

# Invariant prime ideals in equivariant Lazard rings

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## Abstract

Let  $A$  be an abelian compact Lie group. In this paper we compute the spectrum of invariant prime ideals of the  $A$ -equivariant Lazard ring, or equivalently the spectrum of points of the moduli stack of  $A$ -equivariant formal groups. We further show that this spectrum is homeomorphic to the Balmer spectrum of compact  $A$ -spectra, with the comparison map induced by equivariant complex bordism homology.

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## 1 Introduction

Let us pose the question: what algebraic input do we need to develop equivariant versions of chromatic homotopy theory?

Chromatic homotopy theory studies stable homotopy theory through the lens of formal groups, building on Quillen's identification of the complex bordism ring  $\pi_* MU$  with the Lazard ring [Qui71]. Around the same time, tom Dieck introduced for every compact Lie group  $A$  in [tD70] an equivariant analog of  $MU$ , the homotopical  $A$ -equivariant complex bordism  $MU_A$ . Letting  $A$  be abelian, Cole, Greenlees and Kriz [CGK00] found many years later the correct notion of an  $A$ -equivariant formal group law. Recently, the first author generalized work of Hanke–Wiemeler [HW18] and showed that  $\pi_*^A MU_A$  is indeed the universal ring for  $A$ -equivariant formal group laws, thus establishing an equivariant analog of Quillen's theorem for the equivariant Lazard ring  $L_A$ .

Many structural features of stable homotopy theory can be explained through the chromatic perspective. The central notion of chromatic homotopy theory is that of *height*. Honda classified formal groups over a field of characteristic  $p$  in terms of the height  $0 \leq n \leq \infty$ . Thus, the points of the moduli stack of formal groups  $\mathcal{M}_{FG}$  correspond to pairs  $(p, n)$  with  $n = 0$  if and only if  $p = 0$ . Hopkins and Smith [HS98] showed that the same classification pertains to thick subcategories of finite spectra: Given a finite spectrum  $X$ , its  $MU$ -homology  $MU_* X$  defines a coherent sheaf over  $\mathcal{M}_{FG}$ . Taking the support of  $MU_* X$  in the Zariski spectrum  $|\mathcal{M}_{FG}|$  of points, we obtain a support theory on finite spectra. The thick subcategory theorem states that

this support theory is the universal one. In other words, the induced map  $|\mathcal{M}_{FG}| \rightarrow \text{Spec}(Sp^c)$  to the Balmer spectrum of finite spectra (cf. [Bal05, Bal10]) is a homeomorphism.

We show the following equivariant analog (a more precise statement of which we give as Theorem 1.6):

**Theorem 1.1.** *Let  $A$  be an abelian compact Lie group. Then the spectrum of points of the moduli space  $M_{FG}^A$  of  $A$ -equivariant formal groups is homeomorphic to the Balmer spectrum of finite  $A$ -spectra, with the comparison map induced by a support theory based on complex bordism homology  $(MU_A)_*$ .*

This establishes  $MU_A$  and the theory of equivariant formal groups as fundamental tools for building equivariant versions of chromatic homotopy theory.

For abelian groups as above, the Balmer spectrum of finite  $A$ -spectra has been computed completely in the papers [BS17] (the case  $A = C_p$ ), [BHN<sup>+</sup>19] (the finite abelian case) and [BGH20] (the general abelian case). In a surprising turn of history, it had been the algebraic counterpart which had not been computed before. As a set, both  $|\mathcal{M}_{FG}^A|$  and  $\text{Spec}(Sp_A^c)$  decompose as the disjoint union of one copy of  $|\mathcal{M}_{FG}| \cong \text{Spec}(Sp^c)$  for every closed subgroup of  $A$ . Thus, the correct notion of height of an  $A$ -equivariant formal group  $F$  over a field of characteristic  $p$  consists of a pair: a height  $n$  of a non-equivariant formal group and a closed subgroup  $B \subset A$  such that  $F$  is induced along the zig-zag  $A \rightarrow A/B \leftarrow \{1\}$ .

The more subtle information lies in the topology of the spectrum, which encodes on the algebraic level how heights can deform and on the homotopical level the chromatic interdependence of the various geometric fixed points  $\Phi^B X$  of a finite  $A$ -spectrum  $X$ .

We will detail our results below in the language of invariant prime ideals. Crucially, we exhibit equivariant lifts  $\mathbf{v}_n$  of the classical  $v_n$  and show that they provide in many cases a sequence of generators of invariant prime ideals. The non-equivariant  $v_n$  play an important role in many of the highlights of chromatic homotopy theory, like the greek-letter construction [Rav86], the construction of the Morava K-theories or the periodicity theorem [HS98], and we hope that our equivariant lifts open the prospect to generalize these to the equivariant context.

## 1.1 Invariant prime ideals and statement of results

As indicated above, the main theorem can also be stated in terms of invariant prime ideals of the equivariant Lazard ring  $L_A$ , as we now explain. Similarly to the non-equivariant case,  $L_A$  is the ground ring of a flat Hopf algebroid  $(L_A, S_A)$ , classifying  $A$ -equivariant formal group laws and their strict isomorphisms. The associated stack is the moduli stack of  $A$ -equivariant formal groups. Hence, the category of graded  $(L_A, S_A)$ -comodules is equivalent to the category of quasi-coherent sheaves over  $M_{FG}^A$ .

Recall that an ideal  $I$  of  $L_A$  is called *invariant* (in the sense of Hopf algebroids) if it is a sub-comodule, i.e., if  $\eta_L(I)S_A = \eta_R(I)S_A$  for the left and right unit  $\eta_L, \eta_R: L_A \rightarrow S_A$ . Every invariant *prime* ideal  $\mathfrak{p}$  gives rise to a point of the moduli stack of prime ideals via the quotient field of  $L_A/\mathfrak{p}$ . This defines a map from the set of invariant prime ideals  $\text{Spec}^{\text{inv}}(L_A)$  to  $|\mathcal{M}_{FG}^A|$ , which we show to be a homeomorphism in Theorem 4.7.

For the non-equivariant Lazard ring, Morava and Landweber [Lan73] showed that the invariant prime ideals are precisely the ideals  $I_{p,n} = (v_0, \dots, v_{n-1})$  for a prime  $p$  and  $n \in \mathbb{N} \cup \{\infty\}$  (with  $I_{p,0}$  being the 0-ideal for all  $p$ ).

To describe the invariant prime ideals in the equivariant case we recall that  $L_A$  contains universal Euler classes  $e_V$  for all characters  $V \in A^*$ , and that equivariant Lazard rings are contravariantly functorial in continuous group homomorphisms. In particular, all equivariant Lazard rings are algebras over the non-equivariant Lazard ring.

Then, given a non-equivariant invariant prime ideal  $I_{p,n}$  and a closed subgroup  $B$  of  $A$  we obtain an invariant prime ideal  $I_{B,p,n}^A \subseteq L_A$  as the kernel of the composite

$$L_A \xrightarrow{\text{res}_B^A} L_B \rightarrow \Phi^B L_B \rightarrow \Phi^B L_B \otimes_L L/I_{p,n}.$$

Here,  $\Phi^B L$  is defined as the localization of  $L_A$  away from all the Euler classes of non-trivial characters for  $B$ . The ring  $\Phi^B L_B$  is an algebraic version of geometric fixed points and indeed agrees with the coefficient ring of the  $B$ -geometric fixed points of  $MU_B$ .

**Theorem 1.2** (Theorem 4.7). *For every abelian compact Lie group  $A$  the map*

$$\begin{aligned} \text{Sub}(A) \times \text{Spec}^{\text{inv}}(L) &\rightarrow \text{Spec}^{\text{inv}}(L_A) \\ (B, I_{p,n}) &\mapsto I_{B,p,n}^A \end{aligned}$$

*is a bijection.*

Here,  $\text{Sub}(A)$  is the set of all closed subgroups of  $A$ . Hence, as for the Balmer spectrum, the invariant prime ideals of  $L_A$  decompose as a set as one copy of  $\text{Spec}^{\text{inv}}(L)$  for every closed subgroup of  $A$ . And similarly to the Balmer spectrum, the main work then lies in understanding the Zariski topology, in particular in determining the containments between invariant prime ideals associated to different subgroups.

We obtain the following:

**Theorem 1.3** (Theorem 5.1). *There is an inclusion  $I_{B',q,n'}^A \subseteq I_{B,p,n}^A$  if and only if*

1.  $B$  is a subgroup of  $B'$ .
2.  $p = q$  or  $n = 0$  (in which case  $I_{B',q,0}^A = I_{B',p,0}^A$ ), the components  $\pi_0(B'/B)$  are a  $p$ -group and  $n \geq n' - \text{rank}_p(\pi_0(B'/B))$ .

Comparing with [BGH20] we see that these correspond precisely to the inclusions in the Balmer spectrum, but with roles reversed: There is an inclusion  $I_{B',q,n'}^A \subseteq I_{B,p,n}^A$  if and only if there is an inclusion  $P_{B,p,n}^A \subseteq P_{B',q,n'}^A$ . Here,  $P(B, p, n) = \{X \in \text{Sp}_A^c \mid K(n)_*(\Phi^B X) = 0\}$  are the thick subcategories of  $\text{Sp}_A^c$  with  $K(n)$  being Morava K-theory at the prime  $p$ .

To show that  $I_{B',q,n'}^A$  indeed includes into  $I_{B,p,n}^A$  when conditions 1 and 2 are satisfied, one can reduce to the case  $A = \mathbf{T}$  the circle group where it is straightforward to describe explicit generators for the invariant prime ideals. The main step in ruling out further inclusions is the construction of equivariant refinements  $\bar{\mathbf{v}}_{n-1} \in LC_p^n$  of the elements  $v_{n-1} \in L$  which exhibit maximal height shifts (Definition 5.21, Proposition 5.24). Roughly speaking,  $\bar{\mathbf{v}}_{n-1}$  is of height 0 at the top group  $C_p^n$  (i.e., it lies in the ideal  $I_{C_p^n,p,0}^n$ ) while it is of height  $n$  at the trivial group (i.e., it lies in the ideal  $I_{\{1\},p,n}$  but not in  $I_{\{1\},p,n-1}$ ). This is the algebraic analog of the existence of finite  $C_p^n$ -spectra of underlying type  $n$  whose  $C_p^n$ -geometric fixed points are rationally non-trivial as in [BHN<sup>+</sup>19] and [KL20]. More precisely,  $\bar{\mathbf{v}}_n$  is canonically defined only modulo a certain smaller ideal (analogously to  $v_n$  only being defined uniquely up to the ideal  $I_n$ ). More details are given in Section 5.3.

We further show that - at least over elementary abelian  $p$ -groups - the elements  $\bar{\mathbf{v}}_i$  give rise to generators of the invariant prime ideals:

**Theorem 1.4** (Theorem 6.1). *For all primes  $p$  and  $n \in \mathbb{N}$  the elements*

$$p_1^* \bar{\mathbf{v}}_0, p_2^* \bar{\mathbf{v}}_1, \dots, p_{n-1}^* \bar{\mathbf{v}}_{n-2}, \bar{\mathbf{v}}_{n-1}$$

*generate the ideal  $I_{C_p^n,p,0}^{C_p^n}$ . Here,  $p_i: C_p^n \rightarrow C_p^i$  denotes the projection to the first  $i$  coordinates.*

Suitable restrictions of the  $\bar{\mathbf{v}}_n$  then form generators for the ideals  $I_{C_p^n,p,m}^{C_p^n}$  at higher height  $m$ , see Section 6. We emphasize that in contrast to the non-equivariant situation, the sequence of the  $p_i^* \bar{\mathbf{v}}_{i-1}$  is *not* a regular sequence. In fact, since  $I_{C_p^n,p,0}^{C_p^n}$  consists precisely of the Euler-class-power torsion, it does not contain a non-zero divisor and hence cannot be generated by a regular sequence (unless  $n = 0$ ). The torsion in the ring  $LC_p^n$  is closely linked to the torsion in the group of characters  $(C_p^n)^*$ . Hence one might hope that  $I_{B,n}^A$  is generated by a regular sequence whenever  $A$  is a torus, and indeed that is the case in all the cases we understand (cf., Remark 6.7).

Finally, in order to describe the Zariski topology we need one additional ingredient. When  $A$  is infinite, the set of closed subgroups  $\text{Sub}(A)$  contains a non-trivial metric topology, turning it into a totally-disconnected compact Hausdorff space. Together with the inclusions between the invariant prime ideals, this topology determines the Zariski topology on  $\text{Spec}^{\text{inv}}(L_A)$ .

**Theorem 1.5** (Theorem 7.5). *The Zariski topology on  $\text{Spec}^{\text{inv}}(L_A)$  has as basis the closed subsets  $C$  which are*

- (i) closed under upward inclusions, i.e., if  $I_{B',q,n'}^A \in C$  and  $I_{B',q,n'}^A \subseteq I_{B,p,n}^A$ , then  $I_{B,p,n}^A \in C$ , and which
- (ii) are locally constant on  $\text{Sub}(A)$  in the sense that if  $I_{B,p,n}^A \in C$  there exists a neighborhood  $U$  of  $B$  such that  $I_{B',p,n}^A \in C$  for all  $B' \in U$ .

Comparing with [BGH20], we see that this description precisely matches the computation of the topology on the Balmer spectrum, with  $I_{B,p,n}^A$  replaced by  $P_{B,p,n}^A$ . Hence the assignment

$$I_{B,p,n}^A \mapsto P_{B,p,n}^A$$

yields a homeomorphism from  $\text{Spec}^{\text{inv}}(L_A)$  (and hence  $|\mathcal{M}_{FG}^A|$ ) to  $\text{Spec}(\text{Sp}_A^c)$ . In the last section we explain that this comparison map can be obtained less ad hoc via  $MU_A$ -homology:

**Theorem 1.6** (Section 8.1). *Let  $X$  be a finite  $A$ -spectrum and  $I_{B,p,n}^A$  an invariant prime ideal. Then the localization  $(MU_A)_{I_{B,p,n}^A} \wedge X$  is non-trivial if and only if the  $B$ -geometric fixed points  $\Phi^B X$  are of type  $\leq n$  at  $p$ , i.e., if and only if  $P_{B,p,n}^A$  is in the Balmer support of  $X$ .*

*This shows that*

$$X \mapsto \text{supp}((MU_A)_* X) \subseteq \text{Spec}^{\text{inv}}(L_A) \cong |\mathcal{M}_{FG}^A|$$

defines a universal support theory on finite  $A$ -spectra and thus a homeomorphism  $\text{Spec}^{\text{inv}}(L_A) \rightarrow \text{Spec}(\text{Sp}_A^c)$ . Here,  $(MU_A)_* X$  is the Mackey functor recording  $(MU_B)_* \text{res}_B^A X$  for all closed subgroups  $B$  of  $A$ , and  $\text{supp}((MU_A)_* X)$  is defined as the set of invariant prime ideals at which the localization of this Mackey functor is non-trivial.

Our proof that  $X \mapsto \text{supp}((MU_A)_* X)$  is a support theory is independent of [BHN<sup>+</sup>19] and [BGH20]; this already provides a continuous bijection  $\text{Spec}^{\text{inv}}(L_A) \rightarrow \text{Spec}(\text{Sp}_A^c)$ , providing a new proof of one half of their main theorems. This half has been dubbed the chromatic Smith chromatic fixed point theorem in [Kuh21] and [BK23], where other proofs are given. To establish that our support theory is universal, we need to invoke [BHN<sup>+</sup>19] and [BGH20] however.

We show in Proposition 8.12 that the support theory from Theorem 1.6 can alternatively be built by viewing  $(MU_A)_* X$  as defining quasi-coherent sheaves on  $\mathcal{M}_{FG}^B$  for every closed  $B \subseteq A$ ; the union of the supports of these sheaves agrees with  $\text{supp}((MU_A)_* X)$  under the homeomorphism  $\text{Spec}^{\text{inv}}(L_A) \cong |\mathcal{M}_{FG}^A|$ . To show this, we establish in Proposition 8.9 how the adjunction between  $\text{Sp}^A$  and  $\text{Sp}^{A/B}$  defined by geometric fixed points and pullback corresponds on the algebraic level to pullback and pushforward along the open immersion  $\mathcal{M}_{FG}^{A/B} \subseteq \mathcal{M}_{FG}^A$  from Proposition 3.11.

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## 2 Equivariant formal groups

The aim of this section is to recall some basic definitions and properties about equivariant formal groups and equivariant formal group laws from [CGK00, Gre01, Str11, Hau22]. To make our paper more self-contained, we replicate also some of the proofs in our language, and we provide some small extensions of known results. Our treatment of equivariant formal groups is far from exhaustive and especially [Str11] contains a wealth of results we do not touch upon.

## 2.1 Basic definitions

In this subsection, we will recall the notions of an equivariant formal group and an equivariant formal group law over a commutative ring  $k$ . The definition of an equivariant formal group law is due to Cole, Greenlees and Kriz [CGK00] and our definition of an equivariant formal group will be a variant of that of Strickland [Str11].

For us, a *formal  $k$ -algebra* is a complete linearly topologized commutative  $k$ -algebra with a countable system of open ideals generating the topology. By [Yas09, Section 5.2], this is equivalent to the full subcategory of pro-objects in commutative  $k$ -algebras, indexed on a countable directed set with surjective transition maps. For us, the category of *formal  $k$ -schemes* is the opposite of that of formal  $k$ -algebras. It can be viewed as the category of ind-objects in affine  $k$ -schemes, indexed by a countable directed set with closed immersions as transition maps. We will sometimes use the notation  $\mathrm{Spec} R$  or  $\mathrm{Spf} R$  for the formal  $k$ -scheme associated to a formal  $k$ -algebra  $R$ , and  $\mathcal{O}_X$  for the formal  $k$ -algebra associated to a formal  $k$ -scheme  $X$ . The product on affine  $k$ -schemes induces one on formal  $k$ -schemes, and this corresponds to the completed tensor product on formal  $k$ -algebras.

We set  $S = \mathrm{Spec} k$ . Given a countable set  $M$ , we view  $S \times M$  as a formal scheme, namely as the colimit over all  $S \times N$  with  $N \subseteq M$  finite. This corresponds to giving  $k^M$  the product topology. If we just write  $M$ , we will apply this construction to  $\mathrm{Spec} \mathbb{Z}$  instead of  $S$ .

For a compact Lie group  $A$ , we will denote by  $A^* = \mathrm{Hom}(A, \mathbf{T})$  its Pontryagin dual, which is always a discrete group. We use  $\epsilon$  for the unit element in  $A^*$ .

**Definition 2.1.** Given a compact abelian Lie group  $A$ , an  *$A$ -equivariant formal group over  $k$*  consists of a commutative group object  $X$  in formal  $k$ -schemes together with a group homomorphism  $\varphi: S \times A^* \rightarrow X$  of formal  $k$ -schemes satisfying the following two conditions:

1. For the composite  $\varphi_\epsilon: S \xrightarrow{\mathrm{id}_S \times \epsilon} S \times A^* \xrightarrow{\varphi} X$ , the augmentation ideal  $I_\epsilon = \ker(R \rightarrow k)$  of the induced map is fpqc-locally on  $k$  a free  $R$ -module of rank 1, where  $R = \mathcal{O}_X$ .
2. The topology on  $R$  is generated by products of the ideals  $I_V = \ker(R \xrightarrow{\varphi_V^*} k)$  for  $V \in A^*$  and  $\varphi_V: S \xrightarrow{\mathrm{id}_S \times V} S \times A^* \xrightarrow{\varphi} X$ .

**Remark 2.2.** In the above definition, one can easily replace  $\mathrm{Spec} k$  by an arbitrary quasi-compact scheme  $S$ , with formal  $S$ -schemes being a suitable subcategory of the ind-category of  $\mathrm{Aff}_S$ , the category of schemes affine over  $S$ .

Our definition differs in two aspects from that put forward in [Str11, Definition 2.15]. First, Strickland restricts to finite  $A^*$ . Second, Strickland asks  $\ker(\mathcal{O}_X \rightarrow k)$  to be free of rank 1 instead of locally free (cf. [Str11, Proposition 2.10]). We changed it so that our definition satisfies descent. Note that we could have asked equivalently that the augmentation ideal is Zariski locally on  $k$  a free  $R$ -module of rank 1 because line bundles satisfy fpqc-descent.

**Remark 2.3.** If we leave out the second condition in Definition 2.1, we get a notion we call an  *$A$ -equivariant group*. The category of  $A$ -equivariant formal groups embeds into that of  $A$ -equivariant groups and this inclusion has a right adjoint, called *completion*. Concretely, this replaces  $R$  in the notation in Definition 2.1 by the formal  $k$ -algebra  $\lim_{V_1, \dots, V_n \in A^*} R / (I_{V_1} \cdots I_{V_n})$ . We will only use  $A$ -equivariant groups to complete them to  $A$ -equivariant formal groups.

Spelling out what we get in a more algebraic language if we fix a trivialization of the augmentation ideal  $I_\epsilon$  gives us the notion of an equivariant formal group law.

**Definition 2.4.** An  *$A$ -equivariant formal group law over  $k$*  is a quadruple

$$(R, \Delta, \theta, y(\epsilon))$$

of a formal  $k$ -algebra  $R$ , a continuous comultiplication  $\Delta: R \rightarrow R \hat{\otimes} R$ , a map of  $k$ -algebras  $\theta: R \rightarrow k^{A^*}$  and an orientation  $y(\epsilon) \in R$ , such that

- (i) the comultiplication is a map of  $k$ -algebras which is cocommutative, coassociative and counital for the augmentation  $\theta_\epsilon: R \rightarrow k$ ,
- (ii) the map  $\theta$  is compatible with the coproduct, and the topology on  $R$  is generated by finite products of the kernels of the component functions  $\theta_V: R \rightarrow k$  for  $V \in A^*$ , and

(iii) the element  $y(\epsilon)$  is regular and generates the kernel of  $\theta_\epsilon$ .

We refer to [CGK00] and [Hau22] for more information about equivariant formal group laws.

**Remark 2.5.** If we want to remember the base of an equivariant formal group law, we sometimes also write it as a quintuple  $(k, R, \Delta, \theta, y(\epsilon))$ .

**Lemma 2.6.** *An  $A$ -equivariant formal group  $(\text{Spec } R, \varphi)$  over  $k$  together with an  $R$ -linear isomorphism  $I_\epsilon \cong R$  of the augmentation ideal is equivalent datum to an  $A$ -equivariant formal group law over  $k$ .*

*Proof.* The maps  $\Delta$  and  $\theta$  are induced by the multiplication on  $\text{Spec } R$  and  $\varphi$ , respectively. The element  $y(\epsilon)$  corresponds to the trivialization of  $I_\epsilon = \ker(R \rightarrow k)$ .  $\square$

Given any equivariant formal group  $G = (\varphi: A^* \rightarrow X)$  over  $S = \text{Spec } k$ , we obtain for every  $V \in A^*$  a morphism  $\varphi_V: S \xrightarrow{\text{id}_S \times V} S \times A^* \xrightarrow{\varphi} X$ . If  $R = \mathcal{O}_X$ , this corresponds to the morphism  $\theta_V: R \rightarrow k$ . Moreover,  $\varphi$  composed with left multiplication defines an  $A^*$ -action on  $X$ ; for every  $V \in A^*$  this gives a map  $l_V: R \rightarrow R$ . In terms of the data of an equivariant formal group law  $F = (k, R, \Delta, \theta, y(\epsilon))$ , this can explicitly be written as

$$l_V: R \xrightarrow{\Delta} R \hat{\otimes} R \xrightarrow{\theta_V \hat{\otimes} \text{id}_R} R.$$

Given  $V \in A^*$ , we set

$$y(V) = l_V(y(\epsilon)) \in R,$$

which generates the kernel of  $\theta_{V^{-1}}$ . If  $A$  is trivial, we have  $R \cong k[[y]]$ . We want to describe an analog for general  $A$ . A *complete flag* for  $A$  is a sequence of characters  $f = V_1, V_2, \dots \in (A^*)^{\mathbb{N}}$  such that every character appears infinitely often. Given such a flag and  $n \in \mathbb{N}$ , we set

$$y(W_n) = y(V_n)y(V_{n-1}) \cdots y(V_1).$$

Then every element  $x$  of  $R$  can be written uniquely as

$$x = \sum_{n \in \mathbb{N}} a_n^f y(W_n) \tag{2.7}$$

for coefficients  $a_n^f \in k$  [Hau22, Section 2.2]. Hence, as a  $k$ -module,  $R$  is isomorphic to a countable infinite product of copies of  $k$ .

## 2.2 Lazard rings

Our aim in this subsection is to recall the definition of the universal ring for equivariant formal group laws and to clarify its universal property. Let us begin by considering a very strict form of morphisms of equivariant formal group laws.

**Definition 2.8.** A *morphism* between  $A$ -equivariant formal group laws  $(k_1, R_1, \Delta_1, \theta_1, y(\epsilon)_1)$  and  $(k_2, R_2, \Delta_2, \theta_2, y(\epsilon)_2)$  is a pair of maps  $f: k_1 \rightarrow k_2$  and  $g: R_1 \rightarrow R_2$  which are compatible with both the comultiplications  $\Delta$  and the augmentations  $\theta$  and which send  $y(\epsilon)_1$  to  $y(\epsilon)_2$ .

This leads to a category  $A$ -FGL of  $A$ -equivariant formal group laws. In [CGK00] it is shown that this category has an initial object  $F^{\text{uni}}$ , the ground ring of which is called the  *$A$ -equivariant Lazard ring* and denoted  $L_A$ . In fact, the category of  $A$ -equivariant formal group laws is equivalent to the category of commutative rings under  $L_A$ . To discuss this, note first that the forgetful functor

$$A\text{-FGL} \rightarrow \text{CRing}, \quad (k, R, \Delta, \theta, y(\epsilon)) \mapsto k$$

into the category of commutative rings is cofibered in groupoids. Concretely this boils down to the following two observations:

- Every morphism of  $A$ -equivariant formal group laws whose first component  $f: k_1 \rightarrow k_2$  is the identity map is an isomorphism. One observes indeed that the diagram

$$\begin{array}{ccc} R_1 & \xrightarrow{\quad} & R_2 \\ & \searrow \mathbb{R} & \swarrow \mathbb{R} \\ & \prod_{\mathbb{N}} k_1 = \prod_{\mathbb{N}} k_2 & \end{array}$$

obtained from Eq. (2.7) commutes. We call such an isomorphism living over the identity a *very strict isomorphism* between  $A$ -equivariant formal group laws, in order to distinguish from other kinds of isomorphisms.

- Given a morphism  $f: k_1 \rightarrow k_2$  and an  $A$ -equivariant formal group law  $F$  over  $k_1$ , one can define a pushforward  $f_*F$  over  $k_2$  with the usual universal property. Its underlying  $k_2$ -algebra is given by a completion of  $R \otimes_{k_1} k_2$ , where  $R$  is the underlying  $k_1$ -algebra of  $F$  (cf. [Gre01, Section 2.E], [Hau22, Section 2.3]).

Note that the only automorphism of an  $A$ -equivariant formal group law over  $k$  which is also a very strict isomorphism is the identity map. Hence, given two  $A$ -equivariant formal group laws over  $k$ , there either exists a unique very strict isomorphism between them or none at all. For this reason it is usually harmless to identify two very strictly isomorphic  $A$ -equivariant formal group laws, and we will often do so.

Now, given a map  $f: L_A \rightarrow k$  there is an induced  $A$ -equivariant formal group law  $f_*F^{\text{uni}}$  over  $k$  obtained by pushing forward the universal  $A$ -equivariant formal group law. Given an  $A$ -equivariant FGL  $F$  over  $k$ , we can apply this to the first component  $f: L_A \rightarrow k$  of the unique map of  $A$ -equivariant formal group laws  $F^{\text{uni}} \rightarrow F$ . The resulting morphism  $f_*F^{\text{uni}} \rightarrow F$  is necessarily a very strict isomorphism. So, as claimed above, we obtain:

**Corollary 2.9.** *The functor*

$$\text{CAlg}_{L_A} \rightarrow A\text{-FGL}, \quad (f: L_A \rightarrow k) \mapsto f_*F^{\text{uni}}$$

from commutative  $L_A$ -algebras is an equivalence of categories. An inverse is given by sending an  $A$ -equivariant formal group law  $F = (k, R, \Delta, \theta, y(\varepsilon))$  to the first component  $f: L_A \rightarrow k$  of the unique morphism  $F^{\text{uni}} \rightarrow F$ .

**Remark 2.10.** The above proof is an instance of a general characterization of initial objects  $X$  in categories cofibered in groupoids  $\mathcal{C} \xrightarrow{F} \mathcal{D}$ , namely that pushforward defines an equivalence of  $\mathcal{D}_{F(X)/-}$  with  $\mathcal{C}$ .

**Remark 2.11.** Non-equivariantly the  $k$ -algebra  $R$  is often fixed to be the power series ring  $k[[y(\varepsilon)]]$  rather than a ring only isomorphic to it. With this convention, the category of formal group laws is *isomorphic* (not merely equivalent) to the category of commutative rings under  $L$ . In other words, every very strict isomorphism of formal group laws is the identity. Equivariantly one needs to be a little more careful: The statement ‘ $A$ -equivariant formal group laws are represented by the  $A$ -equivariant Lazard ring’ is only true up to this notion of very strict isomorphism.

## 2.3 Global functoriality

In this subsection, we will discuss both a covariant and a contravariant functoriality of the category of  $A$ -equivariant formal groups in  $A$ .

**Definition 2.12.** Let  $\alpha: B \rightarrow A$  be a group homomorphism and let  $G = (B^* \rightarrow X)$  be a  $B$ -equivariant formal group. We define the *corestriction*  $\alpha_*G$  to be the  $A$ -equivariant formal group, which is the completion of the  $A$ -equivariant group  $(A^* \xrightarrow{\alpha^*} B^* \rightarrow X)$

**Proposition 2.13.** *For every injective group homomorphism  $\alpha: B \rightarrow A$ , the functor  $\alpha_*$  from  $B$ -equivariant formal groups to  $A$ -equivariant formal groups is fully faithful. The essential image consists of those  $A$ -equivariant formal groups where the homomorphism from  $A^*$  factors through  $B^*$ .*

*Proof.* For every  $B$ -equivariant formal group  $G$ , by construction  $\alpha_*G$  is the same group object as  $G$  in formal schemes with the structure morphism  $A^* \rightarrow B^* \rightarrow G$  (since  $A^* \rightarrow B^*$  is surjective, no completion is necessary). This implies fully faithfulness.  $\square$

Upon choosing coordinates, Definition 2.12 corresponds to the construction of [Hau22, Section 2.4]: given a  $B$ -equivariant formal group law  $F = (k, R, \Delta, \theta, y(\varepsilon))$  and a group homomorphism  $\alpha: B \rightarrow A$ , there is an induced  $A$ -equivariant formal group law  $\alpha_*F$  over the same ring  $k$ , given by completing  $R$  at products of the ideals  $I_V = \ker(R \xrightarrow{\theta_V} k)$  for those  $V \in B^*$  which are in the image of  $\alpha^*: A^* \rightarrow B^*$ . This defines a functor  $\alpha_*$  from  $B$ -equivariant formal group laws to  $A$ -equivariant formal group laws, which induces a map  $\alpha^*: L_A \rightarrow L_B$  on Lazard rings. Hence we obtain a functor

$$\mathbf{L}: (\text{abelian compact Lie groups})^{op} \rightarrow \text{commutative rings}$$

which we call the *global Lazard ring*. As shown in [Hau22],  $L_A$  is isomorphic to  $\pi_*^A MU_A$  and our map  $\alpha^*$  corresponds to the restriction map on that level, explaining our terminology.

**Remark 2.14.** By Pontryagin duality, the opposite category of abelian compact Lie groups is equivalent to the category of finitely generated abelian groups. Therefore, everything in this paper could alternatively be phrased in terms of finitely generated abelian groups rather than abelian compact Lie groups, and the more algebraically minded reader might prefer to do so.

In addition to this covariant functoriality, there is also a contravariant functoriality.

**Definition 2.15.** Let  $\alpha: A \rightarrow C$  be a surjective group homomorphism and let  $G = (C^* \rightarrow X)$  be a  $C$ -equivariant group. We define the *coinduction*  $\alpha^*G$  to be the  $A$ -equivariant group  $(A^* \rightarrow X \times_{C^*} A^*)$ , where the target denotes the quotient of  $X \times A^*$  by the antidiagonal  $C^*$ -action.

**Lemma 2.16.** For  $\alpha: A \rightarrow C$  a surjective group homomorphism and  $G$  a  $C$ -equivariant formal group,  $\alpha^*G$  is an  $A$ -equivariant formal group, i.e. needs no additional completion.

*Proof.* If  $G = (C^* \rightarrow \text{Spec } R)$  is a  $C$ -equivariant formal group, then

$$\text{Spec } R \times_{C^*} A^* \cong \text{Spec } \text{Map}_{C^*}(A^*, R).$$

The  $k$ -algebra  $\text{Map}_{C^*}(A^*, R)$  is isomorphic to  $\prod_{A^*/C^*} R$ . As products of complete rings are complete, the claim follows.  $\square$

**Proposition 2.17.** Let  $\alpha: A \rightarrow C$  be a surjective group homomorphism. As functors between  $C$ -equivariant formal groups and  $A$ -equivariant formal groups,  $\alpha^*$  is the left adjoint of  $\alpha_*$ .

*Proof.* For an  $A$ -equivariant group  $G = (A^* \rightarrow X)$ , define  $\tilde{\alpha}_*G$  as  $C^* \rightarrow A^* \rightarrow X$ . Then  $\alpha^*$  and  $\tilde{\alpha}_*$  are adjoints between  $C$ -equivariant groups and  $A$ -equivariant groups in the sense of Remark 2.3. Since completion is a right adjoint, the result follows from the previous lemma.  $\square$

## 2.4 Euler classes

Given an  $A$ -equivariant formal group law  $F = (R, \Delta, \theta, y(\varepsilon))$  over  $k$ , we can define Euler classes in  $k$ . Recall that for  $V \in A^*$ , we set  $y(V) = l_V(y(\varepsilon)) \in R$ . The corresponding Euler class is

$$e_V = \theta_\varepsilon(y(V)) = \theta_V(y(\varepsilon)) \in k.$$

In terms of the associated equivariant formal group  $G = (\varphi: A^* \rightarrow X)$ , we have

$$S \times_{\varphi_\varepsilon, X, \varphi_V} S \cong \text{Spec } k \otimes_{\theta_\varepsilon, R, \theta_V} k \cong \text{Spec } k/e_V$$

with  $S = \text{Spec } k$  and  $\varphi_V$  being the composite  $S \cong S \times \{V\} \subseteq S \times A^* \xrightarrow{\varphi} X$ . This implies:

**Lemma 2.18.** For a given equivariant formal group law with notation as above:

1. The Euler class  $e_V$  is invertible iff  $S \times_{\varphi_\varepsilon, X, \varphi_V} S = \emptyset$ .
2. The Euler class  $e_V$  is zero iff  $\varphi_V = \varphi_\varepsilon$ .



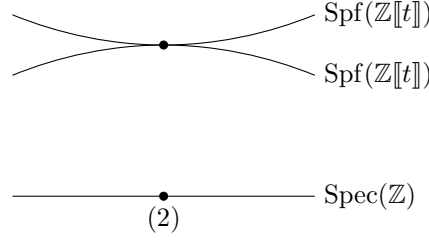


Figure 1: A schematic picture of  $\widehat{\mathbb{G}}_m^{C_2}$  from Example 2.20

Thus, the vanishing or invertibility of Euler classes does not depend on chosen coordinates. This allows us to generalize these concepts to  $A$ -equivariant formal groups in the following way:

**Definition 2.19.** For an  $A$ -equivariant formal group  $G = (\varphi: A^* \rightarrow X)$ , we say that the *Euler class*  $e_V$  is invertible if  $S \times_{\varphi_\epsilon, X, \varphi_V} S = \emptyset$  and that  $e_V$  is zero if  $\varphi_V = \varphi_\epsilon$ .

Informally,  $e_V$  is invertible if and only if the images of  $S \times \{\epsilon\}$  and  $S \times \{V\}$  in  $X$  are disjoint.

**Example 2.20.** Let  $\mathbb{G}_m = \text{Spec } \mathbb{Z}[x^{\pm 1}]$  be the multiplicative group over  $S = \text{Spec } \mathbb{Z}$ . We choose the group homomorphism  $\varphi: C_2 = (C_2)^* \rightarrow \mathbb{G}_m$  picking the units  $\{\pm 1\}$  in  $\mathbb{Z}$ . This defines the structure of a  $C_2$ -equivariant group, and its completion is a  $C_2$ -equivariant formal group we call  $\widehat{\mathbb{G}}_m^{C_2}$ . Let  $V \in (C_2)^*$  be the unique non-trivial character. One computes

$$\text{Spec } \mathbb{Z} \times_{\varphi_\epsilon, \widehat{\mathbb{G}}_m^{C_2}, \varphi_V} \text{Spec } \mathbb{Z} \cong \text{Spec } \mathbb{Z} \times_{\varphi_\epsilon, \mathbb{G}_m, \varphi_V} \text{Spec } \mathbb{Z} \cong \text{Spec } \mathbb{Z}/2.$$

Thus,  $e_V = \pm 2$ , depending on the choice of coordinate. See also [Str11, Section 7] and [Gre01, Section 7] for more information on this and related examples. Note in particular that our example is the pushforward of the true  $C_2$ -equivariant multiplicative formal group (given by a completion of  $\text{Spec } \mathbb{Z}[(C_2 \times \mathbf{T})^*]$ ) along the map  $\mathbb{Z}[C_2^*] \rightarrow \mathbb{Z}$  classifying  $\pm 1$ .

We will use several times the following lemma, taken from [Hau22, Corollary 2.8] and the explanation thereafter:

**Lemma 2.21.** *Let  $B \subseteq A$  be a subgroup. Then the restriction map  $L_A \rightarrow L_B$  is surjective, with kernel  $I_A^B$  generated by the Euler classes  $e_V$  where  $V$  is running over a generating set of  $\ker(A^* \rightarrow B^*)$ .*

Similarly, the following holds on the level of equivariant formal groups.

**Proposition 2.22.** *Let  $G$  be an  $A$ -equivariant formal group.*

1. *Let  $\alpha: A \rightarrow C$  be a surjective group homomorphism. Assume that for  $V \notin \text{im}(C^* \rightarrow A^*)$ , the Euler class  $e_V$  is invertible. Then  $\alpha^* \alpha_* G \rightarrow G$  is an isomorphism.*
2. *Let  $\alpha: B \rightarrow A$  be an injective group homomorphism. Assume that for  $V \in \ker(A^* \rightarrow B^*)$ , we have  $e_V = 0$ . Then  $G$  is in the essential image of  $\alpha_*$ .*

*Proof.* Let  $G = (\varphi: A^* \rightarrow X)$ . Fixing a coordinate  $y(\epsilon)$  Zariski-locally and choosing a complete flag,  $X$  is defined by the directed system  $(\text{Spec } R/(y(V_1) \cdots y(V_n)))_n$  and each of these terms has underlying space  $\bigcup_{i=1}^n \text{im}(\varphi_{V_i}) \subseteq \text{Spec } R$ .

In the first item, Lemma 2.18 implies that  $\text{im}(\varphi_V)$  and  $\text{im}(\varphi_W)$  intersect each other in  $\text{Spec } \mathcal{O}_X$  only if  $[V] = [W] \in A^*/C^*$ . Thus, the underlying space of  $(\text{Spec } R/(y(V_1) \cdots y(V_n)))_n$  decomposes into closed subspaces  $\bigsqcup_{\nu \in A^*/C^*} \bigcup_{V_i \in \nu} \text{im}(\varphi_{V_i})$ , of which only finitely many are non-empty. This induces decompositions of the schemes  $\text{Spec } R/(y(V_1) \cdots y(V_n))$  and we obtain thus an  $A^*$ -equivariant isomorphism  $G \cong \bigsqcup_{A^*/C^*} \alpha_* G$  on the level of formal schemes.

By construction,  $\alpha^* \alpha_* G$  decomposes in the same way. On the unit copy, the map  $\alpha^* \alpha_* G \rightarrow G$  is an isomorphism since  $\alpha_* \alpha^* \alpha_* G \rightarrow \alpha_* G$  is one by Proposition 2.17. For the other copies, this follows by the  $A^*$ -equivariance of the map  $\alpha^* \alpha_* G \rightarrow G$ .

For the second item: by definition, the structure morphism  $\varphi: A^* \rightarrow X$  of  $G$  factors as  $A^* \rightarrow B^* \xrightarrow{\varphi'} X$ . The result follows from Proposition 2.13.  $\square$

The second part is also true in the setting of equivariant formal group laws, as shown in [Hau22, Lemma 2.7]. The analog of the first part becomes more complicated as  $\alpha^* \alpha_* G$  does not have a canonical coordinate; we will talk more about it in Proposition 2.25.

**Corollary 2.23.** *Let  $\alpha: A \rightarrow B$  be a surjective group homomorphism. Then  $\alpha^*$  is a fully faithful embedding from the category of  $B$ -equivariant formal groups to that of  $A$ -equivariant formal groups. The image consists of those  $A$ -equivariant formal groups such that  $e_V$  is invertible for  $V$  not in the image of  $B^* \rightarrow A^*$ .*

*Proof.* If  $G$  is a  $B$ -equivariant formal group, then  $\alpha^* G$  has the property that  $e_V$  is invertible for  $V$  not in the image of  $B^* \rightarrow A^*$  by construction. By the preceding proposition, invertibility of these Euler classes characterizes the image of  $\alpha^*$ . Moreover,  $\alpha^*$  is fully faithful since  $G \rightarrow \alpha_* \alpha^* G$  is an isomorphism by construction.  $\square$

The following proposition provides essentially a classification of equivariant formal groups over fields. The same statement already appears in [Str11, Corollary 8.3].

**Proposition 2.24.** *Let  $k$  be a field and  $G$  be an  $A$ -equivariant formal group over  $k$ . Denote by  $A/B$  the Pontryagin dual of the subgroup  $\{V \in A^* : e_V = 0\} \subseteq A^*$ . Denote further by  $A \xrightarrow{q} A/B \xleftarrow{i} \{1\}$  the obvious morphisms. Then  $G \cong q^* i_* p_* G$ , where  $p_* G$  is the non-equivariant formal group defined by  $p: A \rightarrow \{1\}$ .*

*Proof.* Assume  $e_V = 0$ , i.e.  $\varphi_\epsilon = \varphi_V$ . By the  $A^*$ -action, this implies  $\varphi_W = \varphi_{VW}$  for every  $W \in A^*$ . Setting  $W = V^{-1}$ , this implies  $e_{V^{-1}} = 0$ . Moreover, if  $\varphi_\epsilon = \varphi_W$ , this implies  $\varphi_\epsilon = \varphi_{VW}$ . Thus,  $\{V \in A^* : e_V = 0\} \subseteq A^*$  is indeed a subgroup.

Since  $k$  is a field,  $e_V = 0$  iff  $e_V$  is not invertible. By Proposition 2.22, we thus have  $G \cong q^* i_* \Gamma$  for some non-equivariant formal group  $\Gamma$  over  $k$ . One computes  $\Gamma \cong p_* q^* i_* \Gamma \cong p_* G$ .  $\square$

For any  $A$ , let  $\Phi^A L = L_A[e_V^{-1}]$  be the localization of  $L_A$  away from all Euler classes  $e_V$  for  $V \neq \epsilon$ . Our results above let us compute  $\Phi^A L$  quite explicitly (cf. [Gre01, Corollary 6.4] and [Hau22, Proposition 2.11]).

**Proposition 2.25.** *There is an isomorphism of the form*

$$\Phi^A L \cong L[(b_0^V)^{\pm 1}, b_i^V \mid i > 0, V \in A^* - \{\epsilon\}].$$

*Proof.* The ring  $\Phi^A L$  classifies  $A$ -equivariant formal group laws  $F = (R, \Delta, \theta, y(\epsilon))$  such that  $e_V$  is invertible for all  $V \neq \epsilon$ . By Proposition 2.22,  $R \cong \text{map}(A^*, \widehat{R})$ , where  $\widehat{R}$  is the completion of  $R$  at the augmentation ideal. The structure of  $F$  determines the structure of a non-equivariant formal group law  $p_* F$  on  $\widehat{R}$ , where  $p: A \rightarrow \{1\}$  is the projection; in particular, we obtain an isomorphism  $\widehat{R} \cong k[[y]]$ , where  $y$  is the image of  $y(\epsilon)$ . Vice versa,  $\Delta$  and  $\theta$  are determined by  $p_* F$ . In particular,  $\theta_\epsilon$  is the composite  $\text{map}(A^*, \widehat{R}) \xrightarrow{e_V \epsilon} \widehat{R} \rightarrow k$ . Thus we see that  $F$  is determined by  $p_* F$ , plus a choice of  $y(\epsilon)$  mapping to  $y$  under  $e_V \epsilon$ . Such  $y(\epsilon)$  are exactly those elements  $(y^V) \in \text{map}(A^*, k[[y]])$  such that  $y^\epsilon = y$  and  $y^V = b_0^V + b_1^V y + \dots$  with  $b_0^V \in k^\times$ . This gives the result.  $\square$

**Remark 2.26.** The elements  $b_i^V \in \Phi^A L$  in the previous proposition already come from elements  $\gamma_i^V \in L_A$ , which are uniquely defined by the property that

$$e_{\epsilon \otimes \tau} = \gamma_0^V + \gamma_1^V e_{V^{-1} \otimes \tau} + \gamma_2^V (e_{V^{-1} \otimes \tau})^2 + \dots + \gamma_n^V (e_{V^{-1} \otimes \tau})^n \in L_{A \times \mathbf{T}} / (e_{V^{-1} \otimes \tau})^{n+1},$$

for all  $n \in \mathbb{N}$ . Here,  $\tau \in \mathbf{T}^*$  denotes the tautological character for the circle group  $\mathbf{T}$ . In particular,  $\gamma_0^V$  equals the Euler class  $e_V$ . The elements  $\gamma_i^V$  are natural in the sense that  $\alpha^* \gamma_i^V = \gamma_i^{\alpha^* V}$  for every group homomorphism  $\alpha: B \rightarrow A$ . We refer to [Hau22, Section 2.7] for more details on this construction.

## 2.5 The relationship between Lazard rings at different groups and their completions

While not needed for our *classification* of invariant prime ideals, it will be necessary for our study of *containments* between invariant prime ideals to have a deeper look upon how Lazard rings at different groups relate. These properties are all based on the identification of the global Lazard ring with equivariant complex bordism in [Hau22].

**Proposition 2.27** ([Hau22], Proposition 5.50, Corollary 5.33, Lemma 5.28). *1. For every  $A$  and every non-torsion character  $V \in A^*$ , the sequence*

$$0 \rightarrow L_A \xrightarrow{e_V} L_A \xrightarrow{\text{res}_{\ker(V)}^A} L_{\ker(V)} \rightarrow 0$$

*is exact. In particular, all Euler classes  $e_V \in L_A$  for non-torsion characters  $V$  are non-zero divisors.*

*2. For every  $A$ , the complete  $L_A$ -Hopf algebra  $R$  of the universal  $A$ -equivariant formal group law is canonically isomorphic to the completion*

$$\lim_{n \in \mathbb{N}, V_1, \dots, V_n \in A^*} (L_{A \times \mathbf{T}}) / I_{V_1} \cdots I_{V_n},$$

*where  $I_{V_j}$  is the kernel of the restriction map  $(\text{id}, V_j)^*: L_{A \times \mathbf{T}} \rightarrow L_A$ . More generally,  $R^{\hat{\otimes} n}$  is a completion of  $L_{A \times \mathbf{T}^n}$ . Under this identification*

*(a) the comultiplication  $R \rightarrow R^{\hat{\otimes} R}$  and the augmentations  $\theta_V: R \rightarrow L_A$  are induced by the maps  $(\text{id}_A, m)^*: L_{A \times \mathbf{T}} \rightarrow L_{A \times \mathbf{T} \times \mathbf{T}}$  and  $(\text{id}_A, V)^*: L_{A \times \mathbf{T}} \rightarrow L_A$  on completion, and*

*(b) the elements  $y(V)$  are the image of  $e_{V \otimes \tau} \in L_{A \times \mathbf{T}}$  under the completion map, where  $\tau \in \mathbf{T}^*$  is the tautological character.*

The special case of (2) for  $A$  the trivial group is particularly important: Completing  $L_{\mathbf{T}^n}$  under the kernel  $I$  of the augmentation  $L_{\mathbf{T}^n} \rightarrow L$  yields a power series ring on  $n$  generators  $L[[y_1, y_2, \dots, y_n]]$ , where  $y_i$  is the Euler class of the  $i$ -th projection  $\mathbf{T}^n \rightarrow \mathbf{T}$ . Moreover, the isomorphism

$$(L_{\mathbf{T}^n})_I^{\hat{}} \cong L[[y_1, y_2, \dots, y_n]]$$

is natural in  $\mathbf{T}^n$ , where

- the functoriality of  $(L_{\mathbf{T}^n})_I^{\hat{}}$  is induced by the global functoriality of  $L_{\mathbf{T}^n}$ , and
- the functoriality of  $L[[y_1, y_2, \dots, y_n]]$  is through the universal formal group law over  $L$ .

For example, for any  $n \in \mathbb{N}$  the square

$$\begin{array}{ccc} L_{\mathbf{T}} & \xrightarrow{c} & L[[y]] \\ [n]^* \downarrow & & \downarrow [n]_F \\ L_{\mathbf{T}} & \xrightarrow{c} & L[[y]] \end{array}$$

commutes, where  $[n]: \mathbf{T} \rightarrow \mathbf{T}$  is the  $n$ th power map, and  $[n]_F: L[[y]] \rightarrow L[[y]]$  sends  $y$  to the  $n$ -series  $[n]_F(y)$  with respect to the universal formal group law. This implies that the Euler class  $e_{\tau^n} \in L_{\mathbf{T}}$  for the  $n$ th power map on  $\mathbf{T}$  is sent to the  $n$ -series  $[n]_F(y)$  under the completion map. Similar statements hold for the collection of  $L_{A \times \mathbf{T}^n}$  for fixed  $A$  and varying  $\mathbf{T}^n$ .

We further record, where again  $\tau: \mathbf{T} \rightarrow \mathbf{T}$  denotes the identity character:

**Corollary 2.28.** *Let  $x \in L_{\mathbf{T}}$  be an element whose image in  $L[[y]]$  is of the form  $\lambda y^k +$  higher order terms. Then  $x$  is uniquely divisible by  $e_{\tau}^k$  and the quotient  $x/e_{\tau}^k$  restricts to  $\lambda$  at the trivial group.*

*Proof.* First we note that  $e_{\tau} \in L_{\mathbf{T}}$  is a regular element by Proposition 2.27. Hence division by  $e_{\tau}$  is always unique if possible. By induction on  $k$  the corollary then follows from the following facts, for an element  $z \in L_{\mathbf{T}}$  and its image  $c(z) \in L_{\mathbf{T}}$ :

1.  $z$  is divisible by  $e_\tau$  if and only if  $\text{res}_1^T(z) = 0$ .
2. The leading coefficient of  $c(z)$  is equal to  $\text{res}_1^T(z)$ .
3. If  $\text{res}_1^T(z) = 0$  and hence  $z$  is divisible by  $e_\tau$ , then  $c(z/e_\tau) = c(z)/y$ .  $\square$

Note that the global functoriality makes every equivariant Lazard ring  $L_A$  an algebra over the non-equivariant Lazard ring  $L$ , and that all restriction maps  $\alpha^*: L_A \rightarrow L_B$  are  $L$ -algebra maps. We have the following:

**Proposition 2.29** ([Hau22], [Com96]).  *$L_A$  is free as a module over  $L$ , for every abelian compact Lie group  $A$ .*

**Corollary 2.30.** *The exact sequences of  $L$ -modules in Part 1 of Proposition 2.27 are split exact. In particular, they remain exact after applying any additive functor.*

The following special case is of particular importance to us:

**Corollary 2.31.** *Let  $n \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$  and  $I_n \subseteq L$  be the ideal generated by  $(v_0, \dots, v_{n-1})$ . Then for every  $A$  and every non-torsion character  $V \in A^*$ , the sequence*

$$0 \rightarrow L_A/I_n \xrightarrow{e_V} L_A/I_n \xrightarrow{\text{res}_{\ker(V)}^A} L_{\ker(V)}/I_n \rightarrow 0$$

*is exact. In particular, the Euler classes  $e_V \in L_A$  for non-torsion characters  $V \in A^*$  remain non-zero divisors in  $L_A/I_n$ . In the terminology of [Hau22], the assignment*

$$A \mapsto L_A/I_n$$

*(together with the image of  $e_\tau$  under  $L_{\mathbf{T}} \rightarrow L_{\mathbf{T}}/I_n$ ) is a regular global group law.*

## 3 The Lazard Hopf algebroid and its associated stack

### 3.1 Strict isomorphisms and the Lazard Hopf algebroid

In this subsection, we will introduce one of our main objects of study, the Hopf algebroid  $(L_A, S_A)$  for equivariant formal group laws. There is a hierarchy of notions of isomorphisms between (equivariant) formal group laws, namely

- isomorphisms, which do not need to respect the coordinate and are thus really isomorphisms between the underlying (equivariant) formal groups;
- strict isomorphisms, which respect the coordinate up to quadratic terms;
- very strict isomorphisms as in Section 2.2, which respect the coordinate strictly.

Already classically, strict isomorphisms are especially relevant since the Hopf algebroid modeled on them gives  $(MU_*, MU_*MU)$ .

**Definition 3.1.** A *strict isomorphism* between two  $A$ -equivariant formal group laws

$$(k, R_1, \Delta_1, \theta_1, y(\varepsilon)_1) \quad \text{and} \quad (k, R_2, \Delta_2, \theta_2, y(\varepsilon)_2)$$

over the same ground ring  $k$  is a  $k$ -linear isomorphism

$$\varphi: R_1 \xrightarrow{\cong} R_2$$

of Hopf algebras over  $k^{A^*}$  such that  $y(\varepsilon)_1$  is sent to  $y(\varepsilon)_2$  modulo  $I_\varepsilon^2$ , where  $I_\varepsilon$  is the augmentation ideal in  $R_2$ . Explicitly, this means that  $(\varphi \otimes \varphi) \circ \Delta_1 = \Delta_2 \circ \varphi$ ,  $\theta_2 \circ \varphi = \theta_1$  and  $\varphi(y(\varepsilon)_1) = x \cdot y(\varepsilon)_2$  for some unit  $x \in R_2$  which augments to  $1 \in k$ .

By definition, strict isomorphisms need not preserve the coordinate, hence they are generally not morphisms of formal group laws in the sense of Section 2.2. On the other hand, every very strict isomorphism is both a strict isomorphism and an isomorphism in the category of  $A$ -equivariant formal group laws.

Let  $SI$  be the category of strict isomorphisms of  $A$ -equivariant formal group laws. More precisely, its objects are quadruples  $(k, F^1, F^2, \varphi)$  consisting of a commutative ring  $k$ , two  $A$ -equivariant formal group laws  $F^1$  and  $F^2$  over  $k$  and a strict isomorphism  $\varphi$  between them. Morphisms between two such quadruples  $(k_1, F_1^1, F_1^2, \varphi_\epsilon)$  and  $(k_2, F_2^1, F_2^2, \varphi_2)$  are given by a pair of morphisms  $f_1: F_1^1 \rightarrow F_2^1$  and  $f_2: F_1^2 \rightarrow F_2^2$  with the same underlying map  $k_1 \rightarrow k_2$ , such that

$$\begin{array}{ccc} R_1^1 & \xrightarrow{\varphi_\epsilon} & R_1^2 \\ f_1 \downarrow & & \downarrow f_2 \\ R_2^1 & \xrightarrow{\varphi_2} & R_2^2 \end{array}$$

commutes.

**Proposition 3.2.** *The category  $SI$  has an initial object, whose underlying ring  $S_A$  is a localization of an infinite polynomial ring  $L_A[a_1^f, a_2^f, \dots]$  over the Lazard ring  $L_A$ .*

By Remark 2.10, we can equivalently say that the functor

$$\begin{aligned} \text{CAlg}_{S_A} &\rightarrow SI \\ (f: S_A \rightarrow k) &\mapsto (k, (f_1)_* F^{\text{uni}}, (f_2)_* F^{\text{uni}}, \text{id}) \end{aligned}$$

is an equivalence of categories. Here, we use that  $SI$  is again cofibered in groupoids over commutative rings.

Before we prove the proposition, it will be good to review two general results about the maps  $\theta_V: R \rightarrow k^{A^*} \rightarrow k$  for an  $A$ -equivariant formal group law  $(k, R, \Delta, \theta, y(\epsilon))$ . Recall that after choosing a complete flag  $f$  we can write every element  $x \in R$  uniquely as

$$x = \sum_{n \in \mathbb{N}} a_n^f y(W_n)$$

Given  $V \in A^*$ , we have

$$\theta_V(x) = \sum_{n \in \mathbb{N}} a_n^f \theta_V(y(W_n)) = \sum_{n \in \mathbb{N}} a_n^f e_{V W_n},$$

where  $e_{V W_n}$  is defined as the product  $e_{V \cdot V_n} e_{V \cdot V_{n-1}} \cdots e_{V \cdot V_1}$ . This is a finite sum, since  $e_{V W_n} = 0$  if there exists some  $i \leq n$  where  $V_i = V^{-1}$ . We obtain:

**Lemma 3.3.** *The augmentation  $\theta_V$  is a linear combination of  $a_n^f$  whose coefficients are products of Euler classes.*

Moreover, we have the following:

**Lemma 3.4.** *Let  $F = (k, R, \Delta, \theta, y(\epsilon))$  be an  $A$ -equivariant formal group law. Then an element  $x \in R$  is a unit if and only if  $\theta_V(x)$  is a unit in  $k$  for all  $V \in A^*$ .*

*Proof.* See [Hau22, Lemma 2.3]. □

*Proof of Proposition 3.2.* For every object  $(k, F^1, F^2, \varphi) \in SI$  we can define a new  $A$ -equivariant formal group law  $\tilde{F}^2$  for which the components  $k, R, \Delta$  and  $\theta$  agree with those of  $F^1$ , but  $\tilde{y}(\epsilon)_2$  is defined as  $\varphi^{-1}(y(\epsilon)_2)$ , the preimage of the coordinate of  $F^2$  under  $\varphi$ . Then we obtain a new object  $(k, F^1, \tilde{F}^2, \text{id}_{R_1})$  which is isomorphic in  $SI$  to  $(k, F^1, F^2, \varphi)$  via the commutative square

$$\begin{array}{ccc} F^1 & \xrightarrow{\text{id}} & \tilde{F}^2 \\ \text{id} \downarrow & & \downarrow \varphi \\ F^1 & \xrightarrow{\varphi} & F^2 \end{array}$$

Note that the two vertical maps are in fact very strict isomorphisms of  $A$ -equivariant formal group laws. In summary, every object of  $SI$  is isomorphic to one of the form  $(k, F_1, F_2, \text{id})$ , where  $F_1$  and  $F_2$  are given by the same  $k^{A^*}$ -augmented  $k$ -Hopf algebra and the strict isomorphism

is the identity. This is the same data as a single  $A$ -equivariant formal group law  $F$  together with a second choice of coordinate  $y(\varepsilon)_2 = x \cdot y(\varepsilon)_1$  for some unit  $x \in R$  which augments to 1. Up to very strict isomorphism we can further assume that  $F$  is the push-forward along a map  $L_A \rightarrow k$ . We claim that the functor sending an  $A$ -equivariant formal group law to the set of all units  $x \in R$  augmenting to 1 is representable by an  $L_A$ -algebra  $S_A$ . Indeed, a presentation for  $S_A$  is given by

$$S_A = L_A[a_1^f, a_2^f, \dots][P_V(1, a_1^f, a_2^f, \dots)^{-1} \mid V \in A^* - \{\varepsilon\}] \quad (3.5)$$

where  $f$  is a complete flag starting with  $\varepsilon$  and  $P_V$  is the linear combination expressing  $\theta_V$  in terms of the coefficients with respect to  $f$ ; see Lemma 3.3 and Lemma 3.4. Here we use that the units augmenting to 1 are precisely the elements of the form  $1 + \sum_{n \in \mathbb{N}^+} a_n^f y(W_n)$ , with  $y(W_n)$  as in the end of Section 2.1.

Thus, the functor

$$\begin{aligned} \text{CAlg}_{S_A} &\rightarrow \text{SI} \\ (f: S_A \rightarrow k) &\mapsto (k, (f_1)_* F^{\text{uni}}, (f_2)_* F^{\text{uni}}, \text{id}) \end{aligned}$$

is an equivalence of categories.  $\square$

**Remark 3.6.** The same proof shows that the category of all (not necessarily strict) isomorphisms also has an initial object, whose underlying ring is  $S_A[(a_0^f)^{\pm 1}]$ . In fact, the only difference in the proof is that the equation for the unit  $x$  is now of the form  $\sum_{n \in \mathbb{N}} a_n^f y(W_n)$  so that the presentation for the analog of  $S_A$  becomes

$$L_A[a_0^f, a_1^f, a_2^f, \dots][(a_0^f)^{-1}, P_V(a_0^f, a_1^f, a_2^f, \dots)^{-1} \mid V \in A^* - \{\varepsilon\}].$$

Since  $P_V(a_0^f, a_1^f, a_2^f, \dots) = a_0^f P_V(1, \frac{a_1^f}{a_0^f}, \frac{a_2^f}{a_0^f}, \dots)$ , this ring is indeed isomorphic to

$$L_A[(a_0^f)^{\pm 1}, \frac{a_1^f}{a_0^f}, \frac{a_2^f}{a_0^f}, \dots][P_V(1, \frac{a_1^f}{a_0^f}, \frac{a_2^f}{a_0^f}, \dots)^{-1} \mid V \in A^* - \{\varepsilon\}] \cong S_A[(a_0^f)^{\pm 1}].$$

There are functors

$$s, t: \text{SI} \rightarrow \text{A-FGL}$$

sending a strict isomorphism to its source and target, respectively, as well as an ‘identity’ functor

$$\text{A-FGL} \rightarrow \text{SI},$$

an ‘inverse’ functor

$$i: \text{SI} \rightarrow \text{SI}$$

and a ‘composition’

$$c: \text{SI} \times \text{SI} \rightarrow \text{SI},$$

which restrict to the full subcategory of those objects of SI where the isomorphism  $\varphi$  is given by the identity. By representability we obtain analogous source and target maps  $s, t: L_A \rightarrow S_A$ , an identity map  $S_A \rightarrow L_A$ , an inverse map  $i: S_A \rightarrow S_A$  and a composition map  $c: S_A \rightarrow S_A \otimes_{L_A} S_A$ .

**Corollary 3.7.** *The pair  $(L_A, S_A)$  together with the above structure defines a Hopf algebroid i.e. a cogroupoid object in commutative rings. The associated functor*

$$\text{CRing} \rightarrow \text{Groupoids}$$

*is equivalent to the one sending a commutative ring  $k$  to the groupoid of  $A$ -equivariant formal group laws over  $k$  and strict isomorphisms between them.*

**Remark 3.8.** The Hopf algebroid  $(L_A, S_A)$  has a natural grading. This can be constructed in three equivalent ways:

- [Hau22, Corollary 5.6] shows that the global Lazard ring admits a unique grading such that it defines a graded global group law. By the same arguments, the ‘universal global group law with a strict  $n$ -tuple of coordinates’  $\mathbf{L}^{(n)}$  (cf., [Hau22, Section 5.8]) also carries a unique grading for every  $n$  such that the coordinates have degree  $-2$ . The source-target maps  $s, t: \mathbf{L} \rightarrow \mathbf{L}^{(2)}$ , the identity map  $\mathbf{L}^{(2)} \rightarrow \mathbf{L}$ , the inverse map  $i: \mathbf{L}^{(2)} \rightarrow \mathbf{L}^{(2)}$  and the composition map  $c: \mathbf{L}^{(2)} \rightarrow \mathbf{L}^{(3)}$  all preserve this grading, since they are defined through their effect on the respective coordinates. Evaluating these maps at a group  $A$  yields the Hopf algebroids  $(L_A, S_A)$ , which hence inherit a grading compatible with all restriction and inflation maps.
- The groupoid-valued functor represented by  $(L_A, S_A)$  admits a  $\mathbb{G}_m$ -action from multiplying the coordinate of the equivariant formal group law by a unit  $u$  and acting on strict isomorphisms by multiplying  $a_n^f$  by  $u^n$ . This corresponds to an even grading on  $(L_A, S_A)$ , putting e.g.  $a_n^f$  in degree  $2n$ .
- By its interpretation as representing the groupoid of  $A$ -equivariant formal group laws and isomorphisms between them (see Remark 3.6),  $(L_A, S_A[(a_0^f)^{-1}])$  obtains the structure of a Hopf algebroid. An element  $s$  in  $S_A$  is of degree  $2n$  if applying the composition map  $S_A[(a_0^f)^{-1}] \rightarrow S_A[(a_0^f)^{-1}] \otimes_{L_A} S_A[(a_0^f)^{-1}]$  to  $s$  gives  $(a_0^f)^n c(s)$ , where  $c: S_A \rightarrow S_A \otimes_{L_A} S_A$  is the composition map of  $(L_A, S_A)$ .

One can show that this is the same grading coming from the isomorphism

$$(L_A, S_A) \cong (\pi_*^A MU_A, \pi_*^A MU_A \wedge MU_A)$$

from [Hau22, Theorem E].

Given a strict isomorphism  $\varphi: F_1 \cong F_2$  of  $B$ -equivariant formal group laws and a group homomorphism  $\alpha: B \rightarrow A$ , we obtain an induced strict isomorphism  $\alpha_*\varphi: \alpha_*F_1 \cong \alpha_*F_2$  by completion. This assignment is compatible with composition of strict isomorphisms. Therefore, the functor  $\mathbf{L}$  from Section 2.3 extends to a functor

$$\mathbf{L}: (\text{abelian compact Lie groups})^{op} \rightarrow \text{Hopf algebroids,}$$

which we call the *global Lazard Hopf algebroid*.

### 3.2 The moduli stack of equivariant formal groups

We have discussed above that the Hopf algebroid  $(L_A, S_A)$  represents the functor sending a commutative ring to the groupoid of  $A$ -equivariant formal group laws and strict isomorphisms between them. As discussed in Remark 3.8,  $(L_A, S_A)$  is naturally a graded Hopf algebroid. On the other hand, we have discussed in Remark 3.6 the ungraded Hopf algebroid  $(L_A, S_A[a_0^{\pm 1}])$  classifying  $A$ -equivariant formal group laws and all isomorphisms between them. It is easy to see that this is precisely the ungraded Hopf algebroid associated to the graded Hopf algebroid  $(L_A, S_A)$  in the sense of [MO20, Section 4.1].<sup>1</sup> We will follow [MO20, Definition 4.1] by defining the stack associated to a graded Hopf algebroid as the fpqc-stackification of the groupoid-valued functor corepresented by its associated ungraded Hopf algebroid. In particular, the stack associated to the graded Hopf algebroid  $(L_A, S_A)$  is the same as the stack associated to the ungraded Hopf algebroid  $(L_A, S_A[a_0^{\pm 1}])$ . Equivalently, it is the quotient of the stack associated to  $(L_A, S_A)$  by the  $\mathbb{G}_m$ -action induced by the grading.

**Proposition 3.9.** *Sending an  $A$ -equivariant formal group law to its underlying  $A$ -equivariant formal group defines an equivalence from the stack associated to the graded Hopf algebroid  $(L_A, S_A)$  to  $\mathcal{M}_{FG}^A$ , the (pseudo-)functor sending a commutative ring to the groupoid of  $A$ -equivariant formal groups over it.*

*Proof.* It suffices to show that  $\mathcal{M}_{FG}^A$  is an fpqc-stack. Indeed: denote the stack associated to the graded Hopf algebroid  $(L_A, S_A)$  by  $\mathcal{X}_A$ ; equivalently, this is the stack associated to the ungraded Hopf algebroid  $(L_A, S_A[a_0^{\pm 1}])$ . Since the augmentation ideal of every  $A$ -equivariant formal group  $(X, \varphi)$  is by definition fpqc-locally trivial,  $(X, \varphi)$  comes after fpqc-base change

<sup>1</sup>[MO20] uses an algebraic grading convention, while we use a topological one; thus one has to double all degrees.

from an  $A$ -equivariant formal group law (see Lemma 2.6). Thus,  $\mathcal{X}_A \rightarrow \mathcal{M}_{FG}^A$  is essentially surjective as a functor of stacks. Moreover, it is fully faithful since isomorphisms between  $A$ -equivariant formal group laws are precisely isomorphisms of the underlying  $A$ -equivariant formal groups. Thus, it remains to show that  $\mathcal{M}_{FG}^A$  satisfies fpqc-descent on morphisms and objects.

Given two formal  $k$ -schemes  $X$  and  $Y$ , the functor

$$k\text{-algebras} \rightarrow \text{Set}, \quad K \mapsto \text{Hom}_K(X \times_{\text{Spec } k} \text{Spec } K, Y \times_{\text{Spec } k} \text{Spec } K)$$

is an fpqc-sheaf. Indeed, if we view  $X$  and  $Y$  as ind-objects  $(X_i)$  and  $(Y_j)$ , we can rewrite this Hom as  $\lim_i \text{colim}_j \text{Hom}_K(X_i \times_{\text{Spec } k} \text{Spec } K, Y_j \times_{\text{Spec } k} \text{Spec } K)$  and equalizers commute with filtered colimits in sets. This easily implies that  $\mathcal{M}_{FG}^A$  satisfies fpqc-descent on morphisms.

Let  $\text{Fin}(A^*)$  denote the directed set of finite multi subsets of  $A^*$  (i.e. elements can occur more than once), ordered by inclusion. Sending an  $A$ -equivariant formal group  $(X, \varphi)$  over  $k$  to the system  $(\text{Spec } \mathcal{O}_X/\mathcal{I})$ , where  $\mathcal{I}$  runs over all finite products of the  $\ker(\mathcal{O}_X \xrightarrow{\phi(V)^*} k)$  for  $V \in A^*$ , defines a functor from  $A$ -equivariant formal groups over  $k$  to  $\text{Fun}(\text{Fin}(A^*), \text{Aff}_k)$ .

Given a descent datum for  $A$ -equivariant formal groups for the fpqc-cover  $T \rightarrow S$ , we thus obtain a descent datum for a  $\text{Fin}(A^*)$ -diagram of affine schemes (with closed immersions as transition maps), which descends thus to a  $\text{Fin}(A^*)$ -diagram of closed immersion in  $\text{Aff}_k$ . This diagram defines a formal  $k$ -scheme  $X$ . By descent for morphisms between formal  $k$ -schemes,  $X$  obtains a group structure and also a group homomorphism  $\varphi: S \times A^* \rightarrow X$ . Conditions (1) and (2) for an  $A$ -equivariant formal group are fulfilled by construction.  $\square$

**Remark 3.10.** As every equivariant formal group comes Zariski-locally from an equivariant formal group law, in our case only a Zariski-stackification was necessary to pass from  $(L_A, S_A[u^{\pm 1}])$  to  $\mathcal{M}_{FG}^A$ .

**Proposition 3.11.** *Let  $B \subseteq A$  be a subgroup of an abelian group  $A$ . Denote by  $\alpha: B \rightarrow A$  the inclusion and by  $q: A \rightarrow A/B$  the projection.*

- (i) *The functor  $q^*$  induces an open immersion  $\mathcal{M}_{FG}^{A/B} \rightarrow \mathcal{M}_{FG}^A$  whose image is the common non-vanishing locus of the Euler classes  $e_V$  for all  $V \notin \text{im}((A/B)^* \rightarrow A^*)$ .*
- (ii) *The functor  $\alpha_*$  induces a closed immersion  $\mathcal{M}_{FG}^B \rightarrow \mathcal{M}_{FG}^A$ , inducing an equivalence of  $\mathcal{M}_{FG}^B$  to the common vanishing locus of the  $e_V$  for all  $V \in \ker(A^* \rightarrow B^*)$ .*

*Proof.* The first part is a reformulation of Corollary 2.23. The immersion is open since it is open after pullback to  $\text{Spec } L_A$ .

For the second, note that by Proposition 2.22 every  $A$ -equivariant formal group such that  $e_V = 0$  for all  $V \in \ker(A^* \rightarrow B^*)$  is of the form  $\alpha_* G$  for  $G$  a  $B$ -equivariant formal group. Moreover, by construction, this vanishing of Euler classes is true for all  $A$ -equivariant formal groups of the form  $\alpha_* G$  and thus characterizes the image of  $\alpha_*$ . The substack given by this image is closed since it is closed after pullback to  $\text{Spec } L_A$ . Moreover,  $\alpha_*$  is fully faithful by Proposition 2.13.  $\square$

**Remark 3.12.** With notation as in the preceding proposition, we have  $\alpha^* \alpha_* G \cong G$  for every  $B$ -equivariant formal group  $G$  and  $H \cong q_* q^* H$  for every  $A/B$ -equivariant formal group  $H$ . Thus, the substacks in the preceding proposition are retractive.

**Example 3.13.** Proposition 3.11 gives closed immersions of  $\mathcal{M}_{FG}^{\{1\}}$  and  $\mathcal{M}_{FG}^{C_2}$  into  $\mathcal{M}_{FG}^{C_4}$ . The first is the common vanishing locus of all Euler classes (which equals the vanishing locus of the Euler class of one of the two generators of  $(C_4)^*$ , cf. Proposition 2.24). Its complement is the open substack given by the non-vanishing locus of the Euler class of the generators and hence equivalent to  $\mathcal{M}_{FG}^{C_4/C_2}$ . The second, i.e. the closed immersion of  $\mathcal{M}_{FG}^{C_2}$ , equals the vanishing locus of  $e_{[2]}$  for  $[2]: C_4 \xrightarrow{[2]} C_4 \hookrightarrow \mathbf{T}$ . Its complement is an open substack equivalent to  $\mathcal{M}_{FG}^{C_4/C_4}$ , the non-vanishing locus of all Euler classes of non-trivial characters.

For a non-cyclic group  $A$  the situation is more complicated, and we cannot expect that the complement of the closed substack  $\mathcal{M}_{FG}^B$  in  $\mathcal{M}_{FG}^A$  can be expressed as a single open substack  $\mathcal{M}_{FG}^{A/C}$  in general and vice versa. In general, the complement of  $\mathcal{M}_{FG}^B$  in  $\mathcal{M}_{FG}^A$  can be written as the union of the open substacks  $\mathcal{M}_{FG}^{A/C}$  where  $C$  runs over the minimal subgroups of  $A$  not contained in  $B$ . We have indicated the situation for  $A = C_2 \times C_2$  in Fig. 3.





Figure 2: Decompositions of  $\mathcal{M}_{FG}^{C_4}$  into open and closed substacks, using misty rose for open and lavender for closed

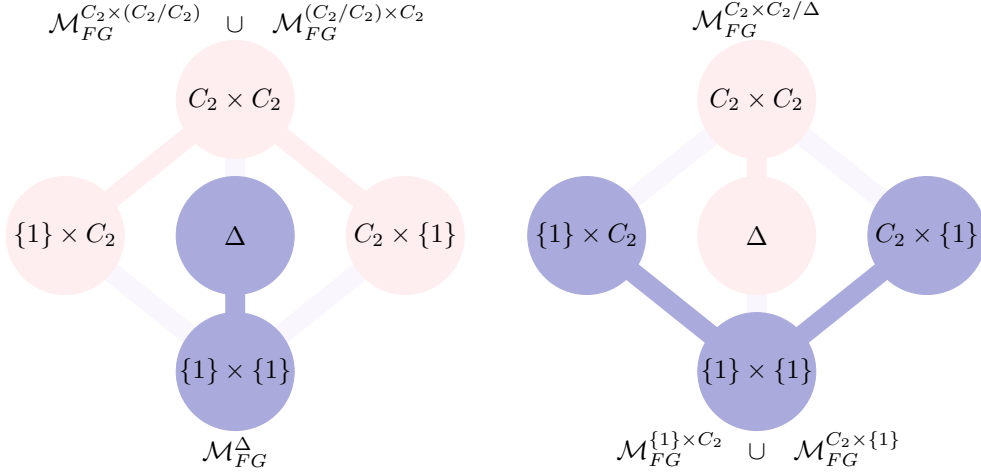


Figure 3: Decompositions of  $\mathcal{M}_{FG}^{C_2 \times C_2}$  into open and closed substacks,  $\Delta$  being the diagonal subgroup

## 4 Points of the moduli stack of equivariant formal groups and invariant prime ideals

The goal of this section is to classify the points of  $\mathcal{M}_{FG}^A$  and the invariant prime ideals of  $(L_A, S_A)$ . Although the latter could be done without the former, we feel that both questions are of the same importance and the stack point of view makes some issues more transparent.

### 4.1 The space associated to a stack

As mentioned above, given a graded flat Hopf algebraoid  $(A, \Gamma)$ , we can associate an ungraded Hopf algebraoid  $(A, \Gamma[u^{\pm 1}])$  to it (see e.g. [MO20, Section 4.1]).<sup>2</sup> The category of comodules over the latter is equivalent to that of graded comodules over  $(A, \Gamma)$ . The stack  $\mathcal{X}$  associated to  $(A, \Gamma)$  is by definition the stack associated to  $(A, \Gamma[u^{\pm 1}])$ , i.e. the fpqc-stackification of the presheaf of groupoids represented by  $(A, \Gamma[u^{\pm 1}])$  on the category of all schemes. We denote the resulting morphism  $\text{Spec } A \rightarrow \mathcal{X}$  by  $\pi$ .

**Definition 4.1.** Let  $(A, \Gamma)$  be a Hopf algebraoid with units  $\eta_L$  and  $\eta_R$ . An ideal  $I \subseteq A$  is called *invariant* if  $\eta_L(I)\Gamma = \eta_R(I)\Gamma$ . If  $(A, \Gamma)$  is graded, we will assume that  $I$  is also graded, i.e. generated by homogeneous elements.

It is easy to check that invariant ideals in a graded Hopf algebraoid  $(A, \Gamma)$  correspond exactly to graded subcomodules of  $A$  and thus to ideal sheaves on  $\mathcal{X}$ . Here, we use that  $\pi^*: \text{QCoh}(\mathcal{X}) \rightarrow \text{QCoh}(\text{Spec } A) \simeq \text{Mod}_A$  refines to an equivalence from quasi-coherent  $\mathcal{O}_{\mathcal{X}}$ -modules to graded  $(A, \Gamma)$ -comodules.

<sup>2</sup>Standard conventions force us to use  $A$  as part of the notation of a general Hopf algebraoid, while  $A$  stands in most of this article for a compact abelian Lie group. We trust that this does not cause confusion.

Following [LMB00, Section 5] and [Aut, Tag 04XL] in the case of Artin stacks, we can associate a topological space to  $\mathcal{X}$ . To that purpose recall that  $\mathcal{U} \rightarrow \mathcal{X}$  is an *open immersion* if the pullback  $\mathcal{U} \times_{\mathcal{X}} \text{Spec } A \rightarrow \text{Spec } A$  is an open immersion.

**Definition 4.2.** For  $\mathcal{X}$  as above, define the underlying set of  $|\mathcal{X}|$  to consist of equivalence classes of morphisms  $x: \text{Spec } K \rightarrow \mathcal{X}$  for  $K$  a field; the equivalence relation is generated by isomorphisms and  $x \sim (\text{Spec } L \rightarrow \text{Spec } K \xrightarrow{x} \mathcal{X})$ , where  $L$  is a field extension of  $K$ . We call a subset of  $|\mathcal{X}|$  *open* if it is the image of  $|\mathcal{U}| \rightarrow |\mathcal{X}|$  for an open immersion  $\mathcal{U} \rightarrow \mathcal{X}$ .

Equivalently, we can characterize the opens as the images of those opens in  $\text{Spec } A$  that are invariant, i.e. have the same preimage along both the left and right unit  $\text{Spec } \Gamma[u^{\pm 1}] \rightarrow \text{Spec } A$ . Indeed: by descent, an open immersion  $\mathcal{U} \rightarrow \mathcal{X}$  corresponds to an open immersion  $\mathcal{V} \rightarrow \text{Spec } A$  with an isomorphism  $\mathcal{V} \times_{\text{Spec } A} \text{Spec } \Gamma[u^{\pm 1}] \cong \text{Spec } \Gamma[u^{\pm 1}] \times_{\text{Spec } A} \mathcal{V}$  over  $\text{Spec } \Gamma[u^{\pm 1}]$  satisfying a cocycle condition. But the category of open immersions into some  $X$  with isomorphisms over  $X$  between them is equivalent to the discrete category of open subsets of  $X$ , yielding the required equivalence. Since invariant opens in  $\text{Spec } A$  form a topology, we deduce that the opens in  $|\mathcal{X}|$  form a topology. One further checks that the map induced by any morphism of stacks is continuous.

Our definition coincides with that of [LMB00] and [Aut] in the intersection of their domains, e.g. when  $\mathcal{X}$  is an affine scheme, where we get the usual topology.

**Proposition 4.3.** *Let  $(A, \Gamma)$  be a graded Hopf algebroid with associated stack  $\mathcal{X}$ . Then:*

1. *For every invariant prime ideal  $I \subseteq A$ , the image of  $V(I) \subseteq |\text{Spec } A|$  is closed in  $|\mathcal{X}|$  and  $\eta_I = |\pi|(\eta)$  for  $\eta$  the generic point of  $V(I)$  (i.e. the point in  $|\text{Spec } A|$  corresponding to  $I$ ) is generic (i.e.  $\overline{\{\eta_I\}} = |\pi|(V(I))$ ).*
2. *Mapping  $I$  to  $\eta_I$  defines an injection from the set of invariant prime ideals  $\text{Spec}^{\text{inv}}(A)$  in  $A$  to  $|\mathcal{X}|$ . Equipping  $\text{Spec}^{\text{inv}}(A)$  with the subspace topology from  $\text{Spec } A$ , this map is continuous; if it is a bijection, it is a homeomorphism.*

*Proof.* For the first point, observe first that for every invariant ideal  $I$ , the set  $V(I)$  is invariant and hence the complement of  $V(I)$  defines an invariant open. Thus the image in  $|\mathcal{X}|$  is open. Moreover,  $|\pi|^{-1}(|\pi|(V(I))) = V(I)$ . Thus  $|\pi|(V(I))$  is the complement of  $|\pi|(|\text{Spec } A| \setminus V(I))$  and thus closed.

Assume now that  $I$  is an invariant prime ideal. Let  $\eta_I$  be the image of the generic point  $\eta$  of  $V(I)$ . Then  $\overline{\{\eta_I\}} \subseteq |\pi|(V(I)) = |\pi|\overline{\{\eta\}} \subseteq \overline{\{\eta_I\}}$  and hence  $\overline{\{\eta_I\}} = |\pi|(V(I))$ .

For the injectivity of  $\text{Spec}^{\text{inv}} A \rightarrow |\mathcal{X}|$ , let  $I, J \subseteq A$  be two invariant prime ideals with  $\eta_I = \eta_J$ . By the first point, this implies that  $|\pi|(V(I)) = |\pi|(V(J))$  and hence  $V(I) = V(J)$ . Thus,  $I = J$ .

The continuity of  $\text{Spec}^{\text{inv}}(A) \rightarrow |\mathcal{X}|$  follows from that of  $\text{Spec } A \rightarrow |\mathcal{X}|$ . An arbitrary closed set of  $\text{Spec}^{\text{inv}} A$  is of the form  $V(I) \cap \text{Spec}^{\text{inv}}(A)$  for some ideal  $I \subseteq A$ . Set  $I' = \bigcap_{I \subseteq J} J$ , where the  $J \subseteq A$  run over all invariant ideals containing  $I$ . Then  $V(I) \cap \text{Spec}^{\text{inv}} A = V(J) \cap \text{Spec}^{\text{inv}} A$ . If  $|\pi|: \text{Spec}^{\text{inv}} A \rightarrow |\mathcal{X}|$  is a bijection,  $|\pi|^{-1}(|\pi|(V(J))) = V(J)$  implies  $|\pi|(V(J) \cap \text{Spec}^{\text{inv}} A) = |\pi|(V(J))$  and this is closed by the first part.  $\square$

We warn the reader that in general, the preimage of an irreducible closed subset of  $|\mathcal{X}|$  won't be irreducible in  $|\text{Spec } A|$  and thus does not correspond to an invariant prime ideal.

**Proposition 4.4.** *Let  $A$  be a compact abelian Lie group. The underlying set of  $|\mathcal{M}_{FG}^A|$  is in bijection with the product of the set  $\text{Sub}(A)$  of closed subgroups of  $A$  and  $|\mathcal{M}_{FG}|$ . For a given  $A$ -equivariant formal group  $G$  over a field, the point in  $|\mathcal{M}_{FG}|$  is the pushforward of  $G$  along  $p: A \rightarrow \{e\}$ , and the subgroup is Pontryagin dual to  $A^*/\{V \in A^* : e_V = 0\}$ .*

*Proof.* Given a closed subgroup  $B$  of  $A$  and a non-equivariant formal group  $\Gamma$  over a field  $k$ , we obtain an  $A$ -equivariant formal group via  $q^*i_*\Gamma$ , where  $A \xrightarrow{q} A/B \xleftarrow{i} \{1\}$ . By Proposition 2.24, every  $A$ -equivariant formal group  $G$  over a field  $k$  is isomorphic to one of this form. Here, the projection  $A \rightarrow A/B$  is Pontryagin dual to the subgroup  $\{V \in A^* : e_V = 0\}$  and thus  $B$  is uniquely defined by  $G$ . Moreover, necessarily  $\Gamma \cong p_*G$ . This implies that two  $A$ -equivariant formal groups  $G_1$  and  $G_2$  defined over the same field  $k$  are isomorphic if and only if they have the same associated closed subgroup and their completions  $p_*G_1$  and  $p_*G_2$  are isomorphic as non-equivariant formal groups, proving the claim.  $\square$

We recall that  $|\mathcal{M}_{FG}|$  has been computed by Honda: its points are classified by a pair  $(p, n)$ , where  $p \geq 0$  is the characteristic of the field and  $n \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$  is the height of the formal group, with  $n = 0$  if  $p = 0$ . We will below always identify  $|\mathcal{M}_{FG}|$  with such pairs.

**Remark 4.5.** By Proposition 3.11, there is an open immersion  $\mathcal{M}_{FG}^{A/B} \rightarrow \mathcal{M}_{FG}^A$  for any closed subgroup  $B \subseteq A$  and hence  $|\mathcal{M}_{FG}^{A/B}|$  is homeomorphic to an open subset of  $|\mathcal{M}_{FG}^A|$ . Given  $C \subseteq A/B$ , the point corresponding to  $(C, p, n)$  is  $(q^{-1}(C), p, n)$  for  $q: A \rightarrow A/B$  the projection. Indeed, for a  $A/B$ -equivariant formal group  $G$  over a field corresponding to  $(C, p, n)$ , we have

$$\{V \in A^* : e_V = 0\} = q^*(\{V \in (A/B)^* : e_V = 0\}) = q^*((A/B)/C)^*.$$

The Pontryagin dual of  $A^*/q^*((A/B)/C)^*$  is precisely  $q^{-1}(C)$  since  $A/q^{-1}(C) \cong (A/B)/C$ .

## 4.2 Invariant prime ideals of $(L_A, S_A)$

As before, let  $\Phi^B L = L_B[e_V^{-1}]$  be the localization of  $L_B$  away from all Euler classes  $e_V$  for  $V \neq \epsilon$  and let  $\Phi^B S = S_B \otimes_{L_B} \Phi^B L$ . Denote by  $I_{p,n}$  the ideal  $(p, v_1, v_2, \dots, v_{n-1}) \subseteq L$ , i.e. the unique invariant prime ideal at height  $n$  containing  $p$ . We also include the case of  $I_{p,0} = 0$ . In this section, we will denote this ideal by  $I_{0,0}$  to uniformize notation with respect to the residue characteristic.

**Construction 4.6.** For every triple  $(B, p, n)$  of a closed subgroup  $B$  of  $A$ , a prime  $p$  and  $n \in \overline{\mathbb{N}}$  we define an invariant ideal  $I_{B,p,n}^A \subseteq L_A$  as the preimage of  $\Phi^B L \cdot I_{p,n} \subseteq \Phi^B L$  along the composite map of Hopf algebroids

$$(L_A, S_A) \rightarrow (L_B, S_B) \rightarrow (\Phi^B L, \Phi^B S).$$

Note that the  $\Phi^B L \cdot I_{p,n}$  are indeed prime ideals since by Proposition 2.25 the quotient ring  $\Phi^B L / \Phi^B L \cdot I_{p,n}$  is of the form  $(L/I_{p,n})[(b_0^V)^{\pm 1}, b_i^V \mid i > 0, V \in A^* - \{\epsilon\}]$  and hence an integral domain. Thus, the  $I_{B,p,n}^A$  are prime as well.

To simplify notation, we will from now on often write  $\Phi^B L / I_{p,n}$  instead of  $\Phi^B L / \Phi^B L \cdot I_{p,n}$  and likewise in similar situations.

**Theorem 4.7.** *The assignment*

$$\begin{aligned} \text{Sub}(A) \times |\mathcal{M}_{FG}| &\rightarrow \text{Spec}^{\text{inv}}(L_A) \\ (B, p, n) &\mapsto I_{B,p,n}^A \end{aligned}$$

is a bijection. In other words, the ideals  $I_{B,p,n}^A$  are pairwise different and constitute all the invariant prime ideals in  $L_A$ .

The map  $\text{Spec}^{\text{inv}}(L_A) \rightarrow |\mathcal{M}_{FG}^A|$  from Proposition 4.3 is a homeomorphism.

*Proof.* By Proposition 4.3, we know that  $\text{Spec}^{\text{inv}}(L_A)$  injects into  $|\mathcal{M}_{FG}^A|$  and the latter we computed to be  $\text{Sub}(A) \times |\mathcal{M}_{FG}|$  as a set. Thus, it suffices to show that the element of  $\text{Sub}(A) \times |\mathcal{M}_{FG}|$  associated to  $I_{B,p,n}^A \in \text{Spec}^{\text{inv}}(L_A)$  is precisely  $(B, (p, n))$ , where  $p$  is a prime number if  $n > 0$  and 0 if  $n = 0$ .

To spell this out concretely, let  $k$  be the field of fractions of  $L_A / I_{B,p,n}^A$ . We denote by  $F$  the pushed forward  $A$ -equivariant formal group law over  $k$  and by  $G$  the corresponding  $A$ -equivariant formal group. Since  $\text{Spec } k \rightarrow \text{Spec } L_A$  hits the point corresponding to the prime ideal  $I_{B,p,n}^A$ , the corresponding point in  $|\mathcal{M}_{FG}^A|$  is represented by  $\text{Spec } k \xrightarrow{G} \mathcal{M}_{FG}^A$ . By the classification in Proposition 4.4, we need to show three things:

1. the set of  $V \in A^*$  such that  $e_V = 0$  in  $k$  is precisely  $\ker(A^* \rightarrow B^*)$ ,
2.  $k$  has characteristic  $p$  (which is clear), and
3. the pushforward  $p_* G$  along  $A \xrightarrow{p} \{1\}$  has height  $n$ .

For the first, recall from Lemma 2.21 that  $I_B^A = \ker(\text{res}: L_A \rightarrow L_B)$  is generated by the Euler classes  $e_V$  for all  $V \in \ker(A^* \rightarrow B^*)$ . These Euler classes must vanish in  $k$  since  $L_A \rightarrow k$  factors through  $L_A / I_B^A$ . If  $V$  is not in  $\ker(A^* \rightarrow B^*)$ , then  $e_V \neq 0$  in  $L_B$  and hence also in  $\Phi^B L / I_{p,n}$

(as else  $\Phi^B L/I_{p,n} = 0$ ). Since  $L_A/I_{B,p,n}^A$  injects into  $\Phi^B L/I_{p,n}$ , the Euler class  $e_V$  is actually nonzero in  $L_A/I_{B,p,n}^A$  and hence invertible in  $k$ . This shows the first point.

The pushforward  $p_*G$  is classified by the composite

$$g: L \rightarrow L_A \rightarrow L_A/I_{B,p,n}^A \rightarrow k.$$

The ideal  $I_{p,n} \cdot L_A$  maps to 0 in  $\Phi^B L/I_{p,n}$  and is hence contained in  $I_{B,p,n}^A$ . Therefore,  $g$  factors through  $L/I_{p,n}$  and the height of  $p_*G$  is at least  $n$ . It remains to show that  $v_n$  is non-zero in  $L_A/I_{B,p,n}^A$  and hence in  $k$ . By definition of  $I_{B,p,n}^A$ , the map  $L_A \rightarrow \Phi^B L/I_{p,n}$  factors over  $L_A/I_{B,p,n}^A$ . We know by Proposition 2.25 that  $\Phi^B L/I_{p,n}$  is an integral domain of the form  $(L/I_{p,n})[(b_0^V)^{\pm 1}, b_i^V]$ . In particular,  $v_n$  is non-trivial in  $\Phi^B L/I_{p,n}$  and hence in  $L_A/I_{B,p,n}^A$ . This shows that  $p_*G$  is of height  $n$  as desired, which finishes the proof.  $\square$

Unraveling the definition, we obtain the following description of the ideal  $I_{B,p,n}^A$ , where  $I_B^A$  still denotes the kernel of the restriction map  $L_B \rightarrow L_A$ .

**Lemma 4.8.** *An element  $x \in L_A$  lies in  $I_{B,p,n}^A$  if and only if there exists an  $A$ -representation  $W$  with  $W^B = 0$ , such that*

$$x \cdot e_W \in (I_B^A, I_{p,n}).$$

*Proof.* By definition,  $I_{B,p,n}^A$  is the kernel of the composition

$$L_A \xrightarrow{\text{res}_B^A} L_B \rightarrow L_B/L_B \cdot I_{p,n} \rightarrow \Phi^B L/\Phi^B L \cdot I_{p,n}.$$

The composition  $L_A \rightarrow L_B \rightarrow L_B/I_{p,n}$  is surjective with kernel  $(I_B^A, I_{p,n})$ , and the map  $L_B/I_{p,n} \rightarrow \Phi^B L/I_{p,n}$  inverts all Euler classes  $e_{\overline{W}}$  for  $B$ -representations  $\overline{W}$  with  $\overline{W}^B = 0$ . The product of Euler classes is an Euler class again. Hence, if  $x \in L_A$  is contained in  $I_{B,p,n}^A$ , its image  $\overline{x}$  in  $L_B/I_{p,n}$  must be annihilated by such an Euler class  $e_{\overline{W}}$ . We can extend  $\overline{W}$  to an  $A$ -representation  $W$  and find that  $x \cdot e_W$  is contained in  $(I_B^A, I_{p,n})$ , as desired.

For the opposite direction, if we assume that  $x \cdot e_W \in (I_B^A, I_{p,n})$  for some  $A$ -representation  $W$  with  $W^B = 0$ , then  $\overline{x} \cdot e_{\text{res}_B^A W} = 0$  in  $L_B/I_{p,n}$ . Since  $\text{res}_B^A W$  has trivial  $B$ -fixed points,  $e_{\text{res}_B^A W}$  becomes invertible in  $\Phi^B L/I_{p,n}$  and hence  $\overline{x}$  is taken to 0 there. Therefore,  $x$  is contained in  $I_{B,p,n}^A$ .  $\square$

When  $B$  is a torus, the ideal  $I_{B,p,n}^A$  is easy to describe explicitly:

**Corollary 4.9.** *If  $B$  is a torus, then  $I_{B,p,n}^A = (I_B^A, I_{p,n})$ .*

*Proof.* When  $B$  is a torus, every non-trivial character is non-torsion and thus the map

$$L_B/I_{p,n} \rightarrow \Phi^B L/I_{p,n}$$

is injective (Corollary 2.31). Hence,  $I_{B,p,n}^A$  is equal to the kernel of

$$L_A \rightarrow L_A/I_{p,n} \rightarrow L_B/I_{p,n},$$

which is generated by  $I_B^A$  and  $I_{p,n}$ .  $\square$

We note the following useful corollary:

**Corollary 4.10.** *Let  $A$  be a torus,  $n \in \mathbb{N}$  and consider the augmentation ideal*

$$I = \ker(L_A/I_{p,n} \rightarrow L/I_{p,n}),$$

*i.e., the ideal generated by all the Euler classes. Then the intersection  $J = \bigcap_{k \in \mathbb{N}} I^k$  equals the 0-ideal.*

For example, this shows that no element in the  $\mathbf{T}$ -equivariant Lazard ring  $L_{\mathbf{T}}$  is infinitely often divisible by the Euler class  $e$ .

*Proof.* Since  $I$  is an invariant ideal of  $L_A/I_{p,n}$ , so are all its powers and the intersection thereof. Moreover,  $J$  equals the kernel of the completion map (cf. Section 2.5)

$$L_A/I_{p,n} \rightarrow L/I_{p,n}[[y_1, \dots, y_r]],$$

where  $r$  is the rank of  $A$  and the  $y_i$  are the images of the Euler classes ranging through a basis of  $A^*$ . Since  $L/I_{p,n}[[y_1, \dots, y_r]]$  is an integral domain,  $J$  must be prime and hence an invariant prime ideal.

Therefore  $J$  must be of the form  $I_{B,p,m}^A$  (or rather its image under the projection  $L_A \rightarrow L_A/I_{p,n}$ ) for some subgroup  $B$  and height  $m \geq n$ . Note that  $v_n$  is not contained in  $I$ , hence in particular not in  $J$ . This means that we must have  $m = n$ . Moreover, given a character  $V = V_1^{\otimes k_1} \otimes \dots \otimes V_r^{\otimes k_r} \in A^*$  expressed in the chosen basis  $V_1, \dots, V_r$  above, the image of  $e_V$  under the completion map is given by

$$F([k_1]_F(y_1), \dots, [k_r]_F(y_r)) \in L/I_{p,n}[[y_1, \dots, y_r]],$$

where  $F$  is the universal formal group law pushed forward to  $L/I_{p,n}$ . Since we assumed  $n$  to be a finite height, the  $[k]$ -series of  $F$  is non-trivial whenever  $k$  is non-zero. It follows that for any non-trivial  $V$  the image of  $e_V$  is non-trivial in  $L/I_{p,n}[[y_1, \dots, y_r]]$ . Hence  $J$  contains no Euler class  $e_V$  and we must have  $B = A$ , i.e.,

$$J = I_{A,p,n}^A = (0) \subseteq L_A/I_{p,n},$$

as desired.  $\square$

## 5 Inclusions between invariant prime ideals

Non-equivariantly the ideals  $I_{p,n} \subseteq L$  form ascending towers

$$(0) = I_{p,0} \subseteq I_{p,1} \subseteq I_{p,2} \subseteq \dots,$$

essentially by definition. Except for the overlap at  $(0)$ , there are no inclusions between the towers for different primes  $p$ . For invariant prime ideals in the equivariant Lazard ring  $L_A$  we saw that we have one tower

$$I_{B,p,0}^A \subseteq I_{B,p,1}^A \subseteq I_{B,p,2}^A \subseteq \dots$$

for every pair of a closed subgroup  $B$  and prime  $p$ . Again, there will be no interplay between the towers associated to different primes (except for the overlap at height 0). However, there are additional inclusions connecting the towers for different subgroups  $B, B'$  at the same prime  $p$ . This relationship between the heights at different subgroups is one of the essential properties of equivariant formal groups. It is closely related to the blue-shift phenomenon in stable homotopy theory. We say more about this in Section 8 below.

To see that there is no inclusion between towers associated with different primes we note that  $p = v_0 \in I_{B,p,n}^A$  whenever  $n \geq 1$ . It is easy to see that  $p$  maps non-trivially under  $L_A \rightarrow \Phi^{B'} L/I_{q,n'}$  whenever  $q \neq p$  (since the target is free over  $L/I_{q,n}$  by Proposition 2.25). Hence there cannot be an inclusion  $I_{B,p,n}^A \subseteq I_{B',q,n'}^A$  for  $n \geq 1$  and  $p \neq q$ . Moreover, if  $n = 0$  we have  $I_{B,p,0}^A = I_{B,q,0}^A$ . Hence we can reduce to studying containments between invariant prime ideals associated to the same prime  $p$ .

**New convention:** For this reason and to simplify notation we from now on and for the rest of the paper implicitly localize at a fixed prime  $p$ . That is, we consider the  $p$ -localized Lazard ring  $L_A$  and denote its invariant prime ideals simply by  $I_{B,n}^A$ , omitting the chosen prime  $p$ . We further sometimes abbreviate  $I_{A,n}^A$  to  $I_{A,n}$ .

Hence our goal is to understand for which pairs of subgroups  $B, B'$  and natural numbers  $n, n'$  there is an inclusion

$$I_{B,n}^A \subseteq I_{B',n'}^A.$$

We will show the following:

**Theorem 5.1.** *There is an inclusion  $I_{B,n}^A \subseteq I_{B',n'}^A$  if and only if the following conditions are satisfied:*

1.  $B'$  is a subgroup of  $B$  and  $\pi_0(B/B')$  is a  $p$ -group.
2. We have  $n' \geq n + \text{rank}_p(\pi_0(B/B'))$ .

Hence, for example there are inclusions  $I_{T,n}^T \subseteq I_{1,n}^T$  and  $I_{C_{p^k},n}^{C_{p^k}} \subseteq I_{1,n+1}^{C_{p^k}}$ , but  $I_{C_{p^k},n}^{C_{p^k}}$  is not contained in  $I_{1,n+k-1}^{C_{p^k}}$ . In fact the theorem can be formally reduced to checking those three special cases, as we will see below. Note also that the theorem in particular says that given a chain of inclusions  $B' \subseteq B \subseteq A$ , the question whether  $I_{B,n}^A$  is contained in  $I_{B',n'}^A$  does not depend on the ambient group  $A$ , but only on  $B, B', n$  and  $n'$ .

Theorem 5.1 can be interpreted as a statement about the heights of geometric fixed points of localizations of  $L_A$ , in the following way. Recall from [HS05, Definition 4.1] that the *height*  $ht(R)$  of an  $L$ -algebra  $R$  is the maximal  $n$  such that  $R/I_n \cdot R \neq 0$ ; equivalently, it is the minimal number  $n$  such that  $I_{n+1} \cdot R = R$ . If there is no such  $n$ , the height is understood to be infinite. If  $R = 0$ , the height is  $-1$ . Then we have the following:

**Corollary 5.2.** *Let  $B \subseteq A$  be a closed subgroup,  $n \in \mathbb{N}$ . If  $\pi_0(A/B)$  is not a  $p$ -group, or if  $\pi_0(A/B)$  is a  $p$ -group but  $\text{rank}_p(\pi_0(A/B)) > n$ , then the geometric fixed points  $\Phi^A((L_A)_{I_{B,n}^A})$  are trivial. Otherwise, their height is given by*

$$ht(\Phi^A((L_A)_{I_{B,n}^A})) = n - \text{rank}_p(\pi_0(A/B)).$$

*Proof.* We have  $I_m \cdot \Phi^A((L_A)_{I_{B,n}^A}) = \Phi^A((L_A)_{I_{B,n}^A})$  if and only if there exists an element of  $L_A$  not contained in  $I_{B,n}^A$  which is mapped to  $I_m \cdot \Phi^A L_A$  under the geometric fixed point map. Since  $I_{A,m}$  is defined precisely as the preimage of  $I_m \cdot \Phi^A L_A$ , this in turn is equivalent to  $I_{A,m}^A$  not being contained in  $I_{B,n}^A$ .

By Theorem 5.1 we know that if  $\pi_0(A/B)$  is not a  $p$ -group or if  $\pi_0(A/B)$  is a  $p$ -group but  $\text{rank}_p(\pi_0(A/B)) > n$ , then  $I_{A,0}^A$  is not contained in  $I_{B,n}^A$ . Since  $I_0 = (0)$ , this implies that the geometric fixed points  $\Phi^A((L_A)_{I_{B,n}^A}) = I_0 \cdot \Phi^A((L_A)_{I_{B,n}^A})$  are trivial.

If  $\pi_0(A/B)$  is a  $p$ -group and  $r = \text{rank}_p(\pi_0(A/B)) \leq n$ , then the theorem tells us that  $I_{A,n-r}^A \subseteq I_{B,n}^A$  and  $I_{A,n-r+1}^A \not\subseteq I_{B,n}^A$ . Hence we have  $I_{n-r} \cdot \Phi^A((L_A)_{I_{B,n}^A}) \neq \Phi^A((L_A)_{I_{B,n}^A})$  and  $I_{n-r+1}^A \cdot \Phi^A((L_A)_{I_{B,n}^A}) = \Phi^A((L_A)_{I_{B,n}^A})$ , as claimed.  $\square$

**Remark 5.3.** The techniques of this paper can be used to compute the height of geometric fixed points for many complex oriented theories. We give one example of this in Proposition 8.6 below.

**Remark 5.4.** There is an inclusion  $I_{B,n}^A \subseteq I_{B',n'}^A$  if and only if in  $|\mathcal{M}_{FG,(p)}^A|$ , the point corresponding to  $I_{B',n'}^A$  lies in the closure of the point corresponding to  $I_{B,n}^A$ . Thus, Theorem 5.1 can be interpreted as a result about the topology of  $|\mathcal{M}_{FG,(p)}^A|$ .

The proof of Theorem 5.1 takes up the remainder of this section.

## 5.1 Formal reduction to the $p$ -toral case

Our first step is the following:

**Lemma 5.5.** *If there is an inclusion  $I_{B,n}^A \subseteq I_{B',n'}^A$ , then  $B'$  is a subgroup of  $B$  and  $n' \geq n$ .*

*Proof.* When  $B'$  is not a subgroup of  $B$  we can choose a character  $V \in A^*$  which is trivial when restricted to  $B$  but non-trivial when restricted to  $B'$ . Its Euler class  $e_V$  then restricts to 0 in  $L_B$ , in particular it is contained in  $I_{B,n}^A$  for all  $n$ . On the other hand, its restriction to  $B'$  becomes an invertible element in the non-trivial ring  $\Phi^{B'} L/I_{n'}$  and is hence non-trivial. Therefore  $e_V$  is not an element of  $I_{B',n'}^A$ . It follows that  $I_{B,n}^A$  cannot be contained in  $I_{B',n'}^A$ .

If  $n' < n$ , then  $v_{n'}$  is contained in  $I_n$  but not in  $I_{n'}$ . This implies that  $v_{n'}$  (now thought of as an element of  $L_A$ ) is contained in  $I_{B,n}^A$  but not in  $I_{B',n'}^A$ , since  $\Phi^{B'} L/I_{n'}$  is a non-trivial free module over  $L/I_{n'}$  by Proposition 2.25. Hence, again,  $I_{B,n}^A$  cannot be contained in  $I_{B',n'}^A$ .  $\square$

**Lemma 5.6.** *Let  $B' \subseteq B$  be an inclusion of subgroups of  $A$  and  $n, n' \in \mathbb{N}$ . Then there is an inclusion*

$$I_{B,n}^A \subseteq I_{B',n'}^A$$

*if and only if there is an inclusion*

$$I_{B,n}^B \subseteq I_{B',n'}^B$$

*if and only if there is an inclusion*

$$I_{B/B',n}^{B/B'} \subseteq I_{B'/B',n'}^{B/B'}.$$

*Proof.* The first two statements are equivalent since the restriction map  $\text{res}_B^A: L_A \rightarrow L_B$  identifies  $L_B$  with a quotient of  $L_A$ , and the ideals  $I_{B,n}^A$  and  $I_{B',n'}^A$  are the preimages of the ideals  $I_{B,n}^B$  and  $I_{B',n'}^B$  under the quotient projection.

Phrased differently, the projection  $L_A \rightarrow L_B$  induces a closed embedding of the stack of  $B$ -equivariant formal groups into the stack of  $A$ -equivariant formal groups. On spectra, the image consists precisely of these  $I_{B'',n}^A$  with  $B'' \subseteq B$ . This implies the desired equivalence by Remark 5.4.

For the second equivalence, recall the open embedding  $|\mathcal{M}_{FG}^{B/B'}| \subseteq |\mathcal{M}_{FG}^B|$  from Remark 4.5, sending the point  $(B''/B', n)$  to  $(B'', n)$  for every  $B' \subseteq B'' \subseteq B$ . Since the closure relation among points in a subspace can be detected in the subspace, Remark 5.4 gives the result.  $\square$

Taken together, the previous two lemmas allow us to reduce to the case  $B = A$  and  $B' = 1$  and understand under what conditions there is an inclusion

$$I_{A,n}^A \subseteq I_{1,n'}^A,$$

or in other words whether the restriction map  $L_A \rightarrow L$  maps  $I_{A,n}^A$  into the ideal  $I_{n'}$ .

Our next goal is to show that we can further reduce to the case where  $\pi_0 A$  is a  $p$ -group. For this we choose a prime  $q$  and consider the Euler class  $e_{\tau q} \in L_{\mathbf{T}}$ , i.e., the pullback of  $e_{\tau} \in L_{\mathbf{T}}$  along the  $q$ th power map  $[q]: \mathbf{T} \rightarrow \mathbf{T}$ . The Euler class  $e_{\tau q}$  restricts to 0 at the trivial group and is hence uniquely divisible by  $e_{\tau}$ . We set  $\tilde{x}_0^{(q)} \in L_{\mathbf{T}}$  to be the unique element satisfying  $e_{\tau q} = \tilde{x}_0^{(q)} \cdot e_{\tau}$  (the reason for this choice of notation will become clear in Section 5.2). Under the completion map  $L_{\mathbf{T}} \rightarrow L[[e_{\tau}]]$ , the Euler class  $e_{\tau q}$  is sent to the  $q$ -series  $[q]_F(e_{\tau})$  of the universal formal group law. Hence,  $\tilde{x}_0^{(q)}$  is sent to the quotient  $[q]_F(e_{\tau})/e_{\tau}$ , whose leading coefficient equals  $q$ . Since the restriction of any element in  $L_{\mathbf{T}}$  to the trivial group equals the leading coefficient of its image in  $L[[e_{\tau}]]$ , we see that  $\text{res}_1^{\mathbf{T}} \tilde{x}_0^{(q)} = q \in L$ . We further set  $x_0^{(q)} \in L_{C_q}$  to be the restriction of  $\tilde{x}_0^{(q)}$ , and find that it satisfies:

$$\begin{aligned} x_0^{(q)} \cdot e_{\overline{\tau}} &= 0 \\ \text{res}_1^{C_q} x_0^{(q)} &= q \end{aligned}$$

Here,  $\overline{\tau}$  denotes the restriction of the tautological character  $\tau \in \mathbf{T}^*$  to  $C_p$ . We are now ready to show:

**Lemma 5.7.** *If  $\pi_0 A$  is not a  $p$ -group, then there is no inclusion of the form  $I_{A,n}^A \subseteq I_{1,n'}^A$ .*

*Proof.* If  $\pi_0 A$  is not a  $p$ -group we can choose a surjection  $f: A \rightarrow C_q$  with  $q \neq p$  a prime. Then the element  $f^* x_0^{(q)} \in L_A$  satisfies the equation  $f^* x_0^{(q)} \cdot e_{f^* \overline{\tau}} = 0$ . Since  $f$  is surjective, the character  $f^* \overline{\tau} \in A^*$  is non-trivial. Hence,  $f^* x_0^{(q)}$  is an element of  $I_{A,0}^A$  and hence also of  $I_{A,n}^A$ . Its restriction to the trivial group equals that of  $x_0^{(q)}$ , which is  $q$  and hence a unit in the ( $p$ -localized) ring  $L/I_{n'}$ . In other words,  $f^* x_0^{(q)}$  is not an element of  $I_{1,n'}^A$ . Hence  $I_{A,0}^A$  does not include into  $I_{1,n'}^A$ .  $\square$

Combined with Lemmas 5.5 and 5.6 we obtain:

**Corollary 5.8.** *If there is an inclusion  $I_{B,n}^A \subseteq I_{B',n'}^A$ , then  $B'$  is a subgroup of  $B$ ,  $n' \geq n$  and the quotient  $B/B'$  is  $p$ -toral, i.e. a product of a  $p$ -group and a torus.*

## 5.2 Proof of inclusions

The next goal is to prove the ‘if’ part of Theorem 5.1, i.e., to show that if the conditions on  $B, B', n$  and  $n'$  stated there are satisfied we do have an inclusion  $I_{B,n}^A \subseteq I_{B',n'}^A$ . We start with the easiest case:

**Lemma 5.9.** *Let  $B' \subseteq B$  be an inclusion of subgroups of  $A$  such that  $B/B'$  is a torus. Then there is an inclusion  $I_{B,n}^A \subseteq I_{B',n}^A$  for all  $n$ .*

*Proof.* By Lemma 5.6 we can reduce to  $B = A$  and  $B' = 1$  the trivial group. Hence  $A$  is a torus. Lemma 4.9 then implies that  $I_{1,n}^A$  is equal to the ideal generated by  $I_n$  and the augmentation ideal  $I_1^A$ , whereas  $I_{A,n}^A$  is generated by  $I_n$  only. Clearly the latter is contained in the former.  $\square$

We now turn to showing that there are inclusions  $I_{C_{p^k},n}^{C_{p^k}} \subseteq I_{1,n+1}^{C_{p^k}}$ . For this we show that  $I_{C_{p^k},n}^{C_{p^k}}$  is generated by  $I_n$  plus one additional element which reduces to  $v_n$  under the restriction map  $L_{C_{p^k}}/I_n \rightarrow L/I_n$ . We start with the case  $k = 1$  and recall again that the Euler class  $e_{\tau p}$  is sent to the  $p$ -series  $[p]_F(e_{\tau})$  under the completion map  $L_{\mathbf{T}} \rightarrow L[[e_{\tau}]]$ . Modulo  $I_n$ , this  $p$ -series is of the form  $v_n e_{\tau}^{p^n} +$  higher order terms. Hence Corollary 2.28 implies that there exists a unique element  $\psi_p^{(n)} \in L_{\mathbf{T}}/I_n$  such that  $e_{\tau p} = \psi_p^{(n)} e_{\tau}^{p^n}$ , and this element satisfies  $\text{res}_1^{\mathbf{T}}(\psi_p^{(n)}) = v_n$ .

We then set  $\psi_{p^k}^{(n)} = [p^{k-1}]^*(\psi_p^{(n)}) \in L_{\mathbf{T}}/I_n$  for all  $k \geq 2$ , where  $[p^{k-1}]^*$  is the multiplication-by- $p^{k-1}$  map on the circle. By functoriality,  $\psi_{p^k}^{(n)}$  also restricts to  $v_n$  at the trivial group. In fact, it already restricts to  $v_n$  at  $L_{C_{p^{k-1}}}/I_n$ . This is because  $C_{p^{k-1}}$  is the kernel of  $[p^{k-1}]^*$  and hence the restriction map factors through the trivial group. Applying  $[p^{k-1}]^*$  to the defining equation for  $\psi_p^{(n)}$ , we also obtain  $e_{p^k} = \psi_{p^k}^{(n)} e_{p^{k-1}}^{p^n}$ ; here and in the following, we will often abbreviate  $e_{\tau m}$  to  $e_m$ .

**Proposition 5.10.** *For every  $k \geq 1$  and  $n \in \mathbb{N}$  the element  $\psi_{p^k}^{(n)}$  generates the kernel of*

$$\phi_{C_{p^k}}^{\mathbf{T}} : L_{\mathbf{T}}/I_n \rightarrow L_{C_{p^k}}/I_n \rightarrow \Phi^{C_{p^k}} L/I_n.$$

Hence,  $I_{C_{p^k},n}^{\mathbf{T}}$  is generated by the regular sequence  $v_0, v_1, \dots, v_{n-1}, \psi_{p^k}^{(n)}$ .

**Corollary 5.11.** *The ideal  $I_{C_{p^k},n}^{C_{p^k}}$  is generated by  $I_n$  and the restriction  $\overline{\psi_{p^k}^{(n)}}$  of  $\psi_{p^k}^{(n)}$  to  $L_{C_{p^k}}/I_n$ .*

*Proof of Proposition 5.10.* Let  $x \in L_{\mathbf{T}}/I_n$  be an element mapping to 0 in  $\Phi^{C_{p^k}} L/I_n$ , i.e. in the image of  $I_{C_{p^k},n}^{\mathbf{T}}$ . By Lemma 4.8 we know that we have an equation of the form

$$x \cdot e_1^{b_1} \cdots e_{p^{k-1}}^{b_{p^{k-1}}} = y \cdot e_{p^k} = y' \cdot \psi_{p^k}^{(n)} \quad (5.12)$$

for some  $b_i \in \mathbb{N}$ ,  $y \in L_{\mathbf{T}}/I_n$  and  $y' = y \cdot (e_{p^k}/\psi_{p^k}^{(n)})$  since  $I_{C_{p^k}}^{\mathbf{T}} = (e_{p^k})$  by Lemma 2.21.

If  $l$  is coprime to  $p$ , then  $e_{p^i l}$  and  $e_{p^i}$  become multiples of one another modulo  $e_{p^k}$ : indeed, the corresponding characters in  $(C_{p^k})^*$  generate the same subgroup and thus Lemma 2.21 implies that  $e_{p^i l}$  and  $e_{p^i}$  generate the same ideal in  $L_{C_{p^k}}$ . It follows that Equation (5.12) gives rise to an equation

$$x \cdot e_1^{a_0} \cdot e_p^{a_1} \cdots e_{p^{k-1}}^{a_{k-1}} = y'' \cdot \psi_{p^k}^{(n)},$$

with only Euler classes for powers of  $p$  appearing on the left hand side. We claim that  $y''$  must be divisible by the entire product  $e_1^{a_0} \cdot e_p^{a_1} \cdots e_{p^{k-1}}^{a_{k-1}}$ , implying that  $x$  is divisible by  $\psi_{p^k}^{(n)}$ , as desired. To see this, recall that  $\psi_{p^k}^{(n)}$  restricts to  $v_n$  in  $L_{C_{p^l}}/I_n$  for all  $l < k$ . Hence we have that  $0 = \text{res}_{C_{p^l}}^{\mathbf{T}}(y'' \cdot \psi_{p^k}^{(n)}) = \text{res}_{C_{p^l}}^{\mathbf{T}}(y'') \cdot v_n$ . Since  $v_n$  is a regular element in  $L_{C_{p^l}}/I_n$  by Proposition 2.29, this implies that  $y''$  is (uniquely) divisible by  $e_{p^l}$ . This argument can be iterated by replacing  $y''$  by  $y''/e_{p^l}$ , and the statement follows.  $\square$



**Remark 5.13.** One can show that more generally there exist elements  $\psi_m^{(n)} \in L_{\mathbf{T}}/I_n$  for all  $m \in \mathbb{N}$  uniquely determined by the equations

$$e_m = \prod_{t|m} (\psi_t^{(n)})^{p^{\nu_p(\frac{m}{t}) \cdot n}}, \quad (5.14)$$

where  $\nu_p(-)$  denotes the  $p$ -adic valuation of a natural number. The element  $\psi_m^{(n)}$  generates the kernel of

$$\phi_{C_m}^{\mathbf{T}} : L_{\mathbf{T}}/I_n \rightarrow L_{C_m}/I_n \rightarrow \Phi^{C_m} L/I_n$$

and its restriction to the trivial group is given by

$$\text{res}_1^{\mathbf{T}}(\psi_m^{(n)}) = \begin{cases} 0 & \text{if } m = 1 \\ v_n & \text{if } m = p^l \text{ and } l > 0 \\ q & \text{if } m = q^l \text{ with } q \neq p \text{ prime and } l > 0 \\ 1 & \text{otherwise.} \end{cases}$$

We note also that every  $\psi_m^{(n)}$  is prime, since the geometric fixed points  $\Phi^{C_m} L/I_n$  are integral domains. The elements  $\psi_m^{(0)}$  were previously considered in [Hau22, Proposition 5.46], denoted  $\psi_m$  there.

**Remark 5.15.** The proof of Proposition 5.10 applies in a more general context. Let  $X$  be a global group law in the sense of [Hau22, Definition 5.1]. As the global Lazard ring  $\mathbf{L}$  is the initial global group law, there is a unique map  $\mathbf{L} \rightarrow X$ . Assume that the map  $L = \mathbf{L}(1) \rightarrow X(1)$  sends  $I_n$  to 0, and that for every  $l = 0, \dots, k$  the Euler classes  $e_{p^l}$  in  $X(\mathbf{T})$  are regular elements and  $v_n$  is a regular element in  $X(C_{p^l}) = X(\mathbf{T})/e_{p^l}$  (for example this is the case if  $v_n$  is regular in  $X(\mathbf{T})$  and  $e_{p^l}$  remains a regular element modulo  $v_n$ ). Then the image of  $\psi_{p^k}^{(n)}$  in  $X(\mathbf{T})$  generates the kernel of the composition

$$\Phi_{C_{p^k}}^{\mathbf{T}} : X(\mathbf{T}) \rightarrow X(C_{p^k}) \rightarrow \Phi^{C_{p^k}} X.$$

For example, this applies to the coefficients of many Borel-equivariant complex oriented spectra, which can be used to compute their blue-shift numbers. We make use of this in Proposition 8.6.

**Corollary 5.16.** *We have an inclusion  $I_{C_{p^k}, n}^{C_{p^k}} \subseteq I_{1, n+1}^{C_{p^k}}$ .*

*Proof.* By Corollary 5.11,  $I_{C_{p^k}, n}^{C_{p^k}}$  is generated by  $I_n$  and  $\overline{\psi}_{p^k}^{(n)}$ . Since  $I_n$  is clearly contained in  $I_{1, n+1}^{C_{p^k}}$  (even in  $I_{1, n}^{C_{p^k}}$ ), we can reduce modulo  $I_n$  and need to show that  $\overline{\psi}_{p^k}^{(n)}$  is taken to 0 under the composition

$$L_{C_{p^k}}/I_n \xrightarrow{\text{res}_1^{\mathbf{T}}} L/I_n \rightarrow L/I_{n+1}.$$

But this is clear, since  $\overline{\psi}_{p^k}^{(n)}$  restricts to  $v_n$  at the trivial group and  $v_n$  lies inside  $I_{n+1}$ .  $\square$

We now have all the ingredients to prove the ‘if’-direction in Theorem 5.1:

**Corollary 5.17.** *Let  $B' \subseteq B$  be an inclusion of subgroups of  $A$  such that  $B/B'$  is  $p$ -toral, and  $n', n \in \mathbb{N}$  such that  $n' \geq n + \text{rank}_p(\pi_0(B/B'))$ . Then there is an inclusion  $I_{B, n}^A \subseteq I_{B', n'}^A$ .*

*Proof.* By Lemma 5.6 we can assume that  $A = B$  and that  $B' = 1$  is the trivial subgroup. Hence,  $A$  is a  $p$ -toral group. Let  $A^0$  denote the path component of the identity. We have  $I_{A^0, n'}^A \subseteq I_{1, n'}^A$  by Lemma 5.9. Hence it suffices to show that  $I_{A, n}^A$  is contained in  $I_{A^0, n'}^A$ . For this, making use of Lemma 5.6 once more, we can assume that  $A^0 = 1$  and hence  $A$  is a finite abelian  $p$ -group. We write  $m = \text{rank}_p(A)$  and choose a filtration of subgroups

$$1 \subseteq A_1 \subseteq A_2 \subseteq \dots \subseteq A_m = A$$

such that every subquotient  $A_i/A_{i-1}$  is a cyclic  $p$ -group. By Corollary 5.16 and Lemma 5.6 we see that

$$I_{A, n}^A \subseteq I_{A_{m-1}, n+1}^A \subseteq I_{A_{m-2}, n+2}^A \subseteq \dots \subseteq I_{1, n+m}^A \subseteq I_{1, n'}^A.$$

The final inclusion follows from the assumption  $n' \geq n + \text{rank}_p(\pi_0(B/B')) = n + m$ . This finishes the proof.  $\square$

### 5.3 Proof of non-inclusions

For the ‘only if’ direction we need to rule out further inclusions between prime ideals by constructing elements whose restrictions exhibit a large ‘height shift’. The goal is for every  $n \in \mathbb{N}$  to construct an element  $x \in L_{C_p^{n+1}}$  which lies in the ideal  $I_{C_p^{n+1},0}$  and whose restriction to the trivial group lies outside of  $I_n \subseteq L$ .

It turns out to be more natural to define such an element modulo a subideal of  $I_{C_p^{n+1},0}$ , namely the inflation of the ideal  $I_{C_p^n,0}$  along the projection  $p_{C_p^n}: C_p^{n+1} \cong C_p^n \times C_p \rightarrow C_p^n$ . That is, we construct an element

$$\bar{\mathbf{v}}_n \in I_{C_p^n \times C_p,0} / p_{C_p^n}^* I_{C_p^n,0}.$$

By Corollary 5.17 the restriction map  $L_{C_p^n} \rightarrow L$  takes  $I_{C_p^n,0}$  into  $I_n$ . Therefore, the restriction map  $L_{C_p^{n+1}} \rightarrow L$  takes  $p_{C_p^n}^* I_{C_p^n,0}$  into  $I_n$  and we obtain an induced restriction map

$$L_{C_p^n \times C_p,0} / p_{C_p^n}^* I_{C_p^n,0} \rightarrow L / I_n.$$

We will see that, under this restriction,  $\bar{\mathbf{v}}_n$  is sent to  $v_n$ . Later in Section 6 we will show that  $\bar{\mathbf{v}}_n$  in fact forms a generator of the quotient  $I_{C_p^n \times C_p,0} / p_{C_p^n}^* I_{C_p^n,0}$  and that suitable inflations and restrictions of these elements generate all the invariant prime ideals at elementary abelian  $p$ -groups.

We now turn to the construction of the element  $\bar{\mathbf{v}}_n$ . We set  $A = C_p^n$  and first define an element  $\mathbf{v}_n$  in the ring  $L_{A \times \mathbf{T}} / p_A^* I_{A,0}$ , whose restriction to  $L_{A \times C_p} / p_A^* I_{A,0}$  then yields  $\bar{\mathbf{v}}_n$ .

For every character  $V \in A^*$ , Proposition 2.27 yields a short exact sequence

$$0 \rightarrow L_{A \times \mathbf{T}} \xrightarrow{e_{V \otimes \tau}} L_{A \times \mathbf{T}} \xrightarrow{(\text{id}, V^{-1})^*} L_A \rightarrow 0,$$

where we use that  $(\text{id}, V^{-1}): A \rightarrow A \times \mathbf{T}$  identifies  $A$  with the kernel of  $(V \otimes \text{id}): A \times \mathbf{T} \rightarrow \mathbf{T}$ . The inflation map  $L_A \rightarrow L_{A \times \mathbf{T}}$  provides an  $L_A$ -linear splitting of this exact sequence if we view  $L_{A \times \mathbf{T}}$  as an  $L_A$ -module via the same inflation map. Thus, we obtain a short exact sequence

$$0 \rightarrow L_{A \times \mathbf{T}} / p_A^* I_{A,0} \xrightarrow{e_{V \otimes \tau}} L_{A \times \mathbf{T}} / p_A^* I_{A,0} \xrightarrow{(\text{id}, V^{-1})^*} L_A / I_{A,0} \rightarrow 0. \quad (5.18)$$

of  $L_A / I_{A,0}$ -modules.

**Remark 5.19.** We will see below in Section 6 that the ideal in  $L_{A \times \mathbf{T}}$  generated by  $p_A^* I_{A,0}^A$  equals the invariant prime ideal  $I_{A \times \mathbf{T},0}^{A \times \mathbf{T}}$ . In particular,  $L_{A \times \mathbf{T}} / p_A^* I_{A,0}$  is again an integral domain. At this point it is only clear that  $p_A^* I_{A,0}^A$  is contained in  $I_{A \times \mathbf{T},0}^{A \times \mathbf{T}}$ .

We now consider the Euler class  $e_{\epsilon \otimes \tau p} \in L_{A \times \mathbf{T}} / I_{A,0}$ . We have  $(\text{id}, V^{-1})^*(e_{\epsilon \otimes \tau p}) = e_{V^{-p}} = 0$ , since  $A$  is an elementary abelian  $p$ -group and hence every character is  $p$ -torsion. By exactness, it follows that  $e_{\epsilon \otimes \tau p}$  is divisible by  $e_{V \otimes \tau}$ , for all  $V \in A^*$ . We want to define  $\bar{\mathbf{v}}_n$  as the quotient of  $e_{\epsilon \otimes \tau p}$  by the product over all the  $e_{V \otimes \tau}$ . For this we first need to check that the different  $e_{V \otimes \tau}$  are coprime. To understand this, we consider the following:

**Lemma 5.20.** *Let  $x, y \in L_{A \times \mathbf{T}} / p_A^* I_{A,0}$ , and assume that  $y$  restricts to a non-zero element under  $L_{A \times \mathbf{T}} / p_A^* I_{A,0} \xrightarrow{(\text{id}, V^{-1})^*} L_A / I_{A,0}$ . Then  $x$  is divisible by  $e_{V \otimes \tau}$  if and only if  $x \cdot y$  is divisible by  $e_{V \otimes \tau}$ .*

*Proof.* By exactness of (5.18),  $x$  is divisible by  $e_{V \otimes \tau}$  if and only if  $(\text{id}, V^{-1})^* x = 0$  in  $L_A / I_{A,0}$ . Since  $(\text{id}, V^{-1})^*(y)$  is non-zero by assumption and  $L_A / I_{A,0}$  is an integral domain, this is the case if and only if

$$(\text{id}, V^{-1})^*(x \cdot y) = (\text{id}, V^{-1})^* x \cdot (\text{id}, V^{-1})^* y = 0,$$

which in turn is equivalent to  $x \cdot y$  being divisible by  $e_{V \otimes \tau}$ .  $\square$

Given a character  $V_2 \neq V$ , the restriction of  $e_{V_2 \otimes \tau}$  along  $(\text{id}, V^{-1})^*$  equals the non-trivial Euler class  $e_{V_2 V^{-1}} \in L_A / I_{A,0}$ . Hence the lemma applies to  $y$  being any product of Euler classes of the form  $e_{V' \otimes \tau}$  with  $V' \neq V$ . We find that in the quotient  $L_{A \times \mathbf{T}} / p_A^* I_{A,0}$ , the Euler class  $e_{\epsilon \otimes \tau p}$  is uniquely divisible by the product  $\prod_{V \in A^*} e_{V \otimes \tau}$ . To summarize:

**Definition 5.21.** Let  $n \in \mathbb{N}$  and  $A = C_p^n$ . We define  $\mathbf{v}_n \in L_{A \times \mathbf{T}}/p_A^* I_{A,0}$  to be the unique element satisfying

$$e_{\epsilon \otimes \tau^p} = \mathbf{v}_n \cdot \prod_{V \in A^*} e_{V \otimes \tau} \in L_{A \times \mathbf{T}}/p_A^* I_{A,0},$$

and we set

$$\bar{\mathbf{v}}_n = \text{res}_{A \times C_p}^{A \times \mathbf{T}}(\mathbf{v}_n) \in L_{A \times C_p}/p_A^* I_{A,0}.$$

**Remark 5.22.** The element  $\bar{\mathbf{v}}_0$  agrees with  $x_0^{(p)} \in L_{C_p}$  as defined in Section 5.1.

**Remark 5.23.** Lemma 5.20 also applies in the ring  $L_{A \times \mathbf{T}}$  itself (i.e., before quotienting by  $p_A^* I_{A,0}$ ) if we demand that  $(\text{id}, V)^* y$  is regular element, rather than just being non-zero. These two conditions are equivalent in  $L_A/I_{A,0}$  since it is an integral domain. However, the Euler classes are not regular elements in  $L_A$ , hence the lemma does not apply for  $y$  a product of the  $e_{V' \otimes \tau}$ . In fact  $e_{\epsilon \otimes \tau^p}$  is not divisible by  $\prod_{V \in A^*} e_{V \otimes \tau}$  before dividing out  $p_A^* I_{A,0}$ , even though it is divisible by each individual  $e_{V \otimes \tau}$ . One can see this by restricting from  $A \times \mathbf{T}$  to  $\mathbf{T}$ : If  $e_{\epsilon \otimes \tau^p}$  was divisible by the product of all the  $e_{V \otimes \tau}$ , this would imply that its restriction  $e_{\tau^p} \in L_{\mathbf{T}}$  was divisible by  $e_{\tau^n}$ , since every  $e_{V \otimes \tau}$  restricts to  $e_{\tau}$ . But  $e_{\tau^p} \in L_{\mathbf{T}}$  is divisible by  $e_{\tau}$  precisely once, since  $e_{\tau^p}/e_{\tau}$  restricts to  $p$  at the trivial group, cf. Section 5. It is for this reason that it is most natural to define  $\mathbf{v}_n$  and  $\bar{\mathbf{v}}_n$  in this quotient. As we will see now, this matches nicely with the fact that  $v_n$  is most naturally defined in the quotient  $L/I_n$ .

**Proposition 5.24.** 1. The element  $\bar{\mathbf{v}}_n$  defines a class in the ideal  $I_{A \times C_p,0}/p_A^* I_{A,0}$ , i.e., it is sent to 0 under the map  $L_{A \times C_p,0}/p_A^* I_{A,0} \rightarrow \Phi^{A \times C_p} L$ .

2. The restriction map

$$L_{A \times C_p,0}/p_A^* I_{A,0} \rightarrow L/I_n$$

takes  $\bar{\mathbf{v}}_n$  to  $v_n$ .

*Proof.* Part 1: The equation

$$e_{\epsilon \otimes \tau^p} = \mathbf{v}_n \cdot \prod_{V \in A^*} e_{V \otimes \tau}$$

reduces to the equation

$$0 = \bar{\mathbf{v}}_n \cdot \prod_{V \in A^*} e_{V \otimes \bar{\tau}}$$

in  $L_{A \times C_p}/p_A^* I_{A,0}$ , where  $\bar{\tau}$  denotes the restriction of  $\tau \in \mathbf{T}^*$  to  $C_p$ . Note that each  $(A \times C_p)$ -character of the form  $V \otimes \bar{\tau}$  is non-trivial. Hence  $\bar{\mathbf{v}}_n$  forms Euler-power torsion and therefore maps to 0 in the geometric fixed points.

For Part 2, we first restrict  $\mathbf{v}_n$  to  $L_{\mathbf{T}}$ . This restriction takes  $p_A^* I_{A,0}$  into  $I_n$  by Corollary 5.17, and sends each  $e_{V \otimes \tau}$  to  $e_{\tau}$ . It follows that, modulo  $I_n$ , we have an equation

$$\text{res}_{\mathbf{T}}^{A \times \mathbf{T}} \mathbf{v}_n \cdot e_{\tau^n} = e_{\tau^p}.$$

Hence, modulo  $I_n$ ,  $\text{res}_{\mathbf{T}}^{A \times \mathbf{T}} \mathbf{v}_n$  equals the element  $\psi_p^{(n)}$  from Proposition 5.10, whose restriction to the trivial group is  $v_n$ . This finishes the proof.  $\square$

**Corollary 5.25.** If  $x_n$  is a preimage of  $\bar{\mathbf{v}}_n$  under the projection  $L_{C_p^{n+1}} \rightarrow L_{C_p^{n+1}}/p_{C_p}^* I_{C_p^n,0}$  and  $B \subseteq C_p^{n+1}$  is a subgroup of rank  $0 \leq m \leq n+1$ , then

$$x_n \in I_{B,n+1-m}^{C_p^{n+1}} - I_{B,n-m}^{C_p^{n+1}}.$$

*Proof.* By the previous proposition,  $x_n$  is an element of  $I_{C_p^{n+1},0}^{C_p^{n+1}}$ . As  $A/B$  has rank  $(n+1-m)$ , we know by Corollary 5.17 that  $x_n$  must lie in  $I_{B,n+1-m}^{C_p^{n+1}}$ .

If  $x_n$  were an element of  $I_{B,n-m}^{C_p^{n+1}}$ , then applying Corollary 5.17 to the inclusion of the trivial group into  $B$  shows that  $x_n$  is also an element of  $I_{1,n}^{C_p^{n+1}}$ . This contradicts the fact that, modulo  $I_n$ , we have  $\text{res}_1^{C_p^{n+1}}(x_n) = \text{res}_1^{C_p^{n+1}}(\bar{\mathbf{v}}_n) = v_n$ .  $\square$

**Corollary 5.26.** *If  $B' \subseteq B$  is a  $p$ -toral inclusion of subgroups of  $A$  (i.e.  $B/B'$  is  $p$ -toral) and  $n' < n + \text{rank}_p(\pi_0(B/B'))$ , then  $I_{B,n}^A$  does not include into  $I_{B',n'}^A$ .*

*Proof.* By Lemma 5.6 we can reduce to the case  $A = B$  and  $B' = 1$ . Let  $r = \text{rank}_p(\pi_0(A))$ , and  $q: A \rightarrow C_p^r$  be a surjection. Let  $x_{n+r-1} \in L_{C_p^{n+r}}$  as in Corollary 5.25. Then, by the corollary, the restriction  $\text{res}_{C_p^r}^{C_p^{n+r}}(x_{n+r-1})$  is an element of  $I_{C_p^n}^r$  but not an element of  $I_{1,n+r-1}^r$ . Therefore  $x = q^*(\text{res}_{C_p^r}^{C_p^{n+r}}(x_{n+r-1}))$  is an element of  $I_{A,n}^A$  whose restriction to the trivial group is not contained in  $I_{n+r-1}$ . In other words,  $x$  is an element of  $I_{A,n}^A$  but not an element of  $I_{1,n+r-1}^A$ . Since by assumption we have  $n' \leq n + r - 1$  and hence  $I_{1,n'}^A \subseteq I_{1,n+r-1}^A$ , this proves that  $I_{A,n}^A$  does not include into  $I_{1,n'}^A$ .  $\square$

Combined with Corollaries 5.8 and 5.17, this proves Theorem 5.1.

## 6 Generators for invariant prime ideals

In this section we show that over elementary abelian  $p$ -groups the elements  $\bar{\mathbf{v}}_n$  – together with the Euler classes – generate all invariant prime ideals under restriction and inflation maps. More precisely, we show the following theorem:

**Theorem 6.1.** *1. For every torus  $B$  and  $n \in \mathbb{N}$ , the ideal  $I_{C_p^n \times B,0} = I_{C_p^n \times B,0}^{C_p^n \times B}$  is generated by the elements*

$$p_1^*(\bar{\mathbf{v}}_0), p_2^*(\bar{\mathbf{v}}_1), \dots, p_{n-1}^*(\bar{\mathbf{v}}_{n-2}), p_n^*(\bar{\mathbf{v}}_{n-1}),$$

*where  $p_i: C_p^n \times B \rightarrow C_p^i$  is the projection to the first  $i$  factors.*

*2. For every  $m \in \mathbb{N}$  and every inclusion  $i: C_p^n \rightarrow C_p^{n+m}$ , the restriction map*

$$(i \times B)^*: L_{C_p^{n+m} \times B} \rightarrow L_{C_p^n \times B}$$

*maps  $I_{C_p^{n+m} \times B,0}$  surjectively onto  $I_{C_p^n \times B,m}$ .*

**Remark 6.2.** Implicit in the statement of the theorem is that each  $p_i^*(\bar{\mathbf{v}}_{i-1})$  is well-defined modulo the ideal generated by  $p_1^*(\bar{\mathbf{v}}_0), \dots, p_{i-1}^*(\bar{\mathbf{v}}_{i-2})$ . By definition,  $p_i^*(\bar{\mathbf{v}}_{i-1})$  is an element of the quotient by the subideal generated by  $p_{i-1}^* I_{C_p^{i-1},0}$ . Applying the theorem to rank  $n-1$  and  $B=0$  we see that this ideal is indeed generated by  $p_1^*(\bar{\mathbf{v}}_0), \dots, p_{i-1}^*(\bar{\mathbf{v}}_{i-2})$ , so the sequence of elements makes sense. Hence, the theorem and sequence should be interpreted in an inductive manner.

Combining both parts it follows that  $I_{C_p^n \times B,m}$  is generated by

$$(i \times B)^* p_1^*(\bar{\mathbf{v}}_0), (i \times B)^* p_2^*(\bar{\mathbf{v}}_1), \dots, (i \times B)^* p_{n+m-1}^*(\bar{\mathbf{v}}_{n+m-2}), (i \times B)^* \bar{\mathbf{v}}_{n+m-1},$$

where  $i: C_p^n \rightarrow C_p^{n+m}$  is any inclusion. The choice of inclusion will generally affect the resulting generators. For example, setting  $n = m = 1$  and  $B$  the trivial group: If we choose  $i_1: C_p \rightarrow C_p^2$  to be the inclusion into the first factor, the composite  $p_1 \circ i_1$  becomes the identity. Hence we obtain that  $I_{C_p,1}$  is generated by the elements  $\bar{\mathbf{v}}_0$  and  $i_1^*(\bar{\mathbf{v}}_1)$ . If we alternatively use the inclusion  $i_2: C_p \rightarrow C_p^2$  into the second factor the composite  $p_1 \circ i_2$  becomes the constant map, yielding the generators  $v_0 = p$  and  $i_2^*(\bar{\mathbf{v}}_1)$  (i.e., the same ones as in Corollary 5.11, as  $\bar{\psi}_p^{(1)}$  equals  $i_2^*(\bar{\mathbf{v}}_1)$ ). Furthermore, it follows that generators for ideals of the form  $I_{C_p^n \times B,m}^A$  with  $B$  a torus can be obtained as the union of Euler classes  $(eV)_{V \in \mathcal{B}}$  for a basis  $\mathcal{B}$  of  $\ker(A^* \rightarrow (C_p^n \times B)^*)$  together with a choice of generators for  $I_{C_p^n \times B,m}^{C_p^n \times B}$ .

We prove Theorem 6.1 by induction on  $n$ . Part 1 of the induction start  $n = 0$  is the statement that  $I_{B,0}^B$  is the 0-ideal for any torus  $B$  (Corollary 4.9). For Part 2 we need to see that the restriction  $L_{C_p^n \times B} \rightarrow L_B$  maps  $I_{C_p^n \times B,0}$  surjectively onto  $I_{B,m}$ , which we know is generated by  $I_m$  by Corollary 4.9. For  $i = 0, \dots, m-1$  we can consider the elements  $\bar{\mathbf{v}}_i \in I_{C_p^{i+1},0}/p_i^* I_{C_p^i,0}$ , which reduce to  $v_i \in L/I_{i-1}$ . It follows that the inflation of  $\bar{\mathbf{v}}_i$  to  $C_p^m$  via any choice of surjection

$C_p^m \times B \rightarrow C_p^{i+1}$  gives an element of a quotient of  $I_{C_p^m \times B, 0}$  which reduces to  $v_i$  in  $L_B/I_{i-1} \cdot L_B$ . Since  $I_i/I_{i-1}$  is generated by  $v_i$ , the claim follows.

We now assume that Theorem 6.1 holds for an elementary abelian  $p$ -group  $A$  of rank  $n$  and show it also holds for  $A \times C_p$ . For any  $m \in \mathbb{N}$  we consider the surjection

$$L_{A \times \mathbf{T} \times B} / p_A^* I_{A, m} \rightarrow L_{A \times C_p \times B} / p_A^* I_{A, m},$$

with kernel generated by  $e_{\tau p}$  for  $\tau$  the tautological  $\mathbf{T}$ -character pulled back to  $A \times \mathbf{T} \times B$ . We first claim that if  $V \in A^*$  is non-trivial, then the Euler class  $p_A^*(e_V)$  is a non-zero divisor in  $L_{A \times C_p \times B} / p_A^* I_{A, m}$ . To see this, we use that since  $\mathbf{T} \times B$  is a torus we can apply the induction hypothesis to  $L_{A \times \mathbf{T} \times B}$ . In particular, we know by Part 1 that  $p_A^* I_{A, m}$  generates the ideal  $I_{A \times \mathbf{T} \times B, m}$  and hence  $L_{A \times \mathbf{T} \times B} / p_A^* I_{A, m}$  is an integral domain. So we have to show that  $p_A^*(e_V)$  still acts regularly modulo  $e_{\tau p}$ . Since both Euler classes are regular, this is equivalent to showing that  $e_{\tau p}$  is regular modulo  $p_A^*(e_V)$ . We have an isomorphism

$$L_{A \times \mathbf{T} \times B} / (p_A^* I_{A, m}, p_A^*(e_V)) \cong L_{\ker(V) \times \mathbf{T} \times B} / p_{\ker(V)}^*(\text{res}_{\ker(V)}^A I_{A, m}).$$

By Part 2 of the induction hypothesis, we know that  $I_{A, m}$  restricts onto  $I_{\ker(V), m+1}$ , hence the latter quotient identifies with  $L_{\ker(V) \times \mathbf{T} \times B} / p_{\ker(V)}^* I_{\ker(V), m+1}$ . Again we know by the induction hypothesis that this quotient is an integral domain, and  $e_{\tau p}$  is clearly a non-trivial element. So the claim follows and we have shown that  $p_A^* I_{A, m}$  generates the Euler power torsion in  $L_{A \times C_p \times B}$  (at height  $m$ ) for characters inflated up from  $A$ .

Hence to understand the full ideal  $I_{A \times C_p \times B, m}$  it suffices to further divide by the Euler-power torsion for the remaining torsion characters in  $(A \times C_p \times B)^*$  (there is no Euler-power torsion for non-torsion characters by Proposition 2.27). These torsion characters are of the form  $V \otimes \bar{\tau}^k$ , where  $V \in A^*$ ,  $\bar{\tau}$  is the restriction of  $\tau \in \mathbf{T}^*$  to  $C_p$  and  $k \in \{1, \dots, p-1\}$ . Furthermore we can assume that  $k=1$ : Any  $V \otimes \bar{\tau}^k$  has some power of the form  $V' \otimes \bar{\tau}$  and hence  $e_{V' \otimes \bar{\tau}}$  is a multiple of  $e_{V \otimes \bar{\tau}^k}$ . Thus,  $I_{A \times C_p \times B, m} / p_A^* I_{A, m}$  is generated by Euler-power torsion for characters of the form  $V \otimes \bar{\tau}$ .

Again it is beneficial to pass to the integral domain  $L_{A \times \mathbf{T} \times B} / p_A^* I_{A, m}$  to understand the Euler-power torsion for these characters. We have the following:

**Lemma 6.3.** *Let  $A = C_p^n$  be an elementary abelian  $p$ -group,  $B$  a torus and  $m \in \mathbb{N}$ . Further let  $x \in L_{A \times \mathbf{T} \times B} / p_A^* I_{A, m}$  be an element satisfying*

$$x \cdot \prod_{V \in A^*} e_{V \otimes \bar{\tau}}^{n_V} = y \cdot e_{\tau p}$$

for some  $y$  and collection of natural numbers  $n_V$ . Then  $x$  lies in the ideal generated by

$$p_{A \times \mathbf{T}}^* \text{res}_{A \times \mathbf{T}}^{A \times C_p^m \times \mathbf{T}}(\mathbf{v}_{n+m}).$$

*Proof.* By applying  $p_{A \times C_p^m \times \mathbf{T}}^*$  to the defining property of  $\mathbf{v}_{n+m}$  (Definition 5.21) we obtain the equation

$$e_{\tau p} = p_{A \times C_p^m \times \mathbf{T}}^*(\mathbf{v}_{n+m}) \cdot \prod_{V \in (A \times C_p^m)^*} e_{V \otimes \bar{\tau}}$$

in  $L_{A \times C_p^m \times \mathbf{T} \times B} / p_{A \times C_p^m}^* I_{A \times C_p^m, 0}$ . Restricting from  $A \times C_p^m$  to  $A = C_p^n$  yields

$$e_{\tau p} = p_{A \times \mathbf{T}}^* \text{res}_{A \times \mathbf{T}}^{A \times C_p^m \times \mathbf{T}}(\mathbf{v}_{n+m}) \cdot \prod_{V \in A^*} e_{V \otimes \bar{\tau}}^{p^m} \quad (6.4)$$

in the quotient  $L_{A \times \mathbf{T} \times B, m} / p_A^* I_{A, m}$ . This uses that every character of  $A$  extends to  $p^m$  different characters of  $A \times C_p^m$  and that the restriction of  $I_{A \times C_p^m, 0} \subseteq L_{A \times C_p^m}$  lands in the ideal  $I_{A, m} \subseteq L_A$ .

For the rest of the proof we write  $z$  for the element  $p_{A \times \mathbf{T}}^* \text{res}_{A \times \mathbf{T}}^{A \times C_p^m \times \mathbf{T}}(\mathbf{v}_{n+m})$ . With  $x$  as in the statement of the lemma, we hence obtain an equation of the form

$$x \cdot \prod_{V \in A^*} e_{V \otimes \bar{\tau}}^{n_V} = z \cdot \prod_{V \in A^*} e_{V \otimes \bar{\tau}}^{p^m} \cdot y \quad (6.5)$$

and we need to show that  $x$  is a multiple of  $z$ . The Euler classes  $e_{V \otimes \tau}$  fit into short exact sequences of the form

$$0 \rightarrow L_{A \times \mathbf{T} \times B} / p_A^* I_{A,m} \xrightarrow{e_{V \otimes \tau}} L_{A \times \mathbf{T} \times B} / p_A^* I_{A,m} \xrightarrow{((\text{id}, V^{-1}) \times B)^*} L_{A \times B} / p_A^* I_{A,m} \rightarrow 0,$$

analogously to Equation 5.18 above. As shown above, the induction hypothesis implies that the quotient  $L_{A \times B} / p_A^* I_{A,m}$  is an integral domain. We know that  $z$  restricts to  $v_{n+m} \in L / I_{n+m}$  at the trivial group. In particular it must restrict non-trivially under each  $((\text{id}, V^{-1}) \times B)^*$ . Hence the above short exact sequence together with the fact that  $L_{A \times B} / p_A^* I_{A,m}$  is an integral domain implies: If an Euler class  $e_{V \otimes \tau}$  divides an element of the form  $z \cdot \alpha$ , then  $e_{V \otimes \tau}$  divides  $\alpha$ . Applying this iteratively to Equation 6.5 (and using that  $L_{A \times \mathbf{T} \times B} / p_A^* I_{A,m}$  is an integral domain by the induction hypothesis) we see that  $\prod_{V \in A^*} e_{V \otimes \tau}^{n_V}$  must divide the term  $\prod_{V \in A^*} e_{V \otimes \tau}^m \cdot y$ . Dividing on both sides shows that  $x$  is a multiple of  $z$ , as desired.  $\square$

**Corollary 6.6.** *The quotient*

$$I_{A \times C_p \times B, m} / p_A^* I_{A,m}$$

*is generated by the element*

$$\text{res}_{A \times C_p^m \times B}^{A \times C_p^m \times C_p \times B} (p_{A \times C_p^m \times C_p}^* \bar{\mathbf{v}}_{n+m}) = p_{A \times C_p}^* (\text{res}_{A \times C_p}^{A \times C_p^m \times C_p} \bar{\mathbf{v}}_{n+m}).$$

*Proof.* We saw above that the quotient  $I_{A \times C_p \times B, m} / p_A^* I_{A,m}$  is generated by Euler-power torsion for characters of the form  $V \otimes \bar{\tau}$ . An element  $\bar{x}$  of  $L_{A \times C_p \times B} / p_A^* I_{A,m}$  is such a torsion element if and only if it is the reduction of an element  $x \in L_{A \times \mathbf{T} \times B} / p_A^* I_{A,m}$  satisfying the conditions of the lemma. Since the reduction of  $p_{A \times \mathbf{T}}^* \text{res}_{A \times \mathbf{T}}^{A \times C_p^m \times \mathbf{T}}(\mathbf{v}_{n+m})$  equals  $p_{A \times C_p}^* \text{res}_{A \times C_p}^{A \times C_p^m \times C_p}(\bar{\mathbf{v}}_{n+m})$ , it follows that  $\bar{x}$  lies in the ideal generated by the latter.

As  $p_{A \times C_p}^* \text{res}_{A \times C_p}^{A \times C_p^m \times C_p}(\bar{\mathbf{v}}_{n+m})$  is Euler-power torsion itself, it hence forms a generator of  $I_{A \times C_p \times B, m} / p_A^* I_{A,m}$ .  $\square$

To finish the proof of Theorem 6.1: Setting  $m = 0$  in the corollary shows that  $\bar{\mathbf{v}}_n$  generates the quotient  $I_{A \times C_p \times B, 0} / p_A^* I_{A,0}$ . By the induction hypothesis we know that  $I_{A,0}$  is generated by  $p_1^*(\bar{\mathbf{v}}_0), \dots, p_{n-1}^*(\bar{\mathbf{v}}_{n-2}), \bar{\mathbf{v}}_{n-1}$ . Combined this proves Part 1 for the group  $A \times C_p$ .

For Part 2 and general  $m$ , we first note that it suffices to show the statement for any choice of injection  $i: A \times C_p \rightarrow C_p^{n+m+1}$  since any two only differ by postcomposition with an automorphism of  $C_p^{n+m+1}$ . We can hence pick the canonical inclusion  $A \times C_p \rightarrow A \times C_p^m \times C_p$ . By the induction hypothesis we know that  $I_{A \times C_p^m, 0}$  surjects onto  $I_{A,m}$ . From the diagram

$$\begin{array}{ccc} I_{A \times C_p^m, 0} & \xrightarrow{\text{res}} & I_{A,m} \\ \downarrow p_{A \times C_p^m}^* & & \downarrow p_A^* \\ I_{A \times C_p^m \times C_p \times B, 0} & \xrightarrow{\text{res}} & I_{A \times C_p \times B, m} \end{array}$$

we see that  $p_A^*(I_{A,m})$  is contained in the image of the lower horizontal arrow. Furthermore, Corollary 6.6 implies that  $I_{A \times C_p \times B, m} / p_A^* I_{A,m}$  is generated by the restriction of an element of  $I_{A \times C_p^m \times C_p \times B, m}$ . This finishes the proof.

**Remark 6.7.** Unlike the sequence  $v_0, \dots, v_{n-1}$ , the sequence

$$p_1^*(\bar{\mathbf{v}}_0), p_2^*(\bar{\mathbf{v}}_1), \dots, p_{n-1}^*(\bar{\mathbf{v}}_{n-2}), \bar{\mathbf{v}}_{n-1}$$

isn't regular. In fact, the ideal  $I_{C_p^n, 0}$  generated by these elements is precisely that of Euler-torsion.

This can be corrected by passing to a torus: The ideal  $I_{C_p^n, 0}^{\mathbf{T}^n}$  is generated by the sequence  $p_1^*(\mathbf{v}_0), p_2^*(\mathbf{v}_1), \dots, p_{n-1}^*(\mathbf{v}_{n-2}), \bar{\mathbf{v}}_{n-1}$ . Here, each  $p_{i+1}^*(\mathbf{v}_i)$  is the element of

$$L^{\mathbf{T}^n} / (p_1^*(\mathbf{v}_0), p_2^*(\mathbf{v}_1), \dots, p_i^*(\mathbf{v}_{i-1})) \cong L_{C_p^i \times \mathbf{T}^{n-i}} / I_{C_p^i \times \mathbf{T}^{n-i}, 0} \quad (6.8)$$

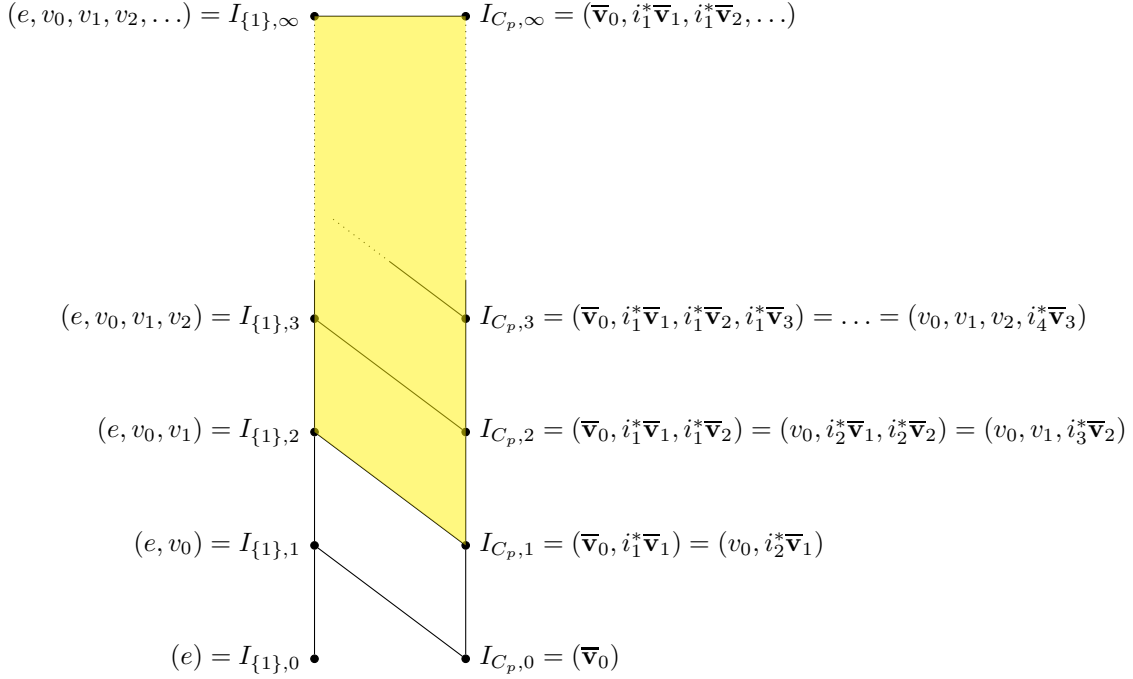


Figure 4: A picture of  $\text{Spec}^{\text{inv}}(L_{C_p})$ , localized at  $p$ , including different choices of generators, with  $i_j: C_p \rightarrow C_p^k$  denoting the  $j$ th canonical inclusion (generators arising from the further inclusions  $C_p \rightarrow C_p^k$  are omitted). The yellow area depicts the closure of the point  $I_{C_p, 1}$ .

obtained as the inflation of  $\mathbf{v}_i \in L_{C_p^i \times \mathbf{T}} / I_{C_p^i \times \mathbf{T}, 0}$  along the projection to the first coordinate of  $\mathbf{T}^{n-i}$ . Since each successive quotient  $L_{C_p^i \times \mathbf{T}^{n-i}} / I_{C_p^i \times \mathbf{T}^{n-i}, 0}$  is an integral domain, the regularity of the sequence is clear once we have demonstrated the isomorphism claimed in (6.8). Similarly one shows that  $I_{C_p^i, m}^{\mathbf{T}^n}$  is generated by a regular sequence of length  $n + m$ .

To establish the isomorphism, first note that  $(e_{\tau^p}) \subseteq (p_{C_p^{i-1} \times \mathbf{T}}^*(\mathbf{v}_{i-1}))$  in the quotient ring  $L_{C_p^{i-1} \times \mathbf{T}^{n-i+1}} / p_{C_p^{i-1}}^* I_{C_p^{i-1}, 0}$  by Lemma 6.3. Thus, Corollary 6.6 with  $m = 0$  gives

$$L_{\mathbf{T}^n} / (p_1^*(\mathbf{v}_0), p_2^*(\mathbf{v}_1), \dots, p_i^*(\mathbf{v}_{i-1})) \cong L_{C_p^i \times \mathbf{T}^{n-i}} / (p_1^*(\bar{\mathbf{v}}_0), p_2^*(\bar{\mathbf{v}}_1), \dots, p_i^*(\bar{\mathbf{v}}_{i-1})),$$

and the claimed isomorphism becomes Theorem 6.1.

## 7 The Zariski topology on the spectrum of invariant prime ideals

The goal of this section is to describe the Zariski topology on  $\text{Spec}^{\text{inv}}(L_A)$ , or equivalently the topology on the space  $|\mathcal{M}_{FG}^A|$  (Theorem 4.7). By definition, the closed subsets of  $\text{Spec}^{\text{inv}}(L_A)$  are the subsets of the form

$$V(X) = \{I_{B, n} \in \text{Spec}^{\text{inv}}(L_A) \mid X \subseteq I_{B, n}\}$$

for some subset  $X$  of  $L_A$ . Hence we need to determine the collections of invariant prime ideals that arise as  $V(X)$  for some  $X$ . We now fix a subset  $X$ . Since a containment  $X \subseteq I_{B, n}^A$  automatically implies  $X \subseteq I_{B, n'}^A$  for all  $n' \geq n$ , it suffices to understand – for every closed subgroup  $B$  of  $A$  – the smallest value of  $n \in \mathbb{N} \cup \{\infty\}$  such that  $X \subseteq I_{B, n}^A$ . In other words we

need to determine the function

$$\text{ht}_X: \text{Sub}(A) \rightarrow \overline{\mathbb{N}}_- = \{-1\} \cup \mathbb{N} \cup \{\infty\}$$

defined by

$$\text{ht}_X(B) = \sup\{n \mid X \not\subseteq I_{B,n}^A\},$$

where we set  $\sup(\emptyset) = -1$ . We note that with this definition the height function of the image of  $v_n \in L$  in  $L_A$  (thought of as a one-element set) is constantly  $n$ . This follows from the fact that  $\Phi^B L$  is a free  $L$ -module by Proposition 2.25; cf. also the proof of Theorem 4.7. Moreover, we have:

**Example 7.1.** Let  $x_n \in L_{C_p^{n+1}}$  be a lift of  $\overline{v}_n \in L_{C_p^{n+1}}/p^*_{C_p^n} I_{C_p^n,0}$ . Then Corollary 5.25 implies that  $\text{ht}_{x_n}(B) = n - \text{rk}(B)$ .

Our goal is to understand which functions  $\text{Sub}(A) \rightarrow \overline{\mathbb{N}}_-$  arise as such height functions. In the previous section we showed that there are inclusions between invariant prime ideals associated to different subgroups of  $A$ . These translate to conditions between the different values of  $\text{ht}_X$ : If  $B' \subseteq B$  is an inclusion of subgroups of  $A$  such that  $B/B'$  is  $p$ -toral, and  $X$  is contained in  $I_{B,n}^A$ , then  $X$  is also contained in  $I_{B',n+\text{rank}_p(\pi_0(B/B'))}^A$ . In terms of the height function this translates to the inequality

$$\text{ht}_X(B') \leq \text{ht}_X(B) + \text{rank}_p(\pi_0(B/B')).$$

This leads us to the following definition:

**Definition 7.2.** A function  $f: \text{Sub}(A) \rightarrow \overline{\mathbb{N}}_-$  is called *admissible* if it satisfies the inequality

$$f(B') \leq f(B) + \text{rank}_p(\pi_0(B/B'))$$

for every  $p$ -toral inclusion  $B' \subseteq B$  of closed subgroups of  $A$ . Here,  $B' \subseteq B$  is  $p$ -toral if  $B/B'$  is a product of a torus and a  $p$ -group.

By the above considerations, any height function  $\text{ht}_X$  is admissible. When the group  $A$  is finite, it turns out that the converse also holds: Any admissible function is realized by a height function  $\text{ht}_X$ . For positive dimensional  $A$  there is an additional condition on top of admissibility. To state this condition, we recall that choosing an invariant Riemannian metric  $d$  on  $A$  also equips the set of closed subgroups  $\text{Sub}(A)$  with the Hausdorff metric, the underlying topology of which does not depend on the chosen metric on  $A$ . This turns  $\text{Sub}(A)$  into a compact totally-disconnected metric space, in which a sequence  $(B_i)_{i \in \mathbb{N}}$  of closed subgroups converges to another closed subgroup  $B \in \text{Sub}(A)$  if and only if almost all  $B_i$  are subgroups of  $B$  and for every element  $b \in B$  the distance function  $d(b, B_i)$  converges to zero (see [tD79, Section 5.6]). If  $B_i \rightarrow B$ , we have the following two implications about representations:

1. Let  $W$  be a representation of  $B$  with  $W^B = 0$ . Thus, writing  $W$  as a sum of characters  $V_i$ , none of the  $V_i$  is trivial. For sufficiently large  $i$ , no  $\ker(V_i)$  contains  $B_i$  (since  $\ker(V_i) \subseteq B$  is a closed proper subgroup) and thus  $W^{B_i} = 0$ .
2. Let  $V$  and  $W$  be two characters of  $B$  such that  $\text{res}_{B_i}^B V = \text{res}_{B_i}^B W$  for all sufficiently large  $i$ . If  $V \neq W$ , then  $(V \cdot W^{-1})^B = 0$ , in contradiction with the previous point. Thus  $V = W$ .

We have the following:

**Proposition 7.3.** *For every finite subset  $X \subseteq L_A$ , the height function  $\text{ht}_X$  is a locally constant function on  $\text{Sub}(A)$ .*

*Proof.* We start by noting that

$$\text{ht}_X(B) = \max\{\text{ht}_{\{x\}}(B) \mid x \in X\}.$$

The maximum of a finite number of locally constant functions is again locally constant. Hence it suffices to understand that  $\text{ht}_{\{x\}}$  is locally constant for any element  $x$  of  $L_A$ .

Now let  $(B_i)_{i \in \mathbb{N}}$  denote a sequence of subgroups of  $A$  converging to a subgroup  $B$ . We need to show that  $\text{ht}_{\{x\}}(B_i) = \text{ht}_{\{x\}}(B)$  for almost all  $i$ . Without loss of generality we can assume



that the  $B_i$  are subgroups of  $B$ . Replacing  $x$  by  $\text{res}_B^A(x)$  if necessary we can further assume that  $A = B$ .

We first assume that  $x \in I_{A,n}^A$  for some  $n$ , and show that then also  $x \in I_{B_i,n}^A$  for almost all  $B_i$ . If  $x \in I_{A,n}^A$ , there exists an  $A$ -representation  $W$  with  $W^A = 0$ , such that  $e_W \cdot x$  lies in the ideal  $L_A \cdot I_n$ . For all  $i$  large enough we have  $W^{B_i} = 0$ , meaning that for these  $i$  the restriction  $e_{\text{res}_{B_i}^A W}$  becomes invertible in  $\Phi^{B_i} L$ . It follows that  $\text{res}_{B_i}^A x$  maps to the ideal generated by  $I_n$  in  $\Phi^{B_i} L$ , in other words  $x$  is contained in  $I_{B_i,n}^A$ .

For the other direction, we assume that  $x$  is not contained in  $I_{A,n}^A$  for some  $n$ , and show that then also  $x \notin I_{B_i,n}^A$  for almost all  $B_i$ . For this we recall from Remark 2.26 the construction of elements  $\gamma_j^V$  for every character  $V \in G^*$  and  $j \in \mathbb{N}$  satisfying the following three properties:

1.  $\alpha^*(\gamma_j^V) = \gamma_j^{\alpha^*(V)}$  for every group homomorphism  $\alpha: G' \rightarrow G$ .
2.  $\gamma_0^V = e_V$ .
3.  $\Phi^G L \cong L[e_V^{\pm 1}, \gamma_j^V \mid V \in G^* - \{\epsilon\}, j > 0]$  for all abelian compact Lie groups  $G$ .

Now, if  $x$  is not contained in  $I_{A,n}^A$ , it maps to a non-trivial element in

$$\Phi^A L/I_n = L/I_n[e_V^{\pm 1}, \gamma_j^V \mid V \in A^* - \{\epsilon\}, j > 0].$$

In other words, there exists an  $A$ -representation  $W$  with  $W^A = 0$  and pairwise different non-trivial characters  $V_1, \dots, V_k$  such that  $e_W \cdot x$  is a polynomial over  $L$  in the classes  $\gamma_j^{V_l}$ ,  $l = 1, \dots, k, j \geq 0$ , not all of whose coefficients are contained in  $I_n$ . For all  $i$  large enough we have that (i)  $W^{B_i} = 0$  and that (ii) all the characters  $V_1, \dots, V_k$  restrict to pairwise different and non-trivial characters of  $B_i$ . It then follows that for these  $i$  the element  $e_{\text{res}_{B_i}^A W} \cdot \text{res}_{B_i}^A x$  equals

the corresponding polynomial in the classes  $\gamma_j^{\text{res}_{B_i}^A V_l}$ , implying that it maps to a non-trivial element in  $\Phi^{B_i} L/I_n$ . In other words,  $x$  is not contained in  $I_{B_i,n}^A$  for  $i$  large enough. This finishes the proof.  $\square$

Together with admissibility, this property characterizes the height functions of finite subsets of  $L_A$ :

**Proposition 7.4.** *Given a function  $f: \text{Sub}(A) \rightarrow \overline{\mathbb{N}}_-$ , the following are equivalent:*

1. *There exists a finite subset  $X \subseteq L_A$  such that  $f = \text{ht}_X$ .*
2. *The function  $f$  is admissible and locally constant.*

*Proof.* We have already shown the implication 1.  $\Rightarrow$  2.

It remains to show that given a locally constant admissible function  $f$ , there exists a finite subset  $X \subseteq L_A$  with  $f = \text{ht}_X$ . We start with the following claim: If  $f$  is admissible, then given any pair of subgroups  $B, B' \subseteq A$ , there exists an element  $x_{B,B'} \in L_A$  such that  $\text{ht}_{x_{B,B'}}(B) = f(B)$  and  $\text{ht}_{x_{B,B'}}(B') \leq f(B')$ . To see this, we distinguish between three cases:

- (i) If  $B$  is not a subgroup of  $B'$ , we can choose a character  $V \in A^*$  which restricts to the trivial character over  $B'$  but to a non-trivial character over  $B$ . Then  $x_{B,B'} = e_V \cdot v_{f(B)}$  has the desired properties, since  $\text{ht}_{e_V \cdot v_n}(B) = n$  and  $\text{ht}_{e_V \cdot v_n}(B') = -1$ . Here and in the following, we set  $v_{-1} = 0$ ,  $v_0 = p$  and  $v_\infty = 1$ .
- (ii) If  $B$  is a subgroup of  $B'$  with  $\pi_0(B'/B)$  not a  $p$ -group, we choose a prime  $q \neq p$  dividing the order of  $\pi_0(B'/B)$  and a surjection  $g: B' \rightarrow C_q$  containing  $B$  in the kernel. Then we set  $y = v_{f(B)} \cdot g^*(x_0^{(q)})$ , where  $x_0^{(q)} \in L_{C_q}$  is the element introduced in Section 5.1. Then  $x_0^{(q)}$  is an element of  $I_{C_q,0}^{C_q}$  and its restriction to the trivial group is given by  $q$  and hence a unit. It follows that  $\text{ht}_y(B') = -1$ , since  $\text{ht}_{g^*(x_0^{(q)})}(B') = -1$ . Moreover,  $\text{ht}_y(B) = f(B)$ , since  $\text{res}_B^{B'}(y) = v_{f(B)} \cdot q$ . Hence, we can set  $x_{B,B'}$  to be any lift of  $y$  to an element of  $L_A$ .
- (iii) The remaining case is when  $B$  is a subgroup of  $B'$  with  $\pi_0(B'/B)$  a  $p$ -group. Let  $r$  be the minimum of the  $p$ -rank of  $\pi_0(B'/B)$  and the number  $f(B) + 1$ , and choose a surjection  $g: B'/B \rightarrow C_p^r$ . By Corollary 5.25 we know for  $f(B) < \infty$  that there exist an element  $x_{f(B)} \in L_{C_p^{f(B)+1}}$  such that  $\text{ht}_{x_{f(B)}}(1) = f(B)$  and  $\text{ht}_{x_{f(B)}}(C_p^{f(B)+1}) = -1$ .

(We set  $x_{-1} = 0$ .) We can choose an embedding  $C_p^r \rightarrow C_p^{f(B)+1}$  and restrict  $x_{f(B)}$  to an element  $y \in LC_p^r$ . Then we have  $\text{ht}_y(1) = f(B)$  and  $\text{ht}_y(C_p^r) = f(B) - r$  (see Corollary 5.25). If  $f(B) = \infty$ , choose  $y = 1$ . It follows that  $\text{ht}_{g^*(y)}(B) = f(B)$  and  $\text{ht}_{g^*(y)}(B') \leq f(B) - r \leq f(B')$ , since  $f$  is admissible. Hence,  $g^*(y)$  has the desired properties.

Now given any such pair  $(B, B')$  there exists an open neighbourhood  $U_{B'}$  of  $B'$  on which both  $\text{ht}_{x_{B, B'}}$  and  $f$  are constant. The  $U_{B'}$  for varying  $B'$  form an open cover of the compact space  $\text{Sub}(A)$ . Let  $U_{B'_1}, \dots, U_{B'_k}$  be a finite subcover. We then set

$$x_B = x_{B, B'_1} \cdot x_{B, B'_2} \cdots x_{B, B'_k}$$

to be the product of the corresponding elements. For any closed subgroup  $B'$  we have

$$\text{ht}_{x_B}(B') = \min(\text{ht}_{x_{B, B'_j}}(B') \mid j = 1, \dots, k).$$

For  $B' = B$  this gives  $\text{ht}_x(B) = f(B)$ , since  $\text{ht}_{x_{B, B'_j}}(B) = f(B)$  for all  $j$ . Any  $B'$  is contained in  $U_{B'_i}$  for some  $i$ , yielding

$$\text{ht}_{x_B}(B') \leq \text{ht}_{x_{B, B'_i}}(B') = \text{ht}_{x_{B, B'_i}}(B'_i) \leq f(B'_i).$$

In summary, the height function  $\text{ht}_{x_B}$  is less than or equal to  $f$  everywhere and agrees with  $f$  at  $B$  itself. Once more we can apply that  $\text{ht}_{x_B}$  and  $f$  are locally constant to find that  $\text{ht}_{x_B}$  and  $f$  in fact agree on a neighborhood  $V_B$  of  $B$ . Letting  $B$  vary, this yields an open cover of  $\text{Sub}(A)$ , for which we can choose a finite subcover  $V_{B_1}, \dots, V_{B_l}$ . Hence, for every  $B \in \text{Sub}(A)$ , there exists some  $i \in \{1, \dots, l\}$  such that  $f(B) = \text{ht}_{x_{B_i}}$ . Finally, we define  $X$  to be the set  $\{x_{B_1}, \dots, x_{B_l}\}$ . Since  $\text{ht}_X$  is given by the maximum of the functions  $\text{ht}_{B_i}$ , it follows that  $X$  has the desired property.  $\square$

**Theorem 7.5.** *The Zariski topology on  $\text{Spec}^{\text{inv}}(L_A)$  has as a basis the closed sets*

$$V_f = \{I_{B, n} \mid n > f(B)\}$$

for all locally constant, admissible functions  $f: \text{Sub}(A) \rightarrow \overline{\mathbb{N}}_-$ .

*Proof.* A basis for the Zariski topology is given by the sets  $V(X)$  for all finite subsets  $X$  of  $L_A$ . As we saw above,  $V(X)$  is determined by its height function  $\text{ht}_X$  as

$$V(X) = \{I_{B, n} \mid n > \text{ht}_X(B)\}.$$

By Proposition 7.4, the functions  $\text{Sub}(A) \rightarrow \overline{\mathbb{N}}_-$  that occur as height functions of finite subsets are precisely the locally constant admissible functions, which finishes the proof.  $\square$

## 8 Comparison with $A$ -spectra

In this final section, we discuss the relationship of the algebraic results of the previous sections with the theory of  $A$ -spectra.

### 8.1 The universal support theory via $MU_A$ -homology

We begin by comparing our classification of invariant prime ideals with the Balmer spectrum of compact  $p$ -local  $A$ -spectra. We recall from [Bal05] that a prime ideal  $\mathfrak{p}$  of a tensor-triangulated category  $\mathcal{C}$  is defined to be a thick tensor-ideal with the additional property that if  $X \otimes Y$  is contained in  $\mathfrak{p}$ , then  $X \in \mathfrak{p}$  or  $Y \in \mathfrak{p}$ . The set of all prime ideals assembles to a topological space  $\text{Spec}(\mathcal{C})$ , the Balmer spectrum, with the topology generated by the closed sets  $\text{supp}(X) = \{\mathfrak{p} \mid X \notin \mathfrak{p}\}$  for all objects  $X \in \mathcal{C}$ .

Here, the support function  $\text{supp}(-)$ , assigning a closed set of the Balmer spectrum to every object of  $\mathcal{C}$ , is the *universal support theory* in the sense of Balmer. This means that it is terminal among pairs  $(T, \sigma)$  of a topological space  $T$  and a function

$$\sigma: \text{ob}(\mathcal{C}) \rightarrow \{\text{closed subsets of } T\}$$

satisfying  $\sigma(0) = \emptyset$ ,  $\sigma(1) = T$ ,  $\sigma(X \oplus Y) = \sigma(X) \cup \sigma(Y)$ ,  $\sigma(\Sigma X) = \sigma(X)$  and  $\sigma(X \otimes Y) = \sigma(X) \cap \sigma(Y)$  for all  $X, Y \in \text{ob}(\mathcal{C})$ , and  $\sigma(X) \subseteq \sigma(Y) \cup \sigma(Z)$  whenever there exists a distinguished triangle  $X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$ . See [Bal05] for more details.

In the case of compact  $p$ -local  $A$ -spectra  $\text{Sp}_A^c$  for an abelian compact Lie group  $A$ , the Balmer spectrum was computed in [BS17, BHN<sup>+</sup>19, BGH20]. Given a closed subgroup  $B$  and  $n \in \mathbb{N} \cup \{\infty\}$ , one defines a prime ideal

$$P(B, n) = \{X \in \text{Sp}_A^c \mid K(n)_*(\Phi^B X) = 0\},$$

where  $K(n)$  denotes the  $n$ th Morava  $K$ -theory, and  $\Phi^B X$  is the  $B$ -geometric fixed point spectrum of  $X$ , a compact  $p$ -local spectrum.

**Theorem 8.1** ([BS17, BHN<sup>+</sup>19, BGH20]). *The map*

$$\begin{aligned} \text{Sub}(A) \times \overline{\mathbb{N}} &\rightarrow \text{Spec}(\text{Sp}_A^c) \\ (B, n) &\mapsto P(B, n) \end{aligned}$$

defines a bijection. Moreover, the topology on  $\text{Spec}(\text{Sp}_A^c)$  has a basis given by the closed sets defined by

$$\{P(B, n) \mid n \geq f(B)\}$$

where  $f$  ranges through all admissible functions  $\text{Sub}(A) \rightarrow \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ .

Here, ‘admissible’ is meant in the sense of Definition 7.2. Hence, comparing to Theorem 7.5, we see that the assignment  $I_{B,n} \mapsto P(B, n)$  defines a homeomorphism from  $\text{Spec}^{\text{inv}}(L_A)$  to  $\text{Spec}(\text{Sp}_A^c)$ .<sup>3</sup> The computation of  $\text{Spec}(\text{Sp}_A^c)$  is also analogous to the one of  $\text{Spec}^{\text{inv}}(L_A)$  in the way that computing the underlying set is relatively straightforward (and is in fact known for all compact Lie groups), with most work going into understanding the topology.

Hence, we can view the universal support theory of  $\text{Sp}_A^c$  to take values in the invariant prime ideals of  $L_A$ . The goal of the remainder of this section is to construct this universal support theory more intrinsically using  $MU_A$ -homology and the structure of equivariant formal group laws described in this paper. The idea is the following: Given a compact  $p$ -local  $A$ -spectrum  $X$ , we can consider its equivariant complex bordism homology  $(\underline{MU}_A)_* X$ . Here, underlining  $MU_A$  indicates that we take the  $A$ -Mackey functor valued homology of  $X$ , i.e., we record the collection of  $(MU_B)_*(\text{res}_B^A X)$  for all closed subgroups  $B$  of  $A$ , together with restriction and transfer maps between them. We will always work at the fixed prime  $p$  and  $p$ -localize everything implicitly.

Since the coefficients  $\pi_*^A MU_A$  are isomorphic to the Lazard ring  $L_A$  and moreover the cooperations  $\pi_*^A MU_A \wedge MU_A$  agree with  $S_A$ , the groups  $(MU_A)_* X$  form a graded  $A$ -Mackey functor in  $(L_A, S_A)$ -comodules. As such, we can take its support in the invariant prime ideals

$$\text{supp}((\underline{MU}_A)_* X) = \{I_{B,n}^A \mid ((\underline{MU}_A)_* X)_{I_{B,n}^A} = 0\} \subseteq \text{Spec}^{\text{inv}}(L_A).$$

**Remark 8.2.** In general,  $\text{supp}((\underline{MU}_A)_* X)$  is different from  $\text{supp}((MU_A)_* X)$ . Take for example  $A = C_2$ ,  $p = 2$  and  $X = S^\sigma$ , the circle with action given by reflection at a line. We have

$$MU_*(\text{res}_1^{C_2} S^\sigma)_{I_{1,0}} = MU_*(S^1)_{\mathbb{Q}} \neq 0.$$

The module  $(MU_{C_2})_*(S^\sigma)_{I_{1,0}^{C_2}}$  is rational as well since  $p \notin I_{1,0}^{C_2}$ . Rationally,  $(MU_{C_2})_*(S^\sigma)$  splits into the coinvariants  $(MU_{C_2})_*(S^\sigma)_{C_2} = 0$  and the geometric fixed points  $(MU_{C_2})_*(S^\sigma)[e^{-1}]$ ,

<sup>3</sup>The reader may have noticed that in this section our admissible functions  $f$  take values in  $\overline{\mathbb{N}}$ , while in the last section they took values in  $\overline{\mathbb{N}}_-$ . Likewise, we consider the condition  $n \geq f(B)$  here, and considered  $n > f(B)$  before. A shift by one shows that the topologies agree. The shift is caused by wanting to have  $v_n$  having height  $n$  in the last section.

for  $e$  the Euler class of the unique non-trivial character. The element  $\bar{\nu}_0$  from 5.21 becomes zero in the geometric fixed points, but restricts to 2 and is thus not in  $I_{1,0}^{C_2}$ ; thus  $(MU_{C_2})_*(S^\sigma)[e^{-1}]$  becomes zero as well after localization at  $I_{1,0}^{C_2}$ , and  $(MU_{C_2})_*(S^\sigma)_{I_{1,0}^{C_2}} = 0$ .

**Remark 8.3** (Transfer maps). The isomorphisms  $L_A \cong \pi_*^A MU_A$  imply that on top of the contravariant restriction maps along group homomorphisms there also exist transfer maps  $L_B \rightarrow L_A$  for inclusions  $B \subseteq A$  of finite index. In other words, the collection of all equivariant Lazard rings  $L_A$  forms a ‘global Green functor’ on the family of abelian compact Lie groups, in the sense of [Sch18, Definition 5.1.3]. While we have no general interpretation of transfers in terms of equivariant formal groups, we can compute them as follows: By Frobenius reciprocity, it suffices to compute  $\text{tr}_B^A: L_B \rightarrow L_A$  on  $1 \in L_B$  since the restriction is surjective. Inductively, we can further assume that  $A/B \cong C_p$ . Furthermore,  $\text{tr}_B^A(1) = q^* \text{tr}_{\{1\}}^{C_p}(1)$  for  $q: A \rightarrow A/B \cong C_p$  since transfers are compatible with inflation maps (see, e.g., [Sch18, Theorem 4.2.6 ff.]). Hence it suffices to identify  $\text{tr}_{\{1\}}^{C_p}(1)$ . We claim that it equals  $\bar{\nu}_0 \in L_{C_p}$ . Indeed, any transfer maps to zero in the geometric fixed points and is thus an element of  $I_{C_p,0}$ . We know that  $\bar{\nu}_0$  generates  $I_{C_p,0}$ . Hence, we can write  $\text{tr}_{\{1\}}^{C_p}(1) = x \cdot \bar{\nu}_0$  for  $x \in L_{C_p}$ . Writing  $x$  as  $x_0 + x' \cdot e$  for  $e$  a non-trivial Euler class and  $x_0 \in L$ , we obtain  $\text{tr}_{\{1\}}^{C_p}(1) = x_0 \cdot \bar{\nu}_0$  since  $e \cdot \bar{\nu}_0 = 0$  by the definition of  $\bar{\nu}_0$ . We obtain  $x_0 \cdot p = \text{res}_{\{1\}}^{C_p} \text{tr}_{\{1\}}^{C_p}(1) = p$  since  $\bar{\nu}_0$  restricts to  $p$  and thus  $x_0 = 1$  and  $\text{tr}_{\{1\}}^{C_p}(1) = \bar{\nu}_0$ .

We will now see that our notion of support is another model for the universal support theory on compact  $A$ -spectra.

We first note a major inconvenience: It is unclear whether  $(MU_A)_*X$  is a finitely generated  $L_A$ -module, even for compact  $X$ . This is in contrast with the non-equivariant situation, where finite generation of  $MU_*X$  for compact  $X$  follows from the fact that  $L$  is a polynomial ring and hence coherent. An analogous statement is unknown for equivariant Lazard rings. In particular, it is a priori unclear that  $\text{supp}((MU_A)_*X)$  is indeed a *closed* subset of  $\text{Spec}^{\text{inv}}(L_A)$  and we have to prove this by hand, see Proposition 8.5 below.

The following proposition gives the relationship between the  $MU_A$ -homology support theory described above and geometric fixed points.

**Proposition 8.4.** *Let  $B$  be a subgroup of  $A$ ,  $n \in \bar{\mathbb{N}}$  and  $X$  a compact  $A$ -spectrum. Then the following are equivalent:*

- (i) *The Mackey functor  $((MU_A)_*(X))_{I_{B,n}^A}$  is trivial.*
- (ii) *The  $(MU_B)_*$ -module  $((MU_B)_*(\text{res}_B^A X))_{I_{B,n}^B}$  is trivial.*
- (iii) *The  $B$ -geometric fixed points  $\Phi^B(X)$  are of chromatic type  $> n$ .*

We give two proofs of this proposition below, one using the results of [BS17, BHN<sup>+</sup>19, BGH20] and one independent of these results. The latter one is complicated by the fact that we don’t know whether  $(MU_A)_*X$  is always finitely generated.

We have the following corollary:

**Proposition 8.5.** *Let  $X$  be a compact  $A$ -spectrum. Then its support  $\text{supp}((MU_A)_*X)$  is a closed subset of  $\text{Spec}^{\text{inv}}(L_A)$ .*

*Moreover, the assignment*

$$\text{ob}(\text{Sp}_A^c) \rightarrow \text{Spec}^{\text{inv}}(L_A) ; X \mapsto \text{supp}((MU_A)_*X)$$

*is a support theory on compact  $A$ -spectra.*

*Proof.* We consider the type function

$$\text{type}_X: \text{Sub}(A) \rightarrow \bar{\mathbb{N}} ; X \mapsto \text{type}(\Phi^B X),$$

which is locally constant by [BGH20, Proposition 4.3]. By the previous proposition,  $I_{B,n}$  is an element of  $\text{supp}((MU_A)_*X)$  if and only if  $n \geq \text{type}_X(B)$ . Since the support  $\text{supp}((MU_A)_*X)$  is closed under inclusion, Theorem 5.1 implies that  $I_{B',n} \in \text{supp}((MU_A)_*X)$  if  $I_{B,n - \text{rk}_p(\pi_0(B/B'))} \in \text{supp}((MU_A)_*X)$  for every  $p$ -toral inclusion  $B' \subseteq B$ . Thus  $\text{type}_X$  is admissible in the sense of Definition 7.2. Theorem 7.5 implies that  $\text{supp}((MU_A)_*X)$  is closed, as desired.

For the second part, all required properties of a support theory follow easily from exactness of localization  $(-)_I_{B,n}^A$  except for the one on the interplay with smash products. This in turn follows from the third characterization in Proposition 8.4, since the type of a smash product of two compact spectra is the maximum of the two types.  $\square$

By the universal property of the Balmer spectrum, we obtain a continuous map

$$\mathrm{Spec}^{\mathrm{inv}}(L_A) \rightarrow \mathrm{Spec}(\mathrm{Sp}_A^c).$$

Proposition 8.4 makes it clear that this map sends  $I_{B,n}^A$  to  $P(B,n)$  and is hence bijective. Therefore we can conclude from the results of this paper that the topology on  $\mathrm{Spec}(\mathrm{Sp}_A^c)$  is at least as coarse as the one on  $\mathrm{Spec}^{\mathrm{inv}}(L_A)$ . In other words, our proof of the existence of inclusions  $I_{B,n} \subseteq I_{B',n'}$  gives another proof of the analogous inclusion  $P(B',n') \subseteq P(B,n)$  on the topological side. The fact that there are no further topological inclusions requires additional arguments, namely the existence of compact  $A$ -spectra with ‘maximal type shifting behaviour’. See [BHN<sup>+</sup>19, Section 4] or [KL20, Section 7]. Knowing this, we see that  $X \mapsto \mathrm{supp}((\underline{MU}_A)_*X)$  is a universal support theory on compact  $A$ -spectra.

It remains to give the proof of Proposition 8.4, which we will do in three steps:

(i)  $\Rightarrow$  (ii): Since the  $L_A$ -action on  $(MU_B)_*(X)$  factors through  $\mathrm{res}_B^A: L_A \rightarrow L_B$  and  $I_{B,n}^A = (\mathrm{res}_B^A)^{-1}(I_{B,n}^B)$ , it follows that we have an isomorphism  $((MU_B)_*(X))_{I_{B,n}^A} \cong ((MU_B)_*(X))_{I_{B,n}^B}$ . In particular the vanishing of the entire Mackey-functor  $(\underline{MU}_*)_*(X)_{I_{B,n}^A}$  implies the vanishing at the subgroup  $B$  as a special case.

(ii)  $\iff$  (iii): Note that  $I_{B,n}^B$  contains none of the non-trivial Euler classes for  $B$ , hence the non-trivial Euler classes act invertibly on  $((MU_B)_*(X))_{I_{B,n}^B}$ . It follows that we have an isomorphism

$$((MU_B)_*(X))_{I_{B,n}^B} \cong ((\Phi^B MU_B)_*\Phi^B X)_{I_{B,n}^B} \cong \Phi^B((MU_B)_{I_{B,n}^B})_*\Phi^B X.$$

Modulo  $I_n$ , the ring  $\Phi^B((MU_B)_{I_{B,n}^B})_*$  embeds into the field of fractions of  $\Phi^B L/I_n$ , which is non-trivial. Moreover,  $v_n$  is not contained in  $I_{B,n}^B$  and hence is invertible in the localization. It follows that the localization  $(\Phi^B MU_B)_{I_{B,n}^B}$  is an  $MU$ -algebra of height  $n$ , i.e., its vanishing detects compact spectra of type  $\geq n+1$ .

(iii)  $\Rightarrow$  (i): Let  $X$  be a compact  $A$ -spectrum such that  $\Phi^B(X)$  is of type  $\geq n$ . By the previous paragraph we know that  $((MU_B)_*(X))_{I_{B,n}^A} = 0$ , and we have to show that  $((MU_{\tilde{B}})_*(X))_{I_{B,n}^A} = 0$  for all other closed subgroups  $\tilde{B}$ , too. We first assume that  $\tilde{B}$  does not contain  $B$ . Then there exists a character  $V \in A^*$  which restricts to the trivial character for  $\tilde{B}$  but to a non-trivial character for  $B$ . It follows that  $e_V$  is not contained in  $I_{B,n}^A$  and hence acts invertibly on  $(MU_{\tilde{B}})_{I_{B,n}^A}$ . On the other hand  $e_V$  restricts to 0 in  $MU_{\tilde{B}}$ , so it follows that  $(MU_{\tilde{B}})_{I_{B,n}^A} = 0$  and in particular  $((MU_{\tilde{B}})_*(X))_{I_{B,n}^A} = 0$ , as desired. Hence we can assume that  $\tilde{B}$  contains  $B$  as a subgroup. Since the statement then no longer depends on the ambient group, we can reduce to the case  $\tilde{B} = A$ .

Hence we need to show that  $(MU_A)_*(X)_{I_{B,n}^A} = 0$ . By induction on the pair (dimension, rank of  $\pi_0(A/B)$ ) it follows that all the localizations at smaller intermediate groups  $B \subseteq \tilde{B} \subseteq A$  vanish, and hence the homotopy groups of  $(MU_A)_{I_{B,n}^A} \wedge X$  are concentrated at  $A$ . In particular this implies that the map  $((MU_A)_{I_{B,n}^A})_*X \rightarrow ((\Phi^A MU_A)_{I_{B,n}^A})_*\Phi^A X$  is an isomorphism and all Euler classes  $e_V$  for non-trivial characters act invertibly on  $((MU_A)_{I_{B,n}^A})_*X$ . Now, if  $V$  restricts to the trivial character in  $B^*$  (such a  $V$  always exists since we can assume that  $B$  is a proper subgroup of  $A$ ), its Euler class  $e_V$  lies in the maximal ideal  $I_{B,n}^A$  of  $((MU_A)_{I_{B,n}^A})_*$ .

If we knew that  $((MU_A)_{I_{B,n}^A})_*X$  is finitely generated over  $(MU_A)_*$  we could apply Nakayama’s lemma to see directly that  $((MU_A)_{I_{B,n}^A})_*X = 0$ , as desired. Since we do not know this, we have to argue differently: By Corollary 5.2, we know that  $\Phi^A((MU_A)_{I_{B,n}^A})_*$  is trivial if  $\pi_0(A/B)$  is not a  $p$ -group, and is of height  $n - \mathrm{rank}_p(\pi_0(A/B))$  otherwise (where negative heights again mean that the theory is trivial). Hence, what we want to show is that if

$\Phi^B X$  is of type  $\geq n$ , then  $\Phi^A X$  is of type  $\geq n - \text{rank}_p(\pi_0(A/B))$ . Indeed, this implies that  $((MU_A)_{J_{B,n}^A})_* X \cong ((\Phi^A MU_A)_{J_{B,n}^A})_* \Phi^A X$  is trivial.

This precise statement about  $\Phi^A X$  is one of the main results of [BGH20], building on [BS17] and [BHN<sup>+</sup>19]. Hence, using these results, Proposition 8.4 follows. Alternatively, rather than importing we can reprove the above statement using the methods from this paper. Note that by induction on the rank of  $A/B$  and by replacing  $X$  by the compact  $A/B$ -spectrum  $\underline{\Phi}^B X$  it suffices to show two special cases:

1. If  $X$  is a compact  $C_{p^k}$ -spectrum of underlying type  $\geq n$ , then the type of  $\Phi^{C_{p^k}} X$  is at least  $n - 1$ .
2. If  $X$  is a compact  $\mathbf{T}$ -spectrum of underlying type  $\geq n$ , then the type of  $\Phi^{\mathbf{T}} X$  is also  $\geq n$ .

For (1) it suffices to find a complex oriented theory  $E$  of height  $n$  such that  $\Phi^{C_{p^k}} \underline{E}$  is of height  $\geq n - 1$ , where  $\underline{E} = F(EG_+, E)$  denotes the Borel theory associated to  $E$  (see the proofs of [BHN<sup>+</sup>19, Corollary 3.12] or [BGH20, Proposition 6.10] for details on this argument). In [BHN<sup>+</sup>19] it was shown that Morava  $E$ -theory  $E = E_n$  has this property, building on results of Hopkins-Kuhn-Ravenel on the  $p$ -divisible group associated to  $E_n$  [HKR00] and extending earlier work of Greenlees, Hovey and Sadofsky. Using Remark 5.15 we obtain similar results more generally:

**Proposition 8.6.** *Let  $E$  be any complex oriented ring spectrum of height  $n$  which is Landweber exact over  $MU/I_{n-1} = MU/(v_0, \dots, v_{n-2})$ , i.e.,  $I_{n-1}$  acts trivially on  $\pi_* E$ ,  $v_{n-1} \in \pi_* E$  is a regular element and  $v_n$  is a unit modulo  $v_{n-1}$ . Then  $\Phi^{C_{p^k}} \underline{E}$  is of height  $n - 1$ .*

*Proof.* We apply Remark 5.15 and check that its assumptions are satisfied. We have  $(\underline{E}^{\mathbf{T}})_* = E_*[[e]]$ , with Euler classes  $e_n$  given by the  $n$ -series  $[n]_F(e)$  for the formal group law associated to  $E$ . If  $n$  is a power of  $p$ , the leading term of this Euler class is a power of  $v_{n-1}$ , which we assumed to be regular. Modulo  $v_{n-1}$ , the leading term becomes  $v_n$  which is a unit since  $E/v_{n-1}$  is of height  $n$ . Hence by Remark 5.15 we find that the pushforward of the element  $\psi_{p^k}^{(n-1)}$  to  $E_*[[e]]$ , i.e., the element  $\sum_{i=0}^{\infty} a_i [p^{k-1}]_F(e)$  for  $\sum_{i=0}^{\infty} a_i e^i = [p]_F(e)/e^{p^{n-1}}$ , generates the kernel of the composite

$$E_*[[e]] \rightarrow E_*[[e]]/[p^k]_F(e) = (\underline{E}^{C_{p^k}})_* \rightarrow (\Phi^{C_{p^k}} \underline{E})_*.$$

The leading term of  $\psi_{p^k}^{(n-1)}$  equals  $v_{n-1}$ , which is not a unit. Hence  $\Phi^{C_{p^k}} \underline{E}$  is non-trivial and since  $I_{n-1}$  acts trivially it must be of height  $\geq n - 1$ . Furthermore, reducing modulo  $v_{n-1}$  the leading coefficient of  $\psi_{p^k}^{(n-1)}$  equals a power of  $v_n$ . Since  $v_n$  is a unit modulo  $v_{n-1}$ , it follows that so is  $\psi_{p^k}^{(n-1)}$ . Therefore  $(\Phi^{C_{p^k}} \underline{E})/v_{n-1}$  is trivial, and hence  $(\Phi^{C_{p^k}} \underline{E})$  is of height exactly  $n - 1$ .  $\square$

**Example 8.7.** Given any Landweber exact complex oriented ring spectrum  $E$  of height  $n$ , its quotient  $E/I_{n-1}$  satisfies the assumptions of the previous proposition. It follows that also in this case the height of  $\Phi^{C_{p^k}} E$  equals  $n - 1$ . For example this applies to Johnson-Wilson spectra, to  $MU[v_n^{-1}]$  or to Morava  $E$ -theories.

**Remark 8.8.** Another approach to blue-shift questions like the above is via the chromatic Nullstellensatz of [BSY22]: Let  $E$  be an  $E_\infty$ -ring of height  $n$ , i.e.  $K(n)_* E$  is non-trivial and  $K(n+1)_* E$  is trivial. If  $E$  is complex-oriented, this is equivalent to our previous definition of height, i.e. to the vanishing of  $E$ -homology detecting compact spectra of type  $\geq n+1$ . We claim that the Tate spectra  $E^{tC_p} = \Phi^{C_{p^k}} \underline{E}$  are of height  $n - 1$ . Indeed, by the results of [BSY22] any such theory  $E$  maps to Morava  $E$ -theory  $E_n$ , and  $\Phi^{C_{p^k}} \underline{E}_n$  is non-trivial  $K(n-1)$ -locally by [BHN<sup>+</sup>19, Theorem 3.4] or the argument above. Furthermore,  $\Phi^{C_{p^k}} \underline{E}$  must be trivial  $K(n)$ -locally since the converse would contradict the existence of compact  $C_{p^k}$ -spectra  $X$  of underlying type  $n$  and  $C_{p^k}$ -geometric fixed points of type  $n - 1$ , by the same argument referenced above.

Similarly, for (2) one needs to find a complex oriented theory  $E$  of height  $n$  such that  $\Phi^{\mathbf{T}} \underline{E}$  is also of height  $n$ . This is more elementary and satisfied by any  $p$ -local complex oriented theory  $E$  of height  $n$ . To see this, note again that the Euler classes  $e_n$  are given by the  $n$ -series  $[n]_F(e) \in E_*[[e]]$ . Writing  $n = p^k \cdot m$  with  $m$  coprime to  $p$ , we find that  $[n]_F(e)$  is a unit multiple

of  $[p^k]_F(e)$ . Modulo  $I_n$ , for  $k > 0$  the leading term of  $[p^k]_F(e)$  is a power of  $v_n$ , which by assumption is a unit in  $E_*/I_n$ . It follows that, modulo  $I_n$ , the coefficients of  $\Phi^T \underline{E}$  are given by  $E_*/I_n((e))$ , which is always non-trivial when  $E_*/I_n$  is.

## 8.2 Change of groups and the structure of $\mathcal{M}_{FG}^A$

For any  $A$ -spectrum  $X$ , the groups  $MU_{2*}^A(X)$  come equipped with the structure of a graded  $(L_A, S_A)$ -comodule. Since the stack  $\mathcal{M}_{FG}^A$  is the stack associated to this graded Hopf algebroid, we obtain an associated quasi-coherent sheaf  $\mathcal{F}^A(X)$  on  $\mathcal{M}_{FG}^A$  (see [MO20, Proposition 4.3]). This is the 0-th graded piece of a  $\text{QCoh}(\mathcal{M}_{FG}^A)$ -valued homology theory on  $A$ -spectra, whose  $i$ -th piece  $\mathcal{F}_i^A(X)$  is given by  $MU_{2*+i}^A(X)$ . We end this section by a closer look upon how the structure of  $\mathcal{M}_{FG}^A$  relates to this homology theory.

Recall from Proposition 3.11 that for a closed subgroup  $B \subseteq A$ , there is an open immersion  $j: \mathcal{M}_{FG}^{A/B} \rightarrow \mathcal{M}_{FG}^A$  and a closed immersion  $i: \mathcal{M}_{FG}^B \rightarrow \mathcal{M}_{FG}^A$ . We obtain corresponding adjunctions

$$\text{QCoh}(\mathcal{M}_{FG}^A) \begin{array}{c} \xleftarrow{j^*} \\ \xrightarrow{j_*} \end{array} \text{QCoh}(\mathcal{M}_{FG}^{A/B}) \quad \text{and} \quad \text{QCoh}(\mathcal{M}_{FG}^A) \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \end{array} \text{QCoh}(\mathcal{M}_{FG}^B).$$

Believing that the structure of  $\mathcal{M}_{FG}^A$  dictates the structure of the  $\infty$ -category  $\text{Sp}_A$  of  $A$ -spectra, we expect a relation to the adjunctions

$$\text{Sp}_A \begin{array}{c} \xleftarrow{\underline{\Phi}^B} \\ \xrightarrow{P_{A/B}^*} \end{array} \text{Sp}_{A/B} \quad \text{and} \quad \text{Sp}_A \begin{array}{c} \xleftarrow{\text{res}_B^A} \\ \xrightarrow{\text{coind}_B^A} \end{array} \text{Sp}_B.$$

Here, denoting by  $\mathcal{F}[B]$  the family of subgroups of  $A$  not containing  $B$  and by  $q: A \rightarrow A/B$  the projection, we define  $\underline{\Phi}^B(X) = (\tilde{E}\mathcal{F}[B] \wedge X)^B$  and  $P_{A/B}^*(Y) = \tilde{E}\mathcal{F}[B] \wedge q^*X$ . Note that the definition of  $\underline{\Phi}^B$  is made so that its underlying spectrum is  $\Phi^B$  and more generally  $\Phi^{C/B} \underline{\Phi}^B \simeq \Phi^C$  for every  $B \subseteq C \subseteq A$ . For more details on the first adjunction, see [LMSM86, Section II.9], [Hil12, Section 4.1] and [MSZ23, Section 2.2]. For the benefit of the reader, we give a brief sketch of its basic properties: The adjunction between  $q^*$  and  $(-)^H$  induces maps

$$\text{id} \rightarrow \underline{\Phi}^B P_{A/B}^* \quad \text{and} \quad P_{A/B}^* \underline{\Phi}^B \rightarrow \tilde{E}\mathcal{F}[B] \wedge (-),$$

which can be checked to be equivalences on geometric fixed points. The inverses

$$\underline{\Phi}^B P_{A/B}^* \xrightarrow{\simeq} \text{id} \quad \text{and} \quad \text{id} \rightarrow \tilde{E}\mathcal{F}[B] \wedge (-) \xrightarrow{\simeq} P_{A/B}^* \underline{\Phi}^B$$

form the counit and unit of the adjunction. In particular,  $P_{A/B}^*$  is fully faithful and its image agrees with that of  $\tilde{E}\mathcal{F}[B] \wedge$ . Since smashing with  $\tilde{E}\mathcal{F}[H]$  is idempotent and hence symmetric monoidal,  $P_{A/B}^*$  is symmetric monoidal as well and so is  $\underline{\Phi}^B$ .

**Proposition 8.9.** *The diagram*

$$\begin{array}{ccc} \text{Sp}_A & \begin{array}{c} \xleftarrow{\underline{\Phi}^B} \\ \xrightarrow{P_{A/B}^*} \end{array} & \text{Sp}_{A/B} \\ \downarrow \mathcal{F}^A & & \downarrow \mathcal{F}^{A/B} \\ \text{QCoh}(\mathcal{M}_{FG}^A) & \begin{array}{c} \xleftarrow{j^*} \\ \xrightarrow{j_*} \end{array} & \text{QCoh}(\mathcal{M}_{FG}^{A/B}) \end{array}$$

commutes, i.e. there are natural isomorphisms  $j^* \mathcal{F}^A(X) \cong \mathcal{F}^{A/B}(\underline{\Phi}^B X)$  for  $X \in \text{Sp}_A$  and  $j_* \mathcal{F}^{A/B}(Y) \cong \mathcal{F}^A(P_{A/B}^* Y)$  for  $Y \in \text{Sp}_{A/B}$ .

Moreover, there is a natural isomorphism  $i_* \mathcal{F}^B(Z) \cong \mathcal{F}^A(\text{coind}_B^A Z)$  for  $Z \in \text{Sp}_B$ .

No such isomorphism can be expected for  $i^*$  and  $\text{res}_B^A$  in general. One reason is that  $i^*$  is not flat, but even a spectral sequence relating  $i^*$  and  $\text{res}_B^A$  seems not to exist for  $A$  not a torus for reasons related to Remark 8.2.

Before we prove the proposition, we need a lemma, in which we will use the Hopf algebroid  $(\underline{\Phi}^B L_A, \underline{\Phi}^B S_A)$ . Here  $\underline{\Phi}^B L_A$  is obtained from  $L_A$  by inverting  $e_V$  for all  $V \notin \text{im}((A/B)^* \rightarrow A^*)$ , and the ring  $\underline{\Phi}^B S_A$  is defined as  $\underline{\Phi}^B L_A \otimes_{L_A} S_A \otimes_{L_A} \underline{\Phi}^B L_A$ . This classifies strict isomorphism between equivariant formal group laws where the relevant Euler classes are invertible on source and target. Since the invertibility of Euler classes only depends on the underlying equivariant formal group and not on the choice of coordinate, this simplifies to  $S_A \otimes_{L_A} \underline{\Phi}^B L_A$ .

**Lemma 8.10.** *For  $Y \in \text{Sp}_B$  and  $q: A \rightarrow A/B$  the projection, there are natural isomorphisms*

$$\underline{\Phi}^B L_A \otimes_{L_A} (MU_A)_*(q^* Y) \xrightarrow{\cong} \underline{\Phi}^B L_A \otimes_{L_A} (MU_A)_*(P_{A/B}^* Y) \xrightarrow{\cong} (\underline{\Phi}^B MU_A)_*(Y)$$

and

$$\underline{\Phi}^B L_A \otimes_{L_{A/B}} (MU_{A/B})_*(Y) \xrightarrow{\cong} (\underline{\Phi}^B MU_A)_*(Y)$$

of graded  $(\underline{\Phi}^B L_A, \underline{\Phi}^B S_A)$ -comodules, where the map  $L_{A/B} \rightarrow \underline{\Phi}^B L_A$  is defined as the composite  $L_{A/B} \xrightarrow{q^*} L_A \rightarrow \underline{\Phi}^B L_A$ .

*Proof.* A model for  $\tilde{E}\mathcal{F}[B]$  is given by  $S^{\infty W}$  for  $W$  the sum of all characters  $V \in \mathcal{V}$ , for  $\mathcal{V}$  the set of characters  $V$  not restricting to 1 in  $B$  or, equivalently,  $V \notin \text{im}((A/B)^* \rightarrow A^*)$ . Indeed,  $W^C = 0$  for  $C \subseteq A$  if and only if  $B \subseteq C$ , as this is equivalent to none of the  $V \in \mathcal{V}$  restricting to 1 in  $C$ .

In other words, smashing with  $\tilde{E}\mathcal{F}[B]$  is the same as inverting all the maps  $S^0 \rightarrow S^V$  for  $V \in \mathcal{V}$ . For an  $MU_A$ -module, this is equivalent to inverting  $e_V$  for  $V \in \mathcal{V}$ .

We have

$$\begin{aligned} \underline{\Phi}^B L_A \otimes_{L_A} (MU_A)_*(q^* Y) &\cong (MU_A)_*(q^* Y)[e_V^{-1} : V \in \mathcal{V}] \\ &\cong \pi_*(MU_A \wedge q^* Y \wedge \tilde{E}\mathcal{F}[B])^A \\ &\cong \pi_*(\underline{\Phi}^B (MU_A \wedge P_{A/B}^* Y))^{A/B} \\ &\cong \pi_*(\underline{\Phi}^B MU_A \wedge Y)^{A/B} \\ &\cong (\underline{\Phi}^B MU_A)_*(Y). \end{aligned}$$

Replacing  $q^* Y$  by  $P_{A/B}^* Y$  in the chain of isomorphisms above yields a similar chain of isomorphisms. All the isomorphisms are isomorphisms of comodules since the isomorphisms are natural in the  $MU_A$ -variable and we can plug into this variable the left and right unit  $MU_A \rightarrow MU_A \wedge MU_A$ .

To construct the second isomorphism, note that as part of the global structure of equivariant  $MU$ , there is a ring map  $q^* MU_{A/B} \rightarrow MU_A$  (cf. [LNP22]). Applying  $\underline{\Phi}^B = (- \wedge \tilde{E}\mathcal{F}[H])^B$  yields a ring map  $MU_{A/B} \rightarrow \underline{\Phi}^B MU_A$ . This induces a morphism

$$\underline{\Phi}^B L_A \otimes_{L_{A/B}} (MU_{A/B})_*(Y) \rightarrow (\underline{\Phi}^B MU_A)_*(Y).$$

It is enough to show that this is an isomorphism for finite  $Y$  and hence for  $Y = (A/B)/(B'/B)_+ \cong A/B'_+$  for  $B \subseteq B' \subseteq A$ . In this case, the map becomes

$$\underline{\Phi}^B L_A \otimes_{L_{A/B}} L_{B'/B} \rightarrow (\underline{\Phi}^B MU_A)_*(A/B'_+). \quad (8.11)$$

The natural map

$$L_A \otimes_{L_{A/B}} L_{B'/B} \rightarrow L_{B'}$$

is an isomorphism since  $L_A \rightarrow L_{B'}$  is a surjection with kernel generated by the Euler classes  $e_V$  for those  $V \in A^*$  restricting trivially to  $B'$ ; these are exactly the images of the Euler classes  $e_W$  for those  $W \in (A/B)^*$  restricting trivially to  $B'/B$ .

Thus, (8.11) becomes

$$\underline{\Phi}^B L_A \otimes_{L_A} MU_A^*(A/B'_+) \cong \underline{\Phi}^B L_A \otimes_{L_A} L_{B'} \rightarrow (\underline{\Phi}^B MU_A)_*(A/B'_+),$$

which is an isomorphism by the first part. Similar to the first part, all isomorphisms are isomorphisms of comodules again.  $\square$



*Proof of Proposition 8.9:* We establish first the isomorphism  $j^* \mathcal{F}^A(X) \cong \mathcal{F}^{A/B}(\underline{\Phi}^B X)$  for  $A$ -spectra  $X$ . Consider the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Spec} L_A & \longleftarrow & \mathrm{Spec} \underline{\Phi}^B L_A & \longrightarrow & \mathrm{Spec} L_{A/B} \\
\downarrow & & \searrow \varphi & & \swarrow \\
\mathcal{M}_{FG}^A & \xleftarrow{j} & \mathcal{M}_{FG}^{A/B} & & 
\end{array}$$

where the down-right arrow comes from applying  $q_*$  to an  $A$ -equivariant formal group classified by a morphism to  $\mathrm{Spec} \underline{\Phi}^B L_A$ , and the right-pointing horizontal morphisms come from the composition  $L_{A/B} \xrightarrow{q^*} L_A \rightarrow \underline{\Phi}^B L_A$ . The square is a pullback square by Proposition 3.11. Thus  $\varphi$  is faithfully flat and hence  $\varphi^*$  induces an equivalence of  $\mathrm{QCoh}(\mathcal{M}_{FG}^{A/B})$  to graded  $(\underline{\Phi}^B L_A, \underline{\Phi}^B S_A)$ -comodules.

The comodule corresponding to  $j^* \mathcal{F}_X^A$  is  $\underline{\Phi}^B L_A \otimes_{L_A} (MU_A)_{2*} X$ . As in (the proof of) the first isomorphism in Lemma 8.10, we observe that this is isomorphic to

$$\begin{aligned}
\underline{\Phi}^B L_A \otimes_{L_A} (MU_A)_{2*} (X \wedge \tilde{E}\mathcal{F}[B]) &\cong \underline{\Phi}^B L_A \otimes_{L_A} (MU_A)_{2*} (P_{A/B}^* \underline{\Phi}^B X) \\
&\cong (\underline{\Phi}^B MU_A)_{2*} (\underline{\Phi}^B X).
\end{aligned}$$

By the second isomorphism in Lemma 8.10, this is isomorphic to the comodule corresponding to  $\mathcal{F}_{\underline{\Phi}^B X}^{A/B}$ , i.e. to  $\underline{\Phi}^B L_A \otimes_{L_{A/B}} (MU_{A/B})_{2*} (\underline{\Phi}^B X)$ . This establishes the first claimed isomorphism of sheaves.

Since the counit  $\underline{\Phi}^B P_{A/B}^* \rightarrow \mathrm{id}_{\mathrm{Sp}_{A/B}}$  is an equivalence, we can assume for the proof of  $j_* \mathcal{F}^{A/B}(Y) \cong \mathcal{F}^A(P_{A/B}^* Y)$  for  $Y \in \mathrm{Sp}_{A/B}$  that  $Y = \underline{\Phi}^B X$  for some  $X \in \mathrm{Sp}_A$  and we obtain from the first isomorphism in this case a natural isomorphism

$$j^* j_* \mathcal{F}^{A/B}(Y) \cong j^* j_* j^* \mathcal{F}^A(X) \cong j^* \mathcal{F}^A(X) \cong \mathcal{F}^{A/B}(Y) \cong j^* \mathcal{F}^A(P_{A/B}^* Y).$$

The Euler classes for characters  $V \notin \mathrm{im}((A/B)^* \rightarrow A^*)$  act invertibly on  $\mathcal{F}^A(P_{A/B}^* Y)$ , and Proposition 3.11 implies that  $\mathcal{F}^A(P_{A/B}^* Y)$  is in the image of  $j_*$ . Since the counit

$$j^* j_* \rightarrow \mathrm{id}_{\mathrm{QCoh}(\mathcal{M}_{FG}^{A/B})}$$

is an isomorphism, the claimed isomorphism follows.

Lastly, we show  $i_* \mathcal{F}^B(Z) \cong \mathcal{F}^A(\mathrm{coind}_B^A Z)$  for  $Z \in \mathrm{Sp}_B$ . This follows from the chain

$$\begin{aligned}
(MU_A)_{2*}(\mathrm{coind}_B^A Z) &\cong \pi_{2*}^A(MU_A \otimes \mathrm{coind}_B^A Z) \\
&\cong \pi_{2*}^A(\mathrm{coind}_B^A(MU_B \otimes Z)) \\
&\cong \pi_{2*}^B(MU_B \otimes Z) \\
&\cong MU_{2*}^B(Z). \quad \square
\end{aligned}$$

of isomorphisms of  $(L_A, S_A)$ -comodules.

We end this section by connecting the stacky point of view with support theories. For a compact  $A$ -spectrum  $X$ , we may consider the support  $\mathrm{supp} \mathcal{F}_i^A(X) \subseteq |\mathcal{M}_{FG}^A|$ , corresponding to those  $x: \mathrm{Spec} k \rightarrow \mathcal{M}_{FG}^A$  such that  $x^* \mathcal{F}_i^A(X)$  is non-trivial. As remarked above, in general the sheaves  $\mathcal{F}_i^A(X)$  do not play well with restrictions. Thus, for an  $A$ -spectrum  $X$ , one should consider all  $\mathcal{F}_i^B(\mathrm{res}_B^A X) \in \mathrm{QCoh}(\mathcal{M}_{FG}^B)$  for  $B \subseteq A$  in conjunction. Using the closed embeddings  $\mathcal{M}_{FG}^B \hookrightarrow \mathcal{M}_{FG}^A$ , the correct notion of support of  $\mathcal{F}^A(X)$  becomes thus  $\mathrm{supp} \underline{\mathcal{F}}_*(X) = \bigcup_{B \subseteq A} \mathrm{supp} \mathcal{F}_*^B(\mathrm{res}_B^A X)$ .

**Proposition 8.12.** *For any compact  $A$ -spectrum  $X$ , the subsets  $\mathrm{supp} \underline{\mathcal{F}}_*(X) \subseteq |\mathcal{M}_{FG}^A|$  and  $\mathrm{supp}((\underline{MU}_A)_* X) \subseteq \mathrm{Spec}^{\mathrm{inv}}(L_A)$  correspond to each other under the bijection from Theorem 4.7. Thus,  $\mathrm{supp} \underline{\mathcal{F}}_*(-)$  is a universal support theory as well.*

*Proof.* Let a point  $[x: \text{Spec } k \rightarrow \mathcal{M}_{FG}^A] \in |\mathcal{M}_{FG}^A|$  correspond to a pair  $(B, n)$ . Then  $x$  factors as  $\bar{x}: \text{Spec } k \rightarrow \mathcal{M}_{FG} = \mathcal{M}_{FG}^{B/B}$  composed with  $\mathcal{M}_{FG}^{B/B} \xrightarrow{j} \mathcal{M}_{FG}^B \xrightarrow{i} \mathcal{M}_{FG}^A$  by Proposition 3.11 and Remark 4.5. By Proposition 8.9, the pullback functor  $j^*$  corresponds to  $\Phi^B$ . Thus,  $x \in \text{supp } \mathcal{F}_*^B(\text{res}_B^A X)$  if and only if  $\bar{x}^* \mathcal{F}_*^{[1]}(\Phi^B X) \neq 0$ , i.e. if  $\Phi^B X$  has chromatic type at least  $n$ . We have seen in Proposition 8.4, this happens if and only if  $I_{B,n} \in \text{supp}((\underline{MU}_A)_* X)$ .

Now suppose  $I_{B,n} \notin \text{supp}((\underline{MU}_A)_* X)$  and let  $B \subseteq \tilde{B} \subseteq A$ . Thus  $(MU_{\tilde{B}})_*(\text{res}_{\tilde{B}}^A(X))_{I_{\tilde{B},n}^{\tilde{B}}}$  vanishes. Since  $[x] \in |\mathcal{M}_{FG}^{\tilde{B}}|$  is the image of the prime ideal  $I_{\tilde{B},n}^{\tilde{B}} \in \text{Spec } L_{\tilde{B}} \cong \text{Spec}(MU_{\tilde{B}})_*$ , we see by Lemma 2.6 that  $x: \text{Spec } k \rightarrow \mathcal{M}_{FG}^{\tilde{B}}$  factors over  $\tilde{x}: \text{Spec } k \rightarrow \text{Spec}(L_{\tilde{B}})_{I_{\tilde{B},n}^{\tilde{B}}}$ . Thus, we can identify  $x^* \mathcal{F}_*^{\tilde{B}}(X)$  with  $\tilde{x}^*(MU_{\tilde{B}})_*(\text{res}_{\tilde{B}}^A(X))_{I_{\tilde{B},n}^{\tilde{B}}} = 0$ , and  $x \notin \text{supp } \mathcal{F}_*(X)$ .  $\square$

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