Redshift and multiplication for truncated Brown-Peterson spectra

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Abstract

We equip $BP\langle n \rangle$ with an $\mathbb{E}_3$-BP-algebra structure, for each prime p and height n. The algebraic K-theory of this $\mathbb{E}_3$-ring is of chromatic height exactly $n+1$. Specifically, it is an fp-spectrum of fp-type $n+1$, which can be viewed as a higher height version of the Lichtenbaum-Quillen conjecture.

Contents

1 Introduction

Our main aim here is to prove the following:

Theorem. *For each prime p and height n, there exists an $\mathbb{E}_3$-BP-algebra structure on $BP\langle n \rangle$. The algebraic K-theory of this $\mathbb{E}_3$-ring has finitely presented cohomology over the mod p Steenrod algebra, and is of fp-type $n+1$.*

The principal connective theories in the chromatic approach to stable homotopy theory are thus more structured than previously known, and they satisfy higher height analogs of the Lichtenbaum–Quillen conjecture. The $\mathbb{E}_3$ forms of $\mathrm{BP}\langle n\rangle$ constructed here give the first known examples, for $n > 1$, of chromatic height n theories with algebraic K-theory provably of height $n + 1$.

Background and results

In his 1974 ICM address, Quillen [Qui75] stated as a 'hope' what is now the proven Lichtenbaum-Quillen conjecture [Voe03, Voe11]. His hope was that the algebraic K-theory of regular noetherian rings can be well approximated by étale cohomology, at least in large degrees. Ten years later, in 1984, Waldhausen [Wal84] was investigating how his K-theory of spaces interacts with the chromatic filtration. He observed that, in the presence of a descent theorem of Thomason [Tho82], the Lichtenbaum-Quillen conjecture could be restated in terms of localization at complex K-theory. Let L_1^f denote the localization that annihilates those finite spectra with vanishing p-adic complex K-theory; then the Lichtenbaum-Quillen conjecture for suitable rings R is equivalent to the statement that

$$\mathrm{K}(R)_{(p)} \to L_1^f \mathrm{K}(R)_{(p)}$$

is an isomorphism on π_* for $* \gg 0$.

Algebraic K-theory is defined not only on rings, but (crucially for applications to smooth manifold theory) on ring spectra. One of the deepest computations of the algebraic K-thory of ring spectra to date is by Ausoni and Rognes [AR02], who for primes $p \geqslant 5$ computed the mod (p, v_1) K-theory of the p-completed Adams summand $\ell_p^\wedge$. Their computations imply that

$$\mathrm{K}(\ell_p^\wedge)_{(p)} \to L_2^f \mathrm{K}(\ell_p^\wedge)_{(p)}$$

is a π_*-isomorphism for $* \gg 0$. Here L_2^f is the next localization in a hierarchy of chromatic localizations L_n^f for each $n \geqslant 0$ (at an implicit prime p). This of course suggests a higher height analog of the Lichtenbaum–Quillen conjecture. In the Oberwolfach lecture [Rog00], Rognes laid out a far-reaching vision of how this higher height analog might go, which is now known as the *chromatic redshift* philosophy. The name *redshift* refers to the hypothesis that algebraic K-theory should raise the chromatic height of ring spectra by exactly 1.

To give a more precise statement, we will need the notion of fp-type, due to Mahowald–Rezk [MR99]: A p-complete, bounded below spectrum X is of fp-type n if the thick subcategory of finite complexes V such that $|\pi_*(V \otimes X)| < \infty$ is generated by a type $(n + 1)$ complex (i.e. a complex with a v_{n+1} self-map).

With this definition, Ausoni-Rognes conjecture that:

Conjecture. For suitable $\mathbb{E}_1$-rings R of fp-type n, $\mathrm{K}(R)_p^\wedge$ is of fp-type $n + 1$.

As we review below (see Corollary 6.0.5), this statement also implies that $\mathrm{K}(R) \to L_{n+1}^f \mathrm{K}(R)$ is a p-local equivalence in large degrees, so we can think of it as a higher height analog of the Lichtenbaum-Quillen conjecture.

In the years since the Ausoni–Rognes computations, redshift has been verified for additional height 1 ring spectra, including $\mathrm{ku}_p^\wedge$, $\mathrm{KU}_p^\wedge$, and ku/p at primes $p \geqslant 5$ [BM08, Aus10, AR12a], and evidence for redshift has accumulated in general [BDR04, Rog14, Wes17, Vee18, AK18, AKQ20, CSY20]. Recent conceptual advances show that the algebraic K-theories of many height n rings are of height *at most* $n + 1$ [LMMT20, CMNN20]. Here, we give the first arbitrary height examples of ring spectra for which redshift provably occurs.

The truncated Brown–Peterson spectra, $\mathrm{BP}\langle n \rangle$, are among the simplest and most important cohomology theories in algebraic topology. There is one such spectrum for every prime p and height $n \geqslant 0$, though we will follow tradition by localizing at the prime and omitting it from notation.[1] The height 1 spectrum $\mathrm{BP}\langle 1 \rangle$ is the Adams summand ℓ, while $\mathrm{BP}\langle 2 \rangle$ is a summand of either topological modular forms (at $p \geqslant 5$), or topological modular forms with level structure (at $p = 2, 3$).

Both ℓ and tmf are extraordinarily structured: they are $\mathbb{E}_\infty$-ring spectra, inducing power operations on the cohomology of spaces. Our first main result is a construction of part of this structure at an arbitrary height n:

Theorem A (Multiplication). *For an appropriate choice of indecomposable generators*

$$v_{n+1}, v_{n+2}, \cdots \in \pi_* \mathrm{BP},$$

the quotient map

$$\mathrm{BP} \to \mathrm{BP}/(v_{n+1}, \cdots) = \mathrm{BP}\langle n \rangle$$

is the unit of an $\mathbb{E}_3$-BP-algebra structure on $\mathrm{BP}\langle n \rangle$.

Our second main theorem establishes the above conjecture conjecture for $R = \mathrm{BP}\langle n \rangle$.

Theorem B (Redshift). *Let $\mathrm{BP}\langle n \rangle$ denote any $\mathbb{E}_3$-BP-algebra such that the unit $\mathrm{BP} \to \mathrm{BP}\langle n \rangle$ is obtained by modding out a sequence of indecomposable generators $v_{n+1}, v_{n+2}, \cdots$. Then both $\mathrm{K}(\mathrm{BP}\langle n \rangle)_p^\wedge$ and $\mathrm{K}(\mathrm{BP}\langle n \rangle_p^\wedge)_p^\wedge$ are of fp-type $n + 1$.*

To prove Theorem B by trace methods, the critical thing to show is that $\pi_*(V \otimes \mathrm{TC}(\mathrm{BP}\langle n \rangle))$ is bounded above for some type $(n + 2)$ complex V. We recall [NS18] that topological cyclic homology can be computed as the fiber:

$$\mathrm{TC}(\mathrm{BP}\langle n \rangle) \simeq \mathrm{fib}(\mathrm{THH}(\mathrm{BP}\langle n \rangle)^{hS^1} \xrightarrow{\varphi^{hS^1} - \mathrm{can}} \mathrm{THH}(\mathrm{BP}\langle n \rangle)^{tS^1}),$$

where the map

$$\varphi : \mathrm{THH}(\mathrm{BP}\langle n \rangle) \to \mathrm{THH}(\mathrm{BP}\langle n \rangle)^{tC_p}$$

is the cyclotomic Frobenius. Our proof of Theorem B will rely on five key results. The first result is a computation of Hochschild homology *relative* to MU, and it is in some ways analogous to the Bökstedt periodicity theorem for $\mathrm{THH}(\mathbb{F}_p)$.

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[1] At each prime p and height $n \geqslant 0$, $\mathrm{BP}\langle n \rangle$ is conjectured to be unique as a p-local spectrum. For $n > 1$, uniqueness is only proved up to p-completion, by work of Angeltveit and Lind [AL17].

Theorem C (Polynomial THH). *The ring* $\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})_*$ *is polynomial on even-degree generators, one of which can be chosen to be the double-suspension class* $\sigma^2 v_{n+1}$.

We then take advantage of the circle action on THH to shift the class $\sigma^2 v_{n+1}$ down to a class detecting v_{n+1}. More precisely, we prove:

Theorem D (Detection). *There is an isomorphism of* $\mathbb{Z}_{(p)}[v_1, ..., v_n]$-*algebras*

$$\pi_*(\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})^{hS^1}) \cong (\pi_*\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU}))\,[\![t]\!],$$

where $|t| = -2$ *and* $\pi_*\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})$ *is a free* $\mathbb{Z}_{(p)}[v_1, \cdots, v_n]$-*algebra. Under the unit map*

$$\pi_*\mathrm{MU} \longrightarrow \pi_*(\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})^{hS^1}),$$

the class v_{n+1} *is sent to a unit multiple of* $t(\sigma^2 v_{n+1})$.

We think of the Detection Theorem as a weak form of redshift. For example, one can use it to deduce that $L_{K(n+1)}\mathrm{K}(\mathrm{BP}\langle n\rangle)$ is nonzero (Corollary 3.0.3).

In order to prove redshift for $\mathrm{BP}\langle n\rangle$, we need to gain some computational understanding of the absolute Hochschild homology $\mathrm{THH}(\mathrm{BP}\langle n\rangle)$ as a cyclotomic object. The Hochschild homology relative to MU does not carry a Frobenius, but we can still use it to approximate the circle action on $\mathrm{THH}(\mathrm{BP}\langle n\rangle)$ by studying the descent spectral sequence for the map

$$\mathrm{THH}(\mathrm{BP}\langle n\rangle) \to \mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU}).$$

In fact, we give a method for mixing the descent filtration with the usual Tate and homotopy fixed point spectral sequences in order to produce 'descent modified spectral sequences,' which may be of independent interest. We use them here to prove the following results about the circle action on $\mathrm{THH}(\mathrm{BP}\langle n\rangle)$:

Theorem E (Canonical Vanishing). *Let* $F(n+1)$ *be any type* $n+2$ *complex. The maps*

$$\mathrm{can}\colon \mathrm{TC}^-(\mathrm{BP}\langle n\rangle) \to \mathrm{TP}(\mathrm{BP}\langle n\rangle)$$
$$\mathrm{can}\colon \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{hC_{p^k}} \to \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tC_{p^k}}$$

induce the zero map on $F(n+1)_*$ *for* $* \gg 0$, *uniformly in* k.

Theorem F (Finiteness). *Let* $F(n+1)$ *be any type* $n+2$ *complex. Then each group*

$$F(n+1)_*\mathrm{TC}^-(\mathrm{BP}\langle n\rangle), F(n+1)_*\mathrm{TP}(\mathrm{BP}\langle n\rangle),$$

$$F(n+1)_*\mathrm{THH}(\mathrm{BP}\langle n\rangle)^{hC_{p^k}}, F(n+1)_*\mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tC_{p^k}}$$

is finite in each dimension.

The previous two results show that, after tensoring with a type $n+2$ complex, topological cyclic homology coincides with the fiber of φ^{hS^1} in large degrees, and so reduces our task to understanding the Frobenius. In this direction we prove:

4

Theorem G (Segal Conjecture). *Let $F(n)$ be any type $n+1$ complex. Then the cyclotomic Frobenius $\mathrm{THH}(\mathrm{BP}\langle n\rangle) \to \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tC_p}$ induces an isomorphism*

$$F(n)_* \mathrm{THH}(\mathrm{BP}\langle n\rangle) \cong F(n)_*(\mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tC_p})$$

in all sufficiently large degrees $* \gg 0$.

We prove the Segal Conjecture by using the Adams filtration on $\mathrm{BP}\langle n\rangle$ to induce a filtration on $\mathrm{THH}(\mathrm{BP}\langle n\rangle)$; this reduces the claim to the Segal conjecture for graded polynomial $\mathbb{F}_p$-algebras, which can be proven directly.

In §6, we explain how to put together the above theorems to prove redshift for $\mathrm{BP}\langle n\rangle$. In fact, we deduce Lichtenbaum-Quillen style results for several invariants of interest:

Corollary (Corollary 6.0.3). Let $F(n)$ be any type $n+1$ complex. The maps

$$F(n)_* \mathrm{TR}(\mathrm{BP}\langle n\rangle) \to v_{n+1}^{-1} F(n)_* \mathrm{TR}(\mathrm{BP}\langle n\rangle),$$

$$F(n)_* \mathrm{TC}(\mathrm{BP}\langle n\rangle) \to v_{n+1}^{-1} F(n)_* \mathrm{TC}(\mathrm{BP}\langle n\rangle),$$

$$F(n)_* \mathrm{K}(\mathrm{BP}\langle n\rangle) \to v_{n+1}^{-1} F(n)_* \mathrm{K}(\mathrm{BP}\langle n\rangle)$$

are isomorphisms in degrees $* \gg 0$.

Corollary (Corollary 6.0.5). The maps

$$\mathrm{K}(\mathrm{BP}\langle n\rangle)_{(p)} \to L_{n+1}^f \mathrm{K}(\mathrm{BP}\langle n\rangle)_{(p)}, \text{ and}$$

$$\mathrm{K}(\mathrm{BP}\langle n\rangle_p^\wedge)_{(p)} \to L_{n+1}^f \mathrm{K}(\mathrm{BP}\langle n\rangle_p^\wedge)_{(p)}$$

are π_*-isomorphisms for $* \gg 0$.

Our computations only apply to forms of $\mathrm{BP}\langle n\rangle$ that are $\mathbb{E}_3$-BP-algebras, and one may wonder how stringent of a condition this is. In fact, work of Basterra and Mandell gives a convenient criterion for producing such structure:

Remark 1.0.1. Suppose that a form of $\mathrm{BP}\langle n\rangle$ is equipped with an $\mathbb{E}_4$-algebra structure. Then there are no obstructions to producing an $\mathbb{E}_4$-ring map from BP to $\mathrm{BP}\langle n\rangle$ [BM13, Corollary 4.4 and Lemma 5.1]. Any such $\mathbb{E}_4$-map is the unit of an $\mathbb{E}_3$-BP-algebra structure, allowing us to apply Theorem B.

Example 1.0.2. At $p = 2$ connective complex K-theory is an $\mathbb{E}_\infty$ form of $\mathrm{BP}\langle 1\rangle$, and it follows that $\mathrm{K}(ku)_2^\wedge$ is of fp-type 2. Even the non-vanishing of $L_{K(2)} \mathrm{K}(ku)$ was previously known only for $p \geqslant 5$ [AR02].

Similarly, we can deduce at $p = 2$ that $\mathrm{K}(tmf_1(3))_2^\wedge$ is of fp-type 3, since $tmf_1(3)$ is the Lawson–Naumann $\mathbb{E}_\infty$ form of $\mathrm{BP}\langle 2\rangle$ [LN12]. Applying algebraic K-theory to the E_∞-ring map $tmf \to tmf_1(3)$, we conclude that $L_{K(3)} \mathrm{K}(tmf) \neq 0$.

Remark 1.0.3. Our methods may help to prove that the algebraic K-theories of many other height n rings are not K$(n+1)$-acyclic, especially when combined with the descent and purity results of [CMNN20, LMMT20]. For example, at the prime 3 these results imply that the non-vanishing of $L_{K(2)} \mathrm{K}(\mathrm{ku})$ is equivalent to the non-vanishing of $L_{K(2)} \mathrm{K}(\mathrm{ko})$ (cf. [Aus05]), and the latter follows from the fact that 3-localized ko is an $\mathbb{E}_\infty$ form of BP$\langle 1 \rangle$.

Our work leaves open many natural questions, chief of which is to determine the homotopy type of K(BP$\langle n \rangle$). Since we show this homotopy type to be closely related to its localization $L^f_{n+1} \mathrm{K}(\mathbb{BP}\langle n \rangle)$, one might hope to assemble an understanding via chromatic fracture squares (c.f. [AR12b]). We would also like to highlight:

Question 1.0.4. For what ring spectra R, other than $R = $ BP$\langle n \rangle$, is it possible to prove a version of the Segal conjecture?

Question 1.0.5. For what ring spectra R, other than $R = $ BP$\langle n \rangle$, is it possible to prove a version of the Canonical Vanishing theorem?

While variants of the Segal conjecture have received much study (see Section 5 for some history), the Canonical Vanishing result does not seem as widely analyzed. It seems plausible that a ring R might satisfy Canonical Vanishing but not the Segal conjecture, or vice versa.

Question 1.0.6. What ring spectra R, other than $R = $ BP$\langle n \rangle$, satisfy redshift, or various less precise forms of the Lichtenbaum–Quillen conjecture?

For an arbitrary BP$\langle n \rangle$-algebra R satisfying the Segal conjecture, Akhil Mathew has deduced (given our work here) various Lichtenbaum–Quillen statements. He has graciously allowed us to reproduce his results at the end of Section 6.

One would also like to make many of the above results *effective*, rather than asserting an isomorphism in degrees above an unspecified dimension. Especially the following question is interesting, since it does not depend on a choice of a finite complex:

Question 1.0.7. In precisely what range of degrees is the map

$$\mathrm{K}(\mathrm{BP}\langle n \rangle)_{(p)} \to L^f_{n+1} \mathrm{K}(\mathrm{BP}\langle n \rangle)_{(p)}$$

a π_*-isomorphism?

Finally, we remark that the spectral sequences (equivalently, filtrations) introduced in Sections 4 & 5 in order to prove Theorems E & G seem to be novel approaches to computing THH$(-)$, TC$^-(-)$, and TP$(-)$. We analyze these spectral sequences in only the crudest possible terms, and would be interested to see them used to make more refined calculations.

Remark 1.0.8. In [AKQ20], Angelini-Knoll and Quigley demonstrate a type of redshift in periodic cyclic homology for a family of spectra $y(n)$ of increasing height; namely that $y(n)$ has the property that $K(i) \otimes y(n) = 0$ when $0 \leqslant i \leqslant n-1$ while a certain 'continuous' $K(i)$-homology of TP$(y(n))$ vanishes for $0 \leqslant i \leqslant n$. These very interesting computations

differ from our results in several respects. First, it is not verified that $L_{K(n+1)}\mathrm{K}(y(n)) \neq 0$, so it's not clear directly from the results of loc. cit. whether nontrivial height $n+1$ phenomena could be detected by $\mathrm{K}(y(n))$ (or by periodic cyclic homology). Second, it is not clear how to deduce the corresponding results about TC or K-theory from these computations (though the authors do show that chromatic height does not *decrease* in this case when taking K-theory). There is clearly more to study in this direction.

Remark 1.0.9. It is not surprising that $\mathbb{E}_3$-algebra structure on $\mathrm{BP}\langle n \rangle$ is useful in the proof of redshift. As far back as 2000, Ausoni and Rognes observed that redshift could be proved whenever $\mathrm{BP}\langle n \rangle$ is $\mathbb{E}_\infty$ and the Smith–Toda complex $V(n)$ exists as a homotopy ring spectrum [Rog00]. Unfortunately, both of these hypotheses are known to generically fail [Nav10, Law18].

Remarks on the Multiplication Theorem

To give context to Theorem A, the question of whether $\mathrm{BP}\langle n \rangle$ can be made $\mathbb{E}_\infty$ was once a major open problem in algebraic topology [May75]. In breakthrough work, Tyler Lawson [Law18] and Andrew Senger [Sen17] showed this to be impossible whenever $n \geqslant 4$.

While the nonexistence of structure is of great theoretical interest, it is the presence of structure that powers additional computations. For example, in this work we use the $\mathbb{E}_3$-algebra structure guaranteed by Theorem A in order to prove the Polynomial THH Theorem (2.5.2), which is the key computational input to many of the remaining results of the paper. Our proof of Theorem A relies on a number of ideas that we have not discussed so far: see §2.1 for an outline of the proof of Theorem A.

Remark 1.0.10. Prior to our work, other authors had succeeded in equipping $\mathrm{BP}\langle n \rangle$ with additional structure. Notably, Baker and Jeanneret produced $\mathbb{E}_1$-ring structures [BJ02] (cf. [Laz01, Ang08]), and Richter produced Robinson $(2p-1)$-stage structures on related Johnson–Wilson theories [Ric06]. Lawson and Naumann equipped $\mathrm{BP}\langle 2 \rangle$ with $\mathbb{E}_\infty$-structure at the prime 2 [LN12], and Hill and Lawson produced an $\mathbb{E}_\infty$ form of $\mathrm{BP}\langle 2 \rangle$ at $p = 3$ [HL10].

Remark 1.0.11. Basterra and Mandell proved that BP admits a unique $\mathbb{E}_4$-algebra structure, a fact which is necessary to make sense of $\mathbb{E}_3$-BP-algebras [BM13]. They also show that BP is an $\mathbb{E}_4$-algebra retract of $\mathrm{MU}_{(p)}$, so a p-local $\mathbb{E}_3$-MU-algebra inherits an $\mathbb{E}_3$-BP-algebra structure. Our proof of Theorem A most naturally produces an $\mathbb{E}_3$-MU-algebra structure on $\mathrm{BP}\langle n \rangle$. In fact, if one also formulates Theorem B in terms of $\mathbb{E}_3$-MU-algebra structures, then none of the statements or proofs in this paper rely on [BM13].

As pointed out by Morava, $\mathrm{BP}\langle n \rangle$ may be equipped with different homotopy ring structures [Mor89]. What we prove here is that *some* form of $\mathrm{BP}\langle n \rangle$ admits an $\mathbb{E}_3$-BP-algebra structure, and it remains an interesting open question to determine exactly which forms admit such structure.

Question 1.0.12. Which forms of $\mathrm{BP}\langle n \rangle$ admit an $\mathbb{E}_3$-BP-algebra structure? Which of these can be built by the procedure in §2?

The subtleties behind Question 1.0.12 are indicated by work of Strickland [Str99, Remark 6.5], who observed at $p = 2$ that neither the Hazewinkel or Araki generators may be used as generators in Theorem A.

Remark 1.0.13. We suspect that our $\mathbb{E}_3$-algebra structure will be of use in additional computations. For example, Ausoni and Richter give an elegant formula for the THH of a height 2 Johnson–Wilson theory, under the assumption that the theory can be made $\mathbb{E}_3$ [AR20]. Our result does not directly feed into their work, for the simple reason that they use a form of $\mathrm{BP}\langle 2 \rangle[v_2^{-1}]$ specified by the Honda formal group. It seems unlikely that their theorem relies essentially on this choice.

Remark 1.0.14. By imitating our construction of an $\mathbb{E}_3$-MU-algebra structure on $\mathrm{BP}\langle n \rangle$, we suspect one could produce an $\mathbb{E}_{2\sigma+1}$-$\mathrm{MU}_{\mathbb{R}}$-algebra structure on $\mathrm{BP}\langle n \rangle_{\mathbb{R}}$. As a result, the fixed points $\mathrm{BP}\langle n \rangle_{\mathbb{R}}^{C_2}$ would acquire an $\mathbb{E}_1$-ring structure. At the moment, these fixed points are not even known to be homotopy associative [KLW18].

Acknowledgements

2   The Multiplication Theorem

Throughout this section we will denote by $x_1, x_2, \ldots$ an unspecified choice of indecomposable generators for the polynomial ring MU_* with the property that x_{p^i-1} is zero in mod p homology, and set $v_i = x_{p^i-1}$. The computations made are independent of this choice, but in the inductive step to produce an $\mathbb{E}_3$-MU-algebra form of $\mathrm{BP}\langle n+1 \rangle$ we will need the flexibility to alter these choices of generators.

We give an outline of our inductive proof in §2.1, and review the necessary background material in §2.2 and §2.3. In §2.4, we make the fundamental calculation of $\mathrm{THH}(\mathbb{F}_p/\mathrm{MU})$ and prove the base case; we then carry out half of the inductive step in §2.5. In §2.6, we take a brief detour to construct a certain MU-algebra used in the second half of the induction step, which we complete in §2.7.

2.1   Outline of Proof

We will prove Theorem A by induction on n. Before giving a more precise outline, we offer some intuition for our approach.

Consider the tower of MU-modules:

$$\mathrm{BP}\langle n+1\rangle \to \cdots \to \mathrm{BP}\langle n+1\rangle/(v_{n+1}^k) \to \cdots \to \mathrm{BP}\langle n\rangle$$

One way to produce an $\mathbb{E}_3$-MU-algebra structure on $\mathrm{BP}\langle n+1\rangle$ would be to refine this whole tower to a tower of $\mathbb{E}_3$-MU-algebras. This, too, could be approached inductively. Given an $\mathbb{E}_3$-MU-algebra structure on $\mathrm{BP}\langle n+1\rangle/(v_{n+1}^k)$, giving a compatible $\mathbb{E}_3$-MU-algebra structure on $\mathrm{BP}\langle n+1\rangle/(v_{n+1}^{k+1})$ is equivalent to refining the k-invariant

$$\mathrm{BP}\langle n+1\rangle/(v_{n+1}^k) \to \Sigma^{k|v_{n+1}|+1}\mathrm{BP}\langle n\rangle$$

to an $\mathbb{E}_3$-MU-derivation. In particular, the very first task would be refining the element

$$\delta v_{n+1} : \mathrm{BP}\langle n\rangle \to \Sigma^{|v_{n+1}|+1}\mathrm{BP}\langle n\rangle$$

to an $\mathbb{E}_3$-MU-derivation. Doing this requires some knowledge about the $\mathbb{E}_3$-MU-André-Quillen cohomology of $\mathrm{BP}\langle n\rangle$, and if we wanted to continue up the tower we would need inductive control over the André-Quillen cohomology of each of the algebras $\mathrm{BP}\langle n+1\rangle/(v_{n+1}^k)$.

Instead of doing this, we show it is possible to realize the entire tower at once as soon as one refines the element δv_{n+1}. One way to phrase what we do is to refine the map δv_{n+1} to a highly structured action of an 'exterior algebra' $\Lambda_{\mathrm{MU}}(\delta v_{n+1})$, and then show

$$\mathrm{BP}\langle n+1\rangle \simeq \mathrm{map}_{\Lambda_{\mathrm{MU}}(\delta v_{n+1})}(\mathrm{MU}, \mathrm{BP}\langle n\rangle).$$

To make this precise, and to make it clear what structure is present on the right hand side, we replace this mapping object with an equivalent one that automatically has extra structure, as we explain below.

Assume that we have already produced an $\mathbb{E}_3$-MU-algebra structure on $\mathrm{BP}\langle n\rangle$. Our formal argument for producing $\mathrm{BP}\langle n+1\rangle$ proceeds in three steps:

Step 1. We compute the $\mathbb{E}_3$-MU-center of $\mathrm{BP}\langle n\rangle$, denoted $\mathfrak{Z}_{\mathbb{E}_3-\mathrm{MU}}(\mathrm{BP}\langle n\rangle)$, and show that its homotopy is polynomial on even generators and contains a class refining δv_{n+1}.

Step 2. We produce an $\mathbb{E}_4$-algebra structure on the free $\mathbb{E}_3$-MU-algebra on an even degree class x, denoted $\mathrm{MU}\{x\}$, and prove that there are no obstructions to producing an $\mathbb{E}_4$-MU-algebra map

$$\mathrm{MU}\{x\} \to \mathfrak{Z}_{\mathbb{E}_3-\mathrm{MU}}(\mathrm{BP}\langle n\rangle)$$

that sends x to δv_{n+1}. This gives $\mathrm{BP}\langle n\rangle$ the structure of an $\mathbb{E}_3$-$\mathrm{MU}\{x\}$-algebra.

Step 3. Finally, we prove that the centralizer of the unit map, i.e.

$$\mathfrak{Z}_{\mathbb{E}_3-\mathrm{MU}\{x\}}(\mathrm{MU} \to \mathrm{BP}\langle n\rangle),$$

is a form of $\mathrm{BP}\langle n+1\rangle$, which completes the proof of the inductive step.

The non-formal inputs here are:

(i) The evenness of the $\mathbb{E}_3$-MU-center of BP$\langle n \rangle$: This ultimately rests on Steinberger's computation of the action of Dyer-Lashof operations on the dual Steenrod algebra and on Kochman's computation of the action of Dyer-Lashof operations on the homology of BU.

(ii) The $\mathbb{E}_4$-algebra structure (and cell decomposition) of MU$\{x\}$, which involves some careful manipulations of Koszul duality in the context of graded algebras.

2.2 Background: Operadic Modules and Enveloping Algebras

Fix an $\mathbb{E}_{n+1}$-algebra k and let $\mathcal{C} = \mathsf{LMod}_k$. If $A \in \mathsf{Alg}_{\mathbb{E}_n}(\mathcal{C})$ is an $\mathbb{E}_n$-algebra, then we can define an $\mathbb{E}_n$-monoidal category, $\mathsf{Mod}_A^{\mathbb{E}_n}(\mathcal{C})$, of $\mathbb{E}_n$-A-modules ([Lur17, 3.3.3.9]). The relevance of this category in our case is the equivalence of $\mathsf{Mod}_A^{\mathbb{E}_n}(\mathcal{C})$ ([Lur17, 7.3.4.14]) with the tangent category $\mathsf{Sp}(\mathsf{Alg}_{\mathbb{E}_n}(\mathcal{C})_{/A})$ controlling deformations of A.

It follows from [Lur17, 7.1.2.1] that we have an equivalence

$$\mathsf{Mod}_A^{\mathbb{E}_n}(\mathcal{C}) \simeq \mathsf{LMod}_{\mathcal{U}^{(n)}(A)}$$

where $\mathcal{U}^{(n)}(A)$ is the endomorphism algebra spectrum of the free $\mathbb{E}_n$-A-module on k.

Remark 2.2.1. It follows from [Lur17, 4.8.5.11] that the assignment $B \mapsto \mathcal{U}^{(n-k)}(B)$ is a lax $\mathbb{E}_k$-monoidal functor of B. In particular, if A is an $\mathbb{E}_n$-algebra in $\mathcal{C}$, then $\mathcal{U}^{(n-k)}(A)$ has a canonical $\mathbb{E}_{k+1}$-algebra structure.

We will need the following standard fact:

Proposition 2.2.2. *There is a canonical equivalence of algebras:*

$$\mathcal{U}^{(n)}(A) \simeq A \otimes_{\mathcal{U}^{(n-1)}(A)} A^{\mathrm{op}}$$

where A^{op} denotes A regarded as an $\mathbb{E}_1$-$\mathcal{U}^{(n-1)}(A)$-algebra with its opposite multiplication.

Proof. The enveloping algebra is obtained by taking the endomorphism algebra of a free object. So it suffices, by [Lur17, 4.8.5.11, 4.8.5.16], to provide an equivalence:

$$\mathsf{Mod}_A^{\mathbb{E}_n}(\mathcal{C}) \simeq \mathsf{LMod}_A(\mathcal{C}) \otimes_{\mathsf{Mod}_A^{\mathbb{E}_{n-1}}(\mathcal{C})} \mathsf{RMod}_A(\mathcal{C})$$

By [Lur17, 4.8.4.6,4.3.2.7], we may identify the right hand side as a category of bimodules:

$$\mathsf{LMod}_A(\mathcal{C}) \otimes_{\mathsf{Mod}_A^{\mathbb{E}_{n-1}}(\mathcal{C})} \mathsf{RMod}_A(\mathcal{C}) \simeq \mathsf{BMod}_A(\mathsf{Mod}_A^{\mathbb{E}_{n-1}}(\mathcal{C})).$$

The result now follows from [HNP19, 1.0.4] by taking tangent categories at A of the equivalence ([Lur17, 5.1.2.2]):

$$\mathsf{Alg}_{\mathbb{E}_n}(\mathcal{C}) \simeq \mathsf{Alg}_{\mathbb{E}_1}(\mathsf{Alg}_{\mathbb{E}_{n-1}}(\mathcal{C})).$$

$\square$

Remark 2.2.3. One can use this result and induction on n to prove that there is an equivalence

$$\mathcal{U}^{(n)}(A) \simeq \int_{\mathbb{R}^n - \{0\}} A$$

of the enveloping algebra with the factorization homology of A over $\mathbb{R}^n - \{0\}$.

2.3 Background: Centers and Centralizers

Let $\mathcal{C}$ be a stable, presentably symmetric monoidal category. Suppose that A is an $\mathbb{E}_{n+1}$-algebra in $\mathcal{C}$. It follows that the pair $(\mathcal{C}, \mathsf{LMod}_A(\mathcal{C}))$ is an algebra over the operad $\mathbb{E}_n \boxtimes \mathsf{LM}$, where LM is the operad whose algebras consist of an associative algebra and a left module over it. By the definition of the Boardman-Vogt tensor product, we have a bifunctor of operads

$$\mathbb{E}_n \times \mathsf{LM} \to \mathbb{E}_n \boxtimes \mathsf{LM}.$$

This puts us in the situation of [Lur17, 5.3.1.12, 5.3.1.14] so that:

- $\mathsf{Alg}_{\mathbb{E}_n}(\mathsf{LMod}_A)$ is a left module over $\mathsf{Alg}_{\mathbb{E}_n}(\mathcal{C})$.

- Given a map $f : R \to S$ of $\mathbb{E}_n$-A-algebras, there is an $\mathbb{E}_n$-algebra $\mathfrak{Z}_{\mathbb{E}_n - A}(f) \in \mathsf{Alg}_{\mathbb{E}_n}(\mathcal{C})$ which is final as an object of

$$\mathsf{Alg}_{\mathbb{E}_n}(\mathcal{C}) \times_{\mathsf{Alg}_{\mathbb{E}_n}(\mathsf{LMod}_A)_{R/}} \mathsf{Alg}_{\mathbb{E}_n}(\mathsf{LMod}_A)_{R//S},$$

  where we are using the map $(-) \otimes R : \mathsf{Alg}_{\mathbb{E}_n}(\mathcal{C}) \to \mathsf{Alg}_{\mathbb{E}_n}(\mathsf{LMod}_A)$ to define the fiber product.

- Given an $\mathbb{E}_n$-A-algebra R, there is an $\mathbb{E}_{n+1}$-algebra $\mathfrak{Z}_{\mathbb{E}_n - A}(R) \in \mathsf{Alg}_{\mathbb{E}_{n+1}}(\mathcal{C})$ with the universal property that specifying an $\mathbb{E}_{n+1}$-algebra map $Z \to \mathfrak{Z}_{\mathbb{E}_n - A}(R)$ in $\mathcal{C}$ is the same as giving R the structure of an $\mathbb{E}_n$-Z-algebra where the action is through A-module maps (see [Lur17, 5.3.1.6] for the precise definition).

We will be interested in the following two cases:

- When $\mathcal{C} = \mathsf{Mod}_{\mathrm{MU}}$ and A is the unit, then we will be computing centers of various algebras in MU-modules.

- When $\mathcal{C} = \mathsf{Mod}_{\mathrm{MU}}$ and A is the $\mathbb{E}_{n+1}$-algebra MU$\{x\}$ constructed in the next section. Then we would like to know that centralizers of maps of $\mathbb{E}_n$-MU$\{x\}$-algebras (which will be $\mathbb{E}_n$-MU-algebras) exist.

The key computational tool is the following:

Theorem 2.3.1 (Theorem 5.3.1.30 [Lur17])**.** *The underlying module of a centralizer can be computed as the mapping object:*

$$\mathfrak{Z}_{\mathbb{E}_n - A}(R \to S) \simeq \mathrm{map}_{\mathsf{Mod}_R^{\mathbb{E}_n}(\mathsf{LMod}_A)}(R, S).$$

Remark 2.3.2. By the previous section, when $\mathcal{C} = \mathsf{LMod}_k$ for an $\mathbb{E}_\infty$-algebra k, the right hand side can be expressed as:

$$\mathrm{map}_{\mathcal{U}_A^{(n)}(R)}(R, S),$$

where $\mathcal{U}_A^{(n)}(R)$ denotes the $\mathbb{E}_n$-enveloping algebra of the $\mathbb{E}_n$-A-algebra R.

Remark 2.3.3. For simplicity assume $A = k$ is $\mathbb{E}_\infty$. Observe that there is a canonical map

$$\mathfrak{Z}_{\mathbb{E}_n-A}(R) \to R.$$

By [Lur17, 7.3.5.1] and the defining property of the cotangent complex, the fiber of this map is a spectrum X with $\Omega^{\infty-j}X$ computed by the space of derivations:

$$\mathrm{Map}_{\mathsf{Alg}_{\mathbb{E}_n}(\mathsf{Mod}_A)/R}(R, R \oplus \Sigma^{j-n}R).$$

In particular, by forgetting the algebra structure, elements in the homotopy of the fiber of $\mathfrak{Z}_k(R) \to R$ in degree $-n - j$ refine elements in $\mathrm{Ext}^j_A(R, R)$.

2.4   Grounding the Induction

The purpose of this section is to compute the higher MU-enveloping algebras of $\mathbb{F}_p$. This will allow us to resolve extension problems when computing the $\mathbb{E}_3$-MU-center of $\mathrm{BP}\langle n \rangle$ during the inductive step.

Lemma 2.4.1. *The $\mathbb{E}_1$-MU-enveloping algebra of $\mathbb{F}_p$ has homotopy given by*

$$\pi_* \mathcal{U}^{(1)}_{\mathrm{MU}}(\mathbb{F}_p) \simeq \Lambda(\sigma v_i) \otimes_{\mathbb{F}_p} \Lambda(\sigma x_j : j \neq p^k - 1)$$

and we have the identities:

$$Q_1 \sigma v_i = \sigma v_{i+1},$$
$$Q_1 \sigma x_j = \sigma x_{jp+p-1}, \mathrm{mod\ decomposables}$$

Proof. The algebra structure follows from [Ang08, Proposition 3.6]. To compute the action of Q_1 we use the two $\mathbb{E}_\infty$ maps:

$$\mathbb{F}_p \otimes \mathbb{F}_p \xrightarrow{f} \mathbb{F}_p \otimes_{\mathrm{MU}} \mathbb{F}_p \xrightarrow{g} \mathbb{F}_p \otimes_{\mathbb{F}_p \otimes \mathrm{MU}} \mathbb{F}_p$$

We have $f(\tau_i) = \sigma v_i$ (independently of our choice of v_i) and $g(\sigma x_j) = \sigma b_j$, where b_j is the Hurewicz image of x_j (for $j \neq p^k - 1$). The first identity now follows from Steinberger's calculation [BMMS86, III.2] that

$$Q_1 \tau_i = \tau_{i+1} \quad \mathrm{mod\ decomposables},$$

together with the fact that σ annihilates decomposables. For the second identity, first recall that the Thom isomorphism is an equivalence

$$\mathbb{F}_p \otimes \mathrm{MU} \simeq \mathbb{F}_p \otimes \mathrm{BU}_+$$

of $\mathbb{E}_\infty$-$\mathbb{F}_p$-algebras, and hence we have an equivalence

$$\mathbb{F}_p \otimes_{\mathbb{F}_p \otimes \mathrm{MU}} \mathbb{F}_p \simeq \mathbb{F}_p \otimes_{\mathbb{F}_p \otimes \mathrm{BU}_+} \mathbb{F}_p \simeq \mathbb{F}_p \otimes \mathrm{B}^2\mathrm{U}_+.$$

of $\mathbb{E}_\infty$-$\mathbb{F}_p$-algebras. The suspension map

$$\sigma : \Sigma_+^\infty \mathrm{BU} \simeq \Sigma_+^\infty \Omega \mathrm{B}^2 \mathrm{U} \longrightarrow S^0 \oplus \Omega \Sigma_+^\infty \mathrm{B}^2 \mathrm{U}$$

has a canonical $\mathbb{E}_\infty$-structure, since Σ_+^∞ is symmetric monoidal, and hence the associated map on homology preserves Dyer-Lashof operations. The result now follows from Kochman's computation [Koc73, Theorem 6] of the action of Dyer-Lashof operations on $H_*(\mathrm{BU})$. $\qquad \square$

Proposition 2.4.2. *The $\mathbb{E}_2$-MU-enveloping algebra of $\mathbb{F}_p$ has homotopy given by:*

$$\pi_* \mathcal{U}_{\mathrm{MU}}^{(2)}(\mathbb{F}_p) \simeq \mathbb{F}_p[\gamma_{p^i}(\sigma^2 v_0)] \otimes_{\mathbb{F}_p} \mathbb{F}_p[\gamma_{p^i}(\sigma^2 x_j) : j \not\equiv -1 \bmod p]$$

Moreover, we have the identities:

$$\gamma_{p^i}(\sigma^2 v_0)^{p^j} = \gamma_{p^i}(\sigma^2 v_j),$$
$$\gamma_{p^i}(\sigma^2 x_j)^p = \gamma_{p^i}(\sigma^2 x_{jp+p-1})$$

Proof. The previous lemma implies that the Künneth spectral sequence for $\mathcal{U}_{\mathrm{MU}}^{(2)}(\mathbb{F}_p)$ collapses as:

$$E_2 = E_\infty = \Gamma\{\sigma^2 v_i, \sigma^2 x_j : i \geqslant 0, j \neq p^k - 1\} \Rightarrow \pi_*(\mathbb{F}_p \otimes_{\mathcal{U}_{\mathrm{MU}}^{(1)}(\mathbb{F}_p)} \mathbb{F}_p) = \pi_* \mathcal{U}_{\mathrm{MU}}^{(2)}(\mathbb{F}_p).$$

By applying [BM13, Theorem 3.6] to the standard representatives of divided powers in the bar complex, we see that there are multiplicative extensions:

$$\gamma_{p^i}(\sigma^2 v_0)^{p^j} = \gamma_{p^i}(\sigma^2 v_j)$$
$$\gamma_{p^i}(\sigma^2 x_j)^p = \gamma_{p^i}(\sigma^2 x_{jp+p-1})$$

This proves the result. $\qquad \square$

Proposition 2.4.3. *The $\mathbb{E}_3$-MU-enveloping algebra of $\mathbb{F}_p$ has homotopy given by:*

$$\pi_* \mathcal{U}_{\mathrm{MU}}^{(3)}(\mathbb{F}_p) \simeq \Lambda(\sigma\gamma_{p^i}(\sigma^2 v_0)) \otimes_{\mathbb{F}_p} \Lambda(\sigma\gamma_{p^i}(\sigma^2 x_j) : j \not\equiv -1 \bmod p).$$

Proof. Immediate from [Ang08, Proposition 3.6]. $\qquad \square$

The spectral sequence

$$\mathrm{Ext}_{\pi_* \mathcal{U}^{(3)}(\mathbb{F}_p)}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \pi_* \mathfrak{Z}_{\mathbb{E}_3 - \mathrm{MU}}(\mathbb{F}_p)$$

then immediately collapses with no possible $\mathbb{F}_p$-algebra extensions, and so proves:

Theorem 2.4.4. *The $\mathbb{E}_3$-MU-center of $\mathbb{F}_p$ has homotopy given by*

$$\pi_* \mathfrak{Z}_{\mathbb{E}_3 - \mathrm{MU}}(\mathbb{F}_p) \simeq \mathbb{F}_p[\delta\sigma\gamma_{p^i}(\sigma^2 v_0)] \otimes_{\mathbb{F}_p} \mathbb{F}_p[\delta\sigma\gamma_{p^i}(\sigma^2 x_j)].$$

In particular, the homotopy of $\mathfrak{Z}_{\mathbb{E}_3 - \mathrm{MU}}(\mathbb{F}_p)$ is concentrated in even degrees. Moreover, under the forgetful map

$$\mathrm{Map}_{\mathsf{Alg}_{\mathbb{E}_3}(\mathsf{Mod}_{\mathrm{MU}})_{/\mathbb{F}_p}}(\mathbb{F}_p, \mathbb{F}_p \oplus \Sigma\mathbb{F}_p) \longrightarrow \mathrm{Map}_{\mathrm{MU}}(\mathbb{F}_p, \Sigma\mathbb{F}_p)$$

the class $\delta(\sigma^3 v_0)$ maps to δv_0.

2.5 Computation of the center

In this section, we will assume that $\mathrm{BP}\langle n\rangle$ has been given the structure of an $\mathbb{E}_3$-MU-algebra and then compute its $\mathbb{E}_3$-MU-center.

Lemma 2.5.1. *The $\mathbb{E}_1$-MU-enveloping algebra of $\mathrm{BP}\langle n\rangle$ has homotopy given, as a $\mathrm{BP}\langle n\rangle_*$-algebra, by:*

$$\pi_* \mathcal{U}_{\mathrm{MU}}^{(1)}(\mathrm{BP}\langle n\rangle) \simeq \Lambda_{\mathrm{BP}\langle n\rangle_*}(\sigma v_i : i \geqslant n+1) \otimes_{\mathrm{BP}\langle n\rangle_*} \Lambda_{\mathrm{BP}\langle n\rangle_*}(\sigma x_j : j \neq p^k - 1).$$

Proof. Immediate from [Ang08, Proposition 3.6]. $\qquad\square$

Proposition 2.5.2. *The $\mathbb{E}_2$-MU-enveloping algebra of $\mathrm{BP}\langle n\rangle$ has homotopy given, as a $\mathrm{BP}\langle n\rangle_*$-algebra, by*

$$\pi_* \mathcal{U}_{\mathrm{MU}}^{(2)}(\mathrm{BP}\langle n\rangle) \simeq \mathrm{BP}\langle n\rangle_*[\gamma_{p^i}(\sigma^2 v_{n+1})] \otimes_{\mathrm{BP}\langle n\rangle_*} \mathrm{BP}\langle n\rangle_*[\gamma_{p^i}(\sigma^2 x_j) : j \not\equiv -1 \bmod p].$$

Proof. The Künneth spectral sequence collapses to a divided power algebra on even classes:

$$\Gamma_{\mathrm{BP}\langle n\rangle_*}(\sigma^2 v_i : i \geqslant n+1) \otimes_{\mathrm{BP}\langle n\rangle_*} \Gamma_{\mathrm{BP}\langle n\rangle_*}(\sigma^2 x_j : j \neq p^k - 1) \Rightarrow \pi_* \mathcal{U}_{\mathrm{MU}}^{(2)}(\mathrm{BP}\langle n\rangle)$$

Since $\mathrm{BP}\langle n\rangle$ is an $\mathbb{E}_3$-MU-algebra, the enveloping algebra $\mathcal{U}_{\mathrm{MU}}^{(2)}(\mathrm{BP}\langle n\rangle)$ is an $\mathbb{E}_2$-algebra (see Remark 2.2.1) and, in particular, its homotopy groups are a graded commutative algebra. We can thus define a map:

$$f : \mathrm{BP}\langle n\rangle_*[\gamma_{p^i}(\sigma^2 v_{n+1})] \otimes_{\mathrm{BP}\langle n\rangle_*} \mathrm{BP}\langle n\rangle_*[\gamma_{p^i}(\sigma^2 x_j) : j \not\equiv -1 \bmod p] \to \pi_* \mathcal{U}_{\mathrm{MU}}^{(2)}(\mathrm{BP}\langle n\rangle)$$

which we would like to be an isomorphism. From the Künneth spectral sequence we already know that $\pi_* \mathcal{U}_{\mathrm{MU}}^{(2)}(\mathrm{BP}\langle n\rangle)$ is a connective, free $\mathrm{BP}\langle n\rangle_*$-module with finitely many generators in each degree. Thus it suffices to prove that f is an isomorphism modulo $(p, v_1, ..., v_n)$.

But now observe that the map

$$\pi_* \mathcal{U}_{\mathrm{MU}}^{(2)}(\mathrm{BP}\langle n\rangle)/(p, v_1, ..., v_n) \longrightarrow \pi_* \mathcal{U}_{\mathrm{MU}}^{(2)}(\mathbb{F}_p)$$

is injective (by our previous calculation of the target and naturality of the Künneth spectral sequence). The result now follows by Proposition 2.4.2. $\qquad\square$

Since $\mathcal{U}_{\mathrm{MU}}^{(2)}(\mathrm{BP}\langle n\rangle)$ coincides with $\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})$ as an $\mathbb{E}_1$-algebra, this is also the computation of Hochschild homology given in the introduction:

Theorem 2.5.3 (Polynomial THH). *There is an isomorphism of $\mathrm{BP}\langle n\rangle_*$-algebras*

$$\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})_* \simeq \mathrm{BP}\langle n\rangle_*[\gamma_{p^i}(\sigma^2 v_{n+1})] \otimes_{\mathrm{BP}\langle n\rangle_*} \mathrm{BP}\langle n\rangle_*[\gamma_{p^i}(\sigma^2 x_j) : j \not\equiv -1 \bmod p].$$

Again, it follows from [Ang08, Proposition 3.6] that the $\mathbb{E}_3$-MU-enveloping algebra has homotopy given by an exterior algebra, and hence that the spectral sequence

$$\mathrm{Ext}_{\pi_* \mathcal{U}_{\mathrm{MU}}^{(3)}(\mathrm{BP}\langle n\rangle)}(\mathrm{BP}\langle n\rangle_*, \mathrm{BP}\langle n\rangle_*) \Rightarrow \pi_* \mathfrak{Z}_{\mathbb{E}_3 - \mathrm{MU}}(\mathrm{BP}\langle n\rangle)$$

collapses. This proves:

14

Theorem 2.5.4. *The $\mathbb{E}_3$-MU-center of $\mathrm{BP}\langle n \rangle$ has homotopy given by*

$$\pi_* \mathfrak{Z}_{\mathbb{E}_3-\mathrm{MU}}(\mathrm{BP}\langle n \rangle) \simeq \mathrm{BP}\langle n \rangle_*[\delta\sigma\gamma_{p^i}(\sigma^2 v_{n+1})] \otimes_{\mathrm{BP}\langle n \rangle_*} \mathrm{BP}\langle n \rangle_*[\delta\sigma\gamma_{p^i}(\sigma^2 x_j)]$$

In particular, the homotopy of $\mathfrak{Z}_{\mathbb{E}_3-\mathrm{MU}}(\mathrm{BP}\langle n \rangle)$ is concentrated in even degrees. Moreover, under the forgetful map

$$\mathrm{Map}_{\mathsf{Alg}_{\mathbb{E}_3}(\mathsf{Mod}_{\mathrm{MU}})_{/\mathrm{BP}\langle n \rangle}}(\mathrm{BP}\langle n \rangle, \mathrm{BP}\langle n \rangle \oplus \Sigma^{|v_{n+1}|+1}\mathrm{BP}\langle n \rangle) \longrightarrow \mathrm{Map}_{\mathrm{MU}}(\mathrm{BP}\langle n \rangle, \Sigma^{|v_{n+1}|+1}\mathrm{BP}\langle n \rangle),$$

the class $\delta\sigma^3 v_{n+1}$ maps to δv_{n+1}.

2.6 Designer MU-algebras

The purpose of this section is to construct a certain $\mathbb{E}_4$-MU-algebra with an even $\mathbb{E}_4$-cell structure. We will assume the contents of Appendix A.

First we recall the construction of an $\mathbb{E}_\infty$-structure on 'polynomial rings' over MU.

Construction 2.6.1. Recall that the J-homomorphism gives a map of $\mathbb{E}_\infty$-spaces:

$$\mathbb{Z} \times \mathrm{BU} \to \mathrm{Pic}(S^0),$$

and hence a symmetric monoidal functor to Sp. Left Kan extending along the map $\mathbb{Z} \times \mathrm{BU} \to \mathbb{Z}$ gives a symmetric monoidal functor

$$\mathbb{Z} \to \mathsf{Sp}$$

(here we view $\mathbb{Z}$ as a discrete category, not as a poset). In other words, we have constructed a graded $\mathbb{E}_\infty$-ring, in fact an $\mathbb{E}_\infty$-MU-algebra, which we will denote by $\mathrm{MU}[\beta^{\pm 1}]$, where β is the class in weight 1 and dimension 2. This is a graded refinement of periodic complex bordism. For any $k \in \mathbb{Z}$, we may precompose with the symmetric monoidal functor $\mathbb{Z} \xrightarrow{\cdot k} \mathbb{Z}$ and thereby produce a graded $\mathbb{E}_\infty$-MU-algebra $\mathrm{MU}[y^{\pm 1}]$ where y has weight 1 and dimension $2k$. Restricting and left Kan extending along $\mathbb{Z}_{\geqslant 0} \to \mathbb{Z}$ yields a weight-connected, graded $\mathbb{E}_\infty$-MU-algebra $\mathrm{MU}[y]$. As the notation suggests, this is a refinement of the free graded $\mathbb{E}_1$-ring on the class y.

Construction 2.6.2. Fix $n, k \geqslant 0$ and denote by $\mathrm{MU}[y]$ the graded $\mathbb{E}_\infty$-MU-algebra constructed above, with y in dimension $2k$ and weight 1. Define

$$\mathrm{MU}\{x\} := \mathbb{D}^{(n)}\mathrm{MU}[y].$$

Here we are using the Koszul duality functor for graded $\mathbb{E}_n$-MU-algebras. Note that the notation $\mathrm{MU}\{x\}$ suppresses both n and k.

Theorem 2.6.3. *Fix $n, k \geqslant 0$. With notation as above, the graded $\mathbb{E}_n$-MU-algebra $\mathrm{MU}\{x\}$ has the following properties:*

(a) MU$\{x\}$ *admits a canonical graded $\mathbb{E}_n$-MU-cell structure with one cell in each bidegree of the form*

$$(\text{dimension}, \text{weight}) = (-2kj - n, -j)$$

for $j \geqslant 1$.

(b) *There is a canonical generator $x : \Sigma^{-2k-n}\mathrm{MU}(-1) \to \mathrm{MU}\{x\}$ which extends to an equivalence of graded $\mathbb{E}_{n-1}$-MU-algebras*

$$\mathrm{Free}_{\mathbb{E}_{n-1}-\mathrm{MU}}(\Sigma^{-2k-n}\mathrm{MU}(-1)) \xrightarrow{\simeq} \mathrm{MU}\{x\}.$$

(c) *Viewing MU as an $\mathbb{E}_{n-1}$-MU$\{x\}$-algebra, we have an equivalence of $\mathbb{E}_1$-algebras:*

$$\mathcal{U}^{(n-1)}_{\mathrm{MU}\{x\}}(\mathrm{MU}) \simeq \mathbb{D}^{(1)}\mathrm{MU}[y] \simeq \mathrm{MU} \oplus \Sigma^{-2k-1}\mathrm{MU}(-1),$$

where this last is the trivial square-zero extension. Under this equivalence, $\sigma^{n-1}x$ corresponds to δy.

Proof. By Lemma A.2.3(v), we have a tower of square-zero extensions of graded $\mathbb{E}_\infty$-MU-algebras:

$$\mathrm{MU}[y] \to \cdots \to \mathrm{MU}[y]/y^{j+1} \to \mathrm{MU}[y]/y^j \to \cdots \to \mathrm{MU}$$

Here we've written $\mathrm{MU}[y]/y^j := \mathrm{MU}[y]_{\leqslant j-1}$ for the weight-truncation. The square-zero extensions are described by pullback squares of the form:

$$\begin{array}{ccc}
\mathrm{MU}[y]/y^{j+1} & \longrightarrow & \mathrm{MU} \\
\downarrow & & \downarrow \\
\mathrm{MU}[y]/y^j & \longrightarrow & \mathrm{MU} \oplus \Sigma^{2kj+1}\mathrm{MU}(j)
\end{array}$$

Every algebra in sight is weight-connected and pointwise-perfect so, by Theorem A.1.5, we deduce that

$$\mathrm{MU}\{x\} := \mathbb{D}^{(n)}(\mathrm{MU}[y]) \simeq \operatorname{colim} \mathbb{D}^{(n)}(\mathrm{MU}[y]/y^j)$$

and that we have pushout squares of graded $\mathbb{E}_n$-MU-algebras:

$$\begin{array}{ccc}
\mathrm{Free}_{\mathbb{E}_n-\mathrm{MU}}(\Sigma^{-2kj-n-1}\mathrm{MU}(-j)) & \longrightarrow & \mathbb{D}^{(n)}(\mathrm{MU}[y]/y^j) \\
\downarrow & & \downarrow \\
\mathrm{MU} & \longrightarrow & \mathbb{D}^{(n)}(\mathrm{MU}[y]/y^{j+1})
\end{array}$$

This proves (a) and provides us with a generator x, from the first cell attachment. Both the source and target of the resulting map

$$\mathrm{Free}_{\mathbb{E}_{n-1}-\mathrm{MU}}(\Sigma^{-2k-n}\mathrm{MU}(-1)) \to \mathrm{MU}\{x\}$$

16

are weight-coconnected and pointwise perfect (since configuration spaces are finite). So it suffices to check this map becomes an equivalence this after applying $\mathbb{D}^{(n-1)}$. But then the map becomes

$$\mathbb{D}^{(n-1)}\mathrm{MU}\{x\} \simeq \mathrm{Bar}(\mathrm{MU}[y]) \to \mathrm{MU} \oplus \Sigma^{2k+1}\mathrm{MU}(1)$$

which is indeed an equivalence. This proves (b).

Finally, to prove (c), observe that, as a *module* (but a priori not as an algebra) we have

$$\mathcal{U}^{(n-1)}_{\mathrm{MU}\{x\}}(\mathrm{MU}) \simeq \mathrm{Bar}^{(n-1)}(\mathrm{MU}\{x\}) \simeq \mathbb{D}^{(1)}(\mathrm{MU}).$$

It follows from Lemma A.2.3(v) (after applying Remark A.1.4) that we have a pullback diagram of $\mathbb{E}_1$-MU-algebras:

$$
\begin{array}{ccc}
\mathcal{U}^{(n-1)}_{\mathrm{MU}\{x\}}(\mathrm{MU}) & \longrightarrow & \mathrm{MU} \\
\downarrow & & \downarrow \\
\mathrm{MU} & \xrightarrow{\;\;d\;\;} & \mathrm{MU} \oplus \Sigma^{-2k}\mathrm{MU}(-1)
\end{array}
$$

But now d must be the trivial derivation, since MU is the unit. This completes the proof. $\square$

Remark 2.6.4. It is possible to enhance $\mathrm{MU}\{x\}$ to a graded $\mathcal{D}\mathrm{isk}_n$-MU-algebra, i.e. an object in the fixed points $\mathsf{Alg}^{hO(n)}_{\mathbb{E}_n}$ for the $O(n)$-action on the category of $\mathbb{E}_n$-algebras. This gives an alternative proof that the enveloping algebra above can be computed as a bar construction, and hence an alternative proof of (c). It would be interesting to check whether our construction of $\mathrm{BP}\langle n \rangle$ actually builds a $\mathcal{D}\mathrm{isk}_3$-MU-algebra, as opposed to just an $\mathbb{E}_3$-MU-algebra.

2.7 Proof of Theorem A

We are now ready to prove Theorem A. We suppose, by induction, that we have constructed $\mathrm{BP}\langle n \rangle$ as an $\mathbb{E}_3$-MU-algebra.

Construction 2.7.1. Let $\mathrm{MU}\{x\}$ be the $\mathbb{E}_4$-algebra with $|x| = -2 - 2p^{n+1}$ that was constructed in §2.6. By Theorem 2.5.4, the homotopy of $\mathfrak{Z}_{\mathbb{E}_3 - \mathrm{MU}}(\mathrm{BP}\langle n \rangle)$ is concentrated in even degrees, and hence, by Theorem 2.6.3(a), there are no obstructions to extending the class $\delta\sigma^3 v_{n+1}$ to an $\mathbb{E}_4$-MU-algebra map

$$\mathrm{MU}\{x\} \to \mathfrak{Z}_{\mathbb{E}_3 - \mathrm{MU}}(\mathrm{BP}\langle n \rangle).$$

Thus we have equipped $\mathrm{BP}\langle n \rangle$ with an $\mathbb{E}_3$-$\mathrm{MU}\{x\}$-algebra structure.

Lemma 2.7.2. *The unit map* $\mathrm{MU} \to \mathrm{BP}\langle n \rangle$ *admits a refinement to a map of $\mathbb{E}_3$-$\mathrm{MU}\{x\}$-algebras.*

Proof. We need to check that the diagram of $\mathbb{E}_3$-MU-algebras

$$
\begin{array}{ccc}
\mathrm{MU}\{x\} & \longrightarrow & \mathfrak{Z}_{\mathbb{E}_3-\mathrm{MU}}(\mathrm{MU}) \\
\downarrow & & \downarrow \\
\mathfrak{Z}_{\mathbb{E}_3-\mathrm{MU}}(\mathrm{BP}\langle n\rangle) & \longrightarrow & \mathfrak{Z}_{\mathbb{E}_3-\mathrm{MU}}(\mathrm{MU} \to \mathrm{BP}\langle n\rangle)
\end{array}
$$

commutes up to homotopy. Since MU is the unit object in $\mathbb{E}_3$-MU-algebras, we can rewrite this diagram as:

$$
\begin{array}{ccc}
\mathrm{MU}\{x\} & \longrightarrow & \mathrm{MU} \\
\downarrow & & \downarrow \\
\mathfrak{Z}_{\mathbb{E}_3-\mathrm{MU}}(\mathrm{BP}\langle n\rangle) & \longrightarrow & \mathrm{BP}\langle n\rangle
\end{array}
$$

By Theorem 2.6.3(b), $\mathrm{MU}\{x\}$ is free on x as an $\mathbb{E}_3$-MU-algebra. Since x is in negative degree, it must map to zero in $\mathrm{BP}\langle n\rangle$, which determines the map up to homotopy. $\qquad\square$

Construction 2.7.3. As recalled in §2.3, we may define an $\mathbb{E}_3$-MU-algebra B by the formula

$$
B := \mathfrak{Z}_{\mathbb{E}_3-\mathrm{MU}\{x\}}(\mathrm{MU} \to \mathrm{BP}\langle n\rangle).
$$

We now complete the induction step.

Proposition 2.7.4. *The $\mathbb{E}_3$-MU-algebra B constructed above is a form of $\mathrm{BP}\langle n+1\rangle$.*

Proof. By Theorem 2.6.3(c), we have an equivalence

$$
B \simeq \mathrm{map}_{\mathrm{MU}\oplus\Sigma^{-|v_{n+1}|-1}\mathrm{MU}}(\mathrm{MU}, \mathrm{BP}\langle n\rangle).
$$

Thus we get a spectral sequence

$$
E_2 = \mathrm{Ext}_{\Lambda_{\mathrm{MU}_*}(\delta v_{n+1})}(\mathrm{MU}_*, \mathrm{BP}\langle n\rangle_*) \Rightarrow B_*
$$

which collapses to

$$
E_2 = E_\infty = \mathrm{BP}\langle n\rangle_*[z].
$$

By design, the generator δv_{n+1} acts by the nontrivial extension

$$
\mathrm{BP}\langle n\rangle \to \Sigma^{|v_{n+1}|+1}\mathrm{BP}\langle n\rangle
$$

defining the MU-module $\mathrm{BP}\langle n+1\rangle/(v_{n+1}^2)$. It follows that $v_{n+1} \in \mathrm{MU}_*$ maps to a lift of z in $\pi_* B$ and hence that $v_1, ..., v_{n+1}$ give indecomposable generators of B_*. So we may alter our choice of indecomposable generators x_r of MU_* so that

(i) x_{p^i-1} maps to the generators v_i in B_* for $1 \leqslant i \leqslant n$,

(ii) $x_{p^{n+1}-1}$ maps to a lift of z, and

18

(iii) x_r maps to zero in B_*, otherwise.

Thus we may form a map of MU-modules

$$\mathrm{BP}\langle n+1 \rangle = \mathrm{MU}/(x_{p^n}) \otimes_{\mathrm{MU}} \mathrm{MU}/(x_{p^n+1}) \otimes_{\mathrm{MU}} \cdots \to B$$

which is an equivalence on homotopy. This proves the proposition. $\qquad\square$

Proof of Theorem A. Recall that Basterra-Mandell [BM13] have produced an $\mathbb{E}_4$-map $\mathrm{BP} \to \mathrm{MU}_{(p)}$ with an $\mathbb{E}_4$-retract $\mathrm{MU}_{(p)} \to \mathrm{BP}$. Restricting along the first map we see that the form of $\mathrm{BP}\langle n \rangle$ constructed above restricts to an $\mathbb{E}_3$-BP-algebra. Under the composite

$$\mathrm{BP} \to \mathrm{MU} \to \mathrm{BP}\langle n \rangle,$$

the elements $v_1, ..., v_n$ map to polynomial generators of $\mathrm{BP}\langle n \rangle_*$, which we will give the same name (possibly altering the names given before). For degree reasons, the elements v_{n+j} for $j \geqslant 1$ must map to decomposables in $\mathrm{BP}\langle n \rangle_*$, i.e. we have $v_{n+j} \equiv 0$ modulo $(v_1, ..., v_n)$, in $\mathrm{BP}\langle n \rangle_*$. So redefine each v_{n+j} by adding the appropriate polynomial in $v_1, ..., v_n$, and we have a new sequence of polynomial generators for BP satisfying the requirements in the theorem. $\qquad\square$

3   The Detection Theorem

Throughout this section, we will use $\mathrm{BP}\langle n \rangle$ to denote a fixed $\mathbb{E}_3$-MU-algebra form $\mathrm{BP}\langle n \rangle$. By $v_{n+1} \in \pi_{2p^{n+1}-2}\mathrm{MU}$ we will refer to a specific indecomposable generator, with:

- trivial mod p Hurewicz image, and

- the key property that the unit map $\mathrm{MU} \to \mathrm{BP}\langle n \rangle$ sends v_{n+1} to 0 in homotopy.

This last assumption ensures that v_{n+1} admits a unique lift to an element in the homotopy of the fiber of the unit map $\mathrm{MU} \to \mathrm{BP}\langle n \rangle$. Our main aim will be to prove Theorem D from the introduction, which we restate for convenience:

Theorem 3.0.1. *Let A denote* $\mathrm{THH}(\mathrm{BP}\langle n \rangle/\mathrm{MU})^{hS^1}$*, with its natural* MU*-algebra structure inducing an* MU_**-algebra structure on $\pi_*(A)$. Then there is an isomorphism of $\mathbb{Z}_{(p)}[v_1, ..., v_n]$-algebras*

$$\pi_* A \cong (\pi_* \mathrm{THH}(\mathrm{BP}\langle n \rangle/\mathrm{MU})) [\![t]\!],$$

where $|t| = -2$ and $\pi_* \mathrm{THH}(\mathrm{BP}\langle n \rangle/\mathrm{MU})$ is a free $\mathbb{Z}_{(p)}[v_1, \cdots, v_n]$-algebra. Furthermore, under the unit map

$$\pi_* \mathrm{MU} \longrightarrow \pi_* A,$$

the class v_{n+1} is sent to a unit multiple of $t(\sigma^2 v_{n+1})$.

Remark 3.0.2. The isomorphism

$$\pi_* A \cong (\pi_* \mathrm{THH}(\mathrm{BP}\langle n \rangle / \mathrm{MU})) \llbracket t \rrbracket$$

of Theorem 3.0.1 depends on an arbitrary choice of power series generator t. What we will show below is essentially that, for any natural choice of t, v_{n+1} is sent to a unit times $t\sigma^2 v_{n+1}$ modulo larger powers of t. However, in a power series ring such as $\pi_* A$, adding a multiple of t^2 to $t\sigma^2 v_{n+1}$ can alternatively be described as multiplying $t\sigma^2 v_{n+1}$ by a unit.

Before turning to the proof, we observe that the Detection Theorem implies a weak form of redshift.

Corollary 3.0.3. *For each $0 \leqslant m \leqslant n+1$, $L_{K(m)}\mathrm{K}(\mathrm{BP}\langle n \rangle) \neq 0$. In particular, $L_{K(n+1)}\mathrm{K}(\mathrm{BP}\langle n \rangle) \neq 0$.*

Proof. By [BGT14], the cyclotomic trace map

$$\mathrm{K}(-) \to \mathrm{TC}(-)$$

is a lax symmetric monoidal natural transformation. It follows that the trace $\mathrm{K}(\mathrm{BP}\langle n \rangle) \to \mathrm{TC}(\mathrm{BP}\langle n \rangle)$ is a map of $\mathbb{E}_2$-rings. Recall that there is a canonical map $\mathrm{TC}(-) \to \mathrm{THH}(-)^{hS^1}$, to negative cyclic homology. Thus we have a sequence of $\mathbb{E}_2$-ring maps:

$$\mathrm{K}(\mathrm{BP}\langle n \rangle) \to \mathrm{TC}(\mathrm{BP}\langle n \rangle) \to \mathrm{THH}(\mathrm{BP}\langle n \rangle)^{hS^1} \to \mathrm{THH}(\mathrm{BP}\langle n \rangle / \mathrm{MU})^{hS^1}$$

and hence an $\mathbb{E}_2$-ring map

$$L_{K(m)}\mathrm{K}(\mathrm{BP}\langle n \rangle) \to L_{K(m)}\mathrm{THH}(\mathrm{BP}\langle n \rangle / \mathrm{MU})^{hS^1}$$

for each height $m \leqslant n+1$. If the source of this map were zero, then the target would be zero as well, since this is a map of rings. The relative negative cyclic homology $\mathrm{THH}(\mathrm{BP}\langle n \rangle / \mathrm{MU})^{hS^1}$ has the structure of an MU-module. It follows from [Hov95, Theorem 1.9] and [Hov97, Theorem 1.5.4] that

$$L_{K(m)}\mathrm{THH}(\mathrm{BP}\langle n \rangle / \mathrm{MU})^{hS^1} = (\mathrm{THH}(\mathrm{BP}\langle n \rangle / \mathrm{MU})^{hS^1})[v_m^{-1}]^{\wedge}_{(p, v_1, \ldots, v_{m-1})}.$$

By Theorem 3.0.1, this completion and localization can be computed algebraically and the result is nonzero. $\qquad\qquad\square$

Remark 3.0.4. In the statement and proof of the theorem we have used that the S^1-action on $\mathrm{THH}(\mathrm{BP}\langle n \rangle / \mathrm{MU})$ is compatible with the algebra structure. One way to see this is to use the generality in which THH is defined. Recall that for any symmetric monoidal category $\mathcal{C}$ with tensor product compatible with sifted colimits, Hochschild homology gives a functor:

$$\mathrm{THH}_{\mathcal{C}} : \mathsf{Alg}_{\mathbb{E}_1}(\mathcal{C}) \to \mathsf{Fun}(\mathrm{B}S^1, \mathcal{C}).$$

For a reference, one could observe that the construction of THH with its circle action in [NS18, §III.2] works just the same for $\mathcal{C}$ in place of Sp. Alternatively, one can use the identification of THH with factorization homology over S^1, which is defined in this generality ([Lur17, §5.5.2], [AF15]). Now apply this in the case $\mathcal{C} = \mathsf{Alg}_{\mathbb{E}_2}(\mathsf{Mod}_{\mathrm{MU}})$ to see that $\mathrm{THH}(\mathrm{BP}\langle n \rangle / \mathrm{MU})$ has a canonical enhancement to an object in $\mathsf{Fun}(\mathrm{B}S^1, \mathsf{Alg}_{\mathbb{E}_2}(\mathsf{Mod}_{\mathrm{MU}}))$.

Let us now proceed with the proof of Theorem 3.0.1.

3.1 Associated graded negative cyclic homology

Recall that $\mathrm{THH}(\mathrm{BP}\langle n \rangle/\mathrm{MU}) \simeq \mathcal{U}^{(2)}_{\mathrm{MU}}(\mathrm{BP}\langle n \rangle)$, and we calculated the homotopy groups of this $\mathbb{E}_2$-MU-algebra as Proposition 2.5.2. An immediate consequence of that calculation is the following proposition:

Proposition 3.1.1. *The homotopy fixed point spectral sequence for* $\mathrm{THH}(\mathrm{BP}\langle n \rangle/\mathrm{MU})^{hS^1}$ *collapses at the E_2-page, with*

$$E_\infty = \mathrm{BP}\langle n \rangle_*[\gamma_{p^i}(\sigma^2 v_{n+1}), \gamma_{p^i}(\sigma^2 x_j) : j \not\equiv -1 \bmod p][t],$$

where $t \in H^2(\mathbb{C}P^\infty)$ is the standard generator and

$$\mathrm{BP}\langle n \rangle_*[\gamma_{p^i}(\sigma^2 v_{n+1}), \gamma_{p^i}(\sigma^2 x_j) : j \not\equiv -1 \bmod p] \cong \pi_* \mathrm{THH}(\mathrm{BP}\langle n \rangle/\mathrm{MU}).$$

Proof. The homotopy fixed point spectral sequence computing $\mathrm{THH}(\mathrm{BP}\langle n \rangle/\mathrm{MU})^{hS^1}$ is concentrated in even degrees, and hence collapses as indicated. $\qquad\square$

Corollary 3.1.2. *Let A denote* $\mathrm{THH}(\mathrm{BP}\langle n \rangle/\mathrm{MU})^{hS^1}$. *Then there is an isomorphism of* $\mathbb{Z}_{(p)}[v_1, \cdots, v_n]$-*algebras*

$$\pi_*(A) \cong (\pi_* \mathrm{THH}(\mathrm{BP}\langle n \rangle/\mathrm{MU}))\,[\![t]\!],$$

where $|t| = -2$.

Proof. The homotopy fixed point spectral sequence produces a short exact sequence

$$0 \to \lim^1 \to \pi_*(A) \to \lim_k \pi_* \mathrm{THH}(\mathrm{BP}\langle n \rangle/\mathrm{MU})[t]/t^k \to 0,$$

and the $\lim^1$ term vanishes because the maps defining the limit are surjective. Here, we use that $\pi_* \mathrm{THH}(\mathrm{BP}\langle n \rangle/\mathrm{MU})$ is a free $\mathbb{Z}_{(p)}[v_1, v_2, \cdots, v_n]$-algebra to check that there are no multiplicative extensions in the terms defining the limit. $\qquad\square$

It remains to check the final sentence of Theorem 3.0.1, which is the result of an interesting MU_*-module extension in the above homotopy fixed point spectral sequence.

3.2 A nontrivial extension

Let A denote the relative negative cyclic homology $\mathrm{THH}(\mathrm{BP}\langle n \rangle/\mathrm{MU})^{hS^1}$, viewed as an MU-algebra. We aim to prove the following proposition:

Proposition 3.2.1. *The element $t\sigma^2 v_{n+1}$ in $\pi_* A/t^2$ detects v_{n+1}, up to multiplication by a unit.*

Recall that the homotopy fixed point spectral sequence for a spectrum X with an S^1-action can be constructed using the cosimplicial object

$$C^\bullet = \{F((S^1)_+^{\times n}, X)\}$$

In particular, the lowest nontrivial piece of the associated tower is described by the fiber sequence

$$\mathrm{Tot}^{\leqslant 1}(C^\bullet) \to X \xrightarrow{B} \Sigma^{-1}X$$

Here $B : \Sigma X \to X$ is obtained from the action map $S^1_+ \wedge X \to X$ by restricting to the ΣX summand of the source.

Examining the filtration, we see that the element $t\sigma^2 v_{n+1}$ is lifted from a class in

$$A' := \mathrm{fib}\left(B : \mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU}) \to \Sigma^{-1}\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})\right)$$

Thus, to prove Proposition 3.2.1, it suffices to prove:

Lemma 3.2.2. *The element $t\sigma^2 v_{n+1}$ in $\pi_* A'$ detects v_{n+1}, up to multiplication by a unit.*

We will deduce this from a more general proposition.

Construction 3.2.3. Let R be an $\mathbb{E}_\infty$-ring and let S be an $\mathbb{E}_1$-R-algebra. Denote by $\overline{R}$ the fiber of the unit $R \to S$. Then we have a canonical map:

$$\sigma : \Sigma\overline{R} \to S \otimes_R S$$

constructed as follows. First, the commutative square

$$
\begin{array}{ccc}
R & \longrightarrow & S \\
\downarrow & & \downarrow {\scriptstyle 1\otimes\mathrm{id}} \\
S & \xrightarrow{\ \mathrm{id}\otimes 1\ } & S \otimes_R S
\end{array}
$$

gives a map

$$S \amalg_R S \to S \otimes_R S.$$

On the other hand, we have a map of diagrams

$$(0 \leftarrow \overline{R} \to 0) \longrightarrow (S \leftarrow R \to S)$$

The induced map on pushouts, composed with the previous map, then gives the usual suspension homomorphism:

$$\Sigma\overline{R} \to S \otimes_R S.$$

Remark 3.2.4. Tracing through the definitions, we see that $\sigma : \Sigma\overline{R} \to S\otimes_R S$ is compatible with the naming of elements in the Künneth spectral sequence. That is, σr is detected by the cycle representative $[r]$ in the bar complex computing Tor-groups, up to a sign.

Construction 3.2.5. The above construction provides a canonical factorization

$$
\begin{array}{ccc}
S & \longrightarrow & \Sigma\overline{R} \\
 & \searrow & \downarrow \sigma \\
1\otimes\mathrm{id}-\mathrm{id}\otimes 1 & & S\otimes_R S
\end{array}
$$

Moreover, each map to $S\otimes_R S$ is canonically null after composition with either multiplication $S\otimes_R S \to S$ or multiplication pre-composed with the swap map. In other words, if we let $\Lambda_0^2 = \{1 \leftarrow 0 \to 2\}$, then we have a diagram

$$
\begin{array}{ccc}
F_S & \longrightarrow & F_{\Sigma\overline{R}} \\
 & \searrow & \downarrow \\
 & & \mathrm{B}^{\mathrm{cyc}}(S)|_{\Lambda_0^2}
\end{array}
$$

of natural transformations between functors on Λ_0^2. Here F_X denotes the span $0 \leftarrow X \to 0$, $\mathrm{B}^{\mathrm{cyc}}(S)$ is the cyclic bar construction, and we are restricting along the map $\phi : \Lambda_0^2 \to \Delta^{\mathrm{op}}$ sending 0 to $[1]$, both 1 and 2 to $[0]$, the map $0 \to 1$ to d_0, and the map $0 \to 2$ to d_1.

It follows that we get a diagram of simplicial objects

$$
\begin{array}{ccc}
\phi_! F_S & \longrightarrow & \phi_! F_{\Sigma\overline{R}} \\
 & \searrow & \downarrow \\
 & & \mathrm{B}^{\mathrm{cyc}}(S/R)
\end{array}
$$

after left Kan extending F_S and $F_{\Sigma\overline{R}}$.[2] Further left Kan extending to a point, we get a diagram on colimits:

$$
\begin{array}{ccc}
\Sigma S & \longrightarrow & \Sigma^2\overline{R} \\
 & \searrow & \downarrow \\
 & & \mathrm{THH}(S/R)
\end{array}
$$

We denote the vertical map by

$$
\sigma^2 : \Sigma^2\overline{R} \to \mathrm{THH}(S/R)
$$

and call it the double suspension.

Remark 3.2.6. Tracing through the definitions, we see that, in terms of the Künneth spectral sequence
$$
\mathrm{Tor}_{*,*}^{\pi_*(S\otimes_R S^{\mathrm{op}})}(\pi_* S, \pi_* S) \Rightarrow S \otimes_{S\otimes_R S^{\mathrm{op}}} S,
$$
the element $\sigma^2 r$ is indeed detected by $[\sigma r]$ in the bar complex, up to a sign.

---

[2]These left Kan extensions are just the usual bar construction computing the pushout.

Lemma 3.2.2 is now an immediate consequence of the following proposition:

Proposition 3.2.7. *For any R and S as above, the diagram*

$$\begin{array}{ccc}
\overline{R} & \longrightarrow & R \\
{\scriptstyle \sigma^2}\downarrow & & \downarrow \\
\Sigma^{-2}\mathrm{THH}(S/R) & \underset{t}{\longrightarrow} & \mathrm{Tot}^{\leqslant 1}
\end{array}$$

commutes up to homotopy and up to sign. In other words, we have $t\sigma^2 x = x$, up to a sign.

Proof. It suffices to produce a commutative diagram

$$\begin{array}{ccccc}
R & \longrightarrow & S & \longrightarrow & \Sigma\overline{R} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Tot}^{\leqslant 1} & \longrightarrow & \mathrm{THH}(S/R) & \underset{B}{\longrightarrow} & \Sigma^{-1}\mathrm{THH}(S/R)
\end{array}$$

The commutativity of the first square is clear, so we need to establish the commutativity of the second square. We will use notation as in Construction 3.2.5. First observe that we have a canonically commutative diagram:

$$\begin{array}{ccc}
S & \longrightarrow & \Sigma\overline{R} \\
{\scriptstyle \begin{bmatrix} 1\otimes\mathrm{id} \\ -1\otimes\mathrm{id} \end{bmatrix}}\downarrow & & \downarrow{\scriptstyle \sigma} \\
(S\otimes_R S)\oplus(S\otimes_R S) & \underset{(\mathrm{id},\mathrm{swap})}{\longrightarrow} & S\otimes_R S
\end{array}$$

Each composition of the left vertical map with the various maps to S built from multiplying, swapping, and summing the answers, is canonically null. It follows that we may produce a commutative diagram of natural transformations

$$\begin{array}{ccc}
F_S & \longrightarrow & F_{\Sigma\overline{R}} \\
\downarrow & & \downarrow \\
j^* j_!(\mathrm{B}^{\mathrm{cyc}}(S/R)|_{\Lambda_0^2}) & \longrightarrow & \mathrm{B}^{\mathrm{cyc}}(S/R)|_{\Lambda_0^2}
\end{array}$$

where $j : \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}}_{\mathrm{cyc}}$ is the inclusion of the simplex category into the cyclic category. It follows that we get a commutative diagram

$$\begin{array}{ccc}
\phi_! F_S & \longrightarrow & \phi_! F_{\Sigma\overline{R}} \\
\downarrow & & \downarrow \\
j^* j_! \mathrm{B}^{\mathrm{cyc}}(S/R) & \longrightarrow & \mathrm{B}^{\mathrm{cyc}}(S/R)
\end{array}$$

24

Taking colimits, we get the diagram

$$\begin{array}{ccc} \Sigma S & \longrightarrow & \Sigma^2 \overline{R} \\ \downarrow & & \downarrow \\ S_+^1 \otimes \mathrm{THH}(S/R) & \longrightarrow & \mathrm{THH}(S/R) \end{array}$$

where $S_+^1 \otimes \mathrm{THH}(S/R) \to \mathrm{THH}(S/R)$ is the circle action map (see, e.g., [Lod98, §7.1]). The result follows. $\qquad\square$

Remark 3.2.8. Rognes has sketched an alternative proof that v_{n+1} is detected in the homotopy of $\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})^{hS^1}$. The strategy is to consider the exact sequence in mod p homology:

$$H_*(\mathrm{Tot}^{\leqslant 1}) \to H_*(\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})) \xrightarrow{B} H_{*+1}(\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})).$$

One computes that $B\overline{\tau}_{n+1}$ is nonzero and hence does not lie in the kernel, i.e. does not arise in the homology of $\mathrm{Tot}^{\leqslant 1}$. It follows from an Adams spectral sequence argument that v_{n+1} must be detected in $\pi_* \mathrm{Tot}^{\leqslant 1}$.

4 The Canonical Vanishing and Finiteness Theorems

The purpose of this section is to prove the following theorems.

Theorem 4.0.1 (The Canonical Vanishing Theorem)**.** *Let $F(n+1)$ be any type $n+2$ complex. The maps*

$$\mathrm{can}\colon \mathrm{TC}^-(\mathrm{BP}\langle n\rangle) \to \mathrm{TP}(\mathrm{BP}\langle n\rangle)$$
$$\mathrm{can}\colon \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{hC_{p^k}} \to \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tC_{p^k}}$$

induce the zero map on $F(n+1)_*$ for $* \gg 0$, uniformly in k.

Theorem 4.0.2 (The Finiteness Theorem)**.** *Let $F(n+1)$ be a type $n+2$ complex. Then each group*

$$F(n+1)_* \mathrm{TC}^-(\mathrm{BP}\langle n\rangle), F(n+1)_* \mathrm{TP}(\mathrm{BP}\langle n\rangle),$$
$$F(n+1)_* \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{hC_{p^k}}, F(n+1)_* \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tC_{p^k}}$$

is finite in each dimension.

By a thick subcategory argument, it suffices to establish each of these results for a specific type $n+2$ complex. Throughout this section, we will choose $F(n)$ and $F(n+1)$ to be generalized Moore spectra of type $n+1$ and $n+2$, respectively, equipped with compatible homotopy commutative ring structures (such complexes exist by [Dev17]).

Both theorems are then an immediate consequence of the following more technical result:

Theorem 4.0.3 (The Approximation Theorem). *There are strongly convergent spectral sequences*

$$E_2 \Rightarrow F(n+1)_* \mathrm{TC}^-(\mathrm{BP}\langle n \rangle)$$

$$\hat{E}_2 \Rightarrow F(n+1)_* \mathrm{TP}(\mathrm{BP}\langle n \rangle)$$

$$E_{2,k} \Rightarrow F(n+1)_* \mathrm{THH}(\mathrm{BP}\langle n \rangle)^{hC_{p^k}}$$

$$\hat{E}_{2,k} \Rightarrow F(n+1)_* \mathrm{THH}(\mathrm{BP}\langle n \rangle)^{tC_{p^k}}$$

with the following properties:

(i) *Each spectral sequence is finite in each homotopy dimension.*

(ii) *The spectral sequences $\{E_r\}$ and $\{E_{r,k}\}$ are supported in nonnegative filtration, while the spectral sequences $\{\hat{E}_r\}$ and $\{\hat{E}_{r,k}\}$ are eventually supported in negative filtration, uniformly in k.*

(iii) *There are maps of spectral sequences*

$$E_r \to \hat{E}_r, \quad and \ E_{r,k} \to \hat{E}_{r,k}$$

converging to the canonical maps from homotopy to Tate fixed points.

Our strategy for proving the Approximation Theorem will be to modify the usual fixed point spectral sequences by incorporating descent data from the situation relative to MU. We review the general set-up for these spectral sequences in §4.1. In §4.2 we establish control over the descent spectral sequence for $F(n)_* \mathrm{THH}(\mathrm{BP}\langle n \rangle)$, and then leverage this knowledge in §4.3 to construct the spectral sequences needed for the Approximation Theorem.

4.1 Descent and fixed point spectral sequences

In this section we assume familiarity with the various notations and constructions in §B.

Construction 4.1.1 (Descent modified fixed point spectral sequences). Let $R \to S$ be a map of $\mathbb{E}_\infty$-rings. For an R-module M, We denote by $\{D_r(M)\}$ (or $\{D_r\}$ if M is understood) the corresponding descent spectral sequence. If R, S, and M are equipped with compatible S^1-actions, then we obtain an action of S^1 on the descent tower $\mathrm{desc}_S^{\geqslant *}(M)$. The towers

$$\mathrm{desc}_S^{\geqslant *}(M)^{hC_{p^k}}, \mathrm{desc}_S^{\geqslant *}(M)^{tC_{p^k}}, \mathrm{desc}_S^{\geqslant *}(M)^{hS^1}, \mathrm{desc}_S^{\geqslant *}(M)^{tS^1}$$

then give rise to spectral sequences denoted $\{D_{r,k}\}$, $\{\hat{D}_{r,k}\}$, $\{D_{r,S^1}\}$, and $\{\hat{D}_{r,S^1}\}$, respectively.

Remark 4.1.2. In the case when the fiber of $R \to S$ is 1-connective, which will hold in our application in this section, these spectral sequences conditionally converge by Propositions B.2.2, B.4.1, and B.4.2.

In practice, the E_2-page of each of these spectral sequences is easy to compute. Indeed, the associated graded of the descent tower is, in each weight, a shift of the totalization of a cosimplicial abelian group. The following lemma then applies:

Lemma 4.1.3. *Let $A^\bullet$ be a cosimplicial abelian group equipped with an S^1-action, and let Y denote its totalization. Then we have canonical equivalences*

$$\pi_*(Y^{hS^1}) \simeq \pi_*Y[t],$$
$$\pi_*(Y^{tS^1}) \simeq \pi_*Y[t^{\pm 1}]$$

Proof. The endomorphism space of $A^\bullet$ is discrete, so S^1 must act trivially on $A^\bullet$ and hence also on Y. The result follows; we do not to complete with respect to t since Y is bounded above. □

Remark 4.1.4. With notation as above, we can also compute $\pi_*(Y^{hC_{p^k}})$ and $\pi_*(Y^{tC_{p^k}})$ using the equivalences

$$Y^{hC_{p^k}} \simeq Y^{hS^1}/p^k,\ Y^{tC_{p^k}} \simeq Y^{tS^1}/p^k,$$

coming from the fact that Y is a $\mathbb{Z}$-module.

For example, if $p^k = 0$ in π_*Y, we get

$$\pi_*(Y^{hC_{p^k}}) = \pi_*Y[t] \otimes \Lambda(s),\ \pi_*(Y^{tC_{p^k}}) = \pi_*Y[t^{\pm 1}] \otimes \Lambda(s)$$

where $|s| = -1$.

4.2   Descent for $\mathrm{THH}(\mathrm{BP}\langle n \rangle)$

Our starting point will be to establish the following result about the descent spectral sequence for $F(n)_*\mathrm{THH}(\mathrm{BP}\langle n \rangle)$, where we are descending along the map

$$\mathrm{THH}(\mathrm{MU}) \to \mathrm{MU}.$$

Proposition 4.2.1. *The descent spectral sequence*

$$D_2 \Rightarrow F(n)_*\mathrm{THH}(\mathrm{BP}\langle n \rangle)$$

contains a permanent cycle $z \in D_2^{0,*}$ such that

   (i) *D_2 is a finitely-generated $\mathbb{Z}_{(p)}[z]$-module, and*

   (ii) *the image of z in $F(n)_*\mathrm{THH}(\mathrm{BP}\langle n \rangle/\mathrm{MU})$ is a power of $\sigma^2 v_{n+1}$.*

Proof. Combine the two lemmas below. □

Lemma 4.2.2. *The descent spectral sequence for* $\mathrm{THH}(\mathrm{BP}\langle n \rangle; \mathbb{F}_p)$ *collapses at the E_2-page as:*

$$\mathbb{F}_p[\sigma^2 v_{n+1}] \otimes \Lambda(\sigma \xi_1, ..., \sigma \xi_{n+1}) \cong \pi_*(\mathbb{F}_p \otimes_{\mathrm{BP}\langle n \rangle} \mathrm{THH}(\mathrm{BP}\langle n \rangle)).$$

There are no extension problems.

Proof. First we compute the answer, independently of the descent spectral sequence. We have

$$\mathrm{THH}(\mathrm{BP}\langle n \rangle; \mathbb{F}_p) \simeq \mathbb{F}_p \otimes_{\mathbb{F}_p \otimes \mathrm{BP}\langle n \rangle} \mathbb{F}_p.$$

The Künneth spectral sequence looks like

$$\Lambda(\sigma \xi_1, \sigma \xi_2, ...) \otimes \Gamma(\sigma \tau_{n+1}, \sigma \tau_{n+2}, ...) \Rightarrow \pi_* \mathrm{THH}(\mathrm{BP}\langle n \rangle; \mathbb{F}_p).$$

We have a pairing with the dual spectral sequence for Hochschild cohomology

$$\Lambda(\delta \xi_i) \otimes \mathbb{F}_p[\delta \tau_j : j \geqslant n + 1].$$

This spectral sequence has Bökstedt differentials given by,

$$d_{p-1}(\delta \xi_{i+1}) = (\delta \tau_i)^p$$

which forces the E_p-page of the Künneth spectral sequence to be

$$\Lambda(\sigma \xi_1, ..., \sigma \xi_{n+1}) \otimes \mathrm{TP}(\sigma \tau_{n+1}, \sigma \tau_{n+2}, ...).$$

But now everything left is a permanent cycle, so the spectral sequence collapses and the extension is resolved by the usual relation $\sigma \tau_i^p = \sigma \tau_{i+1}$.

Next, observe that the descent spectral sequence for $\mathrm{THH}(\mathrm{BP}\langle n \rangle; \mathbb{F}_p)$ along $\mathrm{THH}(\mathrm{MU}) \to \mathrm{MU}$ is isomorphic to the descent along $\mathrm{THH}(\mathrm{MU}; \mathbb{F}_p) \to \mathbb{F}_p$. In this case, the descent Hopf algebra is given by

$$\pi_*(\mathbb{F}_p \otimes_{\mathrm{THH}(\mathrm{MU}; \mathbb{F}_p)} \mathbb{F}_p) \simeq \mathbb{F}_p[\gamma_{p^i}(\sigma^2 \xi_1)] \otimes \mathbb{F}_p[\gamma_{p^i}(\sigma^2 x_j) : j \not\equiv -1 \bmod p],$$

Since the map

$$\mathrm{THH}(\mathrm{BP}\langle n \rangle / \mathrm{MU}; \mathbb{F}_p)_* \to \mathrm{THH}(\mathbb{F}_p / \mathrm{MU})_*$$

is injective, we may reduce the computation of the comodule structure on $\mathrm{THH}(\mathrm{BP}\langle n \rangle / \mathrm{MU}; \mathbb{F}_p)_*$ to the computation of the comodule structure on $\mathrm{THH}(\mathbb{F}_p / \mathrm{MU})_*$.

For this, we use the equivalence:

$$\begin{aligned}
\mathrm{THH}(\mathbb{F}_p / \mathrm{MU}) \otimes_{\mathrm{THH}(\mathbb{F}_p)} \mathrm{THH}(\mathbb{F}_p / \mathrm{MU}) &\simeq \mathrm{THH}(\mathbb{F}_p / \mathrm{MU}) \otimes_{\mathrm{THH}(\mathbb{F}_p)} \mathrm{THH}(\mathbb{F}_p) \otimes_{\mathrm{THH}(\mathrm{MU}; \mathbb{F}_p)} \mathbb{F}_p \\
&\simeq \mathrm{THH}(\mathbb{F}_p / \mathrm{MU}) \otimes_{\mathrm{THH}(\mathrm{MU}; \mathbb{F}_p)} \mathbb{F}_p \\
&\simeq \mathrm{THH}(\mathbb{F}_p / \mathrm{MU}) \otimes_{\mathbb{F}_p} (\mathbb{F}_p \otimes_{\mathrm{THH}(\mathrm{MU}; \mathbb{F}_p)} \mathbb{F}_p).
\end{aligned}$$

Thus, to compute the coaction $\psi(\gamma_{p^i}(\sigma^2 v_0))$, we need to compute where the element $1 \otimes \gamma_{p^i}(\sigma^2 v_0)$ ends up under the above equivalence. When $i = 0$, this is the same as $\sigma^2 v_0 \otimes 1$, which maps to the similarly named element in the target. Otherwise, observe that

$$\mathrm{THH}(\mathbb{F}_p / \mathrm{MU})_* \simeq (\mathrm{THH}(\mathbb{F}_p) \otimes_{\mathrm{THH}(\mathrm{MU}; \mathbb{F}_p)} \mathbb{F}_p)_* \simeq \mathrm{THH}(\mathbb{F}_p)_* \otimes_{\mathbb{F}_p} (\mathbb{F}_p \otimes_{\mathrm{THH}(\mathrm{MU}; \mathbb{F}_p)} \mathbb{F}_p)_*$$

and the elements $\gamma_{p^i}(\sigma^2 v_0)$, for $i > 0$, arise from the second tensor factor, as do the elements $\gamma_{p^i}(\sigma^2 x_j)$.

In other words, we have shown that $\mathrm{THH}(\mathbb{F}_p/\mathrm{MU})_*$ is coinduced from $\mathbb{F}_p[\sigma^2 v_0]$.

Restricting to the subcomodule $\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU})_*$, we can then compute that this is coextended from a smaller Hopf algebra. That is:

$$\mathbb{F}_p[\gamma_{p^i}\sigma^2 v_{n+1}]\otimes\mathbb{F}_p[\gamma_{p^i}(\sigma^2 x_j) : j \not\equiv -1 \bmod p] = \pi_*(\mathbb{F}_p\otimes_{\mathrm{THH}(\mathrm{MU};\mathbb{F}_p)}\mathbb{F}_p)\square_{\Gamma\{\sigma^2\xi_1,...,\sigma^2\xi_{n+1}\}}\mathbb{F}_p[\sigma^2 v_{n+1}]$$

By change of rings we conclude that

$$\mathrm{Ext}_{\pi_*(\mathbb{F}_p\otimes_{\mathrm{THH}(\mathrm{MU};\mathbb{F}_p)}\mathbb{F}_p)}(\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU};\mathbb{F}_p)_*) \simeq \mathrm{Ext}_{\Gamma\{\sigma^2\xi_1,...,\sigma^2\xi_{n+1}\}}[\sigma^2 v_{n+1}]$$
$$\simeq \mathbb{F}_p[\sigma^2 v_{n+1}]\otimes\Lambda(\sigma\xi_1,...,\sigma\xi_{n+1})$$

as desired. $\square$

Lemma 4.2.3. *We have a spectral sequence*

$$\mathbb{F}_p[v_0,...,v_n]/(v_0^{i_0},...,v_n^{i_n})\otimes\Lambda(\sigma\xi_1,...,\sigma\xi_{n+1})\otimes\mathbb{F}_p[\sigma^2 v_{n+1}]\Rightarrow D_2$$

where v_0 detects p and which collapses at a finite page.

Proof. The cobar complex computing

$$\mathrm{Ext}^{*,*}_{\pi_*(\mathrm{MU}\otimes_{\mathrm{THH}(\mathrm{MU})}\mathrm{MU})}(F(n)_*\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU}))$$

is, component-wise, a free, locally finite, $\mathrm{BP}\langle n\rangle_*/(p^{i_0},...,v_n^{i_n})$-module. Filtering by powers of the (nilpotent) ideal $(p,...,v_n)$ gives the indicated spectral sequence. $\square$

4.3 Proof of the Approximation Theorem

We now complete the proof of the approximation theorem by modding out our descent-modified fixed point spectral sequences by a v_{n+1}-element. The Detection Theorem tells us where to find such an element:

Lemma 4.3.1 (Detection Lemma)**.** *In the descent-modified fixed point spectral sequences*

$$D_{2,k} \Rightarrow F(n)_*\mathrm{THH}(\mathrm{BP}\langle n\rangle)^{hC_{p^k}}$$
$$\hat{D}_{2,k} \Rightarrow F(n)_*\mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tC_{p^k}}$$

the v_{n+1}-element $v \in F(n)_*$ is detected by an element of the form $t^j z^{j'}$.

Proof. It suffices by naturality to treat the case $k = \infty$. We have a diagram of descent-modified fixed point spectral sequences:

$$
\begin{array}{ccc}
D_2(F(n)\otimes\mathrm{THH}(\mathrm{MU}))[t] & \longrightarrow & F(n)\otimes\mathrm{TC}^-(\mathrm{MU}) \\
\downarrow & & \downarrow \\
D_2(F(n)\otimes\mathrm{THH}(\mathrm{BP}\langle n\rangle)) & \longrightarrow & F(n)\otimes\mathrm{TC}^-(\mathrm{BP}\langle n\rangle)
\end{array}
$$

In the first spectral sequence, the element $v \in F(n)_*$ is detected by a power $v_{n+1}^j \in \mathrm{MU}_*$, and we have already seen (Theorem 3.0.1) that, under the map $\mathrm{MU} \to \mathrm{TC}^-(\mathrm{BP}\langle n \rangle/\mathrm{MU})$, the element v_{n+1} maps to $t\sigma^2 v_{n+1}$. After possibly replacing v by a power of itself, the result follows. □

Construction 4.3.2. Consider the towers X and $\hat{X}$ used to construct the descent-modified fixed point spectral sequences associated to $F(n) \otimes \mathrm{THH}(\mathrm{BP}\langle n \rangle)$. We have seen that the v_{n+1}-element v is detected by an element in the same filtration degree as $t^j z^{j'}$, and hence, using the multiplicative structure on the tower, gives rise to filtration-shifting endomorphisms of X and $\hat{X}$. We will let $\{E_{r,k}\}$ and $\{\hat{E}_{r,k}\}$ denote the spectral sequences associated to the towers X/v and $\hat{X}/v$, respectively. Notice that, since $X^{\geqslant -j} = \mathrm{colim}\, X$ and $\hat{X}^{\geqslant -j} = \mathrm{colim}\, \hat{X}$ for $j \geqslant 0$, the target of these spectral sequences can be identified with $F(n+1) \otimes \mathrm{THH}(\mathrm{BP}\langle n \rangle)^{hC_{p^k}}$ and $F(n+1) \otimes \mathrm{THH}(\mathrm{BP}\langle n \rangle)^{tC_{p^k}}$, respectively. Both towers converge conditionally, since X and $\hat{X}$ converge conditionally.

Proof of the Approximation Theorem. We let $E_{2,k}$ and $\hat{E}_{2,k}$ be as in the above construction (omitting k when $k = \infty$). The naturality statement (iii) is immediate from the construction. The finiteness statement is immediate from the fact that D_2 is a finitely generated $\mathbb{Z}_{(p)}[z]$-module which is annihilated by a power of p. Moreover, this same finite generation of D_2 implies that, in large homotopy dimension, the element $z^{j'}$ is not a zero divisor. It follows that, in large homotopy dimension, we have

$$\hat{E}_2 = (D_2/z^{j'})[t^{\pm 1}]$$

The module $D_2/z^{j'}$ vanishes in large degrees, and t^{-1} lies in negative filtration, which implies that $\hat{E}_2$ eventually vanishes in nonnegative filtration. The same is true for $\hat{E}_{2,k}$ by Remark 4.1.4. It remains to check that $E_{2,k}$ is supported in nonnegative filtration. We will use notation as in the above construction of $E_{2,k}$. The element v lies in positive filtration, so, by the cofiber sequence defining X/v, it suffices to check that the descent-modified fixed point spectral sequence $D_{2,k}$ lies in nonnegative filtration. This is true for any connective S^1-equivariant module M. Indeed, by our grading conventions (B.1.1) we need to verify that

$$(\mathrm{desc}^{\geqslant t}(M)^{hC_{p^k}})/(\mathrm{desc}^{\geqslant t+1}(M)^{hC_{p^k}}) = \left((\mathrm{desc}^{\geqslant t}(M)/(\mathrm{desc}^{\geqslant t+1}(M))\right)^{hC_{p^k}}$$

is t-truncated. But t-truncated spectra are closed under limits, so it suffices to observe that $(\mathrm{desc}^{\geqslant t}(M)/(\mathrm{desc}^{\geqslant t+1}(M)))$ is t-truncated. This, in turn, is the totalization of a cosimplicial spectrum with each term the tth suspension of an Eilenberg-MacLane object, by construction, so the result follows. □

5 The Segal Conjecture

We fix throughout this section an $\mathbb{E}_3$-BP-algebra form of $\mathrm{BP}\langle n \rangle$, as usual. Our purpose is to prove the Segal Conjecture (Theorem G), which we restate here for convenience.

Theorem 5.0.1. *Let $F(n)$ denote any type $n + 1$ finite complex. Then the cyclotomic Frobenius* $\mathrm{THH}(\mathrm{BP}\langle n \rangle) \to \mathrm{THH}(\mathrm{BP}\langle n \rangle)^{tC_p}$ *induces an isomorphism*

$$F(n)_* \mathrm{THH}(\mathrm{BP}\langle n \rangle) \cong F(n)_*(\mathrm{THH}(\mathrm{BP}\langle n \rangle)^{tC_p})$$

in all sufficiently large degrees $* \gg 0$.

The idea of the proof is to use the (décalage of the) Adams filtration on $\mathrm{BP}\langle n \rangle$ to reduce the claim to a much simpler one about graded polynomial algebras over $\mathbb{F}_p$.

Remark 5.0.2. There are several antecedents to the Segal conjecture. First, the classical Segal conjecture for the group C_p states that the map

$$S^0 = \mathrm{THH}(S^0) \to \mathrm{THH}(S^0)^{tC_p} = (S^0)^{tC_p}$$

is p-completion. For various classes of ordinary commutative rings R, the map

$$\varphi : \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$$

is a p-adic equivalence in large degrees: this is the case for DVRs of mixed characteristic with perfect residue field in odd characteristic [HM03], for smooth algebras in positive characteristic [Hes18, Prop. 6.6], and for p-torsionfree excellent noetherian rings R with R/p finitely generated over its pth powers [Mat20, Cor. 5.3].

When $R = \ell$ is the Adams summand, it is proved in [AR02, Thm. 5.5] for $p \geqslant 5$ that

$$\varphi : \mathrm{THH}(\ell)/(p, v_1) \to \mathrm{THH}(\ell)^{tC_p}/(p, v_1)$$

is an equivalence in degrees larger than $2(p-1)$ (cf. [Lun05]). When $R = \mathrm{MU}$, Lunøe-Nielsen and Rognes show [LR11] that

$$\varphi : \mathrm{THH}(\mathrm{MU}) \to \mathrm{THH}(\mathrm{MU})^{tC_p}$$

is a p-adic equvalence, and in fact our argument for $\mathrm{BP}\langle n \rangle$ gives an alternative proof of this fact. In another direction, Angelini-Knoll and Quigley [AKQ21] have showed that φ is an equivalence for Ravenel's $X(n)$ spectra.

5.1   Polynomial rings over the sphere

In this preliminary section, we fix an even integer $2r$ and an integer w. We will define and develop a few properties of a specific graded $\mathbb{E}_2$-ring $S^0[S^{2r}]$. The underlying graded $\mathbb{E}_1$-ring will be the free graded $\mathbb{E}_1$-ring generated by S^{2r} in weight w, but we suppress w from the notation.

To define this graded $\mathbb{E}_2$-ring in general, we first define it in the case that $2r = 0$ and $w = 1$. In this case, we set $S^0[S^0]$ to be the graded $\mathbb{E}_2$-ring underlying $\Sigma_+^\infty \mathbb{N}$, which is naturally a graded $\mathbb{E}_\infty$-ring. Our definition in general closely follows the constructions over MU from Section 2.6.

We next define the graded $\mathbb{E}_2$-ring $S^0[S^{-2}]$ in the case $2r = -2$ and $w = -1$ to be the $\mathbb{E}_2$-Koszul dual of $\Sigma^\infty_+ \mathbb{N}$. By restricting and regrading, we may then define $S^0[S^{2r}]$ for any negative integer r and weight w. Finally, we define $S^0[S^{2r}]$, for $r > 0$ and any w, to be the $\mathbb{E}_2$-Koszul dual of $S^0[S^{-2-2r}]$, where S^{-2-2r} is considered in weight $-w$. The argument that $S^0[S^{2r}]$ has free underlying graded $\mathbb{E}_1$-ring is the same as that for Theorem 2.6.3(b). We also record for future use the following useful fact, proven in the same way as Theorem 2.6.3(a):

Lemma 5.1.1. *If $2r \geqslant 0$ is non-negative, and w is arbitrary, then $S^0[S^{2r}]$ admits a cell structure as a graded $\mathbb{E}_2$-algebra with all cells in even degrees.*

Finally, we will need the following:

Lemma 5.1.2. *The graded $\mathbb{E}_1$-ring map*

$$S^0[S^{2r}] \to \mathrm{THH}(S^0[S^{2r}])$$

induces on $\mathbb{F}_p$-homology the ring map

$$\mathbb{F}_p[a] \to \mathbb{F}_p[a] \otimes \Lambda_{\mathbb{F}_p}(\sigma a).$$

Here, the weights of a and σa are both w. Furthermore, as a graded C_p-spectrum $\mathrm{THH}(S^0[a])$ is pointwise finite.

Proof. The first part of the claim follows from the expression

$$\mathrm{THH}(S^0[S^{2r}]) \simeq S^0[S^{2r}] \otimes_{S^0[S^{2r}] \otimes S^0[S^{2r}]} S^0[S^{2r}].$$

To see that $\mathrm{THH}(S^0[S^{2r}])$ is pointwise finite as a graded C_p-spectrum, recall that the spectrum $\mathrm{THH}(S^0[S^{2r}])$ may be calculated as the geometric realization of the simplicial object

$$\cdots \substack{\longrightarrow \\ \longleftarrow \\ \Longrightarrow \\ \longleftarrow \\ \longrightarrow} S^0[a]^{\otimes 3p} \substack{\longrightarrow \\ \Longleftarrow \\ \Longrightarrow \\ \longleftarrow} S^0[a]^{\otimes 2p} \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} S^0[a]^{\otimes p},$$

which is obtained as the p-fold subdivision of the cyclic bar construction. The realization of this simplicial object may be calculated as the filtered colimit of partial realizations $S^0[a]^{\otimes p} \to \mathrm{Bar}_{\leqslant 1} \to \mathrm{Bar}_{\leqslant 2} \to \cdots$. Each object in this sequence is a pointwise finite graded C_p spectrum, because it is a finite colimit of pointwise finite graded C_p spectra. Furthermore, at each individual weight the filtered colimit stabilizes at a finite stage. $\square$

5.2   The Segal conjecture for polynomial $\mathbb{F}_p$-algebras

We consider in this section a graded $\mathbb{E}_2$-$\mathbb{F}_p$-algebra R, with homotopy groups a polynomial ring

$$\pi_*(R) \cong \mathbb{F}_p[a_1, a_2, \cdots, a_n].$$

Each a_i will have non-negative even degree $|a_i|$ and weight $\mathrm{wt}(a_i)$, though we supress the weights from the notation. In fact, there is a unique ring R with the above description:

Proposition 5.2.1. *As a graded $\mathbb{E}_2$-$\mathbb{F}_p$-algebra, the ring R above must be equivalent to*

$$\mathbb{F}_p \otimes S^0[S^{|a_1|}] \otimes S^0[S^{|a_2|}] \otimes \cdots \otimes S^0[S^{|a_n|}].$$

Proof. Since each $S^0[S^{|a_i|}]$ admits an even cell decomposition as a graded $\mathbb{E}_2$-ring (Lemma 5.1.1), their tensor product does as well. It is thus unobstructed to produce a map of graded $\mathbb{E}_2$-rings

$$S^0[S^{|a_1|}] \otimes S^0[S^{|a_2|}] \otimes \cdots \otimes S^0[S^{|a_n|}] \to R$$

which on homotopy groups has $a_1, a_2, \cdots, a_n$ in its image. Tensoring this up to a map of graded $\mathbb{E}_2$-$\mathbb{F}_p$-algebras produces the desired equivalence. $\square$

Our main theorem about this $\mathbb{E}_2$-$\mathbb{F}_p$-algebra R is as follows:

Proposition 5.2.2. *The cyclotomic frobenius*

$$\mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$$

induces on homotopy groups the ring map

$$\mathbb{F}_p[x, a_1, a_2, \cdots, a_n] \otimes \Lambda(\sigma a_1, \sigma a_2, \cdots, \sigma a_n) \to \mathbb{F}_p[x^{\pm}, a_1, a_2, \cdots, a_n] \otimes \Lambda(\sigma a_1, \sigma a_2, \cdots, \sigma a_n)$$

that inverts x. Here, x is in degree 2 and weight 0. The degree of σa_i is one more than the degree of a_i, and the weight of σa_i is the same as the weight of a_i.

A version of the above is well-known in the case that all a_i are in degree 0, so R is a classical commutative ring (see, e.g., [Hes18, 6.6] for a much stronger result). Our main observation is that the result extends to the case where not all a_i are in degree 0, in which case R is not Eilenberg–Maclane. Since an exterior algebra on classes of degree $|a_i| + 1$ has no homotopy above degree $n + \sum |a_i|$, the following result immediately follows:

Corollary 5.2.3 (Segal conjecture for graded polynomial $\mathbb{F}_p$-algebras). *The map*

$$\pi_*(\mathrm{THH}(R)/(a_1, \cdots, a_n)) \to \pi_*(\mathrm{THH}(R)^{tC_p}/(a_1, \cdots, a_n))$$

is an equivalence in degrees $* > n + \sum_{i=1}^{n} |a_i|$.

Proof of Propostion 5.2.2. Suppose without loss of generality that $a_1, a_2, \cdots, a_\ell$ are in degree 0, while $a_{\ell+1}, \cdots, a_n$ are in positive degree. By Proposition 5.2.1, we may assume that R is a tensor product of graded $\mathbb{E}_2$-rings

$$R \simeq \mathbb{F}_p[a_1, a_2, \cdots, a_\ell] \otimes S^0[a_{\ell+1}] \otimes \cdots \otimes S^0[a_n].$$

Since THH is symmetric monoidal as a functor to cyclotomic spectra [NS18, p.341], we may compute

$$\mathrm{THH}(R) \simeq \mathrm{THH}(\mathbb{F}_p[a_1, \cdots, a_\ell]) \otimes \mathrm{THH}(S^0[a_{\ell+1}]) \otimes \mathrm{THH}(S^0[a_2]) \otimes \cdots \otimes \mathrm{THH}(S^0[a_n])$$

33

as a C_p-equivariant $\mathbb{E}_1$-ring spectrum. We next compute the cycloctomic Frobenius on each component of the above tensor product.

If $r > 0$, then at least as ungraded $\mathbb{E}_1$-rings $S^0[S^{2r}] \simeq \Sigma_+^\infty \Omega S^{2r+1}$. By e.g. [NS18, IV.3.7], it follows that the cyclotomic Frobenius

$$\mathrm{THH}(S^0[a]) \to \mathrm{THH}(S^0[a])^{tC_p}$$

is p-completion.

Since we assumed $a_1, \cdots, a_\ell$ are in degree 0, the cyclotomic Frobenius

$$\mathrm{THH}(\mathbb{F}_p[a_1, \cdots, a_\ell]) \to \mathrm{THH}(\mathbb{F}_p[a_1, \cdots, a_\ell])^{tC_p}$$

may be read off from [BMS19, Corollary 8.18] to be

$$\mathbb{F}_p[x, a_1, \cdots, a_\ell] \otimes \Lambda(\sigma a_1, \cdots, \sigma a_\ell) \to \mathbb{F}_p[x^\pm, a_1, \cdots, a_\ell] \otimes \Lambda(\sigma a_1, \cdots, \sigma a_\ell).$$

After applying the first part of Lemma 5.1.2, the remainder of our claim here is then simply that $\mathrm{THH}(R)^{tC_p}$ is equivalent to the tensor product

$$\mathrm{THH}(\mathbb{F}_p[a_0, \cdots, a_\ell])^{tC_p} \otimes \mathrm{THH}(S^0[a_{\ell+1}])^{tC_p} \otimes \mathrm{THH}(S^0[a_{\ell+2}])^{tC_p} \otimes \cdots \otimes \mathrm{THH}(S^0[a_n])^{tC_p}.$$

This is not obvious, since in general the Tate construction is only lax symmetric monoidal. However, the Tate construction is exact, and so $(X \otimes Y)^{tC_p} \simeq X^{tC_p} \otimes Y^{tC_p}$ whenever Y is a *finite* C_p-spectrum. The result thus follows from the second part of Lemma 5.1.2, which states in each weight that each $\mathrm{THH}(S^0[a_i])$ is finite. $\square$

5.3 The Segal conjecture for $\mathrm{BP}\langle n \rangle$

The key to the proof of Theorem G is the following:

Theorem 5.3.1. *The map of* BP*-algebras*

$$\mathrm{THH}(\mathrm{BP}\langle n \rangle)/(p, v_1, v_2, \cdots, v_n) \to \mathrm{THH}(\mathrm{BP}\langle n \rangle)^{tC_p}/(p, v_1, v_2, \cdots, v_n)$$

is an equivalence in large degrees.

Proof. Consider the filtered $\mathbb{E}_2$-algebra $\mathrm{desc}_{\mathbb{F}_p}^{\geqslant \bullet} \mathrm{BP}\langle n \rangle$. There is then a map of filtered $\mathbb{E}_1$-$\mathrm{desc}_{\mathbb{F}_p}^{\geqslant \bullet} \mathrm{BP}\langle n \rangle$-algebras

$$\mathrm{THH}(\mathrm{desc}_{\mathbb{F}_p}^{\geqslant \bullet} \mathrm{BP}\langle n \rangle) \to \mathrm{THH}(\mathrm{desc}_{\mathbb{F}_p}^{\geqslant p \bullet} \mathrm{BP}\langle n \rangle)^{tC_p}$$

By Proposition B.5.4, the towers $\mathrm{THH}(\mathrm{desc}_{\mathbb{F}_p}^{\geqslant \bullet} \mathrm{BP}\langle n \rangle)$ and $\mathrm{THH}(\mathrm{desc}_{\mathbb{F}_p}^{\geqslant p \bullet} \mathrm{BP}\langle n \rangle)^{tC_p}$ are conditionally convergent, after v_0-completion. So we can check the claim on associated graded.

Upon taking associated graded, we obtain a map of graded $\mathbb{E}_1$-$\mathbb{F}_p[v_0, v_1, \cdots, v_n]$-algebras

$$\mathrm{THH}(\mathbb{F}_p[v_0, v_1, \cdots, v_n]) \to \mathrm{THH}(\mathbb{F}_p[v_0, v_1, \cdots, v_n])^{tC_p},$$

and it follows from Corollary 5.2.3 that, modulo $v_0, v_1, \cdots, v_n$, this map is an equivalence in large degrees. This completes the proof. $\square$

From a thick subcategory argument in BP-modules, we then learn the following

Corollary 5.3.2. *For any positive integers* $i_0, i_1, \cdots, i_n$, *the map of* BP-*algebras*

$$\mathrm{THH}(\mathrm{BP}\langle n\rangle)/(p^{i_0}, v_1^{i_1}, v_2^{i_2}, \cdots, v_n^{i_n}) \to \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tC_p}/(p^{i_0}, v_1^{i_1}, v_2^{i_2}, \cdots, v_n^{i_n})$$

is an equivalence in large degrees.

In particular, if we let $M(n)$ *denote a generalized Moore spectrum* $S^0/(p^{i_0}, v_1^{i_1}, \cdots, v_n^{i_n})$, *then*

$$M(n)_* \mathrm{THH}(\mathrm{BP}\langle n\rangle) \to M(n)_* \mathrm{THH}(\mathrm{BP}\langle n\rangle)^{tC_p}$$

is an equivalence in large degrees

The Segal conjecture (Theorem G) now follows by a thick subcategory argument in spectra, since any $M(n)$ generates the thick subcategory of type $n+1$ spectra.

6   Redshift

Fix an $\mathbb{E}_3$-BP-algebra form of $\mathrm{BP}\langle n\rangle$. In this section, we will prove, via the conglomeration of all our work above, the following theorem:

Theorem 6.0.1. *For any type* $n+2$ *complex* $F(n+1)$, *the following statements are true:*

(i) $F(n+1)_* \mathrm{TR}(\mathrm{BP}\langle n\rangle)$ *is bounded above.*

(ii) $F(n+1)_* \mathrm{TC}(\mathrm{BP}\langle n\rangle)$ *is finite.*

(iii) $F(n+1)_* \mathrm{K}(\mathrm{BP}\langle n\rangle)$ *is finite.*

(iv) $F(n+1)_* \mathrm{K}(\mathrm{BP}\langle n\rangle_p^\wedge)$ *is finite.*

Remark 6.0.2. We do not expect $F(n+1)_* \mathrm{TR}(\mathrm{BP}\langle n\rangle)$ to be finite. Indeed, in the case of $\mathrm{BP}\langle 1\rangle = \ell$ at primes $p \geqslant 5$, the explicit computations of Ausoni and Rognes show that $V(2)_* \mathrm{TR}(\mathrm{BP}\langle 1\rangle)$ is infinite, even though it is concentrated in finitely many degrees [AR02, Theorem 8.5].

As a corollary, we obtain the following results:

Corollary 6.0.3. *Let* $F(n)$ *be any type* $n+1$ *complex. The maps*

$$F(n)_* \mathrm{TR}(\mathrm{BP}\langle n\rangle) \to v_{n+1}^{-1} F(n)_* \mathrm{TR}(\mathrm{BP}\langle n\rangle),$$

$$F(n)_* \mathrm{TC}(\mathrm{BP}\langle n\rangle) \to v_{n+1}^{-1} F(n)_* \mathrm{TC}(\mathrm{BP}\langle n\rangle),$$

$$F(n)_* \mathrm{K}(\mathrm{BP}\langle n\rangle) \to v_{n+1}^{-1} F(n)_* \mathrm{K}(\mathrm{BP}\langle n\rangle)$$

$$F(n)_* \mathrm{K}(\mathrm{BP}\langle n\rangle_p^\wedge) \to v_{n+1}^{-1} F(n)_* \mathrm{K}(\mathrm{BP}\langle n\rangle_p^\wedge)$$

are isomorphisms in all sufficiently large degrees $* \gg 0$.

Proof. Let v denote a self map of $F(n)$ such that $F(n)/v$ is a type $n+2$ complex; by definiton, $v_{n+1}^{-1}F(n) = v^{-1}F(n)$. The statements can fail only if there are large degree elements in $F(n)_*\mathrm{TR}(\mathrm{BP}\langle n\rangle), F(n)_*\mathrm{TC}(\mathrm{BP}\langle n\rangle)$, or $F(n)_*\mathrm{K}(\mathrm{BP}\langle n\rangle)$ that are v-torsion (in which case injectivity fails) or large degree elements that are not v-divisible (in which case surjectivity fails). Both of these cannot happen because of the long exact sequence calculating $F(n)/v$ homology from $F(n)$ homology. $\square$

Corollary 6.0.4. *As modules over the Steenrod algebra,*

$$H^*\left(\mathrm{K}(\mathrm{BP}\langle n\rangle); \mathbb{F}_p\right), \ \ and$$

$$H^*\left(\mathrm{K}(\mathrm{BP}\langle n\rangle_p^\wedge); \mathbb{F}_p\right)$$

are finitely presented. In particular, the p-completion of $\mathrm{K}(\mathrm{BP}\langle n\rangle)$ is an fp spectrum in the sense of Mahowald and Rezk, of fp-type $n+1$.

Proof. By [MR99, Proposition 3.2], Theorem 6.0.1 implies finite presentation in cohomology. Combining this Corollary 3.0.3, we conclude that both $\mathrm{K}(\mathrm{BP}\langle n\rangle)_p^\wedge$ and $\mathrm{K}(\mathrm{BP}\langle n\rangle_p^\wedge)_p^\wedge$ are of fp-type exactly $n+1$. $\square$

Corollary 6.0.5. *The maps*

$$\mathrm{K}(\mathrm{BP}\langle n\rangle) \to L_{n+1}^f\mathrm{K}(\mathrm{BP}\langle n\rangle), \ \ and$$

$$\mathrm{K}(\mathrm{BP}\langle n\rangle_p^\wedge) \to L_{n+1}^f\mathrm{K}(\mathrm{BP}\langle n\rangle_p^\wedge), \ \ and$$

are p-local equivalences in sufficiently large degrees.

Proof. First observe that the localization $Y \to L_{n+1}^f Y$ is always a rational equivalence, so it suffices to prove the claim after p-completion. But this is true for any connective fp-spectrum of fp-type $n+1$, as we now explain. Let X be of fp-type $n+1$, and consider the fiber, C, of the localization map

$$X \to L_{n+1}^f X$$

Mahowald and Rezk prove [MR99, Theorem 8.2] that C is bounded above, whence the claim. $\square$

Remark 6.0.6. With slightly more computational control over $K(\mathrm{BP}\langle n\rangle)$ one could compute in what range the localization map is an equivalence. If X is an fp-spectrum of fp-type $n+1$ then we can write $H^*(X) = \mathcal{A}^* \otimes_{\mathcal{A}(k)^*} M$ for some k and some finite-dimensional M. Then $X \to L_{n+1}^f X$ is a p-adic equivalence above dimension $a(k) - d$ where $a(k)$ is the dimension of the top cell of $\mathcal{A}(k)^*$ and d is the dimension of the top cell in M.

The key to Theorem 6.0.1, given our work in previous sections, is the following proposition:

Proposition 6.0.7. *Suppose that A is an $\mathbb{E}_1$-ring such that*

36

(i) *(Segal conjecture) The fiber of $F(n) \otimes \mathrm{THH}(A) \to F(n) \otimes \mathrm{THH}(A)^{tC_p}$ is truncated.*

(ii) *(Canonical vanishing) For every $k \geqslant 1$, the canonical map* can : $\mathrm{THH}(A)^{tC_{p^k}} \to \mathrm{THH}(A)^{tC_{p^k}}$ *vanishes in large degrees on $F(n+1)$-homology, independently of k.*

Then $F(n+1) \otimes \mathrm{TR}(A)$ and $F(n+1) \otimes \mathrm{TC}(A)$ are truncated. If, moreover, $F(n+1) \otimes K(\pi_0 A)$ and $F(n+1) \otimes \mathrm{TC}(\pi_0 A)$ are truncated, then $F(n+1) \otimes K(A)$ is truncated.

Proof. We argue as in [Mat20]. Recall that we have pullback squares

$$
\begin{array}{ccc}
\mathrm{THH}(A)^{C_{p^k}} & \longrightarrow & \mathrm{THH}(A)^{hC_{p^k}} \\
{\scriptstyle R}\downarrow & & \downarrow{\scriptstyle \mathrm{can}} \\
\mathrm{THH}(A)^{C_{p^{k-1}}} & \longrightarrow & \mathrm{THH}(A)^{tC_{p^k}}
\end{array}
$$

The bottom horizontal map is an equivalence on $F(n+1)$-homology in large degrees, independent of k, by Tsalidis's theorem [NS18, II.4.9] and (i). The right hand vertical map is zero on $F(n+1)$-homology in large degrees, by (ii). It follows that the left hand vertical map is zero on $F(n+1)$-homology in large degrees, and hence that the limit has vanishing $F(n+1)$-homology in large degrees.

We obtain $\mathrm{TC}(A)$ as the fiber of a self-map of $\mathrm{TR}(A)$, so we conclude that $F(n+1) \otimes \mathrm{TC}(A)$ is also truncated. The final claim follows from the Dundas-Goodwillie-McCarthy theorem [DGM13, Theorem 2.2.1]. $\square$

Proof of Theorem 6.0.1. We combine the above proposition with Corollary 5.3.2, Theorem 4.0.1, and Theorem 4.0.2. The only thing remaining to check is that each of

$$F(n+1)_* K(\mathbb{Z}_p),$$

$$F(n+1)_* K(\mathbb{Z}_{(p)}),$$

$$F(n+1)_* \mathrm{TC}(\mathbb{Z}_p), \text{ and}$$

$$F(n+1)_* \mathrm{TC}(\mathbb{Z}_{(p)})$$

are finite (i.e., finite in each degree and concentrated in finitely many degrees). This follows from the Lichtenbaum–Quillen conjecture for classical rings, proven for $\mathbb{Z}_p$ by Hesselholt and Madsen [HM03] and for $\mathbb{Z}_{(p)}$ by Voevodsky [Voe03, Voe11]. $\square$

It is natural to ask about Lichtenbaum–Quillen style results for more general classes of ring spectra, such as $\mathrm{BP}\langle n \rangle$-algebras. As mentioned in the introduction, it is not clear in what generality one should expect versions of the Segal conjecture or Canonical Vanishing theorems to hold. That said, given our results about $\mathrm{BP}\langle n \rangle$, it turns out that the Segal conjecture alone suffices to prove several Lichtenbaum–Quillen style results. The following two results and their proofs were communicated to us by Akhil Mathew, who has graciously allowed us to reproduce them here. Any errors in transcription are due to the present authors.

Proposition 6.0.8 (Mathew)**.** *Suppose M is p-cyclotomic and that $\mathrm{TR}(M) \in \mathsf{Fun}(\mathrm{B}S^1, \mathsf{Sp})$ is in the thick tensor ideal generated by $\mathbb{F}_p$. Then the following are equivalent:*

(i) *(Segal conjecture) The map $\varphi : M \to M^{tC_p}$ is truncated (i.e. an equivalence in large degrees).*

(ii) $\mathrm{TR}(M)$ *is truncated.*

Proof. That (ii) implies (i) is [AN21, Proposition 2.25].

To prove that (i) implies (ii), it suffices to prove that $V : \mathrm{TR}(M)_{hC_p} \to \mathrm{TR}(M)$ is surjective in large degrees since the homotopy groups $\pi_* \mathrm{TR}(M)$ are derived V-complete by [AN21].

Recall that we also have the map $F : \mathrm{TR}(M) \to \mathrm{TR}(M)^{hC_p}$, which fits into a pullback diagram

$$
\begin{array}{ccc}
\mathrm{TR}(M) & \longrightarrow & M \\
\downarrow & & \downarrow{\scriptstyle \varphi} \\
\mathrm{TR}(M)^{hC_p} & \longrightarrow & M^{tC_p}
\end{array}
$$

Since the right hand vertical arrow is truncated, so is the left hand vertical arrow.

Thus, to prove that V is surjective in large degrees, it suffices to show that the composite

$$FV = \mathrm{trace} : \mathrm{TR}(M)_{hC_p} \to \mathrm{TR}(M)^{hC_p}$$

is surjective in large degrees. Denote by $(-)^{[i]}_{hC_p}$ the skeletal filtration of the functor $(-)_{hC_p}$, and consider the diagram

$$
\begin{array}{ccccc}
(\mathrm{TR}(M))^{[i]}_{hC_p} & \longrightarrow & \mathrm{TR}(M)_{hC_p} & \longrightarrow & \mathrm{TR}(M)^{hC_p} \\
{\scriptstyle F}\downarrow & & {\scriptstyle F}\downarrow & & {\scriptstyle F}\downarrow \\
(\mathrm{TR}(M)^{hC_p})^{[i]}_{hC_p} & \longrightarrow & (\mathrm{TR}(M)^{hC_p})_{hC_p} & \longrightarrow & (\mathrm{TR}(M)^{hC_p})^{hC_p}
\end{array}
$$

Since F is truncated, we have that, for each i, the leftmost and rightmost vertical arrows are truncated. So it suffices to show that, for some i, the bottom horizontal composite is surjective on homotopy in large degrees. In fact, we will show that, for some i, the map

$$(\mathrm{TR}(M)^{hC_p})^{[i]}_{hC_p} \to (\mathrm{TR}(M)^{hC_p})^{hC_p}$$

has a retract. It is enough to show that $\mathrm{TR}(M)^{hC_p}$, as a spectrum with C_p-action, lies in the thick tensor ideal generated by C_{p+}. So we are reduced to proving that: if $X \in \mathsf{Fun}(\mathrm{B}C_{p^2}, \mathsf{Sp})$ lies in the thick tensor ideal generated by $\mathbb{F}_p$, then X^{hC_p} lies in the thick tensor ideal generated by C_{p+} or, equivalently, that $(X^{hC_p})^{tC_p} = 0$.

The assumption implies that X is a retract of $\mathrm{Tot}^{\leqslant N}(X \otimes \mathrm{cb}^\bullet(\mathbb{F}_p))$ so we are reduced to the case when X is an $\mathbb{F}_p$-module, and then X^{hC_p} is an $\mathbb{F}_p^{hC_p}$-module. But then X^{hC_p} is a retract of $\mathbb{F}_p^{hC_p} \otimes X^{hC_p}$, so it suffices to prove that $\mathbb{F}_p^{hC_p}$ lies in the tensor ideal generated by C_{p+}. By [NS18, Lemma I.2.2] we know that $(\mathbb{F}_p^{hC_p})^{tC_p} = 0$, and this implies the result by [MNN17, Theorem 4.19]. $\qquad\square$

Corollary 6.0.9 (Mathew)**.** *Suppose that A is a $\mathrm{BP}\langle n \rangle$-algebra such that $\varphi : F(n+1) \otimes \mathrm{THH}(A) \to F(n+1) \otimes \mathrm{THH}(A)^{tC_p}$ is an equivalence in large degrees. Then $F(n+1) \otimes \mathrm{TR}(A)$ is truncated.*

Proof. It suffices, by the previous proposition, to check that $F(n+1) \otimes \mathrm{TR}(A)$ lies in the thick tensor ideal generated by $\mathbb{F}_p$. Since $\mathrm{TR}(A)$ is a module over $\mathrm{TR}(\mathrm{BP}\langle n \rangle)$, we are reduced to the case $A = \mathrm{BP}\langle n \rangle$. But we have seen that $F(n+1) \otimes \mathrm{TR}(\mathrm{BP}\langle n \rangle)$ is concentrated in finitely many degrees, so it suffices, by considering the Postnikov tower, to show that each homotopy group lies in the thick tensor ideal generated by $\mathbb{F}_p$. This, in turn, follows from the fact that each homotopy group is annihilated by some p^N, and hence is a module over $\mathbb{Z}/p^N$. $\qquad \square$

A Graded Koszul Duality

Given an augmented $\mathbb{E}_1$-MU-algebra R, the bar construction $\mathrm{MU} \otimes_R \mathrm{MU}$ is naturally an $\mathbb{E}_1$-MU-coalgebra. The MU-linear dual of this coalgebra is then an $\mathbb{E}_1$-MU-algebra, said to be *Koszul dual* to the original MU-algebra R.

Similarly, if R is an augmented $\mathbb{E}_n$-MU-algebra, then the n-fold iterated bar construction is an $\mathbb{E}_n$-MU-coalgebra, and its MU-linear dual is an $\mathbb{E}_n$-MU-algebra. In the language of Section 2.3, this Koszul dual algebra is the centralizer

$$\mathfrak{Z}_{\mathbb{E}_n - \mathrm{MU}}(R \to \mathrm{MU})$$

of the augmentation.

In Section 2.6 of the main paper, we construct certain designer $\mathbb{E}_n$-MU-algebras $\mathrm{MU}\{x\}$ as the $\mathbb{E}_n$-Koszul duals of more familiar $\mathbb{E}_\infty$-MU-algebras $\mathrm{MU}[y]$. In this appendix, we recall and develop some basic features of the Koszul duality construction; a key point is that the construction is often better behaved in the presence of an auxiliary weight grading.

A.1 Statement of results

Definition A.1.1. Let k be an $\mathbb{E}_\infty$-ring. The category of graded k-modules is defined by

$$\mathsf{grMod}_k := \mathsf{Fun}(\mathbb{Z}^{\mathrm{ds}}, \mathsf{Mod}_k),$$

where $\mathbb{Z}^{\mathrm{ds}}$ denotes the integers viewed as a 0-category. The ∞-category grMod_k is a presentably symmetric monoidal category under Day convolution. If M is a graded k-module, we will denote by M_i its values at i, and by $M(n)$ the precomposition with addition by $-n$ (so that $M(n)_i = M_{i-n}$). We will refer to the grading as the **weight** throughout.

We now recall the general setup of the duality pairing for algebras over a coherent operad.

Construction A.1.2. Let $\mathcal{C}$ be a presentably symmetric monoidal category, and $\mathcal{O}$ a coherent operad. Let $p_0, p_1 : \mathsf{Alg}_\mathcal{O}(\mathcal{C})^{\times 2} \to \mathsf{Alg}_\mathcal{O}(\mathcal{C})$ be the two projections and let $m : \mathsf{Alg}_\mathcal{O}(\mathcal{C})^{\times 2} \to$

$\mathsf{Alg}_{\mathcal{O}}(\mathcal{C})$ be the tensor product (inherited from $\mathcal{C}$). Since the unit $\mathbf{1}$ is initial, we have natural transformations $p_0 \to m \leftarrow p_1$, and hence a functor

$$\mathsf{Alg}_{\mathcal{O}}(\mathcal{C})^{\times 2} \to \mathsf{Fun}([0 < 1] \amalg_{\{1\}} [0 < 1], \mathsf{Alg}_{\mathcal{O}}(\mathcal{C})).$$

We define the category of pairings by the pullback diagram:

$$
\begin{array}{ccc}
\mathsf{Pair}^{\mathcal{O}} & \longrightarrow & \mathsf{Fun}([0 < 1] \amalg_{\{1\}} [0 < 1], \mathsf{Alg}_{\mathcal{O}}^{\mathrm{aug}}(\mathcal{C})) \\
\downarrow & & \downarrow \\
\mathsf{Alg}_{\mathcal{O}}(\mathcal{C})^{\times 2} & \longrightarrow & \mathsf{Fun}([0 < 1] \amalg_{\{1\}} [0 < 1], \mathsf{Alg}_{\mathcal{O}}(\mathcal{C}))
\end{array}
$$

Informally, an object of $\mathsf{Pair}^{\mathcal{O}}$ cosnsists of $\mathcal{O}$-algebras A and B together with a map $A \otimes B \to \mathbf{1}$ of $\mathcal{O}$-algebras.

It follows from the existence of centralizers, [Lur17, 5.3.1.14], that the forgetful functor

$$\mathsf{Pair}^{\mathcal{O}} \longrightarrow \mathsf{Alg}_{\mathcal{O}}^{\mathrm{aug}}(\mathcal{C}) \times \mathsf{Alg}_{\mathcal{O}}^{\mathrm{aug}}(\mathcal{C})$$

is both left and right representable, in the sense of [Lur17, 5.2.1.8], and hence we may extract a self-right-adjoint functor

$$\mathbb{D}^{\mathcal{O}} : \mathsf{Alg}_{\mathcal{O}}^{\mathrm{aug}}(\mathcal{C})^{\mathrm{op}} \longrightarrow \mathsf{Alg}_{\mathcal{O}}^{\mathrm{aug}}(\mathcal{C}).$$

By construction, we have

$$\mathbb{D}^{\mathcal{O}}(A) = \mathfrak{Z}_{\mathcal{O}}(A \to \mathbf{1}),$$

i.e. $\mathbb{D}^{\mathcal{O}}(A)$ is final amongst $\mathcal{O}$-algebras B equipped with a map $A \otimes B \to \mathbf{1}$ extending the augmentation on A. In the case when $\mathcal{O} = \mathbb{E}_n$, we abbreviate $\mathbb{D}^{\mathbb{E}_n}$ as $\mathbb{D}^{(n)}$.

We will need various (co)connectivity conditions and finiteness conditions.

Definition A.1.3.   • Let A be an augmented, graded $\mathbb{E}_n$-k-algebra. We say that A is **weight-connected** (resp. **weight-coconnected**) if the fiber of the augmentation $A \to k$ is concentrated in positive grading (resp. negative grading). We denote the corresponding categories with superscripts $\mathrm{wt} - \mathrm{cn}$ and $\mathrm{wt} - \mathrm{ccn}$, respectively.

- We denote by $\mathsf{grMod}_k^{\mathrm{wt} \geqslant n}$ (resp. $\mathsf{grMod}_k^{\mathrm{wt} \leqslant n}$) the full subcategory of graded k-modules concentrated in weights at least n (resp. at most n). We will write $M \geqslant n$ (resp. $M \leqslant n$) to indicate that M belongs to this subcategory.

- We say that a graded k-module M is **pointwise perfect** if M_i is a perfect k-module for each $i \in \mathbb{Z}$. We denote the category of pointwise perfect k-modules by $\mathsf{grMod}_k^{\mathrm{pperf}}$.

Remark A.1.4. Observe that the map $-1 : \mathbb{Z}^{\mathrm{ds}} \to \mathbb{Z}^{\mathrm{ds}}$ is a symmetric monoidal equivalence, and hence induces a symmetric monoidal equivalence on the category of graded k-modules, algebras, etc. It follows that any result about weight-connected algebras, or modules of weight bounded below by n, has a counterpart for weight-coconnected algebras or modules of weight bounded above by $-n$. Note also that this regrading intertwines the Koszul duality functor $\mathbb{D}^{(n)}$.

Now we can state the main theorem of this appendix:

Theorem A.1.5. *Koszul duality restricts to an equivalence*

$$\mathbb{D}^{(n)} : \mathsf{Alg}_{\mathbb{E}_n}^{\mathrm{aug,wt-cn}}(\mathsf{grMod}_k^{\mathrm{pperf}})^{\mathrm{op}} \xrightarrow{\simeq} \mathsf{Alg}_{\mathbb{E}_n}^{\mathrm{aug,wt-ccn}}(\mathsf{grMod}_k^{\mathrm{pperf}}).$$

Before proving this we record some corollaries.

Corollary A.1.6. *Let A be an augmented, graded, pointwise perfect $\mathbb{E}_n$-algebra. If A is either weight connected or weight coconnected, and $j \leqslant n$, then the natural map*

$$\mathrm{Bar}^{(j)}(\mathbb{D}^{(n)}(A)) \to \mathbb{D}^{n-j}(A)$$

is an equivalence.

Proof. By Theorem A.1.5 (and Proposition A.2.2 below), it suffices to check this after composing both sides with $\mathbb{D}^{(n-j)}$. Then, by [Lur17, 5.2.3.13], $\mathbb{D}^{(n-j)} \circ \mathrm{Bar}^{(j)} \simeq \mathbb{D}^{(n)}$, so both sides become $\mathbb{D}^{(n)}\mathbb{D}^{(n)}(A)$. $\qquad\square$

Corollary A.1.7. *Let V be a pointwise perfect, graded k-module concentrated in positive weight. Then the natural map*

$$\mathrm{Free}_{\mathbb{E}_n - k}(\Sigma^{-n} V^\vee) \to \mathbb{D}^{(n)}(k \oplus V)$$

is an equivalence.

Proof. By Theorem A.1.5, this follows from the equivalence

$$\mathbb{D}^{(n)}\big(\mathrm{Free}_{\mathbb{E}_n - k}(\Sigma^{-n} V^\vee)\big) \simeq k \oplus V.$$

The proof of [Lur18, 15.3.2.1] applies verbatim to the current setting to prove this equivalence. $\qquad\square$

A.2 Preliminaries

The proof of Theorem A.1.5 will require a few preliminaries about graded modules and algebras.

First we record an analogue of the reduced bar complex in this setting.

Lemma A.2.1. *Let A be an augmented, graded k-algebra and denote by $\overline{A}$ the fiber of the augmentation. Let $M \in \mathsf{LMod}_A$ and $N \in \mathsf{RMod}_A$, then there is a filtration on $M \otimes_A N$:*

$$M \otimes N = F_0 \to F_1 \to \cdots \to \mathrm{colim}\, F_i = M \otimes_A N$$

such that

$$\mathrm{gr}_i(M \otimes_A N) \simeq \Sigma^i M \otimes \overline{A}^{\otimes i} \otimes N.$$

Proof. The relative tensor product is computed by the geometric realization of the standard simplicial object with nth term $M \otimes A^{\otimes n} \otimes N$ ([Lur17, 4.4.2.8]) and hence, by the ∞-categorical Dold-Kan correspondence ([Lur17, 1.2.4.1]), is also computed as the colimit of a filtered object with associated graded corresponding to the normalized complex (which can be computed in the homotopy category), as indicated. $\square$

Proposition A.2.2. *Let A be a weight connected algebra $L, N \in \mathsf{LMod}_A$ and $M \in \mathsf{RMod}_A$.*

(i) *If $M \geqslant \alpha$ and $N \geqslant \beta$ then $M \otimes_A N \geqslant \alpha + \beta$ and $(M \otimes_A N)_{\alpha+\beta} = M_\alpha \otimes N_\beta$.*

(ii) *If $L \geqslant \alpha$ and $N \leqslant \beta$ then $\mathrm{map}_A(L, N) \leqslant \beta - \alpha$.*

(iii) *If M and N are bounded below in weight and pointwise perfect, and A is pointwise perfect, then $M \otimes_A N$ is also pointwise perfect and bounded below in weight.*

(iv) *If A is a weight-connected, augmented, pointwise perfect $\mathbb{E}_n$-algebra, then $\mathrm{Bar}^{(n)}(A)$ is weight-connected and pointwise perfect and $\mathbb{D}^{(n)}(A)$ is weight-coconnected and pointwise perfect.*

Proof. Claims (i), (ii), and (iii) follow from the previous lemma. Claim (iv) follows by induction on n and [Lur17, 5.2.3.13, 5.2.3.14]. $\square$

We will now study a natural filtration on the category of graded modules over a weight-connected algebra. If A is weight-connected, we denote by

$$\mathsf{LMod}_A^{\mathrm{wt} \geqslant j}, \mathsf{LMod}_A^{\mathrm{wt} \leqslant j} \subseteq \mathsf{LMod}_A$$

the full subcategories spanned by those modules which are concentrated in weights at least j and at most j, respectively.

Lemma A.2.3. *Let A be a weight-connected $\mathbb{E}_n$-k-algebra for some $1 \leqslant n \leqslant \infty$.*

(i) *The inclusion $\mathsf{LMod}_A^{\mathrm{wt} \geqslant j} \to \mathsf{LMod}_A$ admits a right adjoint, $(-)_{\geqslant j}$, computed as:*

$$(M_{\geqslant j})_i = \begin{cases} M_i & i \geqslant j \\ 0 & else. \end{cases}$$

(ii) *The inclusion $\mathsf{LMod}_A^{\mathrm{wt} \leqslant j} \to \mathsf{LMod}_A$ admits a left adjoint, $(-)_{\leqslant j}$, computed as $M \mapsto M/M_{\geqslant j}$. In particular,*

$$(M_{\leqslant j})_i = \begin{cases} M_i & i \leqslant j \\ 0 & else. \end{cases}$$

(iii) *The subcategory $\mathsf{LMod}_A^{\mathrm{wt} \geqslant 0}$ inherits an $\mathbb{E}_{n-1}$-monoidal structure.*

(iv) *The localizations $(-)_{\leqslant m}$ are compatible with the $\mathbb{E}_{n-1}$-monoidal structure on $\mathsf{LMod}_A^{\mathrm{wt} \geqslant 0}$.*

(v) *The tower*

$$A \to \cdots \to A_{\leqslant m} \to A_{\leqslant m-1} \to \cdots k$$

of $\mathbb{E}_n$-k-algebras is a tower of square-zero extensions, i.e. we have pullback diagrams of $\mathbb{E}_n$-k-algebras:

$$
\begin{array}{ccc}
A_{\leqslant m} & \longrightarrow & k \\
\downarrow & & \downarrow \\
A_{\leqslant m-1} & \longrightarrow & k \oplus \Sigma A_m(m)
\end{array}
$$

Proof. The existence of these adjoints is immediate since the inclusions preserve all limits and colimits. To compute $M_{\geqslant j}$, observe that, for $i \geqslant j$, the A-module $A(i)$ is in weights at least j, since A is weight-connected. The homogeneous component $(M_{\geqslant j})_i$ is computed as the spectrum of maps of A-modules from $A(i)$ to $M_{\geqslant j}$ which, by the adjunction, is the same as the spectrum of maps from $A(i)$ to M, which is M_i. This proves (i).

Claim (ii) follows formally from the observation that, if $M \geqslant m + 1$ and $N \leqslant m$, then every map $M \to N$ is zero.

Claim (iii) follows, using [Lur17, 2.2.1.1], from the fact that $\mathsf{LMod}_A^{\mathrm{wt} \geqslant 0}$ contains the unit and is closed under tensor products, by Proposition A.2.2(i).

For claim (iv) we must show that, if $M \to M'$ is an equivalence in weights at most m, then so is $Z \otimes_A M \to Z \otimes_A M'$ and $M \otimes_A Z \to M \otimes_A Z'$ for $Z \geqslant 0$. Let F be the fiber of $M \to M'$ so that $F \geqslant m + 1$, and then the result follows from Proposition A.2.2(i) applied to $Z \otimes_A F$ and $F \otimes_A Z$.

For the final claim, we need to produce a derivation, i.e. a map,

$$\mathbf{L}^{(n)}_{A_{\leqslant m-1}} \to \Sigma A_m(m)$$

refining the map $A_{\leqslant m-1} \to \mathrm{cofib}(A_{\leqslant m} \to A_{\leqslant m-1}) = A_m(m)$. Here $\mathbf{L}^{(n)}$ denotes the $\mathbb{E}_n$-cotangent complex. We will produce this refinement as a composite

$$\mathbf{L}^{(n)}_{A_{\leqslant m-1}} \to \mathbf{L}^{(n)}_{A_{\leqslant m-1}/A_{\leqslant m}} \to \Sigma A_m(m)$$

where the first map is the canonical one to the relative cotangent complex and the second is projection onto the first nonzero weight. By [Lur17, 7.5.3.1] applied to the $\mathbb{E}_n$-monoidal category $\mathsf{Mod}^{\mathbb{E}_n}_{A_{\leqslant m}}$ of $\mathbb{E}_n$-$A_{\leqslant m}$-modules, we can compute the relative cotangent complex using the cofiber sequence:

$$\mathcal{U}^{(n)}_{A_{\leqslant m}}(A_{\leqslant m-1}) \to A_{\leqslant m-1} \to \Sigma^n \mathbf{L}^{(n)}_{A_{\leqslant m-1}/A_{\leqslant m}}$$

of $\mathbb{E}_n$-$A_{\leqslant m-1}$-modules. Using the recursive construction of the enveloping algebra, we are reduced to proving the following claim:

(*) If $A \to B$ is a map of weight-connected $\mathbb{E}_1$-algebras with cofiber $C \geqslant j$, denote by C' the cofiber of $B \otimes_A B \to B$. Then $C' \geqslant j$ and $C'_j = \Sigma C_j$.

43

To prove (*), observe that the multiplication map admits a section so that $C' \simeq \Sigma B \otimes_A C$. The result now follows from Proposition A.2.2(i). $\qquad\square$

Remark A.2.4. More generally, a slight variant of the argument for (v) shows that the map

$$A_{\leqslant m+k} \to A_{\leqslant k}$$

is a square-zero extension for any $0 \leqslant k \leqslant m$.

Lemma A.2.5. *Let A be a weight-connected, graded $\mathbb{E}_1$-algebra. Then the thick subcategory of LMod_A generated by the set $\{k(i) : i \in \mathbb{Z}\}$ is the subcategory of weight-bounded, pointwise perfect A-modules.*

Proof. Certainly each module in this thick subcategory is weight-bounded and pointwise perfect, so we need only show the reverse inclusion. If M is bounded, with top degree n, then we have a cofiber sequence

$$M_{\geqslant n} \to M \to M/M_{\geqslant n}$$

By induction, we are reduced to the case where M is concentrated in a single degree, which we may take to be degree zero. We will be done if we can prove that, whenever M is concentrated in a single weight, it must be in the image of the restriction of scalars along $A \to k$. Suppose M is concentrated in weight n. Then, by Proposition A.2.2(i), the lowest weight component of $k \otimes_A M$ is M_n, so we have a map $k \otimes_A M \to M_n(n)$ of graded k-modules which is then adjoint to a map of A-modules $M \to M_n(n)$. Both source and target are concentrated in weight n and the map is an equivalence in that weight, so the claim is proved. $\qquad\square$

A.3 Proof of the Koszul-duality equivalence

Now we turn to the proof of Theorem A.1.5. By Remark A.1.4, it suffices to prove:

Proposition A.3.1. *Let A be an augmented, graded-connected, pointwise perfect $\mathbb{E}_n$-k-algebra. Then the map*

$$A \longrightarrow \mathbb{D}^{(n)}\mathbb{D}^{(n)}(A)$$

is an equivalence.

We will follow the strategy from [Lur18, 15.2.4] and prove the following stronger claim by induction on n:

Proposition A.3.2. *Let $A^1, ..., A^m$ be a collection of augmented, weight-connected, pointwise perfect $\mathbb{E}_n$-k-algebras. Then the map*

$$A^1 \otimes_k \cdots \otimes_k A^m \longrightarrow \mathbb{D}^{(n)}(\mathbb{D}^{(n)}(A^1) \otimes_k \cdots \otimes_k \mathbb{D}^{(n)}(A^m))$$

is an equivalence.

First we establish the case $n = 1$.

Lemma A.3.3. *Let $A^1, ..., A^m$ be a collection of augmented, graded-connected, pointwise perfect $\mathbb{E}_1$-k-algebras. Then the map*

$$A^1 \otimes_k \cdots \otimes_k A^m \longrightarrow \mathbb{D}^{(1)}(\mathbb{D}^{(1)}(A^1) \otimes_k \cdots \otimes_k \mathbb{D}^{(1)}(A^m))$$

is an equivalence.

Proof. Let $B^i = \mathbb{D}^{(1)}(A^i)$ and let $B = B^1 \otimes_k \cdots \otimes_k B^m$. Consider the functors

$$\mathbb{D}^{(1)}_{\mu_i} : \mathsf{LMod}^{\mathrm{op}}_{A^i} \longrightarrow \mathsf{LMod}_{B^i}$$

$$\mathbb{D}^{(1)}_{\mu} : \mathsf{LMod}^{\mathrm{op}}_{B} \longrightarrow \mathsf{LMod}_{A}$$

given informally by

$$M \mapsto \mathrm{map}_{A^i}(M, k), \quad N \mapsto \mathrm{map}_B(N, k).$$

More formally, whenever we have an algebra map $R \otimes S \to k$, we can form the category $\mathsf{Pair}_{R,S}$ of triples $(M, N, f : M \otimes N \to k)$ where f is a map of $A \otimes B$-modules. The forgetful functor $\mathsf{Pair}_{R,S} \to \mathsf{LMod}_R \times \mathsf{LMod}_S$ is then left and right representable, since the tensor product commutes with colimits and the categories of modules are presentable, and this yields the duality functors above.

Notice that $\mathbb{D}^{(1)}_{\mu_i}(A^i) \simeq k$ and $\mathbb{D}^{(1)}_{\mu}(k) \simeq \mathbb{D}^{(1)}(B)$, so the claim is equivalent to showing that

$$A^1 \otimes_k \cdots \otimes_k A^m \longrightarrow \mathbb{D}^{(1)}_{\mu}(\mathbb{D}^{(1)}_{\mu_1}(A^1) \otimes_k \cdots \otimes_k \mathbb{D}^{(1)}_{\mu_m}(A^m))$$

is an equivalence. We'll prove more generally that, given any collection $\{M^i\}$ of A^i-modules, each of which is bounded below in grading and pointwise perfect over k, the map

$$M^1 \otimes_k \cdots \otimes_k M^m \longrightarrow \mathbb{D}^{(1)}_{\mu}(\mathbb{D}^{(1)}_{\mu_1}(M^1) \otimes_k \cdots \otimes_k \mathbb{D}^{(1)}_{\mu_m}(M^m))$$

is an equivalence. Note that this equivalence holds tautologically if each $M^i = k$, and, more generally, if each M^i is a graded-shift of k. Since both sides are exact in each variable, it follows, using Lemma A.2.5, that the result holds whenever each M^i is pointwise perfect and concentrated in finitely many gradings.

Now we prove the general case. Write

$$M^i = \lim M^i/M^i_{\geqslant j}$$

The maps

$$\mathbb{D}^{(1)}_{\mu_i}(M^i/M^i_{\geqslant j}) \to \mathbb{D}^{(1)}_{\mu_i}(M^i)$$

have cofibers given by

$$\mathbb{D}^{(1)}_{\mu_i}(M^i_{\geqslant j}).$$

By Proposition A.2.2(ii),

$$\mathrm{map}_{A^i}(M^i_{\geqslant j}, k)$$

45

is concentrated in gradings $\leqslant -j$. Thus

$$\operatorname{colim} \mathbb{D}^{(1)}_{\mu_i}(M^i/M^i_{\geqslant j}) \to \mathbb{D}^{(1)}_{\mu_i}(M^i)$$

is an equivalence. We are then left to check that the map

$$M^1 \otimes_k \cdots \otimes_k M^m \to \lim_{i_1,\dots,i_m} M^1/M^1_{\geqslant i_1} \otimes_k \cdots M^m/M^m_{\geqslant i_m}$$

is an equivalence when the M^i are weight bounded below. Evaluating the right hand side as an iterated limit, we are reduced to checking that the map

$$N \otimes_k N' \to \lim N \otimes_k N'/N'_{\geqslant i}$$

is an equivalence when N is weight bounded below, say by weight r. By Proposition A.2.2(i), the fibers of each map $N \otimes_k N' \to N \otimes_k N'/N'_{\geqslant i}$ are concentrated in weight $r + i$, and hence the map to the limit is an equivalence. $\qquad\square$

Proof of Proposition A.3.2. The argument is now essentially the same as [Lur18, §15.2.4], but easier.

We proceed by induction on n, the case $n = 1$ having been established above. We have canonical maps:

$$\operatorname{Map}(B, \bigotimes A_i) \to \operatorname{Map}(\operatorname{Bar}(B), \bigotimes \operatorname{Bar}(A_i))$$

$$\operatorname{Map}(\operatorname{Bar}(B), \bigotimes \operatorname{Bar}(A_i)) \to \operatorname{Map}(\bigotimes \mathbb{D}^{(n)}(\operatorname{Bar}(A_i)), \mathbb{D}^{(n)}(\operatorname{Bar}(B))).$$

The functor Bar is monoidal, so to prove the first map is an equivalence it suffices to prove that, for all weight-connected, pointwise perfect $\mathbb{E}_{n+1}$-k-algebras A, the map

$$\operatorname{Cobar}(\operatorname{Bar}(A)) \to A$$

is an equivalence. This can be checked on underlying $\mathbb{E}_1$-k-algebras, so we are reduced to checking that Bar is an equivalence for this class of $\mathbb{E}_1$-k-algebras. But this follows from the induction hypothesis.

The second map is an equivalence by the induction hypothesis combined with [Lur18, 15.2.2.3]. $\qquad\square$

B   Spectral Sequences

In the body of the paper, we use various spectral sequences and maps of spectral sequences obtained by applying certain functors and natural transformations to towers. The purpose of this appendix is to check that these maneuvers produce convergent spectral sequences under certain conditions satisfied in the cases of interest.

Convention B.0.1. Throughout this section, $\mathcal{C}$ will denote a presentably symmetric monoidal stable ∞-category with a t-structure. We will assume that $\mathcal{C}$ satisfies the following properties (all of which are satisfied, for example, by modules over a connective $\mathbb{E}_\infty$-ring, equipped with an action of a group):

(i) The t-structure is compatible with filtered colimits, i.e. $\mathcal{C}_{\leqslant 0}$ is closed under filtered colimits.

(ii) The t-structure is left and right complete, which in this case is equivalent to saying that
$$\operatorname*{colim}_{n \to -\infty} \tau_{\leqslant n} X = 0 = \lim_{n \to \infty} \tau_{\geqslant n} X.$$

(iii) The t-structure is compatible with the symmetric monoidal structure; i.e. $\mathbf{1} \in \mathcal{C}_{\geqslant 0}$ and $X \otimes Y \in \mathcal{C}_{\geqslant n+m}$ whenever $X \in \mathcal{C}_{\geqslant n}$ and $Y \in \mathcal{C}_{\geqslant m}$.

B.1 Towers and convergence

Convention B.1.1. Given a tower $\{X^{\geqslant s}\} \in \mathsf{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{C})$, we index the associated spectral sequence so that
$$E_2^{s,t} = \pi_{t-s}\mathrm{gr}^t X = \pi_{t-s}(\mathrm{cofib}(X^{\geqslant t+1} \to X^{\geqslant t})).$$
We write $X^{-\infty} := \operatorname{colim} X^{\geqslant s}$.

Warning B.1.2. There is not a typo here: we mean $\pi_{t-s}\mathrm{gr}^t$ and not $\pi_{t-s}\mathrm{gr}^s$. The latter would have differentials as in the E_1-term of a spectral sequence, whereas the former will behave as an E_2-term.

Definition B.1.3. Suppose $\{X^{\geqslant s}\}$ is a tower with associated spectral sequence $\{E_r^{s,t}\}$. We say that E_r **converges conditionally** to $\pi_* X^{-\infty}$ if $\operatorname{holim} X^s = 0$. We say that E_r **converges strongly** if the associated filtration $F^s(\pi_{t-s}X^{-\infty}) := \mathrm{im}(\pi_{t-s}X^{\geqslant 2t-s} \to \pi_{t-s}X^{-\infty})$ satisfies
$$\operatorname*{colim}_{s} F^s \pi_{t-s} X^{-\infty} = \pi_{t-s} X^{-\infty}$$
$$\operatorname*{holim}_{s} F^s \pi_{t-s} X^{-\infty} = 0$$

Warning B.1.4. The descent tower shears the filtration in the Adams spectral sequence. If we fix $t - s = n$, then contributions to Adams filtration s come from $\mathrm{desc}^{\geqslant s+n}$. So, for example, a *horizontal* vanishing line on, say, the E_2-term of the Adams spectral sequence would correspond to behavior in the descent filtration that is more like a vanishing line of slope 1. Of course, if one is only interested in a finite range of values of n, there is no difference.

We will content ourselves below with establishing general conditions under which conditional convergence holds. In the body of the paper, when we claim that some spectral sequence actually converges strongly, it is because it also satisfies the conditions of Boardman's theorem [Boa99, Theorem 7.1] for spectral sequences with entering differentials:

Theorem B.1.5 (Boardman). *Suppose that E_r converges conditionally and that, for each fixed (s,t), there are only finitely many nontrivial differentials entering with target in the (s,t) spot. Thus, we eventually have $E_r^{s,t} \supseteq E_{r+1}^{s,t}$. Suppose further that $\lim_r^1 E_r^{s,t} = 0$ for each (s,t). Then E_r converges strongly to $\pi_* X$.*

B.2 Descent towers

Let B be a connective, commutative algebra object in $\mathcal{C}$. Then we may form the descent tower functor (see, e.g., [BHS20, §B-C]):

$$\mathrm{desc}_B : \mathcal{C} \to \mathsf{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathcal{C}),$$

which is lax symmetric monoidal, and specified by

$$\mathrm{desc}_B^{\geqslant j}(X) := \mathrm{holim}_\Delta(\tau_{\geqslant j}(X \otimes B^{\otimes \bullet})).$$

When X is bounded below, this yields a conditionally convergent spectral sequence

$$E_2^{s,t} = H^s(\pi_t(X) \to \pi_t(X \otimes B) \to \cdots) \Rightarrow \pi_{t-s} X_B^\wedge,$$

where $X_B^\wedge = \mathrm{holim}(X \otimes B^{\otimes \bullet})$. When $\pi_*(B \otimes B)$ is flat over $\pi_* B$, we can further identify the E_2-term with Ext in the category of comodules over the Hopf algebroid $(\pi_* B, \pi_*(B \otimes B))$.

Remark B.2.1. The tower $\mathrm{desc}_B^{\geqslant *}(X)$ is not the usual Adams tower, but rather its *décalage*, which is why its associated graded has homotopy groups corresponding to the E_2-page of the Adams spectral sequence rather than the E_1-page.

This story is especially well-behaved when $\mathrm{fib}(\mathbf{1} \to B)$ is 1-connective.

Proposition B.2.2. *Suppose that $I = \mathrm{fib}(\mathbf{1} \to B)$ lies in $\tau_{\geqslant 1}\mathcal{C}$. Then, for any d-connective object X, the descent tower has the following properties:*

(a) *The natural map $X \to X_B^\wedge$ is an equivalence.*

(b) *$E_2^{s,t}$ vanishes when $2s - t \geqslant d$.*

(c) *$\pi_n \mathrm{desc}_B^{\geqslant j}(X) = 0$ whenever $j \geqslant d + 2n$.*

(d) *For each k, there exists an N such that, for $j \geqslant N$, $\mathrm{desc}_B^{\geqslant j}(X)$ is k-connective.*

Proof. Since $\mathrm{holim}_j \mathrm{desc}_B^{\geqslant j}(X) = 0$, we can study the vanishing of the homotopy groups of each $\mathrm{desc}_B^{\geqslant j}(X)$ by establishing a vanishing range in the associated graded. Thus $(b) \Rightarrow (c) \Rightarrow (d)$, so we need only establish (a) and (b). But these claims can be proven using the usual construction of the descent spectral sequence, via the tower $\{\mathrm{Tot}^{\leqslant s}(\mathrm{cb}(B) \otimes X)\}$, where the result is clear. $\square$

B.3 Classical Adams spectral sequence

The classical Adams spectral sequence, given by descent along $S^0 \to \mathbb{F}_p$, has slightly more involved convergence issues since the fiber of the unit map $S^0 \to \mathbb{F}_p$ is no longer 1-connective. We review the classical approach to getting around this issue and leverage this to understand the convergence behavior of the Tate fixed point spectral sequence below.

Throughout this section $\mathrm{desc}(-) = \mathrm{desc}_{\mathbb{F}_p}(-)$.

Construction B.3.1. Since $\mathrm{desc}(-)$ is lax symmetric monoidal, every descent tower is a module over $\mathrm{desc}(S^0)$. Recall that the element $p \in \pi_0(S^0)$ is detected in Adams filtration 1, and hence lifts to an element $v_0 \in \pi_0\mathrm{desc}^{\geqslant 1}(S^0)$. Thus, given any spectrum X, we have a natural map

$$v_0 : \mathrm{desc}(X)(1) \to \mathrm{desc}(X).$$

Remark B.3.2. The composite of the shift operator with v_0 is multiplication by p. It follows that v_0 induces multiplication by p on both $\mathrm{colim}\, X$ and $\lim X$.

Remark B.3.3. There is a canonical identification $\mathrm{desc}(X)/v_0 \simeq \mathrm{desc}(X/p)$. However, when $k \geqslant 2$, $\mathrm{desc}(X)/v_0^k$ and $\mathrm{desc}(X/p^k)$ differ. The former tower has E_2-term computed by the homotopy groups of an object of the derived category of $\mathcal{A}_*$-comodules which does not lie in the heart.

Proposition B.3.4. *Let X be bounded below. Then $\mathrm{desc}(X)/v_0^m$ has the property that, for each k, there is an N such that, for all $j \geqslant N$, $\mathrm{desc}^{\geqslant j}(X)/v_0^m$ is k-connective. Moreover, each term $\mathrm{desc}^{\geqslant j}(X)/v_0^m$ is d-connective.*

Proof. The conclusion about the tower is stable under extensions, so we are reduced to the case when $m = 1$ and $\mathrm{desc}(X)/v_0 = \mathrm{desc}(X/p)$. Since the tower is conditionally convergent, it suffices to establish a vanishing line on the E_2-page, and to show this is concentrated in stems starting in dimension d. But the E_2-page is computed by $\mathrm{Ext}^{s,t}(H_*(X) \otimes \Lambda(\tau_0))$, where the result is classical. $\square$

B.4 Fixed point spectral sequences

Given a tower X in the category of spectra with an action of a group G, we can take homotopy fixed points, orbits, or Tate fixed points levelwise and produce a new tower. In this section we establish some criteria for the conditional convergence of this tower.

Proposition B.4.1. *Suppose $X \in \mathsf{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathsf{Fun}(BG, \mathcal{C}))$ is conditionally convergent (i.e. $\mathrm{holim}\, X = 0$). Then so is X^{hG}.*

Proof. Limits commute with limits. $\square$

Proposition B.4.2. *Suppose $X \in \mathsf{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathsf{Fun}(BG, \mathsf{Sp}))$ is such that, for all n, the tower $\{\pi_n(X^{\geqslant s})\}$ is pro-zero and the terms $X^{\geqslant s}$ are bounded below, uniformly in s. Then the spectral sequence associated to X^{tG} is conditionally convergent.*

Proof. Recall that the Tate construction can be computed from Postnikov truncations as:

$$X^{tG} \simeq \mathrm{holim}(\tau_{\leqslant n}X)^{tG}.$$

So we may assume that X is uniformly bounded above, as well as below. The collection of X for which the conclusion of the proposition holds forms a thick subcategory so we are reduced to the case when each $X^{\geqslant s}$ is Eilenberg-MacLane. But then the hypothesis states this tower is pro-equivalent to zero, so the claim follows. $\square$

Proposition B.4.3. *Suppose Y is a tower in $\mathsf{Fun}(BC_{p^m}, \mathsf{Sp})$, for $1 \leqslant m \leqslant \infty$, with the following properties:*

(i) *The terms $Y^{\geqslant j}$ are uniformly bounded below.*

(ii) *For large j, we have $Y^{\geqslant -j} = \operatorname{colim} Y$.*

(iii) *Y admits a self map $v : Y(1) \to Y$ such that the composite $Y \to Y(1) \to Y$ is p.*

(iv) *Each tower Y/v^ℓ has the property that, for each n, the tower of homotopy groups $\{\pi_n((Y/v^\ell)^{\geqslant j})\}_j$ is pro-zero.*

Then $Y^{tC_{p^m}}$ is conditionally convergent.

Proof. By (i) and the Tate orbit lemma, we may reduce to the case $m = 1$. By (ii) and (iii), we conclude that, for fixed filtration, the map $v(j) : Y(j+1) \to Y(j)$ is multiplication by p on $\operatorname{colim} Y$. But, for any bounded below spectrum Z, the spectrum Z^{tC_p} is p-complete and hence the limit of multiplication by p vanishes. It follows that

$$\lim(\cdots \xrightarrow{v} Y(j)^{tC_p} \xrightarrow{v} Y(j-1)^{tC_p} \to) = 0$$

and hence that

$$Y^{tC_p} = \lim(Y/v^\ell)^{tC_p}.$$

So we need only prove that each $(Y/v^\ell)^{tC_p}$ is conditionally convergent. But this follows from (iv) and the previous proposition. $\qquad\square$

B.5   Hochschild homology of filtered rings

If A is a filtered $\mathbb{E}_1$-ring, one can construct a corresponding filtration of $\mathrm{THH}(A)$ and spectral sequence (see [AKS18]). We will need to understand how this spectral sequence interacts with the Tate-valued Frobenius, and for this we need a construction of $\mathrm{THH}(A)$ as a filtered cyclotomic object. We refer the reader to [AMMN20, §A] for details, and review the relevant definitions here.

Definition B.5.1. Let $L_p : \mathsf{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathsf{Sp}) \to \mathsf{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathsf{Sp})$ denote left Kan extension along multiplication by p.

Proposition B.5.2. *([AMMN20, §A]) Let A be a filtered or graded $\mathbb{E}_1$-ring. Then $\mathrm{THH}(A)$ admits a natural L_p-twisted diagonal, i.e. an S^1-equivariant map*

$$\varphi : L_p\mathrm{THH}(A) \to \mathrm{THH}(A)^{tC_p}.$$

In the filtered case, this map is compatible with passage to the associated graded and, in both cases, the map is compatible with forgetting to underlying objects.

Remark B.5.3. Since L_p is adjoint to restriction along multiplication by p, the Frobenius gives S^1-equivariant maps

$$\varphi : \mathrm{THH}(A)^{\geq j} \to (\mathrm{THH}(A)^{tC_p})^{\geq jp}$$

for all j; and similarly for the graded case.

In particular, this produces maps of spectral sequences (which shear the gradings). We will be using these spectral sequences in the case when we are filtering A by its descent tower for $S^0 \to \mathbb{F}_p$. The following proposition guarantees convergence (after p-completion) when A is connective.

Proposition B.5.4. *Let A be a connective $\mathbb{E}_1$-ring. Then, for each $1 \leq m \leq \infty$, the tower $\mathrm{THH}(\mathrm{desc}_{\mathbb{F}_p}(A))^{tC_{p^m}}$, converges conditionally to $\mathrm{THH}(A)^{tC_{p^m}}$. The tower $\mathrm{THH}(\mathrm{desc}_{\mathbb{F}_p}(A))^{\wedge}_{v_0}$ converges conditionally to $\mathrm{THH}(A)^{\wedge}_p$.*

Proof. We will verify the conditions in Proposition B.4.3. First observe that each $\mathrm{desc}^{\geq j}(A)$ is connective, and hence $\mathrm{THH}(\mathrm{desc}(A))^{\geq j}$ is connective for each j, being a colimit of connective spectra. The same is then true of the v_0-completion. We also have that $\mathrm{desc}^{\geq -j}(A) = \mathrm{desc}^{\geq -j-1}(A)$ for all $j \geq 0$, and hence the same holds after applying $\mathrm{THH}(-)$. This proves (i)-(iii), so we are left with verifying (iv). So fix ℓ and n. Observe that, if $Z_{\bullet}$ is any simplicial spectrum with each Z_i connective, then the inclusion $\mathrm{sk}_{n+1}|Z_{\bullet}| \to |Z_{\bullet}|$ is an equivalence on π_i for $i \leq n$. (Indeed, by the Dold-Kan correspondence, we have that $\mathrm{sk}_m/\mathrm{sk}_{m-1}$ is a summand of $\Sigma^m Z_m$ and hence must be m-connective.) It follows that we may replace $\mathrm{THH}(\mathrm{desc}(A))/v_0^{\ell}$ by its $(n + 1)$-skeleton and then verify that the system of nth homotopy groups is pro-zero. The $(n + 1)$-skeleton arises as a colimit of terms of the form $\mathrm{desc}(A)/(v_0^{\ell}) \otimes \mathrm{desc}(A)^{\otimes p}$ for $p \leq n + 1$. Proposition B.3.4 implies that each of these terms can be made as connective as we like in large filtration, so the result follows.

The second claim is similar and easier. $\square$

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