On model structure for coreflective subcategories of a model category

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1 Introduction

Let $\mathbf{C}$ be a coreflective subcategory of a cofibrantly generated model category $\mathbf{D}$. In this paper we show that under suitable conditions $\mathbf{C}$ admits a cofibrantly generated model structure which is left Quillen adjunct to the model structure on $\mathbf{D}$. As an application, we prove that well-known convenient categories of topological spaces, such as k-spaces, compactly generated spaces, and Δ-generated spaces [3] (called numerically generated in [12]) admit a finitely generated model structure which is Quillen equivalent to the standard model structure on the category **Top** of topological spaces.

2 Coreflective subcategories of a model category

Let $\mathbf{D}$ be a cofibrantly generated model category [7, 2.1.17] with generating cofibrations I, generating trivial cofibrations J and the class of weak equivalences $W_{\mathbf{D}}$. If the domains and codomains of I and J are finite relative to I-cell [7, 2.1.4], then $\mathbf{D}$ is said to be finitely generated.

Recall that a subcategory $\mathbf{C}$ of $\mathbf{D}$ is said to be coreflective if the inclusion functor $i\colon \mathbf{C} \to \mathbf{D}$ has a right adjoint $G\colon \mathbf{D} \to \mathbf{C}$, so that there is a natural isomorphism $\varphi\colon \mathrm{Hom}_{\mathbf{D}}(X,Y) \to \mathrm{Hom}_{\mathbf{C}}(X,GY)$. The counit of this adjunction $\epsilon\colon GY \to Y$ $(Y \in \mathbf{D})$ is called the coreflection arrow.

Theorem 2.1. *Let $\mathbf{C}$ be a coreflective subcategory of a cofibrantly generated model category $\mathbf{D}$ which is complete and cocomplete. Suppose that the unit of the adjunction $\eta\colon X \to GX$ is a natural isomorphism, and that the classes I and J of cofibrations and trivial cofibrations in $\mathbf{D}$ are contained in $\mathbf{C}$. Then $\mathbf{C}$ has a cofibrantly generated model structure with I as the set of generating cofibrations, J as the set of generating trivial cofibrations, and $W_{\mathbf{C}}$ as the class of weak equivalences, where $W_{\mathbf{C}}$ is the class of all weak equivalences contained in $\mathbf{C}$. If $\mathbf{D}$ is finitely generated, then so is $\mathbf{C}$. Moreover, the adjunction $(i, G, \varphi)\colon \mathbf{C} \to \mathbf{D}$ is a Quillen adjunction in the sense of [7, 1.3.1].*

Proof. It suffices to show that $\mathbf{C}$ satisfies the six conditions of [7, 2.1.19] with respect to I, J and $W_{\mathbf{C}}$. Clearly, the first condition holds because $W_{\mathbf{C}}$ satisfies the two out of three property and is closed under retracts. To see that the second and the third conditions hold, let $I_{\mathbf{C}}$-cell and $J_{\mathbf{C}}$-cell be the collections of relative I-cell and J-cell complexes contained in $\mathbf{C}$, respectively. Since $I_{\mathbf{C}}$-cell and $J_{\mathbf{C}}$-cell are subcollections of the collections of relative I-cell and J-cell complexes in $\mathbf{D}$, respectively, the domains of I and J are small relative to $I_{\mathbf{C}}$-cell and $J_{\mathbf{C}}$-cell, respectively. The rest of the conditions are verified as follows. Let $f\colon X \to Y$ be a map in $\mathbf{C}$. Since $\eta\colon X \to GX$ is isomorphic for $X \in \mathbf{D}$, f is I-injective in $\mathbf{C}$ if and only if it is I-injective in $\mathbf{D}$. Similarly, f is J-injective in $\mathbf{C}$ if and only if it is J-injective in $\mathbf{D}$. Let f be an I-cofibration in $\mathbf{D}$. Then it has the left lifting property with respect to all I-injective maps in $\mathbf{C}$. Hence f is an I-cofibration in $\mathbf{C}$. Conversely, let f be an I-cofibration in $\mathbf{C}$. Suppose we are given a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & A \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle p}\big\downarrow \\ Y & \longrightarrow & B \end{array}$$

where p is I-injective in $\mathbf{D}$. Then there is a relative I-cell complex $g\colon X \to Z$ [7, 2.1.9] such that f is a retract of g by [7, 2.1.15]. Since g is an I-cofibration in $\mathbf{D}$, there is a lift $Z \to A$ of g with respect to p. Then the composite $Y \to Z \to A$ is a lift of f with respect to p. Therefore f is an I-cofibration in $\mathbf{D}$. Similarly, f is a J-cofibration in $\mathbf{C}$ if and only if it is a J-cofibration in $\mathbf{D}$. Thus we have the desired inclusions

- $J_{\mathbf{C}}$-cell $\subseteq W_{\mathbf{C}} \cap I_{\mathbf{C}}$-cof,
- $I_{\mathbf{C}}$-inj $\subseteq W_{\mathbf{C}} \cap J_{\mathbf{C}}$-inj, and
- either $W_{\mathbf{C}} \cap I_{\mathbf{C}}$-cof $\subseteq J_{\mathbf{C}}$-cof or $W_{\mathbf{C}} \cap J_{\mathbf{C}}$-inj $\subseteq I_{\mathbf{C}}$-inj.

Here $I_{\mathbf{C}}$-inj and $I_{\mathbf{C}}$-cof denote, respectively, the classes of I-injective maps and I-cofibrations in $\mathbf{C}$, and similarly for $J_{\mathbf{C}}$-inj and $J_{\mathbf{C}}$-cof. Therefore $\mathbf{C}$ is a cofibrantly generated model category by [7, 2.1.19].

It is clear, by the definition, that $\mathbf{C}$ is finitely generated if so is $\mathbf{C}$.

Finally, to prove that (i, G, φ) is a Quillen adjunction, it suffices to show that $G\colon \mathbf{D} \to \mathbf{C}$ is a right Quillen functor, or equivalently, G preserves J-injective maps in $\mathbf{D}$ by [7, 1.3.4] and [7, 2.1.17]. Let $p\colon X \to Y$ be a J-injective map in $\mathbf{D}$. Suppose there is a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & GX \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle Gp}\big\downarrow \\ B & \longrightarrow & GY \end{array}$$

where $f \in J$. Then we have a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & GX & \xrightarrow{\epsilon} & X \\ {\scriptstyle f}\big\downarrow & & & & {\scriptstyle p}\big\downarrow \\ B & \longrightarrow & GY & \xrightarrow{\epsilon} & Y. \end{array}$$

Since p is J-injective in $\mathbf{D}$, there is a lift $h\colon B \to X$ of f. Thus we have a lift $Gh \circ \eta\colon B \cong GB \to GX$ of f with respect to Gp. Therefore $Gp\colon GX \to GY$ is J-injective in $\mathbf{C}$. Similarly, we can show that G preserves I-injective maps in $\mathbf{C}$, and so G preserves trivial fibrations in $\mathbf{C}$. Hence (i, G, φ) is a Quillen adjunction. □

We turn to the case of pointed categories [7, p.4]. Let $\mathbf{D}_*$ be the pointed category associated with $\mathbf{D}$, and let $U\colon \mathbf{D}_* \to \mathbf{D}$ be the forgetful functor. We denote by I_+ and J_+ the classes of those maps $f\colon X \to Y$ in $\mathbf{D}_*$ such that $Uf\colon UX \to UY$ belongs to I and J, respectively. Then we have the following. (Compare [7, 1.1.8], [7, 1.3.5], and [7, 2.1.21].)

Theorem 2.2. *Let $\mathbf{D}$ be a cofibrantly (resp. finitely) generated model category, and let $\mathbf{C}$ be a coreflective subcategory satisfying the conditions of Theorem 2.1. Then the pointed category $\mathbf{C}_*$ has a cofibrantly (resp. finitely) generated model structure, with generating cofibrations I_+ and generating trivial cofibrations J_+, such that the induced adjunction $(i_*, G_*, \varphi_*)\colon \mathbf{C}_* \to \mathbf{D}_*$ is a Quillen adjunction.*

We also have the following Proposition.

Proposition 2.3. *Suppose $\mathbf{C}$ and $\mathbf{D}$ satisfy the conditions of Theorem 2.1. Suppose, further, that the coreflection arrow $\epsilon\colon GY \to Y$ is a weak equivalence for any fibrant object Y in $\mathbf{D}$. Then the adjunctions $(i, G, \varphi)\colon \mathbf{C} \to \mathbf{D}$ and $(i_*, G_*, \varphi_*)\colon \mathbf{C}_* \to \mathbf{D}_*$ are Quillen equivalences.*

Proof. Let X be a cofibrant object in $\mathbf{C}$ and Y a fibrant object in $\mathbf{D}$. Let $f\colon X \to Y$ be a map in $\mathbf{D}$. Then we have $\varphi f = Gf \circ \eta\colon X \cong GX \to GY$. Since f coincides with the composite $X \xrightarrow{\varphi f} GY \xrightarrow{\epsilon} Y$ and ϵ is a weak equivalence in $\mathbf{D}$, φf is a weak equivalence in $\mathbf{C}$ if and only if f is a weak equivalence in $\mathbf{D}$. It follows by [7, 1.3.17] that that the induced adjunction (i_*, G_*, φ_*) is a Quillen equivalence. □

3 On a model structure of the category NG

In [12] we introduced the notion of numerically generated spaces which turns out to be the same notion as Δ-generated spaces introduced by Jeff Smith (cf. [3]) . Let X be a topological space. A subset U of X is numerically open if for every continuous map $P\colon V \to X$, where V is an open subset of Euclidean space, $P^{-1}(U)$ is open in V. Similarly, U is numerically closed if for every such map P, $P^{-1}(U)$ is closed in V. A space X is called a numerically generated space if every numerically open subset is open in X.

Let **NG** denote the full subcategory of **Top** consisting of numerically generated spaces. Then the category **NG** is cartesian closed [12, 4.6]. To any X we can associate the numerically generated space topology, denoted νX, by letting U open in νX if and only if U is numerically open in X. Therefore we have a functor $\nu\colon \mathbf{Top} \to \mathbf{NG}$ which takes X to νX. Clearly, the identity map $\nu X \to X$ is continuous. By the results of [7, §3] the following holds.

Proposition 3.1. *The functor* $\nu\colon \mathbf{Top} \to \mathbf{NG}$ *is a right adjoint to the inclusion functor* $i\colon \mathbf{NG} \to \mathbf{Top}$*, so that* $\mathbf{NG}$ *is a coreflective subcategory of* $\mathbf{Top}$*.*

A continuous map $f\colon X \to Y$ between topological spaces is called a weak homotopy equivalence in **Top** if it induces an isomorphism of homotopy groups

$$f_*\colon \pi_n(X,x) \to \pi_n(Y, f(x))$$

for all $n > 0$ and $x \in X$. Let I be the set of boundary inclusions $S^{n-1} \to D^n$, $n \geq 0$, J the set of inclusions $D^n \times \{0\} \to D^n \times I$, and $W_{\mathbf{Top}}$ the class of weak homotopy equivalences. The standard model structure on **Top** can be described as follows.

Theorem 3.2 ([7, 2.4.19])**.** *There is a finitely generated model structure on* $\mathbf{Top}$ *with* I *as the set of generating cofibraitons,* J *as the set of generating trivial cofibrations, and* $W_{\mathbf{Top}}$ *as the class of weak equivalences.*

The category **NG** is complete and cocomplete by [12, 3.4]. A space X is numerically generated if and only if $\nu X = X$ holds. Thus the unit of the adjunction $\eta\colon X \to \nu X$ is a natural homeomorphism. Moreover, since CW-complexes are numerically generated spaces by [12, 4.4], the classes I and J are contained in **NG**. Let $W_{\mathbf{NG}}$ be the class of maps $f\colon X \to Y$ in **NG** which is a weak equivalence in **Top**. Since the coreflection arrow $\nu Y \to Y$, given by the identity of $Y \in \mathbf{Top}$, is a weak equivalence (cf. [12, 5.4]), we have the following by Theorem 2.1 and Proposition 2.3.

Theorem 3.3. *The category* $\mathbf{NG}$ *has a finitely generated model structure with* I *as the set of generating cofibrations,* J *as the set of generating trivial cofibrations, and* $W_{\mathbf{NG}}$ *as the class of weak equivalences. Moreover the adjunction* $(i, \nu, \varphi)\colon \mathbf{NG} \to \mathbf{Top}$ *is a Quillen equivalence.*

We turn to the case of pointed spaces. Let $\mathbf{Top}_*$ be the category of pointed topological spaces. By [7, 2.4.20], there is a finitely generated model structure on the category $\mathbf{Top}_*$, with generating cofibrations I_+ and generating trivial cofibrations J_+. Then we have the following by Theorem 2.2 and Proposition 2.3.

Corollary 3.4. *There is a finitely generated model structure on the category* $\mathbf{NG}_*$ *of pointed numerically generated spaces, with generating cofibrations* I_+ *and generating trivial cofibrations* J_+*. Moreover, the inclusion functor* $i_*\colon \mathbf{NG}_* \to \mathbf{Top}_*$ *is a Quilen equivalence.*

Remark. (1) The argument of Theorem 3.3 can be applied to the subcategories **K** of k-spaces and **T** of compactly generated spaces. Similarly, the argument of Corollary 3.4 can be applied to the pointed categories $\mathbf{K}_*$ and $\mathbf{T}_*$. Compare [2.4.28], [2.4.25], [2.4.26] of [7].

(2) Let **Diff** be the category of diffeological spaces (cf. [8]). In [12] we introduced a pair of functors $T : \mathbf{Diff} \to \mathbf{Top}$ and $D : \mathbf{Diff} \to \mathbf{Top}$, where T is a left adjoint to D, and showed that the composite TD coincides with

$\nu\colon \mathbf{Top} \to \mathbf{NG}$. Thus **NG** can be embedded as a full subcategory into **Diff**. It is natural to ask whether **Diff** has a model category structure with respect to which the pair (T, D) gives a Quillen adjuntion between **Top** and **Diff**.

Let I be the unit interval, and let $\lambda\colon \mathbf{R} \to I$ be the smashing function, that is, a smooth function such that $\lambda(t) = 0$ for $t \leq 0$ while $\lambda(t) = 1$ for $t \geq 1$. Let $\tilde{I}$ denote the unit interval equipped with the quotient diffeology $\lambda_*(D_{\mathbf{R}})$, where $D_{\mathbf{R}}$ is the standard diffeology of $\mathbf{R}$. In [5] we introduce a finitely generated model category structure on **Diff** with the boundary inclusions $\partial\tilde{I}^{n-1} \to \tilde{I}^n$ as generating cofibrations, and with the inclusions $\partial\tilde{I}^{n-1} \times \tilde{I} \cup \tilde{I}^n \times \{0\} \to \tilde{I}^n \times \tilde{I}$ as generating trivial cofibrations. Its class of weak equivalences consists of those smooth maps $f\colon X \to Y$ inducing an isomorphism $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ for every $n \geq 0$ and $x_0 \in X$. Here, the homotopy set $\pi_n(X, x_0)$ is defined to be the set of smooth homotopy classes of smooth maps $(\tilde{I}^n, \partial\tilde{I}^n) \to (X, x_0)$.

It is expected that with respect to the model structure on **Diff** described above, the pair (T, D) induces a Quillen adjunction between **Top** and **Diff**.

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