purely in dimension $d-1$, and

(b) F_i has at least one face which is not in $F_{i+1} \cup \cdots \cup F_M$.

Then X is contractible.

Proof. The main point is that if σ is an n-simplex and S is any union of codimension one faces forming a *proper* subset of $\partial\sigma$, then there is a deformation retraction of σ onto S. Since the n-simplex F_1 has a face which is not in $F_2 \cup \cdots \cup F_M$, we can therefore deformation-retract X down to $F_2 \cup \cdots \cup F_M$. Now proceed by induction, at each step choosing a deformation retration of $F_k \cup \cdots \cup F_M$ down to $F_{k+1} \cup \cdots \cup F_M$. □

Proposition 18.11. *If $k \leq n$, the nerve of $L_{n,k}$ is contractible.*

Proof. We have already seen that $L_{n,k}$ is purely n-dimensional. The n-simplices of $L_{n,k}$ are indexed by monotone increasing sequences of length $n+1$ with values in $\{0, 1, \ldots, k\}$, and we can order these lexicographically. We claim that this ordering satisfies the conditions of the lemma.

Let F be the n-simplex corresponding to a sequence $a_0a_1 \ldots a_n$. If $a_i = a_{i+1}$ for some i, then the face of F corresponding to $a_0a_1 \ldots a_{i-1}?a_{i+1} \ldots a_n$ only belongs to n-simplices which come before F in the ordering (because such an n-simplex would have the "?" replaced with a number $j \leq a_{i+1}$, and we would then have $j \leq a_i$ as well). On the other hand, if the sequence $a_0a_1 \ldots a_n$ has no repeats then it means that $n = k$ and we are looking at the sequence $0, 1, \ldots, n$. In this case, the face of F corresponding to $0, 1, \ldots, n-1, ?$ only belongs to n-simplices which come before F in the ordering. This proves property (b).

To prove property (a), suppose that F meets an n-simplex G corresponding to the sequence $b_0b_1 \ldots b_n$, where $\{b\}$ is lexicographically greater than $\{a\}$. We need to find a sequence $\{c\}$ which is also lexicographically greater than $\{a\}$, such that the intersection of F with the $\{c\}$-simplex contains an $(n-1)$-simplex which itself contains $F \cap G$.

Let j be the smallest index for which $a_j \neq b_j$. Then $F \cap G$ is contained entirely in the $(n-1)$-simplex $a_0 \ldots a_{j-1}?a_{j+1} \ldots a_n$. Note that we cannot have $a_j = k$, since $a_j < b_j$. If $a_0a_1 \ldots a_{j-1}(a_j+1)a_{j+1} \ldots a_n$ is a monotone increasing sequence then we can take it as our $\{c\}$—the corresponding n-simplex intersects F in the codimension 1 face $a_0 \ldots a_{j-1}?a_{j+1} \ldots a_n$, and this face contains $F \cap G$.

If the sequence $a_0a_1\dots a_{j-1}(a_j+1)a_{j+1}\dots a_n$ is *not* monotone increasing then this means $a_j = a_{j+1} = \cdots = a_{j+p}$ for some $p \geq 1$ (where we choose p as large as possible). Since $a_j < b_j$ we must have $a_i \neq b_i$ for $i \in [j, j+p]$—in particular, $a_{j+p} \neq b_{j+p}$. In this case take $\{c\}$ to be the sequence $a_0a_1\dots a_{j+p-1}(a_{j+p}+1)a_{j+p+1}\dots a_n$. Then the intersection of F and the simplex corresponding to $\{c\}$ contains the codimension one face $a_0a_1\dots a_{j+p-1}?a_{j+p+1}\dots a_n$, which in turn contains $F\cap G$. □

18.12. **Homotopical symmetric products.** This will be the final example of this section. Let (n) denote the finite set $\{1, 2, \dots, n\}$, and let I denote the category whose objects are all such sets (with $n \geq 1$) and where the maps are monomorphisms. The category I is similar to Δ_f, except that we have now expanded the morphisms to include permutations.

Let X be a pointed space. For every map $\sigma\colon (n) \to (k)$ in I, there is an induced map $\sigma_*\colon X^n \to X^k$ sending $(x_1, \dots, x_n)$ to the tuple with x_i in spot $\sigma(i)$ and the basepoint in all other spots. This gives a diagram $X^*\colon I \to \mathcal{T}op$.

Our first goal will be to show that the colimit of X^* is isomorphic to something more familiar, namely the infinite symmetric product of X. The latter is the space $\mathrm{SP}^\infty(X) = X^\infty/\Sigma_\infty$, where X^∞ is the colimit of the sequence

$$X \hookrightarrow X^2 \hookrightarrow X^3 \hookrightarrow \cdots$$

in which each map sends $(x_1, \dots, x_n)$ to $(x_1, \dots, x_n, *)$. To see that $\mathrm{colim}_I X^*$ and $\mathrm{SP}^\infty(X)$ are isomorphic, follow the steps in the exercise below.

Exercise 18.13. Let $\omega = \{1, 2, 3, \dots\}$, and let I_∞ be the subcategory of $\mathcal{S}et$ consisting of the objects (n) (for all $n \geq 1$) and ω, where the maps as follows:

- maps from (n) to (k) are the monomorphisms;
- maps from (n) to ω are the monomorphisms;
- maps from ω to ω are the elements of Σ_∞.

Let $j\colon I \hookrightarrow I_\infty$ be the inclusion. Finally, let I_{std} be the subcategory of I consisting of all objects (n) but where the morphisms are the *standard* inclusions $(n) \hookrightarrow (k)$.

(a) If $D\colon I \to \mathcal{T}op$, let $L_jD = \mathrm{colim}_{I\to I_\infty} D$ be the relative colimit (or left Kan extension) of D along j. Recall that $[L_jD](\omega) \cong \mathrm{colim}_{n\in(j\downarrow\omega)} D_n$. Note that there is an evident functor $I_{std} \to (j \downarrow \omega)$, and prove that this is terminal. Deduce that $[L_jD](\omega) \cong \mathrm{colim}_{I_{std}} D$.

(b) Let $B\Sigma_\infty$ denote the category with one object and endomorphism set Σ_∞. Note that there is an evident functor $B\Sigma_\infty \to I_\omega$ sending the unique object to ω. Prove that this functor is terminal.

(c) If $D\colon I \to \mathcal{T}op$ is any diagram, argue that $\mathrm{colim}\, D$ is isomorphic to $\mathrm{colim}[L_jD]$. Use (b) to deduce that the latter is is isomorphic to $[L_jD](\omega)/\Sigma_\infty$, and use (a) to replace $[L_jD](\omega)$ with $\mathrm{colim}_{I_{std}} D$. When $D = X^*$, deduce that $\mathrm{colim}_I X^* \cong X^\infty/\Sigma_\infty$.

We now wish to consider the homotopy colimit $\mathrm{hocolim}_I X^*$; it is natural to call this the *homotopical infinite symmetric product* of X. We'll use the notation $\mathrm{SP}^h(X) = \mathrm{hocolim}_I X^*$. This construction was probably first considered by Jeff Smith, who used it in the context of symmetric spectra—the first reference I know in print is [Sh, Section 1]. The spaces $\mathrm{SP}^h(X)$ were later intensively studied in [Schl], where it was shown that if X is path-connected then $\mathrm{SP}^h(X) \simeq \Omega^\infty\Sigma^\infty X$; this is related to the Barratt-Priddy-Quillen theorem.

We proved in the previous exercise that $\operatorname{colim}_I X^* \cong (X^\infty)/\Sigma_\infty$, and from this it would be a natural guess that $\operatorname{hocolim}_I X^* \simeq (X^\infty)_{h\Sigma^\infty}$. However, this guess is incorrect (for reasons which we will see below). To correct the guess, recall that $\omega = \{1, 2, 3, \ldots\}$. Let M denote the injective self-maps of ω, which form a monoid under composition. Clearly we have $\Sigma_\infty \subseteq M$, and if X is a pointed space there is a natural action of M on X^∞ which extends the action of Σ_∞. We have the following nice result, which is [Schl, Proposition 3.7].

Proposition 18.14. *If X is a well-pointed CW-complex, then* $\mathrm{SP}^h(X) \simeq (X^\infty)_{hM}$.

We'll outline the proof of this following [Schl], leaving most steps as exercises for the reader. First, let I_ω denote the subcategory of $\mathcal{S}et$ whose objects are the sets (n) together with ω, and where the maps are the monomorphisms. Let $j\colon I \hookrightarrow I_\omega$ denote the evident inclusion. Finally, let BM denote the category with one object and endomorphism set M, and let $i\colon BM \to I_\omega$ denote the inclusion sending the unique object of BM to ω.

The following exercise contains a key result due to J. Smith, which first appeared in [Sh, Lemma 2.2.9]. I owe my understanding of this proof to Stefan Schwede, and the proof we outline below is entirely from [Sch].

Exercise 18.15. Prove that $j\colon BM \hookrightarrow I_\omega$ is homotopy terminal by following the steps below.

(a) Define a functor $c\colon M \to M$ by the following formula: if $f \in M$, then
$$(cf)(i) = \begin{cases} i & \text{if } i \text{ is odd,} \\ 2 \cdot f(i/2) & \text{if } i \text{ is even.} \end{cases}$$
Verify that c is a homomorphism of monoids, and therefore induces a functor $Bc\colon BM_{cat} \to BM_{cat}$.

(b) Construct a natural transformation $id \to Bc$, as well as a natural transformation $Bc \to *$ where $*$ is the functor which sends all morphisms to the identity. Conclude that on classifying spaces one has $id \simeq Bc \simeq *$ as maps $BM \to BM$, and therefore BM is contractible.

(c) Fix $n \geq 1$. For any $\alpha \in M$, let $\alpha + n$ be the element of M which is the identity on the numbers $1, 2, \ldots, n$ and sends $n+i$ to $n+\alpha(i)$ for $i \geq 1$. Define a functor $BM_{cat} \to ((n) \downarrow j)$ which sends the unique object to the standard inclusion $(n) \hookrightarrow \omega$ and which sends the morphism $\alpha \in M$ to $\alpha + n$. Verify that that this is indeed a functor, that it is fully faithful, and that it is surjective on isomorphism classes—so conclude that it as an equivalence of categories.

(d) Deduce that $((n) \downarrow j)$ is contractible, for all $n \geq 1$. Prove that $(\omega \downarrow j)$ has an initial object and is therefore also contractible. Conclude that j is homotopy terminal.

Exercise 18.16. Now let $D\colon I \to \mathcal{T}op$ be any diagram. Let $L_j D$ denote the homotopy left Kan extension of D along the inclusion $j\colon I \hookrightarrow I_\omega$.

(a) Prove that there are weak equivalences
$$\operatorname*{hocolim}_I D \simeq \operatorname*{hocolim}_{I_\omega}(L_j D) \simeq \operatorname*{hocolim}_{BM}(L_j D)(\omega) = [(L_j D)(\omega)]_{hM}.$$

(b) Prove that there is a weak equivalence $(L_j D)(\omega) \simeq \operatorname{hocolim}_{I_{std}} D$.

(c) Prove that if the maps in $D\colon I_{std} \to \mathcal{T}op$ are all cofibrations, then $\operatorname{hocolim}_{I_{std}} D \simeq \operatorname{colim}_{I_{std}} D$.

(d) Conclude that if X is a well-pointed CW-complex then $\operatorname{hocolim}_I X^* \simeq (X^\infty)_{hM}$.

Remark 18.17. The main difference between the work in Exercises 18.13 and 18.15 is that in the latter we must use I_ω instead of I_∞. The reason is that although $B\Sigma_\infty \to I_\infty$ is terminal, it is not homotopy terminal; this is why the monoid M, rather than Σ_∞, appears in Proposition 18.14.

Exercise 18.18. Prove that $B\Sigma_\infty \to I_\infty$ is not homotopy terminal.

19. Homotopical decompositions of spaces

By a "homotopical decomposition" of a space X we mean a diagram $D\colon I \to \mathcal{T}op$ together with a map $\operatorname{colim}_I D \to X$, such that the composite $\operatorname{hocolim}_I D \to \operatorname{colim}_I D \to X$ is a weak equivalence. Note that by Proposition 15.9 a homotopical decomposition yields, in particular, a spectral sequence for computing the cohomology groups $\mathcal{E}^*(X)$ from the groups $\mathcal{E}^*(D_i)$.

We have already seen one example of a homotopical decomposition, back in Section 13.6. If $\{A_1, \dots, A_n\}$ is a closed cover of X then one can form the cubical diagram $A\colon P_n \to \mathcal{T}op$ sending a subset $\{i_1, \dots, i_k\}$ to $A_{i_1} \cap \dots \cap A_{i_k}$. Under the condition of certain inclusions being cofibrations, this is a homotopical decomposition.

Note that giving a diagram $D\colon I \to \mathcal{T}op$ together with a map $\operatorname{colim}_I D \to X$ is the same as giving a diagram $I \to (\mathcal{T}op \downarrow X)$. If we let $\Gamma\colon (\mathcal{T}op \downarrow X) \to \mathcal{T}op$ denote the forgetful functor sending the pair $[Y, Y \to X]$ to Y, then D is just the composite

$$I \longrightarrow (\mathcal{T}op \downarrow X) \xrightarrow{\Gamma} \mathcal{T}op.$$

In many applications I is actually a subcategory of $(\mathcal{T}op \downarrow X)$.

Homotopical decompositions seem to be useful in a variety of situations. In this section we will give a few examples of these decompositions.

Here is one example worth recording:

Proposition 19.1. *Let $\{U_\alpha\}$ be an open cover of X. Let I be the subcategory of $(\mathcal{T}op \downarrow X)$ consisting of the U_α's and all their finite intersections. Then*

$$\operatorname*{hocolim}_I \Gamma \to X$$

is a weak equivalence.

Proof. See [DI]. □

Before proceeding to another important example, we need a new tool. To set this in context, all of the theorems we stated in Parts 1–3 about homotopy colimits are actually generic results which work basically the same in any model category (not just in $\mathcal{T}op$). The following result is very particular to $\mathcal{T}op$, however.

Let $D\colon I \to \mathcal{T}op$ and suppose one has a map $p\colon \operatorname{colim}_I D \to X$, where X is some space. For each n and each map $\sigma\colon \Delta^n \to X$, consider the category $F(D)_\sigma$ whose objects are tuples

$$[i, \alpha\colon \Delta^n \to D_i]$$

such that $p \circ \alpha = \sigma$. A map from this object to $[j, \beta\colon \Delta^n \to D_j]$ is a map $i \to j$ making the evident triangle commute. We call $F(D)_\sigma$ the **fiber category** of D over σ.

Theorem 19.2. *In the above setting, suppose that for each $n \geq 0$ and each $\sigma\colon \Delta^n \to X$, the category $F(D)_\sigma$ is contractible. Then the composite* $\operatorname{hocolim}_I D \to \operatorname{colim}_I D \to X$ *is a weak equivalence.*

Now assume in addition that there is a diagram $\tilde{D}\colon I \to s\mathcal{S}et$ and a natural isomorphism $\phi_i\colon |\tilde{D}_i| \to D_i$. For each $\sigma\colon \Delta^n \to X$ we can define a new category $\tilde{F}(D)_\sigma$ as follows. Objects of this category are pairs $[i, \Delta^n_s \to \tilde{D}_i]$ such that the composite $|\Delta^n_s| \to |\tilde{D}_i| \to D_i \to X$ is equal to α, and maps are the expected things. Here Δ^n_s denote the canonical n-simplex $\Delta^n_s \in s\mathcal{S}et$. Note that there is a map of categories $\tilde{F}(D)_\sigma \to F(D)_\sigma$, but there's no reason to suspect that this is a weak equivalence.

We have the following refinement of the previous theorem:

Theorem 19.3. *In the above setting, suppose that for each* $n \geq 0$ *and each* $\sigma\colon \Delta^n \to X$*, the category* $\tilde{F}(D)_\sigma$ *is contractible. Then the composite* $\operatorname{hocolim}_I D \to \operatorname{colim}_I D \to X$ *is a weak equivalence.*

We will give the proofs of Theorems 19.2 and 19.3 in Section 19.8 below. But first we record some useful applications. These are all inspired by the discussion in [J, Section 2].

Proposition 19.4. *Let* $\Delta \downarrow X$ *denote the overcategory* $(j \downarrow X)$*, where* $j\colon \Delta \to \mathcal{T}op$ *is the usual functor. The functor* j *gives us a map* $(\Delta \downarrow X) \to (\mathcal{T}op \downarrow X)$*, and the natural map*

$$\operatorname*{hocolim}_{(\Delta\downarrow X)} j^*\Gamma \to \operatorname*{colim}_{(\Delta\downarrow X)} j^*\Gamma \to X$$

is a weak equivalence.

Proof. Note that the diagram $j^*\Gamma$ lifts to a diagram $\tilde{\Gamma}\colon (\Delta \downarrow X) \to s\mathcal{S}et$. So we can attempt to use Theorem 19.3.

Let $I = (\Delta \downarrow X)$, and let $\sigma\colon \Delta^n \to X$. An object of $\tilde{F}(\Gamma)_\sigma$ consists of an object $[k]$, a map $f\colon \Delta^k \to X$, and a simplicial map $\Delta^n \to \Delta^k$ whose composite with f is σ. But note that this category has an initial object, given by $[n]$, the map $\sigma\colon \Delta^n \to X$, and the identity map $\Delta^n \to \Delta^n$. So $\tilde{F}(\Gamma)_\sigma$ is contractible, and we are done. □

Proposition 19.5. *Let* $\Delta_c(X)$ *be the full subcategory of* $\mathcal{T}op \downarrow X$ *consisting of all maps whose domain is a simplex. Then* $\operatorname{hocolim}_{\Delta_c(X)} \Gamma \to X$ *is a weak equivalence.*

Proof. This is a consequence of Theorem 19.2. The same kind of argument as in the previous proof shows that the fiber categories $F(\Gamma)_\sigma$ are all contractible. □

Now let $p\colon E \to B$ be a map, and let $\alpha\colon I \to \mathcal{T}op \downarrow B$ be a functor. Let Γ_p denote the diagram $I \to \mathcal{T}op$ sending i to $\alpha(i)^*E$, the pullback of $E \to B$ along the map $\alpha(i)$. Clearly there is a map $\operatorname{colim}_I \Gamma_p \to E$, and so we may consider the composite

$$\operatorname*{hocolim}_I \Gamma_p \to \operatorname*{colim}_I \Gamma_p \to E.$$

Proposition 19.6. *In the above setting, let* $I = (\Delta \downarrow B)$*. Then the map* $\operatorname{hocolim}_I \Gamma_p \to E$ *is a weak equivalence, for any map* p *which is a fibration.*

Proof. Consider the diagram $D\colon I \to \mathcal{T}op$ which sends a pair $([k], \Delta^k \to B)$ to the geometric realization of the simplicial set obtained as the pullback $\Delta^k_s \to SB \leftarrow SE$, where $S(-)$ is the singular functor.

There is an evident map of diagrams $|D| \to \Gamma_p$, and the fact that p is a fibration implies that this map is an objectwise weak equivalence. One uses here that $SE \to SB$ is a fibration of simplicial sets, and that in $s\mathcal{S}et$ a pullback of a weak equivalence along a fibration is another weak equivalence.

So we are reduced to showing that $\operatorname{hocolim}_I |D| \to X$ is a weak equivalence. This is an easy application of Theorem 19.3, very similar to the proof of Proposition 19.4. □

The following corollary is now immediate from Proposition 15.9.

Corollary 19.7. *If $p\colon E \to B$ is a fibration, then for any cohomology theory $\mathcal{E}$ there is a spectral sequence*

$$E_2^{p,q} = H^p(\Delta \downarrow B; \mathcal{E}^q(\Gamma_p)) \Rightarrow \mathcal{E}^{p+q}(E).$$

Note that for each simplex $\sigma\colon \Delta^n \to B$, the space $\Gamma_p(\sigma) = \sigma^* E$ is weakly equivalent to the fiber F of p. So the diagram $\mathcal{E}^q\Gamma_p$ is a diagram of abelian groups where all the abelian groups are isomorphic. One can also check that every map in the diagram is an isomorphism. So this is something which should be called a "local coefficient system". The above spectral sequence is a version of the generalized Atiyah-Hirzebruch/Leray-Serre spectral sequence.

19.8. **Proofs of the two main theorems.** The two proofs are both based on an analogous theorem about simplicial sets. Let $D\colon I \to s\mathcal{S}et$ be a diagram of simplicial sets, let $X \in s\mathcal{S}et$, and suppose there is a map $\operatorname{colim}_I D \to X$. For each simplex $\sigma \in X_n$, let $F(D)_\sigma$ denote the category whose objects are pairs $[i, \alpha \in (D_i)_n]$ such that the map $D_i \to X_i$ sends α to σ. A map in $F(D)_\sigma$ from $[i, \alpha \in (D_i)_n]$ to $[j, \beta \in (D_j)_n]$ is a map $i \to j$ such that $D_i \to D_j$ sends α to β. We call $F(D)_\sigma$ the "fiber category" of D over σ.

The following result is a slight generalization of [J, Lemma 2.7]. The proof, however, is exactly the same.

Proposition 19.9. *Suppose that $D\colon I \to s\mathcal{S}et$ and X are as above, and assume that for every $n \geq 0$ and every $\sigma \in X_n$, the fiber category $F(D)_\sigma$ is contractible. Then the map $\operatorname{hocolim}_I D \to X$ is a weak equivalence of simplicial sets.*

Proof. Consider the simplicial replacement $\operatorname{srep}(D)$, and observe that this is a bisimplicial set. Let us write $\operatorname{srep}(D)_{p,q}$ for the q-simplices in the pth level of $\operatorname{srep}(D)$; that is so say,

$$\operatorname{srep}(D)_{p,q} = \coprod_{i_0 \leftarrow \cdots \leftarrow i_p} D(i_p)_q.$$

When drawing the bisimplicial set we draw the q-direction vertically and the p-direction horizontally.

If $B_{*,*}$ is a bisimplicial set, then there are two geometric realizations of B, depending on whether we realize vertically or horizontally. Define

$$|B|_h = \operatorname{coeq}\left[\coprod_{[n]\to[k]} B_{k,*} \times \Delta^n \rightrightarrows \coprod_n B_{n,*} \times \Delta^n\right]$$

and

$$|B|_v = \operatorname{coeq}\left[\coprod_{[n]\to[k]} B_{*,k} \times \Delta^n \rightrightarrows \coprod_n B_{*,n} \times \Delta^n.\right]$$

Note that $\operatorname{hocolim}_I D = |\operatorname{srep}(D)|_h$ in this notation.

Let $d(B)$ denote the diagonal simplicial set of B. Then we know there are natural maps $|B|_h \to d(B) \leftarrow |B|_v$ and that these are both isomorphisms.

Let $c_h X$ denote the bisimplicial set with $(c_h X)_{p,q} = X_q$, where all the horizontal faces and degeneracies are the identity map. This bisimplicial set is 'horizontally constant'.

There is a natural map of bisimplicial sets $\mathrm{srep}(D) \to c_h X$. This gives a commutative diagram

$$\begin{array}{ccccc} |\,\mathrm{srep}(D)|_h & \xrightarrow{\cong} & d(\mathrm{srep}(D)) & \xleftarrow{\cong} & |\,\mathrm{srep}(D)|_v \\ \downarrow & & \downarrow & & \downarrow \\ |c_h X|_h & \xrightarrow[\cong]{} & d(c_h X) & \xleftarrow[\cong]{} & |c_h X|_v. \end{array}$$

Our goal is to show that the left vertical map is a weak equivalence, and so it will suffice to show that the right vertical map is a weak equivalence.

We will argue that each map of simplicial sets $\mathrm{srep}(D)_{*,q} \to (c_h X)_{*,q}$ is a weak equivalence. This will imply that we get a weak equivalence after applying the vertical geometric realization.

Note that $(c_h X)_{*,q}$ is just the discrete simplicial set corresponding to the set X_q. So it will suffice to prove that the fiber of the map $\pi_q \colon \mathrm{srep}(D)_{*,q} \to X_q$ over any point is contractible. But if $\sigma \in X_q$, then one readily checks that the fiber of π_q over σ is the nerve of the category $F(D)_\sigma$, and hence is contractible by assumption. □

We can now give the proofs of our two theorems:

Proof of Theorem 19.2. Let $\mathrm{Sing}\colon \mathcal{T}op \to s\mathcal{S}et$ denote the usual singular functor. Applying this to D gives a diagram $\mathrm{Sing}\, D \colon I \to s\mathcal{S}et$, together with an induced map $\mathrm{colim}(\mathrm{Sing}\, D) \to \mathrm{Sing}\, X$. An n-simplex of $\mathrm{Sing}\, X$ is just a map $\sigma\colon \Delta^n \to X$, and the fiber category $F(\mathrm{Sing}\, D)_\sigma$ from Proposition 19.9 is precisely the fiber category $F(D)_\sigma$ from the statement of the theorem. Sincee these fiber categories are assumed to be contractible, Proposition 19.9 says that $\mathrm{hocolim}_I(\mathrm{Sing}\, D) \to \mathrm{Sing}\, X$ is a weak equivalence of simplicial sets.

The final step is to apply geometric realization to the above map, and then to use the following commutative diagram:

$$\begin{array}{ccccc} |\,\mathrm{hocolim}(\mathrm{Sing}\, D)| & \longrightarrow & |\,\mathrm{colim}(\mathrm{Sing}\, D)| & \longrightarrow & |\,\mathrm{Sing}\, X| \\ \uparrow\cong & & \uparrow\cong & & \| \\ \mathrm{hocolim}\,|\,\mathrm{Sing}\, D| & \longrightarrow & \mathrm{colim}\,|\,\mathrm{Sing}\, D| & \longrightarrow & |\,\mathrm{Sing}\, X| \\ \downarrow\sim & & \downarrow & & \downarrow\sim \\ \mathrm{hocolim}\, D & \longrightarrow & \mathrm{colim}\, D & \longrightarrow & X. \end{array}$$

We know from the previous paragraph that the composite across the top row is a weak equivalence. The two-out-of-three property then shows that the composite across the bottom row is also a weak equivalence. □

Proof of Theorem 19.3. This proof is similar to the preceeding one. The natural maps $|\tilde{D}_i| \to D_i$ and $D_i \to X$ allow us to consider the composites

$$\tilde{D}_i \to \mathrm{Sing}\,|\tilde{D}_i| \to \mathrm{Sing}\, D_i \to \mathrm{Sing}\, X.$$

These are compatible as i varies, so we have a map $\mathrm{colim}_I\, \tilde{D} \to \mathrm{Sing}\, X$. The assumptions of the theorem say precisely that the fiber categories $F(\tilde{D})_\sigma$ are contractible, for every simplex σ of $\mathrm{Sing}\, X$. By Proposition 19.9 we therefore have that $\mathrm{hocolim}_I\, \tilde{D} \to \mathrm{Sing}\, X$ is a weak equivalence.

To complete the proof one considers the following diagram:

$$\begin{array}{ccccc}
|\operatorname{hocolim} \tilde{D}| & \xrightarrow{\sim} & |\operatorname{hocolim}(\operatorname{Sing} D)| & \longrightarrow & |\operatorname{Sing} X| \\
\downarrow \cong & & \downarrow \cong & & \| \\
\operatorname{hocolim} |\tilde{D}| & \xrightarrow{\sim} & \operatorname{hocolim} |\operatorname{Sing} D| & \longrightarrow & |\operatorname{Sing} X| \\
 & & \downarrow & & \downarrow \sim \\
 & & \operatorname{hocolim} D & \longrightarrow & X.
\end{array}$$

We have proven that $\operatorname{hocolim}_I \tilde{D} \to \operatorname{Sing} X$ is a weak equivalence. Our assumption that the maps $|\tilde{D}_i| \to D_i$ are weak equivalences implies that $\operatorname{hocolim} |\tilde{D}| \to \operatorname{hocolim} D$ is a weak equivalence. The two-out-of-three property, applied several times, now gives that $\operatorname{hocolim} D_i \to X$ is a weak equivalence. $\square$

20. A survey of other applications

20.1. Telescopes and the localization of spaces. ????

20.2. Homotopy decompositions of classifying spaces. ?????

20.3. Homotopical sheaf theory. ?????

20.4. Further directions. In this final section we mention aspects of the theory of homotopy limits and colimits which we have not addressed here. We also suggest some other references.

(1) A very general approach to homotopy limits and colimits, and particularly their role as derived functors, can be found in [DHKS].

(2) Let I be a topological category—that is, a category where the morphism sets have the structure of topological spaces, and where composition is continuous. An **enriched diagram** $X\colon I \to \mathcal{T}op$ consists of a topological space $X(i)$ for every $i \in I$, together with continuous maps of spaces $I(i,j) \to \mathrm{Map}(X(i), X(j))$ which are compatible with composition and identities.

One important example of this is when G is a topological group, and I is the topological category with one object whose endomorphisms are G. An enriched diagram $X\colon I \to \mathcal{T}op$ consists of a space $X(*)$ and a continuous group action $G \times X(*) \to X(*)$.

One can ask for a theory of enriched homotopy colimits and limits. This has been developed recently in [S].

(3) Section 5 of Thomason's paper [T] contains a very compact and appealing treatment of homotopy limits and colimits, their associated spectral sequences, as well as a "Scholium of Great Enlightenment". We highly recommend it.

References

[BK] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lect. Notes in Math., vol. 304, Springer-Verlag, New York, 1972.

[CS] W. Chachólski and J. Scherer, *Homotopy theory of diagrams*, Mem. Amer. Math. Soc. no. 736, 2002.

[C] E. Curtis, *Simplicial homotopy theory*, Advances in Math. **6** (1971), 107–209.

[D] D. Dugger, *Combinatorial model categories have presentations*, Adv. Math. **164** (2001), 177–201.

[DI] D. Dugger, D. C. Isaksen, *Topological hypercovers and $\mathbb{A}^1$-realizations*, Math. Z. **246** (2004), 667–689.

[Dw1] W. G. Dwyer, *Homology decompositions for classifying spaces of finite groups*.

[Dw2] W. G. Dwyer, *Classifying spaces and homology decompositions*, ????

[DHKS] W. G. Dwyer, P. S. Hirschhorn, D. M. Kan, and J.H. Smith, *Model categories and homotopical categories*, Mathematical Surveys and Monographs, vol. 113, Amer. Math. Soc., 2004.

[DK] W. G. Dwyer and D. M. Kan, *Function complexes in homotopical algebra*, Topology **19** (1980), 427–440.

[DS] W. G. Dwyer and A. Spalinski, *Homotopy theories and model categories*. Handbook of algebraic topology, 73–126, North-Holland, Amsterdam, 1995.

[H] P. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, Amer. Math. Soc., 2003.

[HV] J. Hollender and R. M. Vogt, *Modules of topological spaces, applications to homotopy limits and E_∞ structures*, Arch. Math. (Basel) **59** (1992), no. 2, 115–129.

[Ho] M. Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, Amer. Math. Soc., 1999.

[J] J.F. Jardine, *Homotopical foundations of algebraic K-theory*, Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987), 57–82, Contemp. Math., 83, Amer. Math. Soc., Providence, RI, 1989.

[ML] S. Mac Lane, *Categories for the working mathematician, Second edition*, Springer, 1998.

[M] J.P. May, *The geometry of iterated loop spaces*. Lectures Notes in Mathematics, Vol. 271. Springer-Verlag, Berlin-New York, 1972.

[Schl] C. Schlichtkrull, *The homotopy infinite symmetric product represents stable homotopy theory*, Algebr. Geom. Topol. **7** (2007), 1963–1977.

[Sch] S. Schwede, *Untitled book project on symmetric spectra*, preprint.

[Se] G. Segal, *Categories and cohomology theories*. Topology **13** (1974), 293–312.

[Sh] B. Shipley, *Symmetric spectra and topological Hochschild homology*, K-theory **19** (2000), no. 2, 155–183.

[S] M. Shulman, *Homotopy limits and colimits and enriched homotopy theory*, preprint, 2006. arXiv:math.AT/0610194

[Si] D. P. Sinha, *The topology of spaces of knots*, preprint, available at http://front.math.ucdavis.edu/0202.5287.

[T] R. W. Thomason, *Algebraic K-theory and étale cohomology*, Ann. Sci. École Norm. Sup. (4) **18** (1985), no. 3, 437–552.

[V1] R. M. Vogt, *Homotopy limits and colimits*, Math. Z. **134** (1973), 11–52.

[V2] R. M. Vogt, *Commuting homotopy limits*, Math. Z. **153** (1977), no. 1, 59–82.

[W] N. Wahl, *Infinite loop space structure(s) on the stable mapping class group*, Toplogy **43** (2004), no. 2, 343–368.

Department of Mathematics, University of Oregon, Eugene, OR 97403
E-mail address: ddugger@math.uoregon.edu