On the Non-Existence of Kervaire Invariant One Manifolds

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Main Result

Theorem (H.-Hopkins-Ravenel)

There are smooth manifolds of Kervaire invariant one only in dimensions $2, 6, 14, 30, 62$, *and possibly* 126.

Each dimension listed has an associated story.

Corollary (Kervaire-Milnor)

Except in these dimensions, every framed manifold is frame-cobordant to a sphere.

And the manifolds are...

Related to Lie groups

- dim 2 $S^1 \times S^1$ with the left invariant framing.
- dim 6 $SU(2) \times SU(2)$ with the left invariant framing.
- dim 14 $S^7 \times S^7$ with the framing coming from the Caley numbers.
- dim 30 Bökstedt: related to $E_6/\big(Spin(10) \times U(1)\big)$
- dim 62 Known to exist, but...?
- dim 126 ?

Definitions & Conventions

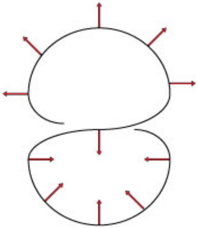
- Manifolds are assumed to be smooth and compact.
- Non-smooth manifolds are PL.

Definition

A framed n-manifold is an n-manifold (embedded in $\mathbb{R}^{n+k}$ for k big) with a choice of framing.

Pontryagin (1930s)

$\{n - \text{manifolds with framing}\}/\text{cobordism} \longleftrightarrow \pi_n^S$
Framed 0-manifolds are "oriented" points $\Rightarrow \pi_0^S = \mathbb{Z}$.
Framed 1-manifolds are framed circles.



$\Rightarrow \pi_1^S = \mathbb{Z}/2\mathbb{Z}$.

An Early Error

Framed 2-manifolds are framed surfaces.
Pontryagin: framed surgery.

- Can cut out circles and glue in disks.
- Can lower the genus of surfaces.

Get a map $\mu\colon H_1(M;\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$.
If we can do surgery: 0, if we can't: 1.

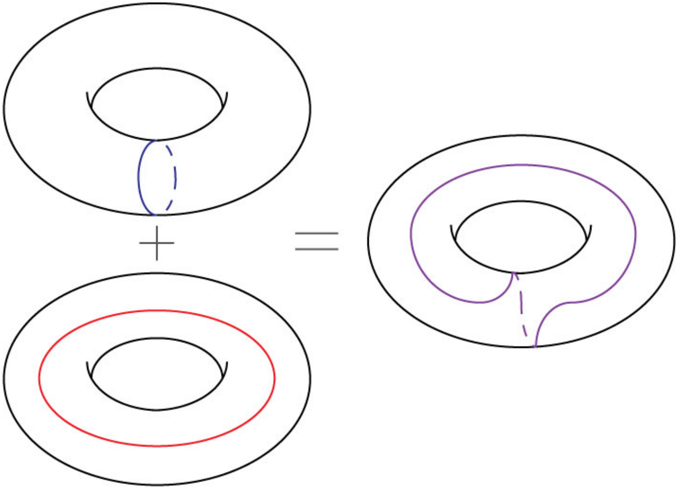
Theorem (False Theorem)

μ is linear, so every surface is framed-cobordant to S^2.

The map is actually a quadratic refinement of the intersection form λ:

$$\mu(x) + \mu(y) + \mu(x+y) = \lambda(x,y).$$

Non-Linearity



Major Changes in Geometry

Theorem (Milnor 1956)

There are manifolds homeomorphic to S^7 but not diffeomorphic to it.

Theorem (Kervaire 1960)

There are 10-dimensional topological manifolds that have no smooth structure.

Kervaire showed this by showing two things

1. Framed 10-dimensional manifolds have a distinguished quadratic refinement of the intersection form.
2. The Arf invariant is zero if the manifold is smooth.

Kervaire Problem

Definition (Kervaire Invariant)

If M is a framed $(4k+2)$-manifold, then the Kervaire invariant is the Arf invariant of a quadratic refinement to the intersection pairing.

Problem (Kervaire Invariant One Problem)

Is there a smooth n-manifold of Kervaire invariant one?

Pontryagin's mistake was that in dimension 2, the answer is yes!

Homotopy Spheres

Let Θ_k denote the group of k-dimensional exotic spheres under connect sum.
Kervaire-Milnor studied the map

$$\rho_k \colon \Theta_k \to \pi_k^S / Im(J).$$

Surgery:

- ρ_k is onto when k is odd.
- ρ_{4k} is onto.

The cokernel of ρ_{4k+2} is at most $\mathbb{Z}/2$ and is generated by a manifold of Kervaire invariant 1.

Browder's Result

Return to Pontryagin's dictionary:

$$\{\text{stable homotopy}\} \leftrightarrow \{\text{framed cobordism}\}$$

Kervaire-Milnor story is one about homotopy elements.

Theorem (Browder 1969)

1. *If $k \neq 2^\ell - 1$, then ρ_{4k+2} is surjective.*
2. *The generator of $coker(\rho_{2^{j+1}-2})$ is represented by h_j^2 in the Adams spectral sequence.*

So the whole game is understanding if h_j^2 survives. h_j is the Hopf invariant one class in dimension $2^j - 1$.

Kervaire Manifolds

$\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ exist, so have h_1^2, h_2^2, and h_3^2.

Theorem (Adams: Hopf Invariant One)

For $i > 3$, h_i does not survive the Adams spectral sequence.

Theorem (Barratt-Jones-Mahowald)

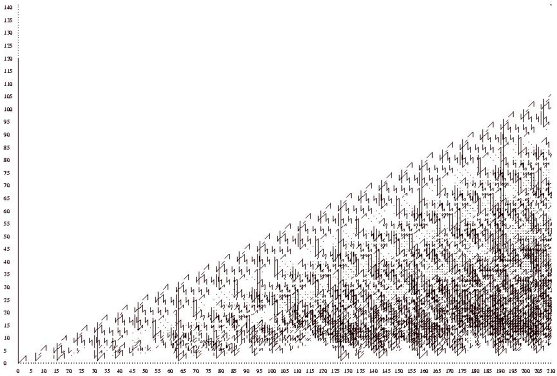
h_4^2 and h_5^2 survive the Adams spectral sequence.

So homotopy tells us the manifolds exist, but not what they are.

Problems with the Adams Spectral Sequence

Two big obstructions to just computing:

1. As the dimension increases, so does the complexity.
2. No known techniques apply.



Computational Reduction

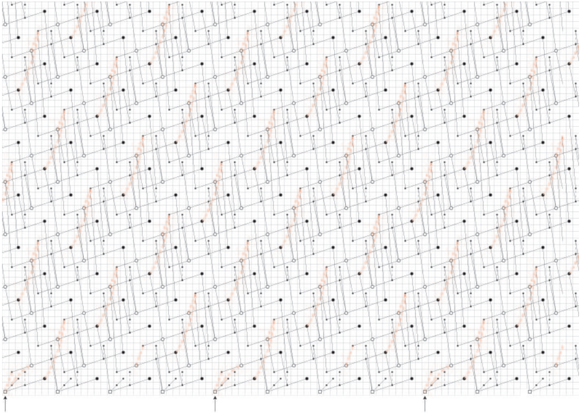
Get around this by comparing with a simpler case:

- case must detect the Kervaire classes
- case must be algebraically simpler.

Our simpler case is 16-periodic so computation is greatly simplified.

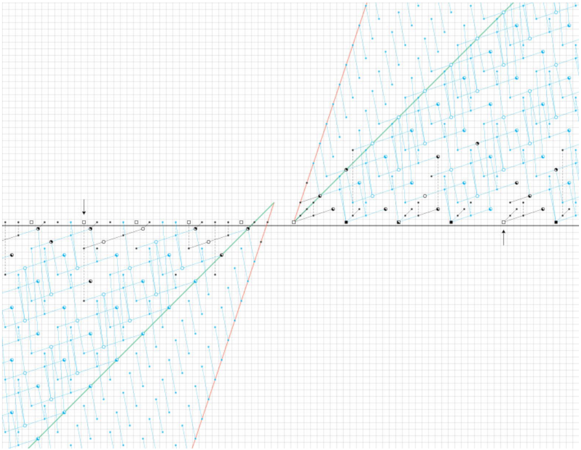
Reduce Computations

Simpler case is still tricky. We are still computing homotopy groups of a spectrum E.



Reduce Computations

Need a new computational tool. The *slice spectral sequence*.



Kervaire Invariant One

Consequences of the slice story:

1. $\pi_{-2}E = 0$.
2. $\pi_{k+256}E = \pi_k E$ for all k.

Together these imply that

$$\pi_{2^{j+1}-2}E = 0$$

for all $j > 6$.
So h_j^2 cannot survive the Adams spectral sequence for $j > 6$.