

Topological modular forms with level structure

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Abstract The cohomology theory known as Tmf , for “topological modular forms,” is a universal object mapping out to elliptic cohomology theories, and its coefficient ring is closely connected to the classical ring of modular forms. We extend this to a functorial family of objects corresponding to elliptic curves with level structure and modular forms on them. Along the way, we produce a natural way to restrict to the cusps, providing multiplicative maps from Tmf with level structure to forms of K -theory. In particular, this allows us to construct a connective spectrum $tmf_0(3)$ consistent with properties suggested by Mahowald and Rezk. This is accomplished using the machinery of logarithmic structures. We construct a presheaf of locally even-periodic elliptic cohomology theories, equipped with highly structured multiplication, on the log-étale site of the moduli of elliptic curves. Evaluating this presheaf on modular curves produces Tmf with level structure.

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1 Introduction

The subject of topological modular forms traces its origin back to the Witten genus. The Witten genus is a function taking String manifolds and producing elements of the power series ring $\mathbb{C}[[q]]$, in a manner preserving multiplication and bordism classes (making it a genus of String manifolds). It can therefore be described in terms of a ring homomorphism from the bordism ring $MO\langle 8 \rangle_*$ to this ring of power series. Moreover, Witten explained that this should factor through a map taking values in a particular subring MF_* : the ring of modular forms.

An algebraic perspective on modular forms is that they are universal functions on elliptic curves. Given a ring R , an elliptic curve \mathcal{E} over R , and a nonvanishing invariant 1-form ω on \mathcal{E} , a modular form g assigns an invariant $g(\mathcal{E}, \omega) \in R$. This is required to respect base change for maps $R \rightarrow R'$ and be invariant under isomorphisms of elliptic curves $\mathcal{E} \rightarrow \mathcal{E}'$. The form is of weight k if it satisfies $g(\mathcal{E}, \lambda\omega) = \lambda^{-k}g(\mathcal{E}, \omega)$. Modular forms can be added and multiplied, making them into a graded ring.

Thus, given a graded ring E_* , we might have two pieces of data.

First, this ring may arise as the coefficient ring of a complex orientable cohomology theory E . This gives E a theory of Chern classes, and from this we construct a formal group \mathbb{G}_E on E_* [44].

Second, the ring E_* may carry an elliptic curve \mathcal{E} with an invariant 1-form ω in a manner appropriately compatible with the grading. This elliptic curve also produces a formal group $\widehat{\mathcal{E}}$ on E_* .

The cohomology theory becomes an *elliptic cohomology theory* if we also specify an isomorphism $\mathbb{G}_E \rightarrow \widehat{\mathcal{E}}$ [2]. The defining properties of modular forms automatically produce a map of graded rings $MF_* \rightarrow E_*$. Witten's work then suggests a topological lift: elliptic cohomology theories should possess a map $MO\langle 8 \rangle_* \rightarrow E_*$. We can ask if this comes from a natural map $MO\langle 8 \rangle \rightarrow E$, and further if the factorization $MO\langle 8 \rangle_* \rightarrow MF_* \rightarrow E_*$ has a natural topological lift. The subjects of elliptic cohomology and topological modular forms were spurred by these developments [19, 28, 32].

However, within this general framework there are several related objects that have been interchangeably described by these names.

The original definitions of elliptic cohomology ($\mathcal{E}ll$) and of topological modular forms (tmf) produced ring spectra, generating multiplicative cohomology theories, that enjoy universal properties for maps to certain elliptic cohomology theories.

Precisely describing this universality leads to a more powerful description as a limit. One would like to take a ring E_* equipped with an elliptic curve \mathcal{E} and solve a realization problem, functorially, to produce a spectrum representing a corresponding elliptic cohomology theory [17]. These representing objects

are called elliptic spectra. After producing a sufficiently large diagram of such elliptic spectra, the homotopy limit is a ring spectrum of topological modular forms.

Even further, these functorial elliptic spectra satisfy a patching property: many limit diagrams of rings equipped with elliptic curves are translated into homotopy limit diagrams of elliptic spectra. This grants one the power to extend to a larger functor: a presheaf of spectra on a moduli stack of elliptic curves. The homotopy limit property is encoded by fibrancy in a Jardine model structure.

There are several versions of this story. The division depends on whether one allows only elliptic spectra corresponding to smooth elliptic curves (represented by the Deligne–Mumford stack \mathcal{M}_{ell}), elliptic curves with possible nodal singularities (represented by the compactification $\overline{\mathcal{M}}_{ell}$), or general cubic curves (represented by an algebraic stack \mathcal{M}_{cub}). These are sometimes informally given the names TMF , Tmf , and tmf respectively, and they represent a progressive decrease in our ability to obtain conceptual interpretations or construct objects. The stack \mathcal{M}_{ell} can be extended to a derived stack representing derived elliptic curves [34]; the so-called “old” construction of topological modular forms due to Goerss–Hopkins–Miller, by obstruction theory, gives the étale site of $\overline{\mathcal{M}}_{ell}$ a presheaf of elliptic spectra [6]; the less-conceptual process of taking a connective cover of Tmf produces a single spectrum tmf . The generalizations of tmf that exist have exceptionally interesting properties, but are constructed in a somewhat ad-hoc manner (however, see [36]).

There are many situations where extra functoriality for elliptic spectra can be a great advantage [7, 37, 51]. In particular, considering elliptic curves equipped with extra structure, such as choices of subgroups or torsion points, leads to a family of generalizations of \mathcal{M}_{ell} . The corresponding modular curves $\mathcal{M}(n)$, $\mathcal{M}_0(n)$, $\mathcal{M}_1(n)$, and more are extensively studied from the points of view of number theory and arithmetic geometry. Away from primes dividing n , these automatically inherit presheaves of elliptic spectra from TMF . For example, the maps $\mathcal{M}(n)[1/n] \rightarrow \mathcal{M}_{ell}[1/n]$ are étale covers with Galois group $\text{GL}_2(\mathbb{Z}/n)$, and so the étale presheaf defined on \mathcal{M}_{ell} can be evaluated on $\mathcal{M}(n)[1/n]$ to produce a spectrum $\text{TMF}(n)$ with a $\text{GL}_2(\mathbb{Z}/n)$ -action, and the homotopy fixed-point spectrum is $\text{TMF}[1/n]$.

However, the compactifications $\overline{\mathcal{M}}(n)$ are not étale over $\overline{\mathcal{M}}_{ell}$. In complex-analytic terms, if $\overline{\mathcal{M}}_{ell}$ is described in terms of a coordinate τ on the upper half-plane, then the compactification point or “cusp” has a coordinate $q = e^{2\pi i\tau}$; the cusps in $\overline{\mathcal{M}}(n)$ have coordinates expressed in terms of $q' = e^{2\pi i\tau/n}$, satisfying $q = (q')^n$. Since the coefficient $(q')^{n-1}$ in the expression $dq = n(q')^{n-1}dq'$ is not a unit, the map is not an isomorphism on cotangent spaces over the cusp

of $\overline{\mathcal{M}}_{ell}$. This ramification complicates the key input to the Goerss–Hopkins–Miller obstruction theory.

The obstruction theory constructing Tmf does generalize to each individual moduli stack, provided that if one imposes level structure at n one must first invert n . This has been carried out in some instances to construct objects $\mathrm{Tmf}(n)$ [51]. However, needing to reconstruct Tmf once per level structure is less than satisfying, and does not provide any functoriality across different forms of level structure. Moreover, it does not give an immediate reason why one might expect a relationship between Tmf and the homotopy fixed-point object for the action of $\mathrm{GL}_2(\mathbb{Z}/n)$ on $\mathrm{Tmf}(n)$.

However, these ramified maps $\overline{\mathcal{M}}(n) \rightarrow \overline{\mathcal{M}}_{ell}$ do possess a slightly less restrictive form of regularity. The cusp determines a “logarithmic structure” on $\overline{\mathcal{M}}_{ell}$ in the sense of Kato (Sect. 2), and the various maps between moduli of elliptic curves are log-étale—this roughly expresses the fact that in the expression $\mathrm{dlog}(q^n) = n \cdot \mathrm{dlog}(q)$, the coefficient is a unit away from primes dividing n . These ramified covers form part of a log-étale site enlarging the étale site of $\overline{\mathcal{M}}_{ell}$. Moreover, for the log schemes we will be considering, the fiber product in an appropriate category of objects with logarithmic structure is geared so that the Čech nerve of the cover $\overline{\mathcal{M}}_{ell}(n) \rightarrow \overline{\mathcal{M}}_{ell}$ is the simplicial bar construction for the action of $\mathrm{GL}_2(\mathbb{Z}/n)$ on $\overline{\mathcal{M}}_{ell}(n)$ (see 3.19).

These precise properties turn out to be useful from the point of view of extending functoriality for topological modular forms. In this paper, we use them to establish the existence of topological modular forms spectra for all modular curves (Theorem 6.1), as follows.

For a fixed integer N and a subgroup $\Gamma < \mathrm{GL}_2(\mathbb{Z}/N)$ we will construct an E_∞ ring spectrum $\mathrm{Tmf}(\Gamma)$, with N a unit in π_0 . It has three types of functoriality:

- inclusions $\Gamma < \Gamma'$ produce maps $\mathrm{Tmf}(\Gamma') \rightarrow \mathrm{Tmf}(\Gamma)$,
- elements $g \in \mathrm{GL}_2(\mathbb{Z}/N)$ produce maps $\mathrm{Tmf}(\Gamma) \rightarrow \mathrm{Tmf}(g\Gamma g^{-1})$, and
- projections $p: \mathrm{GL}_2(\mathbb{Z}/NM) \rightarrow \mathrm{GL}_2(\mathbb{Z}/N)$ produce natural equivalences $M^{-1}\mathrm{Tmf}(\Gamma) \rightarrow \mathrm{Tmf}(p^{-1}\Gamma)$.

These obey straightforward compatibility relations. In addition, for $K \triangleleft \Gamma < \mathrm{GL}_2(\mathbb{Z}/N)$ we find that the natural map

$$\mathrm{Tmf}(\Gamma) \rightarrow \mathrm{Tmf}(K)^{h\Gamma/K}$$

is an equivalence to the homotopy fixed-point spectrum.

(The reader who expects to see PSL_2 rather than GL_2 when discussing modular curves should be aware that this is a difference between the complex-analytic theory and the theory over \mathbb{Z} . These modular curves may have several components when considered over \mathbb{C} , and this may show up in the form of a larger ring of constants. Two modular curves may have the same set of points

over \mathbb{C} because the group $\{\pm 1\}$ acts trivially on isomorphism classes, but this action is nontrivial on modular forms and on the associated spectra).

Our main theorems are consequences of our extending Tmf to a fibrant presheaf of E_∞ ring spectra on the log-étale site of $\overline{\mathcal{M}}_{ell}$ (Theorem 5.17). In rough:

- there is an assignment of E_∞ ring spectra to certain generalized elliptic curves $\mathcal{E} \rightarrow X$ when the scheme X is equipped with a compatible logarithmic structure;
- this is functorial in certain diagrams

$$\begin{array}{ccc} \mathcal{E}' & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

such that the map $\mathcal{E}' \rightarrow \mathcal{E} \times_X X'$ is an isomorphism of elliptic curves over X' ;

- this functor satisfies descent with respect to log-étale covers $\{U_\alpha \rightarrow X\}$ (or hypercovers), in the sense that the value on X is the homotopy limit of the values on the Čech nerve; and
- in the special case where $\mathcal{E} \rightarrow X$ comes from a Weierstrass curve over $\mathrm{Spec}(R)$, the associated spectrum realizes it by an even-periodic elliptic cohomology theory E . (For more general elliptic curves on $\mathrm{Spec}(R)$, it is possible to show that the functor produces a weakly even-periodic object, and the formal group scheme $\mathrm{Spf} E^*(\mathbb{C}\mathbb{P}^\infty)$ will come equipped with a natural isomorphism to the formal group of \mathcal{E}).

The specific property of \mathcal{E}/X needed is that the resulting map $X \rightarrow \overline{\mathcal{M}}_{ell}$ classifying it must be log-étale. In particular, any object étale over a modular curve satisfies this property, and so the theorem simultaneously produces compatible presheaves of E_∞ ring spectra for all the modular curves.

The forms of K -theory from [31, Appendix A] play an important role in our construction, and allow us to generalize the main result of that paper. A generalized elliptic curve $\mathcal{E} \rightarrow X$ has a restriction to the “cusps” $X^c \subset X$. The restriction of \mathcal{E} to X^c is essentially a form of the multiplicative group \mathbb{G}_m , and there is a corresponding form of K -theory [39]. Our proof produces a natural map of E_∞ ring spectra from our functorial elliptic cohomology theory built from X to this form of K -theory built from X^c (Theorem 6.2).

We can then apply this to construct spectra $\mathrm{tmf}_1(3)$ and $\mathrm{tmf}_0(3)$, connective E_∞ ring spectra that realize calculations carried out by Mahowald and Rezk [40], together with a commutative diagram of E_∞ ring spectra

$$\begin{array}{ccc}
 \mathrm{tmf}_0(3) & \longrightarrow & ko[1/3] \\
 \downarrow & & \downarrow \\
 \mathrm{tmf}_1(3) & \longrightarrow & ku[1/3]
 \end{array}$$

(Theorem 6.3). In particular, this E_∞ connective version seems likely to coincide with one of their conjectural models. Carrying out calculations with further level structures should be a very interesting and relatively accessible problem.

There are several directions for further investigation and desirable generalizations of the results of this paper.

One obvious missing component is the connection to elliptic genera. The work of Ando, Hopkins, Rezk, and Strickland produced highly multiplicative lifts of the sigma orientation and the Atiyah–Bott–Shapiro orientation [1, 3], ultimately in the form of E_∞ maps $MO\langle 8 \rangle \rightarrow \mathrm{Tmf}$ and $MO\langle 4 \rangle \rightarrow KO$.

Question 1.1 Does the Ochanine genus of Spin manifolds [42] lift to an E_∞ ring map $MSpin \rightarrow \mathrm{Tmf}_0(2)$?

More generally, it would be very useful to know which objects $\mathrm{Tmf}(\Gamma)$ (such as $\mathrm{Tmf}_1(3)$, which is complex orientable) accept refinements of the sigma orientation to maps from $MSpin$, MSO , MU , or others.

Rognes has recently developed a closely related concept of topological logarithmic structures for applications in algebraic K -theory [45]. The core construction in this paper is built on a map of E_∞ spaces from \mathbb{N} to the multiplicative monoid of Tmf after completion at the cusp, modeling the logarithmic structure of $\overline{\mathcal{M}}_{ell}$ at the cusp. This ultimately imparts a topological logarithmic structure to the completion $KO[[q]]$.

Question 1.2 Does the topological logarithmic structure on $KO[[q]]$ extend to a topological logarithmic structure on Tmf ?

If so, it is natural to suspect that logarithmic obstruction theory is a natural way to construct our presheaf. There does not yet seem to be an obstruction theory for maps between ring spectra with logarithmic structures in the literature that is developed enough to carry out this program, but this is almost entirely due to how recently topological logarithmic structures have appeared.

Currently, Tmf is a functor defined on certain pairs of a scheme together with an elliptic curve, with maps defined for change-of-base and for isomorphisms of elliptic curves. However, separable isogenies of elliptic curves are further maps that produce isomorphisms of formal group laws.

Question 1.3 Does the functor Tmf extend to a presheaf on the moduli object classifying elliptic curves and separable isogenies?

After inverting ℓ , this would allow the construction of two maps $\mathrm{Tmf} \rightrightarrows \mathrm{Tmf}_0(\ell)$ classifying the two canonical isogenous curves over $\mathrm{Tmf}_0(\ell)$, extend to the construction of global versions of Behrens' $Q(\ell)$ spectra [7], and allow an “adelic” formulation of the functoriality of Tmf . However, obstruction theory seems to be an inadequate tool for this job. For example, this would require constructing an action of \mathbb{N}^k on $N^{-1}\mathrm{Tmf}$, in the form of a commuting family of Adams operations $[\ell]$ for primes ℓ dividing N . Since this obstruction theory is not in the category of Tmf -algebras, the relevant obstruction groups are not zero. We hope that a consequence of Lurie's constructive methods for associating spectra to p -divisible groups (as employed in [8]) will be that the smooth object TMF becomes functorial in separable isogenies, and that the construction patching over the cusp in this paper will inherit these isogeny operations from compatibility with the Adams operations on complex K -theory. We have chosen to write this paper without appealing to Lurie's forthcoming work.

Some PEL Shimura varieties at higher heights, such as Picard modular surfaces at height 3, have compactifications (such as Satake–Baily–Borel compactifications and smooth compactifications) that generalize the compactification of the moduli of elliptic curves [14].

Question 1.4 Are there compactifications of other PEL Shimura varieties with presheaves of spectra that extend the known presheaves of topological automorphic forms on the interior?

Finally, the construction of the object tmf by connective cover remains wholly unsatisfactory, and this is even more true when considering level structure. In an ideal world, tmf should be a functor on a category of Weierstrass curves equipped with some form of extra structure. We await the enlightenment following discovery of what exact form this structure should take.

1.1 Outline of the method

In order to minimize the amount of repeated effort, our proof is based heavily on the tools used to construct topological modular forms in [6]. The original method of Hopkins–Miller for the construction of Tmf started with a construction away from the supersingular locus, a construction at the supersingular locus, and an argument in obstruction theory for patching the two together as presheaves. Though the cusp played no special role in their construction, it will need to in this case. Our work follows a similar paradigm to that of Hopkins–Miller: we start with their construction away from the cusp, construct Tmf on the log site explicitly in a formal neighborhood of the cusp, and then patch the two constructions together.

In Sect. 2 we give the necessary background on logarithmic geometry that we need for this paper. Since the objects of importance to us will often be

Deligne–Mumford stacks rather than schemes, we need to express the theory in this generality.

In Sect. 3 we have only two main goals. The first goal is to discuss the moduli objects in question. Coarsely, they parametrize log schemes equipped with an elliptic curve and some data expressing compatibility of the logarithmic structure with the cusps. The second goal is to show that producing our desired derived structure presheaf \mathcal{O} is equivalent to defining functorial elliptic cohomology theories on a much smaller category: log rings equipped with suitable Weierstrass curves. This requires delving into some details about Grothendieck topologies. Our approach to this might be described as utilitarian.

In Sect. 4, we assemble a number of fundamental tools we will need from homotopy theory. The topics include elliptic spectra, obstruction theory, rectification of diagrams where the mapping spaces are homotopy discrete, homotopical versions of sheafification, and background on K -theory and tmf .

We may then begin to construct our presheaf \mathcal{O} in Sect. 5.

In Sect. 5.1, we first use real K -theory to produce a structure presheaf for log-étale maps to a formal neighborhood \mathcal{M}_{Tate} of the cusp, classifying forms of the Tate curve. We start with Tate K -theory, an elliptic cohomology theory with $\mathbb{Z}/2$ -action called $K[[q]]$ that was studied in [2]. We then extend it to a presheaf for log-étale maps to \mathcal{M}_{Tate} by direct construction. In this section we also construct a natural map to forms of K -theory, corresponding to evaluating at the cusps.

In Sect. 5.2, we glue the p -completed structure presheaves on the smooth locus and the cusps. The basic construction at the cusps starts with a map $\mathrm{tmf} \rightarrow KO[[q]]$, an E_∞ map factoring the Witten genus, which is constructed in Appendix A. When we localize, it produces a map $\mathrm{TMF} \rightarrow q^{-1}KO[[q]]$, and degeneration of the Goerss–Hopkins obstruction theory for Tmf -algebras allows us to extend to a map of presheaves.

Finally, Sect. 5.3 uses an arithmetic square to glue the p -complete constructions together with the rational constructions to produce a global, integral, version.

This assembles all the pieces necessary to construct Tmf with level structure in Sect. 6.

2 Logarithmic geometry

2.1 Logarithmic structures

Our primary reference for logarithmic structures is [22], supplemented by [26, 43, 45]. Since our ultimate objects of study will be moduli of elliptic curves, we must also discuss this in the stack case.

Throughout this paper we will assume that monoids have units, maps of monoids are unital, and Deligne–Mumford stacks are separated.

Definition 2.1 A prelogarithmic structure on a scheme or Deligne–Mumford stack X is an étale sheaf of commutative monoids M_X on X , together with a map $\alpha: M_X \rightarrow \mathcal{O}_X$ of sheaves of commutative monoids. (Here \mathcal{O}_X is viewed as a sheaf of monoids under multiplication.) We refer to (X, M_X) as a prelog scheme or prelog stack respectively.

A map of prelog schemes or prelog stacks $(Y, M_Y) \rightarrow (X, M_X)$ is a map $Y \rightarrow X$ together with a commutative diagram

$$\begin{array}{ccc} M_X & \longrightarrow & \mathcal{O}_X \\ \downarrow & & \downarrow \\ f_*M_Y & \longrightarrow & f_*\mathcal{O}_Y \end{array}$$

of sheaves of abelian monoids on X .

Definition 2.2 A prelogarithmic structure $\alpha: M_X \rightarrow \mathcal{O}_X$ is a logarithmic structure if the map α restricts to an isomorphism of sheaves $\alpha^{-1}(\mathcal{O}_X^\times) \rightarrow \mathcal{O}_X^\times$. In this case we refer to (X, M_X) as a log scheme or log stack. A map of log schemes or log stacks is a map of the underlying prelog schemes or stacks.

Definition 2.3 If X has a prelogarithmic structure $\alpha: M_X \rightarrow \mathcal{O}_X$, the associated logarithmic structure $(M_X)^a$ is the pushout of étale sheaves of commutative monoids

$$\begin{array}{ccc} \alpha^{-1}(\mathcal{O}_X^\times) & \longrightarrow & M_X \\ \downarrow & & \downarrow \\ \mathcal{O}_X^\times & \longrightarrow & (M_X)^a, \end{array}$$

equipped with the map $(M_X)^a \rightarrow \mathcal{O}_X$ determined by the universal property.

This logification functor $(X, M_X) \mapsto (X, (M_X)^a)$ is right adjoint to the forgetful functor from log stacks to prelog stacks.

Definition 2.4 If P is a commutative monoid with a map of commutative monoids $P \rightarrow \mathcal{O}_X(X)$, we abuse notation by writing P^a for the logarithmic structure associated to the map from the constant sheaf P on X to \mathcal{O}_X . The induced map $X \rightarrow \text{Spec}(\mathbb{Z}[P])$ then lifts to a map of log schemes.

In particular, if π is a global section of \mathcal{O}_X , there is a unique map of commutative monoids $\mathbb{N} \rightarrow \mathcal{O}_X(X)$ sending the generator to π . This is a prelog structure on X , and we write $\langle \pi \rangle$ for the associated log structure $(X, \langle \pi \rangle)$.

Definition 2.5 Suppose $j : U \subset X$ is an open subobject and U is equipped with a logarithmic structure M_U . Let $j_{\sharp}M_U$ be the pullback of the diagram of sheaves of commutative monoids on X :

$$\begin{array}{ccc} j_{\sharp}M_U & \longrightarrow & \mathcal{O}_X \\ \downarrow & & \downarrow \\ j_*M_U & \longrightarrow & j_*\mathcal{O}_U \end{array}$$

The direct image logarithmic structure on X is the associated logarithmic structure $(j_{\sharp}M_U)^a$.

In the particular case where M_U is the trivial log structure $(\mathcal{O}_U^{\times} \rightarrow \mathcal{O}_U)$ on U , this logarithmic structure consists of the submonoid of \mathcal{O}_X of functions whose restriction to U is invertible.

Definition 2.6 We say that (X, M_X) is:

- integral if each group-completion map on stalks $M_{X,x} \rightarrow (M_{X,x})^{gp}$ is an injective map of monoids;
- quasicoherent if there exists a cover of X by étale neighborhoods U with maps of commutative monoids $P_U \rightarrow M_X(U)$ such that the induced map $(P_U)^a \rightarrow M_X|_U$ is an isomorphism (we refer to such a pair (U, P_U) as a chart for the logarithmic structure on X);
- coherent if there exists a cover of X by charts (U, P_U) such that P_U is finitely generated;
- fine if it is integral and coherent;
- saturated if it is integral, and if M_X contains all sections s of $(M_X)^{gp}$ which have a (positive integer) multiple in M_X .

In this paper, we restrict attention to fine and saturated logarithmic structures. We note that (X, M_X) is fine and saturated if and only if there is a cover of X by charts (U, P_U) such that the monoid P_U itself is fine and saturated: P_U is finitely generated, the group completion map $P_U \rightarrow (P_U)^{gp}$ is injective, and an element in $(P_U)^{gp}$ has a positive multiple in the image of P_U if and only if it is in P_U .

We write **LogSch** for the category of fine and saturated log schemes, with associated forgetful functor **LogSch** \rightarrow **Sch** to the category of schemes. This functor has a fully faithful right adjoint **Sch** \rightarrow **LogSch**, which takes a scheme X and gives it the trivial log structure $(X, \mathcal{O}_X^{\times})$. We will implicitly view **Sch** as a full subcategory of **LogSch** via this map. Similarly, we have a full subcategory **AffLog** of affine log schemes, consisting of objects whose underlying scheme is affine.

2.2 Divisors with normal crossings

We recall the definition of a normal crossings divisor [48, 3.1.5] and give a definition in the stack context.

Definition 2.7 Let X be a scheme. A strict normal crossings divisor on X is a divisor D such that there exist sections $f_1, \dots, f_r \in \mathcal{O}_X(X)$ with the properties that

1. X is regular at all points in the support of D ,
2. $D = \sum \text{div}(f_i)$, and
3. for all x in the support of D , the set of elements f_i in the maximal ideal at x form part of a regular sequence of parameters for the local ring $\mathcal{O}_{X,x}$.

A normal crossings divisor on X is one for which there exists an étale cover by maps $U_\alpha \rightarrow X$ such that $D|_{U_\alpha}$ is a strict normal crossings divisor for all α .

A normal crossings divisor is smooth if we may take $r = 1$ in the above definition.

Definition 2.8 Let X be a Deligne–Mumford stack. A normal crossings divisor D on X is a map associating a normal crossings divisor D_Y to each scheme Y étale over X , compatible with pullbacks for morphisms over X . The complement $X \setminus D$ is the open substack whose restriction to each such Y is $Y \setminus D_Y$. The divisor D is smooth if all its restrictions D_Y are smooth.

Remark 2.9 Suppose $\{U_\alpha \rightarrow X\}$ is an étale cover of X by schemes. The data of a normal crossings divisor (resp. smooth divisor) on X is equivalent to normal crossings divisors (resp. smooth divisors) D_α on U_α such that, on any intersection $U_\alpha \times_X U_\beta$, we have $p_1^* D_\alpha = p_2^* D_\beta$. In particular, for stacks of the form $[Y//G]$ for Y a scheme with G -action, this is equivalent to a choice of G -invariant divisor on Y .

Remark 2.10 Suppose \mathcal{L} is a line bundle on X with a global section $s \in \Gamma(X, \mathcal{L})$ which induces an embedding $\mathcal{O}_X \rightarrow \mathcal{L}$. Tensoring with \mathcal{L}^{-1} , we obtain an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ which defines a compatible family of divisors (specifically, the divisor associated to the section s).

Definition 2.11 If D is a normal crossings divisor on X , then the associated log structure $M_D \rightarrow \mathcal{O}_X$ is the direct image of the trivial log structure on $X \setminus D$.

Proposition 2.12 *If D is a normal crossings divisor on X , then the associated log stack (X, M_D) is fine and saturated.*

Proof By definition, for sufficiently small étale opens $U \rightarrow X$ we have a sequence of sections $f_1, \dots, f_r \in \mathcal{O}_X(U)$ generating the divisor and forming

part of a regular sequence of parameters for the local rings in the support of D . These determine a map $\mathbb{N}^r \rightarrow M_D \subset \mathcal{O}_X(U)$ from the free commutative monoid on r generators, which is fine and saturated. Regularity implies that any function invertible away from D is locally and uniquely a unit times a product of the f_i , and so the resulting logarithmic structure is isomorphic to M_D . \square

2.3 Log-étale maps

We recall from Definition 2.6 that a chart (U, P) of a log stack (X, M_X) is a particular type of map $(U, P) \rightarrow (X, M_X)$ from a prelog scheme to a log stack: an étale map $U \rightarrow X$ and a commutative monoid P mapping to $\mathcal{O}_X(U)$ such that the associated map $P^a \rightarrow M_X|_U$ is an isomorphism. Similarly, a chart of a map $f: (Y, M_Y) \rightarrow (X, M_X)$ is a commutative diagram

$$\begin{CD} (V, Q) @>>> (Y, M_Y) \\ @VVV @VVV \\ (U, P) @>>> (X, M_X) \end{CD}$$

of prelog stacks whose horizontal arrows are charts; these are determined by maps $V \rightarrow U$ and $P \rightarrow Q$ making certain diagrams commute. If both logarithmic structures are fine, any map f has an étale cover by charts.

Definition 2.13 ([26, 3.3]) A map $(Y, M_Y) \rightarrow (X, M_X)$ is log-étale if the underlying map $Y \rightarrow X$ is locally of finite presentation, and there exists a cover of f by charts $(V, Q) \rightarrow (U, P)$ such that

1. the map $P \rightarrow Q$ becomes an injection $P^{gp} \rightarrow Q^{gp}$ whose cokernel has finite order invertible on U , and
2. the induced map $V \rightarrow U \times_{\text{Spec}(\mathbb{Z}[P])} \text{Spec}(\mathbb{Z}[Q])$ is étale.

Remark 2.14 As with ordinary étale maps, these maps can also be defined in terms of an infinitesimal lifting criterion or in terms of inducing an isomorphism on logarithmic cotangent complexes [26, 3.3, 3.12].

Remark 2.15 In particular, any map $P \rightarrow Q$ which induces an isomorphism $P^{gp} \rightarrow Q^{gp}$ produces log-étale maps $(\text{Spec}(k[Q]), Q^a) \rightarrow (\text{Spec}(k[P]), P^a)$. Even in the case of fine and saturated monoids, these can produce a wide variety of phenomena on the underlying schemes.

The Kummer log-étale maps are particularly well-behaved log-étale maps.

Definition 2.16 ([22, 1.6]) A map $h: P \rightarrow Q$ of commutative monoids is of Kummer type if it is injective, and for any $a \in Q$ there exists an $n \geq 1$ such that a^n is in the image of h .

A map $(Y, M_Y) \rightarrow (X, M_X)$ is Kummer log-étale if it is log-étale and, at each point y of Y , the induced map on stalks

$$(M_X/\mathcal{O}_X^*)_{X, \overline{f(y)}} \rightarrow (M_Y/\mathcal{O}_Y^*)_{Y, \overline{y}}$$

is of Kummer type.

Remark 2.17 In particular, if $f: (Y, M_Y) \rightarrow (X, M_X)$ can be covered by log-étale charts $(V, Q) \rightarrow (U, P)$ such that the map of monoids $P \rightarrow Q$ is of Kummer type, the map f is Kummer log-étale.

Proposition 2.18 *Suppose (X, M_X) is a log stack determined by a smooth divisor D on X . Then any log-étale map $(Y, M_Y) \rightarrow (X, M_X)$ is flat and Kummer log-étale, and the log structure on Y is determined by a smooth divisor on Y over D .*

Proof It suffices to work locally in the étale topology, so without loss of generality it suffices to prove this on a coordinate chart of the form $(U, \langle \pi \rangle) \rightarrow (X, M_X)$ for a scheme U , where we view π as a map $\mathbb{N} \rightarrow \mathcal{O}_X(U)$.

Consider the following data:

- a finitely generated commutative monoid Q which is integral and saturated;
- a map of monoids $\mathbb{N} \rightarrow Q$ such that, on group completion, the map $\mathbb{Z} \rightarrow Q^{gp}$ is an injection with cokernel of order $n < \infty$; and
- an étale map of schemes $V \rightarrow U \times_{\text{Spec}(\mathbb{Z}[\mathbb{N}])} \text{Spec}(\mathbb{Z}[Q][1/n])$.

In this situation, the monoid Q generates the log structure $Q^a = M_V$ on V . The associated map $(V, M_V) \rightarrow (U, \langle \pi \rangle)$ is called a standard log-étale map, and it is Kummer log-étale by definition.

By [26, 3.5], any log-étale map $(Y, M_Y) \rightarrow (U, \langle \pi \rangle)$ is, étale-locally on Y , a standard log-étale map.

The hypotheses on Q imply that $Q^{gp} \cong \mathbb{Z} \times F$ where $|F|$ divides n , and the saturation property implies that either $Q = \mathbb{N} \times F$ or $Q = \mathbb{Z} \times F$; both of these generate the same log structure as the submonoid $\mathbb{N} \times 0$.

We obtain a factorization of the map $V \rightarrow U$ as

$$V \rightarrow U \times_{\text{Spec}(\mathbb{Z}[\mathbb{N}])} \text{Spec}(\mathbb{Z}[Q][1/n]) \rightarrow U \times \text{Spec}(\mathbb{Z}[F][1/n]) \rightarrow U.$$

The first and last maps are étale, and so it suffices to show that the middle map is flat and that the log structure is generated by a smooth divisor. Letting $Z = U \times \text{Spec}(\mathbb{Z}[F][1/n])$, the center map is of the form

$$Z \times_{\text{Spec}(\mathbb{Z}[\pi])} \text{Spec}(\mathbb{Z}[(u\pi)^{1/d}]) \rightarrow Z$$

for some unit $u \in F$ and d dividing n . This map is, in particular, finite flat. The log structure on the cover is generated by $(u\pi)^{1/d}$. Since π describes a smooth divisor on Z the element $(u\pi)^{1/d}$ determines a smooth divisor on the cover: at any point of the support of D , any regular sequence on Z containing π lifts to a regular sequence on the cover once we replace π with $(u\pi)^{1/d}$. \square

In particular, this shows that there is a cofinal collection of log-étale covers of $(U, \langle \pi \rangle)$ consisting of étale covers of

$$(Y \times_{\text{Spec}(\mathbb{Z}[\pi])} \text{Spec}(\mathbb{Z}[\pi^{1/n}, 1/n]), \langle \pi^{1/n} \rangle).$$

In Sect. 5.1 we will make use of the special case of a power series ring.

Corollary 2.19 *Any log-étale map $(\mathbb{Z}[[q]], \langle q \rangle) \rightarrow (R, M)$, where R is a formal $\mathbb{Z}[[q]]$ -algebra, has an étale cover by maps*

$$(\mathbb{Z}[[q]], \langle q \rangle) \rightarrow (\mathbb{Z}[[q^{1/m}]], \langle q^{1/m} \rangle) \rightarrow (S, \langle q^{1/m} \rangle),$$

where S is an étale formal $\mathbb{Z}[[q^{1/m}]]$ algebra with m invertible.

Remark 2.20 We note that the integer m is recovered as a ramification index of the ideal (q) , and that the étale extension S of $\mathbb{Z}[[q^{1/m}]]$ is uniquely of the form $\bar{S}[[q^{1/m}]]$, where \bar{S} is the étale extension $S/(q^{1/m})$ of $\mathbb{Z}[1/m]$. In other words, such an extension is determined up to isomorphism by the ramification index and the residue extension, but isomorphisms may differ by a map $q^{1/m} \mapsto \zeta q^{1/m}$.

Remark 2.21 All of the conclusions of Proposition 2.18, except for the final one, remain true if we replace smooth divisors by normal crossings divisors. It is not necessarily the case that a normal crossings divisor locally lifts to a normal crossings divisor.

2.4 Logarithmic topologies

The Kummer log-étale maps give rise to a Kummer log-étale topology [22, §2]. Specifically, it is a Grothendieck topology generated by covers which are collections $\{(U_\beta, M_{U_\beta}) \rightarrow (X, M_X)\}$ of Kummer log-étale maps such that the underlying scheme X is the union of the images of U_β . The representable functors are sheaves for this topology, and quasicohherent sheaves on X automatically extend to sheaves on the Kummer log-étale site of X [41].

Proposition 2.18 has the following consequence which, as in Remark 2.21, generalizes to the case of a normal crossings divisor.

Proposition 2.22 *Suppose (X, M_X) is a log stack defined by a smooth divisor. Then the log-étale and Kummer log-étale topologies are equivalent.*

The Kummer site has the advantage that we can obtain control on cohomology.

Proposition 2.23 ([41, 3.27]) *Suppose (X, M_X) is an affine log scheme with a sheaf \mathcal{F} on its Kummer log-étale site which is locally quasicohherent. Then the cohomology of (X, M_X) with coefficients in \mathcal{F} vanishes in positive degrees.*

Remark 2.24 Quasicohherence is not a local property in the log-étale topology. Log-étale maps are very often not flat, which means that log-étale covers might not satisfy common types of descent.

From a standard hypercover argument relating derived functor cohomology to coherent cohomology, we obtain the following.

Corollary 2.25 *Suppose (X, M_X) is a log stack with a quasicohherent sheaf \mathcal{F} on X . Then the natural map from Zariski cohomology of X to the Kummer log-étale cohomology of (X, M_X) with coefficients in \mathcal{F} is an isomorphism.*

2.5 Log-étale covering maps

The following is analogous to the definition of a covering space.

Definition 2.26 (cf. [22, 3.1]) A map $(Y, M_Y) \rightarrow (X, M_X)$ of log stacks is a Kummer log-étale covering map if there is a cover of (X, M_X) by log schemes (X_β, M_β) , in the Kummer log-étale topology, such that each pullback of (Y, M_Y) in the category of fine and saturated log stacks is isomorphic to a finite union of copies of X_β .

Suppose a finite group G acts on (Y, M_Y) with quotient (X, M_X) . The map $(Y, M_Y) \rightarrow (X, M_X)$ is a (Kummer log-étale) Galois covering map if it is a Kummer log-étale covering map $(Y, M_Y) \rightarrow (X, M_X)$, and the shearing map

$$G \times (Y, M_Y) \rightarrow (Y, M_Y) \times_{(X, M_X)} (Y, M_Y),$$

expressed by $(g, y) \mapsto (gy, y)$, is an isomorphism. Here the source is the log scheme $\coprod_{g \in G} (Y, M_Y)$.

In the case of a normal crossings divisor, we have the following.

Proposition 2.27 *Suppose (X, M_X) is a log stack associated to a normal crossings divisor D , and that X is smooth away from D . Then there is an equivalence of categories between Kummer log-étale covering maps $(Y, M_Y) \rightarrow (X, M_X)$ and étale covering maps of $X \setminus D$ which are tamely ramified over D , given by*

$$(Y, M_Y) \mapsto Y \times_X (X \setminus D).$$

The inverse sends an étale cover $Y' \rightarrow X \setminus D$ to the normalization of X in Y' .

This induces an equivalence between Galois covers with group G and Kummer log-étale Galois covers of $Y \setminus D$ with group G which are tamely ramified over D .

Proof This combines results from [22, 7.3(b), 7.6]. On any log scheme which is an étale open of (X, M_X) , the given maps establish an equivalence of categories. As étale covers and log-étale covers satisfy étale descent, and tame ramification is an étale-local property, the result follows. \square

3 The elliptic moduli

3.1 The logarithmic moduli of elliptic curves

Let $\overline{\mathcal{M}}_{ell}$ be the compactified moduli of generalized elliptic curves, which is a Deligne–Mumford stack. The stack $\overline{\mathcal{M}}_{ell}$ contains a cusp divisor classifying elliptic curves with nodal singularities, and the complement is an open substack \mathcal{M}_{ell} classifying smooth elliptic curves.

Definition 3.1 The log stack $\overline{\mathcal{M}}_{log}$ is the stack $\overline{\mathcal{M}}_{ell}$ equipped with the direct image log structure from \mathcal{M}_{ell} (2.5).

We recall that the category $\mathbf{Sch}/\overline{\mathcal{M}}_{ell}$ is a stack in the fpqc topology on \mathbf{Sch} : maps $X \rightarrow \overline{\mathcal{M}}_{ell}$ classify generalized elliptic curves on X . In particular, $\mathbf{Sch}/\overline{\mathcal{M}}_{ell}$ is a category whose the objects are pairs (X, \mathcal{E}) consisting of a scheme X and a generalized elliptic curve $\mathcal{E} \rightarrow X$, and the morphisms are pullback diagrams

$$\begin{array}{ccc} \mathcal{E}' & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array}$$

respecting the chosen section of the elliptic curve. The functor which forgets \mathcal{E} makes this into a category fibered in groupoids over \mathbf{Sch} .

We denote by $\omega/\overline{\mathcal{M}}_{ell}$ the sheaf of invariant differentials on the universal elliptic curve; global sections of ω^k are modular forms of weight k . The group $\Gamma(\overline{\mathcal{M}}_{ell}, \omega^{12})$ is freely generated over \mathbb{Z} by the modular forms c_4^3 and Δ . These do not vanish simultaneously on $\overline{\mathcal{M}}_{ell}$ and so we can make the following definition.

Definition 3.2 The j -invariant

$$j: \overline{\mathcal{M}}_{ell} \rightarrow \mathbb{P}^1$$

is the map given in homogeneous coordinates by

$$\mathcal{E} \mapsto [c_4^3(\mathcal{E}) : \Delta(\mathcal{E})].$$

We first need a coordinate chart on $\overline{\mathcal{M}}_{ell}$ near the cusps.

Lemma 3.3 *The generalized elliptic curve*

$$y^2 + xy = x^3 - \frac{36}{j-1728}x - \frac{1}{j-1728}$$

defines an étale map $\mathcal{C}: \mathbb{P}^1 \setminus \{0, 1728\} \rightarrow \overline{\mathcal{M}}_{ell}$, covering the complement of the j -invariants 0 and 1728.

Proof This curve has the prescribed j -invariant j [50, III.1.4.c], and on $\text{Spec}(\mathbb{Z}[j^{-1}])$ it has only nodal singularities except when $(1728j^{-1} - 1) \equiv 0$. This curve then describes a map from $\text{Spec}(\mathbb{Z}[j^{-1}, (1728j^{-1} - 1)^{-1}])$ to $\overline{\mathcal{M}}_{ell}$ producing generalized curves of all j -invariants other than 0 and 1728.

Let $U \subset \overline{\mathcal{M}}_{ell}$ be the preimage of $\mathbb{P}^1 \setminus \{0, 1728\}$ under j . By [27, 8.4.3], there are no automorphisms of generalized elliptic curves classified by maps to U other than the identity and the negation automorphism; the automorphism group of the universal elliptic curve on U is the constant group $\mathbb{Z}/2$. As a result, for any elliptic curve $\mathcal{E}: X \rightarrow U$, the fiber product $X \times_U (\mathbb{P}^1 \setminus \{0, 1728\})$ classifying isomorphisms between \mathcal{E} and the pullback of \mathcal{C} is a principal $\mathbb{Z}/2$ -torsor $\text{Iso}_X(\mathcal{E}, \mathcal{C}(j(\mathcal{E})))$. In particular, it is étale over X . The map classifying \mathcal{C} is therefore étale and there is an equivalence of stacks

$$U \cong B\mathbb{Z}/2 \times (\mathbb{P}^1 \setminus \{0, 1728\}).$$

□

As in Remark 2.10, the section Δ of ω^{12} determines a divisor: the cusp of $\overline{\mathcal{M}}_{ell}$.

Corollary 3.4 *The cusp divisor of $\overline{\mathcal{M}}_{ell}$ is a smooth divisor (2.7).*

Proof In the coordinates of Lemma 3.3, the cusp divisor is the smooth divisor described by j^{-1} on an open subset of $\text{Spec}(\mathbb{Z}[j^{-1}])$. □

Proposition 2.12 then has the following consequence.

Proposition 3.5 *The log stack $\overline{\mathcal{M}}_{log}$ is fine and saturated.*

Definition 3.6 The log scheme \mathbb{P}^1_{log} is the log structure on \mathbb{P}^1 defined by the divisor (∞) .

Explicitly, the monoid sheaf $M \subset \mathcal{O}_{\mathbb{P}^1}$ consists of functions whose restriction to $\mathbb{P}^1 \setminus \{\infty\} = \text{Spec}(\mathbb{Z}[j])$ is invertible.

Remark 3.7 For any integer z , viewed as an integral point $z \in \mathbb{A}^1$, there is an isomorphism of log schemes

$$\mathbb{P}^1_{\log} \setminus \{z\} \cong (\text{Spec}(\mathbb{Z}[(j - z)^{-1}]), \langle (j - z)^{-1} \rangle).$$

Proposition 3.8 *The log stack $\overline{\mathcal{M}}_{\log}$ is the pullback*

$$\begin{array}{ccc} \overline{\mathcal{M}}_{\log} & \longrightarrow & \overline{\mathcal{M}}_{\text{ell}} \\ j_{\log} \downarrow & & \downarrow j \\ \mathbb{P}^1_{\log} & \longrightarrow & \mathbb{P}^1. \end{array}$$

In particular, a log scheme over $\overline{\mathcal{M}}_{\log}$ consists of a log scheme (X, M_X) , a generalized elliptic curve \mathcal{E} on X , and a lift of the j -invariant $j(\mathcal{E}): X \rightarrow \mathbb{P}^1$ to a map of log schemes $j_{\log}(\mathcal{E}): (X, M_X) \rightarrow \mathbb{P}^1_{\log}$.

Proof As the logarithmic structure is the trivial logarithmic structure on the complement of the cusps, it suffices to use the coordinate chart of Lemma 3.3 and show that there is a pullback diagram of fine and saturated log schemes

$$\begin{array}{ccc} (\text{Spec}(\mathbb{Z}[j^{-1}, (1728j^{-1} - 1)^{-1}]), \langle j^{-1} \rangle) & \longrightarrow & \text{Spec}(\mathbb{Z}[j^{-1}, (1728j^{-1} - 1)^{-1}]) \\ \downarrow & & \downarrow \\ \mathbb{P}^1_{\log} & \longrightarrow & \mathbb{P}^1. \end{array}$$

However, this is precisely the inclusion of a coordinate chart on the log scheme \mathbb{P}^1_{\log} . □

3.2 Weierstrass curves

We first recall that there is a ring

$$A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$$

parametrizing Weierstrass curves of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

This ring A contains modular quantities c_4 , c_6 , and Δ [50, III.1], and the complement of the common vanishing locus of c_4 and Δ is an open subscheme

$$\text{Spec}(A)^o = \text{Spec}(c_4^{-1}A) \cup \text{Spec}(\Delta^{-1}A) \tag{3.1}$$

parametrizing generalized elliptic curves in Weierstrass form. In particular, there is a universal generalized elliptic curve $\mathcal{E} \rightarrow \text{Spec}(A)^o$. This determines a map from $\text{Spec}(A)^o$ to the Deligne–Mumford stack $\overline{\mathcal{M}}_{ell}$ which is a smooth cover.

Similarly, there is a universal scheme parametrizing pairs of isomorphic Weierstrass curves. Let

$$\Gamma = A[r, s, t, \lambda^{\pm 1}]$$

be the ring parametrizing the change-of-coordinates $y \mapsto \lambda^3 y + rx + s, x \mapsto \lambda^2 x + t$ on Weierstrass curves. The pair (A, Γ) forms a Hopf algebroid with an invariant ideal (c_4, Δ) . Defining

$$\text{Spec}(\Gamma)^o = \text{Spec}(c_4^{-1}\Gamma) \cup \text{Spec}(\Delta^{-1}\Gamma),$$

we obtain a groupoid $(\text{Spec}(A)^o, \text{Spec}(\Gamma)^o)$ in schemes that maps naturally to $\overline{\mathcal{M}}_{ell}$.

As $\text{Spec}(\Gamma)^o$ parametrizes pairs of generalized elliptic curves in Weierstrass form with a chosen isomorphism between them, we have an equivalence to the pullback

$$\text{Spec}(\Gamma)^o \xrightarrow{\sim} \text{Spec}(A)^o \times_{\overline{\mathcal{M}}_{ell}} \text{Spec}(A)^o.$$

For Weierstrass curves, we note that the identity $j^{-1} = \Delta c_4^{-3}$ and the constraint that the units of M_X map isomorphically to \mathcal{O}_X^\times imply that a factorization $(X, M_X) \rightarrow \mathbb{P}^1_{log} \rightarrow \mathbb{P}^1$ of the j -invariant is equivalent to a lift of the element $\Delta \in \mathcal{O}_X(c_4^{-1}X)$ to a section of $M_X(c_4^{-1}X)$. We will casually refer to this as a lift of the elliptic discriminant to M_X .

The divisor defined by the vanishing of Δ determines a log structure on $\text{Spec}(A)^o$. The fiber product $\text{Spec}(A)^o \times_{\mathbb{P}^1} \mathbb{P}^1_{log}$ is, in fact, the log scheme

$$U_{log} = (\text{Spec}(A)^o, \langle \Delta \rangle).$$

This classifies the universal Weierstrass curve in log schemes.

The fiber product of $(\text{Spec}(A)^o, \langle \Delta \rangle)$ with itself over $\overline{\mathcal{M}}_{log}$ is then, by invariance of Δ up to unit,

$$R_{log} = (\text{Spec}(\Gamma)^o, \langle \Delta \rangle).$$

The pair (U_{log}, R_{log}) form a smooth groupoid object in **LogSch** that parametrizes the groupoid of Weierstrass curves with lifts of the log structure.

Proposition 3.9 *The natural map $(U_{log}, R_{log}) \rightarrow \overline{\mathcal{M}}_{log}$ of groupoids induces an equivalence of stacks in the log-étale or Kummer log-étale topologies.*

Proof The induced map of groupoids is fully faithful. In addition, any map $X \rightarrow \overline{\mathcal{M}}_{log}$ can be covered by maps which lift to U_{log} : in fact, any elliptic curve on a scheme is isomorphic to a Weierstrass curve locally in the Zariski topology. □

We remark that $\overline{\mathcal{M}}_{log}$ is merely a prestack, rather than a stack, in the topology on **LogSch** because elliptic curves do not obviously satisfy descent for log-étale covers.

Proposition 3.10 *The map $U_{log} \rightarrow \overline{\mathcal{M}}_{log}$ is representable and a smooth cover.*

Proof Given $(X, M_X) \rightarrow \overline{\mathcal{M}}_{log} \cong \overline{\mathcal{M}}_{ell} \times_{\mathbb{P}^1} \mathbb{P}^1_{log}$, the pullback is

$$(X, M_X) \times_{\overline{\mathcal{M}}_{ell} \times_{\mathbb{P}^1} \mathbb{P}^1_{log}} (\text{Spec}(A)^o \times_{\mathbb{P}^1} \mathbb{P}^1_{log}) \cong (X, M_X) \times_{\overline{\mathcal{M}}_{ell}} \text{Spec}(A)^o.$$

In particular, this follows from the fact that the map $\text{Spec}(A)^o \rightarrow \overline{\mathcal{M}}_{ell}$ is representable and smooth. □

As a result, within the Grothendieck topology on **LogSch**, (U_{log}, R_{log}) gives a presentation of the same stack as $\overline{\mathcal{M}}_{log}$.

Remark 3.11 This smooth cover can be refined to a Kummer log-étale cover. Away from the cusps, for example, there are schemes $\mathcal{M}_1(4)[1/2]$ and $\mathcal{M}_1(3)[1/3]$. These parametrize, respectively, smooth elliptic curves with a chosen 4-torsion point away from the prime 2, and smooth elliptic curves with a chosen 3-torsion point away from the prime 3 [27]. The coordinate chart of Lemma 3.3 gives an étale cover over the cusps.

The following result will be useful in understanding the cohomology rings of objects which are honestly étale over $\overline{\mathcal{M}}_{ell}$ (compare [6, construction of Diagram 9.2]).

Proposition 3.12 *If a map $\text{Spec}(R) \rightarrow \overline{\mathcal{M}}_{ell}$ classifying an elliptic curve \mathcal{E} is étale, then the map of graded rings*

$$\bigoplus_{k \in \mathbb{Z}} H^0(\overline{\mathcal{M}}_{ell}, \omega^k) \otimes \mathbb{Q} \rightarrow \bigoplus_{k \in \mathbb{Z}} H^0(\text{Spec}(R), \omega^k) \otimes \mathbb{Q}$$

is étale.

Proof Let $\text{Spec}(\mathbb{Q}[c_4, c_6])^o$ denote the open complement of the ideal defined by (c_4, Δ) in $\text{Spec}(\mathbb{Q}[c_4, c_6])$. We can form the pullback in the diagram

$$\begin{array}{ccccc}
 Y & \cdots \longrightarrow & \text{Spec}(\mathbb{Q}[c_4, c_6])^o & \longrightarrow & \text{Spec}(\mathbb{Q}[c_4, c_6]) \\
 \downarrow & & \downarrow \mathcal{D} & & \\
 \text{Spec}(R) & \xrightarrow{\mathcal{E}} & \overline{\mathcal{M}}_{ell} & &
 \end{array}$$

Here the center vertical map classifies the curve \mathcal{D} given by the Weierstrass equation $y^2 = x^3 - \frac{c_4}{48}x - \frac{c_6}{864}$, which induces the isomorphism $\bigoplus H^0(\overline{\mathcal{M}}_{ell}, \omega^k) \otimes \mathbb{Q} \rightarrow \mathbb{Q}[c_4, c_6]$. As the map classifying \mathcal{E} is étale, so is the composite upper map $Y \rightarrow \text{Spec}(\mathbb{Q}[c_4, c_6])$. Moreover, this map is equivariant for the action of \mathbb{G}_m by $c_4 \mapsto \lambda^4 c_4, c_6 \mapsto \lambda^6 c_6$.

However, the pullback Y is the universal object over $\text{Spec}(R)$ which is both rational and where the curve \mathcal{E}/R has a chosen isomorphism to \mathcal{D} . As \mathcal{D} has a chosen invariant 1-form, so does the associated elliptic curve on Y . However, away from the primes 2 and 3 any curve \mathcal{E} with a chosen nowhere-vanishing 1-form has a unique isomorphism to a curve $y^2 = x^3 + px + q$ preserving the 1-form.

Therefore, $Y \rightarrow \text{Spec}(R)$ is the object classifying choices of nowhere vanishing 1-form on $\mathcal{E}/(R \otimes \mathbb{Q})$, which is Spec of the ring $\bigoplus_{k \in \mathbb{Z}} H^0(R, \omega^k) \otimes \mathbb{Q}$. As a result, the map of \mathbb{G}_m -equivariant (i.e. graded) rings $\mathbb{Q}[c_4, c_6] \rightarrow \bigoplus_{k \in \mathbb{Z}} H^0(R, \omega^k) \otimes \mathbb{Q}$ is étale. \square

3.3 Log-étale objects over $\overline{\mathcal{M}}_{log}$

Proposition 3.10 allows us to study log-étale objects over $\overline{\mathcal{M}}_{log}$. As the log structure is determined by a smooth divisor by Corollary 3.4, the log-étale covers determine an equivalent topology to the Kummer log-étale topology. It suffices to check that a map $(X, M_X) \rightarrow \overline{\mathcal{M}}_{log}$ is log-étale on the cover defined by the moduli of Weierstrass curves (3.1). (Alternatively, one can check that the restriction to $\overline{\mathcal{M}}_{ell}$ is étale, and check that it is log-étale on the chart of Lemma 3.3).

For any log stack (X, M_X) over $\overline{\mathcal{M}}_{log}$ classifying an elliptic curve \mathcal{E}/X , form the pullback

$$\begin{array}{ccc}
 (X, M_X) \times_{\overline{\mathcal{M}}_{log}} U_{log} & \xrightarrow{p} & U_{log} \\
 \downarrow & & \downarrow \\
 (X, M_X) & \longrightarrow & \overline{\mathcal{M}}_{log}.
 \end{array}$$

This fiber product is universal among log stacks (Y, M_Y) equipped with a map $f: (Y, M_Y) \rightarrow (X, M_X)$ and an isomorphism of $f^*(\mathcal{E})$ with a Weierstrass curve. The object (X, M_X) is (Kummer) log-étale over $\overline{\mathcal{M}}_{log}$ if and only if the map p of log stacks is (Kummer) log-étale.

In particular, if the elliptic curve on X is smooth, then Δ is invertible and there are no restrictions on the logarithmic structure. For such a map to make (X, M_X) log-étale over $\overline{\mathcal{M}}_{log}$, the logarithmic structure on X must be trivial.

Definition 3.13 The small (Kummer) log-étale site of $\overline{\mathcal{M}}_{log}$ is the category of log schemes (X, M_X) equipped with a (Kummer) log-étale map $(X, M_X) \rightarrow \overline{\mathcal{M}}_{log}$, with maps being the (Kummer) log-étale maps over $\overline{\mathcal{M}}_{log}$.

The classical étale site of $\overline{\mathcal{M}}_{ell}$ has a fully faithful embedding into the small log-étale site of $\overline{\mathcal{M}}_{log}$, and so the latter is strictly an enlargement.

The affine examples of log stacks are *log rings*, determined by a ring R and an appropriate étale sheaf of commutative monoids M on $\text{Spec}(R)$. We now give some details about the data needed on a map $f: A \rightarrow R$, classifying a Weierstrass curve over $\text{Spec}(R)$, to get a map $\text{Spec}(R, M) \rightarrow \overline{\mathcal{M}}_{log}$, and when this map is log-étale.

The map f determines a generalized elliptic curve when the ideal $(f(c_4), f(\Delta))$ is the unit ideal of R . As in the previous section, a lift of this to a map of log schemes is a lift of Δ/c_4^3 to a section of M over $\text{Spec}(c_4^{-1}R)$; the fact that M is a logarithmic structure implies that this is equivalent to a lift of Δ to a section $\hat{\Delta}$ of M over $\text{Spec}(R)$.

We then have a composite pullback diagram

$$\begin{array}{ccccc}
 \text{Spec}(R \otimes_A \Gamma, (\eta_L)^*M) & \longrightarrow & R_{log} & \longrightarrow & U_{log} \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{Spec}(R, M) & \longrightarrow & U_{log} & \longrightarrow & \overline{\mathcal{M}}_{log}.
 \end{array}$$

We find that the composite of the lower maps is log-étale if and only if the resulting map

$$(A, \langle \Delta \rangle) \rightarrow (R \otimes_A \Gamma, (\eta_L)^*M),$$

induced by the right unit of Γ , is log-étale. Here $(\eta_L)^*M$ is the pullback logarithmic structure.

In particular, an object $\text{Spec}(R, M)$ can be log-étale over $\overline{\mathcal{M}}_{log}$ only if R is Noetherian. In this case, the Artin–Rees lemma gives us the following.

Proposition 3.14 *For a log ring $\text{Spec}(R, M)$ log-étale over $\overline{\mathcal{M}}_{log}$ and a finitely-generated R -module N , the diagram*

$$\begin{array}{ccc}
 N & \longrightarrow & N_{\Delta}^{\wedge} \\
 \downarrow & & \downarrow \\
 \Delta^{-1}N & \longrightarrow & \Delta^{-1}N_{\Delta}^{\wedge}
 \end{array}$$

is both cartesian and cocartesian in R -modules.

In particular, taking $N = R$ tells us that an object of the log-étale site has its underlying scheme completely determined by an object $\Delta^{-1}X \rightarrow \mathcal{M}_{ell}$ étale over the moduli of smooth elliptic curves, an object $(X_{\Delta}^{\wedge}, M_X)$ over the completion at the cusp, and a patching map $\Delta^{-1}X_{\Delta}^{\wedge} \rightarrow \Delta^{-1}X$ over \mathcal{M}_{ell} .

3.4 The Tate curve

For references for the following material, we refer the reader to [13, VII] or [4, Sect. 2.3].

The Tate curve T is a generalized elliptic curve over $\mathbb{Z}[[q]]$ defined by the formula

$$y^2 + xy = x^3 + a_4(q)x + a_6(q),$$

where

$$a_4(q) = -5 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n}, \quad a_6(q) = -\frac{1}{12} \sum_{n=1}^{\infty} \frac{(5n^3 + 7n^5)q^n}{1 - q^n}.$$

The j -invariant of this curve is

$$j(q) = q^{-1} + 744 + 196884q + \dots \tag{3.2}$$

At $q = 0$, the Tate curve is a curve of genus one with a nodal singularity, whose smooth locus is the group scheme \mathbb{G}_m .

The Tate curve possesses a chosen isomorphism of formal groups $\widehat{T} \cong \widehat{\mathbb{G}}_m$ over $\mathbb{Z}[[q]]$ and a canonical nowhere-vanishing invariant differential. In addition, there are compatible diagrams of group schemes

$$\begin{array}{ccc}
 \mu_n & \longrightarrow & T[n] \\
 \downarrow & & \downarrow \\
 \widehat{\mathbb{G}}_m & \longrightarrow & T
 \end{array}$$

as n varies.

For any $n \in \mathbb{N}$, define $\psi^n(q) = q^n$. This map has a lift $\psi^n_T: T \rightarrow T$ making the following diagram commute:

$$\begin{array}{ccc}
 T & \xrightarrow{\psi^n_T} & T \\
 \downarrow & & \downarrow \\
 \text{Spec}(\mathbb{Z}[[q]]) & \xrightarrow{\psi^n} & \text{Spec}(\mathbb{Z}[[q]])
 \end{array}$$

The resulting map $T \rightarrow (\psi^n)^*T$ is an isogeny with μ_n mapping isomorphically to the kernel.

3.5 The Tate moduli

The Tate curve is classified by a map $\text{Spec}(\mathbb{Z}[[q]]) \rightarrow \overline{\mathcal{M}}_{ell}$. It also has a $\mathbb{Z}/2$ -action by negation.

Equation (3.2) allows us to express j^{-1} and q as power series in each other, so the coefficient q is uniquely determined by j . Both j and $(j - 1728)$ are the inverses of elements in $\mathbb{Z}[[q]]$, and as in Lemma 3.3 this means that there are no isomorphisms between generalized elliptic curves parametrized by the Tate curve other than the identity and negation. In particular, the automorphism scheme $\underline{Aut}(T)$ is the constant group scheme $\mathbb{Z}/2$.

This implies that we have an identification of fiber products

$$\text{Spf}(\mathbb{Z}[[q]]) \times_{\overline{\mathcal{M}}_{ell}} \text{Spf}(\mathbb{Z}[[q]]) \cong \text{Spf}(\mathbb{Z}[[q]]) \times \mathbb{Z}/2.$$

In particular, the substack of $\overline{\mathcal{M}}_{ell}$ parametrizing generalized elliptic curves locally isomorphic to the Tate curve is isomorphic to the quotient stack

$$\mathcal{M}_{Tate} = [\text{Spf}(\mathbb{Z}[[q]]) // \mathbb{Z}/2].$$

The following shows that this identification is compatible with the logarithmic structure.

Proposition 3.15 *Log maps to \mathcal{M}_{Tate} are the same as log maps to $\text{Spf}(\mathbb{Z}[[q]], \langle q \rangle)$, together with a choice of principal $\mathbb{Z}/2$ -torsor.*

Proof Our identification of \mathcal{M}_{Tate} has already shown that maps $X \rightarrow \mathcal{M}_{Tate}$ are equivalent to maps $j^{-1}: X \rightarrow \text{Spf}(\mathbb{Z}[[q]])$, together with a principal $\mathbb{Z}/2$ -torsor $Y = X \times_{\mathcal{M}_{Tate}} \text{Spf}(\mathbb{Z}[[q]])$.

As the j -invariant of the Tate curve has the expression from Eq. (3.2), it is a unit times q . Therefore, the logarithmic structure $\langle j^{-1} \rangle$ on $\text{Spf}(\mathbb{Z}[[q]])$ is the isomorphic to the logarithmic structure $\langle q \rangle$. □

If (X, M_X) is a log scheme with a map $X \rightarrow \mathcal{M}_{Tate}$, an extension to a log map is equivalent to a choice of lift of q to an element $\tilde{q} \in M_X(X)$.

Let $\mathcal{M}_{\mathbb{G}_m} = [\text{Spec}(\mathbb{Z})//\mathbb{Z}/2]$ be the substack of \mathcal{M}_{Tate} defined by $q = 0$.

Definition 3.16 For a log scheme (X, M_X) log-étale over $\overline{\mathcal{M}}_{log}$ carrying the generalized elliptic curve \mathcal{E} , the cusp divisor lifts to a smooth divisor $X^c \subset X$ with a map $X^c \rightarrow \mathcal{M}_{\mathbb{G}_m}$.

The associated form of the multiplicative group scheme is the smooth locus of the restriction $\mathcal{E}|_{X^c}$, classified by the resulting map $X^c \rightarrow \mathcal{M}_{\mathbb{G}_m}$.

We note that the cusp subscheme is functorial, and that there is a natural diagram

$$\begin{array}{ccc} X^c & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathcal{M}_{\mathbb{G}_m} & \longrightarrow & \overline{\mathcal{M}}_{log}. \end{array}$$

If (X, M_X) is log-étale over $\overline{\mathcal{M}}_{log}$, then X^c is étale over $\mathcal{M}_{\mathbb{G}_m}$.

3.6 Modular curves

In this section we will describe why the modular curves, for various forms of level structure, give a natural tower of log-étale maps to $\overline{\mathcal{M}}_{log}$.

In the following, let $\widehat{\mathbb{Z}}$ be the profinite completion of the integers, let $G = \text{GL}_2(\widehat{\mathbb{Z}})$, and for $N \geq 1$ let p_N be the surjection $G \rightarrow \text{GL}_2(\mathbb{Z}/N)$.

Definition 3.17 The category \mathcal{L} of level structures is defined as follows. The objects are pairs (N, Γ) of a positive integer N and a subgroup $\Gamma < \text{GL}_2(\mathbb{Z}/N)$, with

$$\text{Hom}_{\mathcal{L}}((N, \Gamma), (N', \Gamma')) = \begin{cases} \text{Hom}_G(G/(p_N^{-1}\Gamma), G/(p_{N'}^{-1}\Gamma')) & \text{if } N'|N \\ \emptyset & \text{otherwise,} \end{cases}$$

In particular, morphisms in \mathcal{L} are generated by morphisms of the following three types:

1. inclusions $\Gamma < \Gamma'$ for subgroups of $\text{GL}_2(\mathbb{Z}/N)$,
2. conjugation maps $\Gamma \rightarrow g\Gamma g^{-1}$ for elements $g \in \text{GL}_2(\mathbb{Z}/N)/\Gamma$, and
3. changes-of-level $(NM, p^{-1}\Gamma) \rightarrow (N, \Gamma)$, where $p: \text{GL}_2(\mathbb{Z}/NM) \rightarrow \text{GL}_2(\mathbb{Z}/N)$ is the projection.

Proposition 3.18 For any pair (N, Γ) in \mathcal{L} , there is a Deligne–Mumford stack denoted $\overline{\mathcal{M}}(\Gamma)$, parametrizing elliptic curves with level Γ structure over

$\mathbb{Z}[1/N]$. Moreover, this is functorial in the sense that there is a weak 2-functor from \mathcal{L} to the 2-category of Deligne–Mumford stacks.

Proof The existence of a functorial family of smooth Deligne–Mumford stacks $\overline{\mathcal{M}}(\Gamma)$ in **Sch** parametrizing generalized elliptic curves with level Γ structure, away from the primes dividing the level, is from [13, §IV.3]. \square

These have natural open substacks $\mathcal{M}(\Gamma)$ parametrizing smooth elliptic curves, and there is an associated direct image log structure (2.5). Specifically, the submonoid of \mathcal{O}_X of functions invertible away from the cusps defines a logarithmic structure on $\overline{\mathcal{M}}(\Gamma)$, natural in (N, Γ) .

Proposition 3.19 *The functor of Proposition 3.18 extends to log stacks. For a fixed N , any map $(N, \Gamma) \rightarrow (N, \Gamma')$ in \mathcal{L} induces a log-étale covering map $\overline{\mathcal{M}}(\Gamma') \rightarrow \overline{\mathcal{M}}(\Gamma)$. If $K \triangleleft \Gamma < \text{GL}_2(\mathbb{Z}/N)$, then the Čech nerve of the associated covering map in log stacks is the simplicial bar construction for the action of Γ/K on $\overline{\mathcal{M}}(K)$.*

Proof The objects $\overline{\mathcal{M}}(\Gamma)$ are the normalizations of $\overline{\mathcal{M}}_{\text{ell}}[1/N]$ in $\mathcal{M}_{\text{ell}}(\Gamma)[1/N]$ [13, IV.3.10] and have tame ramification over the cusps. Proposition 2.27 establishes that this data is equivalent to a system of Kummer log-étale covering maps of $\overline{\mathcal{M}}_{\text{log}}$. Moreover, Corollary 3.4 and Proposition 2.18 together show that any such cover has logarithmic structure determined by a smooth divisor.

In particular, $\overline{\mathcal{M}}(\Gamma')$ is also the normalization of $\overline{\mathcal{M}}(\Gamma)$ in $\mathcal{M}_{\text{ell}}(\Gamma')[1/N]$, and the map $\overline{\mathcal{M}}(\Gamma') \rightarrow \overline{\mathcal{M}}(\Gamma)$ is also a log-étale covering map.

We now consider the inclusion of a normal subgroup. The action of Γ/K on $\overline{\mathcal{M}}(K)$ over $\overline{\mathcal{M}}(\Gamma)$ gives rise to a map from the simplicial bar construction $(\Gamma/K)^\bullet \times Y$ to the Čech nerve; to show that it is an isomorphism, it suffices by induction to show in degree 2 that the shearing map

$$\Gamma/K \times \overline{\mathcal{M}}(K) \rightarrow \overline{\mathcal{M}}(K) \times_{\overline{\mathcal{M}}(\Gamma)} \overline{\mathcal{M}}(K)$$

is an isomorphism. However, this is a log-étale map covering map which is an isomorphism over \mathcal{M}_{ell} , and so this is true by Proposition 2.27. \square

On any étale open $U \rightarrow \overline{\mathcal{M}}(\Gamma)$, the logarithmic structure is defined by the cusp divisor U^c . As in the previous section, the logarithmic structure determines a natural cusp substack $\overline{\mathcal{M}}(\Gamma)^c$ which is étale over $\mathcal{M}_{\mathbb{G}_m}$.

3.7 Grothendieck sites

For convenience, we consider the following category of elliptic curves. Recall that $(U_{\text{log}}, R_{\text{log}})$ (Sect. 3.2) forms a groupoid in log schemes, parametrizing generalized elliptic curves in Weierstrass form with compatible logarithmic structures.

Definition 3.20 The full subcategory $\mathcal{W} \subset \mathbf{AffLog}/\overline{\mathcal{M}}_{log}$ is defined as follows. The objects of \mathcal{W} are fine and saturated log schemes of the form $\mathrm{Spec}(R, M)$, equipped with a generalized elliptic curve \mathcal{E} in Weierstrass form and a lift of the elliptic discriminant to $\tilde{\Delta} \in M(\mathrm{Spec}(R))$, such that the associated map to $\overline{\mathcal{M}}_{log}$ is log-étale (Sect. 3.3). Maps are maps over $\overline{\mathcal{M}}_{log}$.

Equivalently, the objects of \mathcal{W} are affine log schemes $\mathrm{Spec}(R, M) \rightarrow U_{log}$ such that the composite $\mathrm{Spec}(R, M) \rightarrow U_{log} \rightarrow \overline{\mathcal{M}}_{log}$ is log-étale.

The category \mathcal{W} , while it is not closed under limits in the small étale site of $\overline{\mathcal{M}}_{log}$, still inherits a Grothendieck topology.

Proposition 3.21 *The inclusion from \mathcal{W} to the small log-étale site of $\overline{\mathcal{M}}_{log}$ is an equivalence of Grothendieck sites.*

Proof It suffices, by [24, C.2.2.3] to show that any object Kummer log-étale over $\overline{\mathcal{M}}_{log}$ has an étale cover by objects isomorphic to those from \mathcal{W} . However, this merely expresses the fact that a log scheme can be covered by affine charts, and that elliptic curves on affine schemes are locally isomorphic to elliptic curves in Weierstrass form. □

As our goal is to construct a presheaf \mathcal{O} on the small log-étale site of $\overline{\mathcal{M}}_{log}$ satisfying homotopy descent, it will ultimately be the case that we can equivalently carry out a construction on \mathcal{W} (see Sect. 4.4).

4 Homotopy theory

In this section we will discuss several important tools. We will assume that the reader is familiar with some more fundamental topics in homotopy theory: model categories, homotopy limits, smash products, and the derived smash product. (Appendix A of [35] provides a very convenient reference for much of the material we will be using on combinatorial model categories).

Most of the material in this chapter is the work of other authors, but is compiled here for convenience.

4.1 Elliptic cohomology theories

We begin by recalling the following (see [2, 6, 34]).

Definition 4.1 A homotopy-commutative ring spectrum E is weakly even-periodic if $\pi_n E$ is zero for n odd, and if the tensor product $\pi_p E \otimes_{\pi_0 E} \pi_q E \rightarrow \pi_{p+q} E$ is an isomorphism for p, q even. It is even-periodic if $\pi_2 E \cong \pi_0 E$, or equivalently if $\pi_2 E$ contains a unit.

Proposition 4.2 *If E is a weakly even-periodic spectrum, then $\mathrm{Spf}(E^0(\mathbb{C}\mathbb{P}^\infty))$ is a smooth, 1-dimensional formal group \mathbb{G}_E over the ring $\pi_0 E$. There is a natural identification of $\pi_{2t} E$ with the tensor power $\omega^{\otimes t}$ of sheaf of invariant differentials of \mathbb{G}_E .*

Definition 4.3 An elliptic spectrum consists of a weakly even-periodic spectrum E , an elliptic curve \mathcal{E} over $\pi_0 E$, and an isomorphism $\alpha: \mathbb{G}_E \rightarrow \widehat{\mathcal{E}}$ between the formal group of the complex orientable theory and the formal group of \mathcal{E} over $\pi_0 E$. We say that this elliptic spectrum realizes the elliptic curve \mathcal{E} .

A map of elliptic spectra is a multiplicative map $E \rightarrow E'$ together with a compatible isomorphism $\mathcal{E}' \rightarrow \mathcal{E} \otimes_{\pi_0 E} \pi_0 E'$ of elliptic curves which respects the isomorphisms of formal groups. We will say that a diagram of elliptic spectra realizes the corresponding diagram of elliptic curves.

Remark 4.4 Consider the case where \mathcal{E} is given the structure of a Weierstrass curve. It carries a coordinate $-x/y$ near the unit which trivializes the sheaf ω of invariant differentials, and hence a choice of Weierstrass representation determines a canonical identification $\pi_* E \cong \pi_0 E[u^{\pm 1}]$. (This also gives its formal group a standard lift to a formal group law classified by a map from the Lazard ring to $\pi_0 E$, and E has a corresponding standard orientation).

We will find the following result convenient in showing that many objects defined by pullback naturally remain elliptic spectra (compare [30, 3.9]).

Lemma 4.5 *Suppose that we have a homotopy pullback diagram*

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ S' & \longrightarrow & T \end{array}$$

of maps of homotopy commutative ring spectra, all of which are weakly even-periodic and complex orientable. Suppose $\pi_0 R$ carries an elliptic curve \mathcal{E} , and that the subdiagram $S \rightarrow T \leftarrow S'$ is a diagram of elliptic spectra realizing \mathcal{E} . Then there is a unique way to give R the structure of an elliptic spectrum realizing \mathcal{E} so that the square commutes.

Proof The diagram of elliptic spectra gives us a commutative diagram of formal groups

$$\begin{array}{ccccc} \mathbb{G}_R \otimes \pi_0 S & \longleftarrow & \mathbb{G}_R \otimes \pi_0 T & \longrightarrow & \mathbb{G}_R \otimes \pi_0 S' \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ \widehat{\mathcal{E}} \otimes \pi_0 S & \longleftarrow & \widehat{\mathcal{E}} \otimes \pi_0 T & \longrightarrow & \widehat{\mathcal{E}} \otimes \pi_0 S'. \end{array} \tag{4.1}$$

Here the tensor products are taken over $\pi_0 R$. The hypotheses imply that, for any n , applying π_n to the homotopy pullback diagram gives a bicartesian square. Therefore, taking pullbacks along rows of the coordinate rings from Eq. (4.1) gives us the formal groups \mathbb{G}_R and $\widehat{\mathcal{E}}$, and so there is a unique isomorphism $\mathbb{G}_R \rightarrow \widehat{\mathcal{E}}$ compatible with the given isomorphisms. In addition, the resulting pullback diagram for $\omega^{\otimes n}$ shows that this isomorphism gives R the structure of an elliptic spectrum. \square

Remark 4.6 In this paper, we will often be working with strictly even-periodic ring spectra rather than the weak version. This simplifies discussion of localization and completion with respect to elements in the ring of modular forms.

4.2 Rectification of diagrams

In this section we will describe how to convert homotopy coherent diagrams into strict diagrams. Throughout it we will fix a simplicial model category \mathcal{M} which is combinatorial (locally presentable and cofibrantly generated).

Definition 4.7 For a simplicial category I , let $\pi_0 I$ be the category with the same objects and with $\text{Hom}_{\pi_0 I}(i, i') = \pi_0 \text{Hom}_I(i, i')$.

Definition 4.8 For a small simplicial category I and a simplicial category \mathcal{C} , let \mathcal{C}^I denote the category of simplicial functors $I \rightarrow \mathcal{C}$ and natural transformations.

Theorem 4.9 ([35, A.3.3.2]) *If I is a small simplicial category, there is a projective model structure on \mathcal{M}^I , where a natural transformation $f \rightarrow g$ is a weak equivalence or fibration if and only if $f(i) \rightarrow g(i)$ has the corresponding property for each object $i \in I$.*

Definition 4.10 A map $f: I \rightarrow J$ of simplicial categories is a Dwyer–Kan equivalence if

- $\pi_0 f: \pi_0 I \rightarrow \pi_0 J$ is essentially surjective, and
- for any $i, i' \in I$, the function $\text{Map}_I(i, i') \rightarrow \text{Map}_J(fi, fi')$ is a weak equivalence of simplicial sets.

Theorem 4.11 ([35, A.3.3.8, A.3.3.9]) *For a combinatorial simplicial model category \mathcal{M} and a simplicial functor $f: I \rightarrow J$ between small simplicial categories, the restriction functor $f^*: \mathcal{M}^J \rightarrow \mathcal{M}^I$ is a right Quillen functor between the projective model structures. The left adjoint, denoted by $f_!$, is given by left Kan extension.*

If f is a Dwyer–Kan equivalence, then the Quillen adjunction $(f_!, f^)$ is a Quillen equivalence.*

Definition 4.12 A simplicial category is homotopically discrete if the natural functor $I \rightarrow \pi_0 I$ is a Dwyer–Kan equivalence: all the simplicial sets $\text{Hom}_I(i, i')$ are weakly equivalent to discrete sets.

Corollary 4.13 *Suppose that I is a small simplicial category which is homotopically discrete. Then any simplicial functor $g : I \rightarrow \mathcal{M}$ is naturally weakly equivalent to a functor $\pi_0 I \rightarrow \mathcal{M}$, and any two such are naturally weakly equivalent.*

Proof This follows from the fact that the map $\mathcal{M}^{\pi_0 I} \rightarrow \mathcal{M}^I$ induces an isomorphism on homotopy categories. If we wish, we can be slightly more explicit: writing π for the projection $I \rightarrow \pi_0 I$, we have a diagram of natural weak equivalences

$$g \leftarrow g_{\text{cof}} \rightarrow \pi^* \pi_! g_{\text{cof}},$$

where g_{cof} is a cofibrant replacement of g in \mathcal{M}^I . □

Remark 4.14 Our references are to recent work, but this result and those like it have a long history. Some particularly relevant ones include the following, non-exhaustive catalogue. Rectification for coherent diagrams appeared in work of Vogt and Dwyer–Kan–Smith; Cordier–Porter gave a proof for homotopy classes of diagrams in categories possessing sufficient limits; Segal developed the specific mechanism of using $\pi^* \pi_!$ in diagrams of spaces; Devinatz–Hopkins gave a proof in the case of E_∞ ring spectra; and Elmendorf–Mandell developed this particular adjunction in the multicategory of symmetric spectra.

4.3 Rings and ring maps

In order to make our constructions concrete, we will need to fix a particular model for the theory of E_∞ ring spectra. One convenient place to do so is the category of symmetric spectra [21].

We recall that the category of symmetric spectra has a symmetric monoidal structure \wedge whose unit is the sphere spectrum \mathbb{S} , as well as a symmetric monoidal functor Σ^∞ from based simplicial sets to symmetric spectra.

For a symmetric spectrum X , we take our definition of the n 'th homotopy group $\pi_n(X)$ to be the abelian group $[S^n, X]$ of maps in the homotopy category of symmetric spectra. In particular, equivalences of symmetric spectra are the same as π_* -isomorphisms with this definition.

Definition 4.15 The category of commutative (symmetric) ring spectra is the category of commutative monoids under \wedge .

If R is a commutative ring spectrum, the category of commutative R -algebras is the category of commutative ring spectra under R , and the category

of R -modules is the category of symmetric spectra with a (left) action of R , with its induced symmetric monoidal structure \wedge_R .

The homotopy groups π_*R of a commutative ring spectrum form a graded-commutative ring.

Definition 4.16 For M a commutative monoid, the monoid algebra $\mathbb{S}[M]$ is the commutative ring spectrum $\Sigma^\infty M_+$. Similarly, for M' a based monoid (i.e. a monoid with zero element), the based monoid algebra $\mathbb{S}[M']$ is the commutative ring spectrum $\Sigma^\infty M'$.

Remark 4.17 The ring $\pi_*(\mathbb{S}[M])$ is the monoid algebra $(\pi_*\mathbb{S})[M]$.

Definition 4.18 Let \mathbb{P} be the monad taking a symmetric spectrum X to the free commutative ring spectrum on X :

$$\mathbb{P}(X) = \bigvee_{n \geq 0} X^{\wedge n} / \Sigma_n.$$

The category of commutative ring spectra is equivalent to the category of \mathbb{P} -algebras.

Similarly, if R is a commutative ring spectrum, we write $\mathbb{P}_R(X)$ for the free commutative R -algebra on an R -module X .

Remark 4.19 Upon rationalization, the ring $\pi_*\mathbb{P}_R(X)$ is the free $\pi_*R_{\mathbb{Q}}$ -algebra on the module $\pi_*X_{\mathbb{Q}}$ (assuming that X is cofibrant as an R -module).

Definition 4.20 For commutative ring spectra S and T , let $\text{Map}_{\text{comm}}(S, T) \subset \text{Map}(S, T)$ be the simplicial set of commutative ring maps $S \rightarrow T$. If both S and T are commutative R -algebras for some fixed R , let $\text{Map}_{\text{comm}/R}(S, T) \subset \text{Map}_{\text{comm}}(S, T)$ be the simplicial set of commutative R -algebra maps.

There is a positive flat stable model structure on symmetric spectra (called the positive S -model structure in [49]) so that the following result holds.

Theorem 4.21 *There exists a combinatorial simplicial model structure on commutative ring spectra such that the forgetful functor to symmetric spectra is a right Quillen functor; with adjoint \mathbb{P} . (The same result holds for commutative R -algebras, R -modules, and \mathbb{P}_R).*

Remark 4.22 The existence of the model structure is (3.2) in the referenced paper, and the statement that these model categories are locally presentable is from the second page. The cited reference does not directly show that the categories of symmetric spectra or commutative ring spectra are simplicial model categories. However, as described in [52], it is straightforward to verify that these model categories are simplicial: in all these categories, fibrations

and cotensors with a simplicial set are created in symmetric spectra, and so the pullback formulation of axiom SM7 is inherited from a levelwise verification in symmetric spectra.

In combination with Corollary 4.13, this combinatorial model structure allows us to refine diagrams of commutative ring spectra with simple mapping spaces to strict diagrams. To proceed further in this direction, we will require some results from obstruction theory.

There have been several versions of obstruction theory developed for the classification of objects and maps in commutative ring spectra. Most of these produce obstructions in groups derived from the cotangent complex. As a result, when the cotangent complex vanishes we obtain strong uniqueness results.

Proposition 4.23 *Suppose R is a commutative ring spectrum and $\pi_*R \rightarrow A_*$ is a map of evenly-graded commutative rings which is étale. Then there exists an R -algebra S equipped with an isomorphism $\pi_*S \cong A_*$ of π_*R -algebras, and for any such S the natural transformation*

$$\pi_* : \text{Map}_{\text{comm}/R}(S, -) \rightarrow \text{Hom}_{\text{gr.comm}/\pi_*R}(A_*, -),$$

from the mapping space to the discrete set, is a natural weak equivalence.

Proof Our proof roughly follows [10, §2.2]. The Goerss–Hopkins obstruction theory in the category of R -modules [16], using the auxiliary homology theory $E = R$, gives obstructions to existence of such a commutative R -algebra S in the André–Quillen cohomology groups

$$\text{Der}_{\pi_*R}^{s+2}(A_*, \Omega^s A_*),$$

and for any such S there is a Bousfield–Kan spectral sequence converging to $\pi_{t-s}\text{Map}_{\text{comm}/R}(S, T)$ with

$$E_2^{s,t} = \begin{cases} \text{Hom}_{\text{gr.comm}/\pi_*R}(A_*, \pi_*T) & (s, t) = (0, 0) \\ \text{Der}_{\pi_*R}^s(A_*, \Omega^t \pi_*T) & s > 0. \end{cases}$$

For étale π_*R -algebras the André–Quillen cohomology groups $\text{Der}_{\pi_*R}^s(A_*, -)$ vanish identically for $s > 0$, for example by identifying them with Γ -cohomology [9] and applying vanishing results of Robinson–Whitehouse [46]. □

This result is sometimes referred to as “topological invariance of the étale site.” (A related version with a more restricted notion of étale maps appears in [33, 7.5.0.6, 7.5.4.6].) Combining it with Corollary 4.13, we find the following.

Corollary 4.24 *There exists a functor from étale extensions of \mathbb{Z} to commutative ring spectra, sending a ring A to a ring spectrum $\mathbb{S} \otimes A$ such that $\pi_*(\mathbb{S} \otimes A) \cong \pi_*\mathbb{S} \otimes A$.*

We recall that the Eilenberg–Mac Lane functor H , from graded abelian groups to modules over $H\mathbb{Z}$, extends to a functor from graded-commutative \mathbb{Z} -algebras to commutative $H\mathbb{Z}$ -algebras (corresponding to the underlying differential graded algebra with trivial differential).

Definition 4.25 A commutative ring spectrum R is formal if it is equivalent, in the homotopy category of commutative ring spectra, to $H\pi_*R$.

More generally, a functor from a category I to the category of commutative ring spectra is formal if it is weakly equivalent to a functor factoring through H .

Proposition 4.26 *Suppose R is an evenly-graded commutative ring spectrum which is formal, and let $\acute{E}t/R$ be the category of evenly-graded commutative R -algebras T such that $\pi_*R \rightarrow \pi_*T$ is étale. Then, for any small category I with a functor $f: I \rightarrow \acute{E}t/R$, f is equivalent to the functor $H\pi_*f$.*

Proof Since $H\pi_*R$ is equivalent to R , without loss of generality we may replace both $f: I \rightarrow \acute{E}t/R$ and $H\pi_*f: \pi_0I \rightarrow \acute{E}t/R$ with equivalent functors taking values in cofibrant-fibrant commutative R -algebras.

Let J (resp. J') be the full subcategory of $\acute{E}t/R$ whose object set is the image of f (resp. the union of the images of f and $H\pi_*f$). The composite map

$$\pi_0J \xrightarrow{H} J' \twoheadrightarrow \pi_0J'$$

is an equivalence of categories, and by Proposition 4.23 the right-hand projection is a Dwyer–Kan equivalence. Therefore, H is also a Dwyer–Kan equivalence.

Applying Theorem 4.13 to $H: \pi_0J \rightarrow J'$, we find that the identity functor $J' \rightarrow \acute{E}t/R$ is equivalent to a functor factoring through H . The result follows by precomposition with f . □

Remark 4.27 We might think of this as a consequence of the fact that the functor $\pi_*: \acute{E}t/R \rightarrow \acute{E}t/\pi_*R$ is a Dwyer–Kan equivalence (although the categories in question are not small categories). In particular, we could apply the previous result to the inclusion of a full subcategory of $\acute{E}t/R$ which maps to a skeleton of $\acute{E}t/\pi_*R$, and then use a homotopical Kan extension to construct a homotopy inverse functor to π_* on all of $\acute{E}t/R$.

4.4 Homotopy sheafification

In the following we describe how to construct homotopical sheafifications that we will require. Most of the methods of this section are taken directly, or with slight modification, from [6, §2].

In this chapter, let \mathcal{C} be a small Grothendieck site. Recall that we have a positive flat stable model structure on symmetric spectra from Theorem 4.21.

Definition 4.28 A map $X \rightarrow Y$ of presheaves of symmetric spectra on \mathcal{C} is

- a level cofibration if, for any $U \in \mathcal{C}$, the map $X(U) \rightarrow Y(U)$ is a cofibration,
- a level equivalence if, for any $U \in \mathcal{C}$, the map $X(U) \rightarrow Y(U)$ is an equivalence, and
- a local equivalence if the map of presheaves of homotopy groups $\pi_* X(U) \rightarrow \pi_* Y(U)$ induces an isomorphism on the associated sheaves $\underline{\pi}_* X \rightarrow \underline{\pi}_* Y$.

Proposition 4.29 ([35, A.3.3.2, A.3.3.6]) *The level cofibrations and level equivalences define combinatorial simplicial model structures (the injective model structures) on the categories of presheaves of symmetric spectra and commutative ring spectra. Under these, the forgetful functor from presheaves of commutative ring spectra to presheaves of symmetric spectra is a right Quillen functor whose left adjoint is \mathbb{P} .*

As these model structures are combinatorial, we may then localize with respect to the local equivalences.

Proposition 4.30 ([35, A.3.7.3]) *The level cofibrations and local equivalences define combinatorial simplicial model structures on the categories of presheaves of symmetric spectra and commutative ring spectra. Under these, the forgetful functor from presheaves of commutative ring spectra to presheaves of symmetric spectra is a right Quillen functor whose left adjoint is \mathbb{P} .*

Remark 4.31 For a commutative ring spectrum R , similar results hold in the categories of presheaves of R -modules and R -algebras.

Remark 4.32 Under the ordinary, non-positive stable model structure on symmetric spectra where cofibrations are simply monomorphisms, the existence of this model structure is due to Jardine [23]. It is almost certain that the proof given there, which is axiomatic in flavour, goes through verbatim. Given that, we could instead recover this adjunction from the “pointwise” symmetric monoidal structure on presheaves by work of White [52]. The identity functor is the left adjoint in a Quillen equivalence between the positive model category of presheaves and Jardine’s ordinary model category of presheaves.

Proposition 4.33 *Suppose \mathcal{F} is a fibrant presheaf of symmetric spectra on \mathcal{C} . Then \mathcal{F} satisfies homotopy descent with respect to hypercovers: for any hypercover $U_\bullet \rightarrow X$, the map*

$$\mathcal{F}(X) \rightarrow \operatorname{holim} \mathcal{F}(U_\bullet)$$

is an equivalence.

In this situation, there is also a hypercohomology spectral sequence

$$H^s(X, \underline{\pi}_t(\mathcal{F})) \Rightarrow \pi_{t-s}\mathcal{F}(X)$$

calculating the values of $\pi_\mathcal{F}(X)$.*

Proof We may construct a weak equivalence $\mathcal{F} \rightarrow \mathcal{F}_{Jf}$ to a Jardine fibrant presheaf of symmetric spectra, which is also fibrant in our model structure by Remark 4.32. The map $\mathcal{F} \rightarrow \mathcal{F}_{Jf}$ is an equivalence between fibrant objects, and hence an equivalence in the injective model structure.

Therefore, $\mathcal{F} \rightarrow \mathcal{F}_{Jf}$ is a level equivalence. We therefore recover the homotopy descent property and the stated spectral sequence from the fact that these hold for the presheaf \mathcal{F}_{Jf} . □

Proposition 4.34 *Suppose \mathcal{F} is a presheaf of symmetric spectra on \mathcal{C} which satisfies homotopy descent with respect to hypercovers. Then any fibrant replacement $\mathcal{F} \rightarrow \mathcal{F}_f$ is also a level equivalence.*

Proof This property is independent of the choice of fibrant replacement, so it suffices to show that it holds for one. We take a level equivalence $\mathcal{F} \rightarrow \mathcal{F}_{If}$ which is a fibrant replacement in the ordinary injective model structure. In particular, \mathcal{F}_{If} is injective fibrant in the Jardine model structure and satisfies homotopy descent for hypercovers by comparison with \mathcal{F} . By work of Dugger–Hollander–Isaksen, \mathcal{F}_{If} is Jardine fibrant in the local model structure [12], and therefore also a fibrant replacement for \mathcal{F} in our local model structure. □

Proposition 4.35 *Suppose $i: \mathcal{C} \rightarrow \mathcal{D}$ is a continuous functor between Grothendieck sites. Then the restriction functor i^* on presheaves of commutative ring spectra is a left Quillen functor under the local model structure, and is a Quillen equivalence if i is both fully faithful and an equivalence of sites.*

Proof The functor i^* clearly preserves cofibrations, and possesses a right adjoint i_* (given by right Kan extension). Moreover, sheafification commutes with i^* , and so the restriction of a local equivalence is a local equivalence.

If i is an equivalence of sites, then the map of sheaves $\underline{\pi}_*X \rightarrow \underline{\pi}_*Y$ is isomorphic to the map of sheaves $\underline{\pi}_*(i^*X) \rightarrow \underline{\pi}_*(i^*Y)$, and so i^* preserves and reflects weak equivalences.

If i is also fully faithful, then we have

$$(i_*X)(i(U)) = \lim_{V \rightarrow i(U)} X(V) \cong X(i(U))$$

and so the map $i^*i_*X \rightarrow X$ is always an isomorphism; in particular, the (derived) counit of the adjunction is always an equivalence. \square

Proposition 4.36 *Suppose \mathcal{D} is the Kummer log-étale site of a log stack (X, M_X) , and $i : \mathcal{C} \rightarrow \mathcal{D}$ is a fully faithful equivalence of sites where the objects of \mathcal{C} are affine. Suppose \mathcal{F} is a presheaf of symmetric spectra on \mathcal{C} such that the presheaves $\pi_k\mathcal{F}$ are the restrictions of quasicoherent sheaves to \mathcal{C} . Then any fibrant replacement $\mathcal{F} \rightarrow \mathcal{F}_f$ is a level equivalence.*

Proof Let $\mathcal{F} \rightarrow \mathcal{F}_f$ be a fibrant replacement. The sheaf $\pi_k(\mathcal{F}_f)$ is the quasicoherent sheaf $\pi_k\mathcal{F}$ on \mathcal{C} . By Corollary 2.25, if (U, M_U) is an object of \mathcal{C} , the hypercohomology spectral sequence of Proposition 4.33 calculating the homotopy groups of $\mathcal{F}_f(U, M_U)$ degenerates to an isomorphism

$$\pi_k\mathcal{F}(U, M_U) \xrightarrow{\sim} \pi_k\mathcal{F}_f(U, M_U),$$

as desired. \square

Corollary 4.37 *Suppose \mathcal{D} is the Kummer log-étale site of a log stack (X, M_X) , and $i : \mathcal{C} \rightarrow \mathcal{D}$ is a fully faithful equivalence of sites where the objects of \mathcal{C} are affine. Suppose \mathcal{F} is a presheaf of commutative ring spectra on \mathcal{D} such that the presheaves $\pi_k\mathcal{F}$ are the restrictions of quasicoherent sheaves to \mathcal{C} . Then there exists a fibrant presheaf \mathcal{G} of commutative ring spectra on \mathcal{D} whose restriction $i^*\mathcal{G}$ to \mathcal{C} is equivalent to \mathcal{F} .*

For a log stack (Y, M_Y) log-étale over (X, M_X) , there is a spectral sequence

$$H_{Zar}^s((Y, M_Y), \pi_t\mathcal{F}) \Rightarrow \pi_{t-s}\mathcal{G}(Y, M_Y).$$

Proof By Proposition 3.21, the map i^* is the left adjoint in a Quillen equivalence. Let $\mathcal{F} \rightarrow \mathcal{F}_f$ be a fibrant replacement, which is also a level equivalence. Letting $\mathcal{G} = i_*\mathcal{F}_f$, the adjoint map $i^*\mathcal{G} \rightarrow \mathcal{F}_f$ is an equivalence.

Corollary 2.25 then allows us to interpret the resulting hypercohomology spectral sequence as Zariski cohomology. \square

By applying Proposition 3.19, we can get a homotopy fixed-point description of the values of such a presheaf on modular curves.

Proposition 4.38 *Let \mathcal{G} be a fibrant presheaf of symmetric spectra on $\overline{\mathcal{M}}_{ell}$. If $K \triangleleft \Gamma < \text{GL}_2(\mathbb{Z}/N)$, then the natural map*

$$\mathcal{G}(\overline{\mathcal{M}}(\Gamma)) \rightarrow \mathcal{G}(\overline{\mathcal{M}}(K))^{h\Gamma/K}$$

is an equivalence. In particular, there is a group cohomology spectral sequence

$$H^s(\Gamma/K, \pi_t \mathcal{G}(\overline{\mathcal{M}}(K))) \Rightarrow \pi_{t-s} \mathcal{G}(\overline{\mathcal{M}}(\Gamma)).$$

Remark 4.39 We could, if desired, take the further step of turning presheaves into sheaves, using a further Quillen equivalence to a Joyal model structure on sheaves. This makes comparisons between equivalent Grothendieck sites more direct. However, for the purposes of this paper the homotopy sheaf property is sufficient.

4.5 K -theory and Tmf

We will require some details about the construction of forms of K -theory from [31, Appendix A]. Let K be the periodic complex K -theory spectrum, equipped with its action of the cyclic group C_2 by conjugation.

We recall that, if X is a spectrum with C_2 -action, the homotopy fixed-point spectrum X^{hC_2} is the homotopy limit of the corresponding diagram.

Proposition 4.40 [[31, Appendix A]] *There is a fibrant presheaf \mathcal{O}^{mult} of locally weakly even-periodic commutative ring spectra on the étale site of $\mathcal{M}_{\mathbb{G}_m}$. In particular, given an étale map $\text{Spec}(R) \rightarrow \overline{\mathcal{M}}_{ell}$ classifying a group scheme \mathbb{G} over R locally isomorphic to \mathbb{G}_m , $\mathcal{O}^{mult}(R)$ is a weakly even-periodic spectrum realizing $\widehat{\mathbb{G}}$.*

Remark 4.41 The values of \mathcal{O}^{mult} on affine schemes are forms of K -theory [39]: the analogues of elliptic cohomology theories where forms of \mathbb{G}_m replace elliptic curves.

We will require specific knowledge about how this presheaf is constructed on affine opens of $\mathcal{M}_{\mathbb{G}_m}$.

Proposition 4.42 *Let $\text{Spec}(S) \rightarrow \mathcal{M}_{\mathbb{G}_m}$ be an étale map, corresponding to an element $\alpha \in H_{et}^1(\text{Spec}(S), C_2)$ that classifies a C_2 -Galois extension T/S . Then S and T are étale over \mathbb{Z} , and there is a natural zigzag equivalence of commutative ring spectra*

$$\mathcal{O}^{mult}(S, \alpha) \simeq (K \wedge^{\mathbb{L}} (\mathbb{S} \otimes T))^{hC_2}.$$

Here $\mathbb{S} \otimes T$ is the spectrum of Corollary 4.24, and C_2 has the diagonal action on $K \wedge^{\mathbb{L}} (\mathbb{S} \otimes T)$.

Remark 4.43 The group C_2 acts on the coefficient ring $K_* \otimes T$ with no higher cohomology. Therefore, we have

$$\pi_* \mathcal{O}^{mult}(S, \alpha) = (K_* \otimes T)^{C_2},$$

and rationalization commutes with the homotopy fixed point construction.

The global section object $\mathcal{O}^{mult}(\mathcal{M}_{\mathbb{G}_m})$ is the periodic real K -theory spectrum $KO \simeq K^{hC_2}$.

We now must recall some of the main theorems about topological modular forms.

Theorem 4.44 [[6]] *There exists a fibrant presheaf of elliptic commutative ring spectra \mathcal{O}^{et} on the small étale site of \mathcal{M}_{ell} . In particular, given an étale map $\text{Spec}(R) \rightarrow \overline{\mathcal{M}}_{ell}$ classifying an elliptic curve \mathcal{E} over R , $\mathcal{O}^{et}(R)$ is an elliptic spectrum realizing \mathcal{E} .*

Definition 4.45 Let $\text{Tmf} = \Gamma(\overline{\mathcal{M}}_{ell}, \mathcal{O}^{et})$, and let tmf be its connective cover. The values of \mathcal{O}^{et} naturally take values in commutative tmf -algebras.

Definition 4.46 Let \mathcal{O}^{smooth} be the restriction of \mathcal{O}^{et} to a presheaf of commutative tmf -algebras on the small étale site of \mathcal{M}_{ell} , the open substack parametrizing smooth elliptic curves.

Proposition 4.47 *The rationalized functors $\mathcal{O}_{\mathbb{Q}}^{et}$ and $\mathcal{O}_{\mathbb{Q}}^{smooth}$ are formal.*

Proof By Proposition 3.12, the functor $\mathcal{O}_{\mathbb{Q}}^{et}$ takes values in the category of commutative $\text{tmf}_{\mathbb{Q}}$ -algebras such that map $\pi_* \text{tmf}_{\mathbb{Q}} \rightarrow \pi_* \mathcal{O}_{\mathbb{Q}}^{et}(R)$ is étale. Proposition 4.26 then implies the desired result once we have shown that $\text{tmf}_{\mathbb{Q}}$ is formal.

The ring $\pi_* \text{tmf}_{\mathbb{Q}}$ is a polynomial algebra on the classes c_4 and c_6 . By the universal property of the free algebra, up to equivalence we can construct a diagram

$$H\pi_* \text{tmf} \leftarrow \mathbb{P}(S^8 \vee S^{12}) \rightarrow \text{tmf},$$

and by Remark 4.19 the rationalized diagram

$$H\pi_* \text{tmf}_{\mathbb{Q}} \leftarrow \mathbb{P}(S^8 \vee S^{12})_{\mathbb{Q}} \rightarrow \text{tmf}_{\mathbb{Q}}$$

consists of equivalences of commutative ring spectra. □

The spectrum tmf is explicitly constructed so that the following theorem holds.

Proposition 4.48 *For any prime p , the p -adic K -theory of tmf is Katz’s ring V of p -complete generalized modular functions [25], parametrizing isomorphism classes of elliptic curves \mathcal{E} over p -complete rings with a chosen isomorphism $\widehat{\mathbb{G}}_m \rightarrow \widehat{\mathcal{E}}$.*

The following assembles applications of Goerss-Hopkins obstruction theory from [6, §7].

Proposition 4.49 *Let $\mathrm{Spf}(R)$ be a formal scheme with ideal of definition (p) , and let $\mathrm{Spf}(R) \rightarrow (\overline{\mathcal{M}}_{ell})_p^\wedge$ be an étale formal affine open classifying an elliptic curve \mathcal{E}/R with no supersingular fibers. Then there is a lift of this data to a $K(1)$ -local elliptic commutative tmf-algebra E realizing \mathcal{E} .*

Given a second torsion-free p -complete ring R' with a map $\mathrm{Spf}(R') \rightarrow (\overline{\mathcal{M}}_{ell})_p^\wedge$ classifying an elliptic curve \mathcal{E}' with no supersingular fibers, realized by an elliptic commutative tmf-algebra E' , the space $\mathrm{Map}_{\mathrm{comm}/\mathrm{tmf}}(E, E')$ of $K(1)$ -local commutative tmf-algebra maps is homotopically discrete and equivalent to the set $\mathrm{Hom}_{\overline{\mathcal{M}}_{ell}}(\mathrm{Spf}(R'), \mathrm{Spf}(R))$ of isomorphism classes of pullback diagrams

$$\begin{array}{ccc} \mathcal{E}' & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathrm{Spf}(R') & \longrightarrow & \mathrm{Spf}(R). \end{array}$$

Proof The necessary details for these results are proven within [6, §7, Step 2] (just after Remark 7.15). There it is shown that the étale condition implies that the higher Goerss–Hopkins obstruction groups for tmf-algebras all vanish in this setting. As a result, for such an \mathcal{E}/R there always exists an elliptic commutative tmf-algebra E realizing \mathcal{E} . Moreover, the space $\mathrm{Map}_{\mathrm{comm}/\mathrm{tmf}}(E, E')$ is homotopically discrete: more specifically, it is homotopy equivalent to a particular set of algebraic maps that we will now describe.

Given any torsion-free p -complete ring R with a map $\mathrm{Spf}(R) \rightarrow (\overline{\mathcal{M}}_{ell})_p^\wedge$ classifying an elliptic curve \mathcal{E} with no supersingular fibers, realized by an elliptic cohomology theory E , there is a p -adically complete ring $W = \pi_0(K \wedge E)_p^\wedge$ which is the universal R -algebra where \mathcal{E} comes equipped with a choice of isomorphism $\widehat{\mathbb{G}}_m \rightarrow \widehat{E}$. As V is the universal p -complete ring possessing such an elliptic curve, there is a natural map $\phi: V \rightarrow W$. The ring W comes equipped with natural operators: an action of \mathbb{Z}_p^\times by Adams operations, and with a ring homomorphism $\psi^p: W \rightarrow W$ such that $\psi^p(x) = x^p + p\theta(x)$.

Given elliptic commutative tmf-algebras as in the problem, the Goerss–Hopkins obstruction theory shows that p -completed K -theory gives a homotopy equivalence

$$\mathrm{Map}_{\mathrm{comm}/\mathrm{tmf}}(E, E') \simeq \mathrm{Hom}_{\psi-\theta\text{-alg}/V}(W, W')$$

from the space of maps of p -complete tmf-algebras $E \rightarrow E'$ to the discrete set of maps $W \rightarrow W'$ on p -adic K -theory which are maps of V -algebras that respect the actions of \mathbb{Z}_p^\times and θ .

We will now explain how these imply the statement above.

The map $\mathrm{Spf}(W) \rightarrow \mathrm{Spf}(R)$ classifying isomorphisms $\widehat{\mathbb{G}}_m \rightarrow \widehat{E}$ is a principal torsor for $\mathrm{Aut}(\widehat{\mathbb{G}}_m)$, and hence a principal \mathbb{Z}_p^\times -torsor. If R is étale over $\overline{\mathcal{M}}_{ell}$, the map $V \rightarrow W$ is also étale. As a result, for any U which is p -adically complete there are always unique lifts in diagrams of the following form:

$$\begin{array}{ccc}
 V & \xrightarrow{f} & U \\
 \phi \downarrow & \nearrow g & \downarrow \\
 W & \xrightarrow{\bar{g}} & U/(p)
 \end{array}$$

We will apply this twice.

First, by taking $f = \phi\psi^P$ and $\bar{g} = \mathrm{Frob}$ we find that there is a unique extension of ψ^P (and θ) to W .

Second, if we start with one such diagram where U has a lift ψ^P of Frobenius such that $f\psi^P = \psi^P f$, then we may construct a new diagram by replacing f with $\psi^P f$ and \bar{g} with $\mathrm{Frob} \cdot \bar{g}$. Then both $g\psi^P$ and $\psi^P g$ are lifts making the diagram commute, so they are equal.

As a result, for such algebras W and W' a map of V -algebras automatically commutes with θ , and so the set $\mathrm{Hom}_{\psi-\theta\text{-alg}/V}(W, W')$ is isomorphic to the set $\mathrm{Hom}_{\mathrm{comm}/V}(W, W')^{\mathbb{Z}_p^\times}$ of \mathbb{Z}_p^\times -equivariant V -algebra maps.

The universal property of V implies that the set of V -algebra maps $W \rightarrow W'$ is the same as the set of ring maps $W \rightarrow W'$ together with a chosen isomorphism $\mathcal{E} \otimes_R W' \rightarrow \mathcal{E}' \otimes_{R'} W'$ respecting the isomorphisms of the formal parts with $\widehat{\mathbb{G}}_m$. The universal properties of W and W' , parametrizing isomorphisms of the formal groups of \mathcal{E} and \mathcal{E}' with $\widehat{\mathbb{G}}_m$, together imply that a map is \mathbb{Z}_p^\times -equivariant if and only if it descends to a map $R \rightarrow R'$, together with an isomorphism $\mathcal{E} \otimes_R R' \rightarrow \mathcal{E}'$ over R' expressing the desired pullback. □

5 Construction of the presheaf \mathcal{O}

5.1 Construction at the cusp

Corollary 2.19 will allow us to define our presheaf \mathcal{O} in a formal neighborhood of the cusp by restricting our attention to a subcategory of log schemes over \mathcal{M}_{Tate} (Sect. 3.5).

Definition 5.1 The category \mathcal{C} has, as objects, pairs (m, S) of a nonnegative integer m and a connected, étale $\mathbb{Z}[1/m][[q^{1/m}]]$ -algebra S , complete and

separated in the topology generated by the ideal (q) . We equip S with the logarithmic structure $\langle q^{1/m} \rangle$. The maps in \mathcal{C} are defined so that the forgetful functor to formal schemes over \mathcal{M}_{Tate} , sending (m, S) to $\text{Spec}(S, \langle q^{1/m} \rangle)$, is fully faithful.

We will often leave m implicit when describing objects of \mathcal{C} . It can either be recovered as the ramification index of the ideal (q) or from the logarithmic structure.

Definition 5.2 There is a functor from \mathcal{C} to affine étale opens of $\mathcal{M}_{\mathbb{G}_m}$, sending (m, S) to the quotient $\bar{S} = S/\langle q^{1/m} \rangle$ by the defining ideal.

By Proposition 2.19, the category \mathcal{C} inherits a Grothendieck topology equivalent to the category of formal schemes log-étale over \mathcal{M}_{Tate} . To define the sheaves we are interested in, it then suffices to define them on \mathcal{C} by Corollary 4.37. We will abuse notation by simply writing S for the log ring $(S, \langle q^{1/m} \rangle)$ equipped with a form of the Tate curve and a lift of the discriminant.

Definition 5.3 We define functors from \mathcal{C} to commutative monoids by

$$\mu_m(S) = \{ \zeta \in S^\times \mid \zeta^m = 1 \}$$

and

$$A(m, S) = \mu_m(S) \times \left(\frac{1}{m} \cdot \mathbb{N} \right).$$

We write a generic element in $A(m, S)$ as $\zeta q^{k/m}$, where $\zeta \in \mu_m(S)$ and $k \in \mathbb{N}$.

Remark 5.4 The natural map $\mu_m(S) \rightarrow \mu_m(\bar{S})$ is an isomorphism.

Remark 5.5 We note that a map $(S, \langle q^{1/m} \rangle) \rightarrow (S', \langle q^{1/dm} \rangle)$ over \mathcal{M}_{Tate} is equivalent to a map $\bar{S} \rightarrow \bar{S}'$ over $\mathcal{M}_{\mathbb{G}_m}$, together with an extension in the diagram

$$\begin{array}{ccc} \mu_m(S) \times q^{\mathbb{N}} & \longrightarrow & \mu_{dm}(S') \times q^{\mathbb{N}} \\ \downarrow & & \downarrow \\ A(m, S) & \dashrightarrow & A(dm, S'). \end{array}$$

This is because the natural map of monoids $A(m, S) \rightarrow S$ generates the logarithmic structure on S , and we must have $(f(q^{1/m}))^m = (q^{1/dm})^{dm}$. Once the image of $q^{1/m}$ is defined, the extension to the rest of the étale $\mathbb{Z}[1/m][[q^{1/m}]]$ -algebra S is forced.

Proposition 5.6 *Up to equivalence, there is a natural map*

$$\mathbb{S}[\mu_m(S)] \rightarrow \mathcal{O}^{mult}(\bar{S})$$

of presheaves of commutative ring spectra on \mathcal{C} . On π_0 , this is a realization of the composite map $\mathbb{Z}[\mu_m(S)] \rightarrow S \rightarrow \bar{S}$.

Proof As m acts invertibly on $\pi_*\mathcal{O}^{mult}(\bar{S})$, we have an equivalence of derived spaces of commutative ring maps

$$\text{Map}_{comm}(\mathbb{S}[\mu_m(S)], \mathcal{O}^{mult}(\bar{S})) \simeq \text{Map}_{comm}(\mathbb{S}[\mu_m(S)][1/m], \mathcal{O}^{mult}(\bar{S})).$$

On homotopy groups, the map $\pi_*\mathbb{S} \rightarrow \pi_*\mathbb{S}[\mu_m(S)][1/m]$ is always an étale map because $\mu_m(S)$ has order invertible in $\mathbb{Z}[1/m]$. By Proposition 4.23, the derived space of maps $\Sigma_+^\infty A^\times(S) \rightarrow \mathcal{O}^{mult}(\bar{S})$ is homotopically discrete and equivalent to the space of maps $\mu_m(S) \rightarrow \pi_0(\mathcal{O}^{mult}(\bar{S}))^\times \cong \bar{S}^\times$.

By Corollary 4.13, we can therefore replace these presheaves $\mathbb{S}[\mu_m(S)]$ and $\mathcal{O}^{mult}(\bar{S})$ with equivalent ones possessing a natural transformation as desired. □

Theorem 5.7 *There exists a presheaf \mathcal{O}^{Tate} of elliptic commutative ring spectra on \mathcal{C} , together with a natural homotopy pushout diagram of commutative ring spectra*

$$\begin{array}{ccc} \mathbb{S}[A(m, S)] & \longrightarrow & \mathcal{O}^{Tate}(m, S) \\ \downarrow & & \downarrow \\ \mathbb{S}[\mu_m(S)] & \longrightarrow & \mathcal{O}^{mult}(\bar{S}). \end{array}$$

These satisfy the following properties:

- $\mathcal{O}^{Tate}(m, S)$ realizes the form of the Tate curve on S ;
- the map $A(m, S) \rightarrow \pi_0\mathcal{O}^{Tate}(m, S) \cong S$ is the map inducing the logarithmic structure on S ; and
- the map $\mathbb{S}[A(m, S)] \rightarrow \mathbb{S}[\mu_m(S)]$ is induced by the projection $A(m, S) \rightarrow \mu_m(S)_+$ sending all non-units to the basepoint.

Proof For any $r > 0$, write $A(m, S)/q^r$ for the quotient based monoid sending $\zeta q^{k/m}$ to the zero element for $k/m \geq r$.

We define our presheaf as a homotopy limit of (derived) smash products:

$$\mathcal{O}^{Tate}(S) = \text{holim}_r \left[\mathcal{O}^{mult}(\bar{S}) \underset{\mathbb{S}[\mu_m(S)]}{\overset{\mathbb{L}}{\wedge}} \mathbb{S}[A(m, S)/q^r] \right]$$

On homotopy groups, we have

$$\pi_* \left[\mathcal{O}^{mult}(\bar{S}) \underset{\mathbb{S}[\mu_m(S)]}{\mathbb{L}\wedge} \mathbb{S}[A(m, S)/q^r] \right] \cong \pi_* \mathcal{O}^{mult}(\bar{S})[q^{1/m}]/(q^r),$$

and taking limits gives an isomorphism

$$\pi_* \mathcal{O}^{Tate}(S) \cong \pi_* \mathcal{O}^{mult}(\bar{S})[[q^{1/m}]].$$

Moreover, because $\mathcal{O}^{Tate}(S)$ is an $\mathcal{O}^{mult}(\bar{S})$ -algebra, the formal group of $\mathcal{O}^{Tate}(S)$ carries a canonical isomorphism to the extension of the form of the multiplicative formal group over \bar{S} . We may then compose this with the isomorphism between this formal group and the formal group of the form of the Tate curve over S , making this into a presheaf of elliptic spectra realizing the Tate curve.

The multiplication-by- $q^{1/m}$ map $A(m, S) \rightarrow A(m, S)$ induces a cofiber sequence of $\mathbb{S}[A(m, S)]$ -modules

$$\mathbb{S}[A(m, S)] \xrightarrow{q^{1/m}} \mathbb{S}[A(m, S)] \rightarrow \mathbb{S}[\mu_m(S)],$$

where the right-hand map (but not the identification of the fiber) is natural in (m, S) . This remains a cofiber sequence after completing with respect to the powers of q , and so the homotopy pushout diagram follows by associativity of the derived smash product. □

By construction, the functor \mathcal{O}^{Tate} naturally takes values in the category of algebras over

$$KO[[q]] = \operatorname{holim}_r KO \wedge \{1, q, q^2, \dots, q^{r-1}\}_+.$$

We may then apply Theorem 6.12 from the appendix and Corollary 4.37 to conclude the following.

Corollary 5.8 *The presheaf \mathcal{O}^{Tate} , up to equivalence, extends to a presheaf \mathcal{O}^{Tate} of elliptic commutative tmf-algebras on the small étale site of \mathcal{M}_{Tate} .*

5.2 Patching in the p -complete smooth portion

In this section we describe the p -completion of our desired structure presheaf. We will abuse notation by referring to the composite

$$(R, M) \mapsto \mathcal{O}^{Tate}(R^\wedge_\Delta, M)$$

as a presheaf \mathcal{O}^{Tate} of elliptic commutative tmf-algebras on \mathcal{W} , the category of affine étale opens of $\overline{\mathcal{M}}_{log}$ classifying Weierstrass curves (3.20).

Proposition 5.9 *For objects $\text{Spec}(R, M) \rightarrow \overline{\mathcal{M}}_{log}$ of \mathcal{W} , the natural map*

$$(\Delta^{-1}R)_p^\wedge \rightarrow (\Delta^{-1}R_\Delta^\wedge)_p^\wedge$$

over $\overline{\mathcal{M}}_{log}$ lifts, up to equivalence, to a map of presheaves of p -complete elliptic commutative tmf-algebras on \mathcal{W} :

$$\mathcal{O}^{smooth}(\Delta^{-1}R)_p^\wedge \rightarrow (\Delta^{-1}\mathcal{O}^{Tate}(R, M))_p^\wedge$$

Proof Let $v_1 \in R$ be a lift of the Hasse invariant, which vanishes on supersingular curves; it maps to a unit in $R_{(\Delta, p)}^\wedge$ because nodal elliptic curves are never supersingular. Therefore, we have a factorization

$$(\Delta^{-1}R)_p^\wedge \rightarrow ((v_1\Delta)^{-1}R)_p^\wedge \rightarrow (\Delta^{-1}R_\Delta^\wedge)_p^\wedge,$$

and so this is equivalent to producing a natural map

$$\mathcal{O}^{smooth}((v_1\Delta)^{-1}R)_p^\wedge \rightarrow (\Delta^{-1}\mathcal{O}^{Tate}(R, M))_p^\wedge.$$

The rings $((v_1\Delta)^{-1}R)_p^\wedge$ satisfy the criteria for Proposition 4.49, so the spaces of algebra maps $\mathcal{O}^{smooth}((v_1\Delta)^{-1}R)_p^\wedge \rightarrow (\Delta^{-1}\mathcal{O}^{Tate}(R', M'))_p^\wedge$ are homotopically discrete and equivalent to the set of maps $(R, M) \rightarrow (R', M')$ over $\overline{\mathcal{M}}_{log}$. By Corollary 4.13, up to equivalence we may construct a genuinely commutative diagram as desired. \square

Definition 5.10 The presheaf \mathcal{O}_p^\wedge , of p -complete elliptic commutative tmf-algebras on \mathcal{W} , sends $\text{Spec}(R, M) \rightarrow \overline{\mathcal{M}}_{log}$ to the homotopy pullback in the following diagram:

$$\begin{CD} \mathcal{O}_p^\wedge(R, M) @>>> \mathcal{O}^{Tate}(R, M)_p^\wedge \\ @VVV @VVV \\ \mathcal{O}^{smooth}(\Delta^{-1}R)_p^\wedge @>>> (\Delta^{-1}\mathcal{O}^{Tate}(R, M))_p^\wedge \end{CD}$$

We have the following.

Proposition 5.11 *For any affine log scheme $(R, M) \in \mathcal{W}$ carrying the generalized elliptic curve \mathcal{E} , the spectrum $\mathcal{O}_p^\wedge(R)$ is an elliptic cohomology theory associated to the elliptic curve \mathcal{E} over R_p^\wedge .*

Proof We consider the following diagram:

$$\begin{array}{ccc}
 (\omega^{\otimes n})_p^\wedge & \xrightarrow{\quad\quad\quad} & (\omega^{\otimes n})_{(p,\Delta)}^\wedge \\
 \downarrow & & \downarrow \\
 (\Delta^{-1}(\omega^{\otimes n}))_p^\wedge & \longrightarrow & (\Delta^{-1}(\omega^{\otimes n})_\Delta)^\wedge
 \end{array} \tag{5.1}$$

Before p -adic completion, this square is bicartesian by Proposition 3.14. As the terms consist of torsion-free groups, this property is preserved by p -adic completion.

Taking homotopy groups in Definition 5.10, the Mayer–Vietoris sequence for pullbacks shows that the odd homotopy groups of the pullback are zero and that the even homotopy groups are the tensor powers of ω , as desired. By Lemma 4.5, the result is an elliptic commutative ring spectrum. \square

5.3 Patching in the rational part

In this section, we will use an arithmetic square to construct $\mathcal{O}_\mathbb{Q}$ and \mathcal{O} .

For an object $(R, M) \in \mathcal{W}$, we have a well-defined value of $\mathcal{O}_\mathbb{Q}$ on the restriction $\Delta^{-1}R$ to the smooth locus.

Proposition 5.12 *Up to equivalence, there exists a map*

$$\mathcal{O}^{smooth}(\Delta^{-1}R)_\mathbb{Q} \rightarrow \Delta^{-1}\mathcal{O}^{Tate}(R, M)_\mathbb{Q}$$

of presheaves of elliptic commutative tmf-algebras on \mathcal{W} that fits into the following commutative diagram:

$$\begin{array}{ccc}
 \mathcal{O}^{smooth}(\Delta^{-1}R)_\mathbb{Q} & \xrightarrow{\quad\quad\quad} & \Delta^{-1}\mathcal{O}^{Tate}(R, M)_\mathbb{Q} \\
 \downarrow & & \downarrow \\
 (\prod_p \mathcal{O}^{smooth}(\Delta^{-1}R)_p^\wedge)_\mathbb{Q} & \longrightarrow & \Delta^{-1}(\prod_p \mathcal{O}^{Tate}(R, M)_p^\wedge)_\mathbb{Q}
 \end{array}$$

Proof By Proposition 3.12, the functor $\mathcal{O}_\mathbb{Q}^{smooth}$ takes values in the category of commutative $\text{tmf}_\mathbb{Q}$ -algebras such that map $\pi_*\text{tmf}_\mathbb{Q} \rightarrow \pi_*\mathcal{O}_\mathbb{Q}^{smooth}(R)$ is étale. Proposition 4.23 then implies that the space of maps of elliptic commutative tmf-algebras out of $\mathcal{O}^{smooth}(\Delta^{-1}R)_\mathbb{Q}$ is always homotopically discrete and equivalent to the set of algebraic maps. By Corollary 4.13, we may construct the desired lift up to equivalence. \square

Definition 5.13 The presheaf $\mathcal{O}_{\mathbb{Q}}$ of rational elliptic commutative tmf-algebras on \mathcal{W} sends $\text{Spec}(R, M) \rightarrow \overline{\mathcal{M}}_{log}$ to the homotopy pullback in the diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{Q}}(R, M) & \longrightarrow & \mathcal{O}^{Tate}(R, M)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ \mathcal{O}^{smooth}(\Delta^{-1}R)_{\mathbb{Q}} & \longrightarrow & \Delta^{-1}\mathcal{O}^{Tate}(R, M)_{\mathbb{Q}}. \end{array}$$

Proposition 5.14 For any affine log scheme $(R, M) \in \mathcal{W}$ carrying the generalized elliptic curve \mathcal{E} , the spectrum $\mathcal{O}_{\mathbb{Q}}(R, M)$ is an elliptic spectrum associated to the elliptic curve \mathcal{E} over $R_{\mathbb{Q}}$, and maps in \mathcal{W} become maps of elliptic spectra.

Up to equivalence, there is a map

$$\mathcal{O}_{\mathbb{Q}}(R, M) \rightarrow \left(\prod_p \mathcal{O}_p^{\wedge}(R, M) \right)_{\mathbb{Q}}$$

of presheaves of elliptic commutative tmf-algebras.

Proof For any n , we have a bicartesian square

$$\begin{array}{ccc} \omega_{\mathbb{Q}}^{\otimes n} & \longrightarrow & ((\omega^{\otimes n})_{\Delta}^{\wedge})_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ \Delta^{-1}\omega_{\mathbb{Q}}^{\otimes n} & \longrightarrow & \Delta^{-1}((\omega^{\otimes n})_{\Delta}^{\wedge})_{\mathbb{Q}} \end{array}$$

which is obtained by rationalizing the corresponding bicartesian square for the finitely generated module $\omega^{\otimes n}$. Therefore, the homotopy groups of $\mathcal{O}_{\mathbb{Q}}(R, M)$ are the modules $\omega_{\mathbb{Q}}^{\otimes n}$, and by Lemma 4.5 this is an elliptic spectrum realizing \mathcal{E} .

To construct the given map, we note that we now have a natural diagram of commutative tmf-algebras:

$$\begin{array}{ccccc} \mathcal{O}^{Tate}(R, M)_{\mathbb{Q}} & \longrightarrow & \Delta^{-1}\mathcal{O}^{Tate}(R, M)_{\mathbb{Q}} & \longleftarrow & \mathcal{O}_{\mathbb{Q}}(\Delta^{-1}R) \\ \downarrow & & \downarrow & & \downarrow \\ (\prod_p \mathcal{O}^{Tate}(R, M)_p^{\wedge})_{\mathbb{Q}} & \longrightarrow & \Delta^{-1}(\prod_p \mathcal{O}^{Tate}(R, M)_p^{\wedge})_{\mathbb{Q}} & \longleftarrow & (\prod_p \mathcal{O}_p^{\wedge}(\Delta^{-1}R))_{\mathbb{Q}} \end{array}$$

Taking homotopy pullbacks in rows gives the desired natural map. □

Definition 5.15 The presheaf \mathcal{O} of elliptic commutative tmf-algebras on \mathcal{W} sends $\mathrm{Spec}(R, M)$ to the homotopy pullback in the diagram

$$\begin{CD} \mathcal{O}(R, M) @>>> \prod_p \mathcal{O}_p^\wedge(R, M) \\ @VVV @VVV \\ \mathcal{O}_\mathbb{Q}(R, M) @>>> (\prod_p \mathcal{O}_p^\wedge(R, M))_\mathbb{Q}. \end{CD}$$

As a consequence of Lemma 4.5, we find that $\mathcal{O}(R, M)$ is a functorial elliptic spectrum realizing the elliptic curve \mathcal{E} on R .

The natural map $\mathcal{O}^{Tate} \rightarrow \mathcal{O}^{mult}$ of Proposition 5.6 now allows us to evaluate at the cusps.

Proposition 5.16 *There is a map*

$$\mathcal{O}(R, M) \rightarrow \mathcal{O}^{mult}(\overline{R})$$

of presheaves on \mathcal{W} , where the range is the form of K-theory associated to the cusp subscheme of $\mathrm{Spec}(R)$.

By Proposition 4.36, we may automatically extend \mathcal{O} and \mathcal{O}^{mult} , as well as the map between them, to presheaves of elliptic commutative tmf-algebras on the log-étale site of $\overline{\mathcal{M}}_{log}$ that satisfy homotopy descent.

Theorem 5.17 *There exists a realization of the universal elliptic curve on the small log-étale site of $\overline{\mathcal{M}}_{log}$ by a presheaf \mathcal{O} of elliptic commutative tmf-algebras, satisfying homotopy descent for hypercovers, together with a map $\mathcal{O} \rightarrow \mathcal{O}^{mult}$ realizing evaluation at the cusp.*

The descent property now allows us to take sections on any Deligne–Mumford stack equipped with a logarithmic structure which is log-étale over $\overline{\mathcal{M}}_{log}$.

6 Tmf with level structure

Having constructed our derived structure presheaf \mathcal{O} in the previous section, we can now evaluate it on the modular curves $\overline{\mathcal{M}}(\Gamma)$ from Sect. 3.6.

Theorem 6.1 *There exists a contravariant functor Tmf from the category \mathcal{L} of Definition 3.17 to elliptic commutative tmf-algebras, taking a pair (N, Γ) to an object $\mathrm{Tmf}(\Gamma)$ which is $\mathbb{Z}[1/N]$ -local. When $N = 1$ (and hence Γ is trivial) this recovers the nonconnective, nonperiodic spectrum Tmf .*

For any such Γ , there is a spectral sequence

$$H^s(\overline{\mathcal{M}}(\Gamma); \omega^{\otimes t/2}) \Rightarrow \pi_{t-s}\mathrm{Tmf}(\Gamma),$$

and for any $K \triangleleft \Gamma < \text{GL}_2(\mathbb{Z}/N)$ the natural map

$$\text{Tmf}(\Gamma) \rightarrow \text{Tmf}(K)^{h\Gamma/K}$$

is an equivalence.

If $p: \text{GL}_2(\mathbb{Z}/NM) \rightarrow \text{GL}_2(\mathbb{Z}/N)$ is the natural projection, the map $\text{Tmf}(\Gamma) \rightarrow \text{Tmf}(p^{-1}\Gamma)$ is a localization formed by inverting M .

Proof The functoriality of the modular curves $\overline{\mathcal{M}}(\Gamma)$ as log-étale objects over $\overline{\mathcal{M}}_{\log}$ was discussed in Proposition 3.18. Therefore, we may apply \mathcal{O} to obtain a functorial diagram of commutative tmf-algebras. By definition, the value on the terminal object is Tmf . The statement about localizations is true due to the global section functor commuting with homotopy colimits, since the map $\overline{\mathcal{M}}(\Gamma) \rightarrow \mathcal{M}_{fg}$ is tame [38, 4.12]; the ring of sections over a localization is the localization of the ring of sections.

The spectral sequence for the cohomology of $\text{Tmf}(\Gamma)$ is Corollary 4.37, while the equivalence from $\overline{\mathcal{M}}(K)$ to the homotopy fixed-point object $\overline{\mathcal{M}}(\Gamma)^{h\Gamma/K}$ is Proposition 4.38. □

We can also evaluate at the cusps.

Theorem 6.2 *Let $K(\Gamma)$ be the natural form of K -theory associated to the cusp substack of $\overline{\mathcal{M}}(\Gamma)$. There is a natural transformation of commutative tmf-algebras*

$$\text{Tmf}(\Gamma) \rightarrow K(\Gamma).$$

In particular, we can apply Theorem 6.1 to the specific cover $\overline{\mathcal{M}}_1(3) \rightarrow \overline{\mathcal{M}}_0(3)$, obtaining in particular an (almost) integral lift of the work in [31].

Theorem 6.3 *There exists a commutative tmf-algebra $\text{tmf}_0(3)$ (with 3 inverted) whose homotopy groups form the “positive” portion of the homotopy groups of $\text{TMF}_0(3)$ described in [40, §7]. This fits into a commutative diagram of commutative tmf-algebras*

$$\begin{array}{ccc} \text{tmf}_0(3) & \longrightarrow & ko[1/3] \\ \downarrow & & \downarrow \\ \text{tmf}_1(3) & \longrightarrow & ku[1/3]. \end{array}$$

Proof The cohomology of the moduli stack $\overline{\mathcal{M}}_1(3)$ was essentially determined in [40]; see also [31]. The cohomology groups $H^0(\overline{\mathcal{M}}_1(3); \omega^{\otimes t})$ form the graded ring $\mathbb{Z}[1/3, a_1, a_3]$, which come from the universal cubic curve

$$y^2 + a_1xy + a_3y = x^3$$

carrying a triple intersection with the line $y = 0$. The other cohomology groups $H^s(\overline{\mathcal{M}}_1(3); \omega^{\otimes t})$ are concentrated in $s = 1, t \leq -4$, and so the positive-degree homotopy groups of $\mathrm{Tmf}_1(3)$ form the graded ring $\mathbb{Z}[1/3, a_1, a_3]$. Moreover, the open subscheme $\mathcal{M}_1(3) \subset \overline{\mathcal{M}}_1(3)$ induces a map of commutative tmf-algebras $\mathrm{Tmf}_1(3) \rightarrow \mathrm{TMF}_1(3)$ which, on homotopy groups, inverts the elliptic discriminant $\Delta = a_3^3(a_1^3 - 27a_3)$.

The homotopy fixed-point spectral sequence for the homotopy groups of $\mathrm{Tmf}_0(3)$, in positive degrees, consists of the part of the computation carried out by Mahowald–Rezk which involves no negative powers of Δ . The portion of the spectral sequence with $s > t - s \geq 0$ consists of spurious classes which are annihilated by differentials, and so the portion of the spectral sequence with $s \leq t - s$ converges to the homotopy groups of the connective cover $\mathrm{tmf}_0(3)$, which is the “positive” portion of the computation described in [40, §7].

Evaluating at the cusps, we obtain a diagram of commutative tmf-algebras

$$\begin{array}{ccc} \mathrm{Tmf}_0(3) & \longrightarrow & KO[1/3] \\ \downarrow & & \downarrow \\ \mathrm{Tmf}_1(3) & \longrightarrow & KU[1/3] \times KU^\tau[1/3], \end{array}$$

a global version of the one described in [31], with KU^τ a form of K -theory. Projecting away from the factor $KU^\tau[1/3]$ and taking connective covers, we obtain a diagram of commutative tmf-algebras giving the desired connective lifts. □

Remark 6.4 For sufficiently small subgroups $\Gamma < \mathrm{GL}_2(\mathbb{Z}/N)$, the modular curve $\overline{\mathcal{M}}(\Gamma)$ is genuinely a scheme, and the cohomology of $\overline{\mathcal{M}}(\Gamma)$ is concentrated in degrees 0 and 1. The spectral sequence degenerates to isomorphisms, where the cohomology in the following is implicitly cohomology of $\overline{\mathcal{M}}(\Gamma)$:

$$\begin{aligned} \pi_{2t} \mathrm{Tmf}(\Gamma) &\cong H^0(\omega^{\otimes t}) \\ \pi_{2t-1} \mathrm{Tmf}(\Gamma) &\cong H^1(\omega^{\otimes t}) \end{aligned}$$

The even-degree homotopy groups form the ring of modular forms for Γ over $\mathbb{Z}[1/N]$, and are concentrated in nonnegative degrees. Duality for H^1 (specifically, Grothendieck–Serre duality [18]) takes the form of an exact sequence

$$\begin{aligned} 0 \rightarrow \mathrm{Ext}(H^1(\kappa \otimes \omega^{\otimes(-t)}), \mathbb{Z}[1/N]) &\rightarrow H^1(\omega^{\otimes t}) \\ \rightarrow \mathrm{Hom}(H^0(\kappa \otimes \omega^{\otimes(-t)}), \mathbb{Z}[1/N]) &\rightarrow 0. \end{aligned}$$

In these cases we have an isomorphism of $\omega^{\otimes 2}$ with the logarithmic cotangent complex, which is the twist $\kappa(D)$ of the canonical bundle by the cusp divisor. This allows us to recast H^1 as coming from an exact sequence

$$\begin{aligned}
 0 &\rightarrow \text{Ext}(H^1(\omega^{\otimes(2-t)}(-D)), \mathbb{Z}[1/N]) \rightarrow H^1(\omega^{\otimes t}) \\
 &\rightarrow \text{Hom}(H^0(\omega^{\otimes(2-t)}(-D)), \mathbb{Z}[1/N]) \rightarrow 0.
 \end{aligned}$$

Any p -torsion elements in H^1 arise due to forms of weight $k \bmod p$ that do not lift to integral forms, and outside weight 1 this does not occur as a consequence of the Riemann–Roch formula.

Degree considerations imply that the only possible nonzero homotopy group in positive, odd degree is $\pi_1(\text{Tmf}(\Gamma))$. This has possible contributions from both the dual of the space $H^0(\omega(-D))$, parametrizing cusps of weight 1 and level Γ , and the Pontrjagin dual of its torsion.

Using a Postnikov tower, we can eliminate some of this torsion from the homotopy groups of $\text{Tmf}(\Gamma)$. However, it is not clear if there is a conceptually correct way to do so. The torsion of $H^1(\omega)$ and $H^1(\omega(-D))$ are Pontrjagin dual, and measure the failure of weight-1 forms with level Γ to lift. The story of these non-liftable forms of weight one seems to be just beginning to emerge [11, 47].

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Appendix: The Witten genus

The goal of this section is to construct a map of commutative ring spectra

$$\text{tmf} \rightarrow KO[[q]]$$

which, on homotopy groups, factors the Witten genus $MSpin_* \rightarrow \mathbb{Z}[[q]]$. Here the power series notation $KO[[q]]$ is shorthand for the homotopy limit of the monoid algebras

$$\text{holim}_r KO \wedge \{1, q, \dots, q^{r-1}\}_+$$

where q^r is identified with the basepoint (as in Sect. 5.1). The main result (Theorem 6.12) is well-known and featured prominently in earlier, unpublished, constructions of tmf , but to the knowledge of the authors it does not appear in the literature. The relation of the Tate curve to power operations has

been extensively explored, especially in this context by Baker [5], Ando [4], Ando–Hopkins–Strickland [2], and Ganter [15].

Definition 6.5 For a chosen prime p , the p -adic K -theory of a spectrum X is

$$K_*^\vee(X) = \pi_* L_{K(1)}(K \wedge X) = \pi_* \operatorname{holim}_k (K/p^k \wedge X).$$

In particular, the coefficient ring K_*^\vee is the graded ring $\mathbb{Z}_p[\beta^{\pm 1}]$.

Here K/p^k is the mapping cone of the multiplication-by- p^k endomorphism of K , having a long exact sequence

$$\dots \rightarrow K_*^\vee(X) \xrightarrow{p^k} K_*^\vee(X) \rightarrow \pi_*(K/p^k \wedge X) \rightarrow \dots \tag{6.1}$$

which is natural in X .

Remark 6.6 As K -modules and KO -modules are automatically $E(1)$ -local, $K(1)$ -localizations and p -completions are equivalent on them.

We recall the following result, which was classically used as a definition of $K(1)$ -local tmf at the prime 2.

Proposition 6.7 ([20, 29]) *At $p = 2$, there are homotopy pushout diagrams*

$$\begin{array}{ccc} L_{K(1)}\mathbb{P}(S^{-1}) & \xrightarrow{0} & L_{K(1)}\mathbb{S} & & L_{K(1)}\mathbb{P}(S^0) & \xrightarrow{0} & L_{K(1)}\mathbb{S} \\ \zeta \downarrow & & \downarrow & & \theta(f)-h(f) \downarrow & & \downarrow \\ L_{K(1)}\mathbb{S} & \longrightarrow & T_\zeta & & T_\zeta & \longrightarrow & L_{K(1)}\operatorname{tmf} \end{array}$$

in the category of $K(1)$ -local commutative ring spectra. Here ζ is a topological generator of $\pi_{-1}L_{K(1)}\mathbb{S} \cong \mathbb{Z}_2$; f is an element in π_0T_ζ ; and $h(x)$ is a 2-adically convergent power series such that for any $K(1)$ -local elliptic commutative ring spectrum E , any map of commutative ring spectra $T_\zeta \rightarrow E$ automatically sends $\theta(f)$ and $h(f)$ to the same element.

We first need to identify the p -adic K -theory of $KO[[q]]$.

Proposition 6.8 *For any prime p , the p -adic K -theory of $KO[[q]]$ is the ring*

$$K_*^\vee(KO[[q]]) \cong \operatorname{Map}_c(\mathbb{Z}_p^\times, K_*^\vee[[q]])^{\{\pm 1\}}.$$

Here the group $\{\pm 1\} \subset \mathbb{Z}_p^\times$ acts by conjugation on the group of continuous homomorphisms, and the ring $K_^\vee[[q]]$ is given the p -adic topology.*

This is the universal p -complete $\mathbb{Z}[[q]]$ -algebra with an isomorphism class of pairs of an invariant 1-form on the Tate curve T and an identification $\widehat{\mathbb{G}}_m \xrightarrow{\sim} \widehat{T}$ between the formal multiplicative group and the formal group of the Tate curve. The map $V \rightarrow K_0^\vee(KO[[q]])$ determined by this is a map of ψ - θ -algebras.

Proof We recall from [20] or [1, 9.2] that the map of ψ - θ -algebras

$$K_*^\vee KO \rightarrow K_*^\vee K$$

is the inclusion

$$\text{Map}_c(\mathbb{Z}_p^\times, K_*^\vee)^{\{\pm 1\}} \hookrightarrow \text{Map}_c(\mathbb{Z}_p^\times, K_*^\vee)$$

of sets of continuous maps. The long exact sequence (6.1) gives an identification

$$\begin{aligned} \pi_*(K/p^k \wedge KO) &\cong \text{Map}_c(\mathbb{Z}_p^\times, (K_*)/p^k)^{\{\pm 1\}} \\ &= \text{colim}_m \text{Map}((\mathbb{Z}/p^m)^\times, (K_*)/p^k)^{\{\pm 1\}}. \end{aligned}$$

The graded π_*KO -module $\pi_*(KO[[q]]) \cong \pi_*KO \otimes_{\mathbb{Z}} \mathbb{Z}[[q]]$ is flat, and so the isomorphism

$$K/p^k \wedge KO[[q]] \cong (K/p^k \wedge KO) \underset{KO}{\wedge} KO[[q]]$$

can be re-expressed as an isomorphism

$$\pi_*(K/p^k \wedge KO[[q]]) \cong \text{Map}_c(\mathbb{Z}_p^\times, K_*[[q]]/p^k)^{\{\pm 1\}}$$

by the Künneth formula. Taking limits gives the desired formula for the p -adic K -theory.

The action of the group \mathbb{Z}_p^\times on $\text{Map}_c(\mathbb{Z}_p^\times, K_*^\vee)^{\{\pm 1\}}$, by premultiplication on the source, is compatible with this isomorphism, and therefore determines the action of \mathbb{Z}_p^\times on $K_*^\vee(KO[[q]])$: it is coinduced from the action of the subgroup $\{\pm 1\}$ on $K_*^\vee[[q]]$. The ring $\text{Map}_c(\mathbb{Z}_p^\times, K_*^\vee[[q]])$ is the universal ring classifying isomorphisms $\widehat{\mathbb{G}}_m \rightarrow \widehat{T}$ together with a choice of invariant 1-form, as any such isomorphism differs from the canonical one by a locally constant function to \mathbb{Z}_p^\times . As a $\{\pm 1\}$ -equivariant algebra over the ring of invariants $K_*^\vee KO[[q]]$ it is isomorphic to $K_*^\vee KO[[q]] \times K_*^\vee KO[[q]]$, and so the ring of invariants classifies the quotient by $\text{Aut}(T) \cong \{\pm 1\}$. There is a map $V \rightarrow K_0^\vee KO[[q]]$ determined by the universal property of V .

The element $q \in K_0^\vee KO[[q]]$, since it lifts to an element in $\pi_0 KO[[q]]$, is acted on trivially by \mathbb{Z}_p^\times . Moreover, the extended power operation ψ^p lifts to the corresponding power operation on the discrete monoid \mathbb{N} , which sends q to q^p . The resulting map on $\mathbb{Z}[[q]]$ classifies the quotient of the Tate curve by the canonical subgroup μ_p of its formal group (Sect. 3.4), and thus the map $V \rightarrow K_0^\vee(KO[[q]])$ preserves the operation ψ^p (and hence θ). \square

Proposition 6.9 *At the prime 2, there exists a map of $K(1)$ -local commutative ring spectra*

$$L_{K(1)}\mathrm{tmf} \rightarrow L_{K(1)}KO[[q]]$$

which, on 2-adic K -homology, induces the map

$$V \rightarrow K_0^\vee(KO[[q]])$$

from Proposition 6.8.

Proof We will use the description of $K(1)$ -local tmf from Proposition 6.7 to construct this map.

As $KO[[q]]_2^\wedge$ is the $K(1)$ -localization of $KO[[q]]$ and has trivial π_{-1} , we have a map of commutative ring spectra $T_\zeta \rightarrow KO[[q]]_2^\wedge$. The composite map $T_\zeta \rightarrow K[[q]]_2^\wedge$ detects the effect on π_0 , and is a map to an elliptic cohomology theory, where the latter carries the Tate curve over the power series ring $\mathbb{Z}[[q]]_2^\wedge$. Therefore, the element $\theta(f) - h(f)$ automatically maps to zero, and we obtain an extension $L_{K(1)}\mathrm{tmf} \rightarrow L_{K(1)}KO[[q]]$. \square

Proposition 6.10 *At any odd prime p , there exists a map of $K(1)$ -local commutative ring spectra*

$$L_{K(1)}\mathrm{tmf} \rightarrow L_{K(1)}KO[[q]]$$

which, on p -adic K -theory, induces the map

$$V \rightarrow K_0^\vee(KO[[q]])$$

from Proposition 6.8.

Proof As $KO[[q]]$ is the homotopy fixed-point spectrum of the action of $\{\pm 1\}$ on $K[[q]]$, we have an equivalence

$$\begin{aligned} \mathrm{Map}_{\mathrm{comm}}(L_{K(1)}\mathrm{tmf}, L_{K(1)}KO[[q]]) \\ \simeq \mathrm{Map}_{\mathrm{comm}}(L_{K(1)}\mathrm{tmf}, L_{K(1)}K[[q]])^{h\{\pm 1\}}. \end{aligned}$$

The Goerss–Hopkins obstruction theory computing this space of maps of $K(1)$ -local commutative ring spectra produces obstructions in André–Quillen cohomology groups. There is a fringed spectral sequence with E_2 -term given by

$$E_2^{s,t} = \begin{cases} \text{Hom}_{\psi\text{-}alg/K_*^\vee}(K_*^\vee \text{tmf}, K_*^\vee K[[q]]) & \text{if } (s, t) = (0, 0), \\ H_{\psi\text{-}alg/K_*^\vee}^s(K_*^\vee \text{tmf}, \Omega^t K_*^\vee K[[q]]) & \text{otherwise.} \end{cases}$$

This spectral sequence converges to $\pi_{t-s} \text{Map}_{\text{comm}}(L_{K(1)} \text{tmf}, L_{K(1)} K[[q]])$. By [6, 7.5], the fact that V is formally smooth over \mathbb{Z}_p implies that the obstruction groups

$$H_{\psi\text{-}alg/K_*^\vee}^s(K_*^\vee \text{tmf}, \Omega^t K_*^\vee K[[q]])$$

are trivial for $s > 1$ or $t = 0$.

In particular, the homotopy groups $\pi_t \text{Map}_{\text{comm}}(L_{K(1)} \text{tmf}, L_{K(1)} K[[q]])$ are p -adically complete abelian groups for any choice of basepoint, and so the homotopy fixed-point spectral sequence for the action of the group $\{\pm 1\}$ degenerates. We find that the set of path components is

$$\pi_0 \text{Map}_{\text{comm}}(L_{K(1)} \text{tmf}, L_{K(1)} K[[q]]) \cong \text{Hom}_{\psi\text{-}alg/K_*^\vee}(V, K_*[[q]])^{\{\pm 1\}},$$

and so the map of Proposition 6.8 has a lift which is unique up to homotopy. □

Proposition 6.11 *There exists a map of rational commutative ring spectra*

$$\text{tmf}_{\mathbb{Q}} \rightarrow (KO[[q]])_{\mathbb{Q}}$$

which, on homotopy groups, is given by a map

$$\mathbb{Q}[c_4, c_6] \rightarrow \mathbb{Q} \otimes \mathbb{Z}[[q]][[\beta^{\pm 2}]]$$

sending c_4 and c_6 to their q -expansions. The two maps

$$\text{tmf} \rightarrow \left(\prod_P KO[[q]]_p^\wedge \right)_{\mathbb{Q}},$$

induced by this map and the maps constructed in Propositions 6.9 and 6.10, are homotopic as maps of commutative ring spectra.

Proof The elements c_4 and c_6 can be realized as maps $S^8 \rightarrow \text{tmf}_{\mathbb{Q}}$ and $S^{12} \rightarrow \text{tmf}_{\mathbb{Q}}$ respectively. The induced map of commutative ring spectra $\mathbb{P}_{\mathbb{Q}}(S^8 \vee S^{12}) \rightarrow \text{tmf}_{\mathbb{Q}}$ is a weak equivalence, and so homotopy classes of commutative ring spectrum maps $\text{tmf}_{\mathbb{Q}} \rightarrow (KO[[q]])_{\mathbb{Q}}$ are defined uniquely, up to homotopy, by specifying the images of c_4 and c_6 .

Homotopy classes of maps of commutative ring spectra $\text{tmf} \rightarrow (\prod_p KO[[q]]_p^{\wedge})_{\mathbb{Q}}$ are the same as maps $\text{tmf}_{\mathbb{Q}} \rightarrow (\prod_p KO[[q]]_p^{\wedge})_{\mathbb{Q}}$, and are similarly determined by the images of c_4 and c_6 . Therefore, as the $K(1)$ -local and rational constructions are both obtained by q -expansion in a neighborhood of the Tate curve, the resulting pair of maps are homotopic as maps of commutative ring spectra. \square

Theorem 6.12 *There exists a map of commutative ring spectra*

$$\text{tmf} \rightarrow KO[[q]]$$

compatible with the $K(1)$ -local and rational maps constructed in Propositions 6.9, 6.10, and 6.11.

Proof We can express the spectrum $KO[[q]]$ as a homotopy pullback in the following arithmetic square of commutative ring spectra:

$$\begin{CD} KO[[q]] @>>> \prod_p KO[[q]]_p^{\wedge} \\ @VVV @VVV \\ (KO[[q]])_{\mathbb{Q}} @>>> (\prod_p KO[[q]]_p^{\wedge})_{\mathbb{Q}} \end{CD}$$

However, from Propositions 6.9, 6.10, and 6.11 we obtain maps from tmf to the rational and p -completed entries which are homotopic, and therefore a map from tmf to the homotopy pullback. \square

Remark 6.13 As the spectrum $(\prod_p KO[[q]]_p^{\wedge})_{\mathbb{Q}}$ has trivial homotopy groups in degrees 9 and 13, the path components of the mapping space

$$\text{Map} \left(\text{tmf}, \left(\prod_p KO[[q]]_p^{\wedge} \right)_{\mathbb{Q}} \right)$$

are all simply connected. The Mayer–Vietoris square of mapping spaces shows that there is a unique homotopy class of map of commutative ring spectra from tmf to the pullback.

References

1. Ando, M., Hopkins, M.J., Rezk, C.: Multiplicative orientations of KO -theory and of the spectrum of topological modular forms. Preprint, available at: <http://www.math.uiuc.edu/~mando/papers/koandtmf.pdf>
2. Ando, M., Hopkins, M.J., Strickland, N.P.: Elliptic spectra, the Witten genus, and the theorem of the cube. *Invent. Math.* **146**(3), 595–687 (2001)
3. Ando, M., Hopkins, M.J., Strickland, N.P.: The sigma orientation is an H_∞ map. *Am. J. Math.* **126**(2), 247–334 (2004)
4. Ando, M.: Power operations in elliptic cohomology and representations of loop groups. *Trans. Am. Math. Soc.* **352**(12), 5619–5666 (2000)
5. Baker, A.: Hecke operators as operations in elliptic cohomology. *J. Pure Appl. Algebra* **63**(1), 1–11 (1990)
6. Behrens, M.: Notes on the construction of tmf . available at: <http://www-math.mit.edu/~mbehrens/papers/buildTMF.pdf>
7. Behrens, M.: Buildings, elliptic curves, and the $K(2)$ -local sphere. *Am. J. Math.* **129**(6), 1513–1563 (2007)
8. Behrens, M., Lawson, T.: Topological automorphic forms. *Mem. Am. Math. Soc.* **204**(958), xxiv+141 (2010)
9. Basterra, M., Richter, B.: (Co-)homology theories for commutative (S -)algebras. In: Structured ring spectra, London Math. Soc. Lecture Note Ser., vol. 315, Cambridge Univ. Press, Cambridge, pp. 115–131 (2004)
10. Baker, A., Richter, B.: Realizability of algebraic Galois extensions by strictly commutative ring spectra. *Trans. Am. Math. Soc.* **359**(2), 827–857 (2007). (electronic)
11. Buzzard, K.: Computing weight one modular forms over \mathbb{C} and $\overline{\mathbb{F}}_p$, [arXiv:1205.5077](https://arxiv.org/abs/1205.5077)
12. Dugger, D., Hollander, S., Isaksen, D.C.: Hypercovers and simplicial presheaves. *Math. Proc. Camb. Philos. Soc.* **136**(1), 9–51 (2004)
13. Deligne, P., Rapoport, M.: Les schémas de modules de courbes elliptiques. In: Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), *Lecture Notes in Math.*, vol. 349, Springer, Berlin, 1973, pp. 143–316
14. Faltings, G., Chai, C.-L.: Degeneration of abelian varieties. In: *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*, vol. 22, Springer-Verlag, Berlin (1990) (With an appendix by David Mumford)
15. Ganter, N.: Power operations in orbifold Tate K -theory. *Homol. Homotopy Appl.* **15**(1), 313–342 (2013)
16. Goerss, P.G., Hopkins, M.J.: Moduli spaces of commutative ring spectra. Structured ring spectra, London Math. Soc. Lecture Note Ser., vol. 315, Cambridge Univ. Press, Cambridge, pp. 151–200 (2004)
17. Goerss, P.G.: Realizing families of Landweber exact homology theories. New topological contexts for Galois theory and algebraic geometry (BIRS 2008), *Geom. Topol. Monogr.*, vol. 16, Geom. Topol. Publ. Coventry, 49–78 (2009)
18. Hartshorne, R.: Residues and duality, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. *Lecture Notes in Mathematics*, No. 20, Springer-Verlag, Berlin (1966)
19. Hopkins, M.J., Mahowald, M.: From elliptic curves to homotopy theory. Preprint, <http://hopf.math.purdue.edu/>
20. Hopkins, M.J.: $K(1)$ -local E_∞ -ring spectra. Preprint, available at: <http://www.math.rochester.edu/people/faculty/doug/otherpapers/knlocal.pdf>
21. Hovey, M., Shipley, B., Smith, J.: Symmetric spectra. *J. Am. Math. Soc.* **13**(1), 149–208 (2000)

22. Illusie, L.: An overview of the work of K. Fujiwara, K. Kato, and C. Nakayama on logarithmic étale cohomology. *Astérisque, Cohomologies p -adiques et applications arithmétiques, II*, no. 279, pp. 271–322 (2002)
23. Jardine, J.F.: Presheaves of symmetric spectra. *J. Pure Appl. Algebra* **150**(2), 137–154 (2000)
24. Johnstone, P.T.: *Sketches of an Elephant: a Topos Theory Compendium*. Vol. 2, Oxford Logic Guides, vol. 44. The Clarendon Press, Oxford University Press, Oxford (2002)
25. Katz, N.M.: Higher congruences between modular forms. *Ann. of Math. (2)* **101**, 332–367 (1975)
26. Kato, K.: *Logarithmic structures of Fontaine-Illusie*. Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD pp. 191–224 (1989)
27. Katz, N.M., Mazur, B.: *Arithmetic Moduli of Elliptic Curves*, Annals of Mathematics Studies, vol. 108. Princeton University Press, Princeton (1985)
28. Landweber, P.S. (ed.): *Elliptic curves and modular forms in algebraic topology*. Lecture Notes in Mathematics, vol. 1326. Springer-Verlag, Berlin (1988)
29. Laures, G.: $K(1)$ -local topological modular forms. *Invent. Math.* **157**(2), 371–403 (2004)
30. Lawson, T., Naumann, N.: Commutativity conditions for truncated Brown-Peterson spectra of height 2. *J. Topol.* **5**(1), 137–168 (2012)
31. Lawson, T., Naumann, N.: Strictly commutative realizations of diagrams over the Steenrod algebra and topological modular forms at the prime 2. *Int. Math. Res. Not.* **2014**(10), 2773–2813 (2014)
32. Landweber, P.S., Ravenel, D.C., Stong, R.E.: *Periodic cohomology theories defined by elliptic curves*. The Čech centennial (Boston, MA, 1993), *Contemp. Math.*, vol. 181, Am. Math. Soc., Providence, RI, pp. 317–337 (1995)
33. Lurie, J.: Higher algebra, Draft version available at: <http://www.math.harvard.edu/~lurie/papers/higheralgebra.pdf>
34. Lurie, J.: A survey of elliptic cohomology. *Algebraic Topology, Abel Symp.*, vol. 4, Springer, Berlin, pp. 219–277 (2009)
35. Lurie, J.: *Higher Topos Theory*, Annals of Mathematics Studies, vol. 170. Princeton University Press, Princeton (2009)
36. Mathew, A.: The homology of tmf , [arXiv:1305.6100](https://arxiv.org/abs/1305.6100)
37. Meier, L.: *United elliptic homology*, Ph.D. thesis, Universität Bonn (2012)
38. Mathew, A., Meier, L.: Affineness and chromatic homotopy theory, [arXiv:1311.0514](https://arxiv.org/abs/1311.0514)
39. Morava, J.: Forms of K -theory. *Math. Z.* **201**(3), 401–428 (1989)
40. Mahowald, M., Rezk, C.: Topological modular forms of level 3, *Pure Appl. Math. Q.* 5, no. 2, Special Issue: In honor of Friedrich Hirzebruch. Part 1, 853–872 (2009)
41. Niziol, W.: K -theory of log-schemes. I. *Doc. Math.* **13**, 505–551 (2008)
42. Ochanine, S.: Elliptic genera, modular forms over KO_* and the Brown–Kervaire invariant. *Math. Z.* **206**(2), 277–291 (1991)
43. Ogus, A.: Lectures on logarithmic geometry, Draft version available at: http://math.berkeley.edu/~ogus/preprints/log_book/logbook.pdf
44. Quillen, D.: On the formal group laws of unoriented and complex cobordism theory. *Bull. Am. Math. Soc.* **75**, 1293–1298 (1969)
45. Rognes, J.: Topological logarithmic structures, *New topological contexts for Galois theory and algebraic geometry (BIRS 2008)*, *Geom. Topol. Monogr.*, vol. 16, Geom. Topol. Publ., Coventry, pp. 401–544 (2009)
46. Robinson, A., Whitehouse, S.: Operads and Γ -homology of commutative rings. *Math. Proc. Camb. Philos. Soc.* **132**(2), 197–234 (2002)
47. Schaeffer, G.J.: *The Hecke Stability Method and Ethereal Forms*. ProQuest LLC, Ann Arbor, MI, Thesis (Ph.D.)-University of California, Berkeley (2012)

48. Grothendieck, A.: Cohomologie l -adique et fonctions L . Lecture Notes in Mathematics, Séminaire de Géométrie Algébrique du Bois-Marie 1965–1966 (SGA 5), Edité par Luc Illusie, vol. 589, Springer-Verlag, Berlin, (1977)
49. Shipley, B.: A convenient model category for commutative ring spectra. Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K -theory, *Contemp. Math.*, vol. 346, Am. Math. Soc., Providence, RI, pp. 473–483 (2004)
50. Silverman, J.H.: The arithmetic of elliptic curves, second ed., *Graduate Texts in Mathematics*, vol. 106, Springer, Dordrecht (2009)
51. Stojanoska, V.: Duality for topological modular forms. *Doc. Math.* **17**, 271–311 (2012)
52. White, D.: Model structures on commutative monoids in general model categories, [arXiv:1403.6759](https://arxiv.org/abs/1403.6759)