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Stable homotopy theories and stabilization

Alex Heller

Graduate School, City University of New York, New York, NY 10036 USA

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Abstract

Stable homotopy theories, i.e. pointed theories for which the suspension is an equivalence, are shown to form a reflective sub-2-category. Thus the stabilization $T \rightarrow StabT$ is characterized by a universal property. This permits a perspicuous proof of the existence of the coherent symmetric smash product in the standard stable homotopy theory. It is to be noted that spectra appear only in the proofs, not the statements of theorems.

0. Introduction

Among the principal reasons for the introduction, in [3], of the notion of an abstract homotopy theory, was the hope of subsuming stable homotopy theory under this rubric and thus bypassing the often rebarbative discussions of stable homotopy with operators and the really monstrous treatments of the so-called stable smash product which make the subject so unattractive. This paper effects the subsumption in question.

A "homotopy theory" – the definition is detailed in Section 6 – is designed to incorporate within a single structure all the homotopy categories of diagram-categories of spaces (in the classical example) as well as the changes of index-category and homotopy Kan extensions between them. The homotopy-category of a diagram-category must be distinguished from its underlying diagram-category in the homotopy category of spaces, which is associated to it by a hyperfunctor *dgm*. Within a homotopy category one may carry out all the standard constructions at the homotopy level rather than at the level of spaces. For example, the suspension may be characterized as a homotopy pushout, which is to say a left homotopy Kan extension, yielding a diagram of shape

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$\mathbf{2}^2 = \mathbf{2} \times \mathbf{2}$ ($\mathbf{2}$ is the ordered set $\{0 \to 1\}$), viz.,

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma X \end{array}$$

The homotopy pullback, leading to the diagram

$$\begin{array}{ccc} \Omega Y & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Y, \end{array}$$

is then seen to be right adjoint to the suspension.

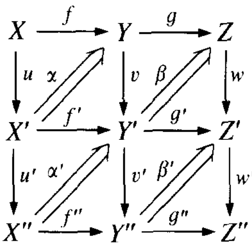
The principal result is the stabilization Theorem 8.1, which characterizes the stabilization of a suitable homotopy theory by means of universal properties, which may then be used to explicate the other constructions generally associated to stable homotopy theory. The price to be paid for this theorem is a stronger dose of the 2-category theory which was, so far as possible, minimised in [3]. Homotopy theories are now to be thought of as 0-cells (i.e. "objects") of a 2-category. In order to do this without excessive violence it seemed appropriate to reduce them in size – to make them set-like rather than class-like. This also suggested a change in some of the terminology of [3]. These procedural matters are discussed in Section 2 below, which is preceded by an aide-mémoire on 2-category theory in Section 1.

1. Aide-mémoire on 2-categories

The category *Cat* of small categories is complete and cocomplete. A *2-category* is a category enriched over *Cat*. Since *Cat* is cartesian closed it is itself the prototypical 2-category. If Γ is a 2-category the set Γ_0 of *0-cells* of Γ contains the objects of its underlying category and $\Gamma(X,Y), X, Y \in \Gamma_0$ is the category of morphisms. The class Γ_1 of objects of all $\Gamma(X,Y)$ coincides with that of the morphisms of the underlying category. These are the 1-cells of Γ. The morphisms of the several $\Gamma(X,Y)$ constitute the class Γ_2 of 2-cells. We shall from now on write *Cat* for the 2-category structure, so that Cat_1 is the class of functors and Cat_2 that of natural transformations.

A *2-functor* $F : \Gamma \to \Theta$ of 2-categories is an enriched functor of the underlying categories and thus consists of maps $\Gamma_i \to \Theta_i$, $i = 0, 1, 2$ preserving all sources, targets and compositions. We shall have no real need for "pseudofunctors" and "lax functors." It will be sufficient for our present purposes to observe that a "pseudofunctor" F differs from a 2-functor in that it need not precisely preserve composition, but is, rather, supplied with an isomorphic 2-cell connecting the two.

In the 2-diagram



(1.1)

the *horizontal composition* $\beta \circ_H \alpha$ is defined by the composition

$$g'f'u \xrightarrow{g'\alpha} g'vf \xrightarrow{\beta f} wgf$$

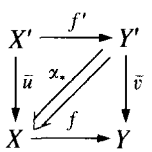
and the *vertical composition* $\alpha' \circ_V \alpha$ is

$$f''u'u \xrightarrow{\alpha' u} v'f'u \xrightarrow{v'\alpha} v'vf$$

Then

$$(\beta' \circ_H \alpha') \circ_V (\beta \circ_H \alpha) = (\beta' \circ_V \beta) \circ_H (\alpha' \circ_V \alpha) \tag{1.2}$$

Suppose, in (1.1), that $\eta_v, \varepsilon_v : \bar{v} \dashv v$, $\eta_u, \varepsilon_u : \bar{u} \dashv u$. The *left transpose* (sometimes called the "mate")



of α, relative to these adjunctions, is defined by the composition

$$\bar{v}f' \xrightarrow{\bar{v}f\eta_v} \bar{v}f'u\bar{u} \xrightarrow{\bar{v}\alpha\bar{u}} \bar{v}vf\bar{u} \xrightarrow{\varepsilon_v f\bar{u}} f\bar{u}$$

If instead we are given adjunctions $f \dashv \bar{f}, f' \dashv \bar{f}'$ the *right transpose* α^* is defined dually.

Let us suppose, in addition to the data just listed, that we are, referring still to (1.1), supplied with an adjunction $\bar{w} \dashv w$.

Proposition 1.3. *The transposes of* α, β *satisfy the following conditions*:

1. $(\beta \circ_H \alpha)_* = \alpha^* \circ_V \beta^*$
2. *If* α *is an isomorphism then* $(\alpha^{-1})_* = (\alpha^*)^{-1}$; α^* *is an isomorphism if and only if* α_* *is.*

In the last case we say that α satisfies the *Beck–Chevalley condition*.

The notion of a sub-2-category Υ of a 2-category Γ is defined in the obvious way. It is *full* if for $X, Y \in \Upsilon$, $\Upsilon(X,Y) = \Gamma(X,Y)$ and *locally full* if each $\Upsilon(X,Y)$ is a full subcategory of $\Gamma(X,Y)$.

In 2-categories there are several varieties of dualization:

$$\Gamma_0^{\mathrm{op}} = \Gamma_0, \Gamma^{\mathrm{op}}(X,Y) = \Gamma(Y,X),$$

$$\Gamma_0^{\mathrm{co}} = \Gamma_0, \Gamma^{\mathrm{co}}(X,Y) = \Gamma(X,Y)^{\mathrm{op}},$$

$$\Gamma^{\mathrm{co-op}} = (\Gamma^{\mathrm{co}})^{\mathrm{op}} = (\Gamma^{\mathrm{op}})^{\mathrm{co}}.$$

If $\Upsilon \subset \Gamma$ is a full sub-2-category a *reflection* of $X \in \Gamma_0$ into Υ is a 1-cell $X \to \bar{X}$ with $\bar{X} \in \Upsilon$ such that for any $W \in \Upsilon$,

$$\Gamma(\bar{X}, W) \to \Gamma(X, W) \tag{1.4}$$

is an equivalence of categories. Υ is *reflective* if each X in Γ has a reflection. We might have defined a *strict* reflection by demanding that (1.4) be an isomorphism; we shall not need this notion.

We may note parenthetically that, given a reflective sub-2-category, a family of reflections into it defines a pseudo-2-functor.

2. Hypercategories

In [3] the notion of a *homotopy theory* was defined as, in effect, a 2-functor with domain Cat^{op} having as values categories, functors and natural transformations, and thus characterized as a "hyperfunctor" and satisfying a number of conditions characterizing, *inter alia*, the behavior of homotopy Kan extensions and thus asserting homotopy completeness.

This definition precludes the construction of a category of homotopy theories. It seems, however, that in order to properly understand the character of stable homotopy theory and stabilization we must in fact have such a category. We may obtain it by being less stringent in our requirement of homotopy completeness, so that the values of our homotopy theories may be taken to be themselves small categories.

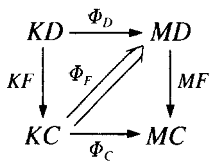
Our restricted notion of completeness will be specified by choosing a small full sub-2-category $\mathrm{CAT} \subset Cat$ as the *regime of completeness* over which homotopy limits and, more generally, homotopy Kan extensions will be required to exist. Thus, for example, we might take CAT to contain representatives of all finite or countable categories, including, without violence, all those categories of that character which we may happen to mention. It will be sufficient for our purposes in this paper to restrict our attention to one such 2-category. However, most of what we shall have to say would be equally valid for other choices of CAT.

Our homotopy theories will then be 2-functors $\mathrm{CAT}^{\mathrm{op}} \to Cat$. We shall, in several ways, make these the 0-cells of 2-categories. At this point it seems desirable to break with the terminology of [3] and call such 2-functors *hypercategories* or, to emphasize the dependence on CAT, CAT-*hypercategories*. The 1-cells will now be styled "hyperfunctors" and the 2-cells "hypernatural transformations".

Hyperfunctors are of several sorts. By a *left weak hyperfunctor*

$$\Phi : K \underset{L}{\longrightarrow} M,$$

where K and M are hypercategories we mean a family of functors $\Phi_C : KC \to MC$, $C \in \mathrm{CAT}_0$, together with a family of natural transformations



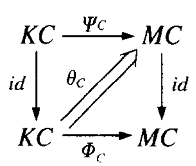
for $F : C \to D$ in CAT, such that if also $G : D \to E$ then

$$\Phi(GF) = (\Phi F) \circ_V (\Phi G) \tag{2.1}$$

If also $\Psi : M \to P$ then $(\Psi\Phi)_C = \Psi_C \Phi_C, (\Psi\Phi)_F = \Psi_F \circ_H \Phi_F$ defines a composition making these left weak hyperfunctors the morphisms of a category. If

$$\Phi, \Psi : K \underset{L}{\longrightarrow} M$$

a *hypernatural transformation* $\theta : \Phi \to \Psi$ is a family of natural transformations $\theta_C : \Phi_C \to \Psi_C$ such that, identifying θ_C with the 2-cell



$\Phi_F \circ_V \theta_D = \theta_C \circ_V \Psi_F$. These then constitute the 2-cells of a 2-category of left weak hyperfunctors. The 2-category of right weak hyperfunctors is defined dually.

A left weak hyperfunctor Φ for which all Φ_F are isomorphisms is a *left strong hyperfunctor*. These evidently form a locally full sub-2-category of the left weak hyperfunctors. This sub-2-category is isomorphic to the 2-category of right strong hyperfunctors, the isomorphism being given by inverting the Φ_F. We shall take the risk of identifying them and write $\mathscr{H}$ for the 2-category of *strong hyperfunctors*.

If $\Phi : K \to M$ is a strong hyperfunctor such that for all F, $(MF)\Phi_D = \Phi_C(KF)$, $\Phi_F = id$ then Φ is a *strict hyperfunctor*. The strict hyperfunctors form a locally full sub-2-category of the strong hyperfunctors.

There is a 2-functor $Cat \to \mathscr{H}$ given by $\mathbf{C} \to \mathbf{C}^{(-)}$, where $\mathbf{C}^{(-)}$ is given by $\mathbf{C}^{(-)}C = \mathbf{C}^C$ for $C \in \mathrm{CAT}_0$. The hypercategory $\mathbf{C}^{(-)}$ is the *representable hypercategory* associated with **C**. There are also two 2-functors $\text{CAT}^{\text{op}} \times \mathscr{H} \to \mathscr{H}$, viz.

$$(C,K) \mapsto K[C], \quad \text{with } K[C]D = K(C \times D),$$
$$(C,K) \mapsto K^C, \quad \text{with } K^C D = (KD)^C.$$

These are in fact cotensorings of $\mathscr{H}$ over CAT. The latter, indeed, extends to $Cat^{\text{op}} \times \mathscr{H}$.

A strict hyperfunctor with values $dgm_C : K[C] \to K^C$ is given by setting

$$(dgm_C X)_\gamma = K(\gamma \times D)X \quad \text{for } \gamma : \mathbf{1} \to C \text{ an object of } C$$

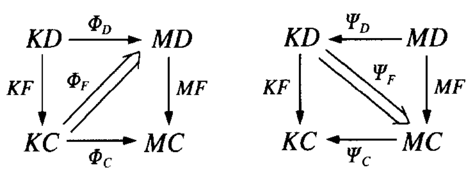
and

$$(dgm_C X)_\theta = K(\theta \times D)X : K(\gamma \times D)X \to K(\gamma' \times D)X$$

for $\theta : \gamma \to \gamma'$, θ being regarded as a natural transformation.

3. Adjoint hyperfunctors

If K, M are hypercategories and $\Phi : K \to M$, $\Psi : M \to K$ are, respectively, left weak and right weak hyperfunctors an *adjunction* $(\eta, \varepsilon) : \Phi \dashv \Psi$ consists of a family of adjunctions $(\eta_C, \varepsilon_C) : \Phi_C \dashv \Psi_C$ such that for each $F : C \to D$ in CAT the squares



are transposes of one another, i.e. the equivalent equations

$$\Phi_F = (\Psi_F)^*, \qquad (\Phi_F)_* = \Psi_F \tag{3.1}$$

hold.

If $(\eta, \varepsilon) : \Phi \dashv \Psi, (\eta', \varepsilon') : \Phi \dashv \Psi'$ then the family

$$\Psi_C \xrightarrow{\eta'_C \Psi_C} \Psi'_C \Phi_C \Psi_C \xrightarrow{\Psi'_C \varepsilon_C} \Psi'_C$$

is a hypernatural equivalence $\Psi \to \Psi'$. Similarly, $\Phi' \dashv \Psi$ gives $\Phi \approx \Phi'$. (But note that Φ and Ψ, which are in different 2-categories, cannot be composed.)

If $\Phi \dashv \Psi$ either one, or both, may be strong. Because of the essential uniqueness of the adjunction which we have just observed the strongness of either adjoint is a property of the other. We may say, for example, that Φ *has a strong right adjoint* to mean that it has a right adjoint and that such an adjoint is strong.

Because of the functoriality of the transpose, (3.1) implies that the existence of an adjoint to, say, Φ is equivalent to the existence of adjoints to all Φ_C; the adjoint itself is determined by the several adjunctions at the $C \in \text{CAT}_0$.

An important special case is that in which $\Phi : K \to M$ is an *equivalence of hypercategories*, by which we mean that it is strong and that each Φ_C is an equivalence of categories. Then each Φ_C has an equivalence up to isomorphism Ψ_C and the isomorphisms $id \approx \Phi_C \Psi_C, id \approx \Psi_C \Phi_C$ become the unit and counit of an adjunction $\Phi \dashv \Psi$, with Ψ being strong as well.

Proposition 3.2. *If $\Phi : M \to K$ is an equivalence of hypercategories then, for any hypercategory W, $\mathscr{H}(W,M) \to \mathscr{H}(W,K)$ is an equivalence of categories.*

4. Bilinearity; cartesian closure of $\mathscr{H}$

The 2-category $\mathscr{H}$ is essentially algebraic and is thus provided with limits and colimits, e.g. $(\prod_i K_i)C = \prod_i (K_i C)$, the projections and diagonals being strict hyperfunctors.

We have furthermore an alternative description of the categories $\mathscr{H}(K \times M, W)$. Let us define a *pairing* of K, M to W as a family of functors $\Gamma_{C,D} : KC \times MD \to W(C \times D)$, together with natural isomorphisms

$$\begin{array}{ccc} KC' \times MD' & \longrightarrow & W(C' \times D') \\ \big\downarrow & \overset{\Gamma_{F,G}}{\nearrow\!\!\!\!\nearrow} & \big\downarrow \\ KC \times MD & \longrightarrow & W(C \times D) \end{array}$$

for $F : C \to C', G : D \to D'$ in CAT, satisfying the obvious analogue of (2.1). Similarly we define *hypernatural transformations* $\Gamma \to \Gamma'$ between pairings in analogy with those between hyperfunctors, thus constructing a category $\mathscr{B}(K,M;W)$ of pairings. If Γ is such a pairing and $X \in KC$ or $Y \in MC$ then $\Gamma(X,-) : M \to W[C]$, $\Gamma(-,Y) : K \to W[C]$ are strong hyperfunctors.

Lemma 4.1. *$\Upsilon : \mathscr{B}(K,M;W) \to \mathscr{H}(K \times M, W)$, with $(\Upsilon\Gamma)_C$ the composition*

$$KC \times MC \xrightarrow{\Gamma_{C,C}} W(C \times C) \xrightarrow{W\Delta} W_C$$

is an isomorphism.

Its inverse is given by the composition of

$$Kp_0 \times Mp_1 : KC \times MD \to K(C \times D) \times M(C \times D)$$

and

$$\Phi_{C \times D} : K(C \times D) \times M(C \times D) \to W(C \times D).$$

Having made this observation we draw the following conclusion.

Theorem 4.2. *$\mathscr{H}$ is cartesian closed.*

That is to say, it is provided with an "internal hom" 2-functor $M, W \mapsto \mathscr{H}^\sharp(M,W)$ with $- \times M \dashv \mathscr{H}^\sharp(M,-)$. By Lemma 4.1 it will be sufficient to see that

$$\mathscr{B}(K,M;W) \approx \mathscr{H}(K, \mathscr{H}^\sharp(M,W)) \tag{4.3}$$

subject, of course, to the pertinent naturality, which will be evident when we set $\mathscr{H}^\sharp(M,W)_C = \mathscr{H}(M,W[C])$. The isomorphism (4.3) is then computed as $\Gamma \leftrightarrow \Phi$ where, for $X \in KC, Y \in MD, (\Phi_C X)Y = \Gamma_{C,D}(X,Y) \in W(C \times D) = W[C]D$.

We may without further ado generalize these observations to the "multilinear" case. If $K_1,\dots,K_n, W \in \mathscr{H}$ then $\mathscr{B}_n(K_1,\dots,K_n;W)$, made up of functors

$$K_1C_1 \times \cdots K_nC_n \to W(C_1,\dots C_n)$$

provided with suitable commutativity isomorphisms for families $\{F_i : C_i \to C_i'\}$ is defined in analogy with $\mathscr{B}$ it satisfies

$$\begin{aligned}\mathscr{B}_n(K_1,\dots,K_n;W) &\approx \mathscr{B}_{n-1}(K_1,\dots,K_{n-1};\mathscr{H}^\sharp(K_n,W))\\ &\approx \mathscr{H}(K_1,\mathscr{H}^\sharp(K_2,\dots,\mathscr{H}^\sharp(K_n,W)\cdots)\end{aligned} \tag{4.4}$$

5. Completeness and continuity

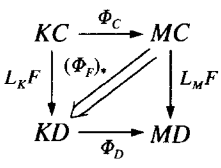
We may now define the notions of completeness and continuity for a hypercategory $K:\text{CAT}^{\text{op}} \to Cat$. K is *(co)complete* if for each $F:C \to D$ in CAT, KF has a right (left) adjoint. We shall adopt the informal notation of [3] and write

$$L_KF \dashv KF \dashv R_KF$$

for such adjoints, with the caution that L_KF, R_KF are not unique and L_K, R_K are not, strictly speaking, functorial. When the identity of K is not in question, we may allow ourselves to elide the subscripts.

The functors L_KF, R_KF are *generalized Kan extensions* along F. When $K = \mathbf{C}^{(-)}$ is representable they are in fact the ordinary Kan extensions $L_KF = Lan_F$, $R_KF = Ran_F$. Thus for such K (co)completeness of K is equivalent to (co)completeness of $\mathbf{C}$ with respect to diagrams indexed by $C \in \text{CAT}_0$

If K, M are complete hypercategories a left hyperfunctor $\Phi:K \to M$ is *cocontinuous* if it preserves all generalized left Kan extensions: that is to say, that for any $F:C \to D$ in CAT and any choice of adjunctions $(\eta_K,\varepsilon_K): L_KF \dashv KF, (\eta_M,\varepsilon_M): L_MF \dashv MF$ the transpose



of Φ_F is an isomorphism. Dually, $\Psi : K \to M$ is *continuous* if all $(\Phi_F)^*$ are isomorphisms. Evidently, (co)continuous hyperfunctors are closed under composition. Thus, the complete hypercategories and (co)continuous hyperfunctors constitute locally full sub-2-categories of, respectively, the 2-categories of left and right weak hyperfunctors. We shall write $\mathscr{H}^{\mathrm{cc}}$ for the locally full sub-2-category of these containing the complete hypercategories and cocontinuous strong hyperfunctors.

The fate of the familiar observation that left adjoint functors preserve colimits is worthy of mention. From (1.3) we deduce easily the following statement.

Proposition 5.1. *Let K, M be complete hypercategories and suppose that*

$$K \xrightarrow[L]{\Phi} M$$

has a right adjoint

$$M \xrightarrow[R]{\Psi} K$$

Then Φ is cocontinuous if and only if Ψ is strong.

In particular, a strong hyperfunctor between complete hypercategories which possesses a right adjoint is cocontinuous if and only if the adjoint is also strong.

We have, of course, suppressed for the sake of economy the mention of the dual assertions.

Let us illustrate this point. Suppose K is a complete hypercategory and that $F : C \to D$. Then the strong hyperfunctor $K[F] : K[D] \to K[C]$ has a left adjoint $L_K[F]$. There is *a priori* no reason why $L_K[F]$ should be strong, and we cannot therefore conclude that $K[F]$ is continuous.

Unlike $\mathscr{H}$, $\mathscr{H}^{\mathrm{cc}}$ is not cartesian closed. It does, however, have the structure of a closed 2-category. For $C \in \mathrm{CAT}_0$ set $\mathscr{H}^{\mathrm{cc}\#}(K,M)C = \mathscr{H}^{\mathrm{cc}}(K, M[C])$. We leave open the question of whether there is a tensor product left adjoint to this, but observe that there is a pairing with the apposite adjointness. If K, M, W are complete let $\mathscr{B}^{\mathrm{cc}}(K,M;W) \subset \mathscr{B}(K,M;W)$ be the full subcategory containing those pairings Γ which are *bicocontinuous*, i.e. such that for each X in K and each Y in M the hyperfunctors $\Gamma(X,-)$, $\Gamma(-,Y)$ are cocontinuous. We then see that, in analogy with (4.4).

Proposition 5.2. *The "internal hom" $\mathscr{H}^{\mathrm{cc}\#}$ has the properties*

$$\mathscr{B}^{\mathrm{cc}}(K,M;W) \approx \mathscr{H}^{\mathrm{cc}}(K, \mathscr{H}^{\mathrm{cc}\#}(M,W)) \approx \mathscr{H}^{\mathrm{cc}}(M, \mathscr{H}^{\mathrm{cc}\#}(K,W)).$$

This generalizes at once to the multicocontinuous case. With

$$\mathscr{B}_n^{\mathrm{cc}}(K_1, \ldots, K_n; W)$$

defined in the obvious way we have

$$\mathscr{B}_n^{cc}(K_1,\ldots,K_n;W) \approx \mathscr{B}_{n-1}^{cc}(K_1,\ldots,K_{n-1};\mathscr{H}^{cc\sharp}(K_n,W))$$
$$\approx \mathscr{H}^{cc}(K_1,\mathscr{H}^{cc\sharp}(K_2,\ldots,\mathscr{H}^{cc\sharp}(K_n,W)\cdots). \tag{5.3}$$

6. Small homotopy theories

We fix, as above, a regime of continuity $\text{CAT} \subset Cat$. A hypercategory T in $\mathscr{H}$ is a *small homotopy theory* or CAT-*homotopy theory* – we shall say, for brevity, merely a *homotopy theory* – if it satisfies the following conditions.

H0: For any countable family $\{C_i\}$ of categories in CAT,

$$T\left(\sum_i C_i\right) \to \prod_i (TC_i)$$

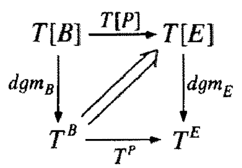
is an equivalence of categories.

H1: For each C in CAT, $dgm_C : TC \to (T\mathbf{1})^C$ reflects isomorphisms.

H2: If F is a finite free category then $dgm_{[F]} : T[F] \to T^F$ is a *weak quotient hyperfunctor*, i.e. each $dgm_{[F],D}$ is full and essentially surjective on objects.

H3: T is complete.

H4: If $P : E \to B$ is a discrete fibration in CAT then the identity 2-cell in



satisfies the Beck–Chevalley condition, i.e. id^* is an isomorphism. If, instead, P is a discrete opfibration then the dual statement is true.

Except for the restriction to categories in CAT these are just the axioms of [3, 4]. The properties of a homotopy theory there adduced, with suitable restrictions on the "argument-categories," thus obtain here as well and will thus be used here without further comment.

"Classical" homotopy theory, as constructed in [3], is of course not small. We may deal with this by substituting for the category of simplicial sets the small subcategory of those with values in some fixed set of uncountable cardinality. The resulting small homotopy theory then depends (functorially) on that set. It should be clear that in this context we can ignore the set and refer with only harmless ambiguity to "the" classical homotopy theory Π.

As in [3] we observe that if T is a (small) homotopy theory then, for any $C \in \text{CAT}$, so also is $T[C]$. A little care is necessary in interpreting the observation that any homotopy theory is tensored and cotensored over Π; for a small homotopy theory, such tensors and cotensors are guaranteed *a priori* only for reasonably small, e.g. countable, objects in Π.

We recall from [3] that a homotopy theory is *regular* if sequential homotopy colimits commute with finite products and homotopy pullbacks. (More precisely, the appropriate transposes of certain identity 2-cells are isomorphisms [3, IV, Section 5]) . This is an analogue of Grothendieck's axiom AB5; indeed for a hypercategory represented by an abelian category it is just a special case of that axiom. Standard homotopy theory is regular.

7. Localization

If T is a homotopy theory a localization of T is defined in [3] in the following way. If X is an object of TC and Y an object of $T[\mathbf{2}]C$ then $Y \perp X$ if $TC(Y_1,X) \to TC(Y_0,X)$ is bijective. This relation is used in the usual way to define a Galois correspondence between subhypercategories of T and $T[\mathbf{2}]$. That is to say, $T \supset S \mapsto S^{\perp}$, where $S^{\perp}$ contains all those Y such that for all $X \in S$, $Y \perp X$, with a dual construction in the other direction. A subhypercategory of T is a *localization* if it is closed under this Galois correspondence and is reflective, i.e. if $S \subset T$ has a left adjoint. An alternative characterization is afforded by the following lemma.

Lemma 7.1. *$S \subset T$ is a localization if and only if S is full and replete, closed under right homotopy Kan extensions and reflective.*

The proof is entirely straightforward.

From [3] we note also

Proposition 7.2. *If $S \subset T$ is a localization then S is a homotopy theory.*

A left adjoint $loc : T \to S$ of the inclusion is called a *localizing hyperfunctor*. Since this inclusion strictly preserves right homotopy Kan extensions such a loc is a strong hyperfunctor.

The dual notions are *colocalization* and *colocalizing hyperfunctor*.

A first example is provided by the *associated pointed homotopy theory* $T^{\bullet}$ of a homotopy theory T. We recall that a homotopy theory T is *pointed* if in $T\mathbf{1}$, and hence in all TC, initial objects $\emptyset$ and terminal object $*$ coincide; we then call them 0-objects and denote them by 0. The theory $T^{\bullet}$ is the localization of $T[\mathbf{2}]$ containing those X such that $X_0 = *$.

Remark 7.3. Suitably construed, pointed homotopy theories are a reflective sub-2-category of a 2-category of homotopy theories, with $T \mapsto T^{\bullet}$ as the reflection.

We omit the details.

The following localizations will be particularly important to us. If T is a pointed homotopy theory and $J : D \subset C$ is a full imbedding in CAT we let $T(C,D)$ be the full replete subcategory of TC containing all those X such that $T[J]X = 0$. Then $T[C,D]E = T(C \times E, D \times E)$ defines a subhypercategory of $T[C]$.

Proposition 7.4. *$T[C,D]$ is both a localization and a colocalization of $T[C]$, and is thus a homotopy theory.*

The localizing hyperfunctor is given, on X, by the homotopy cofibre of the counit $(L[J])(T[J])X \to X$. We may characterize $T[C,D]$ as the *relative cotensor* of T by the pair (C,D). If $F : C \to C'$ takes D into D' then $T[F]$ takes $T[C',D']$ into $T[C,D]$. We shall loosely write $T[F]$ for the restriction as well.

For example, consider the categories (in this case, ordered sets) Λ, defined as the subset

$$\begin{array}{ccc} (0,0) & \longrightarrow & (1,0) \\ \downarrow & & \\ (0,1) & & \end{array}$$

of $\mathbf{Z} \times \mathbf{Z}$, where $\mathbf{Z}$ is the set of integers in their usual ordering, Λ^{op}, its dual, which we shall identify with

$$\begin{array}{ccc} & & (1,0) \\ & & \downarrow \\ (0,1) & \longrightarrow & (1,1) \end{array}$$

and $\dot{\Lambda} = \Lambda \cap \Lambda^{\mathrm{op}}$.

For any pointed homotopy theory T the adjoint hyperfunctors $\Sigma \dashv \Omega : T \to T$, the *suspension* and the *loop-space* are given by the compositions

$$T = T[\mathbf{1},\mathbf{0}] \xrightarrow{L[0,0]} T[\mathbf{2}^2, \dot{\Lambda}] \xrightarrow{T[1,1]} T$$

and

$$T \xleftarrow{T[0,0]} T[\mathbf{2}^2, \dot{\Lambda}] \xleftarrow{R[1,1]} T$$

This may seem more intelligible if we observe that $T[0] : T[\Lambda, \dot{\Lambda}] \to T$ is an equivalence so that for any X, $L[0,0]X$ has the diagram

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma X \end{array}$$

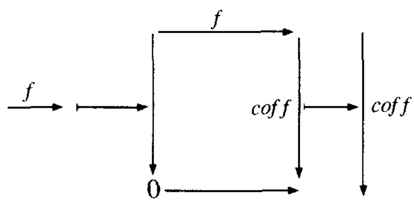
and is a homotopy pushout.

A pointed homotopy theory T is *stable* if either (and thus each) of Σ, Ω is an equivalence. It is a familiar observation that any loop-space (suspension) is a group (cogroup). It follows at once that any stable homotopy theory is *additive*, that is to say, that all its values are additive categories. Furthermore, S being a stable homotopy theory and $F : C \to D$ in CAT, SF, LF, RF, all being adjoints on one side or the other, are additive functors. Similarly continuous and cocontinuous hyperfunctors are additive.

Another example is furnished by the hyperfunctors $cof \dashv fib : T[\mathbf{2}] \to T[\mathbf{2}]$, for T a pointed homotopy theory. The former is defined to be the composition

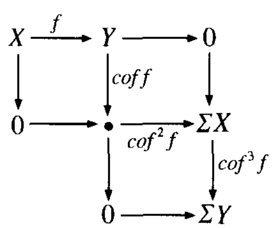
$$T[\mathbf{2}] \xrightarrow{L[\mathbf{2}\times 0]} T[\mathbf{2}\times\mathbf{2},(1,0)] \xrightarrow{T[1\times\mathbf{2}]} T[\mathbf{2}]$$

The effect of this may be understood by looking at the diagrams:



Proposition 7.5. $cof^3 \approx \Sigma$; $fib^3 \approx \Omega$.

The argument goes back, in essence, to [5], see also [2], and may be sufficiently indicated by the diagram



In a stable homotopy theory S this implies that *fib* and *cof* are equivalences and thus, up to isomorphism, inverse to one another. The usual Mayer–Vietoris argument shows that, in fact, homotopy pushouts and homotopy pullbacks in $S[\mathbf{2}^2]$ coincide and thus that sequential homotopy colimits, since they preserve homotopy pushouts, preserve homotopy pullbacks as well. Similarly, since finite products coincide with finite products, the SC being additive, these too are preserved by sequential homotopy colimits. Thus, all stable homotopy theories are regular.

8. The stabilization theorem: spectra

Let us denote by $\mathfrak{H}^{cc}$ the full sub 2-category of $\mathscr{H}^{cc}$, the 2-category of complete hypercategories and cocontinuous strong hyperfunctors, containing the pointed regular homotopy theories and by $\mathfrak{S}^{cc}$ the full sub-2-category containing the stable homotopy theories. The *stabilization theorem*, our principal result, is the following statement, which characterizes the "stabilization" of a regular pointed homotopy theory by means of its universal property.

Theorem 8.1. *$\mathfrak{S}^{cc}$ is reflective in $\mathfrak{H}^{cc}$.*

It is to be noted that the statement of this theorem does not involve spectra, which, in one version or another, are prominent in all constructions of stable homotopy to be found in the literature. Indeed the *stable homotopy category*, which we must now think of as the value at **1** of some stable homotopy theory, is not, properly speaking, ever defined, but is rather characterized by its construction. The original constructions of Boardman, Puppe, Adams, May et al. produce only the category corresponding to the stabilization of $\Pi^\bullet$, more recently additional cases, e.g. those corresponding to $\Pi^\bullet[C]$ have been constructed, at the cost of snowballing complication (cf. e.g. [1]). These constructions, moreover, do not provide for the characterization by a universal property, and thus render extravagantly difficult the proof of the important fact that the "stable smash product" is coherently symmetric monoidal.

This being said, we must nevertheless, in order to prove this stabilization theorem, have recourse to some notion of spectrum. The one we choose is defined within an arbitrary pointed regular homotopy theory, and is thus not immediately comparable with others in the literature cited above.

We denote by **V** the category (i.e. ordered set)

$$\{(i,j) \mid |i-j| \leq 1\} \subset \mathbf{Z} \times \mathbf{Z},$$

where, once again, **Z** stands for the set of integers in their usual ordering. By $\dot{\mathbf{V}}$ we denote the subset with $|i-j| = 1$. Finally, if T is a pointed homotopy theory we denote by *SpecT* the homotopy theory $T[\mathbf{V}, \dot{\mathbf{V}}]$. Thus if X is in *SpecT*, i.e. in $(SpecT)C$ for some $C \in$ CAT then $dgm_{\mathbf{V}}X$ is of the form

$$\begin{array}{cccccccc} & & & 0 & \longrightarrow & X_{1,1} & & \cdots \\ & & & \uparrow & & \uparrow & & \\ & 0 & \longrightarrow & X_{0,0} & \longrightarrow & 0 & & \\ & \uparrow & & \uparrow & & & & \\ \cdots & X_{-1,-1} & \longrightarrow & 0 & & & & \end{array}$$

which we shall condense to

$$(\cdots : X_{-1} : \underline{X_0} : X_1 : \cdots)$$

where the underline indicates the term at $(0,0)$. The automorphism $s(i,j) = (i+1, j+1)$ of $\mathbf{V}$ induces an automorphism $\sigma = T[s]: SpecT \to SpecT$. For X in $SpecT$

$$dgm_{\mathbf{V}}(\sigma X) = (\cdots : X_0 : \underline{X_1} : X_2 : \cdots)$$

We construct also a functor, i.e. an order-preserving map, $w : \mathbf{2}^2 \times \mathbf{V} \to \mathbf{V}$ with the properties

$$w(id \times \sigma) = \sigma w,$$

$$w(p,q,i,i) = (i+p, i+q),$$

$$w(p,q,i,j) \in \dot{\mathbf{V}} \text{ for } i \neq j.$$

These do not characterize w uniquely, but it will be obvious that the choice among such w is irrelevant.

If $X \in SpecT$ then the diagram $dgm_{\mathbf{2}^2}(SpecT)[w]X$ is

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \sigma X \end{array}$$

Thus, the unit of the adjunction $SpecT[\Lambda^{\mathrm{op}} \hookrightarrow \mathbf{2}^2] \dashv R_{SpecT}[\Lambda^{\mathrm{op}} \hookrightarrow \mathbf{2}^2]$ gives a canonical hypernatural transformation $\eta_X : id_{SpecT} \to \Omega\sigma \approx \sigma\Omega$. We shall say that X is an *Ω-spectrum* if η_X is an isomorphism.

There is, of course, a dual morphism $\Sigma \to \sigma$ and a dual notion of a *Σ-spectrum*.

Proposition 8.2. *If T is a regular pointed homotopy theory then the Ω-spectra in T constitute a localization $StabT \subset SpecT$ which is a stable homotopy theory.*

For $StabT$ is clearly replete and closed under right homotopy Kan extensions, while a left adjoint to the inclusion may be constructed by setting, for $X \in SpecT$, $locX = hocolim(X \to \Omega\sigma X \to \Omega^2\sigma^2 X \to \cdots)$. If Y is an Ω-spectrum, the map $TC(\Omega\sigma X, Y) \to TC(X, Y)$ given by $f \mapsto (\eta_Y)^{-1}\Omega\sigma f$ has, by naturality, the inverse $g \mapsto g\eta_X$. Thus $TC(locX, Y) \approx TC(X, Y)$. But, since T is regular, $\Omega\sigma locX \approx locX$, so that $locX$ is an Ω-spectrum.

9. The stabilization theorem: conclusion

The localization $loc_T : SpecT \to StabT$ is, as for any localization, determined by the *stable equivalences*, that is, by the set of morphisms inverted by loc_T. We shall say also that $X \in SpecT$ is *stably trivial* if $loc_T X \approx 0$. Since $StabT$ is stable these notions determine one another.

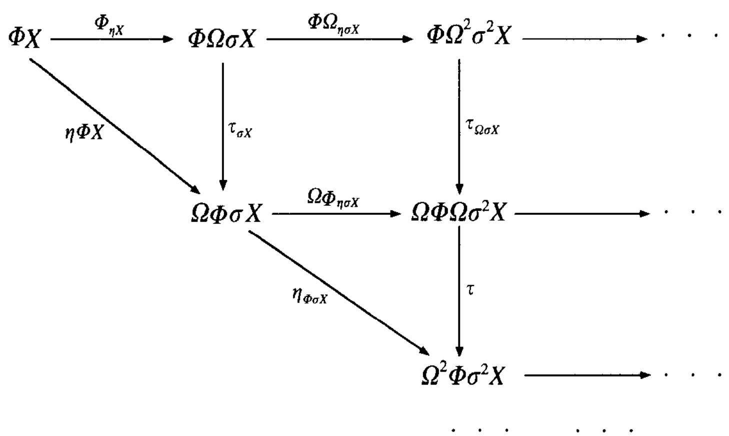
Lemma 9.1. *$f:X \to Y$ in $SpecT$ is a stable equivalence if and only if $(cof\, f)_1$ is stably trivial.*

The central point in the proof of the stabilization theorem is the following *stable equivalence lemma.*

Lemma 9.2. *If T, T' are pointed regular homotopy theories and $\Phi: T \to T'$ is cocontinuous then $Spec\Phi: SpecT \to SpecT'$ preserves stable equivalences.*

Since Φ is cocontinuous it preserves cofibres. Thus by Theorem 8.1 this statement is equivalent to the assertion that Φ preserves stably trivial spectra, that is to say, that if $hocolim(X \to \Omega\sigma X \to \Omega^2\sigma^2 X \to \cdots) = 0$ then also $hocolim(\Phi X \to \Omega\sigma\Phi X \to \cdots) = 0$.

Since Φ is cocontinuous $\Phi X \to \Phi\Omega\sigma X \cdots$ has homotopy colimit 0. We interpolate between these sequences by means of the diagram



(9.3)

indexed by the ordered set $\mathbf{U} = \{(i,j) \mid 0 \leq j \leq i\} \subset \mathbf{Z} \times \mathbf{Z}$, $\tau: \Phi\Omega \to \Omega\Phi$ being the transpose of the composition $\Sigma\Phi\Omega \to \Phi\Sigma\Omega \to \Phi$, where the first arrow comes from the commutation of Φ and Σ and the second from the counit of $\Sigma \dashv \Omega$.

Proceeding as in [3, III Section 3] , we see that the diagram above is $dgm_{\mathbf{U}}W$ for some $W \in SpecT[\mathbf{U}]$. We can evaluate $SpecT\text{-}colim_{\mathbf{U}}W$ in two ways. The horizontal projection $q:\mathbf{U} \to \mathbf{N} \subset \mathbf{Z}$, followed by $\mathbf{N} \to \mathbf{1}$ gives the value $SpecT\text{-}colim_{\mathbf{N}}L[q]W$. But $(L[q]W)_n = SpecT\text{-}colim\Omega^n\Phi\sigma^n(X \to \Omega\sigma X \to \Omega^2\sigma^2 X \to \cdots) = 0$ so that $SpecT\text{-}colimW = 0$. But the diagonal edge of $\mathbf{U}$ is homotopically final, so that the homotopy colimit may also be computed by first restricting to this.

Let us now set $Stab\Phi = loc_{T'}(Spec\Phi)J_T$ where J_T is the inclusion $StabT \subset SpecT$. Then the stable equivalence lemma gives us

$$loc_{T'}Spec\Phi \approx (Stab\Phi)loc_T, \tag{9.4}$$

the isomorphism being given by $loc_{T'}(Spec\Phi)\eta_T$ where, this time, η_T is the unit of the adjunction $loc_T \dashv J_T$.

The operation *Stab* belongs in fact to a pseudofunctor $\mathfrak{H}^{cc} \to \mathfrak{S}^{cc}$; for our purposes it is sufficient to observe that if also $\Phi' : T' \to T''$ then an isomorphism $Stab(\Phi'\Phi) \to (Stab\Phi')(Stab\Phi)$ is obtained by applying the stable equivalence lemma to $Spec\Phi'$ and the unit of the adjunction $loc_{T'} \dashv J_{T'}$.

Finally, for any $T \in \mathfrak{H}^{cc}$ we define the hyperfunctor $stab_T$ as the composition of

$$T = T[1] \xrightarrow{L[0,0]} SpecT \xrightarrow{loc_T} StabT$$

This is of course the left adjoint of $e_T = T[0,0]J_T : StabT \to T$. It is clear that if T is stable then $e_T, stab_T$ are adjoint equivalences and thus inverses up to isomorphism. From (9.4) we get, moreover, for Φ as above, $stab_{T'}\Phi \approx Stab\Phi stab_T$ and thus also, if T, T' are stable, $\Phi e_T \approx e_{T'}Stab\Phi$.

The stabilization theorem may now be expressed by the slightly sharper statement that for any stable homotopy theory S,

$$\mathscr{H}^{cc}(StabT, S) \xrightarrow{\circ stab_T} \mathscr{H}^{cc}(T, S)$$

is an equivalence of categories. But $\Psi \mapsto e_S Stab\Psi$ gives an inverse, up to isomorphism, of $- \circ stab_T$.

10. The stabilization theorem: Corollaries

We shall apply the stabilization theorem to "multilinear" operations. The first step is the observation that its conclusion holds for the "internal hom" as well.

Corollary 10.1. *If T is a regular pointed homotopy theory and S is a stable homotopy theory then*

$$\mathscr{H}^{cc\sharp}(StabT, S) \xrightarrow{\circ stab_T} \mathscr{H}^{cc\sharp}(T, S)$$

is an equivalence of hypercategories.

This observation, together with (5.3), gives us, inductively, the following conclusion.

Corollary 10.2. *For $T_1, \ldots, T_n$ pointed regular and S stable*

$$\mathscr{B}_n^{cc}(StabT_1, \ldots, StabT_n; S) \longrightarrow \mathscr{B}_n^{cc}(T_1, \ldots, T_n; S)$$

where the arrow is induced by the compositions with $stab_{T_1}, \ldots, stab_{T_n}$, is an equivalence of categories.

Now suppose that T is a regular pointed homotopy theory supplied with a bicocontinuous pairing $\Box \in \mathscr{B}^{cc}(T, T; T)$. Then $stab_T \Box \in \mathscr{B}^{cc}(T, T; StabT)$ determines, up to a unique isomorphism, a similar pairing

$$(Stab\Box) \in \mathscr{B}^{cc}(StabT, StabT; StabT)$$

such that $Stab\Box(stab_T \times stab_T) \approx stab_T\Box$. Furthermore, if $\Box$ is supplied with a natural associativity isomorphism $a : \Box(\Box \times id) \approx \Box(id \times \Box)$ then $Stab\Box$ inherits one uniquely, after the same fashion. Commutativity isomorphisms behave analogously. But this uniqueness gives us our principal corollary.

Theorem 10.3. *If the regular pointed homotopy theory T is supplied with a bicocontinuous (braided)(symmetric) coherent associative (monoidal) internal pairing then a similar such pairing is determined up to unique isomorphism on its stabilization StabT by the condition that the stabilization hyperfunctor $stab_T$ preserve the pairing.*

The existence of such pairings on such homotopy theories T deserves more extensive discussion which, however, we do not intend to engage in here. The most notorious case of course is that of standard pointed homotopy theory $\Pi^\bullet$ with the smash-product pairing, whose coherence properties are immediately derivable from the character of the category of pointed sets. It would seem that the stabilization theorem gives, even in this case, a more perspicuous proof of several properties of the stable (or, more properly *stabilized*) smash product than the arguments heretofore available.

Remark 10.4. Throughout Sections 8–10 we have in the interest of brevity systematically omitted any reference to dual notions. We might, perhaps, have called our *Stab left stabilization* and defined the dual notion of *right stabilization* as well. A "*right stabilization theorem*" dual to Theorem 8.1 would then hold for coregular *homotopy theories*, and so forth.

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