GALOIS EQUIVARIANCE AND STABLE MOTIVIC HOMOTOPY THEORY

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Abstract. For a finite Galois extension of fields L/k with Galois group G, we study a functor from the G-equivariant stable homotopy category to the stable motivic homotopy category over k induced by the classical Galois correspondence. We show that after completing at a prime this results in a full and faithful embedding whenever k is real closed and $L = k[i]$. It is integrally a full and faithful embedding if a motivic version of Serre's finiteness theorem is valid. We produce strong necessary conditions on the field extension L/k for this functor to be full and faithful. Along the way, we produce several results on the stable C_2-equivariant Betti realization functor.

1. Introduction

The stable versions of equivariant and motivic homotopy theory play important roles in the geometry of manifolds, algebraic cycles, and quadratic forms. Stable equivariant homotopy theory is the study of topological spaces equipped with a group action up to stable equivariant weak equivalence. It has recently found stunning application [19] to the Kervaire problem, playing an essential role in the proof that there are no smooth framed manifolds of Kervaire invariant one in dimensions greater than 126. Via the work of Devinatz and Hopkins [6], stable equivariant homotopy theory controls the chromatic decomposition of stable homotopy theory. It is also essential to the study of topological Hochschild homology [3].

Motivic homotopy theory is a homotopy theory of schemes in which the affine line plays the role of the unit interval. Its study was initiated by Morel and Voevodsky [37] in work related to Rost and Voevodsky's resolution of the Bloch-Kato conjectures on Milnor K-theory and Galois cohomology [45, 48]. Its stable version plays an essential role in the theory of motives and motivic cohomology [49]. This circle of ideas led to the resolution of the Milnor conjecture on quadratic forms [38] and the Quillen-Lichtenbaum conjecture, a powerful result linking algebraic K-theory and values of Dedekind ζ-functions via a "homotopy limit problem" phrased in the language of stable equivariant homotopy [12]. Stable motivic homotopy theory also opens new vistas, such as the study of algebraic cobordism [44].

The purpose of this paper is to study how equivariant and motivic stable homotopy theory are related via the classical Galois correspondence.

A fundamental computation in stable motivic homotopy theory is the identification of the endomorphism ring of the motivic sphere spectrum by Morel [36]. In *loc. cit.* Morel shows that $\mathrm{End}_{\mathrm{SH}_k}(\mathbb{S}_k)$ is isomorphic to the Grothendieck-Witt group $GW(k)$ of nondegenerate quadratic forms over a perfect field k. It is now a classical fact, going back to Segal and tom

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Dieck, that the endomorphism ring $\mathrm{End}_{\mathrm{SH}_G}(\mathbb{S}_G)$ of the equivariant sphere spectrum in the equivariant stable homotopy category is equal to the Burnside ring $A(G)$ of finite G-sets.

When L/k is a finite Galois extension with Galois group G, Dress [8, Appendix B] (see also, [2, §4]) constructs a ring homomorphism $A(G) \to GW(k)$ relating these two fundamental invariants. In fact, the Galois correspondence can be stabilized to yield a strong symmetric monoidal triangulated functor from the stable G-equivariant homotopy category to the stable motivic homotopy category over k,

$$c^*_{L/k} : \mathrm{SH}_G \to \mathrm{SH}_k.$$

This relies on work of P. Hu [23]. When $L = k$, $c^*_{L/L}$ is simply the functor induced by sending a simplicial set to its associated constant motivic space. When $L = k$ is algebraically closed of characteristic zero, Levine [31] has recently shown that $c^*_{L/L}$ is a full and faithful embedding, but this is not the general case for $c^*_{L/k}$. Indeed, the Burnside ring $A(G)$ is always torsion free while $GW(k)$ can in general contain torsion, which eliminates the possibility of $c^*_{L/k}$ inducing an isomorphism $A(G) \cong GW(k)$. However, if k is a real closed field then $GW(k)$ and $A(C_2)$ are isomorphic so one might still hope that Levine's embedding theorem can be generalized to real closed fields. Our main result, proved in Theorem 2.9 and Theorem 2.10 below, is that this indeed is the case after p-completion for any prime p and it is integrally a full and faithful embedding if $\pi_n(\mathbb{S}_k)_{\mathbb{Q}} = 0$ for $n > 0$. The vanishing of these higher homotopy groups would be a motivic version of the classical result of Serre on the homotopy groups of spheres and is already known to be true when -1 is a sum of squares in the basefield.

Theorem 1.1. *Let k be a real closed field and $L = k[i]$ be its algebraic closure. Then for any prime p the functor*

$$c^*_{L/k} : \mathrm{SH}_{C_2} \to \mathrm{SH}_k$$

is a full and faithful embedding after p-completion. If $\pi_n(\mathbb{S}_k)_{\mathbb{Q}} = 0$ for $n > 0$ it is a full and faithful embedding.

1.1. **Computational ramifications.** Our embedding result has significant implications for (Picard-graded) stable homotopy groups of spheres in the C_2-equivariant and real closed motivic settings. Recall that the representation spheres $S^{m+n\sigma}$ are invertible in SH_{C_2} where $S^{m+n\sigma}$ is the one-point compactification of m copies of the one-dimensional real trivial representation and n copies of the real sign representation. As such $\mathbb{Z} \oplus \mathbb{Z}\{\sigma\}$ is a subgroup of the Picard group of invertible objects in SH_{C_2}, and it is common to consider the bigraded stable homotopy groups $\pi_{m+n\sigma}X = [S^{m+n\sigma}, X]_{C_2}$ of a C_2-spectrum X. When k is real closed and $L = k[i]$, Theorem 1.1 implies that $c^*_{L/k} : \pi_{m+n\sigma}(\mathbb{S}_{C_2})^\wedge_p \cong [c^*_{L/k}S^{m+n\sigma}, (\mathbb{S}_k)^\wedge_p]_k$. We will see that $c^*_{L/k}S^{m+n\sigma} \simeq S^m \wedge (S^L)^{\wedge n}$ where S^L is the unreduced suspension of $\mathrm{Spec}(L)$. By a theorem of P. Hu [23], S^L is invertible and $\mathbb{Z} \oplus \mathbb{Z}\{L\}$ is a subgroup of the Picard group of SH_k. We emphasize that S^L is *not* weakly equivalent to $\mathbb{A}^1 \smallsetminus \{0\}$ and this is *not* the "standard" bigrading in motivic homotopy theory.

Regardless, if we set $S^{m+nL} = S^m \wedge (S^L)^{\wedge n}$ and make the natural definition of π_{m+nL}, we see that $c^*_{L/k}$ induces isomorphisms

$$\pi_{m+n\sigma}(\mathbb{S}_{C_2})^\wedge_p \xrightarrow{\cong} \pi_{m+nL}(\mathbb{S}_k)^\wedge_p$$

for all $m, n \in \mathbb{Z}$ under the conditions of Theorem 1.1. It is an observation of D. Dugger that the same result does not hold if S^L is replaced by $\mathbb{A}^1 \smallsetminus \{0\}$.

The C_2-equivariant stable stems were studied by Araki and Iriye via Toda-style methods. In [1], they compute the groups $\pi_{m+n\sigma}\mathbb{S}_{C_2}$ for $m+n \leq 8$. In particular, they compute the groups $\pi_m\mathbb{S}_{C_2}$ for $m \leq 8$, so Theorem 1.1 implies the following corollary.

Corollary 1.2. *If k is a real closed field and p a prime then $\pi_m(\mathbb{S}_k)^\wedge_p$, $0 \leq m \leq 8$, is the p-completion of the values displayed in the following table. If in addition $\pi_m(\mathbb{S}_k)_\mathbb{Q} = 0$ for all $m > 0$, then $\pi_m\mathbb{S}_k$, $0 \leq m \leq 8$, takes the values displayed in the following table.*

m	0	1	2	3	4	5	6	7	8
$\pi_m\mathbb{S}_k$	$\mathbb{Z}^2$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/24)^2 \oplus \mathbb{Z}/8$	$\mathbb{Z}/2$	0	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/240)^2 \oplus \mathbb{Z}/16 \oplus \mathbb{Z}/2$	$(\mathbb{Z}/2)^7$

In addition, in Corollary 2.13 we show that if $\pi_n(\mathbb{S}_k)_\mathbb{Q} = 0$ for $n > 0$ for real closed fields, then Morel's conjecture on $\pi_1(\mathbb{S}_k)$ also holds for real closed fields. This conjecture says that, for a general base field F, there is a short exact sequence

$$0 \to K_2^M(F)/24 \to \pi_1\mathbb{S}_F \to F^\times/(F^\times)^2 \oplus \mathbb{Z}/2 \to 0.$$

The second-named author and P. Østvær have previously verified Morel's conjecture for fields of cohomological dimension less than three [39].

While these immediate applications transfer information from C_2-equivariant to motivic homotopy over a real closed field, future work should leverage motivic homotopy to produce C_2-equivariant computations. In particular, the dual motivic Steenrod algebra is smaller than its equivariant counterpart, making Adams and Adams-Novikov spectral sequence computations more approachable. The authors plan to apply these tools over the field $\mathbb{R}$ of real numbers (with the above exotic Picard grading) in order to extend our computational understanding of the stable C_2-equivariant homotopy category.

1.2. Galois correspondence and motivic homotopy theory.

An intriguing viewpoint on our embedding theorem is as a generalization of the classical Galois correspondence in the case of real closed fields. Indeed, if L/k is a finite Galois extension with Galois group G then the Galois correspondence is an equivalence between the category of finite G-sets and the category of finite étale k-algebras. Restricting to the orbit category, this correspondence gives the functor

$$c_{L/k} : \mathrm{Or}_G \to \mathrm{Sm}/k$$

to smooth k-schemes which is explicitly given on objects by $c_{L/k}(G/H) = \mathrm{Spec}(L^H)$. As recorded in Theorem 4.6, this functor can be stabilized, yielding a strong symmetric monoidal, triangulated functor

$$c^*_{L/k} : \mathrm{SH}_G \to \mathrm{SH}_k.$$

It is not hard to see that the unstable version of this functor induces a full and faithful embedding from the unstable G-equivariant homotopy category to the unstable motivic homotopy category over k (see Lemma 4.5). Note, though, that the stable equivariant homotopy category is formed by stabilizing with respect to representation spheres while the motivic homotopy category is formed by stabilizing with respect to $\mathbb{P}^1$. Hence there is no reason for this pleasant relationship between the two categories to remain after stabilization, yet it does in special cases. In fact, we can say something slightly more precise. The image of $c^*_{L/k}$ is always contained in the subcategory $\mathcal{E}_k$ of SH_k which is generated by the finite étale

k-algebras.[1] Our result can thus be rephrased as an equivalence of triangulated categories between SH_{C_2} and $\mathcal{E}_k$ when k is real closed.

This translation of stable motivic homotopy over k into stable G-equivariant homotopy for $G = \mathrm{Gal}(L/k)$ will not work for general finite Galois extensions L/k. Indeed, in Theorem 3.4 we show that $c^*_{L/k}$ induces an isomorphism $A(G) \to GW(k)$ if and only if either k is quadratically closed and $L = k$, or k is euclidean[2] and $L = k[i]$. This implies in particular that $c^*_{L/k}$ cannot be full and faithful if L/k is not of this special form.

1.3. **Outline of the proof.** Our main theorem is directly inspired by M. Levine's theorem on full faithfulness of the constant presheaf functor [31], and our methods are, largely, in the same spirit as his. That said, Levine's arguments rely on the convergence of the slice spectral sequence, a result not yet known over fields with infinite cohomological dimension. To remedy this situation, we compare the motivic and equivariant Adams spectral sequences. Ultimately, our proof is independent of Levine's paper and we reproduce his main theorem as Corollary 2.15.

Let k be real closed and set $L = k[i]$ so that $G = C_2$ is cyclic of order 2. By a density argument, to show that $c^*_{L/k}$ is full and faithful it suffices to show that $c^*_{L/k}$ induces isomorphisms

$$[S^n \wedge X, Y]_{C_2} \xrightarrow{\cong} [S^n \wedge c^*_{L/k}X, c^*_{L/k}Y]_k$$

where X, Y take values in the set $\{\mathbb{S}_{C_2}, C_{2+} \wedge \mathbb{S}_{C_2}\}$. The key case is when k admits a real embedding and in this case we can use the C_2-equivariant Betti realization. The computation is broken up into pieces: the p-completed sphere (for any prime p) and the rationalized sphere. The computation concerning the rationalized sphere relies on the conjectural motivic version of Serre's finiteness theorem and so the integral version of the embedding theorem is conditional upon the validity of this conjecture. Of course, the full and faithful embedding of p-completed homotopy categories holds independent of this conjecture. In the p-complete case, we identify the C_2-equivariant Betti realization of the motivic Adams spectral sequence with the C_2-equivariant Adams spectral sequence based on the Bredon cohomology spectrum $\mathrm{H}\underline{\mathbb{Z}/p}$. We establish an equivariant version of Suslin-Voevodsky's theorem on Suslin homology which implies that the realization induces an isomorphism on weight zero components of the E_1-pages from which we deduce the result in this case.

1.4. **Comments on realization and profinite Galois extensions.** We conclude by making a few comments on the role of "realization" functors. M. Levine uses the Betti realization functor $\mathrm{Re}_B : \mathrm{SH}_L \to \mathrm{SH}$ for algebraically closed subfields L of $\mathbb{C}$ to prove his full faithfulness theorem in [31]. Since $\mathrm{Re}_B \circ c^* = \mathrm{id}$, the constant presheaf functor is always faithful for any $k \subseteq \mathbb{C}$. Levine's innovation was to compare the Betti realization of the slice spectral sequence for the motivic sphere spectrum over an algebraically closed field with the Novikov spectral sequence in topology. An isomorphism between the E_2-terms of these spectral sequences implies an isomorphism on stable homotopy groups of spheres which ultimately implies the fullness result.

When k has a real embedding, then there is an associated C_2-equivariant Betti realization $\mathrm{Re}_B^{C_2} : \mathrm{SH}_k \to \mathrm{SH}_{C_2}$. As previously mentioned, we cannot use the slice spectral sequence to prove our embedding theorem, but our arguments still rely on using (equivariant) Betti

[1] Here "generated" means that $\mathcal{E}_k$ is the smallest localizing subcategory of SH_k containing all (suspension spectra of) finite étale k-algebras.

[2] A field k is euclidean if -1 is not a sum of squares in k and $[k^\times : (k^\times)^2] = 2$.

realization to compare some spectral sequences (namely the motivic and equivariant Adams spectral sequences). Again, faithfulness of $c^*_{k[i]/k}$ is easy because $\mathrm{Re}_B^{C_2} \circ c^*_{k[i]/k} = \mathrm{id}$.

Suppose $L = \bar{k}$ is the algebraic closure of k and G is the absolute Galois group $\mathrm{Gal}(L/k)$, which is a profinite group. A natural question is whether the main theorem of this paper extends to a full faithfulness theorem for G-equivariant stable homotopy inside of SH_k. In order to precisely state such a question, though, one would need an appropriate notion of genuine G-spectra and G-stable homotopy when G is profinite. One proposal for this category is contained in [13], and C. Barwick has communicated ideas on an alternate formulation to the authors. Whichever model is chosen, one would hope that it would admit well-behaved functors

$$c^*_{L/k} : \mathrm{SH}_G \to \mathrm{SH}_k \quad \text{and} \quad \mathrm{Re}_B^G : \mathrm{SH}_k \to \mathrm{SH}_G$$

such that $\mathrm{Re}_{\acute{e}t}^G \circ c^*_{L/k}$ is some form of pro-completion of the identity functor. This would result in a pro-faithfulness theorem, at which point one could examine fullness properties as well. The authors hope to pursue this line of inquiry in future research.

1.5. **Organization of the paper.** We prove our main theorem in §2 according to the strategy outlined above. We then deduce several interesting corollaries, including our Picard-graded homotopy comparison (Corollary 2.11), Morel's conjecture on $\pi_1 \mathbb{S}_k$ for real closed fields (Corollary 2.13), and a relative version of our theorem comparing full faithfulness of $c^*_{L/L}$ and full faithfulness of $c^*_{L/k}$ (Corollary 2.15).

In §3, we study the effect of $c^*_{L/k}$ on the endomorphism ring of the sphere spectrum. We show that it induces an isomorphism if and only if either k is quadratically closed and $L = k$, or k is euclidean and $L = k[i]$ (Theorem 3.4); in particular this places strong conditions on L/k necessary in order for $c^*_{L/k}$ to be full and faithful.

We collect several technical constructions and results in §4. In §4.1 and §4.2 we recall some definitions and facts about different model structures we use. With these preliminaries in order, the unstable and stable versions of $c^*_{L/k}$ are constructed in §4.3. In §4.4 we record the construction of and some well-known results on the stable C_2-equivariant Betti realization functor arising from a real embedding of fields. In §4.5 we prove basic compatibility results between $c^*_{L/k}$ and various change-of-group and change-of-base functors. Finally, in §4.6 we study the effect of stable C_2-equivariant Betti realization on motivic cohomology. In particular, we show that the Beilinson-Lichtenbaum conjectures can be rephrased for real closed subfields of $\mathbb{R}$ in terms of Bredon cohomology (Theorem 4.18) and we establish an equivariant version of a theorem of Suslin-Voevodsky for torsion effective motives (Theorem 4.19).

1.6. **Relation to other work.** It is interesting to contrast the subject of this paper with Hu, Kriz, and Ormsby's stable equivariant motivic homotopy theory [26]. In that setup one studies smooth schemes equipped with a G-action, G a finite group. It should be emphasized that this group does not necessarily have any relationship with the automorphisms of a field extension. In contrast, in the present work we study the image of the stable $\mathrm{Gal}(L/k)$-equivariant homotopy category inside the stable *nonequivariant* motivic homotopy category over k. It would be interesting to combine these notions of equivariance and geometry further by studying $(G, \mathrm{Gal}(L/k))$-homotopy inside of the G-motivic homotopy category over k.

1.7. **Notation and conventions.** Throughout k is a perfect field and L/k is a finite Galois extension with Galois group G. For a finite group G we write SH_G for the (genuine) stable equivariant homotopy category. We write Sm/K for the category of smooth schemes of finite type over a base field K and we write SH_K for the stable motivic homotopy category. We use the notation $[-,-]_G = \mathrm{SH}_G(-,-)$ and $[-,-]_k = \mathrm{SH}_k(-,-)$ for morphism sets in respective stable homotopy categories. Our indexing convention for motivic spheres is that $S^{a+b\alpha} := (S^1)^{\wedge a} \wedge (\mathbb{A}^1 \smallsetminus \{0\})^{\wedge b}$. When $G = C_2$ we write S^σ for the sign-representation sphere and set $S^{a+b\sigma} := (S^1)^{\wedge a} \wedge (S^\sigma)^{\wedge b}$. In the special case $a = b = 0$, we write $\mathbb{S}_k$ and $\mathbb{S}_G$ for the sphere spectra in the motivic and equivariant categories, respectively.

For the sake of typographical simplicity, we do not use any special notational device for derived functors in §2 and §3, where we only work on the level of homotopy categories. In §4 we work in both model categories and associated homotopy categories and in this section we use "derived functor notation" (i.e. $\mathbb{L}F$ and $\mathbb{R}F$ respectively for left and right derived functor of F).

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2. Embedding theorem

Let L/k be a Galois extension of fields with Galois group G. As mentioned in the introduction, the functor $\mathrm{Or}_G \to \mathrm{Sm}/k$ which is defined on objects by $G/H \mapsto \mathrm{Spec}(L^H)$, induces a functor $c^*_{L/k} : \mathrm{SH}_G \to \mathrm{SH}_k$ on stable homotopy categories. Details on this construction are given in Section 4.

In this section we prove that if k is a real closed field and $L = k[i]$ then the functor $c^*_{L/k} : (\mathrm{SH}_{C_2})^\wedge_p \to (\mathrm{SH}_k)^\wedge_p$ is full and faithful for any prime p. Additionally if $\pi_n(\mathbb{S}_k)_\mathbb{Q} = 0$ for any $n > 0$ (see Conjecture 2.5) then the functor $c^*_{L/k}$ is full and faithful without completion. This is proved in Theorem 2.9 and Theorem 2.10. The main step is to show that the C_2-equivariant Betti realization induces isomorphisms

(i) $\mathrm{Re}^{C_2}_{B,\phi} : [S^n, (\mathbb{S}_k)^\wedge_p]_k \xrightarrow{\cong} [S^n, (\mathbb{S}_{C_2})^\wedge_p]_{C_2}$, and

(ii) $\mathrm{Re}^{C_2}_{B,\phi} : [\mathrm{Spec}(L)_+ \wedge S^n, (\mathbb{S}_k)^\wedge_p]_k \xrightarrow{\cong} [C_{2+} \wedge S^n, (\mathbb{S}_{C_2})^\wedge_p]_{C_2}$

whenever there is a real embedding $\phi : k \hookrightarrow \mathbb{R}$.

2.1. **Completing at p.** Let p be a prime. We analyze the image, under equivariant Betti realization, of the motivic Adams spectral sequence over a real closed subfield of $\mathbb{R}$. The p-complete motivic sphere spectrum, written $(\mathbb{S}_k)^\wedge_p$, is defined to be the Bousfield localization of $\mathbb{S}_k$ at the mod-p Moore spectrum. As in classical homotopy theory we have that $(\mathbb{S}_k)^\wedge_p = \mathrm{holim}\, \mathbb{S}_k/p^n$. Let $\mathrm{H}\mathbb{Z}/p$ denote the mod-p motivic cohomology spectrum. The motivic Adams spectral sequence for $\mathbb{S}_k$ arises as the totalization spectral sequence of the semi-cosimplicial $\mathbb{P}^1$-spectrum with s-th spectrum $(\mathrm{H}\mathbb{Z}/p)^{\wedge s}$ and co-face maps induced by the unit $\mathbb{S}_k \to \mathrm{H}\mathbb{Z}/p$. We use the following specialization of a theorem of P. Hu, I. Kriz, and the second author.

Theorem 2.1 ([25, Theorem 1]). *Let k be a real closed field, $L = k[i]$, and let Y be either $\mathbb{S}_k$ or $\mathrm{Spec}(L)_+$. The motivic Adams spectral sequence*

$$E_1^{s,t} = [S^t \wedge Y, (\mathrm{H}\mathbb{Z}/p)^{\wedge s}]_k \implies [S^{t-s} \wedge Y, (\mathbb{S}_k)^{\wedge}_p]_k.$$

is strongly convergent.

Proof. This is the weight zero portion of the p-primary motivic Adams spectral sequence constructed in [25] over k (when $Y = \mathbb{S}_k$) or over L (when $Y = \mathrm{Spec}(L)_+$). The form of the E_1-page is immediate from the totalization construction. Convergence follows from [25, Theorem 1] since real closed fields satisfy $cd_p(k) < \infty$ for $p > 2$, $cd_2(k[i]) < \infty$, and algebraically closed fields satisfy $cd_p(L) < \infty$ for all p. □

Fix an embedding $\phi : k \hookrightarrow \mathbb{R}$ and consider the resulting C_2-equivariant Betti realization $\mathrm{Re}^{C_2}_{B,\phi} : \mathrm{SH}_k \to \mathrm{SH}_{C_2}$ (see Section 4.4 for details). By Theorem 4.17, the equivariant Betti realization takes the motivic cohomology spectrum $\mathrm{H}\mathbb{Z}/p$ to the Bredon cohomology spectrum $\mathrm{H}\underline{\mathbb{Z}/p}$ associated to the constant Mackey functor $\underline{\mathbb{Z}/p}$. Since $\mathrm{Re}^{C_2}_{B,\phi}$ is symmetric monoidal and takes the unit for $\mathrm{H}\mathbb{Z}/p$ to the unit for $\mathrm{H}\underline{\mathbb{Z}/p}$, we see that $\mathrm{Re}^{C_2}_{B,\phi}$ takes the semi-cosimplicial $\mathbb{P}^1$-spectrum $(\mathrm{H}\mathbb{Z}/p)^{\wedge\bullet}$ to the semi-cosimplicial C_2-spectrum $(\mathrm{H}\underline{\mathbb{Z}/p})^{\wedge\bullet}$. The totalization spectral sequence for this latter object is the *C_2-equivariant Adams spectral sequence*, which has been studied by P. Hu and I. Kriz.

Theorem 2.2 ([24, Corollary 6.47]). *Let Y be either $\mathbb{S}_{C_2}$ or C_{2+}. The C_2-equivariant Adams spectral sequence*

$$E_1^{s,t} = [S^t \wedge Y, (\mathrm{H}\underline{\mathbb{Z}/p})^{\wedge s}]_{C_2} \implies [S^{t-s} \wedge Y, (\mathbb{S}_{C_2})^{\wedge}_p]_{C_2}$$

is strongly convergent.

Proof. The case $Y = C_{2+}$ is identical to the classical Adams spectral sequence. Hu and Kriz prove convergence for the Adams tower over $\mathbb{S}_{C_2}$ (or, in fact, over any finite C_2-spectrum) when $p = 2$. The case $p > 2$ can be treated as in *loc. cit.* and standard techniques translate this into a theorem about the totalization tower under $\mathbb{S}_{C_2}$. □

By comparing these two Adams spectral sequences, we obtain the following result.

Proposition 2.3. *Let k be real closed, set $L = k[i]$, and let $\phi : k \hookrightarrow \mathbb{R}$ be an embedding of fields. Then the induced maps*

(i) $\mathrm{Re}^{C_2}_{B,\phi} : [S^n, (\mathbb{S}_k)^{\wedge}_p]_k \xrightarrow{\cong} [S^n, (\mathbb{S}_{C_2})^{\wedge}_p]_{C_2}$, *and*

(ii) $\mathrm{Re}^{C_2}_{B,\phi} : [\mathrm{Spec}(L)_+ \wedge S^n, (\mathbb{S}_k)^{\wedge}_p]_k \xrightarrow{\cong} [C_{2+} \wedge S^n, (\mathbb{S}_{C_2})^{\wedge}_p]_{C_2}$

are isomorphisms for any $n \in \mathbb{Z}$.

Proof. We have already noted that $\mathrm{Re}^{C_2}_{B,\phi}(\mathrm{H}\mathbb{Z}/p)^{\wedge s} \simeq \mathrm{H}\underline{\mathbb{Z}/p}^{\wedge s}$, and that we have a map of Adams spectral sequences. The computation of the motivic Steenrod algebra [46, 48] shows that we have a decomposition $\mathrm{H}\mathbb{Z}/p \wedge \mathrm{H}\mathbb{Z}/p \simeq \vee \Sigma^{p_i + q_i\alpha}\mathrm{H}\mathbb{Z}/p$ for appropriate (p_i, q_i) which in particular satisfy $q_i \geq 0$. It follows from Theorem 4.19 that the equivariant Betti realization induces an isomorphism on the weight zero E_1-page of the Adams spectral sequences. By Theorem 2.1 and Theorem 2.2, the proposition follows. □

2.2. **Rational homotopy groups.** For a (motivic or equivariant) spectrum X we write $X_{\mathbb{Q}}$ for the Bousfield localization at $\mathrm{M}\mathbb{Q}$, the rational Moore spectrum. If Y is a compact spectrum, then $[Y, X_{\mathbb{Q}}] = [Y, X] \otimes \mathbb{Q}$.

The homotopy groups of the equivariant rational sphere spectrum are rather simple.

Proposition 2.4. *The homotopy groups of the rational C_2-sphere are $\pi_0(\mathbb{S}_{C_2})_{\mathbb{Q}} = \mathbb{Q} \oplus \mathbb{Q}$ and $\pi_n(\mathbb{S}_{C_2})_{\mathbb{Q}} = 0$ for any integer $n \neq 0$.*

Proof. This follows immediately from the well known fact (see *e.g.* [17, Corollary A.6]) that for any finite group, $(\mathbb{S}_G)_{\mathbb{Q}}$ is weakly equivalent to $\mathrm{H}\underline{A}_{\mathbb{Q}}$, the Eilenberg-MacLane spectrum associated to the rational Burnside Mackey functor. □

Conjecturally the higher homotopy groups of the sphere also vanish.

Conjecture 2.5 (Motivic Serre finiteness)**.** *Let k be a field. Then $\pi_n(\mathbb{S}_k)_{\mathbb{Q}} = 0$ for $n > 0$.*

Rationally (in fact already when 2 is inverted) there are orthogonal idempotents $\epsilon_+ = (\epsilon - 1)/2$ and $\epsilon_- = (\epsilon + 1)/2$ acting on $(\mathbb{S}_k)_{\mathbb{Q}}$, obtained from $\epsilon \in \pi_0\mathbb{S}_k$.[3] We thus obtain a rational decomposition of the sphere spectrum $(\mathbb{S}_k)_{\mathbb{Q}} = (\mathbb{S}_k)_{\mathbb{Q}}^+ \vee (\mathbb{S}_k)_{\mathbb{Q}}^-$ in which the factors correspond respectively to inverting ϵ_+ and ϵ_-. It follows from Morel's description [35] of $(\mathbb{S}_k)_{\mathbb{Q}}^+$ as the rational motivic cohomology spectrum $\mathrm{H}\mathbb{Q}$ (see [4, Theorem 16.2.13]) that Conjecture 2.5 holds whenever -1 is sum of squares in k (in which case $(\mathbb{S}_k)_{\mathbb{Q}}^-$ vanishes). Morel [35] also conjectures a description of $(\mathbb{S}_k)_{\mathbb{Q}}^-$ which would imply that Conjecture 2.5 holds.

Proposition 2.6. *Let k be a real closed field, set $L = k[i]$, and let $\phi : k \hookrightarrow \mathbb{R}$ be an embedding. Assume that Conjecture 2.5 holds. Then the maps*

(i) $\mathrm{Re}_{B,\phi}^{C_2} : [S^n, (\mathbb{S}_k)_{\mathbb{Q}}]_k \xrightarrow{\cong} [S^n, (\mathbb{S}_{C_2})_{\mathbb{Q}}]_{C_2}$, *and*

(ii) $\mathrm{Re}_{B,\phi}^{C_2} : [\mathrm{Spec}(L)_+ \wedge S^n, (\mathbb{S}_k)_{\mathbb{Q}}]_k \xrightarrow{\cong} [C_{2+} \wedge S^n, (\mathbb{S}_{C_2})_{\mathbb{Q}}]_{C_2}$

are isomorphisms for any $n \in \mathbb{Z}$.

Proof. Since $\mathrm{Re}_{B,\phi}^{C_2} \circ c_{L/k}^* = \mathrm{id}$, we know that the map of the proposition is surjective. Since $GW(k) = \mathbb{Z} \oplus \mathbb{Z}$ and $GW(L) = \mathbb{Z}$ for any real closed field k, it follows that the first map is an isomorphism in degree zero. By the previous propositions, these groups are zero in all other degrees. □

2.3. **Full and faithful embedding.** We now assemble the previous computations to deduce our main theorem. We begin by assembling the p-complete and the rational computations into an integral computation, carried out via the standard arithmetic fracture square.

Proposition 2.7. *Let k be a real closed field, set $L = k[i]$, and let $\phi : k \hookrightarrow \mathbb{R}$ be an embedding. Assume that Conjecture 2.5 holds. Then*

(i) $\mathrm{Re}_{B,\phi}^{C_2} : [S^n, \mathbb{S}_k]_k \xrightarrow{\cong} [S^n, \mathbb{S}_{C_2}]_{C_2}$, *and*

(ii) $\mathrm{Re}_{B,\phi}^{C_2} : [\mathrm{Spec}(L)_+ \wedge S^n, \mathbb{S}_k]_k \xrightarrow{\cong} [C_{2+} \wedge S^n, \mathbb{S}_{C_2}]_{C_2}$

are isomorphisms for all $n \in \mathbb{Z}$.

[3] Recall that ϵ is the stable map induced by the permutation $\mathbb{A}^1 \smallsetminus \{0\} \wedge \mathbb{A}^1 \smallsetminus \{0\} \cong \mathbb{A}^1 \smallsetminus \{0\} \wedge \mathbb{A}^1 \smallsetminus \{0\}$.

Proof. By [39, Appendix A] there is a homotopy cartesian square in SH_k

$$\begin{array}{ccc} \mathbb{S}_k & \longrightarrow & \prod_p (\mathbb{S}_k)^\wedge_p \\ \downarrow & & \downarrow \\ (\mathbb{S}_k)_{\mathbb{Q}} & \longrightarrow & \prod_p ((\mathbb{S}_k)^\wedge_p)_{\mathbb{Q}} \end{array}$$

where the products are over prime integers p. The square obtained by applying $\mathrm{Re}^{C_2}_{B,\phi}$ maps to the equivariant arithmetic fracture square. We thus obtain a comparison diagram of associated long exact sequences. The proposition thus follows from Proposition 2.3, Proposition 2.6, and the five lemma. □

It is now straightforward to translate this proposition into one about $c^*_{L/k}$, using a limit argument which is a modification of the one used in [31, Lemma 6.6] to the case of real closed fields.

Proposition 2.8. *Let k be a real closed field and set $L = k[i]$. Assume that Conjecture 2.5 holds. Then for any $n \in \mathbb{Z}$, the maps*

(i) $c^*_{L/k} : [S^n, \mathbb{S}_{C_2}]_{C_2} \xrightarrow{\cong} [S^n, \mathbb{S}_k]_k$, *and*

(ii) $c^*_{L/k} : [C_{2+} \wedge S^n, \mathbb{S}_{C_2}]_{C_2} \xrightarrow{\cong} [\mathrm{Spec}(L)_+ \wedge S^n, \mathbb{S}_k]_k$

are isomorphisms. For any prime p, the maps $c^*_{L/k} : [S^n, (\mathbb{S}_{C_2})^\wedge_p]_{C_2} \xrightarrow{\cong} [S^n, (\mathbb{S}_k)^\wedge_p]_k$, and $c^*_{L/k} : [C_{2+} \wedge (S^n, (\mathbb{S}_{C_2})^\wedge_p]_{C_2} \xrightarrow{\cong} [\mathrm{Spec}(L)_+ \wedge S^n, (\mathbb{S}_k)^\wedge_p]_k$ for p-completed spheres are always isomorphisms.

Proof. If there is an embedding $\phi : k \subseteq \mathbb{R}$, then this is a direct consequence of Proposition 2.3 and Proposition 2.7 and the relation $\mathrm{Re}^{C_2}_{B,\phi} \circ c^*_{L/k} \cong \mathrm{id}$. We treat the case of the uncompleted spheres below, the completed case holds verbatim.

As k is real closed, L is algebraically closed. We may express L as the union $\bigcup_{\alpha \in A} L_\alpha$ of algebraically closed subfields $L_\alpha \subset L$ of finite transcendence degree over $\mathbb{Q}$ indexed by a well-ordered set A. Consider the fields $k_\alpha = L_\alpha \cap k$. We claim that the k_α are isomorphic to real closed subfields of $\mathbb{R}$. If this is the case, then

$$\operatorname*{colim}_\alpha [S^n, \mathbb{S}_{k_\alpha}]_{k_\alpha} \quad \text{and} \quad \operatorname*{colim}_\alpha [\mathrm{Spec}(L_\alpha)_+ \wedge S^n, \mathbb{S}_{k_\alpha}]_{k_\alpha}$$

are colimits of abelian groups with respective constant values $[S^n, \mathbb{S}_{C_2}]_{C_2}$ and $[C_{2+} \wedge S^n, \mathbb{S}_{C_2}]_{C_2}$ by the observation in the first paragraph. Since it is obvious that $k = \bigcup_\alpha k_\alpha$, we conclude by essentially smooth base change [21, Lemma A.7] that these colimits are respectively isomorphic to $[S^n, \mathbb{S}_k]_k$ and $[\mathrm{Spec}(L)_+ \wedge S^n, \mathbb{S}_k]_k$. Thus we may conclude that the maps $[S^n, \mathbb{S}_{C_2}]_{C_2} \to [S^n, \mathbb{S}_k]_k$ and $[C_{2+} \wedge S^n, \mathbb{S}_{C_2}]_{C_2} \to [\mathrm{Spec}(L)_+ \wedge S^n, \mathbb{S}_k]_k$ are isomorphisms for all real closed fields.

It remains to verify the claim that each k_α is isomorphic to a real closed subfield of $\mathbb{R}$. Since L_α is algebraically closed and $[L_\alpha : k_\alpha] = 2$, the Artin-Schreier theorem implies that k_α is real closed. Fix k_α and choose a transcendence basis $x_1, \ldots, x_n$ of k_α over $\mathbb{Q}$ in which each x_i is positive in k_α. By sending each x_i to a positive transcendental real number, we produce an order embedding of $\mathbb{Q}(x_1, \ldots, x_n)$ into $\mathbb{R}$. Since $k_\alpha/\mathbb{Q}(x_1, \ldots, x_n)$ is a finite extension of ordered fields, [30, Proposition VIII.2.16] implies that there is an embedding $k_\alpha \hookrightarrow \mathbb{R}$, as desired. □

We are now ready to prove our main theorem. Recall that a *localizing subcategory* $\mathcal{E}$ of a triangulated category $\mathcal{T}$ is a full triangulated subcategory, containing all direct summands of its objects and closed under arbitrary coproducts.

Theorem 2.9. *Let k be a real closed field and let $L = k[i]$ be its algebraic closure. Assume that Conjecture 2.5 holds. Then*

$$c^*_{L/k} : \mathrm{SH}_{C_2} \to \mathrm{SH}_k$$

is a full and faithful embedding.

Proof. Consider the subcategory $\mathcal{C} \subseteq \mathrm{SH}_{C_2}$ whose objects are C_2-equivariant spectra X such that $c^*_{L/k} : [S^n, X]_{C_2} \to [S^n, c^*_{L/k}(X)]_k$ and $c^*_{L/k} : [C_{2+} \wedge S^n, X]_{C_2} \to [C_{2+} \wedge S^n, c^*_{L/k}(X)]_k$ are isomorphisms for all n. This is a localizing subcategory and by Proposition 2.8 it contains $\mathbb{S}_{C_2}$ and we argue below that $C_{2+} \wedge \mathbb{S}_{C_2}$ is in $\mathcal{C}$ as well. This implies that $\mathcal{C} = \mathrm{SH}_{C_2}$, as this is the smallest localizing subcategory containing $\{\mathbb{S}_{C_2}, C_{2+} \wedge \mathbb{S}_{C_2}\}$.

Now we show that $C_{2+} \wedge \mathbb{S}_{C_2}$ is also in $\mathcal{C}$. Since $c^*_{L/k}$ is strong symmetric monoidal and C_{2+} is dualizable, [14, Proposition 3.12] implies that for any C_2-spectrum X, the natural map $c^*_{L/k}(F(C_{2+}, X)) \to F(\mathrm{Spec}(L)_+, c^*_{L/k}(X))$ is an isomorphism in SH_k, where $F(-,-)$ denotes the function spectrum in the corresponding homotopy category. Now C_{2+} is self dual, *i.e.* there is an isomorphism $C_{2+} \cong D(C_{2+})$ in SH_{C_2} where $D(-) = F(-, \mathbb{S}_{C_2})$ denotes the Spanier-Whitehead dual. As with any dualizable object, there is a natural isomorphism $\nu : D(C_{2+}) \wedge X \cong F(C_{2+}, X)$. Combining these isomorphisms yields the isomorphism $\omega : C_{2+} \wedge X \cong F(C_{2+}, X)$ in SH_{C_2}, which is a simple case of the Wirthmüller isomorphism, and $c^*_{L/k}(\omega)$ is an isomorphism $\mathrm{Spec}(L)_+ \wedge c^*_{L/k}(X) \cong F(\mathrm{Spec}(L)_+, c^*_{L/k}(X))$ in SH_k. This isomorphism together with Proposition 2.8 now implies that the maps

(i) $c^*_{L/k} : [S^n, C_{2+} \wedge \mathbb{S}_{C_2}]_{C_2} \to [S^n, \mathrm{Spec}(L)_+ \wedge \mathbb{S}_k]_k$, and
(ii) $c^*_{L/k} : [C_{2+} \wedge S^n, C_{2+} \wedge \mathbb{S}_{C_{2+}}]_{C_2} \to [\mathrm{Spec}(L)_+ \wedge S^n, \mathrm{Spec}(L)_+ \wedge \mathbb{S}_k]_k$

are isomorphisms for any $n \in \mathbb{Z}$.

Now, for any C_2-spectrum X, let $\mathcal{L}_X$ denote the full subcategory of C_2-spectra Y such that $[S^n \wedge Y, X]_{C_2} \to [S^n \wedge c^*_{L/k}(Y), c^*_{L/k}(X)]_k$ is an isomorphism for all $n \in \mathbb{Z}$. It is clear that $\mathcal{L}_X$ is a localizing subcategory of SH_{C_2}. We have seen that $\mathcal{L}_X$ contains both $\mathbb{S}_{C_2}$ and $C_{2+} \wedge \mathbb{S}_{C_2}$. Therefore $\mathcal{L}_X = \mathrm{SH}_{C_2}$. Since X was arbitrary, we have proved that $c^*_{L/k}$ is full and faithful.

□

Write $(\mathrm{SH}_k)^\wedge_p$ for the p-complete stable motivic homotopy category (i.e. the fullsubcategory $(\mathrm{SH}_k)^\wedge_p \subseteq \mathrm{SH}_k$ consisting of spectra X such that $X \to X^\wedge_p$ is an equivalance) and similarly for $(\mathrm{SH}_{C_2})^\wedge_p$. Note that these are triangulated subcategories. Indepedent of the validity of Conjecture 2.5, the argument in the previous theorem yields the embedding theorem for the p-complete homotopy categories.

Theorem 2.10. *Let k be a real closed field and let $L = k[i]$ be its algebraic closure. Then for any prime p*

$$c^*_{L/k} : (\mathrm{SH}_{C_2})^\wedge_p \to (\mathrm{SH}_k)^\wedge_p$$

is a full and faithful embedding.

As mentioned in the introduction, our main theorem has the following corollary on Picard-graded stable homotopy groups.

Corollary 2.11. *Suppose k is real closed and $L = k[i]$ and let S^L denote the unreduced suspension of* $\mathrm{Spec}(L)$*. Then for all $m, n \in \mathbb{Z}$ and any p-complete C_2-spectrum X, the functor $c^*_{L/k}$ induces an isomorphism of Picard-graded stable homotopy groups*

$$\pi_{m+n\sigma}(X) \xrightarrow{\cong} \pi_{m+nL} c^*_{L/k}(X).$$

If Conjecture 2.5 holds then it is an isomorphism for any C_2-spectrum X. In particular, in this case

$$\pi_{m+n\sigma}(\mathbb{S}_{C_2}) \xrightarrow{\cong} \pi_{m+nL}(\mathbb{S}_k).$$

We can also deduce Morel's conjecture on $\pi_1 \mathbb{S}_k$ for k a real closed field. Recall that for a general field k, Morel's conjecture states that there is a short exact sequence

$$0 \to K^M_2(k)/24 \to \pi_1 \mathbb{S}_k \to K^M_1(k)/2 \oplus \mathbb{Z}/2 \to 0 \tag{2.12}$$

in which the map $\pi_1\mathbb{S}_k \to K^M_1(k)/2 \oplus \mathbb{Z}/2$ is induced by the unit map $\mathbb{S}_k \to \mathrm{KO}$ to Hermitian K-theory and $K^M_2(k)/24 \to \pi_1\mathbb{S}_k$ takes symbols $[a, b]$ to $[a, b]\nu$, ν the motivic quaternionic Hopf map.

Corollary 2.13. *If k is real closed and Conjecture 2.5 holds, then $\pi_1\mathbb{S}_k$ sits in the short exact sequence (2.12).*

Proof. By [1], we have $\pi_1\mathbb{S}_{C_2} = (\mathbb{Z}/2)^3$ with basis $\eta_s, [C_2/e]\eta_s, e^2\nu_{C_2}$ where e is represented by the canonical map $S^0 \to S^\sigma$ and ν_{C_2} is the C_2-equivariant quaternionic Hopf map. By Corollary 2.11, there is an abstract isomorphism $\pi_1\mathbb{S}_k \cong (\mathbb{Z}/2)^3$. By [39, Lemma 5.12], the map $\pi_1\mathbb{S}_k \to K^M_1(k)/2 \oplus \mathbb{Z}/2$ is surjective, taking $\langle u\rangle\eta_s$ to $([u], 1)$. It follows that η_s and $\langle -1\rangle\eta_s$ are linearly independent. The C_2-Betti realization of $\rho^2\nu$ is $e^2\nu_{C_2} \neq 0$, and $\nu = 0 \in \pi_{1+2\alpha}\mathrm{KO} = 0$, so $\rho^2\nu$ is nonzero and linearly independent of $\eta_s, \langle -1\rangle\eta_s$. The corollary follows. □

Remark 2.14. If k is real closed, the map $\pi_1\mathbb{S}_{C_2} \to \pi_1\mathbb{S}_k$ is given by

$$\eta_s \mapsto \eta_s, \quad [C_2/e]\eta_s \mapsto \langle 1, -1\rangle\eta_s, \quad e^2\nu_{C_2} \mapsto \rho^2\nu.$$

Finally we note that an equivariant embedding theorem implies a nonequivariant embedding theorem. Thus our result recovers Levine's result [31] after p-completion and it recovers his result without p-completing if Conjecture 2.5 holds.

Corollary 2.15. *Let L/k be a finite Galois extension with Galois group G. If the functor $c^*_{L/k} : \mathrm{SH}_G \to \mathrm{SH}_k$ is full and faithful, then the constant presheaf functor $c^*_{L/L} : \mathrm{SH} \to \mathrm{SH}_L$ is full and faithful as well.*

Proof. Assume that $c^*_{L/k}$ is full and faithful and consider the commutative diagram

$$\begin{array}{ccc}
[G_+ \wedge S^n, X]_G & \xrightarrow[\cong]{c^*_{L/k}} & [c^*_{L/k}(G_+ \wedge S^n), c^*_{L/k}X]_k \\
\cong\downarrow & & \downarrow\cong \\
[S^n, \mathrm{res}X]_e & \xrightarrow[c^*_{L/L}]{} & [c^*S^n, c^*\mathrm{res}X]_L,
\end{array}$$

obtained using Proposition 4.12. The vertical arrows are isomorphisms, and the top horizontal arrow is an isomorphism by assumption. Thus the bottom horizontal arrow is an isomorphism as well. Since every spectrum is the restriction $\mathrm{res}X$ of some G-spectrum X, we can use a density argument as in the proof of Theorem 2.9 to conclude that $c^*_{L/L}$ is full and faithful.

□

3. The trace homomorphism and necessary conditions for full-faithfulness

In this section, we discuss the possibility of $c^*_{L/k}$ being full and faithful for more general Galois extensions L/k. As noted in the introduction, presence of torsion in the Grothendieck-Witt group is the first obvious obstruction to an isomorphism on π_0 and therefore to $c^*_{L/k}$ inducing a full and faithful embedding. However, there are many fields whose Grothendieck-Witt group is torsion-free and we are able to place strong restrictions on which fields k and L can have the property that $c^*_{L/k}$ induces an isomorphism on π_0.

Recall that the classical map $h_{L/k} : A(G) \to GW(k)$ mentioned in the introduction is the unique ring homomorphism with the property that $G/H \in A(G)$ is mapped to $\mathrm{tr}_{L^H/k}(\langle 1\rangle)$. (See [2, §4] for the basic properties of $h_{L/k}$.) The functor $c^*_{L/k} : \mathrm{SH}_G \to \mathrm{SH}_k$ also induces a map $c^*_{L/k} : A(G) \to GW(k)$. The following is essentially a rephrasing of M. Hoyois's [22] computation of the motivic Euler characteristic of a separable field extension.

Proposition 3.1. *The maps $c^*_{L/k} : A(G) \to GW(k)$ and $h_{L/k}$ are equal.*

Proof. The identification $A(G) \cong \mathrm{End}_{\mathrm{SH}_G}(\mathbb{S}_G)$ is given by sending a finite G-set M to its Euler characteristic $\chi(M)$ (for a recollection of Euler characteristics and their properties see, *e.g.*, [34]). The functor $c^*_{L/k}$ is strong symmetric monoidal and so we have that $c^*_{L/k}\chi(G/H) = \chi(c^*_{L/k}(G/H)) = \chi(\mathrm{Spec}(L^H))$ in $\mathrm{End}_{\mathrm{SH}_k}(\mathbb{S}_k)$. But by [22, Theorem 7], under the identification $\mathrm{End}_{\mathrm{SH}_k}(\mathbb{S}_k) \cong GW(k)$, we have $\chi(\mathrm{Spec}(L^H)) = \mathrm{tr}_{L^H/k}(\langle 1\rangle)$. □

A field k is *pythagorean* if and only if sums of squares in k are squares in k. Since $A(G)$ is always torsion free as an abelian group, the importance of pythagorean fields in our context is illustrated by the following lemma.

Lemma 3.2. *The abelian group underlying $GW(k)$ is torsion free if and only if the field k is pythagorean. If k is pythagorean with finitely many orderings, then the free rank of $GW(k)$ is $1 + x(k)$ where $x(k)$ denotes the number of orderings of k.*

Proof. This is a standard enhancement of [30, Theorem VIII.4.1 & Corollary VIII.6.15] from the Witt ring to Grothendieck-Witt ring case. □

We will also need the following lemma in order to analyze $h_{L/k}$.

Lemma 3.3. *If k is pythagorean and $[k^\times : (k^\times)^2] = 2^n$, then*

$$n \le x(k) \le 2^{n-1}.$$

Proof. This is a specialization of [30, Exercise VIII.16]. □

Recall that k is *euclidean* if -1 is not a sum of squares in k and $[k^\times : (k^\times)^2] = 2$.

Theorem 3.4. *The map $h_{L/k}$ is an isomorphism if and only if either k is quadratically closed and $L = k$, or k is euclidean and $L = k[i]$.*

Proof. If L/k is of one of the prescribed forms, then it is elementary that $h_{L/k}$ is an isomorphism.

If $h_{L/k}$ is an isomorphism, then $GW(k)$ must be torsion free, in which case Lemma 3.2 implies that k is pythagorean. If k is pythagorean and nonreal (*i.e.*, -1 is a sum of squares in k), then k is quadratically closed and $GW(k) \cong \mathbb{Z}$. Thus $A(G)$ has rank 1 and therefore $G = \{e\}$ and $L = k$.

Now assume that k is pythagorean and formally real (so -1 is not a sum of squares in k). By the construction in [2, §4], we know that h factors through the group completion of the monoid of k-quadratic forms q such that $q_L \cong n\langle 1\rangle$ for some natural number n; call this group $GW_L^{\mathbb{Z}}(k)$. Since h is an isomorphism, $GW_L^{\mathbb{Z}}(k) = GW(k)$, whence $\langle a\rangle_L = \langle 1\rangle$ in $GW(L)$ for all $a \in k^\times$. It follows that k is quadratically closed in L.

Choose a basis $\{x_1, x_2, \dots\}$ of $k^\times/(k^\times)^2$ and let $E = k(\sqrt{x_1}, \sqrt{x_2}, \dots)$. We have just proven that E/k is a subextension of L/k, whence G surjects onto $\mathrm{Gal}(E/k)$. Since G is finite, k must have finitely many square classes and $\mathrm{Gal}(E/k) \cong C_2^n$. Recall that the rank of $A(G)$ is the number of conjugacy classes of subgroups of G. We deduce that $\operatorname{rk} A(G) \geq \operatorname{rk} A(C_2^n)$. Just counting the subgroups of C_2^n of order 1 or 2, we find that $\operatorname{rk} A(C_2^n) \geq 2^n$. Since $2^n > 1 + 2^{n-1}$ for $n > 2$, Lemma 3.3 implies that $n = 0$ or 1. Since k is formally real, we can exclude the case $n = 0$, whence k is formally real pythagorean with $[k^\times : (k^\times)^2] = 2$, *i.e.*, k is euclidean. In this case $GW(k)$ has rank 2, so L/k is a quadratic extension. Since k is quadratically closed in L, $L = k[i]$, concluding the proof. □

Corollary 3.5. *If $c^*_{L/k}$ is full and faithful, then k is of the form described in Theorem 3.4.*

Remark 3.6. Algebraically closed and real closed fields are special examples of quadratically closed and euclidean fields, but there are many other examples of these kinds of fields. For instance, the field of real constructible numbers $\widetilde{\mathbb{Q}} \cap \mathbb{R}$ (where $\widetilde{\mathbb{Q}}$ is the quadratic closure of $\mathbb{Q}$) is euclidean but not real closed.

The necessary conditions which we deduced in the previous result were obtained only by analyzing the zeroth homotopy group of the sphere spectrum. The authors expect that torsion phenomena in the higher homotopy groups of $\mathbb{S}_k$ will preclude $c^*_{L/k}$ from being full and faithful unless k is algebraically or real closed.

Conjecture 3.7. *Let L be a field of characteristic zero. The functor $c^*_{L/L} : \mathrm{SH} \to \mathrm{SH}_L$ is full and faithful if and only if L is algebraically closed.*

By Levine's theorem [31] the "if" portion of this conjecture is valid. Observe that the validity of this conjecture together with Corollary 2.15, would imply that $c_{L/k} : \mathrm{SH}_G \to \mathrm{SH}_k$ is full and faithful if and only if $k = L$ is algebraically closed or k is real closed and $L = k[i]$. It is also interesting to ask what happens in positive characteristic.

4. Comparison functors

In this section we construct and analyze the various comparison functors between stable homotopy categories used in our arguments. To avoid potential confusion concerning notation, we point out that a functor on homotopy categories written as the derived functor $\mathbb{L}F$ (or $\mathbb{R}F$) of some functor on model categories in this section would be written simply F in previous sections.

4.1. **Motivic model structures.** Given a base scheme S, the category $\mathrm{Spc}_\bullet(S)$ of based motivic spaces is the category of based simplicial presheaves on Sm/S. There are many different options for a motivic model structure on $\mathrm{Spc}_\bullet(S)$. We will use the so-called closed flasque motivic model structure introduced in [40]. We recall the basic definitions below and refer to *loc. cit.* for full details. The main advantages of this model structure for the present work are that in this model structure all of the standard motivic spheres are cofibrant and all of the various change of base functors as well as the (equivariant) Betti realizations are Quillen functors.

The closed flasque motivic model structure is a Bousfield localization of the global closed flasque model structure. The weak equivalences of the global closed flasque model structure are the schemewise weak equivalences of motivic spaces. A global closed flasque fibration is a map which has the right lifting property with respect to the set J^{gf} defined below. A global closed flasque cofibration is then defined by the appropriate lifting property. This model structure has sets of generating cofibrations I^{gf} and generating acyclic cofibrations J^{gf} as follows. Let $\mathcal{Z} = \{Z_i \to X\}$ be a finite (possibly empty) collection of monomorphisms in Sm/S. Write $\cup\mathcal{Z}$ for the categorical union (i.e. union as presheaves) of the Z_i and write $f : \cup\mathcal{Z} \to X$ for the induced map. Given two maps α and β write $\alpha\Box\beta$ for their pushout product.

(1) The set I^{gf} consists of all maps of the form $f_+\Box\, g_+$ where $f : \cup\mathcal{Z} \to X$ is as above and $g : \partial\Delta^n \to \Delta^n$ is a generating cofibration of simplicial sets.
(2) The set J^{gf} consists of all morphisms of the form $f_+\Box\, h_+$, where $f : \cup\mathcal{Z} \to X$ is as above and $h : \Lambda^n_j \to \Delta^n$ is a generating acyclic cofibration.

The global closed flasque model structure is a proper, cellular, simplicial model structure. Write $\mathrm{Spc}_\bullet(S)_{gf}$ for the category of motivic spaces equipped with the global model structure. Let

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ A & \xrightarrow{\ i\ } & X. \end{array}$$

be a distinguished Nisnevich square, i.e. p is an étale map of smooth schemes, i is an open immersion, and $p^{-1}(X \smallsetminus A)_{red} \to (X \smallsetminus A)_{red}$ is an isomorphism. Write $\mathcal{Q} = \mathcal{Q}(i,p)$ for this distinguished square and write $P_\mathcal{Q}$ for the homotopy pushout in $\mathrm{Spc}_\bullet(S)_{gf}$ of A and Y along B, and write $P_\mathcal{Q} \to X$ for the resulting map. The *motivic closed flasque model structure* is the left Bousfield localization of the global model structure at the set of maps

$$S = \{P_\mathcal{Q} \to X\} \cup \{W \times \mathbb{A}^1 \to W\}$$

where X, W range over all smooth S-schemes and $\mathcal{Q}$ ranges over all distinguished squares.

4.2. **Stable model structures.** The construction of the functor $c^*_{L/k} : \mathrm{SH}_G \to \mathrm{SH}_k$ will involve a nonstandard model for the stable motivic homotopy category. We rely on [20] as needed to equip various categories of spectra (and bispectra) with stable model structures. Recall that if $\mathcal{C}$ is a left proper cellular symmetric monoidal model category whose generating cofibrations have cofibrant domain and K is a cofibrant object of $\mathcal{C}$, then Hovey equips the category $\mathrm{Spt}^\Sigma_K(\mathcal{C})$ of symmetric K-spectra with a stable model structure and it is again a left proper cellular symmetric monoidal model category [20]. Note that $\mathrm{Spc}_\bullet(S)$ satisfies these assumptions and moreover the motivic spheres $\mathbb{P}^1$ (based at ∞), $\mathbb{A}^1/\mathbb{A}^1 \smallsetminus \{0\}$, and $S^\alpha := \mathbb{A}^1 \smallsetminus \{0\}$ (based at 1) are all closed flasque cofibrant.

Let J be a closed flasque cofibrant motivic space over S. We will simply write $\mathrm{Spt}^\Sigma_J(S) := \mathrm{Spt}^\Sigma_J(\mathrm{Spc}_\bullet(S))$ for the category of motivic J-spectra. If J' is another closed flasque cofibrant motivic space we write $\mathrm{Spt}^\Sigma_{J,J'}(F) := \mathrm{Spt}^\Sigma_{J'}(\mathrm{Spt}^\Sigma_J(F))$ for the category of motivic (J,J')-bispectra. As shown in [40] there is a monoidal Quillen equivalence between $\mathrm{Spt}^\Sigma_{\mathbb{P}^1}(S)$ and Jardine's model category of motivic symmetric $\mathbb{P}^1$-spectra [28].

In [20], functoriality of the model categories of symmetric spectra is discussed when $\mathcal{C}$ is fixed (*e.g.* changing the suspension object K in $\mathcal{C}$ or varying the $\mathcal{C}$-model category). We will need slightly more general functoriality, which we record before continuing with the construction of the comparison functors of interest to this paper.

Suppose that $\mathcal{D}$ is another model category satisfying the same hypothesis as $\mathcal{C}$ and K' is a cofibrant object of $\mathcal{D}$. Further suppose that we are given the following:

(1) a Quillen adjoint pair $\Phi : \mathcal{C} \rightleftarrows \mathcal{D} : \Psi$, and
(2) a natural isomorphism $\tau : \Phi(-) \otimes K' \xrightarrow{\cong} \Phi(- \otimes K)$ such that the iterated isomorphisms $\tau^p : \Phi(X) \otimes (K')^{\otimes p} \cong \Phi(X \otimes K^{\otimes p})$ are Σ_p-equivariant, where the actions are the obvious ones given by permuting the respective factors of K and K'.

As seen in the next lemma, (Φ, Ψ) prolong to a Quillen pair $(\mathrm{Sp}(\Phi), \mathrm{Sp}(\Psi))$ of stable model categories of symmetric spectra. In this situation, we usually write Φ and Ψ instead of $\mathrm{Sp}(\Phi)$ and $\mathrm{Sp}(\Psi)$ for the prolongations.

Lemma 4.1. *With notations and assumptions as above, the pair (Φ, Ψ) prolongs to a Quillen adjoint pair on stable model structures*

$$\mathrm{Sp}(\Phi) : \mathrm{Spt}^{\Sigma}_{K}(\mathcal{C}) \rightleftarrows \mathrm{Spt}^{\Sigma}_{K'}(\mathcal{D}) : \mathrm{Sp}(\Psi).$$

If Φ is strong symmetric monoidal then so is $\mathrm{Sp}(\Phi)$.

Proof. Define $\mathrm{Sp}(\Phi)(D)$ by $\mathrm{Sp}(\Phi)(D)_n := \Phi(D_n)$ with structure maps

$$\mathrm{Sp}(\Phi)(D)_n = \Phi(D_n) \otimes K' \cong \Phi(D_n \otimes K) \to \Phi(D_{n+1}) = \mathrm{Sp}(\Phi)(D)_{n+1}.$$

The equivariance assumption on τ implies that the iterations of the structure map $\mathrm{Sp}(\Phi)(D)_n \otimes (K')^{\otimes p} \to \mathrm{Sp}(\Phi)(D)_{n+p}$ are $\Sigma_n \times \Sigma_p$-equivariant and so $\Phi(D)$ is a symmetric K'-spectrum. Define $\mathrm{Sp}(\Phi)$ on morphisms in the obvious way.

Note that τ determines the natural isomorphism $\rho : \Psi\Omega_{K'}(-) \xrightarrow{\cong} \Omega_K\Psi(-)$ and the iterations ρ^p are Σ_p-equivariant. Now define $\mathrm{Sp}(\Psi)(E)$ by setting $\mathrm{Sp}(\Psi)(E)_n := \Psi(E_n)$. The structure maps are defined as the adjoints of

$$\mathrm{Sp}(\Psi)(E)_n = \Psi(E_n) \to \Psi(\Omega_{K'}E_{n+1}) \cong \Omega_K\Psi(E_{n+1}) = \Omega_K\mathrm{Sp}(\Psi)(E)_{n+1}.$$

The equivariance of ρ implies that this is a symmetric K-spectrum. Define $\mathrm{Sp}(\Psi)$ on morphisms in the obvious way.

It is straightforward to verify that $\mathrm{Sp}(\Phi)$ and $\mathrm{Sp}(\Psi)$ are adjoint. The functor $\mathrm{Sp}(\Psi)$ preserves level equivalences and level fibrations. This implies $\mathrm{Sp}(\Phi)$ preserves stable cofibrations and $\mathrm{Sp}(\Psi)$ preserves fibrations between fibrant objects in the stable model structure. It follows from [9, Lemma A.2] that $(\mathrm{Sp}(\Phi), \mathrm{Sp}(\Psi))$ is a Quillen adjoint pair on the stable model structures.

It is immediate that $\mathrm{Sp}(\Phi)$ is symmetric monoidal whenever Φ is. □

4.3. **Galois correspondence.** Let L/k be a finite Galois extension with Galois group G. Define the functor

$$c_{L/k} : \mathrm{Or}_G \to \mathrm{Sm}/k, \tag{4.2}$$

by $c_{L/k}(G/H) = \mathrm{Spec}(L^H)$ on objects and on maps as follows. First recall that

$$\mathrm{Hom}_{\mathrm{Or}_G}(G/H, G/H') = \{gH' \mid g^{-1}Hg \subseteq H'\}.$$

A straightforward check shows that if gH' is such a coset then the corresponding field automorphism $g : L \to L$ restricts to a map of fields $g : L^{H'} \to L^H$ which depends only on the coset gH'. This defines the desired map $c_{L/k}(G/H) \to c_{L/k}(G/H')$.

The category of G-simplicial sets is equivalent to the category of presheaves of simplicial sets on Or_G: the presheaf corresponding to A is given by $G/H \mapsto A^H$. We thus obtain an adjoint pair of functors

$$c^*_{L/k} : G\mathrm{sSet}_\bullet \rightleftarrows \mathrm{Spc}_\bullet(k) : (c_{L/k})_*. \tag{4.3}$$

Remark 4.4. For a G-simplicial set A, the corresponding motivic space $c^*_{L/k}(A)$ isn't in general constant but its possible values are limited to the various fixed points A^H for subgroups $H \subseteq G$. To see this, it suffices to consider the case of a G-set. Every G-set is the disjoint union of orbits and we write this decomposition as $A = \coprod_{\mathrm{Or}_G} \coprod_{A\langle G/H\rangle} G/H$. Then $c^*_{L/k}A$ is the motivic space defined by

$$(c^*_{L/k}A)(X) := \coprod_{\mathrm{Or}_G} \coprod_{A\langle G/H\rangle} \mathrm{Hom}_k(X, \mathrm{Spec}(L^H)).$$

Note that if X is connected, then $\mathrm{Hom}_k(X, \mathrm{Spec}(L^H))$ is either empty or is a set with $|G/H|$ elements and so $c^*_{L/k}(A)(X) = A^H$ for an appropriate subgroup $H \subseteq G$.

Lemma 4.5. *The adjoint pair $c^*_{L/k} : G\mathrm{sSet}_\bullet \rightleftarrows \mathrm{Spc}_\bullet(k) : (c_{L/k})_*$ is a Quillen adjoint pair. Moreover, the induced map on homotopy categories $\mathbb{L}c^*_{L/k} : \mathrm{H}_{\bullet,G} \to \mathrm{H}_{\bullet,k}$ is full and faithful.*

Proof. Note that under the identification $\mathrm{sPre}_\bullet(\mathrm{Or}_G) = G\mathrm{sSet}_\bullet$, the projective model structure on simplicial presheaves corresponds to the usual model structure on based G-simplicial sets. The functor $(c_{L/k})_*$ preserves global weak equivalences and global fibrations and so this pair is a Quillen pair on the global closed flasque model structure. It follows immediately that this is a Quillen pair on the motivic model structure as well.

Using the description of $c^*_{L/k}$ in the previous remark, one sees the following simple facts about $c^*_{L/k}(A)$. If A is fibrant then $c^*_{L/k}(A)(X)$ is fibrant for any X and $c^*_{L/k}(A)$ is $\mathbb{A}^1$-homotopy invariant. If $U \subseteq X$ is a dense open subscheme, then $c^*_{L/k}(A)(X) = c^*_{L/k}(U)$. It is thus easy to see that $c^*_{L/k}(A)$ satisfies Nisnevich descent. Moreover, for G-simplicial sets A and B, we have an equality of simplicial mapping spaces, $\underline{\mathrm{Hom}}_{\mathrm{Spc}_\bullet(k)}(c^*_{L/k}(B), c^*_{L/k}(A)) = \underline{\mathrm{Hom}}_{G\mathrm{sSet}_\bullet}(B, A)$. Now if B is cofibrant and A is fibrant, then we have

$$[S^n \wedge c^*_{L/k}(B), c^*_{L/k}(A)]_k = \pi_n \underline{\mathrm{Hom}}_{\mathrm{Spc}_\bullet(k)}(c^*_{L/k}(B), c^*_{L/k}(A))$$

and $[B, A]_G = \pi_n \underline{\mathrm{Hom}}_{G\mathrm{sSet}_\bullet}(B, A)$ from which the second statement follows. □

Write $S^G = (S^1)^{\wedge G}$ for the G-simplicial set consisting of the $|G|$-fold smash product of S^1 equipped with the obvious permutation action by G. Note also that this is the simplicial representation sphere associated to the regular representation of G. The stable model structure on $\mathrm{Spt}^\Sigma_{S^G}(G) := \mathrm{Spt}^\Sigma_{S^G}(G\mathrm{sSet}_\bullet)$ obtained from [20] agrees with that constructed in [33]. In turn, as shown in *loc. cit.*, the associated homotopy category is tensor triangulated equivalent to the genuine G-equivariant homotopy category as constructed in [32].

To simplify notation below, we sometimes denote the motivic space $c^*_{L/k}(S^G)$ by S^G. Consider the category $\mathrm{Spt}^\Sigma_{S^G,\mathbb{P}^1}(k)$ of motivic $(c^*_{L/k}(S^G), \mathbb{P}^1_k)$-bispectra. This is a model for the stable motivic homotopy category SH_k. Indeed, by [23, Theorem 3.5] the motivic space $c^*_{L/k}(S^G)$ is invertible in SH_k. In particular, by [20, Theorem 9.1], the suspension spectrum functor

$$\Sigma^\infty_{S^G} : \mathrm{Spt}^\Sigma_{\mathbb{P}^1}(k) \to \mathrm{Spt}^\Sigma_{\mathbb{P}^1,S^G}(k)$$

is a left Quillen equivalence and induces a tensor triangulated equivalence on the associated stable homotopy categories.

By Lemma 4.1, the Quillen adjoint pair (4.3) induces a Quillen pair $\mathrm{Spt}^\Sigma_{S^G}(G) \rightleftarrows \mathrm{Spt}^\Sigma_{S^G}(k)$. Combined with the suspension spectrum functor, we have the composite Quillen adjunction

$$\mathrm{Spt}^\Sigma_{S^G}(G) \rightleftarrows \mathrm{Spt}^\Sigma_{S^G}(k) \rightleftarrows \mathrm{Spt}^\Sigma_{S^G,\mathbb{P}^1}(k).$$

We have thus obtained the desired stabilization of $c^*_{L/k}$.

Theorem 4.6. *The Galois correspondence (4.2) induces an adjoint pair*

$$\mathbb{L}c^*_{L/k} : \mathrm{SH}_G \rightleftarrows \mathrm{SH}_k : \mathbb{R}(c_{L/k})_*$$

of triangulated stable homotopy categories. The left adjoint is strong symmetric monoidal.

4.4. **Equivariant Betti realization.** An unstable C_2-equivariant Betti realization functor is constructed for the motivic homotopy category over fields admitting a real embedding in [37], see also [11]. It is well known that this construction stabilizes to yield a C_2-equivariant Betti realization functor. Following the construction of [40] in the complex case, we record here the construction of the stable equivariant Betti realization as a Quillen functor.

Write $(-)^{an} : \mathrm{Sm}/\mathbb{R} \to C_2\mathrm{Top}_\bullet$ for the functor given by $X \mapsto X(\mathbb{C})^{an}_+$, where $X(\mathbb{C})$ is equipped with the involution given by conjugation. It extends to an adjoint pair

$$\mathrm{Re}^{C_2}_B : \mathrm{Spc}_\bullet(\mathbb{R}) \rightleftarrows C_2\mathrm{Top}_\bullet : \mathrm{Sing}^{C_2}_B .$$

The left adjoint $\mathrm{Re}^{C_2}_B$ is defined by the usual left Kan extension formula and the right adjoint $\mathrm{Sing}^{C_2}_B$ is defined by $\mathrm{Sing}^{C_2}_B(K)(X) = \underline{\mathrm{Hom}}_{C_2\mathrm{Top}_\bullet}(X(\mathbb{C})_+, K)$.

Proposition 4.7. *The adjoint pair* $\mathrm{Re}^{C_2}_B : \mathrm{Spc}_\bullet(\mathbb{R}) \rightleftarrows C_2\mathrm{Top}_\bullet : \mathrm{Sing}^{C_2}_B$ *is a Quillen adjoint pair. Moreover* $\mathrm{Re}^{C_2}_B$ *is strong symmetric monoidal.*

Proof. First we show that this is a Quillen pair on global closed flasque model structures. For this we check that $\mathrm{Re}^{C_2}_B$ sends generating closed cofibrations to cofibrations in $C_2\mathrm{Top}_\bullet$ and sends generating global trivial closed fibrations to trivial cofibrations. Note that $\mathrm{Re}^{C_2}_B$ preserves pushout products. It thus suffices to show that $\mathrm{Re}^{C_2}_B(\cup\mathcal{Z}_+) \to \mathrm{Re}^{C_2}_B(X_+)$ is a cofibration for any finite collection $\mathcal{Z} = \{Z_i \hookrightarrow X\}$ of closed immersions in $\mathrm{Sm}/\mathbb{R}$.

Note that $\mathrm{Re}^{C_2}_B(\cup\mathcal{Z})$ is the coequalizer of $\coprod Z_i(\mathbb{C}) \times_{X(\mathbb{C})} Z_j(\mathbb{C}) \rightrightarrows \coprod Z_i(\mathbb{C})$ in in $C_2\mathrm{Top}_\bullet$. One may equivariantly triangulate $X(\mathbb{C})$ such that each $Z_i(\mathbb{C})$ is an equivariant subcomplex and $Z_i(\mathbb{C}) \times_{X(\mathbb{C})} Z_j(\mathbb{C})$ is an equivariant subcomplex for each j, see, *e.g.*, [27]. It follows that $\mathrm{Re}^{C_2}_B(\cup\mathcal{Z}) \to X(\mathbb{C})$ is the inclusion of an equivariant subcomplex. In particular, it is an equivariant cofibration. It follows that $\mathrm{Re}^{C_2}_B$ is a left Quillen functor on the global closed flasque model structure.

Note that $\mathrm{Re}^{C_2}_B$ sends a distinguished Nisnevich square to an equivariant homotopy pushout square, see, *e.g.*, [11]. Also $\mathrm{Re}^{C_2}_B(X \times \mathbb{A}^1) \to \mathrm{Re}^{C_2}_B(X)$ is an equivariant homotopy equivalence. It follows that the adjoint pair of the proposition induces a Quillen pair in the closed flasque motivic structure as well. □

Recall that we write S^σ for the sign representation sphere.

Proposition 4.8. *The above adjoint pair extends to a Quillen adjoint pair*

$$\mathrm{Re}^{C_2}_B : \mathrm{Spt}^{\Sigma}_{\mathbb{P}^1}(\mathbb{R}) \rightleftarrows \mathrm{Spt}^{\Sigma}_{S^{1+\sigma}}(C_2) : \mathrm{Sing}^{C_2}_B$$

on stable model categories. Moreover $\mathrm{Re}^{C_2}_B$ *is strong symmetric monoidal.*

Proof. This follows immediately from Lemma 4.1, noting that $\mathrm{Re}^{C_2}_B(\mathbb{P}^1) = S^{1+\sigma}$. □

Now if k is a field and $\phi : k \hookrightarrow \mathbb{R}$ is a real embedding then the associated C_2-equivariant Betti realization $\mathrm{Re}^{C_2}_{B,\phi}$ is defined to be the composite

$$\mathrm{Re}^{C_2}_{B,\phi} := \phi^* \circ \mathrm{Re}^{C_2}_B : \mathrm{SH}_k \to \mathrm{SH}_{\mathbb{R}} \to \mathrm{SH}_{C_2}.$$

4.5. **Comparing change of group and change of base functors.** It is useful to know that the comparison functors between equivariant and motivic homotopy theory suitably intertwine the standard change of group and change of base functors. We fix as above a Galois extension L/k with Galois group G. Let $H \subseteq G$ be a subgroup and write $K = L^H$ for the corresponding fixed subfield. We denote the corresponding map of schemes by $p : \mathrm{Spec}(K) \to \mathrm{Spec}(k)$. As with any map of schemes we have an induced adjoint pair of functors of motivic spaces $p^* : \mathrm{Spc}_\bullet(k) \rightleftarrows \mathrm{Spc}_\bullet(K) : p_*$. Since p is smooth, the functor p^* has as well a left adjoint $p_\#$, induced by the functor $\mathrm{Sm}/K \to \mathrm{Sm}/k$ which composes the structure map of a K-scheme with p.

Lemma 4.9. *The adjoint pairs* $(p_\#, p^*)$ *and* (p^*, p_*) *are Quillen adjoint pairs.*

Proof. That p^* is a left Quillen adjoint on the motivic closed flasque model structure is verified in [40]. Note that $p_\#$ preserves generating global closed flasque cofibrations and acyclic cofibrations. This is seen by noting that if M is a simplicial set, then we have that $p_\#(X \wedge M_+) = (p_\# X) \wedge M_+$ and since $p_\#$ preserves colimits, it preserves pushouts and since it also preserves closed inclusions of smooth schemes, the claim follows. This implies that $p_\#$ is a left Quillen functor on global closed flasque model structures. The functor $p_\#$ sends Nisnevich distinguished squares to Nisnevich distinguished squares. Furthermore $p_\#(X \times_K \mathbb{A}^1_K) \to p_\#(X)$ is identified with $p_\#(X) \times_k \mathbb{A}^1_k \to p_\#(X)$. It follows that $p_\#$ is also a left Quillen functor on the closed flasque motivic model structure. □

We have the commutative diagram of categories

$$\begin{array}{ccc} \mathrm{Or}_H & \xrightarrow{c_{L/K}} & \mathrm{Sm}/K \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle p_\#} \\ \mathrm{Or}_G & \xrightarrow{c_{L/k}} & \mathrm{Sm}/k, \end{array}$$

where j sends the orbit H/H' to the orbit G/H'. Under the identification $\mathrm{sPre}_\bullet(\mathrm{Or}_G) = G\mathrm{sSet}_\bullet$, the adjoint pair (j^*, j_*) is identified with the adjoint pair $(\mathrm{ind}^G_H, \mathrm{res}^G_H)$ where $\mathrm{ind}^G_H(X) = G \times_H X$ and $\mathrm{res}^G_H(W)$ is W with H-action given by restricting the G-action. The above square thus induces a commutative diagram of Quillen adjoint functors (where we omit the labels for the horizontal right adjoints for typographical reasons)

$$\begin{array}{ccc} H\mathrm{sSet}_\bullet & \underset{\longleftarrow}{\overset{c^*_{L/K}}{\longrightarrow}} & \mathrm{Spc}_\bullet(K) \\ {\scriptstyle \mathrm{ind}^G_H}\downarrow\uparrow{\scriptstyle \mathrm{res}^G_H} & & {\scriptstyle p_\#}\downarrow\uparrow{\scriptstyle p^*} \\ G\mathrm{sSet}_\bullet & \underset{\longleftarrow}{\overset{c^*_{L/k}}{\longrightarrow}} & \mathrm{Spc}_\bullet(k). \end{array} \tag{4.10}$$

We write $\mathrm{H}_{\bullet,G}$ for the homotopy category of based G-spaces and $\mathrm{H}_{\bullet,G}$ for the unstable motivic homotopy category.

Proposition 4.11. *The diagrams of homotopy categories*

$$\begin{array}{ccc} \mathrm{H}_{\bullet,G} & \xrightarrow{\mathbb{L}c^*_{L/k}} & \mathrm{H}_{\bullet,k} \\ {\scriptstyle \mathbb{R}\mathrm{res}^G_H}\downarrow & & \downarrow{\scriptstyle \mathbb{R}p^*} \\ \mathrm{H}_{\bullet,H} & \xrightarrow{\mathbb{L}c^*_{L/K}} & \mathrm{H}_{\bullet,K} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{H}_{\bullet,H} & \xrightarrow{\mathbb{L}c^*_{L/K}} & \mathrm{H}_{\bullet,K} \\ {\scriptstyle \mathbb{L}\mathrm{ind}^G_H}\downarrow & & \downarrow{\scriptstyle \mathbb{L}p_\#} \\ \mathrm{H}_{\bullet,G} & \xrightarrow{\mathbb{L}c^*_{L/k}} & \mathrm{H}_{\bullet,k}. \end{array}$$

induced by (4.10), commute up to natural isomorphism.

Proof. The commutativity of the second diagram follows immediately from the fact that (4.10) commutes and that the adjoint pairs there are Quillen pairs. A direct inspection yields the equality of functors $p^*c^*_{L/k} = c^*_{L/K}\mathrm{res}^G_H$. The commutativity of the first diagram follows since p^* and res^G_H are also left Quillen functors and so $\mathbb{R}p^* \simeq \mathbb{L}p^*$ and $\mathbb{R}\mathrm{res}^G_H \simeq \mathbb{L}\mathrm{res}^G_H$. □

As an H-simplicial set S^G is isomorphic to the $[G:H]$-fold smash product of S^H. This implies that $c^*_{L/K}(S^G) = c^*_{L/k}(S^H)^{\wedge[G:H]}$. We set $d := [G:H]$ and write $S^{dH} = (S^H)^{\wedge d}$ below.

Proposition 4.12. *The adjoint pairs (4.10) induce diagrams of stable homotopy categories*

$$\begin{array}{ccc} \mathrm{SH}_G & \xrightarrow{\mathbb{L}c^*_{L/k}} & \mathrm{SH}_k \\ {\scriptstyle \mathbb{R}\mathrm{res}^G_H}\downarrow & & \downarrow{\scriptstyle \mathbb{R}p^*} \\ \mathrm{SH}_H & \xrightarrow{\mathbb{L}c^*_{L/K}} & \mathrm{SH}_K \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{SH}_H & \xrightarrow{\mathbb{L}c^*_{L/K}} & \mathrm{SH}_K \\ {\scriptstyle \mathbb{L}\mathrm{ind}^G_H}\downarrow & & \downarrow{\scriptstyle \mathbb{L}p_\#} \\ \mathrm{SH}_G & \xrightarrow{\mathbb{L}c^*_{L/k}} & \mathrm{SH}_k \end{array}$$

which commute up to natural isomorphism.

Proof. We have a diagram of model categories and Quillen adjunctions between them

$$\begin{array}{ccccc} \mathrm{Spt}^\Sigma_{S^G}(H) & \overset{c^*_{L/K}}{\rightleftarrows} & \mathrm{Spt}^\Sigma_{S^{dH}}(K) & \overset{\Sigma^\infty_{\mathbb{P}^1}}{\rightleftarrows} & \mathrm{Spt}^\Sigma_{S^{dH},\mathbb{P}^1}(K) \\ {\scriptstyle \mathrm{ind}^G_H}\downarrow\uparrow{\scriptstyle \mathrm{res}^G_H} & & {\scriptstyle p_\#}\downarrow\uparrow{\scriptstyle p^*} & & {\scriptstyle p_\#}\downarrow\uparrow{\scriptstyle p^*} \\ \mathrm{Spt}^\Sigma_{S^G}(G) & \overset{c^*_{L/k}}{\rightleftarrows} & \mathrm{Spt}^\Sigma_{S^G}(k) & \overset{\Sigma^\infty_{\mathbb{P}^1}}{\rightleftarrows} & \mathrm{Spt}^\Sigma_{S^G,\mathbb{P}^1}(k). \end{array}$$

This diagram is commutative and the derived functors of the left adjoints give the functors in the diagrams. The commutativity of the second diagram follows immediately.

For the commutativity of the first square, note that the right adjoints p^* and res^G_H are also left adjoints and the stabilization of these functors considered as a left adjoint agrees with their stabilization as a right adjoint and these are also left Quillen functors. It follows that $\mathbb{R}p^* = \mathbb{L}p^*$ and $\mathbb{R}\mathrm{res}^G_H = \mathbb{L}\mathrm{res}^G_H$. The desired commutativity thus follows from the underived equality $\Sigma^\infty_{\mathbb{P}^1}c^*_{L/K}\mathrm{res}^G_H = p^*\Sigma^\infty_{\mathbb{P}^1}c^*_{L/k}$. □

Now suppose that k is formally real and consider the embedding $p : k \subseteq k[i]$. A real embedding $\phi : k \hookrightarrow \mathbb{R}$ induces a complex embedding $\psi : k[i] \hookrightarrow \mathbb{C}$ and hence an associated Betti realization $\mathrm{Re}_{B,\psi} = \psi^*\mathrm{Re}_B : \mathrm{SH}_{k[i]} \to \mathrm{SH}$.

Proposition 4.13. *With the notations as above we have*

$$\mathbb{R}\mathrm{res}^{C_2}_{\{e\}}\mathbb{L}\mathrm{Re}^{C_2}_{B,\phi} = \mathbb{L}\mathrm{Re}_{B,\psi}\mathbb{R}p^* \quad \text{and} \quad \mathbb{L}\mathrm{Re}^{C_2}_{B,\phi}\mathbb{L}p_\# = \mathbb{L}\mathrm{ind}^{C_2}_{\{e\}}\mathbb{L}\mathrm{Re}_{B,\psi}.$$

Proof. This is a straightforward consequence of the definitions and constructions, as in the previous proposition. □

4.6. **Betti realization and motivic cohomology.** We now turn our attention to the equivariant Betti realization of the motivic cohomology spectrum. Following a similar strategy as in [31] in the nonequivariant case, we show that the equivariant Betti realization takes the motivic cohomology spectrum $\mathrm{H}\mathbb{Z}$ to the Bredon cohomology spectrum $\mathrm{H}\underline{\mathbb{Z}}$. We then reinterpret the Beilinson-Lichtenbaum conjectures and establish an equivariant version of Suslin-Voevodsky's theorem [42] on Suslin homology.

Lemma 4.14. *For any X in Sch/k, the natural map $\mathbb{L}\mathrm{Re}^{C_2}_{B,\phi}(\Sigma^\infty_{\mathbb{P}^1}X_+) \to \Sigma^\infty_{S^{1+\sigma}}X(\mathbb{C})^{an}_+$ is an isomorphism in SH_{C_2}.*

Proof. Since k admits resolution of singularities there is a proper *cdh* hypercover $X_\bullet \to X$ such that each X_n is a smooth k-scheme. It follows from [47] that $|\Sigma^\infty_{\mathbb{P}^1}X_{\bullet\,+}| \to \Sigma^\infty_{\mathbb{P}^1}X_+$ is a stable equivalence in SH_k. Each $X_{n\,+}$ is cofibrant. It follows that we have a natural isomorphism $\mathbb{L}\mathrm{Re}^{C_2}_{B,\phi}\Sigma^\infty_{\mathbb{P}^1}X_+ \cong |\Sigma^\infty_{S^{1+\sigma}}X(\mathbb{C})_{\bullet\,+}|$ in SH_{C_2}.

To see that $|\Sigma^\infty_{S^{1+\sigma}}X(\mathbb{C})^{an}_{\bullet\,+}| \to \Sigma^\infty_{S^{1+\sigma}}X(\mathbb{C})^{an}_+$ is an isomorphism in SH_{C_2} it suffices to check that this map induces an isomorphism in SH after applying the geometric fixed points functors Φ^{C_2} and Φ^e. Recall that in general we have that the geometric fixed points of a suspension spectrum is given by the suspension spectrum of the fixed points: $\Phi^H\Sigma^\infty_{S^G}Y = \Sigma^\infty Y^H$. Therefore we have that the C_2-geometric fixed points of the above map is $|\Sigma^\infty X(\mathbb{R})^{an}_{\bullet\,+}| \to \Sigma^\infty X(\mathbb{R})^{an}_+$. If $W \to Y$ is a proper *cdh*-cover of real varieties then $W(\mathbb{R})^{an} \to Y(\mathbb{R})^{an}$ is a surjective proper map. In particular, it is a map of universal cohomological descent [5, 5.3.5]. It follows that $H^*(|X(\mathbb{R})^{an}_{\bullet\,+}|, A) \to H^*(X(\mathbb{R})^{an}_+, A)$ is an isomorphism for all abelian groups A. In particular, $|X(\mathbb{R})^{an}_{\bullet\,+}| \to X(\mathbb{R})^{an}_+$ induces a stable equivalence on suspension spectra. A similar analysis for the e-geometric fixed points shows that $|\Sigma^\infty X(\mathbb{C})^{an}_{\bullet\,+}| \to \Sigma^\infty X(\mathbb{C})^{an}_+$ is a stable equivalence as well. □

Lemma 4.15. *The natural map*

$$\mathbb{L}\mathrm{Re}^{C_2}_{B,\phi}(\Sigma^\infty_{\mathbb{P}^1}\operatorname{Sym}^N(\Sigma^m_{\mathbb{P}^1}Y_+)) \to \Sigma^\infty_{S^{1+\sigma}}\operatorname{Sym}^N(\Sigma^m_{S^{1+\sigma}}Y(\mathbb{C})^{an}_+)$$

is an isomorphism in SH_{C_2} for any N, m and any Y in Sm/k.

Proof. The argument is identical to [31, Lemma 5.4]. The key point is that there is a homotopy pushout square in $\mathrm{Spc}_\bullet(k)$ of the form

$$\begin{array}{ccc}
\operatorname{Sym}^N(Y_+ \times \mathbb{P}^m, Y_+ \times \mathbb{P}^{m-1}) & \longrightarrow & \operatorname{Sym}^N(Y_+ \times \mathbb{P}^m) \\
\downarrow & & \downarrow \\
\operatorname{Sym}^{N-1} Y_+ \wedge \mathbb{P}^m/\mathbb{P}^{m-1} & \longrightarrow & \operatorname{Sym}^N Y_+ \wedge \mathbb{P}^m/\mathbb{P}^{m-1}
\end{array}$$

where the upper horizontal arrow is a closed inclusion of schemes. The previous lemma applied to the top two vertices and induction on N applied to the lower left vertex yields the result. □

As in [31] we write

$$(\Sigma^\infty_{\mathbb{P}^1}X_+)^{tr}_{eff} := (\operatorname{Sym}^\infty X_+, \operatorname{Sym}^\infty(\Sigma_{\mathbb{P}^1}X_+), \operatorname{Sym}^\infty(\Sigma^2_{\mathbb{P}^1}X_+), \dots)$$

and together with the obvious structure maps. Similarly for a C_2 space W we have the C_2-spectrum $(\Sigma^\infty_{S^{1+\sigma}}W_+)^{tr}_{eff} := \{\operatorname{Sym}^\infty(\Sigma^m_{S^{1+\sigma}}W_+)\}_{m\geq 0}$, equipped with the obvious structure maps.

Proposition 4.16. *For any smooth X there is a natural isomorphism in SH_{C_2}*

$$\mathbb{L}\mathrm{Re}^{C_2}_{B,\phi}(\Sigma^\infty_{\mathbb{P}^1}X_+)^{tr}_{eff} \cong (\Sigma^\infty_{S^{1+\sigma}}X(\mathbb{C})^{an}_+)^{tr}_{eff}.$$

Proof. We have the natural isomorphism $\operatorname{colim}_n(\Sigma^\infty_{\mathbb{P}^1}E_n)[n] \cong E$ in SH_k, where $D[n]$ is the shifted spectrum given by $(D[n])_i = D_{i-n}$. Similarly we have the natural isomorphism $\operatorname{colim}_n(\Sigma^\infty_{S^{1+\sigma}}F_n)[n] \cong F$ in SH_{C_2}. Since $\mathbb{L}\mathrm{Re}^{C_2}_{B,\phi}$ preserves homotopy colimits and shifts, the result follows from the previous lemma. □

Theorem 4.17. *Let Λ be an abelian group. There is an isomorphism in SH_{C_2}*

$$\mathbb{L}\mathrm{Re}^{C_2}_{B,\phi}(\mathrm{H}\Lambda) \cong \mathrm{H}\underline{\Lambda}.$$

Proof. Since $\mathrm{H}\Lambda = \mathrm{H}\mathbb{Z} \wedge \mathrm{M}\Lambda$ and $\mathrm{H}\underline{\Lambda} = \mathrm{H}\underline{\mathbb{Z}} \wedge \mathrm{M}\Lambda$, where $\mathrm{M}\Lambda$ is a Moore spectrum for Λ, and $\mathbb{L}\mathrm{Re}^{C_2}_{B,\phi}(\mathrm{M}\Lambda) = \mathrm{M}\Lambda$, it suffices to establish the result for $A = \mathbb{Z}$. The motivic cohomology spectrum $\mathrm{H}\mathbb{Z}$ is given by $\mathrm{H}\mathbb{Z}_n = \mathbb{Z}^{tr}((\mathbb{P}^1)^{\wedge n})$ and equipped with the obvious structure maps. The natural map $(\mathbb{S}_k)^{tr}_{eff} \to \mathrm{H}\mathbb{Z}$ is an isomorphism in SH_k by [31, Lemma 5.9]. It follows from [7, Proposition 3.7] that the spectrum $\{\mathbb{Z}S^{n(1+\sigma)}\}_{n\geq 0}$ is a model for $\mathrm{H}\underline{\mathbb{Z}}$, *i.e.* it represents Bredon cohomology with coefficients in the constant Mackey functor $\underline{\mathbb{Z}}$. It follows from [10, Corollary A.7] that the natural map $(\mathbb{S}_{C_2})^{tr}_{eff} \to \{\mathbb{Z}S^{n(1+\sigma)}\}_{n\geq 0}$ is an equivariant weak equivalence. By the previous proposition, $\mathbb{L}\mathrm{Re}^{C_2}_{B,\phi}((\mathbb{S}_k)^{tr}_{eff}) = (\mathbb{S}_{C_2})^{tr}_{eff}$ and the result follows. □

The Beilinson-Lichtenbaum conjectures assert that for any smooth variety X over a field k, any $n > 1$, and $q \geq 0$, the generalized cycle map

$$H^{p+q\alpha}_{\mathcal{M}}(X, \mathbb{Z}/n) \to H^{p+q}_{ét}(X, \mu_n^{\otimes q})$$

is an isomorphism for $p \leq 0$ and an injection for $p = 1$. By a theorem of Suslin-Voevodsky [43], these conjectures are equivalent to the Bloch-Kato conjectures. In turn, these have been resolved by Voevodsky in case $n = 2^\ell$ and in general by Voevodsky and Rost. Suppose now that $k = \mathbb{R}$. The étale cohomology (with finite coefficients) of the real variety X can be identified with the Borel cohomology of $X(\mathbb{C})$. On the other hand $\mathrm{Re}^{C_2}_B$ induces a comparison map between motivic cohomology and Bredon cohomology and we would like to reinterpret the Beilinson-Lichtenbaum conjectures as a statement concerning this comparison. When 2 is invertible in the coefficient group this is straightforward. In [18] the first author and M. Voineagu treat the case of coefficient group $\mathbb{Z}/2^\ell$ by carefully comparing various cycle maps together with a computation that Bredon and Borel cohomology agree in the appropriate range. This reinterpretation of Voevodsky's theorem applies more generally to the Betti realization for an embedding of a real closed field into $\mathbb{R}$.

Theorem 4.18. *Let $\phi : k \hookrightarrow \mathbb{R}$ be an embedding with k real closed and X a smooth k-variety. For any $n \geq 1$, and any $q \geq 0$ the map*

$$H^{s+q\alpha}_{\mathcal{M}}(X, \mathbb{Z}/n) \to H^{s+q\sigma}(X(\mathbb{C}), \underline{\mathbb{Z}/n}),$$

induced by $\mathrm{Re}^{C_2}_{B,\phi}$, is an isomorphism for $s \leq 0$ and an injection for $s = 1$.

Proof. Motivic cohomology forms a pretheory with transfers. Applying [41, Theorem 1],[4] we have that the base change $\phi^* : H^{s+q\alpha}_{\mathcal{M}}(X, \mathbb{Z}/n) \to H^{s+q\alpha}_{\mathcal{M}}(X_{\mathbb{R}}, \mathbb{Z}/n)$ is an isomorphism so it suffices to treat the case $k = \mathbb{R}$.

[4] This rigidity result is stated for dense subfields of a henselian valued field. Unfortunately $\mathbb{R}$ can't be equipped with a nontrivial henselian valuation. However, the proof of their result relies only on the density lemma [41, Lemma 1] which is valid for a real closed subfield of $\mathbb{R}$, with the classical topology. This is well-known, see e.g. [29, Lemma 4] for a proof.

Suppose that 2 is invertible in $\mathbb{Z}/n$ and write $p : \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$ for the canonical map. Using Proposition 4.13 we have the commutative diagram induced by $X_{\mathbb{C}} \to X$

$$\begin{array}{ccccc} H^{s+q\alpha}_{\mathcal{M}}(X,\mathbb{Z}/n) & \xrightarrow{p^*} & H^{s+q\alpha}_{\mathcal{M}}(X_{\mathbb{C}},\mathbb{Z}/n) & \xrightarrow{p_\#} & H^{s+q\alpha}_{\mathcal{M}}(X,\mathbb{Z}/n) \\ \downarrow{\scriptstyle \mathrm{Re}^{C_2}_B} & & \downarrow{\scriptstyle \mathrm{Re}_B} & & \downarrow{\scriptstyle \mathrm{Re}^{C_2}_B} \\ H^{s+q\sigma}(X(\mathbb{C}),\underline{\mathbb{Z}/n}) & \longrightarrow & H^{s+q}_{\mathrm{sing}}(X(\mathbb{C}),\mathbb{Z}/n) & \longrightarrow & H^{s+q\sigma}(X(\mathbb{C}),\underline{\mathbb{Z}/n}). \end{array}$$

The middle arrow is an isomorphism for $s \leq 0$ and an injection for $s = 1$. The horizontal maps are multiplication by 2, hence isomorphisms. The result thus follows for coefficient groups in which 2 is invertible.

It remains to treat the case $\mathbb{Z}/2^\ell$. The cycle map $H^{s+q\alpha}_{\mathcal{M}}(X,\mathbb{Z}/2^\ell) \to H^{s+q\sigma}(X(\mathbb{C}),\underline{\mathbb{Z}/2^\ell})$ considered in [18] is induced by the map of simplicial abelian groups (for $q \geq 0$)

$$\frac{\mathrm{Hom}_{\mathbb{R}}(X \times \Delta^\bullet_{\mathbb{R}}, \mathrm{Sym}^\infty \mathbb{P}^n)^+}{\mathrm{Hom}_{\mathbb{R}}(X \times \Delta^\bullet_{\mathbb{R}}, \mathrm{Sym}^\infty \mathbb{P}^{n-1})^+} \to \mathrm{Hom}_{C_2\mathrm{Top}_\bullet}((X(\mathbb{C}) \times \Delta^\bullet_{top})_+, \mathbb{Z}(S^{n(1+\sigma)}))$$

obtained by sending an algebraic map of real varieties to its associated equivariant continuous map of C_2-spaces. This agrees with the map considered here. By [18, Theorem 1.5, Proposition 5.1] it induces an isomorphism for $s \leq 0$ and an injection for $s = 1$. □

We finish with an equivariant version of Suslin-Voevodsky's theorem [42] that over an algebraically closed field Suslin homology agrees with étale homology. To set the stage, fix a real embedding $\phi : k \hookrightarrow \mathbb{R}$ and consider the subcategory of motivic spectra X such that $\mathbb{L}\mathrm{Re}^{C_2}_{B,\phi}$ induces an isomorphism $[S^n, X]_k \cong [S^n, \mathbb{L}\mathrm{Re}^{C_2}_{B,\phi}(X)]_{C_2}$ for all n. This is a localizing subcategory of SH_k and we show that it contains all effective torsion motives. If the motivic slice tower were convergent we would be able to show more generally that it contains all effective torsion motivic spectra (i.e. the localizing subcategory generated by $\Sigma^s_{S^1}\Sigma^t_{\mathbb{P}^1}\Sigma^\infty_{\mathbb{P}^1} X/N$ for any $s \in \mathbb{Z}$, $t \geq 0$, $N > 1$, and smooth X).

Theorem 4.19. *Let k be a real closed field and $\phi : k \hookrightarrow \mathbb{R}$ be an embedding. Let E be in the smallest localizing subcategory of SH_k containing $X_+ \wedge \mathrm{H}\mathbb{Z}/r$ for any smooth projective X and $r > 1$. Then for any n, the equivariant Betti realization induces an isomorphism*

$$\mathrm{Re}^{C_2}_{B,\phi} : [S^n, E]_k \xrightarrow{\cong} [S^n, \mathrm{Re}^{C_2}_{B,\phi}(E)]_{C_2}.$$

Proof. It suffices to show that $[S^n, X \wedge \mathrm{H}\mathbb{Z}/r]_k \to [S^n, X_{\mathbb{R}}(\mathbb{C}) \wedge \mathrm{H}\underline{\mathbb{Z}/r}]_{C_2}$ is an isomorphism for any smooth projective X. As in the previous theorem, using [41, Theorem 1], we are reduced to the case $k = \mathbb{R}$. Tracing through definitions, it suffices to show that the map

$$\mathbb{Z}^{tr}(X)(\Delta^\bullet_{\mathbb{R}}) \otimes \mathbb{Z}/r = \mathrm{Hom}_{\mathbb{R}}(\Delta^\bullet_{\mathbb{R}}, \mathrm{Sym}^\infty X)^+ \otimes \mathbb{Z}/r \to \mathrm{Hom}_{C_2\mathrm{Top}_\bullet}(\Delta^\bullet_{top}, \mathbb{Z}X(\mathbb{C})) \otimes \mathbb{Z}/r$$

of simplicial abelian groups, obtained by sending an algebraic map of real varieties to its associated equivariant continuous map of C_2-spaces, is a homotopy equivalence. Note that this last simplicial abelian group equals $\mathrm{Sing}_\bullet(\mathbb{Z}X(\mathbb{C}))^{C_2} \otimes \mathbb{Z}/r$.

That this map is a homotopy equivalence can be deduced by a variant of some arguments of Friedlander-Walker [16] as follows. First, for a presheaf F on $\mathrm{Sch}/\mathbb{R}$, define $F(\Delta^d_{top}) = \mathrm{colim}_{\Delta^d_{top} \to W(\mathbb{R})} F(X)$ where the colimit ranges over continuous maps and W a finite type real variety. Note that if F is the presheaf represented by a real variety Y then $F(\Delta^\bullet_{top}) = \mathrm{Sing}_\bullet Y(\mathbb{R})$. Consider the presheaf of simplicial abelian groups

$$G(-) := \mathbb{Z}^{tr}(X)(- \times \Delta^\bullet_{\mathbb{R}}) \otimes \mathbb{Z}/r.$$

Note that $G(\Delta^\bullet_{top})$ and $[\mathrm{Hom}_{C_2\mathrm{Top}_\bullet}(\Delta^\bullet_{top}, \mathrm{Sym}^\infty X(\mathbb{C}))]^+ \otimes \mathbb{Z}/r$ are naturally homotopic. Combined with Quillen's theorem [15, Appendix Q] on homotopy group completions of simplicial abelian monoids and the fact that $(\mathbb{Z}X(\mathbb{C}))^{C_2}$ is the homotopy group completion of $(\mathbb{N}X(\mathbb{C}))^{C_2}$, we find that there is a natural homotopy equivalence of simplicial abelian groups $G(\Delta^\bullet_{top}) \simeq \mathrm{Sing}_\bullet(\mathbb{Z}X(\mathbb{C}))^{C_2} \otimes \mathbb{Z}/r$. It thus suffices to show that the map $G(\mathbb{R}) \to G(\Delta^\bullet_{top})$ (induced by the projections $\Delta^d_{top} \to *$) is a homotopy equivalence. This is easily seen via the same argument as in [18, Proposition 5.1]. □

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