THE $\mathbb{Z}/p$-EQUIVARIANT DUAL STEENROD ALGEBRA FOR AN ODD PRIME p

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ABSTRACT. We completely calculate the $RO(\mathbb{Z}/p)$-graded coefficients $H\underline{\mathbb{Z}/p}_\star H\underline{\mathbb{Z}/p}$ for the constant Mackey functor $\underline{\mathbb{Z}/p}$.

1. Introduction

In [8], Hu and Kriz computed the Hopf algebroid

$$(H\underline{\mathbb{Z}/2}_\star, H\underline{\mathbb{Z}/2}_\star H\underline{\mathbb{Z}/2}) \tag{1}$$

where $\underline{\mathbb{Z}/2}$ denotes the constant $\mathbb{Z}/2$-Mackey functor and the subscript $?_\star$ denotes $RO(G)$-graded coefficients. (We direct the reader to [1, 14, 17, 13] for general preliminaries on Mackey functors, and to [15, 16] for their relationship with ordinary equivariant generalized (co)homology theories.) There are several remarkable features of the Hopf algebroid (1), which played an important role in the calculation of coefficients of Real-oriented spectra in [8]. For example, the morphism ring of (1) is a free module over the object ring (in particular, the Hopf algebroid is "flat," see [19]). Another remarkable fact is that the Hopf algebroid (1) is closely related to the motivic dual Steenrod algebra Hopf algebroid [21, 10]. In particular, if we denote by α the real sign representation, there are generator classes ξ_i in degrees $(2^i - 1)(1 + \alpha)$ and τ_i in degrees $(2^i - 1)(1 + \alpha) + 1$. A key point in proving this is the existence of the infinite complex projective space with $\mathbb{Z}/2$-action by complex conjugation.

When trying to generalize this calculation to a prime $p > 2$, i.e. to compute

$$(H\underline{\mathbb{Z}/p}_\star, H\underline{\mathbb{Z}/p}_\star H\underline{\mathbb{Z}/p}) \text{ for } p > 2 \text{ prime,} \tag{2}$$

the first difficulty one encounters is that no $p > 2$ generalization of complex conjugation on $\mathbb{C}P^\infty$ presents itself. Additionally, applications

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of this calculation were not yet known. For those reasons, likely, the question remained open for more than 20 years.

However, recently $\mathbb{Z}/p$-equivariant cohomological operations at $p > 2$ have become of interest, for example due to questions of $p > 2$ analogues of certain computations by Hill, Hopkins, and Ravenel in their solution to the Kervaire invariant 1 problem [6, 7]. In fact, Sankar and Wilson [20] made progress on the calculation of (2), in particular providing a complete decomposition of the spectrum $H\underline{\mathbb{Z}/p} \wedge H\underline{\mathbb{Z}/p}$ and thereby proving that the morphism ring (2) is *not* a flat module over the object ring.

The goal of the present paper is to complete the calculation of the Hopf algebroid (2). The authors note that while the present paper is largely self-contained, comparing with the results of [20] proved to be an step in the present calculation. We also point out the fact that while we do obtain complete characterizations of all the structure formulas of the Hopf algebroid (2), not all of them are presented in a nice explicit fashion, but only by recursion through (explicit) maps into other rings. This, in fact, also occurs in (1), where while the product relations and the coproduct are very elegant, the right unit is only recursively characterized by comparison with Borel cohomology. In the case of (2), this occurs for more of the structure formulas.

The formulas are, in fact, too complicated to present in the introduction, but we outline here the basic steps and results, with references to precise statements in the text. We first need to establish certain general preliminary facts about $\mathbb{Z}/p$-equivariant homology which are given in Section 2 below. The first important realization is that $\mathbb{Z}/p$-equivariant cohomology with constant coefficients is periodic under differences of irreducible real $\mathbb{Z}/p$-representations of complex type. Therefore, since we are only using the additive structure of the real representation ring, we can reduce the indexing from $RO(\mathbb{Z}/p)$ to the free abelian group R on 1 (the real 1-representation) and β, which is a chosen irreducible real representation of complex type.

Considering the full R-grading, on the other hand, is essential, since the pattern is simpler than if we only considered the $\mathbb{Z}$-grading (not to mention the fact that it contains more information). As already remarked, even in the $p = 2$ case, not all the free generators are in integral dimensions. As noted in [20], the dual $\mathbb{Z}/p$-equivariant Steenrod algebra for $p > 2$ is not a free module over the coefficient, but still, the R-graded pattern is a lot easier to describe than the $\mathbb{Z}$-graded pattern.

Additively speaking, for $p > 2$, the R-graded dual $\mathbb{Z}/p$-equivariant Steenrod algebra $A_\star = H\underline{\mathbb{Z}/p}_\star H\underline{\mathbb{Z}/p}$ is a sum of copies of the coefficients of $H\underline{\mathbb{Z}/p}$, and the coefficients of another $H\underline{\mathbb{Z}/p}$-module HM, which, up to shift by $\beta - 2$, is the equivariant cohomology corresponding to the Mackey functor Q which is 0 on the fixed orbit and $\mathbb{Z}/p$ (fixed) on the free orbit. In Propositions 1, 2, we describe the ring $H\underline{\mathbb{Z}/p}_\star$ and the module $HM_\star$.

It is quite interesting that for $p = 2$, this $H\underline{\mathbb{Z}/p}$-module HM is, in fact, an R-shift of $H\underline{\mathbb{Z}/2}$. For $p > 2$, however, this fails, and in fact infinitely many higher $\underline{\mathbb{Z}/p}$-Tor-groups of Q with itself are non-trivial.

How is it possible for $A_\star$ to be manageable, then? As noted in [20], the spectrum $H\underline{\mathbb{Z}/p} \wedge H\underline{\mathbb{Z}/p}$ is, in fact not a wedge-sum of R-graded suspensions of $H\underline{\mathbb{Z}/p}$ and HM. It turns out that instead, the HM-summands of the coefficients form "duplexes," i.e. pairs each of which makes up an $H\underline{\mathbb{Z}/p}$-module which we denote by HT, whose smashes over $H\underline{\mathbb{Z}/p}$ are again R-shifted copies of HT (Proposition 4). We establish this, and further explain the phenomenon, in Section 2 below.

Now our description of the multiplicative properties of $A_\star$ for $p > 2$ must, of course, take into account the smashing rules of the building blocks HT over $H\underline{\mathbb{Z}/p}$. However, we still need to identify elements which play the role of "multiplicative generators." In the present situation, this is facilitated in Section 3 by computing the $H\underline{\mathbb{Z}/p}$-cohomology of the equivariant complex projective space $\mathbb{C}P^\infty_{\mathbb{Z}/p}$ (Proposition 5) and the equivariant infinite lens space $B_{\mathbb{Z}/p}(\mathbb{Z}/p)$ (Proposition 7).

This can be accomplished using the spectral sequence coming from a filtration of the complete complex universe by a regular flag, which was also used in [12] to give a new more explicit proof of Hausmann's theorem [5] on the universality of the equivariant formal group on stable complex cobordism.

The spectral sequences can be completely solved and the multiplicative structure is completely determined by comparison with Borel cohomology. One can then use an analog of Milnor's method [18] to construct elements of $A_\star$. It is convenient to do both the cases of the projective space and the lens space, since in the projective case, the spectral sequence actually collapses, thus yielding elements of $\underline{\xi}_n \in A_\star$ of dimension

$$2p^{n-1} + (p^n - p^{n-1} - 1)\beta$$

and $\underline{\theta}_n$ of dimension

$$2(p^n - 1) + (p - 1)(p^n - 1)\beta.$$

One also obtains analogues of Milnor coproduct relations between these elements by the method of [18]. One notes that the elements $\underline{\theta}_n$ are of the right "slope" $2k + \ell\beta$ where $(k : \ell) = (1 : p - 1)$ and in fact, they turn out to generate $\underline{H\mathbb{Z}/p}_\star$-summands of $A_\star$. It is also interesting to note that for $p > 2$, it is impossible to shift the element $\underline{\xi}_n$ to the "right slope," since the values of $2k$ and ℓ would not be integers.

To completely understand the role of the elements $\underline{\xi}_n$, and also to construct the remaining "generators" of $A_\star$, one solves the flag spectral sequence for $B_{\mathbb{Z}/p}(\mathbb{Z}/p)$. This time, there are some differentials (although not many), which is the heuristic reason why $A_\star$ is not a free $\underline{H\mathbb{Z}/p}_\star$-module. In fact, following the method of [18], one constructs $\underline{\tau}_n$, $\widehat{\tau}_n$, of degree $|\underline{\xi}_n| + 1$, $|\underline{\xi}_n|$, respectively, and also an element $\widehat{\xi}_n$ of degree $|\underline{\xi}_n| - 1$. The "multiplet" of elements

$$\underline{\tau}_n, \widehat{\tau}_n, \underline{\xi}_n, \widehat{\xi}_n, \tag{3}$$

in fact, "generates" a copy of $HT_\star$. Again, Milnor-style coproduct relations on these elements follow using the same method.

The cohomology of the equivariant lens space, in fact, gives rise to one additional set of elements $\underline{\mu}_n$ of degree $|\underline{\theta}_n| + 1$, each of which also generates an $\underline{H\mathbb{Z}/p}_\star$-summand of $A_\star$. Coproduct relations on this element also follow from the method of [18], but this time, there are more terms, so they are hard to write down in an elegant form.

To construct a complete decomposition of $A_\star$ into a sum of copies of $\underline{H\mathbb{Z}/p}_\star$, $HT_\star$, one notes that along with the "multiplet" (3), we have also have a "multiplet"

$$\underline{\tau}_n\underline{\xi}_n^{i-1}, \widehat{\tau}_n\underline{\xi}_n^{i-1}, \underline{\xi}_n^i, \widehat{\xi}_n\underline{\xi}_n^{i-1} \tag{4}$$

for $i = 1, \dots, p-1$, which "generates" another copy of $HT_\star$, in a degree predicted by Proposition 4. (This, in fact, means that some of the linear combination of those elements will be divisible by the class b of degree $-\beta$ which comes from the inclusion $S^0 \to S^\beta$. We will return to this point below.) For now, however, let us remark that for $i = p$, the element $\underline{\xi}_n^p$ is dependent on $\underline{\theta}_n$, which is precisely accounted for by the presence of the additional element $\underline{\mu}_n$.

Tensoring these structures for different n then completely accounts for all elements of $A_\star$, coinciding with the result of [20]. To determine the structure formulas of $A_\star$ completely, we need to determine the divisibility of the linear combinations of the elements (4) by b, and to completely describe all the multiplicative relations. This can, in fact, be done by comparing with the Borel cohomology version $A^{cc}_\star$ of the $\mathbb{Z}/p$-equivariant Steenrod algebra, similarly as in [8]. The divisibility is described in Proposition 8. The final answer is recorded in Theorem 9. We write down explicitly those formulas which are simple enough to state explicitly. The remaining formulas are determined recursively by the specified map into $A^{cc}_\star$.

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2. Preliminaries

Let p be an odd prime. For our present purposes, the ring structure on $RO(\mathbb{Z}/p)$ does not matter. Thus, it can be treated as the free abelian group on the irreducible real representations, which are 1 (the trivial irreducible real representation) and β^i, $i = 1, \dots, (p-1)/2$ where β is the 1-dimensional complex representation where the generator γ acts by ζ_p (since β^i is the dual of β^{p-i}).

However, ordinary $\mathbb{Z}/p$-equivariant cohomology is periodic with period $\beta^i - \beta^j$, $1 \le i, j < p$. This follows from the fact that $S(\beta^i)$ have isomorphic equivariant cell chain complexes C_i given in degrees $0, 1$ as

$$\mathbb{Z}[\mathbb{Z}/p] \to \mathbb{Z}[\mathbb{Z}/p],$$

$$1 \mapsto 1 - \gamma^i.$$

We have an isomorphism $\phi_i : C_i \to C_1$ given by

$$1 \mapsto 1 + \gamma + \dots + \gamma^{i-1}. \tag{5}$$

in degree 1 and by the identity in degree 0. The map (5) can be written as multiplication by $(1-\gamma^i)/(1-\gamma)$, which has, as its inverse, multiplication by $(1-\gamma^{ij})/(1-\gamma^i)$ where $ij \equiv 1 \mod p$.

Therefore, the reduced cell chain complexes of S^{β^i}, which are the unreduced suspensions of $S(\beta^i)$, are also isomorphic, which implies the periodicity by the results of [13, 14] (see also [9, 11]). Thus, when discussing ordinary $\mathbb{Z}/p$-homology, the grading can be by elements of $R = \mathbb{Z}\{1, \beta\}$, without losing information.

Now the Borel cohomology $H\underline{\mathbb{Z}/p}^c_\star = F(E\mathbb{Z}/p_+, H\underline{\mathbb{Z}/p})_\star$ is complex-oriented. Its R-graded coefficients (given by group cohomology) are

given by
$$H\underline{\mathbb{Z}/p}^c_\star = \mathbb{Z}/p[\sigma^2,\sigma^{-2}][b]\otimes\Lambda[u],$$
where the (homological) dimension of the generators is given by
$$|b| = -\beta,\ |\sigma^2| = \beta-2, |u| = -1.$$
The Tate cohomology $H^t_\star(\underline{\mathbb{Z}/p}) = (\widetilde{E\mathbb{Z}/p}\wedge F(E\mathbb{Z}/p_+, H\underline{\mathbb{Z}/p}))_\star$ (where $\widetilde{X}$ denotes the unreduced suspension of X) is given by
$$H\underline{\mathbb{Z}/p}^t_\star = \mathbb{Z}/p[\sigma^2,\sigma^{-2}][b,b^{-1}]\otimes\Lambda[u].$$
(The notation σ^2 is to connect with the notation of [8], where the present question was treated for $p=2$.) The Borel homology $H^b_\star H\underline{\mathbb{Z}/p} = (E\mathbb{Z}/p_+\wedge H\underline{\mathbb{Z}/p})_\star$ then is given by
$$H\underline{\mathbb{Z}/p}^b_\star = \Sigma^{-1}H^t_\star H\underline{\mathbb{Z}/p}/H^c_\star H\underline{\mathbb{Z}/p}.$$
Now similarly as for $p=2$, the $\mathbb{Z}$-graded coefficients imply
$$H\underline{\mathbb{Z}/p}^\phi_\star = \Phi^{\mathbb{Z}/p}H\underline{\mathbb{Z}/p}_\star = (\widetilde{E\mathbb{Z}/p}\wedge H\underline{\mathbb{Z}/p})_\star = \mathbb{Z}/p[\sigma^{-2}][b,b^{-1}]\otimes\Lambda[\sigma^{-2}u],$$
the map into $H\underline{\mathbb{Z}/p}^t_\star$ is given by inclusion.

Composing the inclusion with the quotient map into $\Sigma H\underline{\mathbb{Z}/p}^b_\star$ then gives the connecting map of the long exact sequence associated with the fibration
$$H\underline{\mathbb{Z}/p}^b \to H\underline{\mathbb{Z}/p} \to H\underline{\mathbb{Z}/p}^\phi$$
where $E^\phi = E\wedge\widetilde{E\mathbb{Z}/p}$ for a $\mathbb{Z}/p$-equivariant spectrum E.

Regarding the commutation rule of a (not necessarily coherently) $\mathbb{Z}/p$-equivariant ring spectrum, we note that the commutation rule between a class of degree 1 and a class of degree β is a matter of convention. The commutation rule between two classes of degree β is given by $x\mapsto -x$ on S^β, which is equivariantly homotopic to the identity. Thus, if the dimensions of classes x,y are $k+\ell\beta$, $m+n\beta$, then we can write
$$xy = (-1)^{km}yx.$$

This implies the following

Proposition 1. *Consider the ring*
$$\Gamma = \mathbb{Z}/p[\sigma^{-2}][b]\otimes\Lambda[\sigma^{-2}u]$$
graded as above. Let Γ' be the second local cohomology of Γ with respect to the ideal (σ^{-2},b), desuspended by 1. Then
$$H\underline{\mathbb{Z}/p}_\star = \Gamma\oplus\Gamma'$$
with the abelian ring structure over Γ.

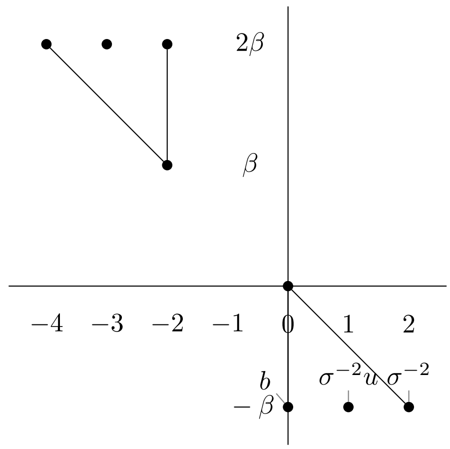

FIGURE 1. The coefficients $H\underline{\mathbb{Z}/p}_\star$

Addtively, therefore, $H\underline{\mathbb{Z}/p}_{k+\ell\beta}$ *is* $\mathbb{Z}/p$ *when* $0 \le k \le -2\ell$ *or* $-2\ell \le k \le -2$*, and* 0 *otherwise.*

□

We shall sometimes call the Γ part of the coefficients as *the good tail* and the Γ' part *the derived tail.*

Generally, the R-graded coefficients of ring spectra X we will consider will split into a "good tail" $X^g_\star$ and a "derived tail" $X^d_\star$. Generally, the coefficient ring will be abelian over the good tail, and the derived tail will be isomorphic to the second local cohomology of Γ with coefficients in the good tail, desuspended by 1. This is convenient, since it describes the multiplication completely. From that point of view, then, we can write

$$\Gamma = H\underline{\mathbb{Z}/p}^g_\star,$$
$$\Gamma' = H\underline{\mathbb{Z}/p}^d_\star = H^2_{(\sigma^{-2},b)}(\Gamma,\Gamma)[-1].$$

There is another $H\underline{\mathbb{Z}/p}$-module spectrum which will play a role in our calculations. Consider the short exact sequence of $\mathbb{Z}/p$-Mackey functors

$$0 \to Q \to \underline{\mathbb{Z}/p} \to \Phi \to 0 \tag{6}$$

where Q resp. Φ are $\mathbb{Z}/p$ on the free resp. fixed orbit, and 0 on the other orbit. This leads to a cofibration sequence of (coherent) $H\underline{\mathbb{Z}/p}$-module spectra. We have $H\Phi^\phi = H\Phi$, which implies

$$H\Phi_\star = \mathbb{Z}/p[b, b^{-1}].$$

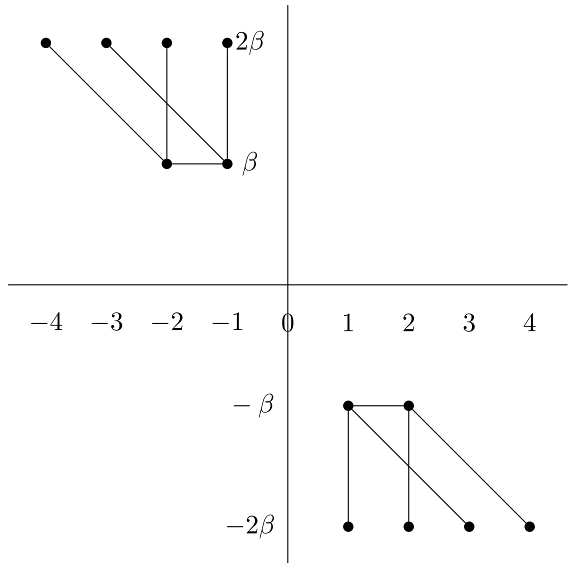

FIGURE 2. The coefficients $HQ_\star$

We also note that

$$HQ^{\phi}_{n} = \begin{cases} \mathbb{Z}/p & \text{for } n \in \{1,2,\dots\} \\ 0 & \text{else.} \end{cases} \tag{7}$$

The following follows from the long exact sequence on $RO(\mathbb{Z}/p)$-graded coefficients associated with (6).

Proposition 2. *The group $HQ_{k+\ell\beta}$ is $\mathbb{Z}/p$ when $1 \leq k \leq -2\ell$ or $-2\ell \leq k \leq -1$ and 0 otherwise. From a $H\underline{\mathbb{Z}/p}_\star$-module point of view, the $\ell < 0$ part of the coefficients (the* good tail*) behaves as an ideal in Γ and the $\ell > 0$ part* the derived tail *behaves as a quotient of Γ'. Additionally, we have, again,*

$$H^d_\star = H^2_{(\beta,\sigma^{-2})}(H\underline{\mathbb{Z}/p}^g_\star, HQ^g_\star)[-1].$$

□

We also put

$$HM = \Sigma^{\beta-2}HQ$$

for indexing purposes, since the element in degree 0 is closest to playing the role of the "generator" of $HM_\star$. (The element in degree -1 is its multiple in Borel cohomology.)

Now there is more to the multiplicative structure, however. We are interested in calculating

$$HQ \wedge_{H\underline{\mathbb{Z}/p}} HQ. \tag{8}$$

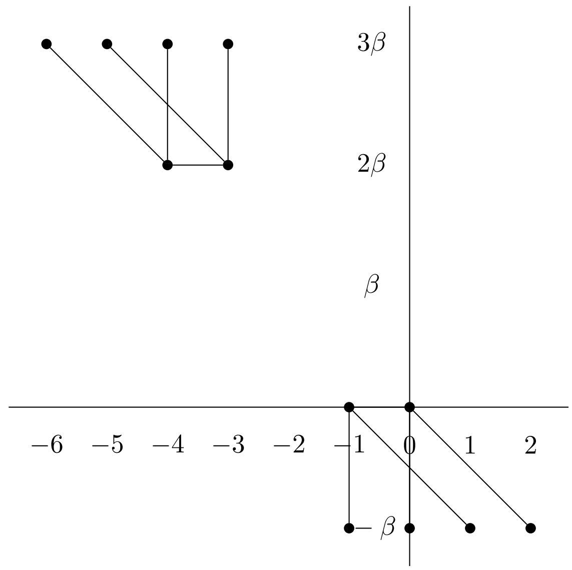

FIGURE 3. The coefficients $HM_\star$

Note that if we denote by $\overline{\mathbb{Z}/p}$ the co-constant $\underline{\mathbb{Z}/p}$-module (i.e. which is $\mathbb{Z}/p$ on both the fixed and free orbit, and the corestriction is 1 while the restriction is 0), then we have

$$\Sigma^{2-\beta} H\underline{\mathbb{Z}/p} = H\overline{\mathbb{Z}/p}. \tag{9}$$

Now note that at $p = 2$, if we denote the real sign representation by α, we have

$$HQ = \Sigma^{1-\alpha} H\underline{\mathbb{Z}/2},$$

so

$$HQ \wedge_{H\underline{\mathbb{Z}/2}} HQ = \Sigma^{2-\beta} H\underline{\mathbb{Z}/2} = H\overline{\mathbb{Z}/2}.$$

For $p > 2$, however, the situation is more complicated. Recall that for $1 \le i \le p$, we have $\mathbb{F}_p[\mathbb{Z}/p]$-modules L_i on which the generator γ of $\mathbb{Z}/p$ acts by a Jordan block of size i. For any $\mathbb{F}_p[\mathbb{Z}/p]$-module, we have a corresponding $\underline{\mathbb{Z}/p}$-module $\underline{V}$ where on the fixed orbit, we have $V^{\mathbb{Z}/p}$, and the restriction is inclusion (see [13, 14]). Recall from [13] that the $\underline{\mathbb{Z}/p}$-modules $\underline{L_p}$, $\underline{L_1} = \underline{\mathbb{Z}/p}$ are projective, so to calculate (8), we can use a projective resolution of Q by these $\underline{\mathbb{Z}/p}$-modules. We have a short exact sequence

$$0 \to \underline{L_{p-1}} \to \underline{L_p} \to Q \to 0 \tag{10}$$

and we also have a short exact sequence

$$0 \to \underline{L_{p+1-j}} \to \underline{L_1} \oplus \underline{L_p} \to \underline{L_j} \to 0. \tag{11}$$

Splicing together short exact sequence (10) with alternating short exact sequences (11) for $j = p-1$, $j = 2$, we therefore obtain a projective resolution of Q of the form

$$\dots \underline{L_1} \oplus \underline{L_p} \to \dots \to \underline{L_1} \oplus \underline{L_p} \to \underline{L_p} \tag{12}$$

We also have

$$\underline{L_p} \otimes_{\underline{\mathbb{Z}/p}} Q = \underline{L_p},$$

so tensoring (12) with Q over $\underline{\mathbb{Z}/p}$ gives a chain complex of $\underline{\mathbb{Z}/p}$-modules of the form

$$\dots Q \oplus \underline{L_p} \to \dots \to Q \oplus \underline{L_p} \to \underline{L_p}, \tag{13}$$

which gives the following

Proposition 3. *For all primes p, we have:*

$$Q \otimes_{\underline{\mathbb{Z}/p}} Q = \overline{\mathbb{Z}/2}.$$

For $p = 2$,

$$Tor_i^{\underline{\mathbb{Z}/2}}(Q, Q) = 0 \textit{ for } i > 0,$$
$$(HQ \wedge_{H\underline{\mathbb{Z}/2}} HQ) = H\overline{\mathbb{Z}/2}.$$

For $p > 2$,

$$Tor_i^{\underline{\mathbb{Z}/p}}(Q, Q) = \Phi \textit{ for } i > 0,$$

and we have

$$(HQ \wedge_{H\underline{\mathbb{Z}/p}} HQ)_\star = \Sigma^{2-\beta} HQ_\star \oplus \Sigma^2 H\underline{\mathbb{Z}/p}^\phi_\star \tag{14}$$

as $H\underline{\mathbb{Z}/p}_\star$-modules.

Proof. All the statements except (14) follow directly from evaluating the homology of (13). The reason the case of $p = 2$ is special is that then we have $2 = p$, $p - 1 = 1$, so there will be additional maps alternatingly canceling the fixed points for higher Tor.

To prove (14), first note that its part concerning the $\mathbb{Z}$-graded line follows from the other statements. The R-graded case, however, needs more attention, since we are no longer dealing with objects in the heart, so a priori we merely have a spectral sequence whose E^2-term is given by the sum of the R-graded coefficients of the $H\underline{\mathbb{Z}/p}$-modules with the Postnikov decomposition given by the Mackey Tor-groups:

$$E^2_{rs} = H(Tor_s^{\underline{\mathbb{Z}/p}}(Q, Q))_{r+*\beta} \Rightarrow (HQ \wedge_{H\underline{\mathbb{Z}/p}} HQ)_{r+s+*\beta} \tag{15}$$

(The indexing may seem reversed from what one would expect. Note, however, that it is induced by a cell filtration on representation sphere spectra.) We need to investigate this spectral sequence.

To this end, it is worthwhile pointing out an alternative approach. Since all the $H\underline{\mathbb{Z}/p}$-modules considered above have very easy Borel homology, essentially the same information can be recovered by working on geometric fixed points. Now for $H\underline{\mathbb{Z}/p}$-modules A, B, one has

$$A^{\phi} \wedge_{H\underline{\mathbb{Z}/p}^{\phi}} B^{\phi} = (A \wedge_{H\underline{\mathbb{Z}/p}} B)^{\phi}. \tag{16}$$

Also, we have a coherent equivalence of categories between $H\underline{\mathbb{Z}/p}^{\phi}$-modules and $(H\underline{\mathbb{Z}/p}^{\phi})^{\mathbb{Z}/p}$-modules. Now we have

$$B := H\underline{\mathbb{Z}/p}^{\phi}_{*} = \mathbb{Z}/p[t] \otimes \Lambda[u]$$

with $|t| = 2$, $|u| = 1$, while

$$J := HQ^{\phi}_{*} = (u, t),$$

(by which we denote that ideal in B). By [3], we therefore have a spectral sequence of the form

$$Tor^{B}_{r}(J, J)_{s} \Rightarrow (HQ \wedge_{H\underline{\mathbb{Z}/p}} HQ)^{\phi}_{r+s}. \tag{17}$$

To calculate the left hand side of (17), we have a B-resolution C of J of the form

$$\begin{tikzcd}
\dots \arrow[r] & B[4] \arrow[r, "-u"] \arrow[dr, "t"] & B[3] \arrow[r, "-u"] \arrow[dr, "t"] & B[2] \\
\cdot \arrow[r] & B[3] \arrow[r, "u"'] & B[2] \arrow[r, "u"'] & B[1].
\end{tikzcd} \tag{18}$$

Tensoring over B with J and taking homology, we get

$$J \otimes_B J = \mathbb{Z}/p\{u \otimes u, u \otimes t, t \otimes u\} \oplus B\{t \otimes t\}$$

(where the braces indicate a sum of copies indexed by the given elements) with the B-module structure the notation suggests, while

$$Tor^{B}_{i}(J, J) = \mathbb{Z}/p\{u, t\}[i + 1] \text{ for } i > 0.$$

Thus, the spectral sequence (18) is given by

$$E^{1}_{rs} = \begin{cases} \mathbb{Z}/p & \text{if } r = 0 \text{ and } s = 2, 4, 5, 6, \dots \\ \mathbb{Z}/p \oplus \mathbb{Z}/p & \text{if } r = 0 \text{ and } s = 3 \\ \mathbb{Z}/p & \text{if } r > 0 \text{ and } s = r + 2, r + 3 \\ 0 & \text{else.} \end{cases}$$

Moreover, one can show that the spectral sequence (17) collapses since the only possible targets of differentials is the $B\{t \otimes t\}$ in filtration degree 0, but by comparison with $H\underline{\mathbb{Z}/p}$, we see that those elements

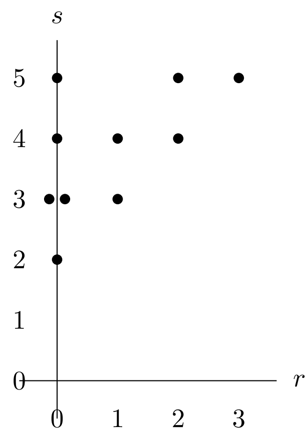

FIGURE 4. The spectral sequence for $(HQ \wedge_{H\underline{\mathbb{Z}/p}} HQ)^{\phi}_{\star}$

cannot be 0, since they inject into Borel homology by the connecting map. Thus, we see that

$$(19) \qquad (HQ \wedge_{H\underline{\mathbb{Z}/p}} HQ)^{\phi}_{n} = \begin{cases} \mathbb{Z}/p & \text{for } n = 2 \\ \mathbb{Z}/p \oplus \mathbb{Z}/p & \text{for } n = 3, 4, \dots \\ 0 & \text{else.} \end{cases}$$

Now the $E2$-term (15) on the $(?)^{\phi}$-level (obtained by inverting b) is "off by one" in the sense that we obtain

$$\begin{array}{ll} \mathbb{Z}/p & \text{for } n = 1 \\ \mathbb{Z}/p \oplus \mathbb{Z}/p & \text{for } n = 2, 3, 4, \dots \\ 0 & \text{else.} \end{array}$$

This, in fact, detects a single d^2-differential in the spectral sequence (15) originating in the $H\overline{\mathbb{Z}/p}_{\star}$-part in degrees

$$2 - n\beta,\ n = 1, 2, 3, \dots$$

and also proves there cannot be any other differentials, thus yielding (14) (see Figures 5 and 6). □

Now let T denote the $\mathbb{Z}/p$-equivariant suspension spectrum of the cofiber of the second desuspension of the $\mathbb{Z}/p$-equivariant based degree p map

$$(20) \qquad S^{\beta} \to S^2.$$

(This spectrum was denoted by $T(\theta)$ in [20].) We also denote $HT = H\underline{\mathbb{Z}/p} \wedge T$. We shall see in Section 3 that the spectrum T (up to suspension) maps into the based suspension spectrum of the $\mathbb{Z}/p$-equivariant

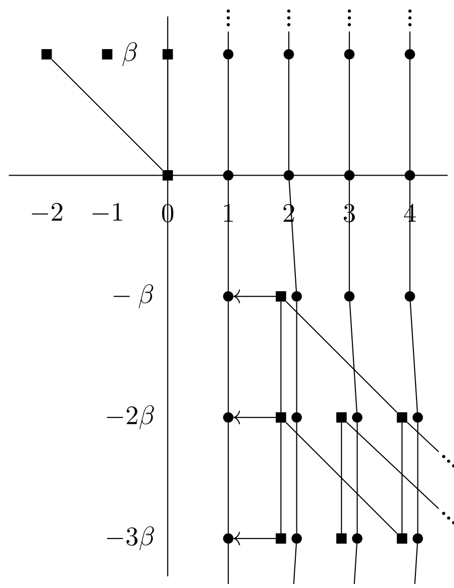

FIGURE 5. The spectral sequence (15)

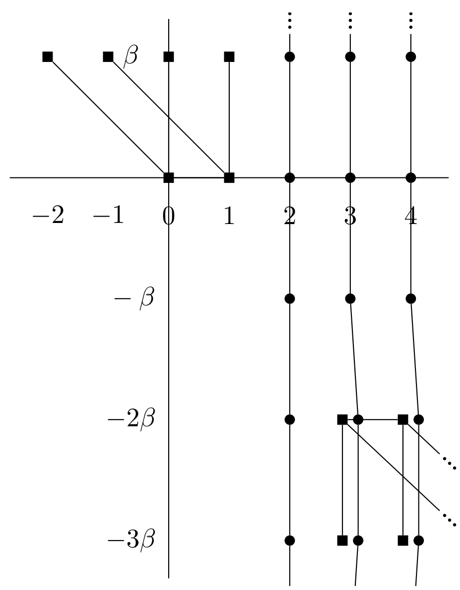

FIGURE 6. $(HQ \wedge_{H\underline{\mathbb{Z}/p}} HQ)_\star$

lens space $B_{\mathbb{Z}/p}(\mathbb{Z}/p)$, and therefore, the connecting map of the $H\underline{\mathbb{Z}/p}_\star$-homology long exact sequence of (20) is an isomorphism on the $\overline{\mathbb{Z}/p}$ in degree β (which is the only degree in which it can be non-trivial for dimensional reason).

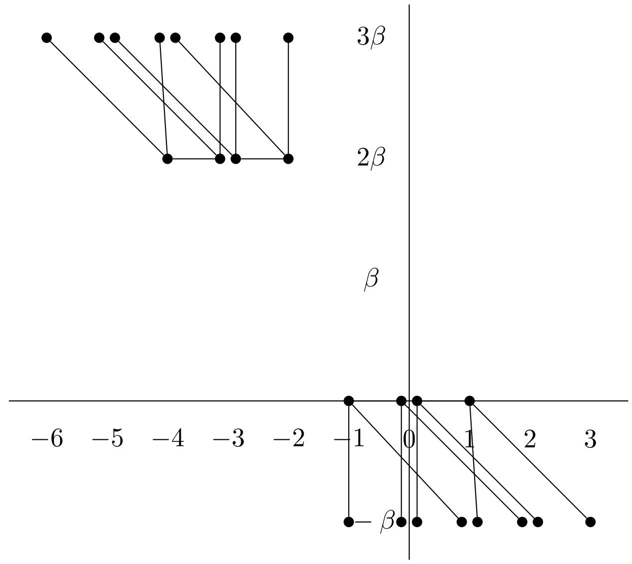

FIGURE 7. The coefficients $HT_\star$

We also see that the coefficients of HT suggest the possibility of a filtration whose associated graded pieces are wedges of suspensions of HM. Indeed, from the universal property, we readily construct a morphism

$$HT \to HM,$$

which leads to a cofibration sequence of the form

$$\Sigma HM \to HT \to HM, \tag{21}$$

which splits additively on R-graded coefficients. The $\mathbb{Z}$-graded coefficients of HT, which are $\mathbb{Z}/p$ in degrees -1 and 1, and $\mathbb{Z}/p \oplus \mathbb{Z}/p$ in degree 0, generate the two HM-copies in (21) in the above sense.

The cofibration (21) can also be seen on the level of Mackey functors. Desuspending (20) by β and smashing with $H\underline{\mathbb{Z}/p}$, we get, by (9), a map of the form

$$H\underline{\mathbb{Z}/p} \to H\overline{\mathbb{Z}/p}.$$

Its cofiber can then be realized by a Mackey chain complex

$$\underline{\mathbb{Z}/p} \to \overline{\mathbb{Z}/p} \tag{22}$$

set in homological degrees $0, 1$, where the differential is 1 on the fixed orbit and 0 on the free orbit (this is, essentially, the only non-trivial possibility). In the derived category of $\underline{\mathbb{Z}/p}$-modules, then, (22) obviously maps to Q, with the kernel quasiisomorphic to $Q[1]$.

The cofibration (21) does not split. To see that, we observe that

Proposition 4. *We have a $\mathbb{Z}/p$-equivariant equivalence*

$$T \wedge T \sim T \vee \Sigma^{\beta-1} T. \tag{23}$$

Proof. The strategy is to smash two copies of the cofibration (20) together. In general, for a spectrum C which is a cofiber of a map of the form

$$f : S' \to S''$$

where S', S'' are $\wedge$-invertible,

$$C \wedge C = (S'' \wedge C) \vee \Sigma(S' \wedge C)$$

if the Euler characteristic of C is 0. To see this, without loss of generality, $S'' = S^0$, and then $\chi(C) = 0$ implies $\chi(S') = 1$, i.e. the homotopy commutativity of the diagram

$$\begin{tikzcd} S' \wedge S' \arrow[r, "tp"] \arrow[dr, "Id\wedge f"'] & S' \wedge S' \arrow[d, "Id\wedge f"] \\ & S', \end{tikzcd}$$

where tp denotes swap of factors, which is equivalent to

$$\begin{tikzcd} S' \wedge S' \arrow[r, "Id"] \arrow[dr, "Id\wedge f"'] & S' \wedge S' \arrow[d, "f\wedge Id"] \\ & S', \end{tikzcd}$$

which can be used to construct the splitting. This is true in our case, since $\chi(S^\beta) = 1$. □

Smashing (23) with $H\underline{\mathbb{Z}/p}$, we see that $HT \wedge_{H\underline{\mathbb{Z}/p}} HT$ additively splits as a direct sum of $HM_\star$ suspended by $0, 1, \beta, \beta - 1$.This is not what would happen if the cofibration of $H\underline{\mathbb{Z}/p}$-modules (21) split: the higher derived terms would appear. This will play a role of our description of the $\mathbb{Z}/p$-equivariant Steenrod algebra in the subsequent sections.

One may, in fact, ask how the "tidy" behavior (23) is even possible on $H\underline{\mathbb{Z}/p}$-homology, given the infinitely many higher Tor's of Q with itself, computed in Proposition 3. We present an explanation in terms of geometric fixed points: The resolution (18) in fact gives a short exact sequence of B-modules of the form

$$0 \to J[1] \to B[1] \oplus B[2] \to J \to 0.$$

On the level of geometric fixed points, the cofibration (21) in fact realizes this extension.

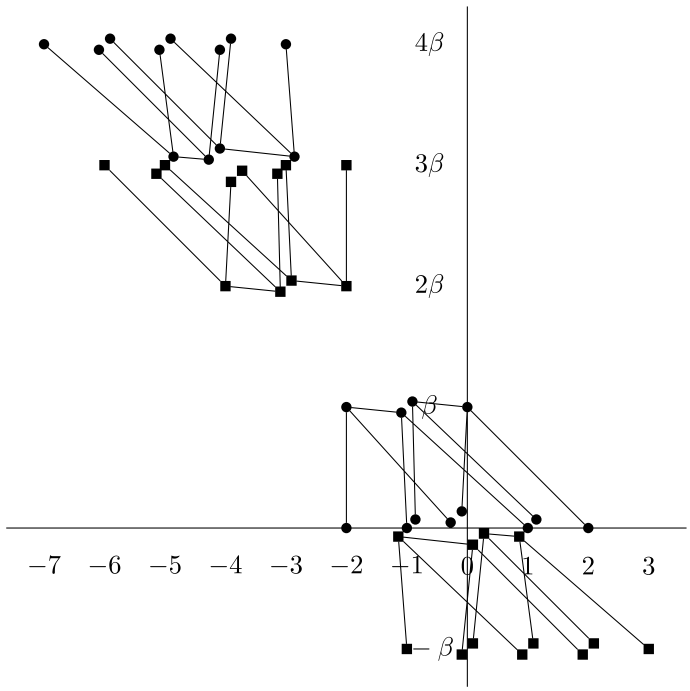

FIGURE 8. The coefficients $(HT \wedge_{H\underline{\mathbb{Z}/p}} HT)_\star$

Comment: It is worth noting that each of the traceless indecomposable modular representaions L_i, $i = 1, \dots, p-1$ gives rise to a Mackey functor (which we also denote by $\widetilde{L_i}$) equal to L_i on the free orbit and 0 on the fixed orbit. Thus, $\widetilde{L_1} = Q$. The $H\mathbb{Z}/p$-modules $H\widetilde{L_i}$, $i = 1, \dots, p-1$ all have additively isomorphic $RO(\overline{\mathbb{Z}/p})$-graded coefficients, even though no morphism of spectra induces this isomorphism (passing to Borel homology, this says there is no map between L_i, L_j for $i \neq j$ which would induce an isomorphism in group homology). Of course, the non-equivariant coefficients of HL_i are L_i in degree 0, so we see they all are of different dimensions. The reason we only encounter $H\widetilde{L_1} = HQ$ in our calculations is that in the Borel homology spectrum of $H\underline{\mathbb{Z}/p} \wedge H\underline{\mathbb{Z}/p}$ is a wedge sum of suspended copies of $H\underline{\mathbb{Z}/p}^b$.

This raises the question as to whether in general a morphism of spectra $f : X \to Y$ which induces an isomorphism in $RO(\mathbb{Z}/p)$-graded coefficients is a weak equivalence. This is obviously true for $p = 2$ by the cofibration sequence

$$\mathbb{Z}/2_+ \to S^0 \to S^\alpha,$$

but it is false for $p > 2$: Consider the Mackey functor which is equal to the traceless complex representation β on the free orbit and 0 on the fixed orbit. (We will also denote it by β.) Then the $\mathbb{Z}/p$-Borel homology spectrum $H\beta^b$ has trivial $RO(\mathbb{Z}/p)$-graded coefficients, since the cohomology theory is $(\beta-2)$-periodic, and the group homology of $\mathbb{Z}/p$ with coefficients in β is 0. (Also note that for $p=2$, this Borel homology will be non-zero in degree $\alpha-1$ where α is the 1-dimensional real sign representation.)

On the other hand, call an equivariant spectrum X *p-complete* when its canonical map into the homotopy inverse limit of $X\wedge M\mathbb{Z}/p^r$ is an equivalence. Then a morphism $f:X\to Y$ of p-complete bounded below $\mathbb{Z}/p$-equivariant spectra which induces an isomorphism of $RO(\mathbb{Z}/p)$-graded coefficients is a weak equivalence. To see this, since we already know f^ϕ is an equivalence, it suffices to consider the case when X,Y are free. Equivalently, we must show that a bounded below free $\mathbb{Z}/p$-spectrum whose fixed point coefficients are 0 is 0. So assume it is not 0, and consider the bottom dimensional degree non-equivariant $\mathbb{Z}[\mathbb{Z}/p]$-module V of its coefficients. But then $V/(1-\gamma)\neq 0$, since on a modular representation of $\mathbb{Z}/p$, $1-\gamma$ is never onto. Thus, the coefficients of the fixed point spectrum are non-zero in the same degree.

We do not know whether the bounded below assumptions can be removed.

3. Cohomology of the equivariant projective spaces and lens spaces

The $\mathbb{Z}/p$-equivariant complex projective space $\mathbb{C}P^\infty_{\mathbb{Z}/p}$ can be identified with the space of complex lines on the complete complex $\mathbb{Z}/p$-universe $\mathcal{U}$. An explicit decomposition

$$\mathcal{U}=\bigoplus_{i\in\mathbb{N}_0}\alpha_i \tag{24}$$

is called a *flag*. It leads to a filtration

$$F_n(\mathbb{C}P^\infty)=P(\alpha_0\oplus\cdots\oplus\alpha_n)$$

We have

$$F_n(\mathbb{C}P^\infty)/F_{n-1}(\mathbb{C}P^\infty)\cong S^{\alpha_n^{-1}(\alpha_0\oplus\cdots\oplus\alpha_{n-1})}.$$

This leads to a spectral sequence

$$E_1=\bigoplus_{n\in\mathbb{N}_0}H\underline{\mathbb{Z}/p}_{\star-\alpha_n^{-1}(\alpha_0\oplus\cdots\oplus\alpha_{n-1})}\Rightarrow H\underline{\mathbb{Z}/p}^\star\mathbb{C}P^\infty_{\mathbb{Z}/p}. \tag{25}$$

Whether or not this collapses depends on the flag. For the *regular flag*

$$\alpha_i=\beta^i,$$

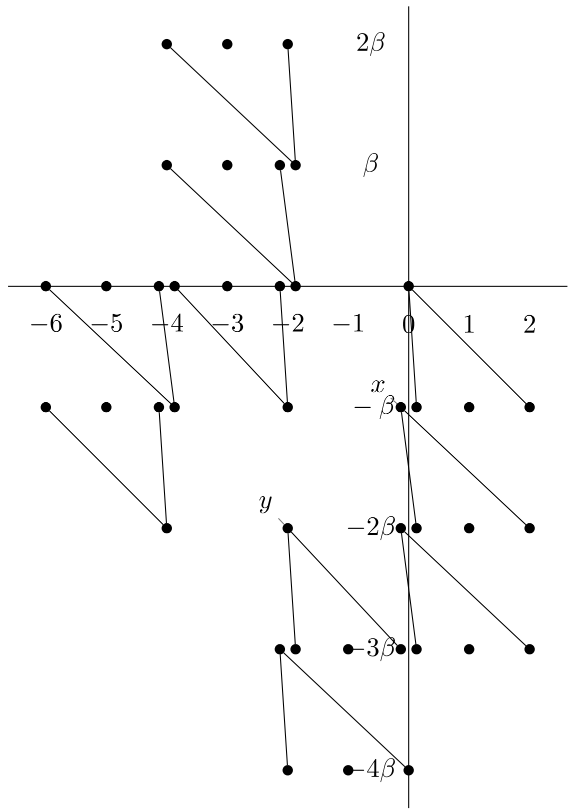

FIGURE 9. The flag spectral sequence for $B_{\mathbb{Z}/3}(S^1)$

(remembering our convention on indexing), the free generators of the copies of $H\underline{\mathbb{Z}/p}_\star$ are in dimensions

$$(26)\qquad \begin{array}{l} 0, -\beta, -2\beta, \dots, -(p-1)\beta, \\ -(p-1)\beta - 2, -p\beta - 2, \dots, -2(p-1)\beta - 2, \\ -2(p-1)\beta - 4, \dots, -3(p-1)\beta - 4, \\ \dots \end{array}$$

We see from Proposition 1 that there is no element in (25) in total dimension $-\beta-1$ or $-3-(p-1)\beta$, and therefore the generator x in dimension $-\beta$ and the generator y in dimension $-2-(p-1)\beta$ are permanent cycles, and thus, the spectral sequence collapses, and thus $H\underline{\mathbb{Z}/p}^\star \mathbb{C}P^\infty_{\mathbb{Z}/p}$ is a free $H\underline{\mathbb{Z}/p}_\star$-module.

Proposition 5. *One has*

$$H\underline{\mathbb{Z}/p}^\star \mathbb{C}P^\infty_{\mathbb{Z}/p} = H\underline{\mathbb{Z}/p}_\star[x, y]/(\sigma^{-2}y - x^p + b^{p-1}x).$$

Proof. The collapse of the spectral sequence was already proved. Thus, what remains to prove is the multiplicative relation. To this end, we work in Borel cohomology. (This could, in principle, generate countert-erms in the Γ' tail, but since it is σ^{-2}-divisible, that could be corrected by a different choice of the generator y.)

Now in Borel cohomology, we are essentially working in the cohomology of the space $\mathbb{C}P^\infty \times B\mathbb{Z}/p$. From this point of view, it is convenient to treat the periodicity σ^2 as the identity, so from this point of view, we have

$$H^*(\mathbb{C}P^\infty \times B\mathbb{Z}/p; \mathbb{Z}/p) = \mathbb{Z}/p[x,b] \otimes \Lambda[u]$$

where x, b have cohomological dimension 2 and u has cohomological dimension 1. The computation of the element y from this point of view then amounts to computing the Euler class of the regular complex representation of $\mathbb{Z}/p$. This is

$$x(x+b)(x+2b)\dots(x+(p-1)b) = x^p - b^{p-1}x.$$

□

Now similarly as in Milnor [18], we have, for a $\mathbb{Z}/p$-space X, a multiplicative map

$$\lambda : H^\star(X) \to H^\star(X) \widehat{\otimes} A_\star \tag{27}$$

where the $\widehat{\otimes}$ denotes the tensor product completed at (b). For $X = \mathbb{C}P^\infty_{\mathbb{Z}/p}$, we get

$$\lambda(x) = x \otimes 1 + \sum_{n \geq 1} y^{p^{n-1}} \otimes \underline{\xi}_n \tag{28}$$

and

$$\lambda(y) = y \otimes 1 + \sum_{n \geq 1} y^{p^n} \otimes \underline{\theta}_n \tag{29}$$

where the dimensions are given by

$$|\underline{\xi}_n| = 2p^{n-1} + (p^n - p^{n-1} - 1)\beta,$$
$$|\underline{\theta}_n| = 2(p^n - 1) + (p-1)(p^n - 1)\beta.$$

From co-associativity, we can further conclude that, writing

$$\widetilde{\psi}(t) = \psi(t) - t \otimes 1 - 1 \otimes t,$$

we have

$$\widetilde{\psi}(\underline{\xi}_n) = \sum \underline{\theta}_i^{p^{n-i-1}} \otimes \underline{\xi}_{n-i}, \tag{30}$$

$$\widetilde{\psi}(\underline{\theta}_n) = \sum \underline{\theta}_i^{p^{n-i}} \otimes \underline{\theta}_{n-i}. \tag{31}$$

The picture becomes a little less tidy when we calculate $H\underline{\mathbb{Z}/p}^{\star}B_{\mathbb{Z}/p}(\mathbb{Z}/p)$. For a model of $B_{\mathbb{Z}/p}(\mathbb{Z}/p)$, we use the quotient of the unit sphere in $\mathcal{U}$ by the action of $\mathbb{Z}/p \subset S^1$. Now a flag (24), we have a filtration with

$$F_{2n+1}B_{\mathbb{Z}/p}(\mathbb{Z}/p) = S(\alpha_0 \oplus \cdots \oplus \alpha_n)/(\mathbb{Z}/p)$$

$$F_{2n}B_{\mathbb{Z}/p}(\mathbb{Z}/p) = \{(x_0, \dots, x_n) \in S(\alpha_0 \oplus \cdots \oplus \alpha_n) \mid Arg(x_n) = 2k\pi/p\}/(\mathbb{Z}/p).$$

We have

$$F_{2n}/F_{2n-1} \cong S^{(\alpha_0 \oplus \cdots \oplus \alpha_{n-1})\alpha_n^{-1}},$$
$$F_{2n+1}/F_{2n} \cong S^{(\alpha_0 \oplus \cdots \oplus \alpha_{n-1})\alpha_n^{-1} \oplus 1_{\mathbb{R}}}.$$

Thus, we have a spectral sequence

$$E_1 = \bigoplus_{n \in \mathbb{N}_0} H\underline{\mathbb{Z}/p}_{\star} F_n/F_{n-1} \Rightarrow H\underline{\mathbb{Z}/p} B_{\mathbb{Z}/p}(\mathbb{Z}/p) \tag{32}$$

If we use the regular flag, the generators of the summands will be in (homological) degrees:

$$\begin{aligned}&0, -1, -\beta, -\beta - 1, \dots, -(p-1)\beta, -(p-1)\beta - 1,\\ &-(p-1)\beta - 2, -(p-1)\beta - 3, \dots, -2(p-1)\beta - 2, -2(p-1) - 3,\\ &-2(p-1) - 4, \dots\end{aligned}$$

The difference however now is that in the spectral sequence (32), the generator z in degree -1 supports a d_1 differential. To see this, otherwise, it would be a permanent cycle, and hence so would its Bockstein. Now the image of x in Borel cohomology is a non-trivial element of

$$H^1(\mathbb{Z}/p \times \mathbb{Z}/p; \mathbb{Z}/p),$$

so the image of $\beta(z)$ in Borel cohomology would be non-zero. However, we see that in the E_1-term of the spectral sequence (32), all the elemenets of dimension -2 have image 0 in Borel cohomology.

There is a unique target of this differential, and all other differentials originate in

$$z \cdot H\underline{\mathbb{Z}/p}^{\star}\mathbb{C}P^{\infty}_{\mathbb{Z}/p}.$$

One also notes that

$$z \cdot x^{p-1} \tag{33}$$

is a permanent cycle. Bookkeeping leads to the following:

Proposition 6. *We have*

(34)
$$\begin{aligned}&H\underline{\mathbb{Z}/p}^{\star}B_{\mathbb{Z}/p}(\mathbb{Z}/p) =\\ &H\underline{\mathbb{Z}/p}_{\star}[y] \otimes \Lambda[z \cdot x^{p-1}] \oplus (HM_{\star}[x,y]/(\sigma^{-2}y - x^p + b^{p-1}x)) \otimes \mathbb{Z}/p\{x, \nu\}.\end{aligned}$$

(See Figure 10 for the element ν.)

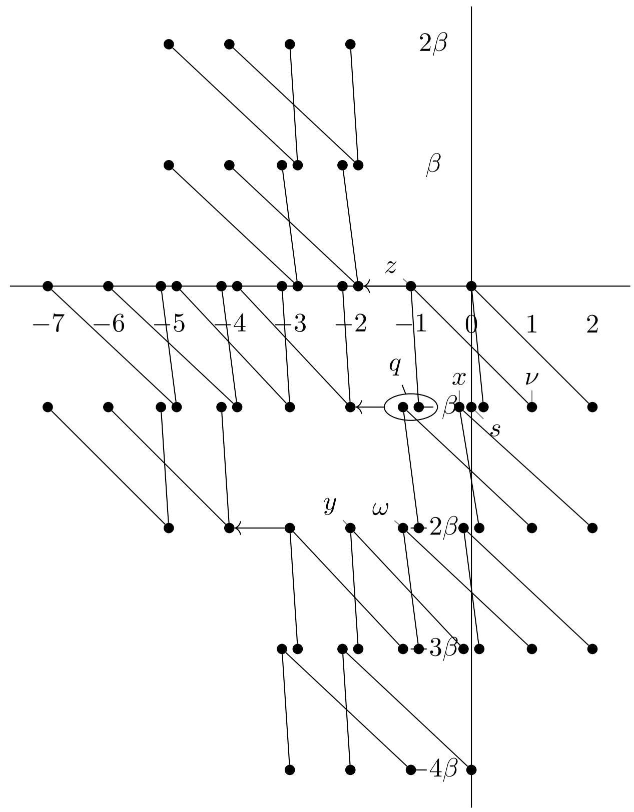

FIGURE 10. The flag spectral sequence for $B_{\mathbb{Z}/3}(\mathbb{Z}/3)$

$\square$

In Proposition 6, the "polynomial generators" at $HM_\star$ just mean suspension by dimensional degree of the given monomial. In particular, we have canonical elements $q \in H^{1+\beta}B_{\mathbb{Z}/p}(\mathbb{Z}/p)$, $s \in H^{\beta}B_{\mathbb{Z}/p}(\mathbb{Z}/p)$, $\nu \in H^{\beta-1}B_{\mathbb{Z}/p}(\mathbb{Z}/p)$ represented by

$$bz - xu \in x \cdot HM_\star,\ z\sigma^{-2}u \in \nu \cdot HM_\star, z\sigma^{-2} \in \nu \cdot HM_\star. \tag{35}$$

(See figure 10; whiskers point to the names of the elements concerned.) In (35), the name of the elements comes from Borel cohomology, which we write as elements of the appropriate terms of (34).

We also have an element $\omega \in H^{(p-1)\beta+1}(B_{\mathbb{Z}/p}(\mathbb{Z}/p))$ which represents (33). One notes, however, that this is only correct in the asociated graded object of our filtration. To determine the exact image in Borel cohomology, we note that we must have

$$\beta(\omega) = y,$$

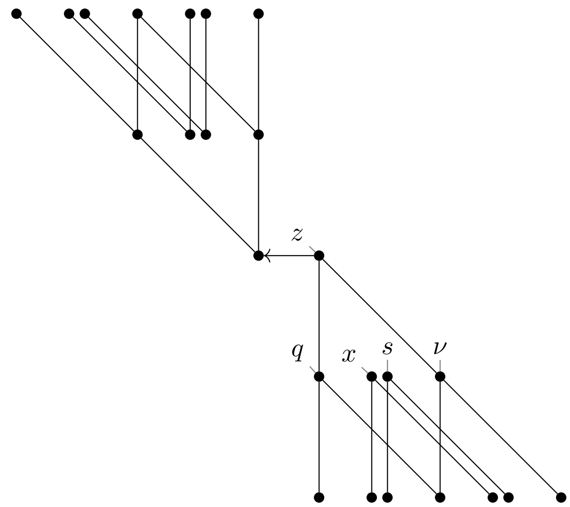

FIGURE 11. The x and z towers in the $B_{\mathbb{Z}/3}(\mathbb{Z}/3)$-flag spectral sequence, p.1

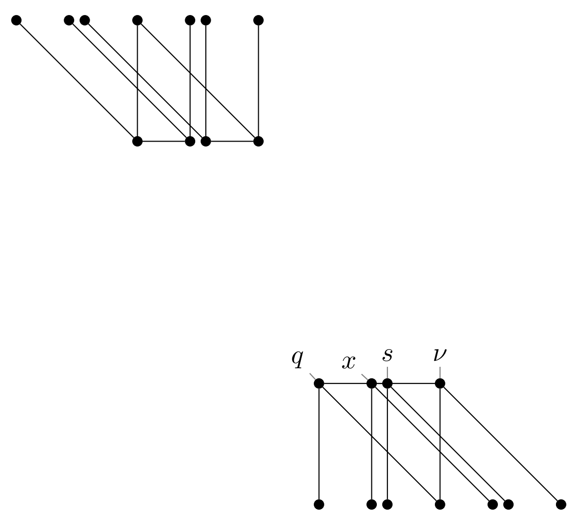

FIGURE 12. The x and z towers in the $B_{\mathbb{Z}/3}(\mathbb{Z}/3)$-flag spectral sequence, p.2

since y is the additive generator of $H\underline{\mathbb{Z}/p}^{(p-1)\beta+2}B_{\mathbb{Z}/p}(\mathbb{Z}/p)$. This gives

$$\omega \mapsto z(\sigma^{2-2p}t^{p-1} - b^{p-1}).$$

Now we can write

$$\lambda(q) = q \otimes 1 + \sum_{n\geq 1} y^{p^{n-1}} \otimes \widehat{\xi}_n,$$

$$\lambda(s) = s \otimes 1 + q \otimes (\tau_0 + b^{p-2}\widehat{\xi}_1) + \sum_{n\geq 1} y^{p^{n-1}} \otimes \widehat{\tau}_n, \tag{36}$$

$$\lambda(\nu) = \nu \otimes 1 - q \otimes b^{p-2}\underline{\xi}_1 + x \otimes (\tau_0 + b^{p-2}\widehat{\xi}_1) + \sum_{n\geq 1} y^{p^{n-1}} \otimes \underline{\tau}_n. \tag{37}$$

The somewhat complicated form of the right hand side of formulas (36), (37) is forced by considering which elements exist in $H\underline{\mathbb{Z}/p}^\star B_{\mathbb{Z}/p}(\mathbb{Z}/p)$. We can also write

$$\lambda(\omega) = \omega \otimes 1 + \sum_{n\geq 1} y^{p^n} \otimes \underline{\mu}_n + \dots,$$

however, the $\dots$ indicate that there will be other summands. The dimensions are then given by

$$|\widehat{\xi}_n| = |\underline{\xi}_n| - 1,$$
$$|\underline{\tau}_n| = |\underline{\xi}_n| + 1,$$
$$|\widehat{\tau}_n| = |\underline{\tau}_n| - 1 = |\underline{\xi}_n|.$$
$$|\underline{\mu}_n| = |\underline{\theta}_n| + 1.$$

Co-associativity then implies

$$\widetilde{\psi}(\widehat{\xi}_n) = \sum \underline{\theta}_i^{p^{n-i-1}} \otimes \widehat{\xi}_{n-i}, \tag{38}$$

$$\widetilde{\psi}(\widehat{\tau}_n) = \sum \underline{\theta}_i^{p^{n-i-1}} \otimes \widehat{\tau}_{n-i} + \widehat{\xi}_n \otimes (\tau_0 + \widehat{\xi}_1 b^{p-2}), \tag{39}$$

$$\widetilde{\psi}(\underline{\tau}_n) = \sum \underline{\theta}_i^{p^{n-i-1}} \otimes \underline{\tau}_{n-i} + \widehat{\xi}_n \otimes \underline{\xi}_1 b^{p-1} + \underline{\xi}_n \otimes (\tau_0 + \widehat{\xi}_1 b^{p-2}). \tag{40}$$

We can now re-state Proposition 6 more precisely, in fact determining the multiplicative structure of $H\underline{\mathbb{Z}/p}^\star B_{\mathbb{Z}/p}(\mathbb{Z}/p)$ completely:

Proposition 7. *The module $H\underline{\mathbb{Z}/p}^\star B_{\mathbb{Z}/p}(\mathbb{Z}/p)$ is additively isomorphic to a direct sum of*

$$H\underline{\mathbb{Z}/p}_\star[y] \otimes \Lambda[\omega]$$

and (suspended) copies of $HM_\star$ generated on monomials of the form

$$y^n x^{i-1}\pi$$

where $n \in \mathbb{N}_0$, $i = 1, \dots, p-1$, and π stands for one of the symbols x, ν. The multiplicative structure is entirely determined by the canonical inclusion of the good tail into

$$H\underline{\mathbb{Z}/p}^\star \mathbb{C}P^\infty_{\mathbb{Z}/p} \otimes \Lambda[z, u].$$

In particular, the good tail is the quotient of

$$H\underline{\mathbb{Z}/p}_\star[x, y] \otimes \Lambda[s, q, \nu, \omega]$$

modulo the relations

$$x^p - b^{p-1}x = y\sigma^{-2},$$
$$\sigma^{-2}q = b\nu - (\sigma^{-2}u)x,$$
$$\sigma^{-2}s = (\sigma^{-2}u)\nu,$$
$$(\sigma^{-2}u)s = 0,$$
$$(\sigma^{-2}u)q = bs,$$
$$qs = 0,$$
$$q\nu = xs,$$
$$\nu s = 0,$$
$$\omega s = \omega\nu = 0,$$
$$\omega q = -sy,$$
$$\omega x = y\nu.$$

Additionally, we have

$$H\underline{\mathbb{Z}/p}^\star B_{\mathbb{Z}/p}(\mathbb{Z}/p)^d = H^2_{(b,\sigma^{-2})}(H\underline{\mathbb{Z}/p}^g_\star, H\underline{\mathbb{Z}/p}^\star B_{\mathbb{Z}/p}(\mathbb{Z}/p)^g)[-1].$$

□

4. Images in Borel cohomology

Similarly as in the $p = 2$ case, it is convenient to consider the *Borel cohomology dual Steenrod algebra*

$$A^{cc}_\star = F(E\mathbb{Z}/p_+, H\underline{\mathbb{Z}/p} \wedge H\underline{\mathbb{Z}/p})_\star.$$

We have

$$A^{cc}_\star = (A_*[\sigma^2, \sigma^{-2}][b] \otimes \Lambda[u])^\wedge_{(b)}.$$

In $A^{cc}_\star$, which makes a formal Hopf algebrioid, the non-equivariant coproduct relations hold, and we have

$$\rho^2 := \eta_R(\sigma^2) = \sigma^2 + \sigma^{2p}b^{p-1}\xi_1 + \cdots + \sigma^{2p^n}b^{p^n-1}\xi_n + \dots \tag{41}$$

$$\overline{u} := \eta_R(u) = u + \sigma^2 b\tau_0 + \sigma^{2p}b^p\tau_1 + \cdots + \sigma^{2p^n}b^{p^n}\tau_n + \dots \tag{42}$$

Multiplying (41) by $\sigma^{-2}\rho^{-2}$, we get

$$\sigma^{-2} = \rho^{-2} + \rho^{-2}\sigma^{2p-2}b^{p-1}\xi_1 + \dots \rho^{-2}\sigma^{2p^n-2}b^{p^n-1}\xi_n + \dots \tag{43}$$

Now $\mathbb{C}P^\infty_{\mathbb{Z}/p}$ has the same $\mathbb{Z}/p$-Borel cohomology as $\mathbb{C}P^\infty$, which is

$$H\underline{\mathbb{Z}/p}^\star_c\mathbb{C}P^\infty = \mathbb{Z}/p[t][\sigma^2, \sigma^{-2}][b] \otimes \Lambda[u]. \tag{44}$$

Further, in Borel cohomology, we can write

$$x = \sigma^{-2}t$$

and thus,

$$y = \sigma^{2-2p}t^p - tb^{p-1}.$$

Combining this with (28), we have

$$\begin{aligned}\lambda(t) = \lambda(\sigma^2 x) &= x \otimes \rho^2 + \sum_{n\geq 1} y^{p^{n-1}} \otimes \underline{\xi}_n \rho^2 \\ &= \sigma^{-2}t \otimes \rho^2 + \sum_{n\geq 1} (\sigma^{2p^{n-1}-2p^n} t^{p^n} - b^{p^n - p^{n-1}} t^{p^{n-1}}) \otimes \underline{\xi}_n \rho^2 \\ &= t \otimes (\sigma^{-2} - b^{p-1}\underline{\xi}_1)\rho^2 + \sum_{n\geq 1} t^{p^n} \otimes (\sigma^{2p^{n-1}-2p^n}\underline{\xi}_n - b^{p^{n+1}-p^n}\underline{\xi}_{n+1})\rho^2\end{aligned}$$

Comparing with the non-equivariant result

$$\lambda(t) = t \otimes 1 + \sum_{n\geq 1} t^{p^n} \otimes \xi_n,$$

we have

$$\begin{cases} 1 & = (\sigma^{-2} - b^{p-1}\underline{\xi}_1)\rho^2 \\ \xi_n & = (\sigma^{2p^{n-1}-2p^n}\underline{\xi}_n - b^{p^{n+1}-p^n}\underline{\xi}_{n+1})\rho^2, \end{cases}$$

and so

$$\begin{cases} \underline{\xi}_1 & = (\sigma^{-2} - \rho^{-2})b^{1-p} \\ \underline{\xi}_{n+1} & = (\sigma^{2p^{n-1}-2p^n}\underline{\xi}_n - \rho^{-2}\xi_n)b^{p^n - p^{n+1}}. \end{cases}$$

(We work in Tate cohomology, into which the Borel cohomology embeds.) Using (43) and induction, we can prove

$$\underline{\xi}_n = \rho^{-2}\sigma^{2p^n-2p^{n-1}}\xi_n + \cdots + \rho^{-2}\sigma^{2p^N-2p^{n-1}}b^{p^N-p^n}\xi_N + \dots \tag{45}$$

Now apply λ to $\sigma^{-2}y = x^p - b^{p-1}x$, we have

$$\begin{aligned}&y \otimes \rho^{-2} + \sum_{n\geq 1} y^{p^n} \otimes \underline{\theta}_n \rho^{-2} \\ &= (x^p \otimes 1 + \sum_{n\geq 1} y^{p^n} \otimes \underline{\xi}_n) - (b^{p-1}x \otimes 1 + \sum_{n\geq 1} y^{p^{n-1}} \otimes b^{p-1}\underline{\xi}_n) \\ &= (\sigma^{-2}y \otimes 1 - y \otimes b^{p-1}\underline{\xi}_1) + \sum_{n\geq 1} y^{p^n} \otimes (\underline{\xi}_n - b^{p-1}\underline{\xi}_{n+1})\end{aligned}$$

Comparing coefficients, we get

$$\underline{\theta}_n \rho^{-2} = \underline{\xi}_n^p - \underline{\xi}_{n+1} b^{p-1}. \tag{46}$$

Note, in fact, that the relation (46) is true on the nose (meaning not just on the image in Borel cohomology, i.e. modulo the derived tail) by Proposition 5, and the Hopf algebroid relation between the product and the coproduct. One should also point out that, in particular,

$$\rho^{-2} = \sigma^{-2} - \underline{\xi}_1 b^{p-1}. \tag{47}$$

Now multiplying (43) by $\overline{u}$, we get

$$\overline{u}\sigma^{-2} = \overline{u}\rho^{-2} + \overline{u}\rho^{-2}\sigma^{2p-2}b^{p-1}\xi_1 + \dots \overline{u}\rho^{-2}\sigma^{2p^n-2}b^{p^n-1}\xi_n + \dots$$

Plugging in (42), we get

$$\begin{aligned}&u\sigma^{-2} - \overline{u}\rho^{-2} + b\tau_0 = \\ &b^{p-1}\sigma^{2p-2}(\overline{u}\rho^{-2}\xi_1 - b\tau_1) + \dots + b^{p^n-1}\sigma^{2p^n-2}(\overline{u}\rho^{-2}\xi_n - b\tau_n) + \dots\end{aligned} \tag{48}$$

In the Borel cohomology, apply λ to $q = bz - xu = bz - \sigma^{-2}tu$, we get

$$\lambda(q) = \lambda(z)(1 \otimes b) - \lambda(t)(1 \otimes \rho^{-2}\bar{u}).$$

Plugging in the formula for $\lambda(q)$, $\lambda(z)$, $\lambda(t)$, we get

$$\begin{aligned}&(bz - \sigma^{-2}tu) \otimes 1 + \sum_{n\geq 0}(\sigma^{2p^{n-1}-2p^n}t^{p^{n+1}} - b^{p^{n+1}-p^n}t^{p^n}) \otimes \widehat{\xi}_{n+1} \\ &= (z \otimes b + \sum_{n\geq 0} t^{p^n} \otimes \tau_n b) - (t \otimes \rho^{-2}\bar{u} + \sum_{n\geq 1} t^{p^n} \otimes \xi_n\rho^{-2}\bar{u}).\end{aligned}$$

The coefficients of z match on both sides. Comparing coefficients of t^{p^n}, we get

$$\begin{cases}-\sigma^{-2}u - b^{p-1}\widehat{\xi}_1 + \rho^{-2}\overline{u} - b\tau_0 & = 0 \\ \sigma^{2p^n-2p^{n-1}}\widehat{\xi}_n - b^{p^{n+1}-p^n}\widehat{\xi}_{n+1} - \tau_n b + \xi_n\rho^{-2}\overline{u} & = 0,\end{cases}$$

and so

$$\begin{cases}\widehat{\xi}_1 & = (-\sigma^{-2}u + \rho^{-2}\overline{u} - b\tau_0)b^{1-p} \\ \widehat{\xi}_{n+1} & = (\sigma^{2p^{n-1}-2p^n}\widehat{\xi}_n - \tau_n b + \xi_n\rho^{-2}\bar{u})b^{p^n-p^{n+1}}.\end{cases}$$

Using (48) and induction, we can prove that

$$\begin{aligned}&\widehat{\xi}_n = \sigma^{2p^n-2p^{n-1}}(b\tau_n - \overline{u}\rho^{-2}\xi_n) + \dots \\ &+\sigma^{2p^N-2p^{n-1}}(b\tau_N - \overline{u}\rho^{-2}\xi_N)b^{p^N-p^n} + \dots\end{aligned} \tag{49}$$

Recall that we have in Borel cohomology

$$\begin{aligned}q &= bz - u\sigma^{-2}t \\ s &= u\sigma^{-2}z \\ \nu &= \sigma^{-2}z\end{aligned}$$

From above, we have

$$b^{p-1}\underline{\xi}_1 = \sigma^{-2} - \rho^{-2}; \tag{50}$$

$$b^{p-1}\widehat{\xi}_1 = -\sigma^{-2}u + \rho^{-2}\overline{u} - b\tau_0. \tag{51}$$

Multiplying (50) by u (resp. $\overline{u}$) and adding to (51), we get

$$ub^{p-1}\underline{\xi}_1 + b^{p-1}\widehat{\xi}_1 + b\tau_0 = \rho^{-2}(\overline{u} - u), \tag{52}$$

$$\overline{u}b^{p-1}\underline{\xi}_1 + b^{p-1}\widehat{\xi}_1 + b\tau_0 = \sigma^{-2}(\overline{u} - u). \tag{53}$$

Plugging in the Borel cohomology expression into formula (37) and comparing coefficients with

$$\lambda(\nu) = \lambda(z)(1 \otimes \rho^{-2}) = z \otimes \rho^{-2} + \sum_{n \geq 0} t^{p^n} \otimes \tau_n \rho^{-2},$$

we must have that

$$\rho^{-2} = \sigma^{-2} - b^{p-1}\underline{\xi}_1 \qquad \text{coefficient of } z \tag{54}$$

$$\rho^{-2}\tau_0 = \sigma^{-2}(ub^{p-2}\underline{\xi}_1 + b^{p-2}\widehat{\xi_1} + \tau_0) - b^{p-1}\underline{\tau}_1 \qquad \text{coefficient of } t \tag{55}$$

$$\rho^{-2}\tau_{n+1} = \sigma^{2p^{n-1}-2p^n}\underline{\tau}_n - b^{p^{n+1}-p^n}\underline{\tau}_{n+1} \qquad \text{coefficient of } t^n \tag{56}$$

Now, (54) is just (50). Plugging (52) into (55), we have

$$b^{p-1}\underline{\tau}_1 = \rho^{-2}(\sigma^{-2}b^{-1}(\overline{u} - u) - \tau_0).$$

Plugging in (42) and inducting based on (56), we get

$$\underline{\tau}_n = \rho^{-2}\sigma^{2p^n-2p^{n-1}}\tau_n + \cdots + \rho^{-2}\sigma^{2p^N-2p^{n-1}}b^{p^N-p^n}\tau_N + \dots\,. \tag{57}$$

Now we shall prove that $\widehat{\tau}_n = \overline{u}\underline{\tau}_n$. From

$$\lambda(s) = s \otimes 1 + q \otimes (\tau_0 + b^{p-2}\widehat{\xi_1}) + \sum_{n \geq 1} y^{p^{n-1}} \otimes \widehat{\tau}_n$$

and $\lambda(s) = \lambda(u\nu) = \lambda(\nu)(1 \otimes \overline{u})$, it remains to verify

$$s \otimes 1 + q \otimes (\tau_0 + b^{p-2}\widehat{\xi_1}) = \nu \otimes \overline{u} - q \otimes b^{p-2}\underline{\xi}_1\overline{u} + x \otimes (\tau_0 + b^{p-2}\widehat{\xi_1})\overline{u}.$$

Using (51) and (53) ($\overline{u}$ is exterior), this is equivalent to

$$\nu \otimes (u - \overline{u}) + q \otimes \sigma^{-2}b^{-1}(\overline{u} - u) + x \otimes b^{-1}\sigma^{-2}u\overline{u} = 0.$$

Plugging in the Borel cohomology expressions, the left hand side is the sum of

$$z \otimes [\sigma^{-2}(u - \overline{u}) + \sigma^{-2}(\overline{u} - u)] = 0$$

and (u is exterior)

$$-\sigma^{-2}t \otimes u\sigma^{-2}b^{-1}(\overline{u} - u) + \sigma^{-2}t \otimes b^{-1}\sigma^{-2}u\overline{u} = 0.$$

Thus, we have

$$\widehat{\tau}_n = \overline{u}\rho^{-2}\sigma^{2p^n-2p^{n-1}}\tau_n + \cdots + \overline{u}\rho^{-2}\sigma^{2p^N-2p^{n-1}}\tau_N b^{p^N-p^n} + \dots\,. \tag{58}$$

From this, we can deduce further multiplicative relations

$$\widehat{\xi}_n\rho^{-2} = -\underline{\xi}_n\overline{u}\rho^{-2} + b\underline{\tau}_n, \tag{59}$$

$$\widehat{\tau}_n\rho^{-2} = \underline{\tau}_n\overline{u}\rho^{-2}, \tag{60}$$

$$b\widehat{\tau}_n = \widehat{\xi}_n\rho^{-2}\overline{u}, \tag{61}$$

and

$$\widehat{\tau}_n \rho^{-2} \overline{u} = 0. \tag{62}$$

Recall, of course, (47), and also

$$\overline{u}\rho^{-2} = u\sigma^{-2} + b\tau_0 + b^{p-1}\widehat{\xi}_1. \tag{63}$$

From this, we deduce additional multiplicative relations

$$\widehat{\tau}_n^2 = \widehat{\tau}_n \underline{\tau}_n = \widehat{\tau}_n \widehat{\xi}_n = \widehat{\xi}_n^2 = 0 \tag{64}$$

$$\widehat{\xi}_n \underline{\tau}_n = \underline{\xi}_n \widehat{\tau}_n. \tag{65}$$

Again, these relations are true on the nose, and not just in Borel cohomology, by Proposition 7 and the compatibility of the product and the coproduct.

Now using the fact that $H\underline{\mathbb{Z}/p} \wedge H\underline{\mathbb{Z}/p}$ is a wedge of R-suspensions of $H\underline{\mathbb{Z}/p}$ and HT, monomials in $A_\star$ which are b-divisible in $A_\star^{cc}$ must also be b-divisible in $A_\star$. Thus, we also get elements of the form

$$\frac{\widehat{\xi}_m \widehat{\xi}_n}{b}, \ \frac{\widehat{\xi}_m \underline{\xi}_n - \widehat{\xi}_n \underline{\xi}_m}{b}, \ \frac{\widehat{\xi}_m \widehat{\tau}_n}{b} = \frac{\widehat{\tau}_m \widehat{\xi}_n}{b}, \frac{\widehat{\xi}_m \underline{\tau}_n}{b} - \frac{\underline{\tau}_m \widehat{\xi}_n}{b}, \tag{66}$$

(note that the last two are related by switching m and n), and elements obtained by iterating this procedure.

In fact, this is related to the Example in Section 2. In the language of [20], the "quadruplet" $\widehat{\xi}_n, \underline{\xi}_n, \widehat{\tau}_n, \underline{\tau}_n$ generate an $HT_\star$ summand of $A_\star$. Thus, the elements (66) are precisely those divisions by b which are allowed in $HT \wedge_{H\underline{\mathbb{Z}/p}} HT_\star$ according to the Example. We obtain the following

Proposition 8. *Put $d\widehat{\xi}_i = \underline{\xi}_i$, $d\widehat{\tau}_i = \underline{\tau}_i$. For given $i_1 < \cdots < i_k$, let $K(i_1, \dots, i_k)$ denote the sub-$\mathbb{Z}/p$-module of $A_\star$ spanned by monomials of the form*

$$\kappa_{i_1} \dots \kappa_{i_k} \underline{\xi}_{i_1}^{s_1} \dots \underline{\xi}_{i_k}^{s_k} \tag{67}$$

where $0 \leq s_\ell \leq p-2$, κ_ℓ can stand for any of the elements $\widehat{\xi}_\ell$, $\underline{\xi}_\ell$, $\widehat{\tau}_\ell$, $\underline{\tau}_\ell$. Let $K_j(i_1, \dots, i_k)$ denote the submodule of $K(i_1, \dots, i_k)$ of elements divisible by b^j. Then $K_j(i_1, \dots, i_k)$ is spanned by elements

$$y, dy$$

where y is of the form (67) so that at least $j+1$ of the elements κ_{i_s} are of the form $\widehat{\xi}_{i_s}$ or $\widehat{\tau}_{i_s}$, only at most one of them being $\widehat{\tau}_{i_s}$. Such y will be called admissible *(otherwise, $y = 0$).*

□

Using the our computation of $H\underline{\mathbb{Z}/p}^\star B_{\mathbb{Z}/p}(\mathbb{Z}/p)$ in the last section, we similarly conclude that

$$\underline{\mu}_n = \frac{\widehat{\xi}_n \underline{\xi}_n^{p-1} - \underline{\theta}_n \rho^{-2}\overline{u}}{b}. \tag{68}$$

By [20], this element generates an $H\underline{\mathbb{Z}/p}_\star$-summand.

5. The $\mathbb{Z}/p$-equivariant Steenrod algebra

To express the additive structure of $A_\star$, we introduce a "Cartan-Serre basis" of elements of the form

$$\tau_0^e \underline{\Theta}_S \underline{M}_Q, \tag{69}$$

and

$$\underline{\Xi}_R \underline{\Theta}_S \underline{\mathrm{T}}_E \tau_0^e \tag{70}$$

where

$$\begin{array}{ll} R = (r_1, r_2, \dots), & r_n \in \mathbb{Z}, 0 \le r_n < p, \\ Q = (q_1, q_2, \dots), & q_n \in \{0,1\}, \\ S = (s_1, s_2, \dots), & s_n \in \mathbb{Z}, 0 \le s_n, \\ E = (e_1, e_1, \dots), & e_n \in \{0,1\}, \\ e \in \{0,1\} & \end{array} \tag{71}$$

As usual, only finitely many non-zero entries are allowed in each sequence. Here we understand

$$\underline{\Theta}_S = \underline{\theta}_1^{s_1} \underline{\theta}_2^{s_2} \dots,$$
$$\underline{M}_Q = \underline{\mu}_1^{q_1} \underline{\mu}_2^{q_2} \dots,$$
$$\underline{\Xi}_R = \underline{\xi}_1^{r_1} \underline{\xi}_2^{r_2} \dots,$$
$$\underline{\mathrm{T}}_E = \underline{\tau}_1^{e_1} \underline{\tau}_2^{e_2} \dots .$$

The elements $\underline{\Theta}_n$ have the dimensions introduced above. Additionally, we recall

$$|\underline{\Xi}_n| = 2p^{n-1} + (p^n - p^{n-1} - 1)\beta,$$
$$|\underline{\mathrm{T}}_n| = |\underline{\Xi}_n| + 1.$$

Additionally, for any sequence of natural numbers P with finitely many non-zero elements, we denote by $|P|$ the number of non-zero elements in P. We will assume $|R + E| \neq 0$ in (70).

Theorem 9. *The dual $\mathbb{Z}/p$-equivariant Steenrod algebra $A_\star$ is, additively, a sum of copies of*

$$H\underline{\mathbb{Z}/p}_\star$$

shifted by the total dimension of elements of the form (69) with $R = E = 0$, and copies of

$$HM_\star$$

with admissible generators of the form y, dy as in Proposition 8 times $b^{-\ell}$ where ℓ is the maximum number so that y is divisible by b^ℓ according to Proposition 8. The dimensions of admissible monomials in the admissible monomials in $\underline{\xi}_m$, $\underline{\tau}_n$ are given by the values $|\underline{\Xi}_R|$, $|\underline{\mathrm{T}}_E|$ given above.

The good tail $A^g_\star$ is multiplicatively generated, as an algebra over

$$\mathbb{Z}/p[\sigma^{-2}, b] \otimes \Lambda[\sigma^{-2}u]$$

by the elements $\underline{\xi}_n$, $\widehat{\xi}_n$, $\underline{\tau}_n$, $\widehat{\tau}_n$, $\underline{\theta}_n, \underline{\mu}_n$, $n \geq 1$, τ_0, subject to all relations valid in $A^{cc}_\star$, including (59)-(65). The derived tail is given by

$$A^d_\star = H^2_{(b,\sigma^{-2})}(H\underline{\mathbb{Z}/p}^g_\star, A^g_\star)[-1]$$

and $A_\star$ is an abelian ring with respect to $A^g_\star$.

Proof. We already checked this on Borel (co)homology. On geometric fixed points (which are determined by the "ideal" part), the elements $\underline{\xi}_R$ match non-negative powers of ρ^{-2}, τ_0 matches $\overline{u}\rho^{-2}$. The elements $\widehat{\xi}_n, \widehat{\tau}_n, \underline{\tau}_n$ and their $\mathbb{Z}/p[\sigma^{-2}] \otimes \Lambda[u\sigma^{-2}]$-multiples represent the $\mathbb{Z}/p[\sigma^{-2}] \otimes \Lambda[u\sigma^{-2}]$-multiples of τ_{n-1}. (The relations guarantee that no element is represented twice.) □

References

[1] A. W. M. Dress: Contributions to the theory of induced representations, in: *"Classical" Algebraic K-Theory, and Connections with Arithmetic*, Lecture Notes in Mathematics, Vol. 342 Springer, Berlin, (1973) 181-240.

[2] L.J.Caruso: Operations in equivariant $\mathbb{Z}_p$-cohomology, *Math. Proc. Cambridge Philos. Soc.* 126 (1999), no. 3, 521–541

[3] A.D.Elmendorf, I.Kriz, M.A.Mandell, J.P.May: *Rings, modules, and algebras in stable homotopy theory*, with an appendix by M. Cole. Mathematical Surveys and Monographs, 47. American Mathematical Society, Providence, RI, 1997. xii+249 pp.

[4] J. P. C. Greenlees: Some remarks on projective Mackey functors, *Journal of Pure and Applied Algebra* 81 (1992) 17-38

[5] M.Hausmann: Global group laws and equivariant bordism rings, to appear in *Annals of Math.* arXiv:1912.07583

[6] M. A. Hill, M. J. Hopkins, and D. C. Ravenel: On the nonexistence of elements of Kervaire invariant one, *Ann. of Math.* (2) 184 (2016), 1-262

[7] M. A. Hill, M. J. Hopkins, and D. C. Ravenel: On the 3-primary Arf-Kervaire Invariant Problem, preprint, https://people.math.rochester.edu/faculty/doug/mypapers/odd.pdf

[8] P.Hu, I.Kriz: Real-oriented homotopy theory and an analogue of the Adams-Novikov spectral sequence, *Topology* 40 (2001), no. 2, 317-399

[9] P.Hu, I.Kriz: $RO(G)$-graded coefficients of the constant Mackey cohomology for cyclic p-groups, preprint, 2010

[10] P.Hu, I.Kriz, K.Ormsby: Convergence of the motivic Adams spectral sequence, *J. K-theory* 7 (2011) 573-596

[11] I.Kriz, Y.Lu: On the $RO(G)$-graded coefficients of dihedral equivariant cohomology, *Math. Res. Lett.* 27, (2020) 1109-1128

[12] I.Kriz, Y.Lu: On the structure of equivariant formal group laws, preprint, available at http://www.math.lsa.umich.edu/~ikriz

[13] S. Kriz: Some remarks on Makey functors, http://www-personal.umich.edu/~skriz/MackeyChains21105.pdf

[14] L.G.Lewis Jr.: The theory of Green functors, Preprint

[15] L. G. Jr. Lewis; J. P. May; M. Steinberger; J. E. McClure: *Equivariant stable homotopy theory*, With contributions by J. E. McClure. Lecture Notes in Mathematics, Vol. 1213. Springer-Verlag, Berlin, (1986), x+538 pp

[16] L. G. Jr. Lewis, J. P. May, J. E. McClure: Ordinary RO(G)-graded cohomology, *Bull. Amer. Math. Soc.* (N.S.) 4 (1981), no. 2, 208-212.

[17] Z. Li: Box Product of Mackey Functors in Terms of Modules, arXiv:1509.07051

[18] J.Milnor: The Steenrod algebra and its dual, *Ann. of Math.* (2) 67 (1958), 150-171

[19] D.C.Ravenel: *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, 121. Academic Press, Inc., Orlando, FL, 1986. xx+413 pp.

[20] K.Sankar, D.Wilson: On the Cp-equivariant dual Steenrod algebra, arXiv 2103.‘6006

[21] V.Voevodsky: On motivic cohomology with $\mathbb{Z}/\ell$-coefficients. *Ann. of Math.* (2) 174 (2011), no. 1, 401-438