Contemporary Mathematics
Volume **729**, 2019
https://doi.org/10.1090/conm/729/14692

The centralizer resolution of the $K(2)$-local sphere at the prime 2

Hans-Werner Henn

Dedicated to Paul Goerss on the occasion of his 60th birthday

Abstract. Let $\mathbb{G}_2$ be the Morava stabilizer group at the prime 2. We construct a resolution of the $K(2)$-local sphere at the prime 2 in terms of certain homotopy fixed point spectra which are closely related to the spectrum of topological modular forms. This resolution is in certain ways analogous to the centralizer resolution of the $K(n)$-local sphere constructed in previous work of the author if p is an odd prime and $n = p - 1$.

Contents

1. Introduction

Let p be a prime, let $n > 0$ be an integer and let $K(n)$ be the n-th Morava K-theory at p. The category of $K(n)$-local spectra is a basic building block of the stable homotopy category of p-local spectra and the $K(n)$-localization of the sphere, $L_{K(n)}S^0$, plays a central role in this category. The homotopy of $L_{K(n)}S^0$ can be studied via the Adams-Novikov spectral sequence, and by [**11**] this spectral sequence can be identified with the homotopy fixed point spectral sequence for the action of the extended Morava stabilizer group $\mathbb{G}_n$ on E_n. Here E_n denotes the 2-periodic Landweber exact spectrum whose coefficients in degree 0 classify

The origin of this paper goes back to the early 2000's and was stimulated by the joint work with P. Goerss, M. Mahowald and C. Rezk [**15**]. Some of the results were announced in [**21**] but proofs of the existence of the centralizer resolutions were never published. Recent work by Beaudry [**2**], [**3**], [**4**], by Bobkova and Goerss [**8**] and a joint project with Beaudry and Goerss [**5**] have underlined the importance of these resolutions. The author apologizes for the delay in making these results available and he is happy to acknowledge helpful discussions with Goerss, Mahowald, Rezk, Beaudry and Bobkova which have led to this research, to improvements of the original results and to simplifications of the proofs.

This work was supported by ANR-16-CE40-0003 ChroK.

deformations (in the sense of Lubin and Tate) of a suitable formal group law Γ_n of height n over $\mathbb{F}_{p^n}$ and $\mathbb{G}_n$ is the automorphism group of Γ_n in the category of formal group laws (cf [**32**]). The E_2-page of this spectral sequence is given by the continuous cohomology $H^*_{cts}(\mathbb{G}_n,(E_n)_*)$ of $\mathbb{G}_n$ with coefficients in $(E_n)_*$. It becomes therefore interesting to find resolutions of the trivial module for the group $\mathbb{G}_n$ from which one can calculate this continuous cohomology.

If p is large with respect to n then the E_2-page satisfies $E_2^{s,*}=0$ for $s>n^2$ and the spectral sequence collapses at its E_2-page. In the sequel we concentrate on the case $n=2$ because the case $n=1$ is well understood and very little is understood in explicit terms if $n>2$.

For $n=2$ the spectral sequence collapses if and only if $p>3$. In these cases the homotopy of $L_{K(2)}S^0$ has been calculated in [**33**] without using the point of view of group cohomology. The results have been reinterpreted in [**6**] and an independent calculation for the Moore space has been carried out in [**27**] by using an explicit projective resolution of length 4 of the trivial $\mathbb{G}_2$-module $\mathbb{Z}_p$.

If $n=2$ and $p\le 3$ the mod-p cohomological dimension of the group $\mathbb{G}_2$ is infinite and there cannot be any projective resolution of the trivial $\mathbb{G}_2$-module $\mathbb{Z}_p$ of finite length. However, for $p=3$ very useful resolutions of the trivial $\mathbb{G}_2$-module $\mathbb{Z}_p$ of length 4 in terms of more general modules and corresponding topological resolutions of $L_{K(2)}S^0$ exist; a "duality resolution" has been constructed in [**15**] and a "centralizer resolution" in [**21**]. These resolutions complement each other and they have been crucial in recent progress of our understanding of $K(2)$-local homotopy theory at the prime 3. In particular they have been used for proving the chromatic splitting conjecture for $n=2$ [**14**], for determining Hopkins' Picard group of $K(2)$-local spectra [**25**], [**16**] and for identifying the Brown-Comenetz dual of the $K(2)$-local sphere [**17**].

If $n=2$ and $p=2$ our understanding is less complete although the chromatic splitting conjecture has already been successfully analyzed in [**4**] and [**5**] by heavily using the algebraic and topological duality resolutions for an important subgroup $\mathbb{S}_2^1$ of $\mathbb{G}_2$. The existence of an algebraic duality and an algebraic centralizer resolution of length 3 for $\mathbb{S}_2^1$ was already announced in [**21**], as well as a topological centralizer resolution for the homotopy fixed point spectrum $E_2^{h\mathbb{S}_2^1}$, in all cases without proofs. For the algebraic duality resolution the construction was finally established in [**2**] and the construction of its topological counterpart was given in [**8**]. The latter paper relied heavily on the existence of both the algebraic and topological centralizer resolution for $\mathbb{S}_2^1$ for which no proof has been published yet. The main purpose of this paper is to fill this gap in the literature and extend the announced results from the group $\mathbb{S}_2^1$ to $\mathbb{S}_2$, which is another important subgroup of $\mathbb{G}_2$, and even to $\mathbb{G}_2$. Such extensions appear to be impossible for the algebraic and topological duality resolutions.

1.1. Preliminaries on Morava stabilizer groups at $n=p=2$. We refer to [**13**] and [**19**] as general references for background on formal group laws.

1.1.1. Let Γ be a formal group law of height n defined over $\mathbb{F}_p$, let $q=p^n$ and assume that the automorphism group $\mathbb{S}_n(\Gamma):=\mathrm{Aut}_{\mathbb{F}_q}(\Gamma)$ is isomorphic to $\mathbb{S}_n:=\mathbb{S}_n(\Gamma_H)$, the automorphism group of the Honda formal group law (cf. Remark 5.2 for more on this condition on $\mathbb{S}_2(\Gamma)$). Because the formal group law is defined over $\mathbb{F}_p$ the Galois group Gal of the extension $\mathbb{F}_p\subset\mathbb{F}_q$ acts on $\mathbb{S}_n(\Gamma)$ and we get extended automorphism groups

$$\mathbb{G}_n(\Gamma) = \mathbb{S}_n(\Gamma) \rtimes \mathrm{Gal}\ .$$

For $n = p = 2$ there are two important candidates for Γ. In fact, there are two particularly interesting formal group laws Γ of height 2 over the prime field $\mathbb{F}_2$: the Honda formal group law Γ_H, i.e. the [2]-typical formal group law with [2]-series $[2]_{\Gamma_H}(x) = x^4$, and the formal group law Γ_E of the supersingular elliptic curve over $\mathbb{F}_2$ with affine equation $y^2 + y = x^3$. In the remainder of this introduction Γ always refers to either Γ_H or to Γ_E.

If $\overline{\mathbb{F}}_2$ denotes the algebraic closure of $\mathbb{F}_2$ then the endomorphism rings of both formal group laws satisfy

$$\mathrm{End}_{\mathbb{F}_4}(\Gamma) \cong \mathrm{End}_{\overline{\mathbb{F}}_2}(\Gamma)\ ,$$

and because both formal group laws become isomorphic over $\overline{\mathbb{F}}_2$ their endomorphism rings are already isomorphic over $\mathbb{F}_4$. Consequently the automorphism groups $\mathbb{S}_2(\Gamma) = \mathrm{Aut}_{\mathbb{F}_4}(\Gamma)$ of these two formal group laws over the field $\mathbb{F}_4$ are abstractly isomorphic. If $\Gamma = \Gamma_H$ this group is the classical second Morava stabilizer group at $p = 2$ and usually denoted $\mathbb{S}_2$, and $\mathbb{G}_2(\Gamma)$ is usually called the extended Morava stabilizer group and denoted $\mathbb{G}_2$. While the groups $\mathbb{S}_2(\Gamma)$ are abstractly isomorphic this ceases to be true for the groups $\mathbb{G}_2(\Gamma)$ (cf. Lemma 2.2).

The endomorphism rings $\mathrm{End}_{\mathbb{F}_4}(\Gamma)$ contain $\mathbb{W}$, the ring of Witt vectors of $\mathbb{F}_4$. They are generated as a non-commutative $\mathbb{W}$-algebra by the endomorphism $\xi_\Gamma \in \mathrm{End}_{\mathbb{F}_4}(\Gamma)$ given by $\xi_\Gamma(x) = x^2$. In order to describe the endomorphism rings more explicitly we denote the image of $w \in \mathbb{W}$ with respect to the lift of the Frobenius automorphism of $\mathbb{F}_4$ by w^σ. Then the canonical algebra map from the free non-commutative $\mathbb{W}$-algebra $\mathbb{W}\langle\xi_\Gamma\rangle$ generated by ξ_Γ to $\mathrm{End}_{\mathbb{F}_4}(\Gamma)$ induces an isomorphism

$$\mathbb{W}\langle\xi_\Gamma\rangle/(\xi_\Gamma w - w^\sigma\xi_\Gamma, \xi_\Gamma^2 - 2u) \cong \mathrm{End}_{\mathbb{F}_4}(\Gamma) \tag{1.1}$$

where

$$u = \begin{cases} 1 & \Gamma = \Gamma_H \\ -1 & \Gamma = \Gamma_E\ . \end{cases} \tag{1.2}$$

An explicit isomorphism between the two rings is given by the $\mathbb{W}$-algebra map which sends ξ_H to $\xi_E y$ where we can take for y any element in $\mathbb{W}$ with the property $y^\sigma y = -1$ (cf. [**2**] for an explicit choice of y).

The ideal generated by ξ_Γ is a two-sided maximal ideal $\mathfrak{m}$ with quotient $\mathbb{F}_4$ and the endomorphism rings are complete with respect to the $\mathfrak{m}$-adic topology. This also defines a decreasing filtration on the group $\mathbb{S}_2(\Gamma)$ indexed by half integers $\frac{i}{2} \geq 0$ given by

$$F_{\frac{i}{2}} := F_{\frac{i}{2}}\mathbb{S}_2(\Gamma) := \{g \in \mathbb{S}_2(\Gamma) \mid g \equiv 1 \mod (\xi_\Gamma^i)\}$$

and successive quotients

$$F_{\frac{i}{2}}/F_{\frac{i+1}{2}} \cong \begin{cases} \mathbb{F}_4^\times & i = 0 \\ \mathbb{F}_4 & i > 0\ . \end{cases}$$

The group

$$S_2(\Gamma) := F_{\frac{1}{2}}\mathbb{S}_2(\Gamma)$$

is a profinite 2-group. It is the normal 2-Sylow subgroup of $\mathbb{S}_2(\Gamma)$. Both groups play an important role in this paper. We use different fonts to clearly distinguish between them.

Inverting 2 in the endomorphism rings gives two isomorphic division algebras which we denote by $\mathbb{D}_2(\Gamma)$. They contain $\mathbb{Q}_2$ as their center and are of dimension 4 over $\mathbb{Q}_2$. The division algebras are equipped with a valuation

$$v : \mathbb{D}_2(\Gamma)^\times \to \frac{1}{2}\mathbb{Z}$$

which extends the valuation on $\mathbb{Q}_2$ which is normalized by $v(2) = 1$.

The group of units $\mathbb{D}_2(\Gamma)^\times$ of $\mathbb{D}_2(\Gamma)$ contains $\mathbb{S}_2(\Gamma)$ as the group of elements of valuation 0 and from (1.1) it is clear that the action of the Galois group on $\mathbb{S}_2(\Gamma)$ is realized by conjugation by ξ_Γ in $\mathbb{D}_2(\Gamma)^\times$. Therefore we get canonical isomorphisms

$$\mathbb{G}_2(\Gamma) \cong \mathbb{D}_2(\Gamma)^\times/\langle \xi_\Gamma^2 \rangle \cong \begin{cases} \mathbb{D}_2(\Gamma)^\times/\langle 2 \rangle & \Gamma = \Gamma_H \\ \mathbb{D}_2(\Gamma)^\times/\langle -2 \rangle & \Gamma = \Gamma_E \ . \end{cases} \tag{1.3}$$

We note that with respect to these isomorphisms the natural projection from $\mathbb{G}_2(\Gamma)$ to Gal corresponds to the homomorphism

$$\mathbb{D}_2(\Gamma)^\times/\langle \xi_\Gamma^2 \rangle \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}$$

induced by the valuation v.

The groups $\mathbb{S}_2(\Gamma)$ and $\mathbb{G}_2(\Gamma)$ contain -1 as unique central element of order 2 and dividing out by the subgroup C_2 generated by it gives us quotient groups which we will denote $P\mathbb{S}_2(\Gamma)$ and $P\mathbb{G}_2(\Gamma)$. From (1.3) it is clear that we have isomorphisms

$$P\mathbb{G}_2(\Gamma_H) \cong \mathbb{D}_2(\Gamma_H)^\times/\langle 2, -1 \rangle \cong \mathbb{D}_2(\Gamma_E)^\times/\langle -2, -1 \rangle \cong P\mathbb{G}_2(\Gamma_E) \ .$$

1.1.2. From (1.1) we see that $\mathrm{End}_{\mathbb{F}_4}(\Gamma)$ is a free $\mathbb{W}$-module with basis 1 and ξ_Γ. Right multiplication induces $\mathbb{W}$-linear maps and the determinant gives a multiplicative homomorphism

$$\det : \mathrm{End}_{\mathbb{F}_4}(\Gamma) \to \mathbb{W}$$

which, in fact, takes its values in $\mathbb{Z}_2$. It is explicitly given as follows: if $a, b \in \mathbb{W}$ then

$$\det(a + b\xi_\Gamma) = aa^\sigma - 2ubb^\sigma$$

with $u = 1$ or $u = -1$ as in (1.2). This determinant induces an epimomorphism

$$\det : \mathbb{S}_2(\Gamma) \to \mathbb{Z}_2^\times$$

which is often also called the reduced norm. Finally we get an epimorphism given as composition

$$\mathbb{G}_2(\Gamma) = \mathbb{S}_2(\Gamma) \rtimes \mathrm{Gal} \stackrel{\det\times id}{\longrightarrow} \mathbb{Z}_2^\times \times \mathrm{Gal} \to \mathbb{Z}_2^\times \to \mathbb{Z}_2^\times/\{\pm 1\}$$

in which the second and third part are given as the obvious projections. Let $\mathbb{G}_2^1(\Gamma)$ be the kernel of this composition and $\mathbb{S}_2^1(\Gamma)$ resp. S_2^1 its intersection with $\mathbb{S}_2(\Gamma)$ resp. $S_2(\Gamma)$. We observe that the action of Gal on $\mathbb{S}_2(\Gamma)$ leaves $\mathbb{S}_2^1(\Gamma)$ invariant and $\mathbb{G}_2^1(\Gamma)$ is equal to the semidirect product $\mathbb{S}_2^1(\Gamma) \rtimes \mathrm{Gal}$. By the definition of $\mathbb{G}_2^1(\Gamma)$ it is clear that every finite subgroup of $\mathbb{G}_2(\Gamma)$ is contained in $\mathbb{G}_2^1(\Gamma)$.

The central element $-1 = 1 - u\xi_\Gamma^2$ (where u is as in (1.2)) is contained in $S_2^1(\Gamma)$ and generates a central subgroup C_2 of order 2. If H is any closed subgroup of $\mathbb{G}_2(\Gamma)$ containing C_2 then we will denote the quotient H/C_2 by PH.

1.1.3. The groups $\mathbb{S}_2^1(\Gamma)$, $P\mathbb{S}_2^1(\Gamma)$, $\mathbb{G}_2^1(\Gamma)$, $P\mathbb{G}_2^1(\Gamma)$ and $PS_2^1(\Gamma)$ contain certain finite subgroups which figure in the statements of our main results. In all cases except that of $\mathbb{G}_2^1(\Gamma)$ the isomorphism type of the ambient group is independent of Γ and only when we discuss finite subgroups of $\mathbb{G}_2^1(\Gamma)$ the choice of Γ matters. In the other cases we will therefore from now on omit Γ from our notation.

If F is a finite subgroup of $\mathbb{G}_2^1(\Gamma)$ which contains the central C_2 and for which $F_0 := F \cap \mathbb{S}_2^1$ is of index 2 in F then we have a commutative diagram of groups with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F_0 & \longrightarrow & F & \longrightarrow & \mathrm{Gal} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 1 & \longrightarrow & PF_0 & \longrightarrow & PF & \longrightarrow & \mathrm{Gal} & \longrightarrow & 1 \end{array} \quad . \tag{1.4}$$

In the following table we give a list of closed subgroups $F \subset \mathbb{G}_2^1(\Gamma)$ and the corresponding groups $PF \subset P\mathbb{G}_2^1$, $F_0 \subset \mathbb{S}_2^1$, $PF_0 \subset P\mathbb{S}_2^1$ and $PF_0 \cap PS_2^1 \subset PS_2^1$ which will be relevant for stating our main results. Subgroups of PS_2^1 will play an important role in Section 4.2.

(1.5)

F	$\mathbb{G}_2^1(\Gamma)$	$G_{48}(\Gamma)$	$G'_{48}(\Gamma)$	$G_{12}(\Gamma)$	C_8	$C_2 \times \mathrm{Gal}$
PF	$P\mathbb{G}_2^1$	$\mathfrak{S}_4$	$\mathfrak{S}'_4$	$\mathfrak{S}_3$	C_4	Gal
F_0	$\mathbb{S}_2^1$	G_{24}	G'_{24}	C_6	C_4	C_2
PF_0	$P\mathbb{S}_2^1$	A_4	A'_4	C_3	C_2	$\{1\}$
$PF_0 \cap PS_2^1$	PS_2^1	E_2	E'_2	$\{1\}$	C_2	$\{1\}$

We refer to Section 2, in particular Lemma 2.2, Lemma 2.3, Lemma 2.4 and Lemma 2.5 for more details on this table. Here we are content to explain that in this table C_n denotes a cyclic group of order n, Gal is the Galois group of the extension $\mathbb{F}_2 \subset \mathbb{F}_4$, $\mathfrak{S}_n$ and $\mathfrak{S}'_n$ denote symmetric groups on n letters, A_4 and A'_4 alternating groups on 4 letters, and E_2 and E'_2 groups isomorphic to $C_2 \times C_2$. The groups G_{24} and G'_{24} are groups of order 24 both isomorphic to $SL_2(\mathbb{F}_3)$. The isomorphism type of the groups $G_{48}(\Gamma)$, $G'_{48}(\Gamma)$ and $G_{12}(\Gamma)$ depends on Γ. The first two are maximal subgroups of $\mathbb{G}_2^1(\Gamma)$ of order 48 which are non-conjugate in $\mathbb{G}_2^1(\Gamma)$ but become conjugate in $\mathbb{G}_2(\Gamma)$. In fact, we have (cf. Lemma 2.2)

$$G_{48}(\Gamma) \cong G'_{48}(\Gamma) \cong \begin{cases} GL_2(\mathbb{F}_3) & \Gamma = \Gamma_E \\ O_{48} & \Gamma = \Gamma_H \end{cases}$$

where O_{48} denotes the binary octahedral group. For the groups $G_{12}(\Gamma)$ we get (cf. Lemma 2.3)

$$G_{12}(\Gamma) \cong \begin{cases} C_2 \times \mathfrak{S}_3 & \Gamma = \Gamma_E \\ C_3 \rtimes C_4 & \Gamma = \Gamma_H \end{cases}$$

where $C_3 \rtimes C_4$ denotes the semidirect product of C_3 with C_4 acting non-trivially on C_3.

1.2. Main results. Let G be a profinite group, let X be a profinite G-set such that $X = \lim_i X_i$ with finite G-sets X_i and let $\mathbb{W}$ be the ring of Witt vectors for a finite field k of order $q = p^n$ for a prime p and an integer $n > 0$. We define

$$\mathbb{W}[[X]] = \lim_{i,k} \mathbb{W}/p^k[[X_i]] \ . \tag{1.6}$$

Suppose that G is equipped with a continuous homomorphism $\phi : G \to \mathrm{Gal}$ to the Galois group Gal of the extension $\mathbb{F}_p \subset \mathbb{F}_q$.

The *Galois-twisted completed group ring* $\mathbb{W}_\phi[[G]]$ of G is the $\mathbb{W}$-module $\mathbb{W}[[G]]$ with multiplication induced by $(w_1g_1)(w_2g_2) = w_1{}^{g_1}w_2g_1g_2$ if $g_1, g_2 \in G$, $w_1, w_2 \in \mathbb{W}$ and if ${}^{g_1}w_2$ is the result of the Galois action of $\phi(g_1)$ on w_2. A p-profinite $\mathbb{W}_\phi[[G]]$-module will also be called a *Galois-twisted p-profinite G-module*, or simply a *Galois-twisted profinite G-module* if p is understood from the context. In order to keep notation simple we will write $\mathbb{W}[[G]]$ instead of $\mathbb{W}_\phi[[G]]$.

Analogous to [**21**] we introduce relative homological algebra in the context of Galois-twisted p-profinite G-modules. Let $\mathcal{F}(G)$ be the set of conjugacy classes of finite subgroups of G and assume that $\mathcal{F}(G)$ is a finite set. A Galois-twisted p-profinite G-module P will be called $\mathcal{F}$-projective if it is a direct summand in a module of the form $\bigoplus_{(F)} \mathbb{W}[[G]]\widehat{\otimes}_{\mathbb{W}[F]}M_F$ where each M_F is a Galois-twisted p-profinite[1] $\mathbb{W}[F]$-module, the direct sum is indexed by conjugacy classes of finite subgroups of G and the tensor product is the completed tensor product. In the sequel we will also write $M\uparrow_F^G$ instead of $\mathbb{W}[[G]]\widehat{\otimes}_{\mathbb{W}[F]}M$.

The class of $\mathcal{F}$-projective Galois-twisted p-profinite G-modules determines in the usual way a class of $\mathcal{F}$-exact sequences: a sequence of Galois-twisted p-profinite G-modules $M' \to M \to M''$ is called $\mathcal{F}$-exact if the composition $M' \to M''$ is trivial and

$$\mathrm{Hom}_{\mathbb{W}[[G]]}(P, M') \to \mathrm{Hom}_{\mathbb{W}[[G]]}(P, M) \to \mathrm{Hom}_{\mathbb{W}[[G]]}(P, M'')$$

is an exact sequence of abelian groups for each $\mathcal{F}$-projective Galois-twisted p-profinite G-module P.

An $\mathcal{F}$-resolution of a Galois-twisted p-profinite G-module M is a sequence of Galois-twisted p-profinite G-modules

$$P_\bullet : \ldots \to P_1 \to P_0 \to M \to 0$$

where each P_i is $\mathcal{F}$-projective and each 3-term subsequence is $\mathcal{F}$-exact. We note that $\mathcal{F}$-exactness is equivalent to the complex being split when restricted to any finite subgroup of G.

Here is the main algebraic result of this paper in which $\mathbb{W}$ is now the ring of Witt vectors of $\mathbb{F}_4$ and the subgroups of $\mathbb{G}_2(\Gamma)$ are those of table (1.5).

THEOREM 1.1. *Let Γ be either Γ_H or Γ_E. There exists an $\mathcal{F}$-resolution of the trivial Galois-twisted profinite $\mathbb{G}_2(\Gamma)$-module $\mathbb{W}$*

$$P_\bullet : 0 \to P_4 \to P_3 \to P_2 \to P_1 \to P_0 \to \mathbb{W} \to 0$$

with

$$\begin{aligned}
P_0 &= \mathbb{W}\uparrow_{G_{48}(\Gamma)}^{\mathbb{G}_2(\Gamma)} \oplus \mathbb{W}\uparrow_{G'_{48}(\Gamma)}^{\mathbb{G}_2(\Gamma)}\\
P_1 &= \mathbb{W}\uparrow_{G_{12}(\Gamma)}^{\mathbb{G}_2(\Gamma)} \oplus \mathbb{W}\uparrow_{C_8}^{\mathbb{G}_2(\Gamma)} \oplus \mathbb{W}\uparrow_{G_{48}(\Gamma)}^{\mathbb{G}_2(\Gamma)} \oplus \mathbb{W}\uparrow_{G'_{48}(\Gamma)}^{\mathbb{G}_2(\Gamma)}\\
P_2 &= \mathbb{W}\uparrow_{C_2\times Gal}^{\mathbb{G}_2(\Gamma)} \oplus \mathbb{W}\uparrow_{G_{12}(\Gamma)}^{\mathbb{G}_2(\Gamma)} \oplus \mathbb{W}\uparrow_{C_8}^{\mathbb{G}_2(\Gamma)}\\
P_3 &= \mathbb{W}\uparrow_{G_{12}(\Gamma)}^{\mathbb{G}_2(\Gamma)} \oplus \mathbb{W}\uparrow_{C_2\times \mathrm{Gal}}^{\mathbb{G}_2(\Gamma)}\\
P_4 &= \mathbb{W}\uparrow_{G_{12}(\Gamma)}^{\mathbb{G}_2(\Gamma)}\ .
\end{aligned}$$

The main work in establishing Theorem 1.1 lies in the proof of the following analogous result for the group $\mathbb{G}_2^1(\Gamma)$.

[1] The assumption that M_F is p-profinite was regrettably missing in [**21**].

THEOREM 1.2. *Let* Γ *be either* Γ_H *or* Γ_E. *There exists an* $\mathcal{F}$*-resolution of the trivial Galois-twisted profinite* $\mathbb{G}_2^1(\Gamma)$*-module* $\mathbb{W}$

$$P_\bullet : 0 \to P_3 \to P_2 \to P_1 \to P_0 \to \mathbb{W} \to 0$$

with

$$\begin{aligned} P_0 &= \mathbb{W}\uparrow_{G_{48}(\Gamma)}^{\mathbb{G}_2^1(\Gamma)} \oplus \mathbb{W}\uparrow_{G'_{48}(\Gamma)}^{\mathbb{G}_2^1(\Gamma)} \\ P_1 &= \mathbb{W}\uparrow_{G_{12}(\Gamma)}^{\mathbb{G}_2^1(\Gamma)} \oplus \mathbb{W}\uparrow_{C_8}^{\mathbb{G}_2^1(\Gamma)} \\ P_2 &= \mathbb{W}\uparrow_{C_2\times Gal}^{\mathbb{G}_2^1(\Gamma)} \\ P_3 &= \mathbb{W}\uparrow_{G_{12}(\Gamma)}^{\mathbb{G}_2^1(\Gamma)} . \end{aligned}$$

REMARK 1.3. a) The resolutions for the group $\mathbb{G}_2^1(\Gamma)$ resp. for $\mathbb{G}_2(\Gamma)$ are really obtained from resolutions for $P\mathbb{G}_2^1$ resp. $P\mathbb{G}_2$ via the obvious projections $\mathbb{G}_2^1(\Gamma) \to P\mathbb{G}_2^1$ resp. $\mathbb{G}_2(\Gamma) \to P\mathbb{G}_2$. In terms of the table (1.5) in the corresponding resolutions for $\mathbb{G}_2^1(\Gamma) \to P\mathbb{G}_2^1$ resp. $\mathbb{G}_2(\Gamma) \to P\mathbb{G}_2$ a summand of the form $\mathbb{W}\uparrow_F^{\mathbb{G}_2(\Gamma)}$ resp. $\mathbb{W}\uparrow_F^{\mathbb{G}_2^1(\Gamma)}$ gets replaced by $\mathbb{W}\uparrow_{PF}^{P\mathbb{G}_2^1}$ resp. $\mathbb{W}\uparrow_{PF}^{P\mathbb{G}_2^1}$ (cf. Theorem 4.5). We note that the form of the resolutions for $P\mathbb{G}_2^1$ resp. $P\mathbb{G}_2$ is independent of the choice of Γ.

b) Restricted to $\mathbb{S}_2^1$ the resolution for $\mathbb{G}_2^1(\Gamma)$ is an untwisted $\mathcal{F}$-resolution of $\mathbb{W}$ which is a $\mathbb{W}$-linear extension of the algebraic centralizer resolution announced in [**21**] and used in [**8**]. We refer to Remark 3.3 for a justification of the terminology centralizer resolution.

Next we will describe the topological analogues of these algebraic resolutions. As in [**21**] we call a sequence of spectra

$$X_\bullet : * \to X_{-1} \xrightarrow{\alpha_0} X_0 \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} \dots \tag{1.7}$$

a *complex of spectra* if the composite of two consecutive maps is null-homotopic. Such a complex is called a *resolution* of X_{-1} if in addition each of the maps $\alpha_i : X_{i-1} \to X_i$, $i > 0$, can be factored as $X_{i-1} \xrightarrow{\beta_i} W_i \xrightarrow{\gamma_i} X_i$ such that $W_{i-1} \xrightarrow{\gamma_{i-1}} X_{i-1} \xrightarrow{\beta_i} W_i$ is a cofibration for every $i \geq 0$ (with W_0 the cofibre of α_0). We say that the resolution is of length n if $W_n \simeq X_n$ and $X_i \simeq *$ if $i > n$.

Here are the main topological results of this paper. In their statements E_2 should really read $E_2(\Gamma)$ where $E_2(\Gamma)$ is the 2-periodic Landweber exact spectrum whose coefficients in degree 0 classify deformations (in the sense of Lubin and Tate) of Γ. In order to keep notation readable we will nevertheless simply write E_2 instead of $E_2(\Gamma)$. By the Goerss-Hopkins-Miller theorem (see [**18**], [**32**]) Γ acts on E_2, in particular there exist homotopy fixed point spectra E_2^{hF} for all finite subgroups F of $\mathbb{G}_2(\Gamma)$ and by [**11**] also for all closed subgroups.

THEOREM 1.4. *Let* Γ *be either* Γ_H *or* Γ_E. *Then there exists a resolution of* $E_2^{h\mathbb{G}_2^1(\Gamma)}$ *of the form*

$$* \to E_2^{h\mathbb{G}_2^1(\Gamma)} \xrightarrow{\alpha_0} X_0 \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} X_2 \xrightarrow{\alpha_3} X_3 \to *$$

with

$$\begin{aligned} X_0 &= E_2^{hG_{48}(\Gamma)} \vee E_2^{hG'_{48}(\Gamma)} \\ X_1 &= E_2^{hG_{12}(\Gamma)} \vee E_2^{hC_8} \\ X_2 &= E_2^{C_2\times \mathrm{Gal}} \\ X_3 &= E_2^{hG_{12}(\Gamma)} \ . \end{aligned}$$

THEOREM 1.5. *Let Γ be either Γ_H or Γ_E. Then there exists a resolution of $L_{K(2)}S^0 \simeq E_2^{h\mathbb{G}_2(\Gamma)}$ of the form*

$$* \to L_{K(2)}S^0 \xrightarrow{\alpha_0} X_0 \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} X_2 \xrightarrow{\alpha_3} X_3 \xrightarrow{\alpha_4} X_4 \to *$$

with

$$\begin{aligned} X_0 &= E_2^{hG_{48}(\Gamma)} \vee E_2^{hG'_{48}(\Gamma)} \\ X_1 &= E_2^{hG_{12}(\Gamma)} \vee E_2^{hC_8} \vee E_2^{hG_{48}(\Gamma)} \vee E_2^{hG'_{48}(\Gamma)} \\ X_2 &= E_2^{C_2\times \mathrm{Gal}} \vee E_2^{hG_{12}(\Gamma)} \vee E_2^{hC_8} \\ X_3 &= E_2^{hG_{12}(\Gamma)} \vee E_2^{C_2\times \mathrm{Gal}} \\ X_4 &= E_2^{hG_{12}(\Gamma)} \end{aligned}$$

REMARK 1.6. a) Because $G_{48}(\Gamma)$ and $G'_{48}(\Gamma)$ are conjugate subgroups of $\mathbb{G}_2(\Gamma)$, the homotopy fixed point spectra $E_2^{hG_{48}(\Gamma)}$ and $E_2^{hG'_{48}(\Gamma)}$ have the same homotopy type.

b) There are corresponding resolutions for $E_2^{h\mathbb{S}_2^1}$ and $E_2^{h\mathbb{S}_2}$ which are obtained by replacing E_2^{hF} by $E_2^{hF_0}$ where F and F_0 are the finite subgroups of table (1.5).

The paper is organized as follows. In Section 2 we discuss the finite subgroups of the Morava stabilizer groups at $n = p = 2$ which figure in our main results and in Section 3 we study the mod-2 cohomology algebra of PS_2^1 via its restriction to the cohomology of elementary abelian 2-subgroups. The calculation of this cohomology algebra is crucial input for the construction of the algebraic centralizer resolutions in Section 4. In Section 5 we show how to realize the algebraic resolutions topologically.

2. Important finite subgroups for Morava stabilizer groups at $n = p = 2$

In this section we will elaborate on table (1.5) and describe more explicitly the relevant finite subgroups. We remark that in the general case of any prime p and any height n finite subgroups of $\mathbb{S}_n$ have been studied by Hewett in [**23**] and [**24**] and finite subgroups of $\mathbb{G}_n(\Gamma)$ have been studied by Bujard [**10**].

We will start by recalling from [**2**] the description of explicit maximal subgroups G_{24} and G'_{24} of $\mathbb{S}_2$ and we prefer to work with $\mathbb{S}_2(\Gamma_H)$ and write S instead of ξ_H.

Let ω be a third root of unity in $\mathbb{W}^\times$ and let

$$\pi := 1 + 2\omega \ . \tag{2.1}$$

By Hensel's Lemma the element $-7 \in \mathbb{Z}_2$ has two square roots in $\mathbb{Z}_2$. We pick the one which satisfies $\sqrt{-7} \equiv 1 + 4 \mod (8)$ and let

$$\alpha := \frac{1 - 2\omega}{\sqrt{-7}} \ . \tag{2.2}$$

We note that π and α both belong to $\mathbb{S}_2$ and the reduced norm of α is -1 while the reduced norm of π is 3.

The following lemma is proved by direct calculation (cf. Lemma 2.4.3 of [**2**]).

LEMMA 2.1. *Let*

$$i := \frac{1}{3}(1+2\omega^2)(1-\alpha S), \quad j := \frac{1}{3}(1+2\omega^2)(1-\alpha\omega^2 S), \quad k := \frac{1}{3}(1+2\omega^2)(1-\alpha\omega S)\ .$$

Then the elements $\{\pm 1, \pm i, \pm j, \pm k\}$ form a subgroup of $\mathbb{S}_2^1$ which is isomorphic to the quaternion group Q_8. This subgroup is invariant by conjugation by ω, more precisely

$$j = \omega i \omega^{-1}, \quad k = \omega j \omega^{-1} \quad i = \omega k \omega^{-1}\ .$$

Furthermore

$$\omega = -\frac{1}{2}(1+i+j+k)\ . \quad \square$$

We let G_{24} be the subgroup generated by Q_8 and ω. It is isomorphic to the semidirect product of Q_8 with C_3,

$$G_{24} \cong Q_8 \rtimes C_3\ . \tag{2.3}$$

It is easy to verify that the 16 elements of G_{24} which are not in Q_8 are the elements of the form $\frac{1}{2}(\pm 1 \pm i \pm j \pm k)$ so that

$$G_{24} = \{\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k)\} \tag{2.4}$$

We also note that the center of G_{24} is the subgroup $\{\pm 1\}$ and Q_8 is a characteristic subgroup.

LEMMA 2.2. *Let Γ be either Γ_E or Γ_H.*

a) The subgroup of $\mathbb{G}_2(\Gamma)$ generated by G_{24} and the image of $1+i \in \mathbb{D}_2^\times$ in $\mathbb{G}_2(\Gamma)$ is a maximal finite subgroup $G_{48}(\Gamma)$ of $\mathbb{G}_2(\Gamma)$ of order 48.

b) $G_{48}(\Gamma)$ is a subgroup of $\mathbb{G}_2^1(\Gamma)$.

c) The quotient $PG_{48}(\Gamma)$ is isomorphic to $\mathfrak{S}_4$ independent of Γ.

d) There are isomorphisms $G_{48}(\Gamma_E) \cong GL_2(\mathbb{F}_3)$ and $G_{48}(\Gamma_H) \cong O_{48}$. The groups $GL_2(\mathbb{F}_3)$ and O_{48} are not isomorphic.

e) The intersection $G_{48}(\Gamma) \cap \mathbb{S}_2^1$ is G_{24}, PG_{24} is isomorphic to A_4 and $PG_{24} \cap PS_2^1$ is the 2-Sylow subgroup of A_4, isomorphic to $C_2 \times C_2$.

PROOF. a) It is easy to see, for example from (2.4), that the element $1+i$ normalizes the group G_{24}. The order of $1+i$ as element of $\mathbb{D}_2^\times$ is clearly infinite. However, because of $(1+i)^2 = 2i$ and because of (1.3), its square in $\mathbb{G}_2(\Gamma)$ is an element of $\mathbb{S}_2(\Gamma)$, equal to i if $\Gamma = \Gamma_H$ and equal to $-i$ if $\Gamma = \Gamma_E$. Because G_{24} is a maximal finite subgroup of $\mathbb{S}_2$ of order 24 it follows that $G_{48}(\Gamma)$ is a maximal finite subgroup of $\mathbb{G}_2(\Gamma)$ and is of order 48.

b) Any finite subgroup of $\mathbb{G}_2(\Gamma)$ is contained in $\mathbb{G}_2^1(\Gamma)$.

c) For F a subgroup of G let $N_G(F)$ resp. $C_G(F)$ denote the normalizer resp. centralizer of F in G. Conjugation in $\mathbb{D}_2^\times$ induces a monomorphism from $N_{\mathbb{D}_2^\times}(Q_8)/C_{\mathbb{D}_2^\times}(Q_8)$ to $\mathrm{Aut}(Q_8)$, the group of automorphisms of Q_8. The latter group is well known to be isomorphic to $\mathfrak{S}_4$ and the subgroup A_4 of $\mathfrak{S}_4$ is realized by conjugation in $G_{24}/C_2 = PG_{24}$. The element $1+i$ belongs to $N_{\mathbb{D}_2^\times}(Q_8)$ and it is easy to check that conjugation by it does not belong to A_4. Hence conjugation induces an epimorphism $PG_{48}(\Gamma) \to \mathrm{Aut}(Q_8) \cong \mathfrak{S}_4$ which for cardinality reasons has to be an isomorphism.

d) The automorphism group of the elliptic curve with equation $y^2+y=x^3$ over $\mathbb{F}_4$ is isomorphic to G_{24} (cf. Appendix A of [**34**] and Section 2.4 of [**3**]). This group injects into the automorphism group of the formal group law over $\mathbb{F}_4$. Because the elliptic curve is already defined over $\mathbb{F}_2$ we get an injection $G_{24} \rtimes \mathrm{Gal} \to \mathbb{G}_2(\Gamma_E)$ and the image is $G_{48}(\Gamma_E)$. It is elementary to verify that the group of $\mathbb{F}_4$-points of the elliptic curve is of order 9, isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}/3$ and that $G_{24} \rtimes \mathrm{Gal}$ realizes all automorphisms of this subgroup. Hence $G_{48}(\Gamma_E)$ is isomorphic to $\mathrm{Aut}(\mathbb{Z}/3 \oplus \mathbb{Z}/3) \cong GL_2(\mathbb{F}_3)$.

Next it is easy to construct an isomorphism between O_{48} and $G_{48}(\Gamma_H)$ which restricts to the identity on G_{24}; in fact, O_{48} can be realized within the classical unit quaternions such that G_{24} corresponds to the subgroup which contains the elements of (2.4) and the element $(1+i) \in G_{48}(\Gamma_H)$ corresponds to the element $\frac{1}{\sqrt{2}}(1+i) \in O_{48}$.

In order to see that $GL_2(\mathbb{F}_3)$ and O_{48} are not isomorphic it is enough to see that their 2-Sylow subgroups are not isomorphic. In the case of $GL_2(\mathbb{F}_3)$ this is the semidihedral group of order 16 while in the case of O_{48} this is the generalized quaternion group of order 16 and these two groups of order 16 are not isomorphic.

e) This is now obvious. $\square$

Then we define

$$G'_{24} := \pi G_{24} \pi^{-1}, \quad G'_{48}(\Gamma) := \pi G_{48}(\Gamma) \pi^{-1} \ . \tag{2.5}$$

The groups G_{24} and G'_{24} are known to be non-conjugate in $\mathbb{S}_2^1$ and, up to conjugacy, they are the two maximal finite subgroups of $\mathbb{S}_2^1$ (cf. [**2**]). Consequently $G_{48}(\Gamma)$ and $G'_{48}(\Gamma)$ are non-conjugate in $\mathbb{G}_2^1(\Gamma)$ and, up to conjugacy, they are the two maximal finite subgroups of $\mathbb{G}_2^1(\Gamma)$. Likewise, $\mathfrak{S}_4$ and $\mathfrak{S}'_4$ are non-conjugate in $P\mathbb{G}_2^1(\Gamma)$ and, up to conjugacy, they are the two maximal finite subgroups of $P\mathbb{G}_2^1(\Gamma)$.

LEMMA 2.3. *Let Γ be either Γ_E or Γ_H.*

a) The subgroup of $\mathbb{G}_2(\Gamma)$ generated by $C_6 = \langle -\omega \rangle$ and the image of $j-k \in \mathbb{D}_2^\times$ in $\mathbb{G}_2(\Gamma)$ is a subgroup $G_{12}(\Gamma)$ of $\mathbb{G}_2(\Gamma)$ of order 12.

b) $G_{12}(\Gamma)$ is a subgroup of $\mathbb{G}_2^1(\Gamma)$.

c) The quotient $PG_{12}(\Gamma)$ is isomorphic to $\mathfrak{S}_3$ independent of Γ.

d) There are isomorphisms $G_{12}(\Gamma_E) \cong C_2 \times \mathfrak{S}_3$ and $G_{12}(\Gamma_H) \cong C_3 \rtimes C_4$.

e) The intersection $G_{12}(\Gamma) \cap \mathbb{S}_2^1$ is C_6, PC_6 is isomorphic to C_3 and $PC_6 \cap PS_2^1$ is the trivial group.

PROOF. a) The element $j-k$ normalizes the subgroup C_6 generated by $-\omega$. In fact, a direct calculation in the division algebra using that $\omega = -\frac{1}{2}(1+i+j+k)$ shows

$$(j-k)\omega(j-k)^{-1} = \omega^2 \ .$$

The order of $j-k$ as element of $\mathbb{D}_2^\times$ is clearly infinite. However, because of $(j-k)^2 = -2$, its square in $\mathbb{G}_2(\Gamma)$ is an element of $\mathbb{S}_2(\Gamma)$, equal to 1 if $\Gamma = \Gamma_E$ and equal to -1 if $\Gamma = \Gamma_H$. Then it is clear that $G_{12}(\Gamma)$ is of order 12.

b) Any finite subgroup of $\mathbb{G}_2(\Gamma)$ is contained in $\mathbb{G}_2^1(\Gamma)$.

c) This is immediate from the calculation in part a). The image of ω in $PG_{12}(\Gamma)$ generates a normal subgroup of order 3 and the image of $j-k$ is of order 2 and acts non-trivially on the image of ω.

d) This follows because the image of $j-k$ in $G_{12}(\Gamma)$ is of order 2 in the case of $\Gamma=\Gamma_E$ and of order 4 in the case of $\Gamma=\Gamma_H$.

e) This is again obvious. □

The following two lemmas are elementary and their proof is left to the reader.

LEMMA 2.4. *Let Γ be either Γ_E or Γ_H.*

a) The subgroup of $\mathbb{G}_2(\Gamma)$ generated by the image of $1+i\in\mathbb{D}_2^\times$ in $\mathbb{G}_2(\Gamma)$ is a subgroup G_8 of $\mathbb{G}_2(\Gamma)$ of order 8 which contains $C_4=\langle i\rangle$ and is, up to isomorphism, independent of Γ.

b) G_8 is a subgroup of $\mathbb{G}_2^1(\Gamma)$.

c) The quotient PG_8 is isomorphic to C_4.

d) The intersection $G_8\cap\mathbb{S}_2^1$ is the subgroup C_4 generated by i, PC_4 is isomorphic to C_2 and $PC_4\cap PS_2^1=PC_4$. □

LEMMA 2.5. *Let Γ be either Γ_E or Γ_H.*

a) The subgroup of $\mathbb{G}_2(\Gamma)$ generated by -1 and the Galois group is a subgroup G_4 of $\mathbb{G}_2(\Gamma)$ of order 4 which is isomorphic to $C_2\times\mathrm{Gal}$ independent of Γ.

b) G_4 is a subgroup of $\mathbb{G}_2^1(\Gamma)$.

c) The quotient PG_4 is isomorphic to Gal.

d) The intersection $G_4\cap\mathbb{S}_2^1$ is the subgroup C_2 generated by -1 and $PC_2=PC_2\cap PS_2^1$ is the trivial group. □

3. The mod-2 cohomology algebra of PS_2^1

Our approach to analyze the mod-2 cohomology algebra of PS_2^1 depends crucially on Quillen's F-isomorphism theorem. We start by briefly recalling Quillen's theory.

3.1. Quillen's F-isomorphism for the mod-p cohomology of a profinite group. Let G be a profinite group and let p be a fixed prime. The continuous cohomology $H_c^*(G;\mathbb{F}_p)$ of G with coefficients in the trivial module $\mathbb{F}_p$ will be abbreviated by $H^*(G;\mathbb{F}_p)$, or simply by H^*G if p is understood from the context. We recall that if G is the (inverse) limit of finite groups G_i then $H^*G=\mathrm{colim}_iH^*G_i$.

We will assume that H^*G is finitely generated as $\mathbb{F}_p$-algebra. By work of Lazard [**26**] it is known that this holds for many interesting profinite groups, for example for profinite p-analytic groups like $GL(n,\mathbb{Z}_p)$, the general linear groups over the p-adic integers, or the automorphism groups of formal group laws over finite fields.

In case H^*G is finitely generated as $\mathbb{F}_p$-algebra Quillen has shown [**30**] that there are only finitely many conjugacy classes of elementary abelian p-subgroups of G (i.e. groups isomorphic to $(\mathbb{Z}/p)^n$ for some natural number n). In other words, the following category $\mathcal{A}(G)$ is equivalent to a finite category: objects of $\mathcal{A}(G)$ are all elementary abelian p-subgroups of G, and if E_1 and E_2 are elementary abelian p-subgroups of G, then the set of morphisms from E_1 to E_2 in $\mathcal{A}(G)$ consists precisely of those homomorphisms $\alpha:E_1\longrightarrow E_2$ of abelian groups for which there exists an element $g\in G$ with $\alpha(e)=geg^{-1}$ for all $e\in E_1$. The assignment $E\mapsto H^*E$ determines a functor from the opposite category $\mathcal{A}_*(G)^{op}$ to graded $\mathbb{F}_p$-algebras.

Theorem 3.1. **(Quillen)** [**30**] *Let G be a profinite group and assume H^*G is a finitely generated $\mathbb{F}_p$-algebra. Then the canonical map*

$$q_G : H^*G \to \lim_{\mathcal{A}(G)^{op}} H^*E$$

is an F-isomorphism, in other words q has the following properties.

- *If $x \in \mathrm{Ker} q_G$, then x is nilpotent.*
- *If $y \in \lim_{\mathcal{A}(G)^{op}} H^*E$ then there exists an integer n with $y^{p^n} \in Imq$.*

In the sequel we will call $\mathcal{A}(G)$ the Quillen category of G.

Let $\mathcal{A}_*(G)$ be the full subcategory of $\mathcal{A}(G)$ whose objects are all elementary abelian p-subgroups except the trivial subgroup. The centralizer $C_G(E)$ of an elementary abelian p-subgroup E is a closed subgroup and hence inherits a natural profinite structure from G. The assignment $E \mapsto H^*C_G(E)$ extends to a functor from $\mathcal{A}_*(G)$ to graded $\mathbb{F}_p$-algebras and the restriction homomorphisms $H^*G \longrightarrow H^*C_G(E)$ (for E running through the non-trivial elementary abelian p-subgroups of G) induce a canonical map $\rho : H^*G \longrightarrow \lim_{\mathcal{A}_*(G)} H^*C_G(E)$. The main result of [**20**] reads as follows.

Theorem 3.2. *Let G be a profinite group and assume H^*G is a finitely generated $\mathbb{F}_p$-algebra. Then the canonical map $\rho : H^*G \longrightarrow \lim_{\mathcal{A}_*(G)} H^*C_G(E)$ has finite kernel and cokernel.*

Remark 3.3. a) In the current approach this theorem is no longer needed. However, it played a crucial role in the initial approach to construct resolutions for $P\mathbb{S}_2^1$ and is ultimately the reason for naming our resolutions centralizer resolutions. Furthermore, in [**21**] the theorem played a crucial role for constructing algebraic centralizer resolutions at odd primes, which as the algebraic resolutions of this paper are $\mathcal{F}$-resolutions in the sense of Section 1.2.

b) Theorem 3.2 is not useful if G contains central elements of order p, because then H^*G appears in the limit. In these cases one can use the theorem to study the quotient of G by the maximal central elementary abelian p-subgroup of G and this was the origin for considering the groups PS_2^1 and $P\mathbb{S}_2^1$.

3.2. The Quillen category of PS_2^1. We recall from Section 2 that $\mathbb{S}_2^1$ contains two subgroups isomorphic to Q_8 and they give rise to two elementary abelian 2-subgroups E_2 and E_2' in $P\mathbb{S}_2^1$ which are contained in the normal 2-Sylow subgroup PS_2^1.

The following result has a significant overlap with Section 2.4 of [**2**].

Proposition 3.4. *a) Up to conjugacy PS_2^1 contains three elementary abelian 2-subgroups of rank 1 and two of rank 2.*

b) All automorphism groups of the category $\mathcal{A}(PS_2^1)$ are trivial and there is exactly one morphism from each rank 1 group to each of the rank 2 groups.

Proof. a) If E is an elementary abelian 2-subgroup of PS_2^1 then its inverse image $\widetilde{E}$ in S_2^1 is an extension of E by $\mathbb{Z}/2$. The structure of the possible finite 2 subgroups of the division algebra $\mathbb{D}_2$ is explicitly known: in fact, any finite abelian subgroup of a field must be cyclic and generates in the division algebra a cyclotomic extension the degree of which must divide 2. Hence any abelian 2-subgroup is cyclic of order 2 or 4 and this implies that any finite 2-subgroup is isomorphic to a subgroup of Q_8 (cf. Theorem 4.3 of Chapter IV in [**9**]). In particular we see that the 2-rank of E is either 1 or 2.

Now suppose that F_1 and F_2 are two elementary abelian 2-subgroups of rank 1 of PS_2^1. Then $\widetilde{F}_1$ and $\widetilde{F}_2$ are two subgroups isomorphic to $\mathbb{Z}/4$ and by the Skolem Noether theorem any isomorphism $\varphi : \widetilde{F}_1 \to \widetilde{F}_2$ can be realized by conjugation by an element of $u \in \mathbb{D}_2^\times$, i.e. $\varphi(x) = uxu^{-1}$ for any $x \in \widetilde{F}_1$. If we denote a generator of $\widetilde{F}_1$ by i then $1+i \in \mathbb{D}_n^\times$ centralizes F_1, so we can change u by any power of $(1+i)$ and conjugation by $u(1+i)^n$ will still give φ. Because the valuation of $1+i$ is $\frac{1}{2}$ we can choose an integer n such that $(1+i)^n u$ is of valuation 0. In other words, we can suppose that u is an element of $\mathbb{S}_2$. Furthermore, the element $1+2i \in \mathbb{S}_2$ has reduced norm 5 and is thus a topological generator of $\mathbb{S}_2/\mathbb{S}_2^1$. It also centralizes $\widetilde{F}_1$ and by multiplying u by a suitable p-adic power of $1+2i$ we can even assume that u is in $\mathbb{S}_2^1$. This implies that all rank 1 subgroups of $P\mathbb{S}_2^1$ are conjugate and therefore the quotient group $\mathbb{S}_2^1/S_2^1 \cong \mathbb{F}_4^\times$ which is generated by the image of ω acts transitively on the PS_2^1-conjugacy classes of elementary abelian 2-subgroups of rank 1.

Thus there are either three or one PS_2^1-conjugacy classes of elementary abelian 2-subgroups of rank 1. If there was only one then conjugation by ω would have to be the same as conjugation by an element in PS_2^1 and this would mean that there is an element in $\mathbb{S}_2^1$ of the form $\omega u'$ with $u' \in S_2^1$ whose image in $P\mathbb{S}_2^1$ centralizes F_1, and hence $\omega u'$ normalizes $\widetilde{F}_1$. However, $N_{\mathbb{S}_2^1}(\widetilde{F}_1)/C_{\mathbb{S}_2^1}(\widetilde{F}_1)$ is isomorphic to a subgroup of

$$N_{\mathbb{D}_2^\times}(\widetilde{F}_1)/C_{\mathbb{D}_2^\times}(\widetilde{F}_1) \cong \mathrm{Aut}(\widetilde{F}_1) \cong C_2 \ ,$$

hence $N_{\mathbb{S}_2}(\widetilde{F}_1)$ contains the centralizer $C_{\mathbb{S}_2}(\widetilde{F}_1) \cong \mathbb{Z}_2[i]^\times$ as a subgroup of index at most 2. This implies that $N_{\mathbb{S}_2}(\widetilde{F}_1)$ is a profinite 2-group and cannot contain an element as $\omega u'$which would have non-trivial image in $\mathbb{F}_4^\times$.

Next suppose F_1 and F_2 are two elementary abelian 2-subgroups of rank 2 of PS_2^1. Then $\widetilde{F}_1$ and $\widetilde{F}_2$ are two subgroups of $\mathbb{S}_2^1$ isomorphic to Q_8 and again by the Skolem Noether theorem any isomorphism $\varphi : \widetilde{F}_1 \to \widetilde{F}_2$ can be realized by conjugation by an element of $u \in \mathbb{D}_2^\times$, i.e. $\varphi(x) = uxu^{-1}$ for any $x \in \widetilde{F}_1$. In particular, we have an isomorphism

$$N_{\mathbb{D}_2^\times}(\widetilde{F}_1)/C_{\mathbb{D}_2^\times}(\widetilde{F}_1) \cong \mathrm{Aut}(Q_8) \cong \mathfrak{S}_4 \ .$$

In order to determine the number of conjugacy classes of elementary abelian 2-subgroups of rank 2 of PS_2^1 we need to know something about the structure of the normalizer $N_{\mathbb{S}_2}(Q_8)$. The centralizer $C_{\mathbb{D}_2^\times}(Q_8)$ is isomorphic to $\mathbb{Q}_2^\times$ and the quotient $N_{\mathbb{D}_2^\times}(Q_8)/C_{\mathbb{D}_2^\times}(Q_8)$ is generated by the image of the group G_{24} and the element $1+i$ (cf. the proof of part a) of Lemma 2.2). Furthemore the centralizer $C_{\mathbb{S}_2}(Q_8)$ is isomorphic to $\mathbb{Z}_2^\times$ and we get an isomorphism

$$N_{\mathbb{S}_2}(Q_8) \cong \mathbb{Z}_2^\times \times_{C_2} G_{24} \tag{3.1}$$

between $N_{\mathbb{S}_2}(Q_8)$ and the central product $\mathbb{Z}_2^\times \times_{C_2} G_{24}$ as well as an isomorphism

$$N_{\mathbb{S}_2}(Q_8)/C_{\mathbb{S}_2}(Q_8) \cong PG_{24} = A_4 \ . \tag{3.2}$$

Because the normalizer $N_{\mathbb{D}_2^\times}(\widetilde{F}_1)$ always contains an element y of valuation $\frac{1}{2}$, we can assume by changing u, if necessary, by a suitable power of y that there is an isomorphism $\psi : \widetilde{F}_1 \to \widetilde{F}_2$ which is realized by conjugation in $\mathbb{S}_2$. In particular, in $\mathbb{S}_2$ there is only one conjugacy class of subgroups isomorphic to Q_8 and in $P\mathbb{S}_2$ there is only one conjugacy class of elementary abelian 2-subgroups of rank 2. This means that the group $\mathbb{S}_2/\mathbb{S}_2^1$ acts transitively on the set of conjugacy classes of subgroups of $\mathbb{S}_2^1$ which are isomorphic to Q_8. Because the center acts trivially on the set of conjugacy classes and the image of the center in $\mathbb{S}_2/\mathbb{S}_2^1$ is of index 2 there are at most two conjugacy classes of Q_8's in $\mathbb{S}_2^1$. We claim that there are two of them given by Q_8 and $\pi Q_8 \pi^{-1}$ where Q_8 is the 2- Sylow subgroup of the group G_{24} of Section 2 and $\pi \in \mathbb{S}_2$ is the element defined in (2.1). If they were conjugate then π could be written as product xn with $x \in \mathbb{S}_2^1$ and $n \in N_{\mathbb{D}_2^\times}(\widetilde{F}_1)$. However, from the description of $N_{\mathbb{D}_2^\times}(\widetilde{F}_1)$ given above, we see that the reduced norm of such a product can never be equal to 3, the reduced norm of π.

b) It is clear that the automorphism groups of elementary abelian 2-subgroups of rank 1 are trivial. For the automorphisms of a rank 2 subgroup we note that (3.2) implies that the automorphism group $\mathrm{Aut}_{\mathcal{A}(P\mathbb{S}_2)}(PQ_8)$ is C_3 because conjugation by any element of the subgroup Q_8 of G_{24} induces the trivial automorphism. This in turn implies that $\mathrm{Aut}_{\mathcal{A}(PS_2^1)}(PQ_8)$ is trivial.

It remains to show that there is exactly one morphism from each rank 1 to each rank 2 object, or equivalently, that the three non-trivial elements in a rank 2 object are non-conjugate in PS_2^1. If they were conjugate in PS_2^1, then there would be an element in $P\mathbb{S}_2^1$ of the form $\omega^{\pm 1}x$ with x in PS_2^1 which centralizes the rank 1 subgroup generated by one of these elements, respectively its preimage in $\mathbb{S}_2^1$ would normalize the preimage, and this contradicts what we have seen in the proof of part a) above. □

Remark 3.5. We can choose representatives E_2 and E_2' for the two conjugacy classes of elementary abelian 2-subgroups of rank 2 such that $E_2 \cap E_2'$ is cyclic of order 2. In fact, if E_2 is such that $\widetilde{E}_2$ is the subgroup of G_{24} generated i and j then conjugation by $1+2i$ fixes i and carries $\widetilde{E}_2$ to $\widetilde{E}_2'$. In $\mathbb{S}_2^1$ the group $\widetilde{E}_2$ is not conjugate to $\widetilde{E}_2'$, hence in the quotient $P\mathbb{S}_2^1$ we get that E_2 and E_2' are non-conjugate and intersect in the subgroup generated by the image of i.

3.3. Quillen's F-isomorphism for PS_2^1 and the mod-2 cohomology algebra of PS_2^1. The inverse limit in Quillen's Theorem 3.1 is always a subalgebra of the product $\prod_E H^*E$ where E runs through the maximal elementary abelian subgrous of G, up to conjugacy. By Theorem 3.4 there are, in the case of $G = PS_2^1$, two of them, both or rank 2 with mod-2 cohomology both given by $\mathbb{F}_2[x,y]$ with x and y of cohomological degree 1.

Proposition 3.6. *There is an isomorphism of graded $\mathbb{F}_2$-algebras*

$$\lim_{\mathcal{A}(PS_2^1)} H^*E \cong \{(p_1,p_2) \in \mathbb{F}_2[x,y] \times \mathbb{F}_2[x,y] \mid p_1 - p_2 \text{ is divisible by } xy(x+y)\}$$

Proof. If E_1 and E_2 are two non-conjugate elementary abelian 2-subgroups of rank 2 of PS_2^1 then the non-trivial elements of E_1 and E_2 belong to the three non-conjugate elementary abelian 2-subgroups F_1, F_2 and F_3 of rank 1. If we choose the non-trivial elements of E_1 and E_2 as e_1^j, e_2^j and e_3^j for $j = 1,2$ then the morphisms in $\mathcal{A}(PS_2^1)$ are given by the homomorphisms $\alpha_{i,j} : F_i \to E_j$ which send the nontrivial element of F_i to the element e_i^j of E_j.

Then the inverse limit is given by pairs of polynomials $(p_1,p_2) \in H^*E_1 \times H^*E_2$ such that $\alpha_{i,1}^* p_1 = \alpha_{i,2}^* p_2$ for $i = 1,2,3$, or if we identify H^*E_1 with H^*E_2 via the abstract group isomorphism which sends e_i^1 to e_i^2 for $i = 1,2,3$ then $p_1 - p_2$

is divisible by each of the three non-trivial degree one elements in $\mathbb{F}_2[x, y]$ and the claim follows. $\square$

The quotient homomorphism $S_2^1 \to S_2^1/F_1S_2^1 \cong \mathbb{F}_4 \cong C_2 \times C_2$ induces a surjection $PS_2^1 \to \mathbb{F}_4$ and the explicit form of the elements i, j and k given in Lemma 2.1 shows that both subgroups E_2 and E_2' map isomorphically to this quotient.

COROLLARY 3.7. *As a module over* $H^*(S_2^1/F_1S_2^1) \cong \mathbb{F}_2[x, y]$ *the inverse limit is the free submodule of* $\mathbb{F}_2[x, y] \times \mathbb{F}_2[x, y]$ *generated by the classes* $(1, 1)$ *and* $(xy(x+y), 0)$. $\square$

The following result describes the algebraic duality resolution of the trivial $\mathbb{S}_2^1$-module $\mathbb{Z}_2$. This resolution has been established in Theorem 1.2.1 and 1.2.6 of [**2**]. The subgroups of $\mathbb{S}_2^1$ occurring in the statement are those of (1.5) and IS_2^1 is the augmentation ideal of the completed group algebra $\mathbb{Z}_2[[\mathbb{S}_2^1]]$. The notation used is analogous to that of Section 1.2. In other words, if G is a profinite group and X is a profinite G-set such that $X = \lim_i X_i$ with finite G-sets X_i then we define

$$\mathbb{Z}_2[[X]] = \lim_{i,k} \mathbb{Z}/2^k[[X_i]] \ , \tag{3.3}$$

and if F is a finite subgroup of G and M is a profinite $\mathbb{Z}_2[F]$-module then $M \uparrow_F^G$ denotes the $\mathbb{Z}_2[[G]]$-module $\mathbb{Z}_2[[G]]\widehat{\otimes}_{\mathbb{Z}_2[F]} M$.

THEOREM 3.8. *a) There is an exact complex of profinite* $\mathbb{Z}_2[[\mathbb{S}_2^1]]$*-modules*

$$0 \to \mathbb{Z}_2 \uparrow_{G_{24}'}^{\mathbb{S}_2^1} \xrightarrow{\partial_3} \mathbb{Z}_2 \uparrow_{C_6}^{\mathbb{S}_2^1} \xrightarrow{\partial_2} \mathbb{Z}_2 \uparrow_{C_6}^{\mathbb{S}_2^1} \xrightarrow{\partial_1} \mathbb{Z}_2 \uparrow_{G_{24}}^{\mathbb{S}_2^1} \xrightarrow{\varepsilon} \mathbb{Z}_2 \to 0$$

b) The maps ∂_1, ∂_2 *and* ∂_3 *are trivial modulo* $(2, IS_2^1)$. $\square$

REMARK 3.9. a) In fact, the central subgroup C_2 acts trivially in this complex and according to table (1.5) the complex can be considered as a complex of profinite $P\mathbb{S}_2^1$-modules

$$0 \to \mathbb{Z}_2 \uparrow_{A_4'}^{P\mathbb{S}_2^1} \xrightarrow{\partial_3} \mathbb{Z}_2 \uparrow_{C_3}^{P\mathbb{S}_2^1} \xrightarrow{\partial_2} \mathbb{Z}_2 \uparrow_{C_3}^{P\mathbb{S}_2^1} \xrightarrow{\partial_3} \mathbb{Z}_2 \uparrow_{A_4}^{P\mathbb{S}_2^1} \xrightarrow{\varepsilon} \mathbb{Z}_2 \to 0 \tag{3.4}$$

or even as a complex of profinite PS_2^1-modules

$$0 \to \mathbb{Z}_2 \uparrow_{E_2'}^{PS_2^1} \xrightarrow{\partial_3} \mathbb{Z}_2 \uparrow_{\{1\}}^{PS_2^1} \xrightarrow{\partial_2} \mathbb{Z}_2 \uparrow_{\{1\}}^{PS_2^1} \xrightarrow{\partial_3} \mathbb{Z}_2 \uparrow_{E_2}^{PS_2^1} \xrightarrow{\varepsilon} \mathbb{Z}_2 \to 0 \ . \tag{3.5}$$

b) For every profinite PS_2^1-module M there is a duality spectral sequence associated to the complex of (3.5)

$$E_1^{s,t} = \mathrm{Ext}^t_{\mathbb{Z}_2[[PS_2^1]]}(C_s, M) \Longrightarrow \mathrm{Ext}^{s+t}_{\mathbb{Z}_2[[PS_2^1]]}(\mathbb{Z}_2, M) \cong H^{s+t}(PS_2^1, M)$$

with

$$C_s = \begin{cases} \mathbb{Z}_2 \uparrow_{E_2}^{PS_2^1} & s = 0 \\ \mathbb{Z}_2 \uparrow_{\{1\}}^{PS_2^1} & s = 1, 2 \\ \mathbb{Z}_2 \uparrow_{E_2'}^{PS_2^1} & s = 3 \\ 0 & \text{else} \ . \end{cases}$$

If $M = \mathbb{F}_2$ we can identify the E_1-term via the usual Shapiro-type isomorphisms with

$$(3.6) \qquad E_1^{s,*} = \begin{cases} H^*(E_2) \cong \mathbb{F}_2[x,y] & s=0 \\ H^*(\{1\}) \cong \mathbb{F}_2 & s=1,2 \\ H^*(E_2') \cong \mathbb{F}_2[x,y] & s=3 \\ 0 & \text{else} . \end{cases}$$

PROPOSITION 3.10. *The duality spectral sequence for the group PS_2^1 and $M = \mathbb{F}_2$ collapses at its E_1-term.*

PROOF. By part b) of Theorem 3.8 we have $d_1 = 0$ and by (3.6) any higher differential would have to originate at the vertical edge. However, as we have noted before the composition of the inclusion of E_2 into PS_2^1 followed by the quotient map $PS_2^1 \to \mathbb{F}_4$ is an isomorphism. This implies that the vertical edge of the duality spectral sequence survives to E_∞, in particular all differentials originating at the vertical edge are trivial. □

THEOREM 3.11. *a) The map of Theorem* 3.1

$$q_{PS_2^1} : H^*(PS_2^1) \to \lim_{\mathcal{A}(PS_2^1)^{op}} H^*(E)$$

is surjective with kernel $\Sigma\mathbb{F}_2 \oplus \Sigma^2\mathbb{F}_2$ where $\Sigma^k\mathbb{F}_2$ is the graded $\mathbb{F}_2$-module $\mathbb{F}_2$ concentrated in degree k.

b) The Poincaré series $\chi := \sum_{n\geq 0} \dim_{\mathbb{F}_2} H^n(PS_2^1)t^n$ is given by

$$\chi = \frac{1+t^3}{(1-t)^2} + t + t^2 \ .$$

c) The Bockstein homomorphism induces an isomorphism between the kernel of $q_{PS_2^1}$ in cohomological degree 1 *and* 2.

PROOF. a) The spectral sequence (3.6) is a spectral sequence of modules over $H^*(PS_2^1/F_1PS_2^1, \mathbb{F}_2) \cong \mathbb{F}_2[x,y]$. By Proposition 3.10 we get a filtration

$$0 \subset G_3 \subset G_2 \subset G_1 \subset G_0 = H^*(PS_2^1; \mathbb{F}_2)$$

by $H^*(PS_2^1/F_1PS_2^1; \mathbb{F}_2) \cong \mathbb{F}_2[x,y]$-modules G_i with associated graded given by

$$G_0/G_1 = \mathbb{F}_2[x,y], \quad G_1/G_2 \cong \Sigma\mathbb{F}_2, \quad G_2/G_3 = \Sigma^2\mathbb{F}_2, \quad G_3 \cong \Sigma^3\mathbb{F}_2[x,y] \ .$$

Because both inclusions $E_2 \subset PS_2^1$ and $E_2' \subset PS_2^1$ split the projection map from PS_2^1 to $PS_2^1/F_1PS_2^1$ the image of q_G maps onto the diagonal in $H^*(E_2) \times H^*(E_2')$. Furthermore G_1 maps trivially to $H^*(E_2; \mathbb{F}_2)$. By linearity with respect to $H^*(PS_2^1/F_1PS_2^1; \mathbb{F}_2)$ the quotient G_1/G_3 maps also trivially to $H^*(E_2'; \mathbb{F}_2)$ and then the generator of the $\mathbb{F}_2[x,y]$-module G_3 must map to $(0, xy(x+y)) \in \lim_{\mathcal{A}(PS_2^1)^{op}} H^*(E; \mathbb{F}_2)$ because otherwise Quillen's map could not be an F-isomorphism. Part a) follows.

b) This is an immediate consequence of part a) and Corollary 3.7.

c) It is enough to show that the reduction homomorphism $H^1(PS_2^1, \mathbb{Z}/4) \to H^1(PS_2^1, \mathbb{Z}/2)$ is trivial. The cohomology of the group S_2 has been first investigated by Ravenel [**31**]. For a recent acount see [**22**]. In particular, it follows from Proposition 3.5.3 of [**22**] that the quotient of PS_2^1 by the topological closure of its commutator subgroup is isomorphic to $(\mathbb{Z}/2)^3$. As a consequence we find that mod-2 reduction

$$H^1(PS_2^1, \mathbb{Z}/4) = \mathrm{Hom}_{cts}(PS_2^1, \mathbb{Z}/4) \to \mathrm{Hom}_{cts}(PS_2^1, \mathbb{Z}/2) = H^1(PS_2^1, \mathbb{Z}/2)$$

is trivial and hence the mod-2 Bockstein is injective. □

4. Algebraic centralizer resolutions

4.1. Galois-twisted modules. We take up the notions introduced in Section 1.2. So we assume that $n \geq 1$ is an integer, p is a prime and Gal is the Galois group of the field extension $\mathbb{F}_p \subset \mathbb{F}_q$ where $q = p^n$, and $\mathbb{W}$ denotes the ring of Witt vectors of $\mathbb{F}_q$. Furthermore let G be a profinite group equipped with a continuous homomorphism $\phi : G \to \mathrm{Gal}$ and let S be the kernel of ϕ. As before we consider the *Galois-twisted completed group ring* $\mathbb{W}_\phi[[G]]$ of G and *Galois-twisted p-profinite* $\mathbb{W}_\phi[[G]]$*-modules*. In order to keep notations simple we will, as before, simply write $\mathbb{W}[[G]]$ instead of $\mathbb{W}_\phi[[G]]$.

Note that for $n = 1$ we recover the usual completed group ring $\mathbb{Z}_p[[G]]$. In general, the action of S on a Galois-twisted profinite G-module is $\mathbb{W}$-linear while the action of G is only $\mathbb{Z}_p$-linear. The groups we have in mind are $\mathbb{G}_n$, $\mathbb{G}_n^1$, $P\mathbb{G}_n$ and in particular the closed subgroups F of (1.5) of these groups in the case $n = p = 2$. In all these cases the homomorphism ϕ is surjective.

A crucial input for the sequel is the following result.

PROPOSITION 4.1. *Suppose that G is a finite group equipped with a continuous homomorphism $\phi : G \to Gal$ and let P be a Galois-twisted p-profinite G-module.*
a) If P is $\mathbb{W}[[S]]$-projective then P is $\mathbb{W}[[G]]$-projective.
b) If P is $\mathbb{F}_q[[S]]$ projective then P is $\mathbb{F}_q[[G]]$-projective.

PROOF. We give the proof of part a). The proof of part b) is completely analogous. Let $\varphi : P \to M$ be a homomorphism of Galois-twisted profinite G-modules and let $\pi : M \to N$ be an epimorphism of Galois-twisted profinite G-modules. Because P is $\mathbb{W}[[S]]$-projective there exists a $\mathbb{W}[[S]]$-linear homomorphism $\widetilde{\varphi} : P \to M$ such that $\pi\widetilde{\varphi} = \varphi$.

If $h \in S$, $g \in G$, $x \in P$ and $\lambda \in \mathbb{W}$ then

$$^{gh}(\lambda)gh\widetilde{\varphi}(h^{-1}g^{-1}x) = (^g\lambda)g\widetilde{\varphi}(g^{-1}x) \ ,$$

hence $(^g\lambda)g\widetilde{\varphi}(g^{-1}x)$ is constant on S-orbits for the translation action of S on G on the right and

$$\psi : P \to M, \quad x \mapsto \sum_{g \in G/S} (^g\lambda)g\widetilde{\varphi}(g^{-1}x)$$

is well-defined. Furthermore, ψ is a $\mathbb{W}[[G]]$-linear map. In fact, if h is in G then

$$\begin{aligned} h\psi(x) &= \textstyle\sum_{g\in G/S} {}^h(^g\lambda)hg\widetilde{\varphi}(g^{-1}x) = \sum_{hg \in G/S}(^{hg}\lambda)hg\widetilde{\varphi}(g^{-1}x) \\ &= \textstyle\sum_{hg\in G/S}(^{hg}\lambda)hg\widetilde{\varphi}((hg)^{-1}hx) = \psi(hx) \ . \end{aligned}$$

Because π is a homomorphism of Galois-twisted profinite G-modules, we have

$$\pi\psi(x) = \sum_{g\in G/S} (^g\lambda)g\pi\widetilde{\varphi}(g^{-1}x) = \sum_{g\in G/S} (^g\lambda)g\varphi(g^{-1}x) = \sum_{g\in G/S} (^g\lambda)\varphi(x) \ .$$

Finally, G/S can be identified with a subgroup of Gal and by Hilbert 90 there exists $\lambda \in \mathbb{W}$ such that $\sum_{g\in G/S} {}^g\lambda = 1$ and this shows that P is projective. □

Corollary 4.2. *Suppose that G is a finite group equipped with a homomorphism $\phi : G \to Gal$, and suppose that $S := \mathrm{Ker}\phi$ is of order prime to p. Then the trivial Galois-twisted profinite G-module $\mathbb{W}$ resp. the trivial Galois-twisted profinite G-module $\mathbb{F}_q$ is a projective $\mathbb{W}[[G]]$-module resp. $\mathbb{F}_q[[G]]$-module.* □

Lemma 4.3. *Suppose that G is a finite group equipped with a homomorphism $\phi : G \to Gal$ and let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of Galois-twisted p-profinite G-modules which is split as sequence of $\mathbb{W}$-modules. If M_1 is projective then the sequence is split as a sequence of Galois-twisted profinite G-modules.*

Proof. This proof is actually extracted from the proof of Lemma 16 of [**21**]. We begin with the following observation. If M_1 is projective as Galois-twisted profinite G-module then M_1 is a direct summand in the induced module $\mathbb{W}[G] \otimes_{\mathbb{W}} M_1$. Furthermore for a finite group the induced module and the coinduced module $\mathrm{Hom}_{\mathbb{W}}(\mathbb{W}[G], M_1)$ are isomorphic.

The existence of a $\mathbb{W}$-linear splitting of the monomorphism $M_1 \to M_2$ implies that any $\mathbb{W}[G]$-linear map φ from M_1 to the coinduced module $\mathrm{Hom}_{\mathbb{W}}(\mathbb{W}[G], M_1)$ can be extended to a $\mathbb{W}[G]$-linear map $\widetilde{\varphi} : M_2 \to \mathrm{Hom}_{\mathbb{W}}(\mathbb{W}[G], M_1)$. Now we take for φ any $\mathbb{W}[G]$-split inclusion of M_1 into $\mathrm{Hom}_{\mathbb{W}}(\mathbb{W}[G], M_1)$. Then the composition of $\widetilde{\varphi}$ with a $\mathbb{W}[G]$-linear splitting of φ provides the desired splitting of the monomorphism $M_1 \to M_2$. □

In the sequel the kernel of the augmentation $\mathbb{W}[[G]] \to \mathbb{F}_q$ is denoted by I_pG, or simply by I if p and G are clear from the context. The other crucial input in the construction of the centralizer resolution is the following Nakayama type lemma which is analogous to Lemma 4.3 of [**15**]. A Galois-twisted p-profinite $\mathbb{W}[[G]]$-module M is called I-complete, if the canonical map $M \to \lim_n M/I^nM$ is an isomorphism. We note that for a profinite p-group S any finitely generated profinite $\mathbb{W}[[S]]$-module is I-complete (cf. Proposition 5.2.17 of [**29**]).

Lemma 4.4. *Let G be a profinite group equipped with a continuous homomorphism $\phi : G \to Gal$ such that $\mathrm{Ker}\phi$ contains a profinite p-subgroup S of finite index. Let $f : M \to N$ be a morphism of finitely generated Galois-twisted p-profinite $\mathbb{W}[[G]]$-modules.*

a) If $\mathrm{Tor}_0(f) : \mathrm{Tor}_0^{\mathbb{W}[[S]]}(M, \mathbb{F}_q) \to \mathrm{Tor}_0^{\mathbb{W}[[S]]}(N, \mathbb{F}_q)$ is surjective, then f is surjective.

b) If $Tor_q(f) : \mathrm{Tor}_q^{\mathbb{W}[[S]]}(M, \mathbb{F}_q) \to \mathrm{Tor}_q^{\mathbb{W}[[S]]}(N, \mathbb{F}_q)$ is an isomorphism for $q = 0$ and surjective for $q = 1$ then f is an isomorphism. □

4.2. The algebraic centralizer resolution for $\mathbb{G}_2^1(\Gamma)$ and $\mathbb{G}_2(\Gamma)$. The following theorem establishes Theorem 1.2 of the introduction. The finite subgroups of $P\mathbb{G}_2^1$ and $\mathbb{G}_2^1(\Gamma)$ occurring in this theorem are those specified in table (1.5) and Section 2.

Theorem 4.5. *Let Γ be either Γ_H or Γ_E.*

a) There is an $\mathcal{F}$-resolution of the trivial Galois-twisted profinite $P\mathbb{G}_2^1$-module $\mathbb{W}$ of the following form

$$0 \to \mathbb{W}\uparrow_{\mathfrak{S}_3}^{P\mathbb{G}_2^1} \xrightarrow{\partial_3} \mathbb{W}\uparrow_{\mathrm{Gal}}^{P\mathbb{G}_2^1} \xrightarrow{\partial_2} \mathbb{W}\uparrow_{\mathfrak{S}_3}^{P\mathbb{G}_2^1} \oplus \mathbb{W}\uparrow_{C_4}^{P\mathbb{G}_2^1} \xrightarrow{\partial_1} \mathbb{W}\uparrow_{\mathfrak{S}_4}^{P\mathbb{G}_2^1} \oplus \mathbb{W}\uparrow_{\mathfrak{S}_4'}^{P\mathbb{G}_2^1} \xrightarrow{\varepsilon} \mathbb{W} \to 0 \ .$$

b) There is an $\mathcal{F}$-resolution of the trivial Galois-twisted profinite $\mathbb{G}_2^1(\Gamma)$-module $\mathbb{W}$ of the following form

$$0 \to \mathbb{W}\uparrow_{G_{12}(\Gamma)}^{\mathbb{G}_2^1(\Gamma)} \xrightarrow{\partial_3} \mathbb{W}\uparrow_{C_2\times \mathrm{Gal}}^{\mathbb{G}_2^1(\Gamma)} \xrightarrow{\partial_2} \mathbb{W}\uparrow_{G_{12}(\Gamma)}^{\mathbb{G}_2^1(\Gamma)} \oplus \mathbb{W}\uparrow_{C_8}^{\mathbb{G}_2^1(\Gamma)}$$
$$\xrightarrow{\partial_1} \mathbb{W}\uparrow_{G_{48}(\Gamma)}^{\mathbb{G}_2^1(\Gamma)} \oplus \mathbb{W}\uparrow_{G'_{48}(\Gamma)}^{\mathbb{G}_2^1(\Gamma)} \xrightarrow{\varepsilon} \mathbb{W} \to 0\ .$$

Proof. The second complex is obtained from the first one by simply considering a complex of profinite $\mathbb{W}[[P\mathbb{G}_2^1]]$-modules as a complex of profinite $\mathbb{W}[[\mathbb{G}_2^1(\Gamma)]]$-modules via the canonical projection $\mathbb{G}_2^1(\Gamma) \to P\mathbb{G}_2^1$. The property of being an $\mathcal{F}$-resolution is preserved by the analogue of Lemma 14 of [**21**]. So we concentrate on constructing the first complex. In order to simplify notation we will write this complex in the sequel as

$$0 \to P_3 \to P_2 \to P_1 \to P_0 \to \mathbb{W} \to 0\ .$$

The existence of an exact complex follows from splicing the exact sequences of Lemma 4.6, Lemma 4.9 and Lemma 4.10 below. The $\mathcal{F}$-projectivity of the resolution will be established in Lemma 4.11 below. □

Our strategy for the remainder of the proof of Theorem 4.5 is analogous to the strategy used in Section 4 of [**15**] in the construction of the duality resolution for $p = 3$. In the following computations we will abbreviate $\mathrm{Ext}^*_{\mathbb{W}[[PS_2^1]]}(M, \mathbb{F}_4)$ simply by $\mathrm{Ext}^*(M)$ and $\mathrm{Tor}_*^{\mathbb{W}[[PS_2^1]]}(\mathbb{F}_4, M)$ simply by $\mathrm{Tor}_*(M)$. We observe that we have isomorphisms

$$\mathrm{Ext}^q(M) \cong \mathrm{Tor}_q(M)^* \tag{4.1}$$

for any profinite $\mathbb{W}[[PS_2^1]]$-module M if $(-)^*$ denotes the $\mathbb{F}_4$-linear dual.

We also note that $\mathrm{Ext}^*(-)$ and $\mathrm{Tor}_*(-)$ define functors from the category of Galois-twisted profinite $\mathbb{W}[[P\mathbb{G}_2^1]]$-modules to Galois-twisted $\mathbb{F}_4[[P\mathbb{G}_2^1/PS_2^1]]$-modules and this will be important in the proof of Lemma 4.8, Lemma 4.9 and Lemma 4.10 below. Note that the quotient group $P\mathbb{G}_2^1/PS_2^1$ is isomorphic to $P\mathbb{S}_2^1/PS_2^1 \rtimes \mathrm{Gal} \cong \mathbb{F}_4^\times \rtimes \mathrm{Gal} \cong \mathfrak{S}_3$.

As input for our construction we will use Lemma 4.4 and the isomorphisms (4.1) together with the calculation of $\mathrm{Ext}^*(\mathbb{W}) = H^*(PS_2^1; \mathbb{F}_4)$. The latter is given by Theorem 3.11 with coefficients extended from $\mathbb{F}_2$ to $\mathbb{F}_4$.

Lemma 4.6. *a) There is a short exact sequence of Galois-twisted profinite $P\mathbb{G}_2^1$-modules*

$$0 \to N_1 \to P_0 := \mathbb{W}\uparrow_{\mathfrak{S}_4}^{P\mathbb{S}_2^1} \oplus \mathbb{W}\uparrow_{\mathfrak{S}'_4}^{P\mathbb{S}_2^1} \xrightarrow{\varepsilon} \mathbb{W} \to 0 \tag{4.2}$$

where ε is given by augmentation.

b) We have the following Poincaré series

$$\chi_{C,\varepsilon} := \sum_{n\geq 0} \dim_{\mathbb{F}_4} \mathrm{Coker}(\mathrm{Ext}^n(\varepsilon))t^n = \frac{1+t+t^2}{1-t}$$
$$\chi_{K,\varepsilon} := \sum_{n\geq 0} \dim_{\mathbb{F}_4} \mathrm{Ker}(\mathrm{Ext}^n(\varepsilon))t^n = t + t^2$$
$$\chi_1 := \sum_{n\geq 0} \dim_{\mathbb{F}_4} \mathrm{Ext}^n(N_1)t^n = \frac{1+t+t^2}{1-t} + 1 + t\ .$$

PROOF. It is clear that ε is surjective. As modules over $\mathbb{W}[[PS_2^1]]$ we have

$$P_0 = \mathbb{W}\uparrow_{E_2}^{PS_2^1} \oplus \mathbb{W}\uparrow_{E_2'}^{P\mathbb{S}_2^1}$$

where E_2 and E_2' are the elementary abelian 2-subgroups of rank 2 of table (1.5). By the Shapiro lemma there is an isomorphism

$$\mathrm{Ext}^*(P_0) \cong H^*(E_2, \mathbb{F}_4) \oplus H^*(E_2'; \mathbb{F}_4)$$

and $\mathrm{Ext}^*(\varepsilon)$ corresponds via this isomorphism to the restriction homomorphism

$$H^*(PS_2^1, \mathbb{F}_4) \to H^*(E_2, \mathbb{F}_4) \oplus H^*(E_2', \mathbb{F}_4) \ .$$

The long exact sequence in $\mathrm{Ext}^*(-)$ associated to the short exact sequence (4.2) gives a short exact sequence of $\mathbb{F}_4[P\mathbb{G}_2^1/PS_2^1]$-modules

$$0 \to \mathrm{Coker}(\mathrm{Ext}^*(\varepsilon)) \to \mathrm{Ext}^*(N_1) \to \mathrm{Ker}(\mathrm{Ext}^{*+1}(\varepsilon)) \to 0 \tag{4.3}$$

and by Theorem 3.11 the Poincaré series of $\mathrm{Coker}(\mathrm{Ext}^*(\varepsilon))$ is given by

$$\frac{2}{(1-t)^2} - \frac{1+t^3}{(1-t)^2} = \frac{1-t^3}{(1-t)^2} = \frac{1+t+t^2}{1-t}$$

while that of $\mathrm{Ker}(\mathrm{Ext}^*(\varepsilon))$ is given by $t + t^2$. The result follows. □

REMARK 4.7. (cf. Remark 3.5) We can and will choose $\mathfrak{S}_4$ and $\mathfrak{S}_4'$ such that $\mathfrak{S}_4 \cap \mathfrak{S}_4' = C_4$. In fact, if we choose as generator of the subgroup C_4 the image of $1+i$ (as in Lemma 2.4) and for $\mathfrak{S}_4$ the group PG_{48} of Lemma 2.2 then conjugation by $1 + 2i$ fixes C_4 and we can take the conjugate copy of $\mathfrak{S}_4$ as $\mathfrak{S}_4'$. Then it is elementary to check that $\mathfrak{S}_4 \cap \mathfrak{S}_4' = C_4$.

LEMMA 4.8. *There is a homomorphism of Galois-twisted profinite $P\mathbb{G}_2^1$-modules*

$$\psi : \mathbb{W}\uparrow_{C_4}^{P\mathbb{G}_2^1} \to N_1 \tag{4.4}$$

with the following properties.

a) The Poincaré series

$$\chi_{K,\psi} := \sum_{n\geq 0} \dim_{\mathbb{F}_4} \mathrm{Ker}(\mathrm{Ext}^n(\psi))t^n, \quad \chi_{C,\psi} := \sum_{n\geq 0} \dim_{\mathbb{F}_4} \mathrm{Coker}(\mathrm{Ext}^n(\psi))t^n$$

are given by

$$\chi_{K,\psi} = 1 + t, \quad \chi_{C,\psi} = 2 + t \ .$$

b) As a Galois-twisted $\mathfrak{S}_3 = P\mathbb{G}_2^1/PS_2^1$-module $\mathrm{Coker}(\mathrm{Ext}^0(\psi))$ *is isomorphic to the cokernel of the inclusion $\mathbb{F}_4 \cong \mathbb{F}_4[\mathfrak{S}_3/\mathrm{Gal}]^{C_3} \to \mathbb{F}_4[\mathfrak{S}_3/\mathrm{Gal}]$ of Galois-twisted $\mathfrak{S}_3$-modules.*

PROOF. Consider the map of Galois-twisted profinite $P\mathbb{G}_2^1$-modules

$$\varphi : \mathbb{W}\uparrow_{C_4}^{P\mathbb{G}_2^1} \to P_0 = \mathbb{W}\uparrow_{\mathfrak{S}_4}^{P\mathbb{G}_2^1} \oplus \mathbb{W}\uparrow_{\mathfrak{S}_4'}^{P\mathbb{G}_2^1} \tag{4.5}$$

which sends the generator e_1 to $(e_0, -e_0')$.

Here e_1 is given as $e \otimes 1 \in \mathbb{W}[[P\mathbb{G}_2^1]]\widehat{\otimes}_{\mathbb{W}[[C_4]]}\mathbb{W}$ and e_0 resp. e_0' as the analogous element in $\mathbb{W}[[P\mathbb{G}_2^1]]\widehat{\otimes}_{\mathbb{W}[[\mathfrak{S}_4]]}\mathbb{W}$ resp. in $\mathbb{W}[[P\mathbb{G}_2^1]]\widehat{\otimes}_{\mathbb{W}[[\mathfrak{S}_4']]}\mathbb{W}$. Clearly $\varepsilon\varphi$ is trivial, hence φ factors as composition $\psi : \mathbb{W}\uparrow_{C_4}^{P\mathbb{G}_2^1} \to N_1$ followed by the inclusion of N_1 into P_0. In order to analyze $\mathrm{Ext}^*(\psi)$ we start by analyzing $\mathrm{Ext}^*(\varphi)$.

As modules over PS_2^1 there are isomorphisms

$$\mathbb{W}\uparrow_{C_4}^{P\mathbb{G}_2^1} \cong \mathbb{W}\uparrow_{C_2}^{PS_2^1} \oplus \mathbb{W}\uparrow_{\omega C_2\omega^{-1}}^{PS_2^1} \oplus \mathbb{W}\uparrow_{\omega^2 C_2\omega^{-2}}^{PS_2^1}$$
$$\mathbb{W}\uparrow_{\mathfrak{S}_4}^{P\mathbb{G}_2^1} \oplus \mathbb{W}\uparrow_{\mathfrak{S}_4'}^{P\mathbb{G}_2^1} \cong \mathbb{W}\uparrow_{E_2}^{PS_2^1} \oplus \mathbb{W}\uparrow_{E_2'}^{PS_2^1}$$

and the induced map $\mathrm{Ext}^*(\varphi)$ becomes a homomorphism

$$H^*(E_2, \mathbb{F}_4) \oplus H^*(E_2', \mathbb{F}_4) \to \bigoplus_{i=0}^{2} H^*(\omega^i C_2 \omega^{-i}, \mathbb{F}_4)$$

whose kernel is the inverse limit of Corollary 3.7 with $\mathbb{F}_2$-coefficients replaced by $\mathbb{F}_4$-coefficients. Therefore $K(\varphi) := \mathrm{Ker}(\mathrm{Ext}^*(\varphi))$ has Poincaré series $\frac{1+t^3}{(1-t)^2}$ and the exact sequence

$$0 \to K(\varphi) \to H^*(E_2, \mathbb{F}_4) \oplus H^*(E_2', \mathbb{F}_4) \to \bigoplus_{i=0}^{2} H^*(\omega^i C_2 \omega^{-i}, \mathbb{F}_4) \to C(\varphi) \to 0$$

shows that $C(\varphi) := \mathrm{Coker}(\mathrm{Ext}^*(\varphi))$ has Poincaré series $\chi_{C,\varphi}$ given by

$$\chi_{C,\varphi} := \frac{1+t^3}{(1-t)^2} + \frac{3}{1-t} - \frac{2}{(1-t)^2} = \frac{t^3 + 3(1-t) - 1}{(1-t)^2} = \frac{t^3 - 3t + 2}{(1-t)^2} = t + 2\ .$$

Now we turn towards analyzing $\mathrm{Ext}^*(\psi)$ and we consider the exact sequence (4.3)

$$0 \to \mathrm{Coker}(\mathrm{Ext}^*(\varepsilon)) \to \mathrm{Ext}^*(N_1) \to \mathrm{Ker}(\mathrm{Ext}^{*+1}(\varepsilon)) \to 0\ .$$

Because $\varepsilon\varphi$ is trivial $\mathrm{Ext}^*(\varphi)$ factors through $\mathrm{Coker}(\mathrm{Ext}^*(\varepsilon))$ and the restriction of $\mathrm{Ext}^*(\psi)$ to the submodule $\mathrm{Coker}(\mathrm{Ext}^*(\varepsilon))$ of $\mathrm{Ext}^*(N_1)$ is induced by $\mathrm{Ext}^*(\varphi)$. If χ_K is the Poincaré series of the kernel of this restriction then we have an identity

$$\chi_K + \sum_{n\geq 0} \dim_{\mathbb{F}_4} \mathrm{Ext}^n(\mathbb{W}\uparrow_{C_4}^{P\mathbb{G}_2^1}) t^n = \chi_{C,\varphi} + \sum_{n\geq 0} \dim_{\mathbb{F}_4} \mathrm{Coker}(\mathrm{Ext}^n(\varepsilon)) t^n$$

We have just seen that $\chi_{C,\varphi} = t + 2$ and therefore Lemma 4.6 implies the following identity of Poincaré series

$$\begin{aligned}\chi_{C,\varphi} + \sum_{n\geq 0} \dim_{\mathbb{F}_4} \mathrm{Coker}(\mathrm{Ext}^n(\varepsilon)) t^n &= t + 2 + \frac{1+t+t^2}{1-t} \\ &= \frac{3}{1-t} = \sum_{n\geq 0} \dim_{\mathbb{F}_4}(\mathrm{Ext}^n(\mathbb{W}\uparrow_{C_4}^{P\mathbb{G}_2^1})) t^n\ ,\end{aligned}$$

and this shows that $\chi_K = 0$.

In other words, the restriction of $\mathrm{Ext}^*(\psi)$ to $\mathrm{Coker}(\mathrm{Ext}^*(\varepsilon))$ is injective. Part a) will therefore follow if we can show that there are elements $\widetilde{x} \in \mathrm{Ext}^0(N_1)$ respectively $\widetilde{y} \in \mathrm{Ext}^1(N_1)$ which are both in the kernel of $\mathrm{Ext}^*(\psi)$ and which project in the exact sequence (4.3) to non-trivial elements in $\mathrm{Ext}^1(\mathrm{Ker}(\varepsilon)) = \mathbb{F}_4$ respectively in $\mathrm{Ext}^2(\mathrm{Ker}(\varepsilon)) = \mathbb{F}_4$.

The short exact sequence (4.3) is a sequence of Galois-twisted $P\mathbb{G}_2^1/PS_2^1 = \mathfrak{S}_3$-modules and we know from Lemma 4.6 that $\mathrm{Ker}(\mathrm{Ext}^{*+1}(\varepsilon))$ is trivial unless $* = 0$ or $* = 1$ and in this case its value is $\mathbb{F}_4$. The Galois-twisting arises from the canonical homomorphism $\mathbb{G}_2^1 \to \mathrm{Gal}$ which induces a homomorphism $P\mathbb{G}_2^1/PS_2^1 \cong \mathfrak{S}_3 \to \mathrm{Gal}$ with kernel C_3 cyclic of order 3. Therefore Corollary 4.2 shows that the short exact sequences

$$0 \to \mathrm{Coker}(\mathrm{Ext}^0(\varepsilon)) = \mathbb{F}_4 \to \mathrm{Ext}^0(N_1) \to \mathrm{Ker}(\mathrm{Ext}^1(\varepsilon)) = \mathbb{F}_4 \to 0$$

and

$$0 \to \mathrm{Coker}(\mathrm{Ext}^1(\varepsilon)) = \mathbb{F}_4^2 \to \mathrm{Ext}^1(N_1) \to \mathrm{Ker}(\mathrm{Ext}^2(\varepsilon)) = \mathbb{F}_4 \to 0$$

are split exact as sequences of Galois-twisted $\mathbb{F}_4[\mathfrak{S}_3]$-modules. Hence $\mathrm{Ext}^0(N_1)$ is isomorphic to the trivial Galois-twisted module $\mathbb{F}_4 \oplus \mathbb{F}_4$. On the other hand it is clear that as $\mathbb{F}_4[\mathfrak{S}_3]$-module $\mathrm{Ext}^0(\mathbb{W}\uparrow_{C_4}^{P\mathbb{G}_2^1})$ is isomorphic to $\mathbb{F}_4[\mathfrak{S}_3/\mathrm{Gal}]$ and therefore its C_3-invariants are isomorphic to $\mathbb{F}_4$. The image of $\mathrm{Ext}^0(\varphi)$ is isomorphic to $\mathbb{F}_4$ and therefore necessarily equal to these invariants. Hence the image of $\mathrm{Ext}^0(\varphi)$ must agree with the image of $\mathrm{Ext}^0(\psi)$ because otherwise $\mathrm{Ext}^0(\mathbb{W}\uparrow_{C_4}^{P\mathbb{G}_2^1})$ would contain a $\mathbb{F}_4[C_3]$-submodule of dimension 2 with trivial action of C_3. This shows the existence of $\widetilde{x}$ and also proves part (b) because we have just seen that $\mathrm{Coker}(\mathrm{Ext}^0(\psi))$ is isomorphic to the cokernel of the inclusion

$$\mathbb{F}_4 = \mathbb{F}_4[\mathfrak{S}_3/\mathrm{Gal}]^{C_3} \to \mathbb{F}_4[\mathfrak{S}_3/\mathrm{Gal}] \ .$$

Finally we use that for each profinite $\mathbb{W}[[P\mathbb{G}_2^1]]$-module M the short exact sequence of trivial Galois-twisted $\mathbb{W}[[P\mathbb{G}_2^1]]$-modules

$$0 \to \mathbb{W}/(2) \xrightarrow{\times 2} \mathbb{W}/(4) \to \mathbb{W}/(2) \to 0$$

induces a connecting homomorphism

$$\delta : \mathrm{Ext}^*(M) = \mathrm{Ext}^*_{\mathbb{W}[[PS_2^1]]}(M, \mathbb{W}/(2)) \to \mathrm{Ext}^{*+1}_{\mathbb{W}[[PS_2^1]]}(M, \mathbb{W}/(2)) = \mathrm{Ext}^{*+1}(M)$$

which is functorial in M and commutes with the connecting homomorphisms associated to short exact sequences $0 \to M_1 \to M_2 \to M_3 \to 0$ in the first variable of Ext. If $M = \mathbb{W}$ is the trivial module then part c) of Theorem 3.11 says that this connecting homomorphism induces an isomorphism between the kernel of $\mathrm{Ext}^1(\varepsilon)$ and the kernel of $\mathrm{Ext}^2(\varepsilon)$. In the exact sequence (4.3) for $* = 0$ we have just seen that there is a lift of a generator $x \in \mathrm{Ext}^1(\mathrm{Ker}(\varepsilon))$ to an element $\widetilde{x} \in \mathrm{Ext}^0(N_1)$ such that $\widetilde{x}$ is in the kernel of $\mathrm{Ext}^0(\psi)$. Then $\widetilde{y} := \delta(\widetilde{x})$ is in the kernel of $\mathrm{Ext}^1(\psi)$ and projects to a non-trivial element in $\mathrm{Ker}(\mathrm{Ext}^2(\varepsilon))$. □

LEMMA 4.9. *There is a short exact sequence of Galois-twisted profinite $P\mathbb{G}_2^1$-modules*

$$0 \to N_2 \to P_1 := \mathbb{W}\uparrow_{C_4}^{P\mathbb{G}_2^1} \oplus \mathbb{W}\uparrow_{\mathfrak{S}_3}^{P\mathbb{G}_2^1} \xrightarrow{\rho} N_1 \to 0 \qquad (4.6)$$

such that the Poincaré series $\chi_2 := \sum_{n\geq 0} \dim_{\mathbb{F}_4}(\mathrm{Ext}^n(N_2))t^n$ is given by

$$\chi_2 = 3 + t \ ,$$

and such that there is an isomorphism of Galois-twisted $\mathbb{F}_4[P\mathbb{G}_2^1/PS_2^1] = \mathbb{F}_4[\mathfrak{S}_3]$-modules

$$\mathrm{Ext}^0(N_2) \cong \mathbb{F}_4[\mathfrak{S}_3/\mathrm{Gal}] \ .$$

PROOF. By the isomorphisms of (4.1) we have $\mathrm{Coker}(\mathrm{Tor}_0(\psi)) \cong \mathrm{Ker}(\mathrm{Ext}^0(\psi))$ and by the previous lemma this is isomorphic to the necessarily trivial Galois-twisted module $\mathbb{F}_4$.

By Corollary 4.2 $\mathbb{W}$ is a projective Galois-twisted $\mathbb{W}[\mathfrak{S}_3]$-module. Hence, the canonical epimorphism $\mathbb{W} \to \mathbb{F}_4$ can be lifted against the canonical projection

$$N_1 \to \mathrm{Tor}_0(N_1) \to \mathrm{Coker}(\mathrm{Tor}_0(\psi)) \cong \mathbb{F}_4$$

to a $\mathbb{W}[[P\mathbb{G}_2^1]]$-linear homomorphism

$$\psi' : \mathbb{W}\uparrow_{\mathfrak{S}_3}^{P\mathbb{G}_2^1} \to N_1 \ .$$

Then the homomorphism

$$\rho : P_1 = \mathbb{W}\uparrow_{C_4}^{P\mathbb{G}_2^1} \oplus \mathbb{W}\uparrow_{\mathfrak{S}_3}^{P\mathbb{G}_2^1} \to N_1$$

is defined via its restriction to the two summands given by ψ and ψ'. By construction the map ρ induces an epimorphism on $\mathrm{Tor}_0(-)$. By Lemma 4.4 it is therefore surjective and N_2 is defined as its kernel and we have established the short exact sequence (4.6).

The long exact sequence in $\mathrm{Ext}^*(-)$ associated to the short exact sequence (4.6) gives a short exact sequence

$$0 \to \mathrm{Coker}(\mathrm{Ext}^*(\rho)) \to \mathrm{Ext}^*(N_2) \to \mathrm{Ker}(\mathrm{Ext}^{*+1}(\rho)) \to 0 \ . \tag{4.7}$$

Because $\mathbb{W}\uparrow_{\mathfrak{S}_3}^{P\mathbb{G}_2^1}$ is isomorphic to $\mathbb{W}[[PS_2^1]]$ as PS_2^1-module we get $\mathrm{Ext}^*(\mathbb{W}\uparrow_{\mathfrak{S}_3}^{P\mathbb{G}_2^1}) \cong \mathbb{F}_4$ concentrated in degree 0 and hence $\mathrm{Ext}^*(\rho)$ agrees with $\mathrm{Ext}^*(\psi)$ for $* > 0$. For $* = 0$ the difference is that

$$\mathrm{Ext}^0(\rho) : \mathrm{Ext}^0(N_1) \cong \mathbb{F}_4 \oplus \mathbb{F}_4 \to \mathrm{Ext}^0(P_1) = (\mathbb{F}_2)^4$$

is injective with cokernel isomorphic to the cokernel of $\mathrm{Ext}^0(\psi)$ while

$$\mathrm{Ext}^0(\psi) : \mathrm{Ext}^0(N_1) \cong \mathbb{F}_4 \oplus \mathbb{F}_4 \to \mathrm{Ext}^0(\mathbb{W}\uparrow_{C_4}^{P\mathbb{G}_2^1}) = (\mathbb{F}_4)^3$$

has kernel $\mathbb{F}_4$. It follows that $\chi_2 = 3 + t$ as claimed.

For the last statement we use that the short exact sequence (4.7) for $* = 0$ is one of Galois-twisted $\mathbb{F}_4[\mathfrak{S}_3]$-modules and identifies by part b) of Lemma 4.8 with the sequence

$$0 \to \mathrm{Coker}(\mathbb{F}_4 \to \mathbb{F}_4[\mathfrak{S}_3/\mathrm{Gal}]) \to \mathrm{Ext}^0(N_2) \to \mathrm{Ker}(\mathrm{Ext}^1(\rho)) = \mathbb{F}_4 \to 0 \ .$$

By Proposition 4.1 the sequence is split just as the sequence

$$0 \to \mathbb{F}_4 \cong \mathbb{F}_4[\mathfrak{S}_3/\mathrm{Gal}]^{C_3} \to \mathbb{F}_4[\mathfrak{S}_3/\mathrm{Gal}] \to \mathrm{Coker}(\mathbb{F}_4 \to \mathbb{F}_4[\mathfrak{S}_3/\mathrm{Gal}]) \to 0$$

and this implies $\mathrm{Ext}^0(N_2) \cong \mathbb{F}_4 \oplus \mathrm{Coker}(\mathbb{F}_4 \to \mathbb{F}_4[\mathfrak{S}_3/\mathrm{Gal}]) \cong \mathbb{F}_4[\mathfrak{S}_3/\mathrm{Gal}]$. □

LEMMA 4.10. *There is a short exact sequence of Galois-twisted profinite $P\mathbb{G}_2^1$-modules*

$$0 \to P_3 := \mathbb{W}\uparrow_{\mathfrak{S}_3}^{P\mathbb{G}_2^1} \to P_2 := \mathbb{W}\uparrow_{\mathrm{Gal}}^{P\mathbb{S}_2^1} \to N_2 \to 0 \ . \tag{4.8}$$

PROOF. By Proposition 4.1 the Galois-twisted $\mathfrak{S}_3$-module $\mathbb{W}[\mathfrak{S}_3/\mathrm{Gal}]$ is projective. Hence the canonical epimorphism $\mathbb{W}[\mathfrak{S}_3/\mathrm{Gal}] \to \mathbb{F}_4[\mathfrak{S}_3/\mathrm{Gal}]$ can be lifted against the canonical projection

$$N_2 \to \mathrm{Tor}_0(\mathbb{N}_2) \cong \mathbb{F}_4[\mathfrak{S}_3/\mathrm{Gal}]$$

to a $\mathbb{W}[[P\mathbb{G}_2^1]]$-linear homomorphism

$$\mathbb{W}\uparrow_{\mathrm{Gal}}^{P\mathbb{G}_2^1} = (\mathbb{W}\uparrow_{\mathrm{Gal}}^{\mathfrak{S}_3}) \uparrow_{\mathfrak{S}_3}^{P\mathbb{G}_2^1} \to N_2 \ .$$

Then the homomorphism $P_2 \to N_2$ induces an epimorphism on $\mathrm{Tor}_0(-)$. By Lemma 4.4 it is therefore surjective. Let P_3 be its kernel so that (4.8) is a short exact sequence. By the isomorphisms of (4.1) and by the previous lemma we get

$$\mathrm{Tor}_q(P_3) = \begin{cases} \mathbb{F}_4 & q = 0 \\ 0 & q > 0 \ . \end{cases}$$

By Corollary 4.2 the Galois-twisted $\mathfrak{S}_3$-module $\mathbb{W}$ is again projective. Hence the canonical epimorphism of Galois twisted $\mathfrak{S}_3$-modules $\mathbb{W} \to \mathbb{F}_4$ can be lifted against the canonical projection

$$P_3 \to \mathrm{Tor}_0(P_3) \cong \mathbb{F}_4$$

to a $\mathbb{W}[[P\mathbb{G}_2^1]]$-linear homomorphism of Galois-twisted profinite $P\mathbb{G}_2^1$-modules

$$\psi' : \mathbb{W}\uparrow_{\mathfrak{S}_3}^{P\mathbb{G}_2^1} \to P_3 \ .$$

By construction the map induces an isomorphism on $\mathrm{Tor}_q(-)$ for all q and by Lemma 4.4 it is therefore an isomorphism. □

LEMMA 4.11. *The exact complexes of Galois-twisted profinite $P\mathbb{G}_2^1$-modules respectively $\mathbb{G}_2^1(\Gamma)$-modules of Theorem* 4.5 *are $\mathcal{F}$-resolutions of the trivial Galois-twisted module* $\mathbb{W}$.

PROOF. By the obvious analogue of Lemma 14 of [**21**] it suffices to consider the case of $P\mathbb{G}_2^1$. Furthermore it suffices to show that the complex is split after restriction to any finite subgroup F and for this it is enough that the short exact sequences of Lemma 4.6, Lemma 4.9 and Lemma 4.10 are split exact after restriction to F. Furthermore it suffices to consider to restrict attention to maximal finite 2-subgroups which are given by $\mathfrak{S}_4$ and $\mathfrak{S}_4'$ (cf. the discussion before Lemma 2.3). The maps

$$\mathbb{W} \to \mathbb{W}\uparrow_{\mathfrak{S}_4}^{P\mathbb{G}_2^1} \oplus \mathbb{W}\uparrow_{\mathfrak{S}_4'}^{P\mathbb{G}_2^1}$$

given by sending x to $(x\mathfrak{S}_4, 0)$ resp. x to $(0, x\mathfrak{S}_4')$ are clearly $\mathbb{W}[\mathfrak{S}_4]$ resp. $\mathbb{W}[\mathfrak{S}_4']$-linear and provide splittings of ε. Lemma 4.3 shows that for any finite subgroup F of $P\mathbb{G}_2^1$ the exact sequence of Lemma 4.10 is split as sequence of $\mathbb{W}[F]$-modules. In particular, as $\mathbb{W}[F]$-module N_2 is a direct summand in $\mathbb{W}\uparrow_{\mathrm{Gal}}^{P\mathbb{G}_2^1}$ and therefore N_2 is projective as $\mathbb{W}[F]$-module. So Lemma 4.3 also applies to the exact sequence of Lemma 4.9 and this completes the proof. □

Finally we turn towards the construction of an $\mathcal{F}$-resolution for the trivial Galois-twisted profinite G-module $\mathbb{W}$ for $G = \mathbb{G}_2(\Gamma)$ resp. $G = P\mathbb{G}_2$. We observe that because of $\mathbb{G}_2(\Gamma)/\mathbb{G}_2^1(\Gamma) \cong P\mathbb{G}_2/P\mathbb{G}_2^1 \cong \mathbb{Z}_2$ there is a short exact sequence of Galois-twisted profinite $\mathbb{G}_2(\Gamma)$ resp. $P\mathbb{G}_2$-modules

$$0 \to \mathbb{W}[[\mathbb{Z}_2]] \xrightarrow{g-e} \mathbb{W}[[\mathbb{Z}_2]] \to \mathbb{W} \to 0 \tag{4.9}$$

where g is a topological generator of the group $\mathbb{Z}_2$. In fact, for every prime p there is a well known isomorphism $\mathbb{Z}_p[[T]] \to \mathbb{Z}_p[[\mathbb{Z}_p]]$ which sends T to $g-e$ where g is any topological generator of $\mathbb{Z}_p$ and this extends to an isomorphism $\mathbb{W}[[T]] \to \mathbb{W}[[\mathbb{Z}_p]]$. Via this isomorphism the augmentation is just given by the map which sends T to 0.

By the analogue of Lemma 15 of [**21**] induction of the $\mathcal{F}$-resolutions of Theorem 4.5 from $P\mathbb{G}_2^1$ to $P\mathbb{G}_2$ resp. from $\mathbb{G}_2^1(\Gamma)$ to $\mathbb{G}_2(\Gamma)$ gives an $\mathcal{F}$-resolution of the $\mathbb{W}[[P\mathbb{G}_2]]$-module $\mathbb{W}\uparrow_{P\mathbb{G}_2^1}^{P\mathbb{G}_2}$ resp. of the $\mathbb{W}[[\mathbb{G}_2(\Gamma)]]$-module $\mathbb{W}\uparrow_{\mathbb{G}_2^1(\Gamma)}^{\mathbb{G}_2(\Gamma)}$ of the form

$$0 \to Q_3 \xrightarrow{\partial_3} Q_2 \xrightarrow{\partial_2} Q_1 \xrightarrow{\partial_1} Q_0 \xrightarrow{\varepsilon} \mathbb{W}[[\mathbb{Z}_2]] \to 0 \ . \tag{4.10}$$

In the case of $\mathbb{G}_2(\Gamma)$ the modules in this resolution are given as

$$\begin{aligned} Q_3 &= \mathbb{W}\uparrow_{G_{12}(\Gamma)}^{\mathbb{G}_2(\Gamma)} \\ Q_2 &= \mathbb{W}\uparrow_{C_2\times \mathrm{Gal}}^{\mathbb{G}_2(\Gamma)} \\ Q_1 &= \mathbb{W}\uparrow_{G_{12}(\Gamma)}^{\mathbb{G}_2(\Gamma)} \oplus \mathbb{W}\uparrow_{C_8}^{\mathbb{G}_2(\Gamma)} \\ Q_0 &= \mathbb{W}\uparrow_{G_{48}(\Gamma)}^{\mathbb{G}_2(\Gamma)} \oplus \mathbb{W}\uparrow_{G'_{48}(\Gamma)}^{\mathbb{G}_2(\Gamma)} \end{aligned}$$

and the monomorphism of the exact sequence (4.9) can be covered by a map of complexes

$$\begin{array}{ccccccccccccc} 0 & \to & Q_3 & \xrightarrow{\partial_3} & Q_2 & \xrightarrow{\partial_2} & Q_1 & \xrightarrow{\partial_1} & Q_0 & \xrightarrow{\varepsilon} & \mathbb{W}\uparrow_{\mathbb{G}_2^1(\Gamma)}^{\mathbb{G}_2(\Gamma)} & \to 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow g-e & \\ 0 & \to & Q_3 & \xrightarrow{\partial_3} & Q_2 & \xrightarrow{\partial_2} & Q_1 & \xrightarrow{\partial_1} & Q_0 & \xrightarrow{\varepsilon} & \mathbb{W}\uparrow_{\mathbb{G}_2^1(\Gamma)}^{\mathbb{G}_2(\Gamma)} & \to 0\ . \end{array} \tag{4.11}$$

In the case of $P\mathbb{G}_2$ we need to make the appropriate replacements indicated in Remark 1.3.

The following result is a more precise form of Theorem 1.1.

THEOREM 4.12. *The total complex of the double complex $Q_{*,*}$ of (*4.11*) is a Galois-twisted $\mathcal{F}$-resolution of the trivial $\mathbb{W}[[\mathbb{G}_2(\Gamma)]]$-module $\mathbb{W}$.*

PROOF. It is clear that $Tot(C)_*$ is a complex of $\mathcal{F}$-projective modules. The fact that the complexes $\mathrm{Hom}_{\mathbb{W}[[\mathbb{G}_2(\Gamma)]]}(P, Q_{i,*})$ are exact for $i = 1, 0$ and each $\mathcal{F}$-projective module P implies that the complex $\mathrm{Hom}_{\mathbb{W}[[\mathbb{G}_2(\Gamma)]]}(P, Tot(Q)_*)$ is exact for each $\mathcal{F}$-projective module P. ☐

5. Topologically realizing the algebraic centralizer resolutions

5.1. Preliminaries on Morava modules. Let Γ be a formal group of height n defined over the field $\mathbb{F}_q$ with $q = p^n$. By the Goerss-Hopkins-Miller theorem the extended Morava stabilizer group $\mathbb{G}_n(\Gamma)$ acts on the spectrum $E_n(\Gamma)$ (see [**18**], [**32**]); we recall that $E_n(\Gamma)$ is a Landweber exact spectrum given by a 2-periodic theory with coefficients $\pi_*(E_n(\Gamma)) = \pi_0(E_n(\Gamma))[u^{\pm 1}]$ (with $u \in \pi_{-2}(E_n(\Gamma))$) whose associated formal group law over $\pi_0(E_n(\Gamma))$ is a universal deformation of Γ in the sense of Lubin and Tate. In particular there is a (non-canonical) isomorphism between $\pi_0(E_n(\Gamma))$ and $\mathbb{W}[[u_1, \ldots, u_{n-1}]]$, the ring of formal power series over the ring of Witt vectors $\mathbb{W} = \mathbb{W}(\mathbb{F}_q)$ of $\mathbb{F}_q$, in the variables $u_1, \ldots, u_{n-1}$. The maximal ideal $(p, u_1, \ldots, u_{n-1})$ of this power series ring will be denoted $\mathfrak{m}$. To keep notation simple we will abbreviate in this section $E_n(\Gamma)$ by E_n and $\mathbb{G}_n(\Gamma)$ by $\mathbb{G}_n$.

For the purposes of this paper, a Morava module is an $\mathfrak{m}$-adically complete $(E_n)_*$-module M equipped with a continuous $\mathbb{G}_n$-action (continuous with respect to the $\mathfrak{m}$-adic topology on M and the profinite topology on $\mathbb{G}_n$) such that for $g \in \mathbb{G}_n$, $a \in \pi_* E_n$, $\lambda \in \mathbb{W}$ and $x \in M$ we have

$$g(ax) = g(a)g(x) \quad \text{and} \quad g(\lambda x) = {}^{g}\lambda g(x)\ .$$

So a Morava module is a Galois-twisted module but it need not be p-profinite and therefore not be a module over the (twisted) group algebra $\mathbb{W}[[\mathbb{G}_n]]$. The category of Morava modules will be denoted $\mathcal{EG}_n$. A morphism in this category is an $(E_n)_*$-linear map $M \to M'$ which commutes with the action of $\mathbb{G}_n$. We note that such a map will be automatically continuous with respect to the $\mathfrak{m}$-adic topologies on M and M'.

The Morava module of a spectrum X is defined as

$$(E_n)_*X = \pi_* L_{K(n)}(E_n \wedge X) \ .$$

This is an $(E_n)_*$-module which is complete but not necessarily Hausdorff with respect to the $\mathfrak{m}$-adic topology if $\mathfrak{m}$ denotes the maximal ideal of $\pi_0(E_n)$. All Morava modules in this section will be Hausdorff, in fact they will all be pro-discrete (cf. Corollary 5.5).

For the Honda formal group law the following result is well-known and can be found in [**11**] or in [**35**]. We give a proof which is very close to that in [**35**].

PROPOSITION 5.1. *Let E_n be the Lubin-Tate spectrum associated to a deformation of a formal group law Γ over $\mathbb{F}_q$ which is already defined over $\mathbb{F}_p$. Assume that the Frobenius endomorphism ξ_Γ defined by $\xi_\Gamma(x) = x^p$ satisfies an equation $\xi_\Gamma^n = pu$ in the endomorphism ring of Γ (over $\mathbb{F}_p$) where u is a p-adic unit. Then there is an isomorphism*

$$\phi : \pi_* L_{K(n)}(E_n \wedge E_n) \cong \mathrm{map}_{cts}(\mathbb{G}_n, (E_n)_*) \tag{5.1}$$

which is adjoint to the map

$$\mathbb{G}_n \times \pi_* L_{K(n)}(E_n \wedge E_n) \to \pi_*(E_n)$$

given by

$$(g \in \mathbb{G}_n, x : S^m \to E_n \wedge E_n) \mapsto (S^m \xrightarrow{x} E_n \wedge E_n \xrightarrow{1\wedge g} E_n \wedge E_n \xrightarrow{\mu} E_n)$$

where μ is multiplication on E_n.

We prepare the proof of the proposition with two remarks, one on formal group laws and another one on q-Boolean algebras.

REMARK 5.2. a) Let $q = p^n$ and let k be a field which contains $\mathbb{F}_q$. The endomorphism ξ_F^n commutes with an endomorphism $\sum_i a_i x^i \in \mathrm{End}_k(\Gamma)$ if and only if $a_i^q = a_i$ for all i, i.e. $a_i \in \mathbb{F}_q$ for all i. Hence the canonical map

$$\mathrm{End}_{\mathbb{F}_q}(\Gamma) \to \mathrm{End}_k(\Gamma)$$

is an isomorphism if and ony if ξ_F^n is central in which case it must satisfy an equation $\xi_\Gamma^n = pu$ in the endomorphism ring of Γ (over $\mathbb{F}_p$) for some p-adic unit u.

b) More generally, if k is a finite field of order p^m and K its algebraic closure then the endomorphism ring over k is isomorphic to the centralizer of ξ_F^m in $\mathrm{End}_K(\Gamma)$.

REMARK 5.3. a) Let Γ be any formal group law over $\mathbb{F}_q$ and consider as in the proof of Theorem 12 of [**35**] the functor which sends an $\mathbb{F}_q$-algebra A to the set of pairs (β, f) where $\beta : \mathbb{F}_q \to A$ is any ring homomorphim and f is an isomorphism $\alpha_*\Gamma \to \beta_*\Gamma$ of formal group laws where α is the ring homomorphism defining the $\mathbb{F}_q$-algebra structure on A. This functor is corepresented by the $\mathbb{F}_q$-algebra

$$B(\Gamma) := \mathbb{F}_q \otimes_L L[b_0^{\pm 1}, b_1, \ldots] \otimes_L \mathbb{F}_q$$

where the $\mathbb{F}_q$-algebra structure comes from the first tensor factor $\mathbb{F}_q$, L is the Lazard ring, $\mathbb{F}_q$ is considered as an L-algebra via the homomorphism classifying Γ and $L[b_0^{\pm 1}, b_1, \ldots]$ is an L-algebra via the usual units η_L and η_R in the Hopf-algebroid $(L, L[b_0^{\pm 1}, b_1, \ldots])$. The algebra B is generated over $\mathbb{F}_q \otimes \mathbb{F}_q$ by the elements

b_i, $i = 0, 1, \ldots$, with respect to complicated relations determined by η_R and the homomorphism $L \to \mathbb{F}_q$ classifying Γ.

Now assume that the formal group law Γ is already defined over $\mathbb{F}_p$. Then we have $\alpha_*\Gamma = \beta_*\Gamma$ because there is only one algebra homomorphism $\mathbb{F}_p \to A$ and then f is an automorphism of $\alpha_*\Gamma = \beta_*\Gamma$. If furthermore $\zeta_\Gamma^n = pu$ then for any endomorphism $\sum_i a_i x^i \in \mathrm{End}_A(\alpha_*\Gamma)$ we have $a_i^q = a_i$ and the complicated relations must include the relations $b_i^q = b_i$ for all i. Therefore $B(\Gamma)$ is a q-Boolean algebra, i.e. an $\mathbb{F}_q$-algebra which satisfies $x^q = x$ for any $x \in B$.

b) Let B be a q-Boolean algebra. A q-Boolean algebra which is an integral domain only has q solutions to the equation $x^q = x$, hence any prime ideal in such a B is maximal and is the kernel of a unique $\mathbb{F}_q$-algebra morphism $B \to \mathbb{F}_q$. So we can identify the spectrum $\mathrm{spec}(B)$ with $\mathrm{Hom}_{\mathbb{F}_q-alg}(B, \mathbb{F}_q)$. Furthermore B is the colimit of its finite $\mathbb{F}_q$-subalgebras and this defines a profinite topology on its spectrum $\mathrm{spec}(B)$. The structure theorem for q-Boolean algebras says that the evaluation map from B to the algebra of continuous functions on its spectrum

$$B \to \mathrm{map}_{cts}(\mathrm{spec}(B), \mathbb{F}_q), \quad x \mapsto (\varphi \mapsto \varphi(x))$$

is an isomorphism.

In fact, if $x \in B$ is any element then x^{q-1} is idempotent. Hence, if $x^{q-1} \neq 1$ then B is the product of the ideals generated by x^{q-1} and $1 - x^{q-1}$. From this one sees immediately that the evaluation map is an isomorphism if B is finite. The general case follows by observing that for a profinite set $S = \lim_i S_i$ with S_i finite, the set of continuous functions $\mathrm{map}_{cts}(S, \mathbb{F}_q)$ is equal to $\mathrm{colim}_i \mathrm{map}(S_i, \mathbb{F}_q)$.

We are now ready for the proof of Proposition 5.1.

PROOF. It is enough to prove the isomorphism in degree 0 after reducing modulo the ideal generated by the maximal ideal $\mathfrak{m}$ in $\pi_0(E)$. The $\mathbb{F}_q$-algebra $(E_n)_0 E_n/\mathfrak{m}$ agrees with the algebra $B(\Gamma)$ considered in part a) of the previous remark. By the assumption on Γ and part b) of the preceeding remark $(E_n)_0 E_n/\mathfrak{m}$ is an $\mathbb{F}_q$-Boolean algebra and is therefore isomorphic to the ring of continuous functions on its spectrum. The spectrum of $(E_n)_0 E_n/\mathfrak{m}$ identifies with the profinite set of pairs (β, f) where β is a ring homomorphism from $\mathbb{F}_q$ to itself and f is an automorphism of $\alpha_*\Gamma = \beta_*\Gamma$, and this is exactly equal to $\mathbb{G}_n$. □

In the remainder of this section we assume that Γ satisfies the assumption of Proposition 5.1.

The group $\mathbb{G}_n \times \mathbb{G}_n$ acts on $\pi_*(L_{K(n)}(E_n \wedge E_n))$. The action of the left hand factor is the one used in the definition of the Morava module of $\pi_*(L_{K(n)}(E_n \wedge E_n))$. We will also need the action of the right hand factor and we need to know how this action translates to the right hand side of the isomorphism (5.1). We record this in the following lemma whose proof is straightforward.

LEMMA 5.4. *Let g, h_1 and h_2 be elements of $\mathbb{G}_n$ and x be an element of $\pi_*(L_{K(n)}(E_n \wedge E_n))$. Then*

$$\phi((h_1, h_2)x)(g) = h_1 \phi(x)(h_1^{-1} g h_2) \ .$$

In other words, the action on the left hand copy of E_n corresponds to the diagonal action on the set of continuous maps while the action on the right hand copy of E_n corresponds to the action on $\mathbb{G}_n$ on the set of continuous maps given by the action on the source on the right. □

The results of [**11**] on homotopy fixed points will now carry over to the case of $E_n = E_n(\Gamma)$ if Γ satisfies the assumptions of Proposition 5.1. In particular we have the following result.

COROLLARY 5.5. *Let K be a closed subgroup of $\mathbb{G}_n$. Then there is an isomorphism of Morava modules*

$$(E_n)_*(E_n^{hK}) \cong \mathrm{map}_{cts}(\mathbb{G}_n/K, (E_n)_*) \cong \mathrm{Hom}_{\mathbb{W}[\mathrm{Gal}]-cts}(\mathbb{W}[[\mathbb{G}_n/K]], (E_n)_*)$$

if $\mathbb{G}_n$ acts diagonally on the set of continuous maps respectively continuous Galois-twisted homomorphisms. □

Let M be a Morava module and let $x \in M$, $g \in \mathbb{G}_n$. If $\alpha : M \to (E_n)_*$ is an $(E_n)_*$-linear map let $\Phi(\alpha) : M \to \mathrm{map}_{cts}(\mathbb{G}_n, (E_n)_*)$ be given by

$$(\Phi(\alpha)(x))(g) = g\alpha(g^{-1}x) .$$

Conversely, let $\beta : M \to \mathrm{map}_{cts}(\mathbb{G}_n, (E_n)_*)$ be a morphism of Morava modules where $\mathbb{G}_n$ acts on $\mathrm{map}_{cts}(\mathbb{G}_n, (E_n)_*)$ diagonally. Then let $\Psi(\beta) : M \to (E_n)_*$ be given by

$$\Psi(\beta)(x) = (\beta(x))(e)$$

where $e \in \mathbb{G}_n$ is the unit. For $h \in \mathbb{G}_n$ let $h * \beta : M \to \mathrm{map}_{cts}(\mathbb{G}_n, (E_n)_*)$ and $h * \alpha : M \to (E_n)_*$ be given by

$$(h * \beta(x))(g) = (\beta(x))(gh) \quad \text{and} \quad (h * \alpha)(x) = h\alpha(h^{-1}x) .$$

The proof of the following lemma is straightforward and left to the reader.

LEMMA 5.6.
a) $\Phi(\alpha)$ is a homomorphism of Morava modules.
b) $\Psi(\beta)$ is $(E_n)_*$-linear.
c) The map

$$\Phi : \mathrm{Hom}_{(E_n)_*}(M, (E_n)_*) \to \mathrm{Hom}_{\mathcal{E}\mathcal{G}_n}(M, \mathrm{map}_{cts}(\mathbb{G}_n, (E_n)_*)), \quad \alpha \mapsto \Phi(\alpha)$$

is an isomorphism with inverse given by

$$\Psi : \mathrm{Hom}_{\mathcal{E}\mathcal{G}_n}(M, \mathrm{map}_{cts}(\mathbb{G}_n, (E_n)_*)) \to \mathrm{Hom}_{(E_n)_*}(M, (E_n)_*), \quad \beta \mapsto \Psi(\beta) .$$

d) The action of $\mathbb{G}_n$ on $\pi_* L_{K(n)}(E_n \wedge E_n)$ on the right hand smash factor translates via the isomorphism of Proposition 5.1 and via the isomorphisms Φ and Ψ of part c) into the diagonal action on $\mathrm{Hom}_{(E_n)_*}(M, (E_n)_*)$, i.e. for $h \in \mathbb{G}_n$ we have $\Phi(h * \alpha) = h * \Phi(\alpha)$ and $\Psi(h * \beta) = h * \Psi(\beta)$. □

The following two results are analogous to Proposition 2.6 and Proposition 2.7 of [**15**]. There they were crucial in realizing the duality resolution at $n = 2$ and $p = 3$ and here they are crucial for constructing the centralizer resolution for $n = p = 2$. In these results we use the following notation: if E is a spectrum and $X = \lim_i X_i$ is an inverse limit of a sequence of finite sets then $E[[X]]$ is defined as $\mathrm{holim}_i E \wedge (X_i)_+$. We observe that if X is such a profinite set with a continuous action of a finite group K and if E is a K-spectrum then $E[[X]]$ is a K-spectrum via the diagonal action. If X and Y are spectra then we denote the function spectrum by $F(X, Y)$.

PROPOSITION 5.7. *Let K_1 be a closed subgroup and K_2 a finite subgroup of $\mathbb{G}_n$.*

a) Then there is a natural equivalence (where the homotopy fixed points on the left hand side are formed with respect to the diagonal action of K_2)

$$E_n[[\mathbb{G}_n/K_1]]^{hK_2} \simeq F(E_n^{hK_1}, E_n^{hK_2}) \ .$$

b) If K_1 is also an open subgroup then there is a natural decomposition

$$E_n[[\mathbb{G}_n/K_1]]^{hK_2} \simeq \prod_{K_2\backslash\mathbb{G}_n/K_1} E_n^{hK_x}$$

where $K_x = K_2 \cap xK_1x^{-1}$ is the isotropy subgroup of the coset xK_1 and $K_2\backslash\mathbb{G}_n/K_1$ is the finite set of double cosets.

c) If K_1 is a closed subgroup and $K_1 = \bigcap_i U_i$ for a decreasing sequence of open subgroups U_i then

$$F(E_n^{hK_1}, E_n^{hK_2}) \simeq \mathrm{holim}_i E_n[[\mathbb{G}_n/U_i]]^{hK_2} \simeq \mathrm{holim}_i \prod_{K_2\backslash\mathbb{G}_n/U_i} E_n^{hK_{x,i}}$$

where $K_{x,i} = K_2 \cap xU_ix^{-1}$ is, as before, the isotropy subgroup of the coset xU_i. □

The following remark is taken from Section 1.3 of [**21**].

REMARK 5.8. If $U_i \subset U_j$ then the map

$$\prod_{K_2\backslash\mathbb{G}_n/U_i} E_n^{hK_{x,i}} \to \prod_{K_2\backslash\mathbb{G}_n/U_j} E_n^{hK_{x,j}}$$

in the inverse system of part (c) of the proposition can be described as follows: if $x \in \mathbb{G}_n/U_i$ has isotropy group $K_{x,i}$ and its image $x' \in \mathbb{G}_n/U_j$ has isotropy group $K_{x',j}$ then the restriction of the map to the factor determined by x sends $E_n^{hK_{x,i}}$ via the transfer to the factor $E_n^{hK_{x',j}}$ determined by x. Because K_2 is finite this implies that on homotopy groups the inverse system is Mittag-Leffler.

PROPOSITION 5.9. *Let K_1 and K_2 be closed subgroups of $\mathbb{G}_n$ and suppose that K_2 is finite. Then there is an isomorphism*

$$\big((E_n)_*[[\mathbb{G}_n/K_1]]\big)^{K_2} \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{EG}_n}((E_n)_*E_n^{hK_1}, (E_n)_*E_n^{hK_2})$$

such that the following diagram commutes

$$\begin{array}{ccc} \pi_*\big(E_n[[\mathbb{G}_n/K_1]]^{hK_2}\big) & \longrightarrow & \big((E_n)_*[[\mathbb{G}_n/K_1]]\big)^{K_2} \\ \downarrow \cong & & \downarrow \cong \\ \pi_*F(E_n^{hK_1}, E_n^{hK_2}) & \longrightarrow & \mathrm{Hom}_{\mathcal{EG}_n}((E_n)_*E_n^{hK_1}, (E_n)_*E_n^{hK_2}) \end{array}$$

where the top horizontal map is the edge homomorphism in the homotopy fixed point spectral sequence, the left-hand vertical map is the isomorphism given by Proposition 5.7 and the bottom horizontal map is the E_n-Hurewicz homomorphism. □

We will also need the following result from Section 1 of [**8**].

LEMMA 5.10. *Let $K \subset \mathbb{G}_n$ be a closed subgroup and let $K_0 = K \cap \mathbb{S}_n$. Suppose the canonical map*

$$K/K_0 \to \mathbb{G}_n/\mathbb{S}_n \cong \mathrm{Gal}$$

is an isomorphism.

a) There is a Gal*-equivariant equivalence*

$$\mathrm{Gal}_+ \wedge E^{hK} \to E^{hK_0} \ .$$

b) For any profinite Morava module M we have isomorphisms

$$H^*(K,M)\cong H_*(K_0,M)^{\mathrm{Gal}},\quad H^*(K_0,M)\cong \mathbb{W}\otimes_{\mathbb{Z}_p} H^*(K,M)\ .\quad \square$$

5.2. Realizing the centralizer resolution for $\mathbb{G}_2^1(\Gamma)$ and for $\mathbb{G}_2(\Gamma)$. The construction of the topological centralizer resolutions of Theorem 1.4 and Theorem 1.5 comes in two steps. In Proposition 5.11 we first construct a complex of spectra $X_\bullet$ such that the complex of Morava modules $(E_2)_*(X_\bullet)$ is isomorphic to the complex $\mathrm{Hom}_{\mathbb{W}[\mathrm{Gal}]-cts}(P_\bullet,(E_2)_*)$ if $P_\bullet$ denotes the complexes of Theorem 1.1. Here $\mathrm{Hom}_{\mathbb{W}[\mathrm{Gal}]-cts}$ denotes continuous homomorphisms of Galois-twisted continuous Gal-modules. This part is analogous to the first step in the construction of the duality resolution resp. centralizer resolution at $n=2$ and $p=3$ in [**15**] resp. [**21**]. In the second step we refine the complex of spectra to a resolution, i.e. we construct the necessary factorisations of the maps α_i. This step follows the strategy used in the proof of Theorem 25 and Theorem 26 of [**21**]. The crucial result is Proposition 5.12. We give details of the proof of Theorem 1.5. The proof of Theorem 1.4 is completely analogous with details which are less involved.

PROPOSITION 5.11. *Let Γ be either Γ_H or Γ_E. Then there is a complex of spectra*

$$X_\bullet : * \to L_{K(2)}S^0 \simeq E_2^{h\mathbb{G}_2(\Gamma)} \xrightarrow{\alpha_0} X_0 \xrightarrow{\alpha_1} X_1 \xrightarrow{\alpha_2} X_2 \xrightarrow{\alpha_3} X_3 \xrightarrow{\alpha_4} X_4 \to *$$

with

$$\begin{aligned}
X_0 &= E_2^{hG_{48}(\Gamma)} \vee E_2^{hG'_{48}(\Gamma)}\\
X_1 &= E_2^{hG_{12}(\Gamma)} \vee E_2^{hC_8} \vee E_2^{hG_{48}(\Gamma)} \vee E_2^{hG'_{48}(\Gamma)}\\
X_2 &= E_2^{C_2\times \mathrm{Gal}} \vee E_2^{hG_{12}(\Gamma)} \vee E_2^{hC_8}\\
X_3 &= E_2^{hG_{12}(\Gamma)} \vee E_2^{C_2\times\mathrm{Gal}}\\
X_4 &= E_2^{hG_{12}(\Gamma)}\ .
\end{aligned}$$

such that the complex of Morava modules $(E_2)_*(X_\bullet)$ is isomorphic to the complex of Morava modules $\mathrm{Hom}_{\mathbb{W}[\mathrm{Gal}]-cts}(P_\bullet,(E_2)_*)$ if $P_\bullet$ denotes the complex given by Theorem 1.1.

PROOF. By Corollary 5.5 we can choose X_i as the explicit (wedges of) homotopy fixed point spectra appearing in the statement of the proposition such that we have isomorphisms of Morava modules

$$(E_2)_*(X_i)\cong \mathrm{Hom}_{\mathbb{W}[\mathrm{Gal}]-cts}(P_i,(E_2)_*)$$

for $i=-1,0,1,2,3,4$. It is therefore enough to show that the E_2-Hurewicz homomorphisms

$$\pi_0F(X_i,X_{i+1})\to \mathrm{Hom}_{\mathcal{E}\mathcal{G}_2}((E_2)_*X_i,(E_2)_*X_{i+1}) \tag{5.2}$$

are surjective for $i=-1,0,1,2,3$ and the E_2-Hurewicz homomorphisms

$$\pi_0F(X_i,X_{i+2})\to \mathrm{Hom}_{\mathcal{E}\mathcal{G}_2}((E_2)_*X_i,(E_2)_*X_{i+2}) \tag{5.3}$$

are injective for $i=-1,0,1,2$. In fact, we will see that in most cases these homomorphisms are even isomorphisms.

By the explicit nature of the spectra X_i it is enough to show that the E_2-Hurewicz homomorphisms

$$\pi_0 F(E_2^{hK_1}, E_2^{hK_2}) \to \mathrm{Hom}_{\mathcal{E}\mathcal{G}_2}((E_2)_* E_2^{hK_1}, (E_2)_* E_2^{hK_2}) \tag{5.4}$$

are isomorphisms for every combination of K_1 and K_2 with K_1 running through $\mathbb{G}_2(\Gamma)$, $G_{48}(\Gamma)$, $G'_{48}(\Gamma)$, $G_{12}(\Gamma)$, C_8, $C_2 \times \mathrm{Gal}$ and K_2 running through $G_{48}(\Gamma)$, $G'_{48}(\Gamma)$, $G_{12}(\Gamma)$, C_8, $C_2 \times \mathrm{Gal}$, except possibly in the case that K_1 and K_2 are equal to either $G_{48}(\Gamma)$ or $G'_{48}(\Gamma)$. In this case we will see that we still have at least a surjection and this is good enough.

In fact, by Proposition 5.9 it is enough to show that the edge homomorphisms

$$\pi_0(E_2[[\mathbb{G}_2/K_1]])^{hK_2} \to \pi_0(E_2[[\mathbb{G}_2/K_1]])^{K_2}$$

of the descent spectral sequences are isomorphisms for every combination of K_1 and K_2, except if $K_1, K_2 \in \{G_{48}(\Gamma), G'_{48}(\Gamma)\}$, and that in this case it is still surjective.

For this we use Proposition 5.7 and the usual lim-$\lim^1$-sequence for the homotopy groups of holim_i. First we note that the $\lim^1$-terms $\lim^1 \prod_{K_2\backslash\mathbb{G}_2/U_i} \pi_1(E_2^{hK_{x,i}})$ arising from part c) of Proposition 5.7 are trivial. In fact, this follows from Remark 5.8 because the inverse system satisfies the Mittag-Leffler condition. Furthermore, because of

$$\pi_0(E_2[[\mathbb{G}_2/K_1]])^{K_2} = \lim \prod_{K_2\backslash\mathbb{G}_2/U_i} \pi_0(E_2)^{K_{x,i}}$$

it is enough to show that for every $K_{x,i}$ the edge homomorphism

$$\pi_0(E_2^{hK_{x,i}}) \to \pi_0(E_2)^{K_{x,i}} \tag{5.5}$$

of the homotopy fixed point spectral sequence is an isomorphism in degree 0 respectively surjective in degree 0. The groups $K_{x,i}$ always contain the central C_2. Furthermore, by Lemma 5.10 it is enough to assume that K_2 is contained in $\mathbb{S}_2$. The relevant groups are then G_{24}, Q_8, C_6, C_4 and C_2. By Lemma 5.10 the edge homomorphism is a surjection for G_{24} if and only if this is the case for $G_{48}(\Gamma)$.

The relevant calculations can be found in [**1**] and [**12**] in the case of $G_{48}(\Gamma_E)$, in [**28**] in the case of C_6, in [**7**] in the case of C_4, and in [**8**] in the case of C_2. In [**8**] there is also a nice discussion how the calculations of [**1**], [**12**], [**7**] and [**28**] which concern the homotopy groups of spectra of topological modular forms with various level structures translate into calculations of the homotopy groups of our homotopy fixed point spectra.

In all these cases the edge homomorphism is always an isomorphism. So it remains to consider the case of Q_8. The homotopy fixed point spectrum $E_2^{hG_{24}}$ is 192-periodic with periodicity generator given by Δ^8 where the modular form Δ is the algebraic periodicity generator for the G_{24}-module $(E_2)_*$. With respect to the action of Q_8 there is an invariant $\widetilde{\Delta}$ such that $\widetilde{\Delta}^3 = \Delta$ (cf. Theorem A.4 of [**3**]). Then $E_2^{hQ_8}$ will be 64-periodic with periodicity generator $\widetilde{\Delta}^8$ and there is an equivalence

$$E_2^{hQ_8} \cong E_2^{hG_{24}} \vee \Sigma^{64} E_2^{hG_{24}} \vee \Sigma^{128} E_2^{hG_{24}} \ .$$

So we need to understand the edge homomorphism

$$\pi_k(E_2^{hG_{24}}) \to \pi_k(E_2)^{G_{24}}$$

not only for $k = 0$ but also for $k = 64$ and $k = 128$. The calculations in [**1**] and [**12**] show that this is still an isomorphism for $k = 64$ while for $k = 128$ it is only surjective with kernel isomorphic to $\mathbb{F}_4$ and given by a class denoted $\Delta^5\varepsilon$. The case of Q_8 can only arise if $K_1, K_2 \in \{G_{48}(\Gamma), G'_{48}(\Gamma)\}$; in case $K_1 = \mathbb{G}_n$ all $K_{x,i}$ are equal to K_2. □

To complete the proof of Theorem 1.5 it remains to construct the factorizations $X_{i-1} \xrightarrow{\beta_i} W_i \xrightarrow{\gamma_i} X_i$ of α_i for $i = 1, 2, 3, 4$, such that each $W_{i-1} \xrightarrow{\gamma_{i-1}} X_{i-1} \xrightarrow{\beta_i} W_i$ is a cofibration (with W_0 being the cofibre of α_0). We note that these factorisations will realize the splitting of the exact complex of Morava modules $(E_2)_*(X_\bullet)$ into the usual short exact sequences. In particular, this will show that $\gamma_4 : (E_2)_*W_4 \to (E_2)_*X_4$ is an isomorphism, hence γ_4 is an equivalence and the resolution is of length 4.

The factorizations are constructed inductively. In the case $i = 1$ we simply use that the composition $\alpha_1\alpha_0$ is null homotopic. Now let $2 \le i \le 4$ and suppose that for $0 \le r < i$ we have already constructed factorizations $X_{r-1} \xrightarrow{\beta_r} W_r \xrightarrow{\gamma_r} X_r$ of α_r such that $W_{r-1} \xrightarrow{\gamma_{r-1}} X_{r-1} \xrightarrow{\beta_r} W_r$ is a cofibration for $1 \le r < i$. In order to factor α_i it is enough to show that the composition $\alpha_i\gamma_{i-1}$ considered as element in $\pi_0 F(W_{i-1}, X_i)$ is null.

The already constructed cofibrations can be used to analyze $\pi_* F(W_{i-1}, X_i)$. In fact, if Z is any spectrum then these cofibrations determine a finitely convergent Adams type spectral sequence which has the form

$$E_1^{s,t}(W_{i-1}, Z) \Longrightarrow \pi_{t-s} F(W_{i-1}, Z) \tag{5.6}$$

with differentials $d_r^{s,t} : E_r^{s,t}(W_{i-1}, Z) \to E_r^{s+r,t+r-1}(W_{i-1}, Z)$ and

$$E_1^{s,t}(W_{i-1}, Z) = \begin{cases} \pi_t F(X_{i-2-s}, Z) & 0 \le s < i \\ 0 & s \ge i\ . \end{cases}$$

Clearly $\alpha_i\gamma_{i-1}$ is in the kernel of $\beta_{i-1}^* : \pi_0(F(W_{i-1}, X_i)) \to \pi_0(F(X_{i-2}, X_i))$. This implies that $\alpha_i\gamma_{i-1}$ must be detected in higher filtration, i.e. in one of the groups $E_\infty^{s,s}(W_{i-1}, X_i)$ for $s > 0$. By the following result these groups are trivial, so the factorization exists and the induction step is complete.

Proposition 5.12. *Let $i \in \{2, 3, 4\}$. If $Z = X_i$ then in the spectral sequences (*5.6*) we have $E_2^{s,s}(W_{i-1}, X_i) = 0$ for all $s > 0$.*

Proof. If $s \ge i$ then we already have $E_1^{s,s}(W_{i-1}, X_i) = 0$. The d_1-differentials are induced by the maps α_i and hence we need to show that for $1 \le s < i - 1$ the sequences

$$\pi_s F(X_{i-1-s}, X_i) \xrightarrow{\alpha_{i-1-s}^*} \pi_s F(X_{i-2-s}, X_i) \xrightarrow{\alpha_{i-2-s}^*} \pi_s F(X_{i-3-s}, X_i) \tag{5.7}$$

are exact in the middle and that for $s = i - 1$

$$\alpha_0^* : \pi_s F(X_0, X_i) \to \pi_s F(X_{-1}, X_i) \tag{5.8}$$

is onto. The following two lemmas imply Proposition 5.12. □

Lemma 5.13. *$E_2^{3,t}(W_3, X_4) = 0$ for every t.*

Proof. In this case Proposition 5.7 shows that

$$F(X_{-1}, X_4) \simeq E_2^{hG_{12}(\Gamma)}\ ,\ \ F(X_0, X_4) \simeq (E_2[[\mathbb{G}_2/G_{48}(\Gamma)]] \vee E_2[[\mathbb{G}_2/G'_{48}(\Gamma)]])^{hG_{12}(\Gamma)}$$

and the map $F(X_0, X_4) \to F(X_{-1}, X_4)$ corresponds to the map

$$(E_2[[\mathbb{G}_2/G_{48}(\Gamma)]] \vee E_2[[\mathbb{G}_2/G'_{48}(\Gamma)]])^{hG_{12}(\Gamma)} \to E_2[[\mathbb{G}_2/\mathbb{G}_2]]^{hG_{12}(\Gamma)}$$

induced by the unique map $\mathbb{G}_2/G_{48}(\Gamma)\coprod \mathbb{G}_2/G'_{48}(\Gamma) \to \mathbb{G}_2/\mathbb{G}_2$. The latter map has a $G_{12}(\Gamma)$-equivariant section and this implies that the map

$$\pi_*((E_2[[\mathbb{G}_2/G_{48}(\Gamma)]] \vee E_2[[\mathbb{G}_2/G'_{48}(\Gamma)]])^{hG_{12}(\Gamma)}) \to \pi_*(E_2[[\mathbb{G}_2/\mathbb{G}_2]]^{hG_{12}(\Gamma)})$$

is split surjective and hence

$$\alpha_0^* : \pi_t F(X_0, X_4) \to \pi_t F(X_{-1}, X_4)$$

is surjective for every t. □

LEMMA 5.14. *Let $i \in \{2,3,4\}$, $t \in \{0,1,2\}$ and $s > 0$. Then*

$$E_2^{s,t}(W_{i-1}, X_i) = 0 \ .$$

PROOF. The spectrum X_i is a wedge of homotopy fixed point spectra $E_2^{hF_{i,j}}$ for certain explicitly given finite subgroups $F_{i,j}$, $j = 1,\dots,m_i$,

$$X_i = \bigvee_{j=1}^{m_i} E_2^{hF_{i,j}} \ .$$

Then, for any spectrum Y we have a homotopy equivalence natural in Y

$$F(Y, X_i) \simeq \bigvee_{j=1}^{m_i} F(Y, E_2^{hF_{i,j}}) \simeq \bigvee_{j=1}^{m_i} F(Y, E_2)^{hF_{i,j}} \ . \tag{5.9}$$

If $Y = X_{i-2-s}$ then Y is again a finite wedge of homotopy fixed point spectra $E_2^{hG_{s,j}}$ with explicitly given closed subgroups $G_{s,j}$, $j = 1,\dots,n_s$. For each of these wedge summands Proposition 5.7 and Proposition 5.9 (with K_2 the trivial group) give an isomorphism

$$\pi_* F(E_2^{hG_{s,j}}, E_2) \cong \mathrm{Hom}_{\mathcal{E}\mathcal{G}_2}((E_2)_* E_2^{hG_{s,j}}, (E_2)_* E_2)$$

and because $(E_2)_*E_2 \cong \mathrm{Hom}_{cts}(\mathbb{G}_2, (E_2)_*)$ is coinduced this simplifies by Lemma 5.6 to an isomorphism

$$\pi_* F(E_2^{hG_{s,j}}, E_2) \cong \mathrm{Hom}_{(E_2)_*}((E_2)_* E_2^{hG_{s,j}}, (E_2)_*) \ .$$

These isomorphisms combine to give an isomorphism

$$\pi_* F(X_{i-2-s}, E_2) \cong \mathrm{Hom}_{(E_2)_*}((E_2)_* X_{i-2-s}, (E_2)_*) \ . \tag{5.10}$$

Again by Lemma 5.6 this isomorphism is compatible with the action of $\mathbb{G}_2$ which acts on the left hand side via its action on E_2 and on the right hand side diagonally.

From (5.9) and (5.10) we get a (direct sum of) descent spectral sequence(s) converging to $\pi_{q-p}F(X_{i-2-s}, X_i)$ with E_2-term given by

(5.11)

$$E_2^{p,q}(s,i) := \bigoplus_{j=1}^{m_i} H^p\big(F_{i,j}, \mathrm{Hom}_{(E_2)_*}((E_2)_*\Sigma^q X_{i-2-s}, (E_2)_*)\big) \Rightarrow \pi_{p-q}F(X_{i-2-s}, X_i).$$

From the isomorphism of complexes $(E_2)_*(X_\bullet) \cong \mathrm{Hom}_{\mathbb{W}[\mathrm{Gal}]-cts}(P_\bullet, (E_2)_*)$ and the fact that the appropriately truncated complex

$$0 \to N_i \to P_i \to \ldots \to P_0 \to \mathbb{W} \to 0$$

is $\mathbb{W}[F]$-split[2] for every finite subgroup of $\mathbb{G}_2$ we deduce that the complex

$$0 \to (E_2)_* X_{-1} \to (E_2)_* X_0 \to \ldots \to (E_2)_* X_{i-1} \to (E_2)_* W_i \to 0$$

is $(E_2)_*[F]$-split.

Now let $i \in \{2,3,4\}$, $0 \le s \le i-1$ and $t \in \{0,1,2\}$. By Lemma 5.15 below there are isomorphisms

$$E_1^{s,t}(W_{i-1}, X_i) = \pi_t F(X_{i-2-s}, X_i) \cong E_\infty^{t,2t}(s,i) \cong E_2^{t,2t}(s,i)$$

and the differential $d_1^{s,t} : E_1^{s,t}(W_{i-1}, X_i) \to E_1^{s+1,t}(W_{i-1}, X_i)$ is induced by the map $\alpha_{i-2-s} : X_{i-3-s} \to X_{i-2-s}$ (with $X_{-2} = *$). Because the complex

$$0 \to (E_2)_* X_{-1} \to (E_2)_* X_0 \to \ldots \to (E_2)_* X_{i-2} \to (E_2)_* W_{i-1} \to 0$$

is $\pi_* E_2[F_{i,j}]$-split we deduce that for $t \in \{0,1,2\}$

$$E_2^{s,t}(W_{i-1}, X_i) \cong \begin{cases} \bigoplus_{j=1}^{m_i} H^t\big(F_{i,j}, \mathrm{Hom}_{(E_2)_*}((E_2)_*\Sigma^{2t} W_{i-1}, (E_2)_*)\big) & s = 0 \\ 0 & s > 0 \end{cases}$$

as claimed. □

LEMMA 5.15. *Let $i \in \{2,3,4\}$, s and t be integers, $0 \le s < i$ and $t \in \{0,1,2\}$. In the spectral sequences (*5.11*) we have*

$$E_2^{t,2t}(s,i) \cong E_\infty^{t,2t}(s,i) \cong \pi_t F(X_{i-2-s}, X_i)\ .$$

PROOF. If K_1 is a closed subgroup and K_2 is a finite subgroup of $\mathbb{G}_2$ then Proposition 5.7 gives an equivalence

$$F(E_2^{hK_1}, E_2^{hK_2}) \simeq E_2[[\mathbb{G}_2/K_1]]^{hK_2}\ ,$$

in particular an isomorphism

$$\pi_* F(E_2^{hK_1}, E_2) \cong \pi_* E_2[[\mathbb{G}_2/K_1]])\ .$$

The spectrum X_{i-2-s} is a wedge of homotopy fixed point spectra with respect to explicitly known closed subgroups $G_{s,j}$, $j = 1, \ldots, n_s$ and X_i is also a wedge of homotopy fixed point spectra with respect to explicitly known closed subgroups $F_{i,j}$, $j = 1, \ldots, n_i$. Therefore the E_2-term of the spectral sequence (5.11) can be rewritten as

$$E_2^{p,q}(s,i) = \bigoplus_{j=1}^{m_i} \bigoplus_{k=1}^{n_s} H^p(F_{i,j}, \pi_q E_2[[\mathbb{G}_2/G_{s,k}]]) \Longrightarrow \pi_{q-p} F(X_{i-2-s}, X_i)$$

This spectral sequence is the direct sum of spectral sequences indexed by j and k.

If K_1 is an open subgroup of $\mathbb{G}_2$ and K_2 is a finite subgroup of $\mathbb{G}_2$ then by part b) of Proposition 2.6 of [**15**] the function spectrum $F(E_2^{hK_1}, E_2^{hK_2})$ is identified with $E_2[[\mathbb{G}_2/K_1]]^{hK_2}$ and this is a finite product of homotopy fixed point spectra of the form E_2^{hF} where F is always a subgroup of K_2. In our case K_2 is one of the groups $C_2 \times \mathrm{Gal}$, C_8 or $G_{12}(\Gamma)$ and the homotopy groups π_t for $t = 0,1,2$ of the homotopy fixed point spectra E_2^{hF} are always given by $H^{t,2t}(F, (E_2)_*)$. In fact, as in the proof of Proposition 5.11 one sees that because of Lemma 5.10 it is enough to consider the cases that F is either C_2, C_4 or C_6 and then the necessary information is provided by [**28**], [**7**] and [**8**].

[2] We use this opportunity to point out an annoying typo on line 2 of page 164 of [**21**]. Instead of "$\mathbb{Z}_p$-split" it should have read "$\mathbb{Z}_p[F]$-split".

Therefore for $t \in \{0,1,2\}$, K_1 open and $K_2 \in \{C_2 \times \mathrm{Gal}, C_8, G_{12}(\Gamma)\}$ we get isomorphisms

$$\pi_t(F(E_2^{hK_1}, E_2^{hK_2})) \cong H^{t,2t}(K_2; \pi_*(E_2[[\mathbb{G}_2/K_1]])) \ . \tag{5.12}$$

A general closed subgroup K_1 can be written as an intersection of a decreasing sequence of open subgroups U_i and then

$$F(E_2^{hK_1}, E_2^{hK_2}) \simeq \mathrm{holim}_i F(E_2^{hU_i}, E_2^{hK_2})$$

and Remark 5.8 show that for $t = 0,1,2$ the sources of the isomorphisms of (5.12) are given as the obvious inverse limit. For the target one uses that

$$\pi_*(E_2[[\mathbb{G}_2/K_1]]) = \lim_i \pi_*(E_2[[\mathbb{G}_2/U_i]])$$

if $K_1 = \bigcap_i U_i$ for U_i a decreasing sequence of open subgroups of $\mathbb{G}_2$. Because the inverse limit is an exact functor on the category of profinite abelian groups we find that $H^*(F,-)$ commutes with inverse limits of profinite coefficient modules and thus the case of open subgroups implies the case of closed subgroups. □

References

[1] T. Bauer, *Computation of the homotopy of the spectrum* `tmf`, Groups, homotopy and configuration spaces, Geom. Topol. Monogr., vol. 13, Geom. Topol. Publ., Coventry, 2008, pp. 11–40, DOI 10.2140/gtm.2008.13.11. MR2508200

[2] A. Beaudry, *The algebraic duality resolution at $p = 2$*, Algebr. Geom. Topol. **15** (2015), no. 6, 3653–3705, DOI 10.2140/agt.2015.15.3653. MR3450774

[3] A. Beaudry, *Towards the homotopy of the $K(2)$-local Moore spectrum at $p = 2$*, Adv. Math. **306** (2017), 722–788, DOI 10.1016/j.aim.2016.10.020. MR3581316

[4] A. Beaudry, *The chromatic splitting conjecture at $n = p = 2$*, Geom. Topol. **21** (2017), no. 6, 3213–3230, DOI 10.2140/gt.2017.21.3213. MR3692966

[5] A. Beaudry, P. Goerss and H.-W.Henn, "Chromatic splitting for the $K(2)$-local sphere at $p = 2$", arXive 2017

[6] M. Behrens, *The homotopy groups of $S_{E(2)}$ at $p \geq 5$ revisited*, Adv. Math. **230** (2012), no. 2, 458–492, DOI 10.1016/j.aim.2012.02.023. MR2914955

[7] M. Behrens and K. Ormsby, *On the homotopy of $Q(3)$ and $Q(5)$ at the prime 2*, Algebr. Geom. Topol. **16** (2016), no. 5, 2459–2534, DOI 10.2140/agt.2016.16.2459. MR3572338

[8] I. Bobkova and P. Goerss, *Topological resolutions in $K(2)$-local homotopy theory at the prime 2*, J. Topol. **11** (2018), no. 4, 917–956, DOI 10.1112/topo.12076.

[9] K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York-Berlin, 1982. MR672956

[10] C. Bujard, "Finite subgroups of extended Morava stabilizer groups", Ph.D.thesis, Université de Strasbourg (2012). Available at arXiv:1206.1951v2.

[11] E. S. Devinatz and M. J. Hopkins, *Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups*, Topology **43** (2004), no. 1, 1–47, DOI 10.1016/S0040-9383(03)00029-6. MR2030586

[12] C. L. Douglas, J. Francis, A. G. Henriques, and M. A. Hill (eds.), *Topological modular forms*, Mathematical Surveys and Monographs, vol. 201, American Mathematical Society, Providence, RI, 2014. MR3223024

[13] A. Fröhlich, *Formal groups*, Lecture Notes in Mathematics, No. 74, Springer-Verlag, Berlin-New York, 1968. MR0242837

[14] P. G. Goerss, H.-W. Henn, and M. Mahowald, *The rational homotopy of the $K(2)$-local sphere and the chromatic splitting conjecture for the prime 3 and level 2*, Doc. Math. **19** (2014), 1271–1290. MR3312144

[15] P. Goerss, H.-W. Henn, M. Mahowald, and C. Rezk, *A resolution of the $K(2)$-local sphere at the prime 3*, Ann. of Math. (2) **162** (2005), no. 2, 777–822, DOI 10.4007/annals.2005.162.777. MR2183282

[16] P. Goerss, H.-W. Henn, M. Mahowald, and C. Rezk, *On Hopkins' Picard groups for the prime 3 and chromatic level 2*, J. Topol. **8** (2015), no. 1, 267–294, DOI 10.1112/jtopol/jtu024. MR3335255

[17] P. G. Goerss and H.-W. Henn, *The Brown-Comenetz dual of the $K(2)$-local sphere at the prime 3*, Adv. Math. **288** (2016), 648–678, DOI 10.1016/j.aim.2015.08.024. MR3436395

[18] P. G. Goerss and M. J. Hopkins, *Moduli spaces of commutative ring spectra*, Structured ring spectra, London Math. Soc. Lecture Note Ser., vol. 315, Cambridge Univ. Press, Cambridge, 2004, pp. 151–200, DOI 10.1017/CBO9780511529955.009. MR2125040

[19] M. Hazewinkel, *Formal groups and applications*, Pure and Applied Mathematics, vol. 78, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1978. MR506881

[20] H.-W. Henn, *Centralizers of elementary abelian p-subgroups and mod-p cohomology of profinite groups*, Duke Math. J. **91** (1998), no. 3, 561–585, DOI 10.1215/S0012-7094-98-09121-9. MR1604171

[21] H.-W. Henn, *On finite resolutions of $K(n)$-local spheres*, Elliptic cohomology, London Math. Soc. Lecture Note Ser., vol. 342, Cambridge Univ. Press, Cambridge, 2007, pp. 122–169, DOI 10.1017/CBO9780511721489.008. MR2330511

[22] H.-W. Henn, *A mini-course on Morava stabilizer groups and their cohomology*, Algebraic topology, Lecture Notes in Math., vol. 2194, Springer, Cham, 2017, pp. 149–178. MR3790894

[23] T. Hewett, *Finite subgroups of division algebras over local fields*, J. Algebra **173** (1995), no. 3, 518–548, DOI 10.1006/jabr.1995.1101. MR1327867

[24] T. Hewett, *Normalizers of finite subgroups of division algebras over local fields*, Math. Res. Lett. **6** (1999), no. 3-4, 271–286, DOI 10.4310/MRL.1999.v6.n3.a2. MR1713129

[25] N. Karamanov, *On Hopkins' Picard group* Pic_2 *at the prime 3*, Algebr. Geom. Topol. **10** (2010), no. 1, 275–292, DOI 10.2140/agt.2010.10.275. MR2602836

[26] M. Lazard, *Groupes analytiques p-adiques* (French), Inst. Hautes Études Sci. Publ. Math. **26** (1965), 389–603. MR0209286

[27] O. Lader, "Une résolution projective pour le second groupe de Morava pour $p \geq 5$ et applications", Ph.D.thesis, Université de Strasbourg (2013). Available at hal.archives-ouvertes.fr/tel-00875761

[28] M. Mahowald and C. Rezk, *Topological modular forms of level 3*, Pure Appl. Math. Q. **5** (2009), no. 2, Special Issue: In honor of Friedrich Hirzebruch., 853–872, DOI 10.4310/PAMQ.2009.v5.n2.a9. MR2508904

[29] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of number fields*, 2nd ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 323, Springer-Verlag, Berlin, 2008. MR2392026

[30] D. Quillen, *The spectrum of an equivariant cohomology ring. I, II*, Ann. of Math. (2) **94** (1971), 549–572; ibid. (2) 94 (1971), 573–602, DOI 10.2307/1970770. MR0298694

[31] D. C. Ravenel, *The cohomology of the Morava stabilizer algebras*, Math. Z. **152** (1977), no. 3, 287–297, DOI 10.1007/BF01488970. MR0431168

[32] C. Rezk, *Notes on the Hopkins-Miller theorem*, Homotopy theory via algebraic geometry and group representations (Evanston, IL, 1997), Contemp. Math., vol. 220, Amer. Math. Soc., Providence, RI, 1998, pp. 313–366, DOI 10.1090/conm/220/03107. MR1642902

[33] K. Shimomura and A. Yabe, *The homotopy groups* $\pi_*(L_2S^0)$, Topology **34** (1995), no. 2, 261–289, DOI 10.1016/0040-9383(94)00032-G. MR1318877

[34] J. H. Silverman, *The arithmetic of elliptic curves*, Graduate Texts in Mathematics, vol. 106, Springer-Verlag, New York, 1986. MR817210

[35] N. P. Strickland, *Gross-Hopkins duality*, Topology **39** (2000), no. 5, 1021–1033, DOI 10.1016/S0040-9383(99)00049-X. MR1763961

Institut de Recherche Mathématique Avancée, C.N.R.S. - Université de Strasbourg, F-67084 Strasbourg, France