All about $Tmf_1(3)$

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ALL ABOUT $Tmf_1(3)$

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Abstract. We explore the C_2-equivariant spectra $Tmf_1(3)$ and $TMF_1(3)$. In particular, we compute their C_2-equivariant Picard groups and the C_2-equivariant Anderson dual of $Tmf_1(3)$. This implies corresponding results for the fixed point spectra $TMF_0(3)$ and $Tmf_0(3)$. Furthermore, we prove a Real Landweber exact functor theorem.

1. Introduction

The spectrum TMF of topological modular forms comes in many variants. While TMF itself arises from the moduli stack of elliptic curves $\mathcal{M}_{ell}$, there is also a spectrum Tmf associated with the compactification $\overline{\mathcal{M}}_{ell}$. Finally, tmf is defined as the connective cover of Tmf. It has been the spectrum tmf and its cohomology that have been so far most relevant to applications (see for example [BHHM08] and [BP04]).

It is often simpler to work with topological modular forms with level structures. Among the many possibilities, the most relevant for us will be $TMF_1(n)$ and $TMF_0(n)$ corresponding to the moduli stacks $\mathcal{M}_1(n)$ and $\mathcal{M}_0(n)$. The former stack classifies elliptic curves with a chosen point of exact order n and the latter elliptic curves with a chosen subgroup of order n. Note that for $n \geq 2$, the spectrum $TMF_1(n)$ is Landweber exact, while $TMF_0(n)$ is not.

In [HL13], Lawson and the first-named author overcame certain technical obstacles to define E_∞-ring spectra $Tmf_1(n)$ and $Tmf_0(n)$ corresponding to the compactified moduli stacks $\overline{\mathcal{M}}_1(n)$ and $\overline{\mathcal{M}}_0(n)$. One can then define $tmf_1(n)$ and $tmf_0(n)$ as the connective covers of the spectra and hope that they form good connective models for $TMF_1(n)$ and $TMF_0(n)$. The aim of this article is to explore these spectra in the case $n = 3$ with methods from Real homotopy theory.

Real homotopy theory is the study of genuinely equivariant C_2-spectra, also known as Real spectra. The theory has its origins in Atiyah's article [Ati66] on Real K-theory and came to new prominence through the work of Hu–Kriz [HK01] and the work of Hill–Hopkins–Ravenel on the Kervaire invariant 1 problem [HHR09].

The spectra $TMF_1(3)$ and $Tmf_1(3)$ inherit C_2-actions from an algebro-geometrically defined C_2-action on $\overline{\mathcal{M}}_1(3)$. We will view them as cofree C_2-spectra so that

$$TMF_1(3)^{C_2} \simeq TMF_1(3)^{hC_2} \simeq TMF_0(3)$$

and

$$Tmf_1(3)^{C_2} \simeq Tmf_1(3)^{hC_2} \simeq Tmf_0(3).$$

We define the C_2-spectrum $tmf_1(3)$ as the C_2-equivariant connective cover of $Tmf_1(3)$.

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Mahowald and Rezk [MR09] have already computed the homotopy groups of $TMF_0(3)$ and a similar computation produces actually the $RO(C_2)$-graded C_2-equivariant homotopy groups of $tmf_1(3)$ and hence $TMF_1(3)$. Using this computation, we show that $tmf_1(3)$ has a Real orientation and is more precisely a form of $BP\mathbb{R}\langle 2\rangle$. Moreover, we show that $TMF_1(3)$ is Real Landweber exact in the sense that there is an isomorphism

$$M\mathbb{R}_\bigstar(X) \otimes_{MU_*} TMF_1(3)_* \to TMF_1(3)_\bigstar X,$$

natural in a C_2-spectrum X. Here $M\mathbb{R}_\bigstar(X)$ denotes the $RO(C_2)$-graded C_2-equivariant homology groups of X with respect to the Real bordism spectrum $M\mathbb{R}$ and similarly for $TMF_1(3)_\bigstar X$.

As $\overline{\mathcal{M}}_1(3)$ is proper over $\operatorname{Spec}\mathbb{Z}\left[\frac{1}{3}\right]$, one expects a manifestation of Serre duality in $Tmf_1(3)$. A suitable duality to look for in the topological setting is *Anderson duality*, an integral version of Brown–Comenetz duality. For example, Stojanoska computed in [Sto12] that $Tmf[\frac{1}{2}]$ is Anderson self dual in the sense that $I_{\mathbb{Z}[\frac{1}{2}]}Tmf[\frac{1}{2}] \simeq \Sigma^{21}Tmf[\frac{1}{2}]$. We want to compute the C_2-equivariant Anderson dual $I_{\mathbb{Z}\left[\frac{1}{3}\right]}Tmf_1(3)$ of $Tmf_1(3)$. While it is an easy calculation that non-equivariantly $I_{\mathbb{Z}\left[\frac{1}{3}\right]}Tmf_1(3) \simeq \Sigma^9 Tmf_1(3)$, this equivalence does *not* hold C_2-equivariantly. We rather get the following:

Theorem. *There is a C_2-equivariant equivalence*

$$I_{\mathbb{Z}\left[\frac{1}{3}\right]}Tmf_1(3) \simeq \Sigma^{5+2\rho}Tmf_1(3),$$

where ρ denotes the regular representation of C_2. It follows that

$$I_{\mathbb{Z}\left[\frac{1}{3}\right]}Tmf_0(3) \simeq (\Sigma^{5+2\rho}Tmf_1(3))^{hC_2}.$$

Thus, the self-duality of $Tmf_0(3)$ is not fully apparent in the integer-graded homotopy groups

$$\pi_* Tmf_0(3) \cong \pi_*^{C_2} Tmf_1(3),$$

but only in the $RO(C_2)$-graded homotopy groups $\pi_\bigstar^{C_2} Tmf_1(3)$. We prove this theorem by an application of the slice spectral sequence. There has been similar work by Ricka [Ric14] on Anderson duality of integral versions of Morava K-theory; our results have been obtained independently.

Next we turn to the topic of Picard groups. Given an E_∞-ring spectrum R, its Picard group $\operatorname{Pic}(R)$ is defined as the group of invertible R-module spectra up to weak equivalence. From the perspective of [BN14], these are the global twists of the associated cohomology theory. The Picard group was first introduced into stable homotopy theory by Hopkins; recent work of Mathew and Stojanoska [MS14] then significantly extended our toolbox for its computation. They show that all invertible TMF-modules are suspensions of TMF so that $\operatorname{Pic}(TMF) \cong \mathbb{Z}/576$. In contrast, they show that $\operatorname{Pic}(Tmf)$ contains exotic elements that are not suspensions of Tmf and compute $\operatorname{Pic}(Tmf) \cong \mathbb{Z} \oplus \mathbb{Z}/24$.

We will use their methods to understand $\operatorname{Pic}(TMF_0(3))$ and $\operatorname{Pic}(Tmf_0(3))$, but add a dash of equivariant homotopy theory. The maps

$$Tmf_0(3) \to Tmf_1(3)$$

and

$$TMF_0(3) \to TMF_1(3)$$

are faithful C_2-Galois extensions in the sense of Rognes. Galois descent then shows that

$$\mathrm{Pic}(Tmf_0(3)) \cong \mathrm{Pic}_{C_2}(Tmf_1(3))$$

and

$$\mathrm{Pic}(TMF_0(3)) \cong \mathrm{Pic}_{C_2}(TMF_1(3)),$$

where $\mathrm{Pic}_{C_2}(Tmf_1(3))$ denotes the group of invertible C_2-module spectra over $Tmf_1(3)$ and similarly for $\mathrm{Pic}_{C_2}(TMF_1(3))$. First we prove:

Theorem. *Every invertible $TMF_0(3)$-module is an (integral) suspension of $TMF_0(3)$. Thus,*

$$\mathrm{Pic}_{C_2}(TMF_1(3)) \cong \mathrm{Pic}(TMF_0(3)) \cong \mathbb{Z}/48.$$

The analogous theorem for $Tmf_0(3)$ is not true, but we have the following equivariant refinement:

Theorem. *Every invertible C_2-equivariant $Tmf_1(3)$-module is an equivariant suspension $\Sigma^V Tmf_1(3)$, for an element $V \in RO(C_2)$. The corresponding homomorphism*

$$RO(C_2) \to \mathrm{Pic}_{C_2}(Tmf_1(3)), \quad V \mapsto \Sigma^V Tmf_1(3)$$

is thus surjective and has kernel generated by $8 - 8\sigma$, for σ the sign representation. Therefore,

$$\mathrm{Pic}(Tmf_0(3)) \cong \mathrm{Pic}_{C_2}(Tmf_1(3)) \cong \mathbb{Z} \oplus \mathbb{Z}/8.$$

We give a short overview of the structure of this article. Section 2 contains some basics about C_2-spectra, discusses Real orientability and the Real Landweber exact functor theorem; it concludes with the definition and basic properties of forms of $BP\mathbb{R}\langle n\rangle$ and $E\mathbb{R}(n)$. Section 3 recalls basic properties about the C_2-equivariant slice filtration. Section 4 introduces the main characters $Tmf_0(3)$ and $Tmf_1(3)$ and their variants, discusses their relationship and computes the $RO(C_2)$-graded homotopy groups of $tmf_1(3)$. Section 5 computes the slices of $Tmf_1(3)$ and applies this to compute its equivariant Anderson dual. Section 6 is about Picard groups, especially those of $TMF_0(3)$ and $Tmf_0(3)$.

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2. C_2-spectra and Real Landweber exactness

In this section, we will first treat some basics about C_2-spectra and Real orientations. Then we will prove a Real version of the Landweber exact functor theorem, both in classical and in stack language. In the last subsection, we define what we mean by forms of $BP\mathbb{R}\langle n\rangle$ and $E\mathbb{R}(n)$ and apply the Real Landweber exact functor theorem to the latter.

2.1. **Basics.** We define for a finite group G the category Sp_G of G-spectra to be the category of orthogonal spectra with G-action; weak equivalences are those morphisms that induce isomorphisms under π_*^H for all subgroups $H \subseteq G$, where the homotopy groups are defined via stabilization with respect to the regular representation. This is explained in detail in [Sch14].

Denote by Sp_G^u the same category but with the underlying equivalences as weak equivalences. The functor

$$\mathrm{Sp}_G^u \to \mathrm{Sp}_G, \quad X \mapsto F(EG_+, X)$$

preserves weak equivalences and defines therefore a derived functor for $\mathrm{id}\colon \mathrm{Sp}_G^u \to \mathrm{Sp}_G$. G-Spectra in the image are called *cofree*. If we come across a spectrum with a G-action whose homotopy type is only well-defined up to underlying equivalence, we will view it in this way as a cofree G-spectrum.

Since the hom objects in $\mathrm{Ho}(\mathrm{Sp}_G)$ are naturally Mackey functor valued, we will make frequent use of this additional structure. As usual, Mackey functor valued homotopy groups will be denoted by $\underline{\pi}_*$. If we want to stress that we consider the homotopy groups of the underlying non-equivariant spectrum, we write π_*^u. Furthermore, we will use $RO(G)$-gradings and denote the regular representation of C_2 by ρ and the sign representation by σ.

We will only use the case $G = C_2$ in this paper. Motivated by Atiyah's Real K-theory, C_2-spectra are also sometimes called *Real spectra*. Given a C_2-spectrum $E\mathbb{R}$, we denote by $E\mathbb{R}_\bigstar(X)$ the value of the associated $RO(C_2)$-graded homology theory on a C_2-spectrum X and we set $E\mathbb{R}_\bigstar = E\mathbb{R}_\bigstar(\mathrm{pt})$. This is the value at C_2/C_2 of the associated Mackey functor valued homology.

Definition 2.1. A C_2-spectrum $E\mathbb{R}$ is *even* if $\underline{\pi}_{k\rho-1}E\mathbb{R} = 0$ for all $k \in \mathbb{Z}$. It is called *strongly even* if additionally $\underline{\pi}_{k\rho}E\mathbb{R}$ is a constant Mackey functor for all $k \in \mathbb{Z}$, i.e. if the restriction $\pi_{k\rho}^{C_2}E\mathbb{R} \to \pi_{k\rho}^u E\mathbb{R}$ is an isomorphism.

For example, by [HK01, Theorem 4.11], $BP\mathbb{R}$ and $M\mathbb{R}$ are strongly even.

Recall the following definition:

Definition 2.2. Let X be a C_2-spectrum. A *Real orientation* for $E\mathbb{R}$ is a class

$$x \in E\mathbb{R}^\rho(\mathbb{CP}^\infty) = \underline{[\mathbb{CP}^\infty, \Sigma^\rho E\mathbb{R}]}(C_2/C_2),$$

restricting to the class in $E\mathbb{R}^\rho(\mathbb{CP}^1) \cong [\mathbb{CP}^1, S^\rho \wedge X]^{C_2}$ corresponding to

$$1 \in [S^0, E\mathbb{R}]^{C_2} \cong [S^\rho, S^\rho \wedge E\mathbb{R}]^{C_2}$$

under the (chosen) isomorphism $S^\rho = \mathbb{CP}^1$. Here, we view $\mathbb{CP}^n$ as a C_2-space via complex conjugation.

By [HK01, Theorem 2.25], Real orientations of commutative C_2-ring spectra are in one-to-one correspondence with homotopy classes of maps $M\mathbb{R} \to E\mathbb{R}$ of C_2-ring spectra, where ring spectra are understood to be up to homotopy.

Lemma 2.3. *Every even C_2-spectrum $E\mathbb{R}$ is Real orientable.*

Proof. We have cofiber sequences

$$S^{(n+1)\rho-1} \to \mathbb{CP}^n \to \mathbb{CP}^{n+1}.$$

The long exact sequence in cohomology then shows that the map

$$E\mathbb{R}^\rho(\mathbb{CP}^{n+1}) \to E\mathbb{R}^\rho(\mathbb{CP}^n)$$

is surjective. The Milnor sequence gives the result. $\square$

It is part of our philosophy that the Mackey functor $\underline{\pi}_{k\rho}$ behaves often much nicer than the integral Mackey functor $\underline{\pi}_{2k}$. The following is a weak version of a Whitehead theorem using $\underline{\pi}_{k\rho}$:

Lemma 2.4. *Let $f\colon E\mathbb{R} \to F\mathbb{R}$ be a natural transformation of $RO(C_2)$-graded homology theories. Denote the underlying homology theories by E and F. Assume that f induces isomorphisms*

$$E\mathbb{R}_{k\rho} \to F\mathbb{R}_{k\rho} \quad \text{and} \quad E_k \to F_k$$

for all $k \in \mathbb{Z}$. Assume furthermore that $E\mathbb{R}_{k\rho-1} = 0$ for all $k \in \mathbb{Z}$. Then f is a natural isomorphism.

Proof. It is well known that it is enough to show that $E_k \to F_k$ and $E\mathbb{R}_k \to E\mathbb{R}_k$ are isomorphisms for all $k \in \mathbb{Z}$. As the former is true by assumption, it is in particular enough to show that $f_{a+b\sigma}\colon E\mathbb{R}_{a+b\sigma} \to F\mathbb{R}_{a+b\sigma}$ is an isomorphism for all $a, b \in \mathbb{Z}$. This is true for $a = b$ again by assumption.

Smashing the cofiber sequence

$$(C_2)_+ \to S^0 \to S^\sigma$$

with $S^{a+b\sigma}$ gives the cofiber sequence

$$(C_2)_+ \wedge S^{a+b\sigma} \to S^{a+b\sigma} \to S^{a+(b+1)\sigma}.$$

We have a map between the associated long exact sequences:

$$\begin{tikzcd}
E_{a+b+1} \arrow[r] \arrow[d, "\cong"] & E\mathbb{R}_{a+(b+1)\sigma} \arrow[r] \arrow[d, "f_{a+(b+1)\sigma}"] & E\mathbb{R}_{a+b\sigma} \arrow[r] \arrow[d, "f_{a+b\sigma}"] & E_{a+b} \arrow[r] \arrow[d, "\cong"] & E\mathbb{R}_{(a-1)+(b+1)\sigma} \arrow[d, "f_{(a-1)+(b+1)\sigma}"] \\
F_{a+b+1} \arrow[r] & F\mathbb{R}_{a+(b+1)\sigma} \arrow[r] & F\mathbb{R}_{a+b\sigma} \arrow[r] & F_{a+b} \arrow[r] & F\mathbb{R}_{(a-1)+(b+1)\sigma}
\end{tikzcd}$$

We have the following implications from the weak 5-lemmas:

(M1) If $f_{a+(b+1)\sigma}$ is mono, then $f_{a+b\sigma}$ is mono.
(M2) If $f_{(a+1)+b\sigma}$ is epi and $f_{a+b\sigma}$ is mono, then $f_{a+(b+1)\sigma}$ is mono.
(E1) If $f_{a+b\sigma}$ is epi, then $f_{a+(b+1)\sigma}$ is epi.
(E2) If $f_{(a-1)+(b+1)\sigma}$ is mono and $f_{a+(b+1)\sigma}$ is epi, then $f_{a+b\sigma}$ is epi.

This implies the following two statements:

(1) $f_{a+b\sigma}$ is epi for $b \geq a$ by repeated application of (E1)
(2) $f_{a+b\sigma}$ is mono for $b \leq a+1$ by repeated application of (M1) as $f_{(a-1)+a\sigma} = f_{a\rho-1} = 0$ is mono..
(3) $f_{a+b\sigma}$ is epi if $b \leq a$ by repeated application of (E2). Thus, $f_{a+b\sigma}$ is epi for all $a, b \in \mathbb{Z}$.
(4) $f_{a+b\sigma}$ is mono for all $a, b \in \mathbb{Z}$ by repeated application of (M2). $\square$

2.2. **Real Landweber Exactness.** We have isomorphisms $M\mathbb{R}_{k\rho} = MU_{2k}$ by [HK01, Theorem 2.28]. This defines a graded ring morphism $MU_{2*} \to M\mathbb{R}_\bigstar$ along the morphism

$$2\mathbb{Z} \to RO(C_2), \qquad 2k \mapsto k\rho$$

of the monoids indexing the grading. In particular, $M\mathbb{R}_\bigstar$ becomes a graded MU_{2*}-module in a suitable sense.

Theorem 2.5. (a) *Let E_* be a graded Landweber exact MU_*-module, concentrated in even degrees. Then*

$$X \mapsto M\mathbb{R}_\bigstar(X) \otimes_{MU_{2*}} E_{2*}$$

is a $RO(C_2)$-graded homology theory.

(b) *Let $E\mathbb{R}$ be a strongly even C_2-spectrum whose underlying spectrum E is Landweber exact. Choosing a Real orientation induces a map*

$$M\mathbb{R}_\bigstar(X) \otimes_{MU_{2*}} E_{2*} \to E\mathbb{R}_\bigstar(X)$$

that is an isomorphism for every C_2-spectrum X. Under these conditions, we call $E\mathbb{R}$ Real Landweber exact.

The gradings can be parsed in the following way: For every $k \in \mathbb{Z}$, $M\mathbb{R}_{k+*\rho}(X)$ is a $2\mathbb{Z}$-graded MU_{2*}-module in the way described above so that the expression $M\mathbb{R}_{k+*\rho}(X) \otimes_{MU_{2*}} E_{2*}$ makes sense in the world of $2\mathbb{Z}$-graded MU_{2*}-modules. As a $RO(C_2)$-graded abelian group is equivalent datum to a $\mathbb{Z}$-graded $\mathbb{Z}\rho$-graded abelian group, this expresses what $M\mathbb{R}_\bigstar(X) \otimes_{MU_{2*}} E_{2*}$ means.

The proof of (a) is completely analogous to Landweber exactness in the motivic setting as in [NSØ09] and the crucial ingredient is the next proposition. Before we state it, we want to remark that the underlying ungraded rings of $M\mathbb{R}_\bigstar$ and $M\mathbb{R}_\bigstar M\mathbb{R}$ are commutative. Indeed, it follows from [HK01, Theorem 4.11] that everything outside of degrees of the form $k\rho + l$ with l even is 2-torsion. The claim follows now from the fact that in a Real-orientable theory, we have $xy = (-1)^{k_1l_2+k_2l_1+l_1l_2}yx$ for $|x| = k_1\rho + l_1$ and $|y| = k_2\rho + l_2$ by [HK01, Lemma 2.17].

Proposition 2.6. *The following square of stacks is cartesian:*

$$\begin{tikzcd}
\operatorname{Spec} M\mathbb{R}_\bigstar \arrow[r] \arrow[d] & \operatorname{Spec} MU_* \arrow[d] \\
{[\operatorname{Spec} M\mathbb{R}_\bigstar / \operatorname{Spec} M\mathbb{R}_\bigstar M\mathbb{R}]} \arrow[r] & {[\operatorname{Spec} MU_* / \operatorname{Spec} MU_*MU] \simeq \mathcal{M}_{FG}^s}
\end{tikzcd}$$

Here, Spec *always refers to the spectrum of the underlying ungraded rings. Likewise,* $[\operatorname{Spec} M\mathbb{R}_\bigstar / \operatorname{Spec} M\mathbb{R}_\bigstar M\mathbb{R}]$ *denotes the stack associated to the ungraded Hopf algebroid* $(M\mathbb{R}_\bigstar, M\mathbb{R}_\bigstar M\mathbb{R})$ *and $\mathcal{M}_{FG}^s$ is the moduli stack of formal groups with strict isomorphisms.*

Proof. By [NSØ09, Corollary 2.2], this follows from the isomorphism

$$M\mathbb{R}_\bigstar \otimes_{MU_*} MU_*MU \xrightarrow{\cong} M\mathbb{R}_\bigstar M\mathbb{R}.$$

It is enough to show this after localization at every prime p, where it follows from the fact

$$BP\mathbb{R}_\bigstar BP\mathbb{R} \cong BP\mathbb{R}_\bigstar[t_1, t_2, \dots] \cong BP\mathbb{R}_\bigstar \otimes_{BP_*} BP_*BP,$$

which is contained in [HK01, Theorem 4.11]. □

The rest of the proof of (a) is as in [NSØ09].

For the proof of (b), note first that $E\mathbb{R}$ carries a Real orientation by Lemma 2.3 and that this corresponds to a map $M\mathbb{R} \to E\mathbb{R}$ of C_2-"ring spectra. By Lemma 2.4 it is now enough to show that the induced maps

$$M\mathbb{R}_{*\rho} \otimes_{MU_{2*}} E_{2*} \to E\mathbb{R}_{*\rho}$$

and

$$MU_* \otimes_{MU_*} E_* \to E_*$$

are isomorphisms. The latter is clear and the former is true since both $\underline{\pi}_{*\rho} M\mathbb{R}$ and $\underline{\pi}_{*\rho} E\mathbb{R}$ are constant.

2.3. **Stacky reformulations.** We will in this subsection explicitly reformulate Real Landweber exactness in stack terms. This philosophy is well-known in the non-equivariant case.

Let E_* be an evenly graded commutative ring carrying a graded formal group (see [NSØ09, Section 6.3] for a discussion of graded formal groups). The grading defines a $\mathbb{G}_m$-action on $\operatorname{Spec} E_*$ and the graded formal group defines a map

$$f\colon\ \operatorname{Spec} E_*/\mathbb{G}_m \to \mathcal{M}_{FG},$$

where $(\operatorname{Spec} E_*)/\mathbb{G}_m$ denotes the stack quotient. Recall that the category of quasi-coherent sheaves on $\mathcal{M}_{FG}$ is equivalent to that of evenly graded (MU_*, MU_*MU)-comodules (see for example [Nau07, Remark 34]). The graded comodule MU_{*-2} corresponds to a line bundle ω on $\mathcal{M}_{FG}$. Likewise, the category of quasi-coherent sheaves on $(\operatorname{Spec} E_*)/\mathbb{G}_m$ is equivalent to that of graded modules over E_*. The line bundle $f^*\omega^{\otimes k}$ corresponds to the graded module E_{*-2k}.

Lemma 2.7. *Assume that the graded formal group over E_* has a coordinate, corresponding to a graded ring morphism $MU_* \to E_*$. Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{M}_{FG}$. Then*

$$\mathcal{F}(\operatorname{Spec} MU_*) \otimes_{MU_*} E_* \cong \Gamma(\mathcal{M}_{FG}; \mathcal{F} \otimes_{\mathcal{O}_{\mathcal{M}_{FG}}} f_*f^*\omega^{\otimes *}),$$

where we view $\mathcal{F}(\operatorname{Spec} MU_*)$ as an evenly graded MU_*-module.

Proof. We have a commutative diagram

$$\begin{tikzcd} & \operatorname{Spec} MU_*/\mathbb{G}_m \arrow[d, "q"] \\ \operatorname{Spec} E_*/\mathbb{G}_m \arrow[ur, "g"] \arrow[r, "f"] & \mathcal{M}_{FG} \end{tikzcd}$$

By definition, $q^*\mathcal{F}$ corresponds to the evenly graded MU_*-module $\mathcal{F}(\operatorname{Spec} MU_*)$. Thus, $f^*\mathcal{F} \cong g^*q^*\mathcal{F}$ correspond to the graded E_*-module $\mathcal{F}(\operatorname{Spec} MU_*) \otimes_{MU_*} E_*$. We see that the degree $2k$-part of $\mathcal{F}(\operatorname{Spec} MU_*) \otimes_{MU_*} E_*$ is

$$\Gamma((Spec\, E_*)/\mathbb{G}_m; f^*\mathcal{F} \otimes f^*\omega^{\otimes k}) \cong \Gamma(\mathcal{M}_{FG}; f_*(f^*\mathcal{F} \otimes f^*\omega^{\otimes k})).$$

By a version of the projection formula (see e.g. [Mei12, Lemma 2.3.13]), we have isomorphisms

$$f_*(f^*\mathcal{F} \otimes f^*\omega^{\otimes k}) \cong f_*f^*(\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{M}_{FG}}} \omega^{\otimes k}) \cong \mathcal{F} \otimes_{\mathcal{O}_{\mathcal{M}_{FG}}} f_*f^*\omega^{\otimes k}.$$

The result follows. □

Given a spectrum X, define quasi-coherent sheaves $\mathcal{F}_i^X$ for $i = 0, 1$ on $\mathcal{M}_{FG}$ corresponding to the graded (MU_*, MU_*MU)-comodules $MU_{2*-i}X$. Given a C_2-spectrum X, define quasi-coherent sheaves $\mathcal{F}_i^X$ for $i \in \mathbb{Z}$ on $\mathcal{M}_{FG}$ corresponding to the graded (MU_*, MU_*MU)-comodules $M\mathbb{R}_{*\rho-i}X$. Here, we use again the identification $MU_{2*} \cong M\mathbb{R}_{*\rho}$ and $MU_*MU \cong M\mathbb{R}_{*\rho}M\mathbb{R}$.

Applying the previous lemma to $\mathcal{F} = \mathcal{F}_*^X$, we obtain the following proposition:

Proposition 2.8. (a) *Let E be a Landweber exact spectrum. The associated graded formal group on E_* defines f as above. Then given a spectrum X, we have*

$$E_*(X) \cong \Gamma(\mathcal{M}_{FG}; \mathcal{F}_*^X \otimes_{\mathcal{O}_{\mathcal{M}_{FG}}} f_*f^*\omega^{\otimes *}),$$

where we view $\omega^{\otimes k}$ to have degree $2k$.

(b) *Let $E\mathbb{R}$ be a Real Landweber exact spectrum. The associated graded formal group on $E\mathbb{R}_{*\rho} \cong E_{2*}$ (for E the underlying spectrum) defines f as above. Then given a C_2-spectrum X, we have*

$$E_\bigstar(X) \cong \Gamma(\mathcal{M}_{FG}; \mathcal{F}_*^X \otimes_{\mathcal{O}_{\mathcal{M}_{FG}}} f_*f^*\omega^{\otimes *}),$$

where we view $\omega^{\otimes k}$ to have degree $k\rho$.

In particular, we see that the values of a Landweber exact theory do not depend on the MU_*-module structure of E_*, but only on the graded quasi-coherent sheaf $f_*f^*\omega^{\otimes *}$ on $\mathcal{M}_{FG}$ defined by E_*. This sheaf has an alternative description:

Lemma 2.9. *Let E be Landweber exact and f and $\mathcal{O}_*$ as above. Then we have an isomorphism $f_*f^*\omega^{\otimes k} \cong \mathcal{F}_0^E \otimes_{\mathcal{O}_{\mathcal{M}_{FG}}} \omega^{\otimes k}$.*

Proof. This was proven in the even-periodic context in the proof of [MM15, Proposition 2.4]. The general case is similar. □

All these statements also hold after p-completing everything in sight.

2.4. Forms of $BP\mathbb{R}\langle n\rangle$ and $E\mathbb{R}(n)$.

Fix a prime p.

Definition 2.10. Let E be a complex oriented p-local commutative and associative ring spectrum (up to homotopy). The p-typification of its formal group law defines a map $BP_* \to E_*$.

(a) We call E a *form of $BP\langle n\rangle$* if the map

$$\mathbb{Z}_{(p)}[v_1, \ldots, v_n] \subset BP_* \to E_*$$

is an isomorphism. This does not depend on the choice of v_i.

(b) We call E a *form of* $E(n)$ if there is a choice of indecomposables $v_1, \dots, v_n \in BP_*$ with $|v_i| = 2(p^i - 1)$ such that the image of v_n under the homomorphism

$$\mathbb{Z}_{(p)}[v_1, \dots, v_n] \subset BP_* \to E_*$$

is invertible and the induced morphism $\mathbb{Z}_{(p)}[v_1, \dots, v_n, v_n^{-1}] \to E_*$ is an isomorphism.

There is an equivariant analogue:

Definition 2.11. Let $E\mathbb{R}$ be a Real oriented p-local commutative and associative C_2-ring spectrum (up to homotopy). The p-typification of its formal group law defines a map $BP_{2*} \cong BP\mathbb{R}_{*\rho} \to E\mathbb{R}_{*\rho}$.

(a) We call $E\mathbb{R}$ a *form of* $BP\mathbb{R}\langle n\rangle$ if the map

$$\underline{\mathbb{Z}_{(p)}[v_1, \dots, v_n]} \subset \underline{\pi}_{*\rho} BP\mathbb{R} \to \underline{\pi}_{*\rho} E\mathbb{R}$$

is an isomorphism of constant Mackey functors. This does not depend on the choice of v_i.

(b) We call $E\mathbb{R}$ a *form of* $E\mathbb{R}(n)$ if there is a choice of indecomposables $v_1, \dots, v_n \in BP_*$ with $|v_i| = 2(p^i - 1)$ such that the image of v_n under the homomorphism

$$\mathbb{Z}_{(p)}[v_1, \dots, v_n] \subset BP\mathbb{R}_{*\rho} \to E\mathbb{R}_{*\rho}$$

is invertible and the induced morphism $\underline{\mathbb{Z}_{(p)}[v_1, \dots, v_n, v_n^{-1}]} \to \underline{\pi}_{*\rho} E\mathbb{R}$ is an isomorphism of constant Mackey functors.

Proposition 2.12. *If for two forms of $E\mathbb{R}(n)$ their underlying spectra are equivalent, then they are equivalent as Real spectra.*

Likewise if for two forms of $E\mathbb{R}(n)$ the p-completions of their underlying spectra are equivalent, then the p-completions of the forms of $E\mathbb{R}(n)$ are equivalent as Real spectra.

Proof. Let $E\mathbb{R}$ and $F\mathbb{R}$ be two forms of $E\mathbb{R}(n)$ with underlying spectra E and F.

Assume $E \simeq F$. Then they define isomorphic graded quasi-coherent sheaves $\mathcal{F}_*^E$ and $\mathcal{F}_*^F$ on $\mathcal{M}_{FG}$. The first statement follows by Theorem 2.5, Proposition 2.8 and Lemma 2.9.

If $\widehat{E}_p \simeq \widehat{F}_p$, the argument is similar. □

3. The C_2-equivariant slice filtration

The C_2-equivariant slice filtration was introduced by Dugger in his study of Atiyah's Real K-theory. This was generalized by Hopkins, Ravenel, and the first author to arbitrary finite groups in the solution to the Kervaire Invariant One problem [HHR09]. We will recall some of the basic properties here. A more detailed treatment can be found in [HHR09] or [Hil12].

Proposition 3.1 ([HHR09, Proposition 4.20 & Lemma 4.23])**.** *For any C_2-equivariant spectrum E, the odd slices are determined by the formula*

$$P_{2n-1}^{2n-1}(E) = \Sigma^{n\rho-1} H\underline{\pi}_{n\rho-1} E.$$

Corollary 3.2. *If R is an even C_2-spectrum, then all odd slices of R vanish.*

For the even slices, there is a similar formula involving homotopy Mackey functors of E.

Definition 3.3. If $\underline{M}$ is a C_2 Mackey functor, let $P^0\underline{M}$ denote the maximal quotient of $\underline{M}$ in which the restriction map $\underline{M}(C_2/C_2) \to \underline{M}(C_2/e)$ is injective.

There are several equivalent formulations. One of which is to notice that we can build a Mackey functor out of the kernel of the restriction by declaring that the value at C_2/C_2 is the kernel of the restriction map and that the value at $C_2/\{e\}$ is trivial. The functor $P^0\underline{M}$ is then the quotient of $\underline{M}$ by this subMackey functor.

The second formulation requires an endofunctor on Mackey functors.

Definition 3.4. If T is a finite C_2-set and $\underline{M}$ is a Mackey functor, then let $\underline{M}_T$ be the Mackey functor defined by
$$S \mapsto \underline{M}(T \times S).$$

The restriction map defines a map of Mackey functors
$$\underline{M} \to \underline{M}_{C_2},$$
and $P^0\underline{M}$ is simply the image of this map.

Proposition 3.5. *For any C_2-equivariant spectrum E, the even slices are determined by the formula*
$$P^{2n}_{2n}(E) = \Sigma^{n\rho} H P^0 \underline{\pi}_{n\rho}(E).$$
In particular, if $\underline{\pi}_{n\rho}(E)$ is constant, we have
$$P^{2n}_{2n}(E) = \Sigma^{n\rho} H \underline{\pi_{2n}(E)}.$$

We need several Mackey functors. We will define them via a Lewis diagram, stacking the value of the Mackey functor at C_2/C_2 over that of $C_2/\{e\}$ and then drawing in the restriction map, the transfer map, and the action of the non-trivial element of the Weyl group.

Definition 3.6. Let $\underline{G}$, $\underline{\mathbb{Z}}_-$ and $\underline{\mathbb{Z}}^*$ be the Mackey functors defined by

$$\begin{tikzcd} \mathbb{Z}/2 \arrow[d, bend right=60] \\ 0 \arrow[u, bend right=60] \arrow[loop below] \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[d, bend right=60] \\ \mathbb{Z} \arrow[u, bend right=60] \arrow[loop below, "-"] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z} \arrow[d, bend right=60, "2"'] \\ \mathbb{Z} \arrow[u, bend right=60, "1"'] \arrow[loop below, "1"] \end{tikzcd}$$
$$\underline{G} \qquad\qquad \underline{\mathbb{Z}}_- \qquad\qquad \underline{\mathbb{Z}}^*$$

Lemma 3.7. *If X is a strongly even C_2-spectrum without 2-torsion in π_*X, then we have*
$$\begin{aligned} \underline{\pi}_{k\rho+1}X &= \underline{G} \otimes_{\mathbb{Z}} \pi_{2k+2}X \\ \underline{\pi}_{k\rho}X &= \underline{\mathbb{Z}} \otimes_{\mathbb{Z}} \pi_{2k}X, \\ \underline{\pi}_{k\rho-1}X &= 0, \ \textit{and} \\ \underline{\pi}_{k\rho-2}X &= \underline{\mathbb{Z}}_- \otimes_{\mathbb{Z}} \pi_{2k-2}X. \end{aligned}$$

Proof. We have $P^{2k-1}_{2k-1}X \simeq *$ and

$$P^{2k}_{2k}X \simeq S^{k\rho} \wedge H\underline{\pi_{2k}X}.$$

Smashing the slice tower with $S^{-k\rho}$ gives a tower of the same form. It therefore suffices to prove this for $k = 0$. The positions in question are those for which the slice spectral sequence is especially simple: they are all non-bounding permanent cycles due to the vanishing of the homology and cohomology groups of regular representation spheres. □

4. $TMF_1(3)$ and friends

4.1. **Basics.** Denote by $\mathcal{M}_{ell}$ the moduli stack of elliptic curves and by $\overline{\mathcal{M}}_{ell}$ its compactification. Mapping an elliptic curve to its formal group defines a flat map $\overline{\mathcal{M}}_{ell} \to \mathcal{M}_{FG}$ to the moduli stack of formal groups. By [HL13] (extending earlier work by Goerss, Hopkins and Miller), the induced presheaf of even-periodic Landweber exact homology theories refines to a sheaf of E_∞-ring spectra $\mathcal{O}^{top}$ on the log-étale site of $\overline{\mathcal{M}}_{ell}$.

Denote by $\mathcal{M}_1(3)$ the moduli stack of elliptic curves with one chosen point of exact order 3 and by $\overline{\mathcal{M}}_1(3)$ its compactification, whose definition we will review below. We define

$$TMF_1(3) = \mathcal{O}^{top}(\mathcal{M}_1(3))$$
$$Tmf_1(3) = \mathcal{O}^{top}(\overline{\mathcal{M}}_1(3))$$
$$tmf_1(3) = \tau_{\geq 0} Tmf_1(3)$$

Sending the point of order 3 to its negative defines a C_2-action on $\mathcal{M}_1(3)$ that also extends to $\overline{\mathcal{M}}_1(3)$. This induces C_2-actions on $TMF_1(3)$ and $Tmf_1(3)$. Thus we will view these spectra as cofree C_2-spectra. Likewise, $tmf_1(3)$ refines to a C_2-spectrum that is the equivariant connective cover of $Tmf_1(3)$. Note that this spectrum is not cofree, but $tmf_1(3)^{C_2} \simeq \tau_{\geq 0} tmf_1(3)^{hC_2}$.

Denote by $\mathcal{M}_0(3)$ the moduli stack of elliptic curves with a chosen subgroup of order 3 and by $\overline{\mathcal{M}}_0(3)$ its compactification. We define

$$TMF_0(3) = \mathcal{O}^{top}(\mathcal{M}_0(3))$$
$$Tmf_0(3) = \mathcal{O}^{top}(\overline{\mathcal{M}}_0(3))$$

The forgetful maps

$$\mathcal{M}_1(3) \to \mathcal{M}_0(3) \quad \text{and} \quad \overline{\mathcal{M}}_1(3) \to \overline{\mathcal{M}}_0(3)$$

are C_2-Galois coverings. This implies that $Tmf_0(3) \simeq Tmf_1(3)^{hC_2}$ and $TMF_0(3) \simeq Tmf_1(3)^{hC_2}$.

Before we proceed, we have to study the algebraic geometry of these moduli stacks. To give a precise definition of $\overline{\mathcal{M}}_1(3)$, we want to review first the notion of normalization of Deligne–Mumford stacks.

Definition 4.1. Let $f\colon Y \to X$ be a quasi-compact and quasi-separated map of Deligne–Mumford stacks (over some base scheme S). Analogously to [Aut, Lemma

50.43.1], there exists a quasi-coherent $\mathcal{O}_X$-subalgebra $\mathcal{O}' \subseteq f_*\mathcal{O}_Y$ such that for every map $U \to X$ from an affine scheme, $\mathcal{O}'(U)$ is the integral closure of $\mathcal{O}_X(U)$ in $(f_*\mathcal{O}_X)(U)$. The *normalization* of X in Y is the morphism

$$X' = \underline{\operatorname{Spec}}_X(\mathcal{O}') \to X.$$

Following Deligne–Rapoport [DR73, Definition IV.3.3], we define $\overline{\mathcal{M}}_1(n)$ to be the normalization of $\overline{\mathcal{M}}_{ell}$ in $\mathcal{M}_1(n)$ and $\overline{\mathcal{M}}_0(n)$ to be the normalization of $\overline{\mathcal{M}}_{ell}$ in $\mathcal{M}_0(n)$. Note that there also exist moduli interpretations, for example as in [Con07].

The following lemma essentially says that any reasonable compactification of $\mathcal{M}_1(n)$ must be $\overline{\mathcal{M}}_1(n)$.

Lemma 4.2. *Let $f\colon Y \to X$ be a quasi-compact and quasi-separated map of Deligne–Mumford stacks (over some base scheme S). Let $\overline{f}\colon \overline{Y} \to X$ be a finite morphism from a normal Deligne–Mumford stack (over S) such that $Y \subset \overline{Y}$ is a dense open substack and $\overline{f}|_Y = f$. Then $\overline{Y} \to X$ is the normalization of X in Y.*

Proof. It is easy to see that $\overline{f}\colon \overline{Y} \to X$ is the normalization of X in Y if and only if $\overline{Y} \times_X U \to U$ is the normalization of U in $Y \times_X U$ for every morphism $U \to X$ for an affine scheme U. Thus, we can assume that $X = \operatorname{Spec} A$ and $\overline{Y} = \operatorname{Spec} B$ are affine schemes. Set $C = f_*\mathcal{O}_Y(X)$. Then we have to show that B is the integral closure of A in C. As $A \to B$ is finite, all elements of B are integral over A so that we just have to check that B is integrally closed in C. But B is even integrally closed in its fraction field as B is normal and C is inside the fraction field of B as $Y \subset \overline{Y}$ is dense. □

The following result is well-known (see e.g. [LN14]), but has to the knowledge of the authors not appeared with full proof in print.

Proposition 4.3. *We have equivalences*

$$\mathcal{M}_1(3) \simeq \operatorname{Spec} \mathbb{Z}\left[\tfrac{1}{3}\right][a_1, a_3][\Delta^{-1}]/\mathbb{G}_m$$
$$\overline{\mathcal{M}}_1(3) \simeq \operatorname{Spec}(\mathbb{Z}\left[\tfrac{1}{3}\right][a_1, a_3] \setminus \{0\})/\mathbb{G}_m = \mathcal{P}_{\mathbb{Z}\left[\frac{1}{3}\right]}(1,3)$$

Here,

- *the $\mathbb{G}_m$-action on $\operatorname{Spec} \mathbb{Z}\left[\frac{1}{3}\right][a_1, a_3]$ is induced by the grading with $|a_1| = 1$ and $|a_3| = 3$,*
- *all the quotients are meant to be stack quotients,*
- $\Delta = a_3^3(a_1^3 - 27a_3)$,
- $\{0\}$ *denotes the common vanishing locus of a_1 and a_3*
- $\mathcal{P}_{\mathbb{Z}\left[\frac{1}{3}\right]}(1,3)$ *is the notation for the weighted (stacky) projective line with weights* 1 *and* 3.

Proof. The first equivalence follows from [MR09, Proposition 3.2].

Set $A = \mathbb{Z}\left[\frac{1}{3}\right][a_1, a_3]$. The equality $\mathcal{P}_{\mathbb{Z}\left[\frac{1}{3}\right]}(1,3) = (\operatorname{Spec} A \setminus \{0\})/\mathbb{G}_m$ is just the definition of the weighted projective line. This is a proper and smooth Deligne–Mumford stack over $\operatorname{Spec} \mathbb{Z}\left[\frac{1}{3}\right]$ by [Mei13, Proposition 2.1, Remark 2.2]. Note furthermore that $\mathcal{M}_1(3) \subset \mathcal{P}_{\mathbb{Z}\left[\frac{1}{3}\right]}(1,3)$ is a dense open substack.

To check that $\mathcal{P}_{\mathbb{Z}\left[\frac{1}{3}\right]}(1,3)$ is the normalization of $\overline{\mathcal{M}}_{ell}$ in $\mathcal{M}_1(n)$, we need to construct a finite morphism

$$\mathcal{P}_{\mathbb{Z}\left[\frac{1}{3}\right]}(1,3) \to \overline{\mathcal{M}}\left[\tfrac{1}{3}\right]$$

that extends the morphism

$$\mathcal{M}_1(3) \to \mathcal{M}_{ell}\left[\tfrac{1}{3}\right] \subset \overline{\mathcal{M}}_{ell}\left[\tfrac{1}{3}\right].$$

The equation $y^2+a_1xy+a_3y = x^3$ defines a cubic curve over $\operatorname{Spec} A/\mathbb{G}_m$. We want to show that this equation actually defines a generalized elliptic curve E over $\mathcal{P}_{\mathbb{Z}\left[\frac{1}{3}\right]}(1,3)$. For this, we have to check that for no map $f\colon \operatorname{Spec} k \to \mathcal{P}_{\mathbb{Z}\left[\frac{1}{3}\right]}(1,3)$ for k a field (of characteristic $\neq 3$), the pullback f^*E has a cusp. Equivalently, we have to show that for any values $a_1, a_3 \in k$ for which $c_4 = a_1^4 - 24a_1a_3$ and $\Delta = a_3^3(a_1^3 - 27a_3)$ vanish, also a_1 and a_3 vanish. First observe that if $c_4 = \Delta = 0$, then $a_1 = 0$ implies $a_3 = 0$ and vice versa. If $\Delta = 0$, either $a_3 = 0$ or $a_1^3 = 27a_3$. In the second case, $27a_1a_3 = a_1^4 = 24a_1a_3$ and thus $a_1 = 0$ or $a_3 = 0$.

Thus, we obtain a map $p\colon \operatorname{Spec} A/\mathbb{G}_m \to \mathcal{M}_{cub}\left[\frac{1}{3}\right]$ to the moduli stack of cubic curves that restricts to a map $\mathcal{P}_{\mathbb{Z}\left[\frac{1}{3}\right]}(1,3) \to \overline{\mathcal{M}}_{ell}\left[\frac{1}{3}\right]$, which in turn extends the map $\mathcal{M}_1(3) \to \mathcal{M}_{ell}\left[\frac{1}{3}\right] \subset \overline{\mathcal{M}}_{ell}\left[\frac{1}{3}\right]$.

As computed in the beginning of Section 7 of [Bau08], the map p is surjective and we have $\operatorname{Spec} A/\mathbb{G}_m \times_{\mathcal{M}_{cub}} \operatorname{Spec} A/\mathbb{G}_m \simeq (\operatorname{Spec} A[s,t]/(f,g))/\mathbb{G}_m$, where f and g are polynomials in s and t such that $A[s,t]/(f,g)$ is a finite flat A-module. As finiteness can be checked after fpqc-base change, the map p is finite and hence also its restriction $\mathcal{P}_{\mathbb{Z}\left[\frac{1}{3}\right]}(1,3) \to \overline{\mathcal{M}}_{ell}\left[\frac{1}{3}\right]$, which is the base change $p \times_{\mathcal{M}_{cub}\left[\frac{1}{3}\right]} \overline{\mathcal{M}}_{ell}\left[\frac{1}{3}\right]$. Thus, the result follows by Lemma 4.2. □

By checking the gradings, we see that $p^*\omega \cong \mathcal{O}(1)$ for $p\colon \mathcal{P}_{\mathbb{Z}\left[\frac{1}{3}\right]}(1,3) \to \overline{\mathcal{M}}_{ell}\left[\frac{1}{3}\right]$ the restriction of the morphism constructed in the proof above. (Here, ω denotes as usual the line bundle $\pi_2\mathcal{O}^{top}$ on $\overline{\mathcal{M}}_{ell}$.) Thus, we have

$$H^s(\overline{\mathcal{M}}_1(3);\omega^{\otimes *}) \cong \begin{cases} \mathbb{Z}\left[\frac{1}{3}\right][a_1,a_3] & \text{for } s=0 \\ \mathbb{Z}\left[\frac{1}{3}\right][a_1,a_3]/(a_1^\infty, a_3^\infty) & \text{for } s=1 \\ 0 & \text{for } s \geq 2 \end{cases}$$

as shown, for example, in [Mei13, Proposition 2.5]. Here, $\mathbb{Z}\left[\frac{1}{3}\right][a_1,a_3]/(a_1^\infty, a_3^\infty)$ denotes the $\mathbb{Z}\left[\frac{1}{3}\right][a_1,a_3]$-torsion module with $\mathbb{Z}\left[\frac{1}{3}\right]$-basis given by the monomials $\frac{1}{a_1^i a_3^j}$, $i,j \geq 1$. Thus, the descent spectral sequence for $Tmf_1(3)$ collapses. In particular, we see that $\pi_* tmf_1(3) = \mathbb{Z}\left[\frac{1}{3}\right][a_1,a_3]$.

4.2. $RO(C_2)$**-graded homotopy of** $tmf_1(3)$**.** Our goal in this subsection is to understand the C_2-equivariant $RO(C_2)$-graded homotopy groups of $tmf_1(3)$. We will compute this via an $RO(C_2)$-graded homotopy fixed point spectral sequence. This can be described for a general group G, and we start with a brief description of this.

4.2.1. *The $RO(G)$-graded homotopy fixed point spectral sequence.* If V is a virtual representation of G, then by tracing through the adjunctions, we see that

$$\pi_V^G F(EG_+, X) \cong [S^V \wedge EG_+, X]^G \cong \pi_0^G F(EG_+, S^{-V} \wedge X).$$

The skeletal filtration of EG then gives us a spectral sequence computing these. Moreover, if E is a homotopy ring spectrum, we have a natural multiplication map

$$(S^{-V} \wedge E) \wedge (S^{-W} \wedge E) \to S^{-V-W} \wedge E,$$

and the standard argument then shows that this spectral sequence is a spectral sequence of $RO(G)$-graded algebras. We summarize in the following proposition.

Proposition 4.4. *If E is a G-spectrum with a multiplication up to homotopy, then there is an $RO(G)$-graded spectral sequence of algebras*

$$E_2^{s,V} = H^s\big(G; \pi_0(S^{-V} \wedge E)\big) \Rightarrow \pi^G_{V-s} F(EG_+, E).$$

When $G = C_2$ there are two important simplifications. The first allows us to identify the E_2 term more transparently:

Lemma 4.5. *Let E be a C_2-spectrum. Then*

$$\pi_*(E \wedge S^{\sigma-1}) \cong \pi_* E \otimes \mathrm{sgn}$$

as C_2-modules.

Proof. We have two commuting C_2-actions on $E \wedge S^{\sigma-1}$ corresponding to the two factors and we denote the action of the non-trivial element by s and t, respectively. As the diagonal action we are using in the lemma equals $t \circ s$, it is enough to show that t acts via sign on $\pi_*(E \wedge S^{\sigma-1}) = E_*(S^{\sigma-1})$. This is true as t acts as -1 on $S^{\sigma-1}$. $\square$

Corollary 4.6. *If E is a C_2-spectrum, then the $RO(C_2)$-graded homotopy fixed point spectral sequence has the form*

$$H^s(C_2; \pi_t(E) \otimes \mathrm{sgn}^r) \Rightarrow \pi^{C_2}_{t-s+(\sigma-1)r} F\big(EC_{2+}, E\big).$$

The second C_2 simplification is a recasting of the $RO(C_2)$-graded homotopy fixed points spectral sequence in a way that allows us to read off permanent cycles. Recall that there is a C_2-equivariant map

$$a_\sigma \colon S^0 \to S^\sigma$$

which is essential but for which the restriction is null. The following is undoubtedly well-known to experts.

Lemma 4.7. *The $RO(C_2)$-graded homotopy fixed points spectral sequence for a C_2-spectrum X coincides with the a_σ-Bockstein spectral sequence for X.*

Proof. The map a_σ^n fits in a cofiber sequence

$$S(n\sigma)_+ \to S^0 \xrightarrow{a_\sigma^n} S^{n\sigma},$$

where $S(n\sigma)$ is the unit sphere in the representation $n\sigma$. Applying $F(-, X)$, we deduce a cofiber sequence of spectra

$$\Sigma^{-n\sigma} X \xrightarrow{a_\sigma^n} X \to F\big(S(n\sigma)_+, X\big).$$

The space $S(n\sigma)_+$ is also the $(n-1)$-skeleton of the standard model for EC_{2+} as the infinite sign sphere, and the map on function spectra induced by the inclusion of the $(n-1)$-skeleton into the n-skeleton coincides with the obvious map of cofibers:

$$\begin{tikzcd}
\Sigma^{-(n+1)\sigma}X \arrow[r, "a_\sigma^{n+1}"] \arrow[d, "a_\sigma"'] & X \arrow[r] \arrow[d, "1"'] & F\big(S((n+1)\sigma)_+, X\big) \arrow[d] \\
\Sigma^{-n\sigma}X \arrow[r, "a_\sigma^n"'] & X \arrow[r] & F\big(S(n\sigma)_+, X\big).
\end{tikzcd}$$

Thus the filtration by powers of a_σ and the filtration by the skeleton of EC_{2+} coincide. $\square$

4.2.2. *The $RO(C_2)$-graded homotopy groups of $tmf_1(3)$.* Recall from Section 4.1 that non-equivariantly $\pi_*tmf_1(3) \cong \mathbb{Z}\left[\frac{1}{3}\right][a_1, a_3]$ and $\pi_*TMF_1(3) \cong \mathbb{Z}\left[\frac{1}{3}\right][a_1, a_3, \Delta^{-1}]$ with $|a_1| = 2$ and $|a_3| = 6$.

As a first step, we will determine the C_2-action on these groups. By [MR09, Proposition 3.4], the group C_2 acts by -1 on a_1 and a_3 in $\pi_*TMF_1(3)$ and hence also in $\pi_*tmf_1(3)$ (as $tmf_1(3) \to TMF_1(3)$ is a C_2-map that is an injection on underlying homotopy groups).

By Corollary 4.6, the $RO(C_2)$-graded homotopy fixed point spectral sequence E_2 term for $tmf_1(3)^{hC_2}$ can be written as follows:

$$E_2^{*,*} = \mathbb{Z}\left[\tfrac{1}{3}\right][a_\sigma, u_{2\sigma}^{\pm 1}, \bar{a}_1, \bar{a}_3]/(2a_\sigma)$$

with degrees $|a_\sigma| = (-\sigma, 1)$, $|u_{2\sigma}| = (2-2\sigma, 0)$, $|\bar{a}_1| = (1+\sigma, 0)$ and $|\bar{a}_3| = (3+3\sigma, 0)$.

We start by identifying the permanent cycles corresponding to η and ν in the Hurewicz image in $\pi_*tmf_1(3)^{hC_2}$. By [HL13, Theorem 6.2], there is a C_2-equivariant map $Tmf_1(3) \to KU$ of E_∞-ring spectra into K-theory, inducing a map between the homotopy fixed point spectral sequences for $Tmf_1(3)^{hC_2}$ and $KO \simeq KU^{hC_2}$. In the latter, η is of filtration 1, so it has to be of filtration ≤ 1 in the former. As the homotopy fixed point spectral sequences of $Tmf_1(3)^{hC_2}$ and $tmf_1(3)^{hC_2}$ agree in nonnegative degrees, η is also filtration 1 in the homotopy fixed point spectral sequence for $tmf_1(3)^{hC_2}$ and is detected by $a_\sigma\bar{a}_1$.

To identify ν, we observe the following lemma:

Lemma 4.8. *The composite $Tmf\left[\frac{1}{3}\right] \xrightarrow{\text{res}} Tmf_0(3) \xrightarrow{\text{tr}} Tmf\left[\frac{1}{3}\right]$ is multiplication by 4.*

Proof. This is true on the level of E_2-terms of homotopy fixed point spectral sequences, expressing $Tmf_0(3)$ and $Tmf\left[\frac{1}{3}\right]$ as homotopy fixed points of $Tmf(3)$. The $Tmf\left[\frac{1}{3}\right]$-linear self-maps of $Tmf\left[\frac{1}{3}\right]$ are in one-to-one correspondence to elements in $\pi_0Tmf\left[\frac{1}{3}\right]$. These are all of filtration 0 in the descent spectral sequence by [Kon12] and thus detected by their action on $\pi_0Tmf\left[\frac{1}{3}\right] = H^0(GL_2(\mathbb{Z}/3); \pi_0Tmf(3))$. $\square$

As 4ν in $\pi_3Tmf\left[\frac{1}{3}\right]$ is non-zero and of filtration 3, we know that $\nu = \text{res}(\nu) \in \pi_3Tmf_0(3)$ is of filtration ≤ 3 and non-zero. For degree reasons, it has to be detected by the image of $a_\sigma^3\bar{a}_3$. As the homotopy fixed point spectral sequences for $tmf_1(3)^{hC_2}$ and $Tmf_1(3)^{hC_2}$ agree in this range, the same is true for $tmf_1(3)^{hC_2}$.

Corollary 4.9. *The classes $\bar{a}_1$ and $\bar{a}_3$ are permanent cycles.*

Proof. Since the homotopy fixed point spectral sequence and a_σ-Bockstein spectral sequences coincide, we learn that if an a_σ-multiple of a class is a permanent cycle, then the class is a permanent cycle. This in particular implies to $\eta = \bar{a}_1 a_\sigma$ and $\nu = \bar{a}_3 a_\sigma^3$. □

Corollary 4.10. *The only generator of the E_2 term for the $RO(C_2)$-graded homotopy fixed point spectral sequences for $tmf_1(3)$ and for $Tmf_1(3)$ which is not a permanent cycle is $u_{2\sigma}$.*

Similarly, the transfer of any element in the underlying homotopy is a permanent cycle. In particular, we conclude immediately that the classes

$$v_0(k) := 2u_{2\sigma}^k$$

for $k \in \mathbb{Z}$ are all permanent cycles which generate copies of $\mathbb{Z}$. These satisfy an obvious multiplicative relation

$$v_0(k)v_0(j) = 2v_0(j+k).$$

Next, we will determine the differentials. Note first that for degree reasons all d_{2k} are 0 for $k \geq 1$. The other differentials follow easily from work of Mahowald-Rezk [MR09].

Proposition 4.11. *We have differentials*

$$d_3(u_{2\sigma}) = a_\sigma^3 \bar{a}_1$$
$$d_7(u_{2\sigma}^2) = a_\sigma^7 \bar{a}_3.$$

The torsion produced by the first differential yields new d_7-cycles:

$$\bar{a}_1(k) := \bar{a}_1 u_{2\sigma}^{2k},$$

for $k \in \mathbb{Z}$. These also participates in the expected multiplicative relations:

$$\bar{a}_1(k)\bar{a}_1(j) = \bar{a}_1 \cdot \bar{a}_1(j+k),$$
$$\bar{a}_1(j)v_0(k) = \bar{a}_1 \cdot v_0(k+2j).$$

Remark 4.12. The classes $v_0(k)$ and $\bar{a}_1(j)$ form families exactly like the families $v_0(k)$ and $v_1(j)$ described by Hu-Kriz is the computation of the homotopy of $BP\mathbb{R}$.

There is no room for further differentials in E_8, which is the subalgebra of

$$\mathbb{Z}\left[\tfrac{1}{3}\right][a_\sigma, u_{2\sigma}, \bar{a}_1, \bar{a}_3]/(2a_\sigma, \bar{a}_1 a_\sigma^3, \bar{a}_3 a_\sigma^7)$$

generated by a_σ, $\bar{a}_1$, $\bar{a}_3$, $v_0(1)$, $v_0(2)$, $v_0(3)$, $\bar{a}_1(1)$ and $u_{2\sigma}^{\pm 4}$. So this is also the E_∞-term.

Observe that there can be no additive extension issues for degree reasons. Also for degree reasons, we have no hidden multiplicative extensions. Thus, we get the following explicit presentation of the homotopy groups:

Theorem 4.13. *We have*

$$\begin{aligned}\pi_\star^{C_2} F\big(EC_{2+}, tmf_1(3)\big) \cong \mathbb{Z}\left[\tfrac{1}{3}\right][a_\sigma, u_{2\sigma}^{\pm 4}, \bar{a}_1, \bar{a}_3, v_0(k), \bar{a}_1(1)]/\Big(&a_\sigma v_0(k), a_\sigma^3\big(\bar{a}_1, \bar{a}_1(1)\big), a_\sigma^7 \bar{a}_3,\\ v_0(k+4) = v_0(k)u_{2\sigma}^4, \bar{a}_1(k+2) = \bar{a}_1(k)u_{2\sigma}^4, &v_0(k)v_0(j) = 2v_0(j+k),\\ &\bar{a}_1(k)\bar{a}_1(j) = \bar{a}_1\bar{a}_1(j+k), \bar{a}_1(j)v_0(k) = \bar{a}_1 v_0(k+2j)\Big).\end{aligned}$$

Remark 4.14. We have $\pi^{C_2}_{a+b\sigma}F\big((EC_{2+}, tmf_1(3)\big) \cong \pi^{C_2}_{a+b\sigma}tmf_1(3)$ for all $a, b \geq 0$ and $\pi^{C_2}_{a+b\sigma}tmf_1(3) = 0$ for $a < 0$ and $a + b < 0$; this follows from the cofiber sequence

$$S^{a+(b-1)\sigma} \to S^{a+b\sigma} \to S^{a+b} \wedge (C_2)_+.$$

Is it possible, but more complicated, to describe also the other homotopy groups.

Corollary 4.15. *The spectrum $tmf_1(3)$ is strongly even as a C_2-spectrum. In particular, it is Real orientable and thus a form of $BP\mathbb{R}\langle 2\rangle$.*

Proof. It is immediate from 4.13 and the remark thereafter that $tmf_1(3)$ is even as a C_2-spectrum and also that the Mackey functor $\underline{\pi}_{k\rho}tmf_1(3)$ is constant for all $k \in \mathbb{Z}$. The map $BP_* \to tmf_1(3)_*$ induced by the p-typification of the formal group law associated to the Weierstrass equation $y^2 + a_1xy + a_3y = x^3$ sends the Hazewinkel generators v_1 and v_2 exactly to a_1 and a_3. This implies that $tmf_1(3)$ is a form of $BP\mathbb{R}\langle 2\rangle$. $\square$

This implies actually the following strengthening of a result by Romie Banerjee in [Ban14]:

Corollary 4.16. *After 2-completion, there is an (additive) equivalence $tmf_1(3)[\bar{a}_3^{-1}] \simeq E\mathbb{R}(2)$. For concreteness, $E\mathbb{R}(2)$ means here the one defined by the Hazewinkel generators.*

Proof. By the main result of [AL15], there is after 2-completion an equivalence $tmf_1(3) \simeq BP\langle 2\rangle$ of underlying spectra, where $BP\langle 2\rangle$ is, say, the form of $BP\langle 2\rangle$ defined by killing the Hazewinkel generators $v_3, v_4, \dots$ from BP. This induces an equivalence after 2-completion of the underlying spectra $tmf_1(3)[a_3^{-1}] \simeq E(2)$.

By Proposition 2.12, the result follows. $\square$

Conjecturally, this holds also before 2-completion.

4.3. **The relationship between $tmf_1(3)$, $Tmf_1(3)$ and $TMF_1(3)$.** The following is proven in [MM15, Theorem 7.12].

Proposition 4.17. *The map $Tmf_0(3) \to Tmf_1(3)$ is a faithful C_2-Galois extension in the sense of Rognes.*

By [Rog08, Proposition 6.3.3], this implies that the norm map $Tmf_1(3)_{hC_2} \to Tmf_1(3)^{hC_2}$ is an equivalence. Equivalently, $\Phi^{C_2}Tmf_1(3) \simeq Tmf_1(3)^{tC_2} \simeq *$ as $Tmf_1(3)$ is cofree. By [HHR09, Corollary 10.6], it follows that every $Tmf_1(3)$-module spectrum is cofree.

Lemma 4.18. *Given a C_2-spectrum X, its geometric fixed points $\Phi^{C_2}X$ can be computed as the C_2-fixed points of*

$$X[a_\sigma^{-1}] = \operatorname{hocolim}\left(X \cong S^0 \wedge X \xrightarrow{a_\sigma \wedge \mathrm{id}} S^\sigma \wedge X \xrightarrow{a_\sigma \wedge \mathrm{id}} S^{2\sigma} \wedge X \xrightarrow{a_\sigma \wedge \mathrm{id}} \cdots\right).$$

Proof. We have $\Phi^{C_2}X \simeq (\widetilde{E\mathcal{P}} \wedge X)^{C_2}$, where $\widetilde{E\mathcal{P}}$ is a contractible pointed space such that $\widetilde{E\mathcal{P}}^{C_2} \simeq S^0$. A model for $\widetilde{E\mathcal{P}}$ is

$$S^{\infty\sigma} = \operatorname{hocolim}\left(S^0 \xrightarrow{a_\sigma} S^\sigma \xrightarrow{a_\sigma} S^{2\sigma} \xrightarrow{a_\sigma} \cdots\right).$$ $\square$

Lemma 4.19. *Let $\overline{f}$ be a homogeneous polynomial in $\bar{a}_1$ and $\bar{a}_3$ of positive degree. Then*

$$tmf_1(3)[\overline{f}^{-1}] \to Tmf_1(3)[\overline{f}^{-1}]$$

is an equivalence.

Proof. For some $k > 0$, we have $a_\sigma^7\overline{f}^k = 0$ in $\pi_\bigstar F((EC_2)_+, tmf_1(3))$ and $|a_\sigma^7\overline{f}^k|$ positive. Thus also $a_\sigma^7\overline{f}^k = 0$ in $\pi_\bigstar^{C_2} tmf_1(3)$ and $\Phi^{C_2}(tmf_1(3)[\overline{f}^{-1}]) = 0$. By [HHR09, Corollary 10.6], $tmf_1(3)[\overline{f}^{-1}]$ is then cofree. Thus, we have only to show that $tmf_1(3)[\overline{f}^{-1}] \to Tmf_1(3)[\overline{f}^{-1}]$ is an equivalence of underlying spectra. As every element of negative degree in $\pi_*^u Tmf_1(3)$ is killed by a_1 and a_3, the result follows. $\square$

Lemma 4.20. *Let $\overline{f}$ be a homogeneous polynomial in $\bar{a}_1$ and $\bar{a}_3$ of positive degree. Denote by $D(f)$ the non-vanishing locus of the underlying element $f \in H^0(\overline{\mathcal{M}}_1(3); \omega^*)$. Then there is an equivalence*

$$Tmf_1(3)[\overline{f}^{-1}] \to \mathcal{O}^{top}(D(f))$$

of C_2-spectra.

Proof. Note first that $D(f)$ is C_2-invariant as f^2 is an invariant section. There is a C_2-map of ring spectra $Tmf_1(3) = \mathcal{O}^{top}(\overline{\mathcal{M}}_1(3)) \to \mathcal{O}^{top}(D(f))$. The image of $\overline{f}$ is invertible as it is detected by $fu_{2\sigma}^k$ for some k and f and $u_{2\sigma}$ are invertible. Thus, we get an induced map

$$Tmf_1(3)[\overline{f}^{-1}] \to \mathcal{O}^{top}(D(f))$$

of C_2-spectra.

By [MM15, Theorem 7.2] and the proof of [MM15, Theorem 7.12], the global sections functor

$$\Gamma\colon\ \mathrm{QCoh}(\overline{\mathcal{M}}_1(3), \mathcal{O}^{top}) \to Tmf_1(3)\text{-mod}$$

is an equivalence.[1] Thus, we can apply [MM15, Lemma 3.20] to see that

$$Tmf_1(3)[\overline{f}^{-1}] \to \mathcal{O}^{top}(D(f))$$

is an equivalence of underlying spectra. As both spectra are cofree, the result follows. $\square$

This applies in particular to $\overline{f} = \overline{\Delta}$. Thus,

$$tmf_1(3)[\Delta^{-1}] \simeq Tmf_1(3)[\overline{\Delta}^{-1}] \simeq TMF_1(3)$$

as C_2-spectra (with $\overline{\Delta} = \overline{a_3}^3(\overline{a_1}^3 - 27\overline{a_3})$). In particular, $TMF_1(3)$ is strongly even. Thus, Theorem 2.5 implies:

Proposition 4.21. *The C_2-spectrum $TMF_1(3)$ is Real Landweber exact in the sense that there is a natural isomorphism*

$$M\mathbb{R}_\bigstar(X) \otimes_{MU_{2*}} TMF_1(3)_{2*} \to TMF_1(3)_\bigstar(X)$$

for all C_2-spectra X.

---

[1] We only really need that Γ commutes with homotopy colimits. As observed in the proof of [MM15, Proposition 3.8], this is automatic when the stack has finite cohomological dimension as $\overline{\mathcal{M}}_1(3)$ does. This circumvents the use of most of the heavy machinery in [MM15].

The following fiber square will be useful later.

Proposition 4.22. *We have a fiber square*

$$\begin{tikzcd} Tmf_1(3) \arrow[r] \arrow[d] & tmf_1(3)[\bar{a}_1^{-1}] \arrow[d] \\ tmf_1(3)[\bar{a}_3^{-1}] \arrow[r] & tmf_1(3)[(\bar{a}_1\bar{a}_3)^{-1}] \end{tikzcd}$$

Proof. The square

$$\begin{tikzcd} \overline{\mathcal{M}}_1(3) & D(a_1) \arrow[l] \\ D(a_3) \arrow[u] & D(a_1a_3) \arrow[l] \arrow[u] \end{tikzcd}$$

induces a fiber square

(1)
$$\begin{tikzcd} \mathcal{O}^{top}(\overline{\mathcal{M}}_1(3)) \arrow[r] \arrow[d] & \mathcal{O}^{top}(D(a_1)) \arrow[d] \\ \mathcal{O}^{top}(D(a_3)) \arrow[r] & \mathcal{O}^{top}(D(a_1a_3)) \end{tikzcd}$$

as

$$\overline{\mathcal{M}}_1(3) \simeq \mathcal{P}_{\mathbb{Z}\left[\frac{1}{3}\right]}(1,3) = D(a_1) \cup D(a_3)$$

and $\mathcal{O}^{top}$ is a sheaf (see [MM15, Appendix B]).

By the last two lemmas, this is equivalent to

$$\begin{tikzcd} Tmf_1(3) \arrow[r] \arrow[d] & tmf_1(3)[\bar{a}_1^{-1}] \arrow[d] \\ tmf_1(3)[\bar{a}_3^{-1}] \arrow[r] & tmf_1(3)[(\bar{a}_1\bar{a}_3)^{-1}] \end{tikzcd}$$

as a square of C_2-spectra. ☐

5. Slices and Anderson Dual

5.1. **Slices.** We can apply the computations of the regular representation homotopy groups of $tmf_1(3)$ and its localizations to determine their slices.

Since all of the odd slices vanish and the even slices are regular representation suspensions of $H\underline{\mathbb{Z}}\left[\frac{1}{3}\right]$, the homotopy groups "near multiples of regular representations" are easy to compute since the slice spectral sequence is especially simple here.

As a bit of notation, if R is a graded localization of a polynomial ring, let $M_n(R)$ denote the monic monomials of degree n in R. Then Lemma 3.7 implies:

Corollary 5.1. *Let M be one of $tmf_1(3)$, $tmf_1(3)[\bar{a}_1^{-1}]$, $tmf_1(3)[\bar{a}_3^{-1}]$, or $tmf_1(3)[(\bar{a}_1\bar{a}_3)^{-1}]$.*

For all k, we have

$$\begin{aligned}\underline{\pi}_{k\rho+1}M &= \underline{G}\cdot\{M_{2k+2}(\pi_*^u M)\},\\ \underline{\pi}_{k\rho}M &= \underline{\mathbb{Z}}\left[\tfrac{1}{3}\right]\{M_{2k}(\pi_*^u M)\},\\ \underline{\pi}_{k\rho-1}M &= 0, \text{ and}\\ \underline{\pi}_{k\rho-2}M &= \underline{\mathbb{Z}}\left[\tfrac{1}{3}\right]_-\{M_{2k-2}(\pi_*^u M)\}.\end{aligned}$$

Similarly, naturality of the slice spectral sequence implies that we understand the effect of the localization maps on homotopy groups in dimensions $k\rho-2,\dots,k\rho+1$.

Corollary 5.2. *For $k\in\mathbb{Z}$ and for $j=-2,\dots,1$, the localization maps*

$$\underline{\pi}_{k\rho+j}tmf_1(3)[\bar{a}_i^{-1}]\to\underline{\pi}_{k\rho+j}tmf_1(3)[(\bar{a}_1\bar{a}_3)^{-1}]$$

are induced by the obvious inclusions of graded pieces of these graded rings.

Remark 5.3. We could also have read off these results from the homotopy fixed point spectral sequence, but the slice spectral sequence approach is both more conceptual and is easier for Mackey functor computations.

We want now to compute the slices of $Tmf_1(3)$. To that purpose, we denote by $M[\bar{a}_1,\bar{a}_3]$ the monic monomials in $\mathbb{Z}\left[\frac{1}{3}\right][\bar{a}_1,\bar{a}_3]$.

Proposition 5.4. *The associated slice graded to $Tmf_1(3)$ is*

$$\bigvee_{P\in M[\bar{a}_1,\bar{a}_3]} S^{|P|}\wedge H\underline{\mathbb{Z}}\left[\tfrac{1}{3}\right]\quad\vee\bigvee_{P\in M[\bar{a}_1,\bar{a}_3]} S^{-|P|-4\rho-1}\wedge H\underline{\mathbb{Z}}\left[\tfrac{1}{3}\right].$$

Proof. Since the slice (-1)-connective cover is the ordinary (-1)-connective cover, the non-negative slices of $Tmf_1(3)$ are those of its connective cover $tmf_1(3)$.

For the remaining slices, we use Propositions 3.1 and 3.5. The long exact sequence in homotopy associated to the fiber square 4.22 and Corollary 5.1 identify the needed homotopy groups. For $k<0$, let R_k denote the degree $2k$ piece of

$$\mathbb{Z}\left[\tfrac{1}{3}\right][a_1^{\pm1},a_3^{\pm1}]/\left(\mathbb{Z}\left[\tfrac{1}{3}\right][a_1^{\pm1},a_3]+\mathbb{Z}\left[\tfrac{1}{3}\right][a_1,a_3^{\pm1}]\right).$$

We then have isomorphisms

$$\underline{\pi}_{k\rho}Tmf_1(3)=\underline{G}\otimes R_k$$

and

$$\underline{\pi}_{k\rho-1}Tmf_1(3)=\underline{\mathbb{Z}}\otimes R_k.$$

The functor P^0 applied to the Mackey functor $\underline{G}$ yields zero, so we conclude by Proposition 3.5 that there are no negative even slices, and and by Proposition 3.1 that all of the negative odd slices are of the desired form. □

This allows us to compute the E_2-term of the slice spectral sequence

$$E_2^{s,t}=\pi_{t-s}^{C_2}P_t^tTmf_1(3)\Rightarrow\pi_{t-s}^{C_2}Tmf_1(3),$$

where P_t^t denotes the t-slice of $Tmf_1(3)$. For $t \geq 0$ even, we get:

$$\begin{aligned}\pi_{t-s}^{C_2} P_t^t Tmf_1(3) &= \bigoplus_{P \in M[\bar{a}_1, \bar{a}_3]_{\frac{1}{2}t\rho}} \pi_{t-s}^{C_2} S^{\frac{1}{2}t\rho} \wedge H\underline{\mathbb{Z}}\left[\tfrac{1}{3}\right] \\ &= \bigoplus_{P \in M[\bar{a}_1, \bar{a}_3]_{\frac{1}{2}t\rho}} H_{t-s}^{C_2}\left(S^{\frac{1}{2}t\rho}, \underline{\mathbb{Z}}\left[\tfrac{1}{3}\right]\right) \\ &= \bigoplus_{P \in M[\bar{a}_1, \bar{a}_3]_{\frac{1}{2}t\rho}} H_{\frac{1}{2}t-s}^{C_2}\left(S^{\frac{1}{2}t\sigma}, \underline{\mathbb{Z}}\left[\tfrac{1}{3}\right]\right)\end{aligned}$$

By [HHR09, Example 3.16], we have:

$$H_{\frac{1}{2}t-s}^{C_2}\left(S^{\frac{1}{2}t\sigma}, \underline{\mathbb{Z}}\left[\tfrac{1}{3}\right]\right) = \begin{cases} \mathbb{Z}\left[\frac{1}{3}\right] & \text{if } t-s \text{ divisible by 4 and } s=0 \\ \mathbb{Z}/2 & \text{if } 0 < s \leq t-s \text{ and } (t-s)-s \text{ divisible by 4} \\ 0 & \text{else} \end{cases}$$

Similarly, one can reduce the computation for $t < 0$ to Bredon *co*homology and use that

$$H_{C_2}^k\left(S^{d\sigma}, \underline{\mathbb{Z}}\left[\tfrac{1}{3}\right]\right) = \begin{cases} \mathbb{Z}\left[\frac{1}{3}\right] & \text{if } d \text{ even and } k=d \\ \mathbb{Z}/2 & \text{if } k \text{ odd and } 1 < k \leq d \\ 0 & \text{else} \end{cases}$$

In the chart on the next page, the unboxed number n denotes n copies of $\mathbb{Z}/2$, a box denotes a copy of $\mathbb{Z}\left[\frac{1}{3}\right]$ and a boxed n denotes n copies of $\mathbb{Z}\left[\frac{1}{3}\right]$. The vertical coordinate is s and the horizontal one is $t-s$.

5.2. **Anderson Duality.** Let G be a finite group. For an injective abelian group J, the functor

$$(\text{genuine})\ G-\text{Spectra} \to \text{graded abelian groups}, \quad X \mapsto \mathrm{Hom}_{\mathbb{Z}}(\pi_{-*}^G X, J)$$

is representable by a G-spectrum I_J, as follows from Brown representability. If A is an abelian group and $A \to J^0 \to J^1$ an injective resolution, we define the G-spectrum I_A to be the fiber of $I_{J^0} \to I_{J^1}$. Given a G-spectrum X, we define its *A-Anderson dual* $I_A X$ by $F(X, I_A)$. It satisfies for all $k \in \mathbb{Z}$ the following functorial short exact sequence:

$$0 \to \mathrm{Ext}_{\mathbb{Z}}^1(\pi_{-k-1}^G X, A) \to \pi_k^G I_A X \to \mathrm{Hom}_{\mathbb{Z}}(\pi_{-k}^G X, A) \to 0$$

For $G = \{e\}$ we get non-equivariant Anderson duality as explored in [Sto12]. If G is (possibly) non-trivial, denote by $\mathcal{A}_G$ the *Burnside category*, by which we mean the full subcategory of $\mathrm{Ho}(\mathrm{Sp}_G)$ on the cosets $\Sigma^\infty(G/H)_+$. Precomposing for a G-spectrum X with the functor

$$\mathcal{A}_G \to \mathrm{Sp}_G, \quad \Sigma^\infty(G/H)_+ \mapsto \Sigma^\infty(G/H)_+ \wedge X$$

we see that the short exact sequence above refines to a short exact sequence of Mackey functors

$$0 \to \mathrm{Ext}_{\mathbb{Z}}^1(\underline{\pi}_{-k-1} X, A) \to \underline{\pi}_k I_A X \to \mathrm{Hom}_{\mathbb{Z}}(\underline{\pi}_{-k}^G X, A) \to 0.$$

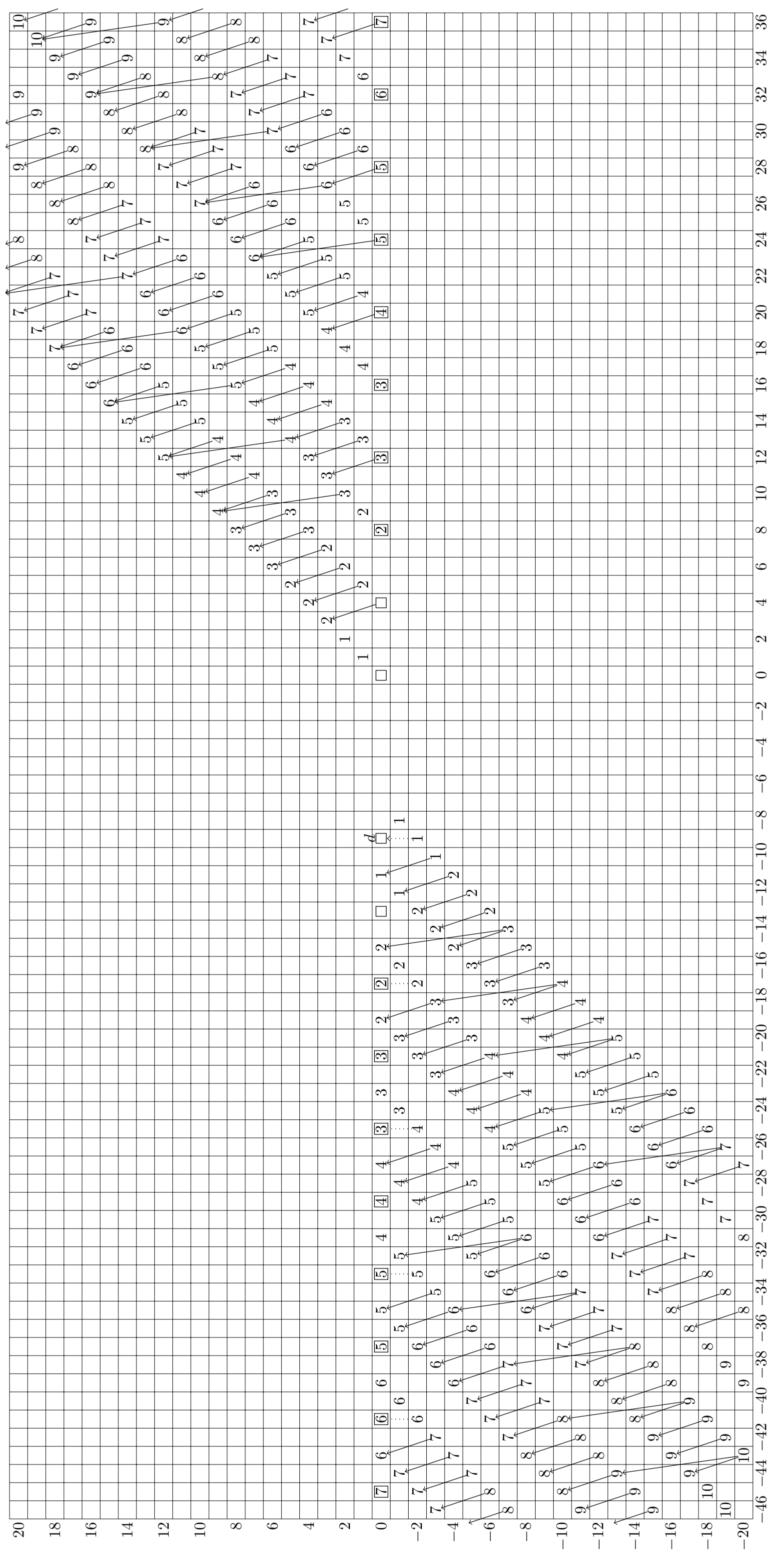

FIGURE 1. The slice spectral sequence for $Tmf_1(3)$.

By smashing X with representation spheres, we see that it even refines to an $RO(G)$-graded sequence. Equivariant Anderson duality in the case $G = C_2$ has been explored in some detail in [Ric14].

One reason to be interested in Anderson (self) duality is the universal coefficient sequence, relating homology and cohomology, which we will state for simplicity only non-equivariantly. Let E be a spectrum, X be another spectrum and A be an abelian group. Then $I_A(X \wedge E) \simeq F(X, I_A E)$ implies the short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(E_{k-1}X, A) \to (I_A E)^k X \to \mathrm{Hom}_{\mathbb{Z}}(E_k X, A) \to 0.$$

In particular, Anderson self-duality implies useful universal coefficient theorems; for example, $I_{\mathbb{Z}} KO \simeq \Sigma^4 KO$ implies one of the main theorems of [And70].

Our goal in this section is to compute the $\mathbb{Z}\left[\frac{1}{3}\right]$-Anderson dual of $Tmf_1(3)$ as a C_2-spectrum and then deduce a computation of the $\mathbb{Z}\left[\frac{1}{3}\right]$-Anderson dual of $Tmf_0(3)$.

Observe that $H\underline{\mathbb{Z}}^* \simeq S^{4-2\rho} \wedge H\underline{\mathbb{Z}}$. Thus, Proposition 5.4 implies that the associated slice graded of $Tmf_1(3)$ is

$$\bigvee_{P \in M[\bar{a}_1, \bar{a}_3]} S^{|P|} \wedge H\underline{\mathbb{Z}}\left[\tfrac{1}{3}\right] \quad \vee \bigvee_{P \in M[\bar{a}_1, \bar{a}_3]} S^{-|P|-2\rho-5} \wedge H\underline{\mathbb{Z}}\left[\tfrac{1}{3}\right]^*.$$

This suggests the following theorem:

Theorem 5.5. *There is a C_2-equivariant equivalence $I_{\mathbb{Z}\left[\frac{1}{3}\right]} Tmf_1(3) \simeq \Sigma^{5+2\rho} Tmf_1(3)$.*

To prove this, we will start with two lemmas.

Lemma 5.6. *We have non-equivariantly $I_{\mathbb{Z}\left[\frac{1}{3}\right]} Tmf_1(3) \simeq \Sigma^9 Tmf_1(3)$.*

Proof. For shortness, we will write $R = Tmf_1(3)$.

By 4.3, the moduli stack $\overline{\mathcal{M}}_1(3)$ is equivalent to the weighted projective stack $\mathcal{P}(1,3) = \mathcal{P}_{\mathbb{Z}\left[\frac{1}{3}\right]}(1,3)$ and the sheaf ω on $\overline{\mathcal{M}}_1(3)$ corresponds to $\mathcal{O}(1)$ on $\mathcal{P}(1,3)$. This weighted projective stack has Serre duality in the sense that there is a class $D = \frac{1}{a_1 a_3} \in H^1(\mathcal{P}(1,3); \mathcal{O}(-4))$ such that

$$H^s(\mathcal{P}(1,3); \mathcal{F}) \otimes H^{1-s}(\mathcal{P}(1,3); \mathcal{F}^* \otimes \mathcal{O}(-4)) \to H^1(\mathcal{P}(1,3); \mathcal{O}(-4)) \cong \mathbb{Z}\left[\tfrac{1}{3}\right] \cdot D$$

is a perfect pairing for $s = 0, 1$ for an arbitrary coherent sheaf $\mathcal{F}$.

As $\mathcal{P}(1,3)$ has cohomological dimension 1, the element D is a permanent cycle in the descent spectral sequence for $R = Tmf_1(3)$ and is represented by a unique element in $\pi_{-9} R \cong \mathbb{Z}\left[\frac{1}{3}\right]$, which we will also denote by D. Denote by δ the element in $\pi_9 I_{\mathbb{Z}\left[\frac{1}{3}\right]} R$ with $\phi(\delta)(D) = 1$, where $\phi\colon \pi_9 I_{\mathbb{Z}\left[\frac{1}{3}\right]} R \xrightarrow{\cong} \mathrm{Hom}(\pi_{-9} R, \mathbb{Z}\left[\frac{1}{3}\right])$. The element δ induces a R-linear map $\widehat{\delta}\colon \Sigma^9 R \to I_{\mathbb{Z}\left[\frac{1}{3}\right]} R$.

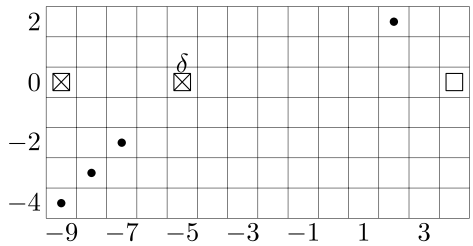

FIGURE 2. The E^2-term of the Slice Spectral Sequence for $\pi_{k-2\rho}Tmf_1(3)$

We obtain a commutative diagram

$$\begin{tikzcd}
\pi_{k-9}R \otimes \pi_{-k}R \arrow[r, "\widehat{\delta}_* \otimes \mathrm{id}"] \arrow[d] & \pi_k I_{\mathbb{Z}\left[\frac{1}{3}\right]} R \otimes \pi_{-k}R \arrow[r, "\phi\otimes\mathrm{id}", "\cong"'] & \mathrm{Hom}(\pi_{-k}R, \mathbb{Z}\left[\frac{1}{3}\right]) \otimes \pi_{-k}R \arrow[d] \\
\pi_{-9}R \arrow[rr, "\phi(\delta)", "\cong"'] & & \mathbb{Z}\left[\frac{1}{3}\right]
\end{tikzcd}$$

The left vertical map is a perfect pairing because of Serre duality (as described above), as is the right vertical map by definition. Thus, the map $\widehat{\delta}_*\colon \pi_{k-9}R \to \pi_k I_{\mathbb{Z}\left[\frac{1}{3}\right]} R$ is an isomorphism for all k. This shows that $\widehat{\delta}$ is an equivalence. $\square$

The following key lemma uses our information about the slices of $Tmf_1(3)$:

Lemma 5.7. *The transfer*

$$\pi_{-9}Tmf_1(3) = \pi^u_{-5-2\rho}Tmf_1(3) \to \pi^{C_2}_{-5-2\rho}Tmf_1(3)$$

is an isomorphism.

Proof. The slice spectral sequence for $\Sigma^{2\rho}Tmf$ (as shown in Figure 2, dots meaning the Mackey functor G and boxtimes meaning $\underline{\mathbb{Z}}^*$) gives an isomorphism of Mackey functors

$$\underline{\pi}_{-5-2\rho}Tmf_1(3) \cong \underline{\pi}_{-5-2\rho}S^{-4\rho-1} \wedge H\underline{\mathbb{Z}}\left[\tfrac{1}{3}\right] \cong H^2(S^{2\sigma}; \underline{\mathbb{Z}}\left[\tfrac{1}{3}\right]) \cong \underline{\mathbb{Z}}\left[\tfrac{1}{3}\right]^*. \qquad \square$$

Proof of Theorem. Consider the commutative diagram

$$\begin{tikzcd}
\pi^{C_2}_{5+2\rho} I_{\mathbb{Z}\left[\frac{1}{3}\right]} Tmf_1(3) \arrow[r, two heads] \arrow[d, "\mathrm{res}"] & \mathrm{Hom}(\pi^{C_2}_{-5-2\rho}Tmf_1(3), \mathbb{Z}\left[\frac{1}{3}\right]) \arrow[d, "\mathrm{res}=\mathrm{tr}^*", "\cong"'] \\
\pi_9 I_{\mathbb{Z}\left[\frac{1}{3}\right]} Tmf_1(3) \arrow[r, "\cong"] & \mathrm{Hom}(\pi_{-9}Tmf_1(3), \mathbb{Z}\left[\frac{1}{3}\right])
\end{tikzcd}$$

By the last lemma, tr^* is an isomorphism. This implies that we can refine the element $\delta \in \pi_9 I_{\mathbb{Z}\left[\frac{1}{3}\right]} Tmf_1(3)$ corresponding to the equivalence $\Sigma^9 Tmf_1(3) \to I_{\mathbb{Z}\left[\frac{1}{3}\right]} Tmf_1(3)$

from Lemma 5.6 to an element $\tilde{\delta} \in \pi^{C_2}_{5+2\rho} I_{\mathbb{Z}[\frac{1}{3}]} Tmf_1(3)$. This induces a C_2-equivariant $Tmf_1(3)$-linear map

$$\Sigma^{5+2\rho} Tmf_1(3) \to I_{\mathbb{Z}[\frac{1}{3}]} Tmf_1(3)$$

that is an equivalence of underlying spectra. By the discussion after Proposition 4.17, $Tmf_1(3)$ and $I_{\mathbb{Z}[\frac{1}{3}]} Tmf_1(3)$ are cofree C_2-spectra. Thus, the theorem follows. □

This allows us also to compute the Anderson dual of $Tmf_0(3)$. As in [Sto12], we will use the following lemma:

Lemma 5.8. *Let A be an abelian group and X be a spectrum with an action by a finite group G. Assume that the norm map $X_{hG} \to X^{hG}$ is an equivalence. Then there is an equivalence $(I_A X)^{hG} \simeq I_A(X^{hG})$.*

Proof. We have the following chain of equivalences:

$$(I_A X)^{hG} \simeq F(X, I_A)^{hG} \simeq F(X_{hG}, I_A) \simeq F(X^{hG}, I_A) \simeq I_A(X^{hG})$$ □

By Proposition 4.17, we get:

Corollary 5.9. *There is an equivalence $I_{\mathbb{Z}[\frac{1}{3}]} Tmf_0(3) \simeq (\Sigma^{5+2\rho} Tmf_1(3))^{hC_2}$.*

6. The Picard Groups

In this section we will compute the Picard groups of $TMF_0(3)$, $Tmf_0(3)$ and related spectra. We recommend to have a look at [MS14] as it contains a good introduction to Picard groups and our techniques are very similar to theirs.

6.1. **Generalities.** If $\mathcal{C}$ is a monoidal (∞-)category, we denote by $\mathrm{Pic}(\mathcal{C})$ the (possibly large) group of isomorphism classes of invertible spectra. If $\mathcal{C}$ is a monoidal ∞-category, we denote by $\mathcal{P}ic(\mathcal{C})$ the maximal ∞-subgroupoid (Kan complex) of the full subcategory of invertible objects. Clearly, $\pi_0 \mathcal{P}ic(\mathcal{C}) \cong \mathrm{Pic}(\mathrm{Ho}(\mathcal{C}))$. If $\mathcal{C}$ is a symmetric monoidal ∞-category, $\mathcal{P}ic(\mathcal{C})$ inherits the structure of a group-like E_∞-space; thus, there is a connective spectrum $\mathfrak{pic}(\mathcal{C})$ with $\Omega^\infty \mathfrak{pic}(\mathcal{C}) \simeq \mathcal{P}ic(\mathcal{C})$.

Given an E_2-ring spectrum R, its ∞-category R-mod of (left) R-modules has the structure of a monoidal ∞-category ([Lur14, Proposition 7.1.2.6]). We define the *Picard group* $\mathrm{Pic}(R)$ of R to be $\mathrm{Pic}(\mathrm{Ho}(R\text{-mod}))$ and the *Picard space* $\mathcal{P}ic(R)$ to be $\mathcal{P}ic(R\text{-mod})$. If R is an E_∞-ring spectrum, then R-mod is even a *symmetric* monoidal ∞-category. We define then $\mathfrak{pic}(R)$ to be $\mathfrak{pic}(R\text{-mod})$.

For us, a *derived stack* will be a pair $\mathcal{X} = (X, \mathcal{O}^{top})$, where X is a Deligne–Mumford stack and $\mathcal{O}^{top}$ is a sheaf of even-periodic E_∞-ring spectra with $\pi_0 \mathcal{O}^{top}$ isomorphic to the structure sheaf $\mathcal{O}_X$ of X. For example, X might be a moduli stack of elliptic curves. For a derived stack $\mathcal{X} = (X, \mathcal{O}^{top})$, we write $\mathrm{Pic}(\mathcal{X})$ etc. for the Picard group, space or spectrum of the symmetric monoidal ∞-category of quasi-coherent $\mathcal{O}^{top}$-modules $\mathrm{QCoh}(\mathcal{X})$ on $\mathcal{X}$. For a short treatment of quasi-coherent sheaves in this context see [MM15, Section 2.3] and for a full-blown treatment see [Lur11b].

Definition 6.1. We call a derived stack $\mathcal{X} = (X, \mathcal{O}^{top})$ 0-*affine* if the global sections functor

$$\Gamma\colon \ \mathrm{QCoh}(\mathcal{X}) \to \mathcal{O}^{top}(X)\text{-mod}$$

is an equivalence of symmetric monoidal ∞-categories.

Clearly, $\mathfrak{pic}(\mathcal{X}) \simeq \mathfrak{pic}(\mathcal{O}^{top}(X))$ if $\mathcal{X}$ is 0-affine. It was shown in [MM15] that the (compactified) moduli stack of elliptic curves with arbitary level structure together with its derived structure sheaf $\mathcal{O}^{top}$ is 0-affine.

The following Mayer–Vietoris principle will be useful later.

Lemma 6.2. *Let $\mathcal{X} = (X, \mathcal{O}_{\mathcal{X}})$ be a* 0*-affine derived stack and $U, V \subset \mathcal{X}$ be a covering by open substacks. Then we have a long exact sequence*

$$\cdots \to GL_1\pi_0\mathcal{O}_{\mathcal{X}}(U\cap V) \to \mathrm{Pic}(\mathcal{O}_{\mathcal{X}}(X)) \to \mathrm{Pic}(\mathcal{O}_{\mathcal{X}}(U))\times\mathrm{Pic}(\mathcal{O}_{\mathcal{X}}(V)) \to \mathrm{Pic}(\mathcal{O}_{\mathcal{X}}(U\cap V)).$$

The boundary map $GL_1\pi_0\mathcal{O}_{\mathcal{X}}(U\cap V) \to \mathrm{Pic}(\mathcal{O}_{\mathcal{X}}(X))$ can be described as follows: An element of $GL_1\pi_0\mathcal{O}_{\mathcal{X}}(U\cap V)$ corresponds to an $\mathcal{O}_{\mathcal{X}}$-linear self-equivalence of $\mathcal{O}_{\mathcal{X}}|_{U\cap V}$. Gluing $\mathcal{O}_{\mathcal{X}}|_U$ and $\mathcal{O}_{\mathcal{X}}|_V$ along this self-equivalence gives an invertible $\mathcal{O}_{\mathcal{X}}$-module on X.

Proof. As shown in [MS14, Section 3.1], the presheaf $\mathcal{P}ic$ defined by

$$\mathcal{P}ic(\operatorname{Spec} R \to \overline{\mathcal{M}}_1(3)) = \mathcal{P}ic(\mathcal{O}^{top}(\operatorname{Spec} R \to \overline{\mathcal{M}}_1(3))$$

(where $\operatorname{Spec} R \to \overline{\mathcal{M}}_1(3)$ is étale) is actually a sheaf. Thus, we have a homotopy pullback square

$$\begin{tikzcd}
\mathcal{P}ic(X, \mathcal{O}_{\mathcal{X}}) \arrow[r] \arrow[d] & \mathcal{P}ic(U, \mathcal{O}_{\mathcal{X}}|_U) \arrow[d] \\
\mathcal{P}ic(V, \mathcal{O}_{\mathcal{X}}|_V) \arrow[r] & \mathcal{P}ic(U\cap V, \mathcal{O}_{\mathcal{X}}|_{U\cap V})
\end{tikzcd}$$

The identification of these Picard spaces with those of $\mathcal{O}_{\mathcal{X}}(X)$ etc. follows from the fact that X, U, V and $U\cap V$ are 0-affine (see [MM15, Proposition 3.27]). This fiber square induces then the long exact sequence in the lemma. The identification of the boundary map follows directly from the usual description of the map

$$GL_1\mathcal{O}_{\mathcal{X}}(U\cap V) \simeq \Omega\mathcal{P}ic(\mathcal{O}_{\mathcal{X}}(U\cap V)) \to \mathcal{P}ic(\mathcal{O}_{\mathcal{X}}(U)) \times^h_{\mathcal{P}ic(\mathcal{O}_{\mathcal{X}}(U\cap V))} \mathcal{P}ic(\mathcal{O}_{\mathcal{X}}(V)). \quad \square$$

Let now $A \to B$ be a faithful G-Galois extension in the sense of [Rog08]. Then by [MS14, Section 3.3], we have the following theorem:

Theorem 6.3. *There is an equivalence $\mathfrak{pic}(A) \simeq \tau_{\geq 0}\mathfrak{pic}(B)^{hG}$.*

There is also another equivariant interpretation of the Picard group of A if $A \to B$ is a faithful G-Galois extension. View $B \simeq F(EG_+, B)$ as a cofree G-spectrum. Denote the category of equivariant B-modules by G-B-mod. As B is cofree and $A \to B$ is a faithful Galois extension, $\Phi^G B \simeq B^{tG}$ is contractible. By [HHR09, Corollary 10.6] every (equivariant) B-module is thus cofree again. Therefore, a map in G-B-mod is a weak equivalence if it is an underlying weak equivalence. It is then a consequence of Galois descent in the form [Mei12, Lemma 6.1.4, Proposition 6.2.6] that there is a monoidal equivalence $\mathrm{Ho}(A\text{-mod}) \simeq \mathrm{Ho}(G\text{-}B\text{-mod})$. Thus, $\mathrm{Pic}(R) \cong \mathrm{Pic}(\mathrm{Ho}(G\text{-}B\text{-mod}))$, the group of equivariant invertible B-modules. We will denote the latter group by $\mathrm{Pic}_G(B)$.

6.2. **A generalized Baker–Richter theorem.** Baker and Richter proved in [BR05] that the Picard group of an E_∞-ring spectrum R is completely algebraic if R is even periodic and $\pi_0 R$ is a regular complete local ring. This applies, for example, to the Lubin–Tate spectra E_n. Mathew and Stojanoska generalized this in [MS14] by dropping the condition that $\pi_0 R$ is complete and local (and also weakened the periodicity requirement). The main purpose of this subsection is to show that the assumption of periodicity is superfluous.

Let R be an E_2-ring spectrum. Let $\overline{L}$ be an invertible $\pi_* R$-module. Then $\overline{L}$ is projective over $\pi_* R$. Thus, there is an R-module L with $\pi_* L \cong \overline{L}$ and this module L is well-defined up to isomorphism in $\mathrm{Ho}(R\text{-mod})$. This defines a map $\mathrm{Pic}(\pi_* R) \to \mathrm{Pic}(R)$. By the degenerated Künneth spectral sequence, this is a homomorphism.

Let R_* be a commutative graded ring. By an element $x \in R_*$ we will always mean a *homogeneous* element and by an ideal $I \subset R_*$ we will always mean a *homogeneous* ideal. We call R_* *local* if it has a unique maximal ideal $\mathfrak{m}$. We call a graded local ring *regular* if the maximal ideal is generated by a finite regular sequence. We call a graded local ring *complete* if the map $R_* \to \lim_k R/\mathfrak{m}^k$ is an isomorphism. We call an arbitrary commutative graded ring *regular* if every localization of it at a prime ideal is regular.

We have the following generalization of [BR05, Theorem 38].

Theorem 6.4. *Let R be an E_2-ring spectrum. Assume that $\pi_* R$ is concentrated in even degrees and regular. Then the morphism $\mathrm{Pic}(\pi_* R) \to \mathrm{Pic}(R)$ is an isomorphism.*

This is not really new as this generalization is just a combination of [BR05, Remark 39] and [MS14, Theorem 2.4.6]. We will sketch a proof anyhow as we introduce one simplification, avoiding the use of obstruction theory for A_∞-structures.

Let M be an invertible R-module with $M \wedge_R N \simeq R$ for some R-module N. It is enough to show that $\pi_* M$ is a projective $\pi_* R$-module. For this, it is enough to show that the completion $\widehat{(\pi_* M)}_{\mathfrak{m}}$ is a projective $\widehat{\pi_* R}_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m} \subset \pi_* R$.

The theory from [Lur11a, Section 4.2] implies that there is actually an R-module $\widehat{M}_{\mathfrak{m}}$ with $\pi_* \widehat{M}_{\mathfrak{m}} \cong \widehat{(\pi_* M)}_{\mathfrak{m}}$.[2] We have $\widehat{M}_{\mathfrak{m}} \wedge_R \widehat{N}_{\mathfrak{m}} \simeq \widehat{R}_{\mathfrak{m}}$ by [Lur11a, Remark 4.2.6].

Let $x_1, \dots, x_n$ be a regular sequence generating $\mathfrak{m}$. Consider the $\widehat{R}_{\mathfrak{m}}$-module $\widehat{R}_{\mathfrak{m}}/\underline{x}$, obtained by killing the regular sequence $x_1, \dots, x_n$. By [EKMM97, Theorem V.2.6][3] $\widehat{R}_{\mathfrak{m}}/\underline{x}$ has the structure of an R-ring spectrum in the sense that there exists a map

$$\widehat{R}_{\mathfrak{m}}/\underline{x} \wedge_{\widehat{R}_{\mathfrak{m}}} \widehat{R}_{\mathfrak{m}}/\underline{x} \to \widehat{R}_{\mathfrak{m}}/\underline{x}$$

that is unital up to homotopy.[4] For an arbitrary $\widehat{R}_{\mathfrak{m}}$-module X, set $X/\underline{x} = X \wedge_{\widehat{R}_{\mathfrak{m}}} \widehat{R}_{\mathfrak{m}}/\underline{x}$.

Claim 6.5. *The map*

$$\pi_*(X_1/\underline{x}) \otimes_{\pi_* \widehat{R}_{\mathfrak{m}}/\underline{x}} \pi_*(X_2/\underline{x}) \to \pi_*((X_1 \wedge_{\widehat{R}_{\mathfrak{m}}} X_2)/\underline{x})$$

[2]Lurie only considers ideals in $\pi_0 R$, but the theory also works for homogeneous ideals in $\pi_* R$ under our assumptions.

[3]While the source states the result only for E_∞-ring spectra, the same proof works also for E_2-ring spectra.

[4]For our argument, this naive result suffices, while Baker and Richter use that $\widehat{R}_{\mathfrak{m}}/\underline{x}$ has an A_∞-structure.

is an isomorphism for all $\widehat{R}_{\mathfrak{m}}$-modules X_1 and X_2.

Proof. This is clearly true if $X_1 = \widehat{R}_{\mathfrak{m}}$. Both sides are homological in X_1 – as $\pi_*(\widehat{R}_{\mathfrak{m}}/\underline{x})$ is a graded field – and compatible with arbitrary coproducts. Thus, it is true for all $X_1 \in \widehat{R}_{\mathfrak{m}}$-mod. □

In particular $\pi_*(\widehat{M_{\mathfrak{m}}}/\underline{x})$ is in the Picard group of $\pi_*(\widehat{R}_{\mathfrak{m}}/\underline{x})$. Thus, $\pi_*(\widehat{M_{\mathfrak{m}}}/\underline{x})$ is a free $\pi_*\widehat{R}_{\mathfrak{m}}/\underline{x}$-module of rank 1.

As in [BR05], one can show that $\pi_*(\widehat{M_{\mathfrak{m}}}/(x_1^{i_1}, \ldots, x_n^{i_n}))$ is a cyclic $\pi_*\widehat{R}_{\mathfrak{m}}$-module for $i_1, \ldots, i_n \geq 1$, using the Nakayama lemma for graded rings. Using the completeness of $\pi_*\widehat{R}_{\mathfrak{m}}$, one can show as in [BR05] that $\pi_*\widehat{M_{\mathfrak{m}}}$ is a shift of $\pi_*\widehat{R}_{\mathfrak{m}}$. In particular, $\pi_*\widehat{M_{\mathfrak{m}}}$ is projective over $\pi_*\widehat{R}_{\mathfrak{m}}$ as we wanted to show.

6.3. The case of $TMF_1(3)$ and $Tmf_1(3)$.

Lemma 6.6. *We have isomorphisms*

$$\begin{aligned} \operatorname{Pic} TMF_1(3) &\cong \mathbb{Z}/6 \\ \operatorname{Pic} tmf_1(3)[a_1^{-1}] &\cong \mathbb{Z}/2 \\ \operatorname{Pic} tmf_1(3)[a_3^{-1}] &\cong \mathbb{Z}/6 \\ \operatorname{Pic} tmf_1(3)[a_1^{-1}\bar{a}_3^{-1}] &\cong \mathbb{Z}/2 \end{aligned}$$

In all the cases, all the invertible modules are equivalent to suspensions of the ground ring spectrum.

Proof. We will just prove the lemma for $TMF_1(3)$ – the other cases are analogous. By Theorem 6.4, $\operatorname{Pic} TMF_1(3) \cong \operatorname{Pic}(\pi_*TMF_1(3))$. A graded $\pi_{2*}TMF_1(3)$-module is an equivalent datum to a quasi-coherent sheaf on $\mathcal{M}_1(3) \simeq \operatorname{Spec} \mathbb{Z}\left[\frac{1}{3}\right][a_1, a_3][\Delta^{-1}]/\mathbb{G}_m$. Thus, there is a short exact sequence

$$0 \to \operatorname{Pic}(\mathcal{M}_1(3)) \to \operatorname{Pic}(\pi_*TMF_1(3)) \to \mathbb{Z}/2 \to 0.$$

Given a line bundle $\mathcal{L}$ on $\mathcal{M}_1(3)$, we can extend it to the weighted projective stacky line $\overline{\mathcal{M}}_1(3)$. Indeed, by [Mei13, Lemma 3.2], we can extend $\mathcal{L}$ to a reflexive sheaf on $\overline{\mathcal{M}}_1(3)$ and every reflexive sheaf of rank 1 is a line bundle by [Har80, Proposition 1.9]. Every line bundle on a weighted projective stacky line is of the form $\mathcal{O}(k)$ for some $k \in \mathbb{Z}$ as can be seen, for example, along the lines of [Mei13, Proposition 3.4].

As a_3 is invertible on $\mathcal{M}_1(3)$, it defines a trivialization of $\mathcal{O}(3)|_{\mathcal{M}_1(3)}$. We see that

$$\operatorname{Pic}(\mathcal{M}_1(3)) \cong \mathbb{Z}/3.$$

As the subgroup of $\operatorname{Pic} TMF_1(3)$ spanned by the $\Sigma^k TMF_1(3)$ is isomorphic to $\mathbb{Z}/6$, the lemma follows. □

Proposition 6.7. *The extensions*

$$\begin{aligned} TMF_0(3) &\to TMF_1(3) \\ (tmf_1(3)[\bar{a}_1^{-1}])^{hC_2} &\to tmf_1(3)[\bar{a}_1^{-1}] \\ (tmf_1(3)[\bar{a}_3^{-1}])^{hC_2} &\to tmf_1(3)[\bar{a}_3^{-1}] \\ (tmf_1(3)[\bar{a}_1^{-1}\bar{a}_3^{-1}])^{hC_2} &\to tmf_1(3)[\bar{a}_1^{-1}\bar{a}_3^{-1}] \end{aligned}$$

are faithful C_2-Galois extensions in the sense of Rognes.

Proof. We obtain these maps of E_∞-ring spectra by applying $\mathcal{O}^{top}$ to the following C_2-Galois covers of stacks

$$\begin{aligned} \mathcal{M}_1(3) &\to \mathcal{M}_0(3) \\ D(a_1) &\to D(a_1)/C_2 \\ D(a_3) &\to D(a_3)/C_2 \\ D(a_1a_3) &\to D(a_1a_3)/C_2 \end{aligned}$$

as follows from the results in Section 4.3. By the main result of [MM15], the derived stack $(\mathcal{M}_{ell}, \mathcal{O}^{top})$ is 0-affine and by [MM15, Proposition 3.28] the same is true for the targets of the above four Galois covers. Then [MM15, Theorem 5.6] implies the result. $\square$

Theorem 6.8. *We have isomorphisms*

$$\begin{aligned} \operatorname{Pic}_{C_2} TMF_1(3) &\cong \operatorname{Pic}(TMF_0(3)) \cong \mathbb{Z}/48 \\ \operatorname{Pic}_{C_2} tmf_1(3)[\bar{a}_1^{-1}] &\cong \operatorname{Pic}((tmf_1(3)[\bar{a}_1^{-1}])^{hC_2}) \cong \mathbb{Z}/8 \\ \operatorname{Pic}_{C_2} tmf_1(3)[\bar{a}_3^{-1}] &\cong \operatorname{Pic}((tmf_1(3)[\bar{a}_3^{-1}])^{hC_2}) \cong \mathbb{Z}/48 \\ \operatorname{Pic}_{C_2} tmf_1(3)[\bar{a}_1^{-1}\bar{a}_3^{-1}] &\cong \operatorname{Pic}((tmf_1(3)[\bar{a}_1^{-1}\bar{a}_3^{-1}])^{hC_2}) \cong \mathbb{Z}/8 \end{aligned}$$

In all the cases, all the (equivariant) invertible modules are equivalent to (trivial) suspensions of the ground ring spectrum.

Proof. We will only prove this in the first case. The other cases are similar. The first equivalence follows directly from Proposition 6.7 and the discussion at the end of the previous subsection.

In the following, we will denote by *HFPSS* the homotopy fixed point spectral sequence for the C_2-action on $TMF_1(3)$ and differentials in it will be denoted by d^{HF}. We will always use the Adams convention that the k-th column consists of the groups $H^s(C_2; \pi_t TMF_1(3))$ with $k = t - s$.

We have $TMF_1(3) \simeq_{C_2} tmf_1(3)[\overline{\Delta}^{-1}]$ with $\overline{\Delta} = \bar{a}_3^3(\bar{a}_1^3 - 27\bar{a}_3)$ by the results of Section 4.3. As $\overline{\Delta}$ is a permanent cycle, this allows us to deduce from the results of Section 4.2 all differentials in the HFPSS. For example, $\gamma = \frac{\bar{a}_3^4}{\overline{\Delta}}$ is a permanent cycle.

It is easy to see that the (-1)-st column of the HFPSS for $TMF_1(3)$ is in cohomological degrees ≤ 7 isomorphic to $\mathbb{F}_2[\gamma] \cdot b_3 \oplus \mathbb{F}_2[\gamma] \cdot b_7$ with $b_3 = \bar{a}_1 u_{2\sigma}^{-1} a_\sigma^3$ of cohomological degree 3 and $b_7 = \bar{a}_3 u_{2\sigma}^{-2} a_\sigma^7$ of degree 7. We have the differentials

$$d_3^{HF}(\gamma^k b_3) = \gamma^k b_3^2$$

and

$$d_7^{HF}(\gamma^k b_7) = \gamma^k b_7^2$$

in the HFPSS.

From the equivalence $\mathfrak{pic}(TMF_0(3)) \simeq \tau_{\geq 0}(\mathfrak{pic}(TMF_1(3)))^{hC_2}$, we get the *Picard spectral sequence*

$$H^s(C_2; \pi_t\, \mathfrak{pic}\, TMF_1(3))$$

that converges to $\pi_{t-s}\mathfrak{pic}\, TMF_0(3)$ for $t - s \geq 0$. Differentials in it will be denoted by d^{Pic}

The Picard group of $TMF_1(3)$ is $\mathbb{Z}/6$ by Lemma 6.6 and $GL_1\pi_0 TMF_1(3)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}/2$, generated by $\frac{1}{3}$ and -1. Thus:

$$\pi_t \mathfrak{pic} TMF_1(3) = \begin{cases} \mathbb{Z}/6 & \text{for } t = 0 \\ \mathbb{Z} \times \mathbb{Z}/2 & \text{for } t = 1 \\ \pi_{t-1} TMF_1(3) & \text{for } t \geq 2. \end{cases}$$

We are interested in the 0-th column of the Picard spectral sequence. We have

$$H^0(C_2; \mathbb{Z}/6) = \mathbb{Z}/6$$

and

$$H^1(C_2; \mathbb{Z} \times \mathbb{Z}/2) \cong \mathbb{Z}/2;$$

for $s \geq 2$ the 0-th column of the Picard spectral sequence agrees with the (-1)-st column of the HFPSS. For an element x in the (-1)-st column of the HFPSS, denote the corresponding element in the 0-th column of the Picard spectral sequence by $\underline{x}$.

If $x \in E^s_{0,s}$ is in cohomological degree s, then $d_s^{Pic}(\underline{x}) = \underline{d_s^{HF}(x) + x^2}$ by [MS14, Theorem 6.1.1]. For degree reasons, the first possible differential for $\underline{\gamma^k b_3}$ is a d_3^{Pic} and this equals $\underline{(\gamma^k + \gamma^{2k}) b_3^2}$. This is zero only if $k = 0$. Likewise for degree reasons, the first possible differential for $\underline{\gamma^k b_7}$ is a d_7^{Pic} and this equals $\underline{(\gamma^k + \gamma^{2k}) b_7^2}$. This is again zero only if $k = 0$, so that $\underline{b_3}$ and $\underline{b_7}$ are the only permanent cycles in the 0-th column of the Picard spectral sequence in cohomological degrees $2 \leq s \leq 7$.

It is easy to check that each element in the (-1)-st column of the HFPSS of cohomological degree ≥ 8 either supports a d_3- or d_7-differential or is hit by a d_3- or d_7-differential from an element of degree ≥ 8. By [MS14, Comparison Tool 5.2.4], this implies that all non-trivial elements in the 0-th column of the Picard spectral sequence in cohomological degrees ≥ 8 support non-trivial differentials or are hit by differentials.

Thus, $\mathrm{Pic}(TMF_0(3))$ has at most $6 \cdot 2 \cdot 2 \cdot 2 = 48$ elements. We just need to show that the image of the morphism

$$\mathbb{Z} \to \mathrm{Pic}(TMF_1(3)), \quad k \mapsto \Sigma^k TMF_0(3)$$

has order 48. This follows easily from the fact that 48 is the smallest period of $\pi_* TMF_0(3)$ as Δ is not a permanent cycle in the HFPSS. $\square$

Question 6.9. *Let E be an E_2 C_2-spectrum. Assume that E is strongly even and that π_*E is a regular graded ring and an integral domain. Is under these conditions every invertible E-module of the form $S^V \wedge L$, where $V \in RO(C_2)$ and L is a strongly even E-module with $\pi_*L \in \mathrm{Pic}(\pi_*E)$?*

Note that every even projective π_*E module can be realized by a strongly even E-module in a unique way up to homotopy, giving a well defined homomorphism

$$\mathrm{Pic}_{even}(\pi_*E) \to \mathrm{Pic}^{C_2}(E).$$

The question above could thus be restated as asking for the surjectivity of the induced homomorphism

$$RO(C_2) \oplus \mathrm{Pic}_{even}(\pi_*E) \to \mathrm{Pic}^{C_2}(E).$$

A positive answer to this question would be a Real generalization of the theorem by Baker and Richter given here as Theorem 6.4.

We could provide a similar spectral sequence argument as above for the computation of $\mathrm{Pic}_{C_2}(Tmf_1(3))$, but we prefer to use a Mayer–Vietoris style argument instead. This will demonstrate how the computation of $\mathrm{Pic}_{C_2}(Tmf_1(3))$ follows essentially formally from the fact that the Picard groups $\mathrm{Pic}_{C_2}(tmf_1(3)[\bar{a}_1^{-1}])$ and $\mathrm{Pic}_{C_2}(tmf_1(3)[\bar{a}_3^{-1}])$ are generated by the suspension of the ground ring spectrum.

Theorem 6.10. *The morphism*

$$RO(C_2) \to \mathrm{Pic}_{C_2}(Tmf_1(3)), \quad V \mapsto S^V \wedge Tmf_1(3)$$

is surjective. Its kernel is generated by $8 - 8\sigma$. Thus,

$$\mathrm{Pic}(Tmf_0(3)) \cong \mathrm{Pic}_{C_2}(Tmf_1(3)) \cong \mathbb{Z} \oplus \mathbb{Z}/8.$$

Proof. By Lemmas 4.19, 4.20 and 6.2, we have an exact sequence:

$$GL_1\pi_0^{C_2}tmf_1(3)[\bar{a}_1^{-1}] \times GL_1\pi_0^{C_2}tmf[\bar{a}_3^{-1}] \xrightarrow{f} GL_1\pi_0^{C_2}tmf_1(3)[\bar{a}_1^{-1}\bar{a}_3^{-1}] \xrightarrow{\partial}$$
$$\mathrm{Pic}_{C_2}(Tmf_1(3)) \longrightarrow \mathrm{Pic}_{C_2}(tmf_1(3)[\bar{a}_1^{-1}]) \times \mathrm{Pic}_{C_2}(tmf_1(3)[\bar{a}_3^{-1}]) \xrightarrow{g} \mathrm{Pic}_{C_2}(tmf_1(3)[\bar{a}_1^{-1}, \bar{a}_3^{-1}])$$

By Corollary 5.1, we have $\pi_0^{C_2}tmf_1(3)[\bar{a}_1^{-1}\bar{a}_3^{-1}] \cong \mathbb{Z}\left[\frac{1}{3}\right][(\bar{a}_1^3\bar{a}_3^{-1})^{\pm 1}]$. Thus,

$$GL_1\pi_0^{C_2}tmf_1(3)[\bar{a}_1^{-1}\bar{a}_3^{-1}] \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2,$$

generated by $\frac{1}{3}$, $\bar{a}_1^3\bar{a}_3^{-1}$ and -1 and $\mathrm{coker}(f) \cong \mathbb{Z}$ generated by $[\bar{a}_1^3\bar{a}_3^{-1}]$.

We claim that $\partial(\bar{a}_1^3\bar{a}_3^{-1}) \simeq S^{3\rho} \wedge Tmf_1(3)$. Indeed, we have trivializations

$$\bar{a}_3\colon\ S^{3\rho} \wedge tmf_1(3)[\bar{a}_3^{-1}] \to tmf_1(3)[\bar{a}_3^{-1}]$$

and

$$\bar{a}_1^3\colon\ S^{3\rho} \wedge tmf_1(3)[\bar{a}_1^{-1}] \to tmf_1(3)[\bar{a}_1^{-1}].$$

Thus, we get $S^{3\rho} \wedge Tmf_1(3)$ by gluing $tmf_1(3)[\bar{a}_3^{-1}]$ and $tmf_1(3)[\bar{a}_1^{-1}]$ by the map $\bar{a}_1^3\bar{a}_3^{-1}$ on $tmf_1(3)[\bar{a}_1^{-1}\bar{a}_3^{-1}]$.

By Theorem 6.8, $\ker(g) \cong \mathbb{Z}/48$. Furthermore, $\Sigma^{8-8\sigma}Tmf_1(3) \simeq_{C_2} Tmf_1(3)$ as $u_{2\sigma}^4$ is a permanent cycle. Thus, we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "3\rho"] \arrow[d, "\cong"] & RO(C_2)/(8-8\sigma) \arrow[r] \arrow[d] & RO(C_2)/(8-8\sigma, 3\rho) \cong \mathbb{Z}/48 \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \operatorname{coker}(f) \arrow[r] & \operatorname{Pic}_{C_2}(Tmf_1(3)) \arrow[r] & \ker(g) \arrow[r] & 0
\end{tikzcd}$$

Thus, the middle map is also an isomorphism. □

Remark 6.11. Using the technique of the last proof, it seems plausible to extend Question 6.9 to certain derived stacks whose underlying stack is regular and that have a C_2-action.

Remark 6.12. The map $\operatorname{Pic}(Tmf) \to \operatorname{Pic}(Tmf_0(3))$ is *not* surjective. The former has been identified with $\mathbb{Z} \oplus \mathbb{Z}/24$ in [MS14, Theorem B and Construction 8.4.2], where the summands are generated by ΣTmf and by the global sections $\mathcal{O}^{top}(\mathcal{I})$. Here, $\mathcal{I}$ is a line bundle on the derived compactified moduli stack of elliptic curves $(\overline{\mathcal{M}}_{ell}, \mathcal{O}^{top})$ obtain by gluing $\Sigma^{24}\mathcal{O}^{top}$ on $\mathcal{M}_{ell}$ and $\Sigma^{24}\mathcal{O}^{top}$ on $\overline{\mathcal{M}}_{ell}[c_4^{-1}]$ via the clutching function $j = \frac{c_4^3}{\Delta}$.

We claim that the module $\mathcal{O}^{top}(\mathcal{I}) \wedge_{Tmf} Tmf_0(3)$ is 2-torsion in $\operatorname{Pic}(Tmf_0(3))$. Indeed, for $p\colon \overline{\mathcal{M}}_0(3) \to \overline{\mathcal{M}}_{ell}$, we have for an arbitrary locally-free sheaf $\mathcal{F}$ on $(\overline{\mathcal{M}}, \mathcal{O}^{top})$ an equivalence

$$\begin{aligned}
\Gamma(\mathcal{F}) \wedge_{Tmf} Tmf_0(3) &\simeq \Gamma(\overline{\mathcal{M}}_{ell}; \mathcal{F} \wedge_{\mathcal{O}^{top}} p_*\mathcal{O}^{top}_{\overline{\mathcal{M}}_0(3)}) \\
&\simeq \Gamma(\overline{\mathcal{M}}_{ell}; p_*(p^*\mathcal{F} \wedge_{\mathcal{O}^{top}_{\overline{\mathcal{M}}_0(3)}} \mathcal{O}^{top}_{\overline{\mathcal{M}}_0(3)}) \\
&\simeq \Gamma(\overline{\mathcal{M}}_0(3); p^*\mathcal{F})
\end{aligned}$$

In the first equivalence, we use that $(\overline{\mathcal{M}}, \mathcal{O}^{top})$ is 0-affine and in the second we use the projection formula. Thus, $\mathcal{O}^{top}(\mathcal{I}) \wedge_{Tmf} Tmf_0(3)$ can be constructed as $\mathcal{O}^{top}(p^*\mathcal{I})$, where $p^*\mathcal{I}$ can be constructed by an analogous gluing construction on $\overline{\mathcal{M}}_0(3)$, gluing $\Sigma^{24}\mathcal{O}^{top}$ on $\mathcal{M}_0(3)$ and $\Sigma^{24}\mathcal{O}^{top}$ on $\overline{\mathcal{M}}_0(3)[c_4^{-1}]$ via the clutching function $j = \frac{c_4^3}{\Delta}$ with $c_4 = a_1^4 - 24a_1a_3$. There is an equivalence of gluing data

$$(\mathcal{O}^{top}, \mathcal{O}^{top}, \mathrm{id}) \to (\Sigma^{48}\mathcal{O}^{top}, \Sigma^{48}\mathcal{O}^{top}, j^2)$$

given by $\Delta^2\colon \mathcal{O}^{top} \to \mathcal{O}^{top}$ on $\mathcal{M}_0(3)$ and $c_4^6\colon \mathcal{O}^{top} \to \Sigma^{48}\mathcal{O}^{top}$ on $\overline{\mathcal{M}}_0(3)[c_4^{-1}]$. Note here that Δ^2 is a permanent cycle for $TMF_0(3)$. Thus, $2 \cdot [\mathcal{I}] = 0 \in \operatorname{Pic}(\overline{\mathcal{M}}_0(3), \mathcal{O}^{top}) \cong \operatorname{Pic}(Tmf_0(3))$.

As not every torsion in $\operatorname{Pic}(Tmf_0(3))$ is 2-torsion, $\operatorname{Pic}(Tmf) \to \operatorname{Pic}(Tmf_0(3))$ is indeed not surjective.

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