Summer Seminar - Mike Hill - Computing with Slice Spectral Sequence 7/15/09

① Equivariant Homotopy Theory

② Sketch construction of Slice Spectral Sequence (SSS)

③ SSS for MUR & $\mathbb{Z}/2^n \underset{\mathbb{Z}/2}{\otimes} MUR$

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① G finite group. Can consider two types of homotopy groups:

1. $H \subseteq G$, $\pi_n X^H = \pi_n^H X = [G_+ \underset{H}{\wedge} S^n, X]_G = [S^n, X]_H$

(trivial action on S^n)

Taken all together, these form a Mackey functor, ie a functor

$$\mathcal{O}(G) = \{ G/H, \ \& \ G\text{-maps} \}$$

Conjugation & restriction & transfers

We will call it $\underline{\pi_*}(X)$

2. So if V is a G-representation, have a sphere S^V,

$$\pi_V(X) = [S^V, X]_G$$

S^{V-W} (virtual representation) $\pi_{V-W}(X) = [S^V, S^W \wedge X]_G$

$RO(G)$-graded homotopy gps of X. (will use $\pi_\star X = [S^\star, X]_G$, $\star \in RO(G)$)

If V is a representation of G, then define

$$a_V : S^0 \longrightarrow S^V$$
$$\{0, \infty\} \hookrightarrow V^+ \quad \text{equivariant}$$

$a_V = 0$ if V contains a trivial representation

$u: S^{|V|} \longrightarrow S^V$ (not an equivariant map)

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② Constructing the SSS

Build a "Postnikov" tower

Slice cells spheres $\{ G_+ \underset{H}{\wedge} S^{k\rho_H}, \ G_+ \underset{H}{\wedge} S^{k\rho_H - 1} \}$ $k \in \mathbb{Z}$.

ρ_H regular representation for $H \subseteq G$

$\dim(G_+ \wedge_H S^{k\rho_H}) = k\cdot|H|$, $\dim(G_+ \wedge_H S^{k\rho_{H}-1}) = k\cdot|H|-1$

Note: only take one desuspension.

The slice tower is formed by killing all maps from slice spheres of dim $\geq n$

$$\begin{array}{ccc} P^n_n X & \longrightarrow & P^n X \\ & & \downarrow \\ & & P^{n-1}X \\ & & \downarrow \\ & & \vdots \end{array}$$

n-th slice of X

Apply $\underline{\pi}_*(\ )$ to the tower or $\pi_\star(-)$ to the tower & get a slice spectral sequence. Sometimes just do $\pi^G_\star(-)$.

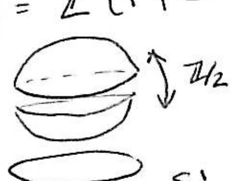
Proposition $H\mathbb{Z}\wedge(G_+\wedge_H S^{k\rho_H})$ is a $k\cdot|H|$ slice

$H\mathbb{Z}$ Eilenberg-MacLane object for the constant Mackey functor $\underline{\mathbb{Z}}$.

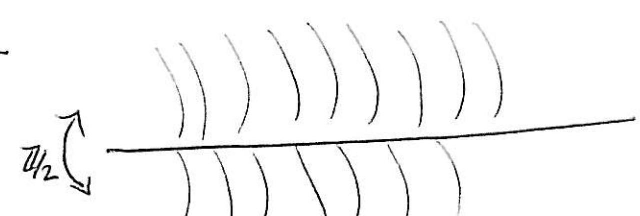
Ex. $G=\mathbb{Z}/2$, $\rho_2=\rho_{\mathbb{Z}/2}$ regular rep. $=1\oplus\sigma$

$RO(\mathbb{Z}/2)=\mathbb{Z}\{1\}\oplus\mathbb{Z}\{\sigma\}$

~~[illegible]~~ sign



$\mathbb{Z}/2\times D^2$



S^1 (fixed by entire group) $= \mathbb{Z}/2\times_{\mathbb{Z}/2} S^1$

boundary of is identified w/ common S^1.

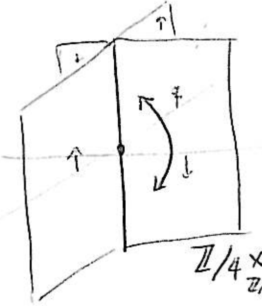
		fixed pts of $\{1\}\subset\mathbb{Z}/2$	fixed pts for $\mathbb{Z}/2$
dim 2	$\mathbb{Z}[\mathbb{Z}/2] \xrightarrow{[1\ \ 1]} \mathbb{Z}$ (trivial action)	$\mathbb{Z}^2 \xrightarrow{[1\ \ 1]} \mathbb{Z}$	$\mathbb{Z}\{(1,1)\}\subset\mathbb{Z}^2 \xrightarrow{[1\ \ 1]}$; $\mathbb{Z} = \mathbb{Z}$
dim 1			
	$H_*(-)$: 2, 1	$H_*(-)$: $\mathbb{Z}$, 0	$H_*(-)$: 0, $\mathbb{Z}/2$

(This gives $\underline{\pi}_*(H\mathbb{Z}\wedge(G_+\wedge_G S^{k\rho_G}))$)

(homology of 2-sphere)
fixed pt sphere

Ex $G = \mathbb{Z}/4$ $\quad S^{\rho_{\mathbb{Z}/4}-1}$

$$\rho_{\mathbb{Z}/4} = 1 \oplus \underbrace{\sigma \oplus L}$$



$\mathbb{Z}/4 \times D^3$

$\mathbb{Z}/4 \times D^2$

$\mathbb{Z}/4 \times_{\mathbb{Z}/2} D^1$

$\downarrow$

$S^0 = \{0, \infty\}$

$$\begin{array}{ll} \mathbb{Z}^4 & 3 \\ \downarrow & \\ \mathbb{Z}^4 = \mathbb{Z}[\mathbb{Z}/4] & 2 \\ \downarrow & \\ \mathbb{Z}^2 = \mathbb{Z}[{}^{\mathbb{Z}/4}/_{\mathbb{Z}/2}] & 1 \\ \downarrow & \\ \mathbb{Z} & 0 \end{array}$$

Know that when we take $\{1\}$-fixed points, we should recover $H_*(S^3)$ (non-equivariant),

so w/ this info + equivariance, can recover all maps

$$\begin{array}{l} \mathbb{Z}^4 \\ \downarrow 1+g \\ \mathbb{Z}^4 \\ \downarrow \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix} \\ \mathbb{Z}^2 \\ \downarrow [1 \ 1] \\ \mathbb{Z} \end{array}$$

g gen of $\mathbb{Z}/2$

In general, for $G = \mathbb{Z}/2^n$, $k\rho_G$

$$\begin{array}{l} \vdots \\ \mathbb{Z}^4 \\ \downarrow \\ \mathbb{Z}^4 \\ \downarrow \\ \mathbb{Z}^2 = \mathbb{Z}[G/\mathbb{Z}/2^{n-1}] \\ \downarrow \\ \mathbb{Z}^2 \\ \downarrow \\ \mathbb{Z}^2 \\ \downarrow \\ \mathbb{Z} \end{array}$$

$4k$

(sign rep

in dim $(k\rho_G)^{\mathbb{Z}/2^{n-1}} = 2k$

in dim $(k\rho_G)^G = k$

Knowing that the $\{1\}$-fixed pts should recover $H_*(S^{2^n k})$ gives most of the maps.

$G = \mathbb{Z}/2$ $\quad S^{2\rho_{\mathbb{Z}/2}}$ $\quad$ dim 4

$$\begin{array}{l} \mathbb{Z}^2 \\ \downarrow 1-g \\ \mathbb{Z}^2 \\ \downarrow [1 \ 1] \\ \mathbb{Z} \end{array}$$

$\pi_*^{\{1\}}(-) = H_*$(underlying complex) $= \begin{cases} \mathbb{Z} & 4 \\ 0 & 3 \\ 0 & 2 \end{cases}$,

dim 2

$$\begin{array}{l} \mathbb{Z} = \mathbb{Z}\{[1\ 1]\} \subset \mathbb{Z}^2 \\ \downarrow 0 \qquad\qquad\qquad\quad \downarrow \\ \mathbb{Z} = \mathbb{Z}\{[1\ 1]\} \subset \mathbb{Z}^2 \\ \downarrow 2 \\ \mathbb{Z} \end{array}$$

$\mathbb{Z}/2$ - fixed pts

$\pi_*^{\mathbb{Z}/2}(-) = \begin{cases} \mathbb{Z} & 4 \\ 0 & 3 \\ \mathbb{Z}/2 & 2 \end{cases}$

Gen of $\pi_*^{\mathbb{Z}/2}(-) = \begin{cases} \mathbb{Z} & * = 4 \leftrightarrow S^4 \to H\mathbb{Z} \wedge S^{2\rho_2} \quad u_{2\rho_2} \text{ (Hurewicz image)} \\ \mathbb{Z}/2 & * = 2 \leftrightarrow S^2 \to H\mathbb{Z} \wedge S^{2\rho_2} \end{cases}$

$\leftrightarrow S^0 \to H\mathbb{Z} \wedge S^{2\rho_2} = H\mathbb{Z} \wedge S^{2\sigma}$

is the Hurewicz image of $a_{2\sigma}$

Any homotopy class can be written as $a_{\sqcup} u_{\sqcup}$.

$S^{-k\rho_G}$ is the equivariant Spanier-Whitehead dual to $S^{k\rho_G}$

$\Rightarrow$ to compute $\underline{\pi}_* H\mathbb{Z} \wedge S^{-k\rho_G}$, we dualize the equivariant complex for $H\mathbb{Z} \wedge S^{k\rho_G}$

$G = \mathbb{Z}/2$, $H\mathbb{Z} \wedge S^{-\rho_2}$ $\quad {}_2\ \mathbb{Z}^2 \xrightarrow{[1\ 1]} {}_1\ \mathbb{Z} \quad \rightsquigarrow \quad {}_{-1}\ \mathbb{Z} \xrightarrow{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} {}_{-2}\ \mathbb{Z}^2 \quad \supset \quad \mathbb{Z}\{[1\ 1]\} \xleftarrow{\cong} \mathbb{Z}$

fixed pts

$\pi_{-2}^{\mathbb{Z}/2} H\mathbb{Z} \wedge S^{-\rho_2} = 0$

$\quad \| \quad \pi_*^{\mathbb{Z}/2}$

We always see something like this in all complexes:

Thm (Gaps) $\quad \pi_{-2}^{G} H\mathbb{Z} \wedge (G_+ \wedge_H S^{k\rho_H}) = 0 \quad$ if $H \neq \{e\}$

Thm The odd slices of $MU\mathbb{R}$ & $\mathbb{Z}/2^n \otimes_{\mathbb{Z}/2} MU\mathbb{R}$ are all contractible and the even slices are of the form $\bigvee H\mathbb{Z} \wedge (G_+ \wedge_H S^{k\rho_H})$ $\quad \{e\} \neq H$.

$\overline{(-)}$ cx conjugation

$\mathbb{Z}/2^n \otimes_{\mathbb{Z}/2} MU\mathbb{R} = MU^{\wedge 2^{n-1}}$, eg $\mathbb{Z}/4 \otimes_{\mathbb{Z}/2} MU\mathbb{R} = MU \wedge MU$

underlying htpy type

$\mathbb{Z}/2^n$-equivariant E_∞ ring spectrum

$MU\mathbb{R} \longleftrightarrow MU \qquad \pi_* MU = \mathbb{Z}[x_1, \dots] \qquad S^{2i} \xrightarrow{x_i} MU$

$\mathbb{Z}/4 \underset{\mathbb{Z}/2}{\otimes} MU\mathbb{R} \longleftrightarrow \pi_* MU \wedge MU = \mathbb{Z}[x_1, \dots, b_1, \dots]$

$S^{i\rho_2} \xrightarrow{\bar{x}_i} MU\mathbb{R} \longleftrightarrow x_i : S^{2i} \longrightarrow MU$

equivariant lift

$b_1 \longmapsto b_1 + x_1$

Can identify sub $\mathbb{Z}/4$ - representations inside $\pi_* MU \wedge MU$, ie $\pi_2 MU \wedge MU =$ $\mathrm{Ind}_{\mathbb{Z}/2}^{\mathbb{Z}/4} \sigma$, and it corresponds to something in the equivariant side

$\mathbb{Z}/4_+ \wedge_H S^{k\rho_H} \longleftrightarrow \mathrm{Ind}_H^{\mathbb{Z}/4} \begin{cases} 1 \\ \sigma \end{cases}$

$k \cdot |H| \longleftrightarrow$ dim in which $\mathrm{Ind}_H^{\mathbb{Z}/4} \begin{matrix} 1 \\ \sigma \end{matrix}$ lives.

Each copy of $\mathbb{Z}/2^n{}_+ \wedge_H S^{k\rho_H}$ contributes a wedge summand.

$\mathbb{Z}[x, y] \qquad x \longmapsto y \longmapsto -x \qquad |x| = |y| = 2$

$\pi_2 \quad x \rightleftarrows y = \mathrm{Ind}_{\mathbb{Z}/2}^{\mathbb{Z}/4} \sigma \longleftrightarrow H\mathbb{Z} \wedge (\mathbb{Z}/4_+ \wedge_{\mathbb{Z}/2} S^{\rho_2})$

$\pi_4 \quad \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} = \mathrm{Ind}_{\mathbb{Z}/2}^{\mathbb{Z}/4} 1 \longleftrightarrow H\mathbb{Z} \wedge (\mathbb{Z}/4_+ \wedge_{\mathbb{Z}/2} S^{2\rho_2})$

$xy = \sigma \longleftrightarrow H\mathbb{Z} \wedge (S^{\rho_{\mathbb{Z}/4}})$

$\pi_8 \quad \begin{pmatrix} x^4 \\ y^4 \end{pmatrix} \quad \mathrm{Ind}_{\mathbb{Z}/2}^{\mathbb{Z}/4} 1 \longleftrightarrow H\mathbb{Z} \wedge (\mathbb{Z}/4_+ \wedge_{\mathbb{Z}/2} S^{4\rho_2})$

$\begin{pmatrix} x^3 x \\ y^3 x \end{pmatrix} \quad " \longleftrightarrow \quad "$

$x^2 y^2 = 1 \longleftrightarrow S^{2\rho_{\mathbb{Z}/4}}$

This gives me an algorithm to compute E_2 -term of SSS, reading it off the non-equivariant htpy gps.