EILENBERG-MACLANE SPECTRA AS EQUIVARIANT THOM SPECTRA

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Abstract. We prove that the G-equivariant mod p Eilenberg–MacLane spectrum arises as an equivariant Thom spectrum for any finite, p-power cyclic group G, generalizing a result of Behrens and the second author in the case of the group C_2. We also establish a construction of $\mathrm{H}\underline{\mathbb{Z}}_{(p)}$, and prove intermediate results that may be of independent interest. Highlights include constraints on the Hurewicz images of equivariant spectra that admit norms, and an analysis of the extent to which the non-equivariant $\mathrm{H}\mathbb{F}_p$ arises as the Thom spectrum of a more than double loop map.

Contents

Fix a prime p, $n \geqslant 1$ an integer, and let $G = C_{p^n}$ denote a cyclic group of order p^n. Let λ denote the *standard* representation of $G = C_{p^n}$ on the complex plane, where the generator acts by $e^{2\pi i/p^n}$. There is a corresponding representation sphere S^λ, with underlying sphere S^2 and G-fixed points equivalent to S^0.

The action of C_{p^n} on the plane defines an action on the configuration spaces of little disks in the plane, leading to the notion of an $\mathbb{E}_\lambda$-algebra. If X is any G-space, then $\Omega^\lambda X$ is naturally an $\mathbb{E}_\lambda$-algebra. Our main result is:

Theorem A. *The free* $\mathbb{E}_\lambda$*-algebra in* G*-spectra with* $p = 0$ *is* $\mathrm{H}\underline{\mathbb{F}}_p$.

A somewhat more explicit version of the above theorem, generalizing a non-equivariant result of Hopkins–Mahowald, is as follows:

Theorem B (Theorem 7.1). *Let*

$$\mu : \Omega^\lambda S^{\lambda+1} \longrightarrow \mathrm{BGL}_1(S^0_{(p)})$$

denote the $\mathbb{E}_\lambda$*-map extending the map*

$$S^1 \longrightarrow \mathrm{BGL}_1(S^0_{(p)})$$

corresponding to $1 - p \in \pi_0^G S^0_{(p)}$. *Then the Thom spectrum* $\left(\Omega^\lambda S^{\lambda+1}\right)^\mu$ *of* μ *is* $\mathrm{H}\underline{\mathbb{F}}_p$.

We also establish a (p-local) integral variant.

Theorem C (Theorem 8.1). *Let* $S^{\lambda+1}\langle\lambda+1\rangle$ *denote the fiber of the unit*

$$S^{\lambda+1} \longrightarrow \Omega^\infty\left(\Sigma^{\lambda+1}\mathrm{H}\underline{\mathbb{Z}}\right).$$

Then there is an equivalence of $\mathbb{E}_\lambda$*-algebras*

$$\left(\Omega^\lambda(S^{\lambda+1}\langle\lambda+1\rangle)\right)^\mu \simeq \mathrm{H}\underline{\mathbb{Z}}_{(p)}.$$

Remark. This work constitutes an equivariant generalization of the non-equivariant fact that $\mathrm{H}\mathbb{F}_p$ is a Thom spectrum over $\Omega^2 S^3$ [MRS01, Lemma 3.3]. When $G = C_2$, the second author and Mark Behrens [BW] constructed $\mathrm{H}\underline{\mathbb{F}}_2$ as a Thom spectrum over $\Omega^\rho S^{\rho+1}$. Since C_2 is the only group with 2-dimensional regular representation, and many groups have no nontrivial, one-dimensional real representations, it was far from clear how the results of [BW] might generalize to larger groups. In Section 4, we note a *non-obvious equivalence* of C_2-spaces

$$\Omega^\rho S^{\rho+1} = \Omega^{\sigma+1} S^{\sigma+2} \simeq \Omega^{2\sigma} S^{2\sigma+1} = \Omega^\lambda S^{\lambda+1}.$$

Observation of the above equivalence kick-started the present project.

Even in the case $G = C_2$, our work here involves entirely new techniques. Indeed, the main tool applied in [BW] was deep computational knowledge of the C_2-equivariant mod 2 dual Steenrod algebra, knowledge that is unavailable at larger primes or larger groups. In fact, work in progress suggests that the theorems we prove here may be used to deduce information about the G-equivariant Steenrod algebra, providing an important application by reversing the flow of information.

Remark. Theorem B identifies $\mathrm{H}\underline{\mathbb{F}}_p$ as a Thom $\mathbb{E}_\lambda$-algebra. In fact, our proof will show that μ is an H-space map for a certain H-space structure on $\Omega^\lambda S^{\lambda+1}$. With a bit more work, we thus equip the Thom spectrum with the additional structure of an $\mathbb{A}_2$-algebra in $\mathbb{E}_\lambda$-algebras. Verifying that this extra structure exists is somewhat subtle, and may be of independent interest even in the non-equivariant setting.

In §1 we review and expand on the non-equivariant case. In §2 we give an outline of the proof of the main theorem, which includes an explanation of the content of the remaining sections. We offer some concluding remarks, including a proof of Theorem C, in §8.

Conventions. We freely use the language of ∞-categories (alias quasicategories, alias weak Kan complexes) in the form developed in [Lur17]. We denote by Spaces the ∞-category of spaces, and append decorations to obtain the ∞-category of pointed spaces or of G-spaces for a finite group G, the latter being defined as the ∞-category of presheaves $\mathsf{Psh}(\mathcal{O}_G)$ on the 1-category of transitive G-sets. We denote by Sp^G the ∞-category of (genuine) G-spectra, and assume the reader is familiar with the standard notations of equivariant homotopy theory. We could not improve on the summary given in [HHR16, §2-3], and recommend it to the reader. In particular, we will require the Eilenberg-MacLane spectrum associated to the constant Mackey functor $\underline{\mathbb{F}}_p$, and the notion of the geometric fixed points $X^{\Phi G}$ of a G-spectrum. Finally, we denote the G-space classifying equivariant stable spherical fibrations with fiber of type S^0 by $\mathrm{BGL}_1(S^0)$. (See §4 for more details).

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1. The non-equivariant story

Both authors are fond of the following result of Mahowald [Mah77]:

Theorem 1.1 (Mahowald)**.** *The Thom spectrum of the unique non-trivial double loop map*

$$\Omega^2 S^3 \longrightarrow \mathrm{BO} \tag{1}$$

is $\mathrm{H}\mathbb{F}_2$.

Remark 1.2. This result and several variants have enjoyed many subsequent proofs in the literature. Examples include [Mah79, CMT81, Pri78, MRS01, MNN15].

In modern language, Mahowald's theorem states that the free $\mathbb{E}_2$-algebra with $2 = 0$ is $\mathrm{H}\mathbb{F}_2$ (cf. [AB14, MNN15]), and so proves the nilpotence of all simple 2-torsion classes in the stable homotopy groups of spheres [Nis73].

Less well-known is that Mahowald's map (1) is in fact a triple loop map, as revealed by the following observation:

Observation 1.1. Mahowald's map (1) may be obtained by thrice looping the composite

$$\mathbb{H}P^\infty \simeq \mathrm{BSp}(1) \longrightarrow \mathrm{BSp} \simeq \mathrm{B}^5\mathrm{O} \xrightarrow{\eta} \mathrm{B}^4\mathrm{O}$$

Remark 1.3. This observation, combined with one of the main theorems of Klang in [Kla16], allows a quick computation of the $\mathbb{E}_3$-center (or $\mathbb{E}_3$ topological Hochschild cohomology) of $\mathrm{H}\mathbb{F}_2$:

$$\mathrm{map}_{\mathsf{Mod}^{\mathbb{E}_3}_{\mathrm{H}\mathbb{F}_2}}(\mathrm{H}\mathbb{F}_2, \mathrm{H}\mathbb{F}_2) \cong F(\mathbb{H}P^\infty_+, \mathrm{H}\mathbb{F}_2).$$

In fact, a Borel equivariant precursor to our main theorem at the prime 2 appears as Lemma 3.1 in [Kla16], where it is used in an essential manner to compute the factorization homology of Eilenberg–Maclane spectra.

Remark 1.4. The authors first learned of Observation 1.1 from Mike Hopkins. We give a C_2-equivariant generalization in Section 4.

The remainder of the section will be about the case $p > 2$, where there is the following theorem of Hopkins [MRS01, Lemma 3.3]:

Theorem 1.5 (Hopkins)**.** *Let*

$$S^3 \xrightarrow{1-p} \mathrm{B}^3\mathrm{GL}_1(S^0_{(p)})$$

denote the class $1 - p \in \pi_0(S_{(p)})^\times = \pi_3(\mathrm{B}^3\mathrm{GL}_1(S_{(p)}))$*. Then, applying* Ω^2*, one obtains a map*

$$\Omega^2 S^3 \longrightarrow \mathrm{BGL}_1(S^0_{(p)})$$

with Thom spectrum equivalent to $\mathrm{H}\mathbb{F}_p$*.*

In light of the situation at $p = 2$, it is natural to wonder if the map

$$1 - p : S^3 \longrightarrow \mathrm{B}^3\mathrm{GL}_1(S^0_{(p)})$$

may be delooped. We record here the fact that it *cannot*, which the authors could not locate in the literature. At the prime $p = 3$, the map is not even an H-space map for the standard H-space structure on $S^3 = \mathrm{SU}(2)$. Crucially, we will see in Section 5 that the map *is* an H-space map for a certain exotic H-space structure on S^3.

Theorem 1.6. *Let* S^0_p *denote the p-complete sphere spectrum, and suppose that* $p > 2$*. Then there is no triple loop map*

$$X \longrightarrow \mathrm{BGL}_1(S^0_p),$$

for any triple loop space X*, that makes* $\mathrm{H}\mathbb{F}_p$ *as a Thom spectrum.*

Proof. To avoid a lengthy digression, we defer the proof to Appendix A. ☐

Remark 1.7. The above theorem also implies that there can be no triple loop map $X \longrightarrow \mathrm{BGL}_1(S^0_{(p)})$ making $\mathrm{H}\mathbb{F}_p$ as a Thom spectrum.

2. OUTLINE OF THE PROOF

The Thom spectrum of the map

$$S^1 \xrightarrow{1-p} \mathrm{BGL}_1(S^0_{(p)})$$

is the mod p Moore spectrum, which admits a Thom class for $\mathrm{H}\underline{\mathbb{F}}_p$. It follows formally (see the argument for Proposition 5.3 in [BW]) that there is a Thom class

$$\alpha : \left(\Omega^\lambda S^{\lambda+1}\right)^\mu \longrightarrow \mathrm{H}\underline{\mathbb{F}}_p.$$

Our goal is to show that this map is an equivalence of G-spectra. By induction on the order of the group (the base case being supplied by Hopkins-Mahowald), it will suffice to prove that the map on geometric fixed points

$$\alpha^{\Phi G} : \left(\left(\Omega^\lambda S^{\lambda+1}\right)^\mu\right)^{\Phi G} \longrightarrow \mathrm{H}\underline{\mathbb{F}}_p^{\Phi G}$$

is an equivalence.

The proof proceeds in several steps.

Step 1. Compute the homotopy groups of $\left(\left(\Omega^\lambda S^{\lambda+1}\right)^\mu\right)^{\Phi G}$.
Step 2. Compute the homotopy groups of $\mathrm{H}\underline{\mathbb{F}}_p^{\Phi G}$.
Step 3. Show that $\alpha^{\Phi G}$ is a ring map (for some ring structure on the source).
Step 4. Show that $\alpha^{\Phi G}$ hits algebra generators in the target.

The computation in Step 2 is well-known, and is stated as Lemma 7.2 below. At odd primes, one learns that

$$\pi_* \mathrm{H}\underline{\mathbb{F}}_p^{\Phi G} = \mathbb{F}_p[t] \otimes \Lambda(s),\ |t| = 2, |s| = 1.$$

Step 1 is more difficult. In §3 we show that $\left(\Omega^\lambda S^{\lambda+1}\right)^G = \Omega^2 S^3 \times \Omega S^2$, and, after developing some more properties of this Thom spectrum, we verify (Lemma 7.3) that the homotopy groups of $\left(\left(\Omega^\lambda S^{\lambda+1}\right)^\mu\right)^{\Phi G}$ are *additively* given by

$$\pi_* \left(\left(\Omega^\lambda S^{\lambda+1}\right)^\mu\right)^{\Phi G} = \mathbb{F}_p[x] \otimes \Lambda(y),\ |x| = 2, |y| = 1.$$

There is a serious problem to be addressed in Step 3: by construction, $\left(\Omega^\lambda S^{\lambda+1}\right)^\mu$ is an $\mathbb{E}_\lambda$-spectrum and the Thom class is represented by an $\mathbb{E}_\lambda$-map. When we take geometric fixed points, we are left without even an $\mathbb{A}_2$-structure on either the source or the map. Nonetheless, we will show that $\alpha^{\Phi G}$ is a ring map for a certain $\mathbb{A}_2$-structure on the source. To do so requires a different argument at the prime 2 than at odd primes.

<u>Step 3a)</u> In §3 we show that $S^{\lambda+1} = \Omega \mathbb{H}P^\infty$ for a certain G-action on $\mathbb{H}P^\infty$. We then show (§4) that, when $p = 2$, the map

$$S^{\lambda+1} \longrightarrow \mathrm{B}^\lambda \mathrm{BGL}_1(S^0_{(2)})$$

deloops once. Thus, $\alpha^{\Phi G}$ is an $\mathbb{A}_\infty$-map in this case. This section contains some material that may be of independent interest, such as a description of one of the spaces in the equivariant K-theory spectrum in terms of bundles of (twisted) G-$\mathbb{H}$-modules.

<u>Step 3b)</u> In §5 we produce an exotic $\mathbb{A}_2$-structure on $S^{\lambda+1}$ at odd primes with respect to which the map μ is an $\mathbb{A}_2$-map in $\mathbb{E}_\lambda$-spaces. Our proof uses a small dose of unstable equivariant homotopy theory, in particular the EHP sequence.

Finally, we come to Step 4. The element $s \in \pi_1 \mathrm{H}\underline{\mathbb{F}}_p^{\Phi G}$ arises as a witness to the fact that the composite

$$S^0 \xrightarrow{\nabla} G_+ \xrightarrow{1} \mathrm{H}\underline{\mathbb{F}}_p$$

is null. Said differently, s witnesses the vanishing of the element $[G] \in \pi_0^G S^0$ in the Hurewicz image. Nonequivariantly, the zeroth homotopy group of $\left(\Omega^2 S^3\right)^\mu$ is already detected by the map

$$(S^1)^\mu = M(p) \longrightarrow \left(\Omega^2 S^3\right)^\mu .$$

Equivariantly, this is no longer true. Recall [Seg71, tD75] that $\pi_0^G S^0 = A(G)$ is the Burnside ring of finite G-sets. While the element p dies in the Moore space $(S^1)^\mu$, the same is not true of the elements $[G/K]$ for $K \subsetneq G$. The proof that these elements vanish in the Hurewicz image of $\left(\Omega^\lambda S^{\lambda+1}\right)^\mu$ is a consequence of the vanishing of p together with the existence of *norms* supplied by the $\mathbb{E}_\lambda$-structure. This is proved in §6 and the final pieces of the proof of Step 4 and the main result are then spelled out in §7.

We end in §8 with some miscellaneous concluding remarks, including:

- An explanation of how to produce $\mathrm{H}\underline{\mathbb{Z}}_{(p)}$ as a Thom spectrum.
- A proposed definition for equivariant Brown-Gitler spectra.
- A brief outline of a construction of Milnor operations, which is the subject of work in progress by the authors.
- Some unanswered questions.

3. Quaternionic projective space

We begin by examining a very natural action of S^1 on quaternionic projective space. It is in some ways analogous to the natural action of C_2 on $\mathbb{C}P^\infty$ by complex conjugation.

Construction 3.1. Consider the action of $\mathrm{Sp}(1)$ on the quaternions $\mathbb{H}$ by conjugation. The center $\{\pm 1\}$ acts trivially, so this produces an action of $\mathrm{Sp}(1)/\{\pm 1\} = \mathrm{SO}(3)$ on $\mathbb{H}$. This produces an action of $S^1 \subseteq SO(3)$ on $\mathbb{H}$ which can be described in two equivalent ways:

- $z \in S^1$ acts on $q \in \mathbb{H}$ by $(\sqrt{z})q(\sqrt{z})^{-1}$;
- $z \in S^1$ acts on $q = u + vj$ by $u + (zv)j$, where $u, v \in \mathbb{C}$.

From the first description it's clear that $\mathbb{H}$ is an S^1-equivariant algebra. From the second description it's clear that $\mathbb{H} = 2 + \lambda$ as a representation.

Definition 3.2. Let $\mathbb{H}P^\infty$ denote the S^1-space obtained from $\mathbb{H}^\infty$ by imposing the relation

$$[q_0 : q_1 : \cdots] \sim [q_0 h : q_1 h : \cdots],\ h \in \mathbb{H}^\times$$

and acting by S^1 componentwise as in Construction 3.1.

We will, without comment, also denote by $\mathbb{H}P^\infty$ the G-space obtained by restricting the action to a finite subgroup $G \subseteq S^1$. We will assume that G is a fixed, *nontrivial* finite subgroup of S^1 for the remainder of this section.

Remark 3.3. It follows from the definition that the natural inclusion $\mathbb{C}P^\infty \subseteq \mathbb{H}P^\infty$ is equivariant for the *trivial* action on complex projective space and identifies the fixed points

$$\mathbb{C}P^\infty = (\mathbb{H}P^\infty)^G .$$

The reader might compare this to the equivalence $\mathbb{R}P^\infty = (\mathbb{C}P^\infty)^{C_2}$ where C_2 acts by complex conjugation on projective space.

We now study the loop spaces of $\mathbb{H}P^\infty$.

Proposition 3.4. $\Omega\mathbb{H}\mathbb{P}^\infty \simeq S^{\lambda+1}$.

Proof. The usual inclusion $S^{\lambda+2} \simeq \mathbb{H}P^1 \to \mathbb{H}P^\infty$ is equivariant for the action above and induces a map $S^{\lambda+1} \to \Omega\mathbb{H}P^\infty$. This is an underlying equivalence, and on fixed points for nontrivial subgroups we have the standard map (see Remark 3.3) $S^1 \to \Omega\mathbb{C}P^\infty$, which is also an equivalence. □

Remark 3.5. As a corollary of the proof, we record that the map $\nu : S^{2\lambda+3} \to S^{\lambda+2}$ becomes $\eta : S^3 \to S^2$ upon passage to fixed points.

Proposition 3.6. *If $G = C_2$, then $\Omega^\sigma\mathbb{H}P^\infty \simeq S^{\rho+1}$.*

Proof. Again, the usual inclusion $S^{2\sigma+2} \to \mathbb{H}P^\infty$ is equivariant and we get a map $S^{\sigma+2} = S^{\rho+1} \to \Omega^\sigma\mathbb{H}P^\infty$ which is an underlying equivalence. To compute the fixed points of the right hand side, we use the fiber sequence of spaces

$$(\Omega^\sigma\mathbb{H}P^\infty)^{C_2} \to \mathbb{C}P^\infty \to \mathbb{H}P^\infty$$

arising from the cofiber sequence $C_{2+} \to S^0 \to S^\sigma$ of C_2-spaces. But the fiber of $\mathrm{B}S^1 \to \mathrm{B}S^3$ is $S^3/S^1 \simeq S^2$, by the Hopf fibration, which completes the proof. □

Corollary 3.7. *If $G = C_2$, then*

$$\Omega^\lambda S^{\lambda+1} \simeq \Omega^\rho S^{\rho+1}.$$

Moreover, this equivalence respects the inclusion of S^1 up to homotopy.

Proof. There is a chain of equivalences

$$\Omega^\lambda S^{\lambda+1} \simeq \Omega^{2\sigma}\Omega\mathbb{H}P^\infty \simeq \Omega^\rho\Omega^\sigma\mathbb{H}P^\infty \simeq \Omega^\rho S^{\rho+1}.$$

□

Remark 3.8. We can reword this equivalence as the statement that the free $\mathbb{E}_{2\sigma}$-space and free $\mathbb{E}_\rho$-space on the pointed C_2-space S^1 coincide. A prototype of this result is that the free, group-like $\mathbb{E}_\sigma$-space and free $\mathbb{E}_1$-space on the pointed C_2-space S^0 also coincide, i.e. $\Omega S^1 \simeq \Omega^\sigma S^\sigma \simeq \mathbb{Z}$ (with trivial action). This can be proved in much the same way, using the equivalences:

$$\Omega^\rho\mathbb{C}P^\infty \simeq \mathbb{Z}, \quad \Omega\mathbb{C}P^\infty \simeq S^\sigma, \quad \Omega^\sigma\mathbb{C}P^\infty \simeq S^1,$$

where C_2 acts on $\mathbb{C}P^\infty$ by complex conjugation.

As a consequence of the equivalence $\Omega\mathbb{H}P^\infty \simeq S^{\lambda+1}$, we observe that $\Omega^\lambda S^{\lambda+1}$ inherits an $\mathbb{E}_{\lambda+1}$-structure. This gives the fixed points $\left(\Omega^\lambda S^{\lambda+1}\right)^G$ an $\mathbb{E}_1$-algebra structure.

Proposition 3.9. *As an $\mathbb{E}_1$-space,*

$$\left(\Omega^{\lambda+1}\mathbb{H}P^\infty\right)^G \simeq \Omega^2 S^3 \times \Omega S^2.$$

Warning 3.10. At odd primes, the classifying map μ^G is not an $\mathbb{E}_1$-map.

Proof of the proposition. Define $S^{\lambda/2}$ by the cofiber sequence

$$G_+ \to S^0 \to S^{\lambda/2},$$

and notice that we have a cofiber sequence

$$G_+ \wedge S^1 \to S^{\lambda/2} \to S^\lambda.$$

From the second cofiber sequence, we learn that there is a fiber sequence

$$\Omega^\lambda \mathbb{H}P^\infty \to \mathrm{map}_*(S^{\lambda/2}, \mathbb{H}P^\infty) \to \mathrm{map}_*(G_+ \wedge S^1, \mathbb{H}P^\infty).$$

Taking fixed points, we get

$$\left(\Omega^\lambda \mathbb{H}P^\infty\right)^G \to \mathrm{map}_*(S^{\lambda/2}, \mathbb{H}P^\infty)^G \to \Omega\mathbb{H}P^\infty \simeq S^3.$$

The first cofiber sequence identifies the middle term as the fiber of the inclusion of fixed points:

$$\mathrm{map}_*(S^{\lambda/2}, \mathbb{H}P^\infty)^G \to \mathbb{C}P^\infty \to \mathbb{H}P^\infty.$$

In other words, $\mathrm{map}_*(S^{\lambda/2}, \mathbb{H}P^\infty)^G \simeq S^2$ and we can identify our previous fiber sequence with:

$$\left(\Omega^\lambda \mathbb{H}P^\infty\right)^G \to S^2 \to S^3.$$

The second map is null, so we learn that there is an equivalence:

$$\left(\Omega^\lambda \mathbb{H}P^\infty\right)^G \simeq \Omega S^3 \times S^2,$$

and hence an equivalence of loop spaces:

$$\left(\Omega^{\lambda+1} \mathbb{H}P^\infty\right)^G = \Omega\left(\Omega^\lambda \mathbb{H}P^\infty\right)^G \simeq \Omega(\Omega S^3 \times S^2).$$

□

We stress that the above result does not concern multiplicative structure on the Thom spectrum in question. This is the subject of the next section at the prime two, and of the subsequent section at odd primes.

4. Extra structure at the prime 2

Hopkins observed that the map $\mu : \Omega^2 S^3 \to \mathrm{BO}$ admits a *triple* delooping as the composite:

$$\mathbb{H}P^\infty = \mathrm{BSp}(1) \longrightarrow \mathrm{BSp} \simeq \mathrm{B}^5\mathrm{O} \xrightarrow{\eta} \mathrm{B}^4\mathrm{O}.$$

We would like to establish an equivariant version of this result. The statement requires a few preliminaries.

The first results of this section hold for any finite subgroup $G \subseteq S^1$. We will indicate later when we must restrict attention to $G = C_{2^n}$.

Definition 4.1. A **G-$\mathbb{H}$-module** is a real G-representation V equipped with a G-equivariant algebra map $\mathbb{H} \to \mathrm{End}(V)$. Here $\mathrm{End}(V)$ is the G-representation of all endomorphisms and we use the G-action on $\mathbb{H}$ constructed in (3.1). More generally, a **G-$\mathbb{H}$-bundle** on a G-space X is a G-equivariant, real vector bundle $E \to X$ together with a G-equivariant algebra map $\mathbb{H} \to \mathrm{End}(E)$.

Construction 4.2. For $F = \mathbb{R}$ or $\mathbb{H}$, let $\mathcal{U}_F$ be a **complete G-F-universe**. That is: $\mathcal{U}_F$ is a direct sum of infinitely many finite dimensional G-F-modules which contains every finite dimensional G-F-module as a summand (up to isomorphism). Let $\mathrm{Gr}^F(\mathcal{U}_F)$ denote the infinite grassmanian with its induced G-action. Then we define:

$$\mathrm{BO}_G := \mathrm{Gr}^{\mathbb{R}}(\mathcal{U}_{\mathbb{R}}),$$
$$\mathrm{BGL}(\mathbb{H}) := \mathrm{Gr}^{\mathbb{H}}(\mathcal{U}_{\mathbb{H}}).$$

The G-space BO_G is well-known, and there is an equivalence

$$\Omega^\infty \mathrm{KO}_G = \mathbb{Z} \times \mathrm{BO}_G.$$

Warning 4.3. The G-space $\mathrm{BGL}(\mathbb{H})$ is *not* the same as the space BSp_G associated to equivariant, symplectic K-theory. The latter does not incorporate a nontrivial action of G on $\mathbb{H}$.

Warning 4.4. Neither BO_G nor $\mathrm{BGL}(\mathbb{H})$ are equivariantly connected when G is nontrivial. For example, $\pi_0^G \mathrm{BO}_G$ is the group of virtual real representations of virtual dimension zero.

Theorem 4.5 (Karoubi)**.**

$$\Omega^\infty\Sigma^{\lambda+2}\mathrm{KO}_G \simeq \mathbb{Z}\times\mathrm{BGL}(\mathbb{H})$$

where $\mathrm{BGL}(\mathbb{H})$ *is the* G*-space constructed above.*

Proof sketch. We indicate how to recover this result from the much more general work of Karoubi. First, if we endow λ with the standard negative definite quadratic form, then the Clifford algebra $C\ell(\lambda)$ is G-equivariantly isomorphic to $\mathbb{H}$ as an algebra. It follows from the 'fundamental theorem' [Kar70, Theorem 1.1] that, when X is compact, $\mathrm{KO}_G^{2+\lambda}(X)$ is given by Karoubi's (graded) K-theory of the graded Banach category of G-$C\ell(2+\lambda)$-bundles, in the sense of [Kar68, Definition 2.1.6]. From the interpretation of this K-theory group explained on [Kar70, p.192], we learn that a class in $\mathrm{KO}_G^{2+\lambda}(X)$ is specified by a G-$C\ell(2+\lambda)$-bundle E on X together with two extensions to a G-$C\ell(3+\lambda)$-bundle structure on E. Such a triple is declared trivial if the two extensions give isomorphic bundles, and two triples are equivalent if they become isomorphic after adding a trivial triple.

As in the classical computation of Clifford algebras, we have equivariant isomorphisms:

$$C\ell(2+\lambda)\simeq \mathrm{M}_2(\mathbb{H})$$
$$C\ell(3+\lambda)\simeq \mathrm{M}_2(\mathbb{H})\times\mathrm{M}_2(\mathbb{H})$$

By Morita invariance, we may reinterpret elements in $\mathrm{KO}_G^{2+\lambda}(X)$ as equivalence classes of G-$\mathbb{H}$-bundles E equipped with two decompositions $\eta_1 : E\simeq E_0\oplus E_1$ and $\eta_2 : E\simeq E_0'\oplus E_1'$. Arguing as in [Kar08, Proposition 4.26] and [Seg68, Proposition 2.4], one can show that every such datum (E,η_1,η_2) is equivalent to one of the form $(X\times(M_0\oplus M_1),\mathrm{id},\eta)$ where M_0 and M_1 are G-$\mathbb{H}$-modules. After supplying a metric, we may replace η by the data of a sub-G-$\mathbb{H}$-module of $M_0\oplus M_1$. For fixed M_0 and M_1, this data is equivalent to an equivariant map $X\to\coprod_{k\geqslant 0}\mathrm{Gr}_k^{\mathbb{H}}(M_0\oplus M_1)$. Now the result follows by the definition of a complete G-$\mathbb{H}$-universe and the construction of $\mathrm{BGL}(\mathbb{H})$. □

At this point we may form the composite

$$\mathbb{H}P^\infty \xrightarrow{\mathcal{O}(-1)-1} \mathbb{Z}\times\mathrm{BGL}(\mathbb{H})\simeq\Omega^\infty\Sigma^{\lambda+2}\mathrm{KO}_G \xrightarrow{\eta} \Omega^\infty\Sigma^{\lambda+1}\mathrm{KO}_G$$

where

- $\mathcal{O}(-1)$ is the tautological G-$\mathbb{H}$-bundle on $\mathbb{H}P^\infty$.
- $\eta\in\pi_1^G KO_G$ is the image of $\eta\in\pi_1 S^0\subseteq\pi_1^G S^0$.

To complete the construction, we will need an equivariant version of the J-homomorphism. The J-homomorphism and equivariant spherical fibrations have been studied previously (e.g. [Seg71, McC83, Wan80, Wan82]) and it is shown in [CW91] and [Shi92] that the classifying space of equivariant stable spherical fibrations is an equivariant infinite loop space. For the reader's convenience, we prove this here, as well as the corresponding notion and results regarding Picard spectra, which provides the target for the J-homomorphism. The construction below is natural from the point of view of [BDG+16], from whom we draw inspiration.

Construction 4.6 (Picard G-spectrum)**.** The existence of norms for spectra produces a product preserving functor

$$\underline{\mathsf{Sp}}^G : \mathsf{A}^{\mathit{eff}}(G)\longrightarrow\mathsf{CAlg}(\mathsf{Cat}_\infty), G/H\mapsto\mathsf{Sp}^H,$$

where the left-hand side denotes the (effective) Burnside (2,1)-category of finite G-sets and spans [Bar17]. The formation of Picard spectra [MS16, 2.2] for symmetric monoidal ∞-categories is product preserving. We define $\mathrm{pic}(S^0)$ as the composite

$$\mathrm{pic}(S^0) : \mathsf{A}^{\mathit{eff}}(G)\longrightarrow\mathsf{CAlg}(\mathsf{Cat}_\infty)\longrightarrow\mathsf{Sp}.$$

This is a spectral Mackey functor, so we may regard it as an object in Sp^G [GM11, Nar16], called the **Picard G-spectrum** of S^0. We denote by $\mathcal{P}\mathrm{ic}(S^0)$ the 0th space of this spectrum, which is a group-like G-$\mathbb{E}_\infty$-space. We note that this G-space may be obtained directly from $\underline{\mathsf{Sp}}^G$ by assigning to the orbit G/H the maximal subgroupoid of the full subcategory of Sp^H consisting of invertible objects. If one further restricts to the full subcategory consisting of objects equivalent to S^0, this is a model for $\mathrm{BGL}_1(S^0)$, so there is a G-$\mathbb{E}_\infty$-map

$$\mathrm{BGL}_1(S^0)\to\mathcal{P}\mathrm{ic}(S^0).$$

More generally, given any virtual G-representation V, restricting, for each H, to the full subcategory of objects equivalent to $S^{\mathrm{res}_H(V)}$ produces such a map.

Warning 4.7. The space $\mathcal{P}\mathrm{ic}(S^0)$ does not decompose into a disjoint union of copies of $\mathrm{BGL}_1(S^0)$ when G is nontrivial.

Construction 4.8 (Equivariant J-homomorphism)**.** Let $\mathbf{Vect}_G$ denote the topological category of finite-dimensional G-representations. We use the same notation for the associated ∞-category. Consider the product preserving functors

$$\underline{\mathbf{Vect}}_G, \underline{\mathsf{Spaces}}^G_* : \mathsf{A}^{\mathit{eff}}(G) \longrightarrow \mathsf{CAlg}(\mathsf{Cat}_\infty)$$

given by:

- $\underline{\mathbf{Vect}}_G(G/H) := \mathbf{Vect}_H$ with direct sum, functoriality by restriction and coinduction;
- $\underline{\mathsf{Spaces}}^G_*(G/H) := \mathsf{Spaces}^H_* = \mathsf{Psh}(\mathcal{O}_H, \mathsf{Spaces})$ with $\wedge$, functoriality by restriction and norm defined by

$$N^G_H(X) := \mathrm{map}_H(G, X)/\{f : * \in f(G)\}.$$

The assignment $V \mapsto S^V$ produces a natural transformation

$$\underline{\mathbf{Vect}}_G \to \underline{\mathsf{Spaces}}^G_* \xrightarrow{\Sigma^\infty} \underline{\mathsf{Sp}}^G.$$

Restricting to maximal subgroupoids, and noting that each S^V is invertible, we get a natural transformation

$$\underline{\mathbf{Vect}}^{\simeq}_G \longrightarrow \mathcal{P}\mathrm{ic}(S^0)$$

which we may regard as a map of G-$\mathbb{E}_\infty$-spaces. The target is group-like, so this map factors through the group-completion of the source. One can identify the underlying space of that group-completion with $\mathbb{Z} \times \mathrm{BO}_G$, so we have produced a G-$\mathbb{E}_\infty$-map

$$J : \mathbb{Z} \times \mathrm{BO}_G \longrightarrow \mathcal{P}\mathrm{ic}(S^0).$$

We also denote by J the restriction to $\{0\} \times \mathrm{BO}_G = \mathrm{BO}_G$ as well as any deloopings.

Warning 4.9. Unlike the classical case, the restriction to virtual dimension zero representations

$$\mathrm{BO}_G \to \mathcal{P}\mathrm{ic}(S^0)$$

does *not* factor through $\mathrm{BGL}_1(S^0)$ when G is nontrivial.

Remark 4.10. Since $\Omega^{\lambda+1}\mathbb{H}P^\infty = \Omega^\lambda S^{\lambda+1}$ is equivariantly connected, the map $\mathbb{H}P^\infty \to \Omega^\infty\Sigma^{\lambda+1}\mathrm{KO}_G$ constructed above factors through $\mathrm{B}^{\lambda+1}\mathrm{BO}_G$.

Now we specialize to the case $G = C_{2^n}$.

Proposition 4.11. *Let g denote the composite*

$$\mathbb{H}P^\infty \longrightarrow \mathrm{B}^{\lambda+1}\mathrm{BO}_G \longrightarrow \mathrm{B}^{\lambda+1}\mathcal{P}\mathrm{ic}(S^0)$$

Then $\Omega^{\lambda+1}g$ factors through $\mathrm{BGL}_1(S^0)$ and is homotopic to μ under the equivalence $\Omega^{\lambda+1}\mathbb{H}P^\infty \simeq \Omega^\lambda S^{\lambda+1}$.

Proof. Since $\Omega^{\lambda+1}\mathbb{H}P^\infty$ is equivariantly connected, $\Omega^{\lambda+1}g$ automatically factors through $\mathrm{BGL}_1(S^0)$. To complete the proof, we need only identify the map

$$S^{\lambda+2} \to \mathbb{H}P^\infty \to \mathrm{B}^{\lambda+1}\mathcal{P}\mathrm{ic}(S^0).$$

To begin, notice that the map

$$S^{\lambda+2} \to \mathrm{B}^{\lambda+1}\mathrm{BO}_G$$

corresponds to an element of KO^{-1}_G. By [AS69, p.17], this group is $RO(G)/R(G)$. For $G = C_{2^n}$ we have

$$RO(G)/R(G) = \begin{cases} \mathbb{F}_2\{1, \sigma\} & n = 1 \\ \mathbb{F}_2 & n > 1 \end{cases}$$

Even when $n = 1$, the bundle we started with was restricted from a bundle defined for $n > 1$, so the element in question is either 0 or 1. But we know that the *underlying* class is nonzero, so we must be looking at the nonzero element in $RO(G)/R(G)$. Moreover, this class corresponds precisely to the Möbius bundle on S^1, whence the claim. □

5. An equivariant H-space orientation

The purpose of this section is to prove the following theorem.

Theorem 5.1. *There is an H-space structure on $S^{\lambda+1}_{(p)}$ such that*

(i) *The element $1-p \in \pi_0(S^0_{(p)})$ defines an H-map*

$$S^{\lambda+1}_{(p)} \to \mathrm{B}^{\lambda+1}\mathrm{GL}_1(S^0_{(p)}).$$

(ii) *The composite*

$$S^{\lambda+1}_{(p)} \to \mathrm{B}^{\lambda+1}\mathrm{GL}_1(S^0_{(p)}) \to \mathrm{B}^{\lambda+1}\mathrm{GL}_1(\mathrm{H}\underline{\mathbb{F}}_p)$$

is nullhomotopic through H-maps.

Before doing so, we record a corollary.

Corollary 5.2. *The Thom class*

$$\left(\Omega^\lambda S^{\lambda+1}_{(p)}\right)^\mu \longrightarrow \mathrm{H}\underline{\mathbb{F}}_p$$

has the structure of a map of $\mathbb{A}_2$-algebras in $\mathbb{E}_\lambda$-algebras.

Remark 5.3. The H-space structure we define is necessarily p-local. We only use this structure to produce a multiplication on the homology of $\left(\Omega^\lambda S^{\lambda+1}_{(p)}\right)^\mu$ which is preserved by the map induced by the Thom class. The localization map

$$\Omega^\lambda S^{\lambda+1} \longrightarrow \Omega^\lambda S^{\lambda+1}_{(p)}$$

induces an isomorphism on mod p homology and using this one can show that the map

$$\left(\Omega^\lambda S^{\lambda+1}\right)^\mu \longrightarrow \left(\Omega^\lambda S^{\lambda+1}_{(p)}\right)^\mu$$

is a p-local equivalence. On the other hand, the left hand side is automatically p-local, being a (albeit, equivariant) homotopy colimit of p-local spectra by construction. We can transport the $\mathbb{A}_2$-structure from the target to the source and get a map

$$\left(\Omega^\lambda S^{\lambda+1}\right)^\mu \to \mathrm{H}\underline{\mathbb{F}}_p$$

of $\mathbb{A}_2$-algebras in $\mathbb{E}_\lambda$, even though the $\mathbb{A}_2$-structure does not 'de-Thomify'.

We will need to recall a few facts about H-space objects, mainly to establish notation.

Definition 5.4. Let $\mathcal{C}$ be an ∞-category which admits products. For $0 \leqslant k \leqslant \infty$, an $\mathbb{A}_k$**-monoid** X in $\mathcal{C}$ is a truncated simplicial object

$$\mathbf{B}_{\leqslant k}X : \Delta^{op}_{\leqslant k} \longrightarrow \mathcal{C}$$

such that

- The object $(\mathbf{B}_{\leqslant k}X)_0$ is final.
- If $k \geqslant 1$, then for $1 \leqslant j \leqslant k$, the maps

  $$(\mathbf{B}_{\leqslant k}X)_j \to (\mathbf{B}_{\leqslant k}X)_1,$$

  induced by $\{i, i+1\} \to [j]$, exhibit the source as the j-fold product of $(\mathbf{B}_{\leqslant k}X)_1$.

In this case we denote $(\mathbf{B}_{\leqslant k}X)_1$ by X. If $\mathcal{C}$ admits homotopy colimits, we denote the homotopy colimit of the diagram $\mathbf{B}_{\leqslant k}X$ by $\mathrm{B}_{\leqslant k}X$.

The ∞-category of $\mathbb{A}_k$-monoids, $\mathsf{Mon}_{\mathbb{A}_k}(\mathcal{C})$, is the full subcategory of $\mathsf{Fun}(\Delta^{op}_{\leqslant k}, \mathcal{C})$ spanned by the $\mathbb{A}_k$-monoids. Note that restriction defines forgetful functors

$$\mathsf{Mon}_{\mathbb{A}_k}(\mathcal{C}) \to \mathsf{Mon}_{\mathbb{A}_j}(\mathcal{C})$$

for $j \leqslant k$, and we have natural maps

$$\mathrm{B}_{\leqslant j}X \longrightarrow \mathrm{B}_{\leqslant k}X.$$

Example 5.5 ($k = 0$)**.** An $\mathbb{A}_0$-monoid is a final object of $\mathcal{C}$. The natural maps above provide each $\mathrm{B}_{\leqslant j}X$ with a basepoint.

Example 5.6 ($k = 1$)**.** An $\mathbb{A}_1$-monoid in $\mathcal{C}$ is specified by the data of an object X and a map $* \to X$, where $*$ is a final object in $\mathcal{C}$. The object $\mathrm{B}_{\leqslant 1}X$ is computed as the colimit of

$$\begin{tikzcd} X \arrow[r, shift left=2] \arrow[r, shift right=2] & * \arrow[l] \end{tikzcd}$$

which is the suspension ΣX.

Example 5.7 ($k = 2$)**.** Suppose $\mathcal{C}$ admits limits and colimits. By [Lur17, A.2.9.16], extending a diagram $\mathbf{B}_{\leqslant 1}X$ to a diagram $\mathbf{B}_{\leqslant 2}X$ is equivalent to specifying a factorization

$$\begin{tikzcd}
& C \arrow[dr, "{(d_0,d_1,d_2)}"] & \\
X \vee X \arrow[ur] \arrow[rr] & & X \times X \times X
\end{tikzcd}$$

where $C \in \mathcal{C}$ is some object and $X \vee X$ denotes the pushout $X \amalg_* X$ where $* = (\mathbf{B}_{\leqslant 1}X)_0$ is a final object. In order for this extended diagram to define an $\mathbb{A}_2$-monoid, the maps d_0 and d_2 must give an equivalence $C \simeq X \times X$. Under this equivalence, the map $X \vee X \to X \times X$ is the standard one. The only additional data is the map $d_1 : X \times X \simeq C \to X$. In summary: an $\mathbb{A}_2$-monoid in $\mathcal{C}$ is precisely the data of a pointed object X together with a map $m : X \times X \to X$ which extends the fold map $X \vee X \to X$. It follows that the cofiber of $\mathrm{B}_{\leqslant 1}X \to \mathrm{B}_{\leqslant 2}X$ is given by $\Sigma^2(X \wedge X)$.

Example 5.8 (Loop spaces)**.** If $\mathcal{C} = \mathsf{Spaces}^G$ and Y is a pointed G-space, then ΩY has a natural $\mathbb{A}_\infty$-structure.

Now we will restrict attention to the ∞-category Spaces^G of G-spaces. The following is proved just as in the classical case, for which there are many references. The earliest appears to be [Sta61, Proposition 3.5].

Lemma 5.9. *If X is an $\mathbb{A}_2$-algebra, then the map $X \to \Omega\mathrm{B}_{\leqslant 2}X$ adjoint to*

$$\Sigma X = \mathrm{B}_{\leqslant 1}X \to \mathrm{B}_{\leqslant 2}X$$

extends to an $\mathbb{A}_2$-map.

We now return to the case of interest. We begin by establishing the existence of the $\mathbb{A}_2$-structure we need on $S^{\lambda+1}$.

Proposition 5.10. *For p an odd prime, there is an H-space structure on $S^{\lambda+1}_{(p)}$ with the property that the map $\Sigma X \to \mathrm{B}_{\leqslant 2}X$ stably splits.*

We will deduce this proposition from the following calculation, which is an equivariant version of a classical result (see, e.g. [Jam57]).

Proposition 5.11. *Let E denote the suspension homomorphism. Then, after localization at p,*

$$E(2\nu) \in E^2(\pi_{2\lambda+2}S^{\lambda+1}).$$

Proof of Proposition 5.10 assuming Proposition 5.11. In general, if one modifies an H-space structure on X by an element $d \in [X \wedge X, X]$, then the attaching map

$$\Sigma(X \wedge X) \to \Sigma X$$

for $\mathrm{B}_{\leqslant 2}X$ is altered by $E(d)$. In our case, denote the attaching map for the standard H-space structure on $S^{\lambda+1}$ by $\nu : S^{2\lambda+3} \to S^{\lambda+2}$. After inverting 2, Proposition 5.11 implies that $E(\nu) = E^2(x)$ for some $x \in \pi_{2\lambda+2}S^{\lambda+1}$. So alter the standard H-space structure by x and the suspension of the attaching map for $\mathrm{B}_{\leqslant 2}S^{\lambda+1}$ becomes null, proving the result. ☐

Now we turn to the proof of Proposition 5.11. We will deduce this theorem from a slightly stronger result. Recall that, given classes $x \in [\Sigma A, X]$ and $y \in [\Sigma B, Y]$, the Whitehead product $[x, y] \in [\Sigma(A \wedge B), X]$ is induced by the commutator of $\pi_A x$ and $\pi_B y$ in the group $[\Sigma(A \times B), X]$.

Lemma 5.12. *Let $\iota_{\lambda+2} \in \pi_{2\lambda+3}S^{\lambda+2}$ be the fundamental class. Then, after localization at p,*

$$[\iota_{\lambda+2}, \iota_{\lambda+2}] \equiv 2\nu \mod E(\pi_{\lambda+2}S^{\lambda+1}).$$

Proof of Proposition 5.11 assuming Lemma 5.12. The suspension of a Whitehead product vanishes (the proof for Spaces applies in general), which immediately implies the result. ☐

In order to prove Lemma 5.12 we will establish an exact sequence of the form

$$\pi_{2\lambda+2}S^{\lambda+1} \xrightarrow{E} \pi_{2\lambda+3}S^{\lambda+2} \xrightarrow{H} \pi_{2\lambda+3}S^{2\lambda+3}$$

and then identify the image of the Whitehead product in the last group. To that end, we note that the James splitting

$$\Sigma\Omega\Sigma X \simeq \Sigma\left(\bigvee_{k\geqslant 1} X^{\wedge k}\right)$$

holds for pointed objects in any ∞-topos[1], and, in particular, in Spaces_*^G. (For a direct argument, see [Kro10]). This provides a natural transformation

$$H : \Omega\Sigma X \to \Omega\Sigma X^{\wedge 2}$$

which induces a map

$$H : \pi_{\star+1}\Sigma X \to \pi_{\star+1}\Sigma X^{\wedge 2}$$

for any X.

Lemma 5.13. *The sequence*

$$S^{\lambda+1} \xrightarrow{E} \Omega S^{\lambda+2} \xrightarrow{H} \Omega S^{2\lambda+3}$$

is a fiber sequence when localized at p.

Proof. Let F denote the homotopy fiber of H so that we have a natural map $S^{\lambda+1} \to F$. We would like to show this is an equivalence. Since restriction to underlying spaces and fixed points preserves homotopy limits and colimits, we are reduced to the nonequivariant statement that

$$S^{2n+1} \xrightarrow{E} \Omega S^{2n+2} \xrightarrow{H} \Omega S^{4n+3}$$

is a fiber sequence when localized at p for $n = 0, 1$. In fact, it is a classical result of James [Jam57] that this is a p-local fiber sequence for any $n \geqslant 0$. ☐

We will need some control over the last term in this sequence, which is provided by an equivariant version of the Brouwer-Hopf degree theorem. For us, the only fact we need is that the homomorphism

$$\pi_{2\lambda+3}S^{2\lambda+3} \to \bigoplus_{K\subseteq G} \mathbb{Z},$$

recording each of the degrees of a map on K-fixed points, is an injection. See, e.g., [tD79, 8.4.1]. We now prove the only remaining lemma necessary for producing the exotic H-space structure on $S^{\lambda+1}$.

Proof of lemma 5.12. The formation of Whitehead products commutes with passage to fixed points and restriction to underlying classes, as does the map H. From the remarks above, it then suffices to check the nonequivariant formulas:

$$H([\iota_4, \iota_4]) = 2H(\nu),$$
$$H([\iota_2, \iota_2]) = 2H(\nu^K), \quad K \neq \{e\}.$$

But ν and $\nu^K = \eta$ (see Remark 3.5) have Hopf invariant 1, while $[\iota_{2n}, \iota_{2n}]$ has Hopf invariant 2 for any $n \geqslant 1$, whence the result. ☐

Since the attaching map in $\mathrm{B}_{\leqslant 2}S^{\lambda+1}_{(p)}$ is stably null, the following lemma is immediate.

Lemma 5.14. *There exists a dotted map making the diagram below commute up to homotopy in Sp^G:*

$$\begin{tikzcd}
S^{\lambda+2}_{(p)} \arrow[r, "1-p"] \arrow[d] & \Sigma^{\lambda+2}\mathrm{gl}_1 S^0_{(p)} \\
\Sigma^\infty \mathrm{B}_{\leqslant 2}S^{\lambda+1}_{(p)} \arrow[ur, dashed] &
\end{tikzcd}$$

We will eventually need to produce a Thom isomorphism in mod p cohomology which respects our extra structure. For that we require the next lemma.

Lemma 5.15. *Choose a dotted map $\tilde{f}$ as in the previous lemma. Then the composite*

$$\Sigma^\infty \mathrm{B}_{\leqslant 2}S^{\lambda+1}_{(p)} \xrightarrow{\tilde{f}} \Sigma^{\lambda+2}\mathrm{gl}_1(S^0_{(p)}) \longrightarrow \Sigma^{\lambda+2}\mathrm{gl}_1(\mathrm{H}\underline{\mathbb{F}}_p)$$

is null.

---

[1] The second author learned this from Gijs Heuts, who attributes the essential idea to Mike Hopkins, who in turn attributes the essential idea to Tudor Ganea [Gan68].

Proof. The composite

$$\begin{tikzcd} S^{\lambda+2}_{(p)} \arrow[r] & \Sigma^{\infty}\mathrm{B}_{\leqslant 2}S^{\lambda+1}_{(p)} \arrow[r, "\tilde{f}"] & \Sigma^{\lambda+2}\mathrm{gl}_1(S^0_{(p)}) \arrow[r] & \Sigma^{\lambda+2}\mathrm{gl}_1(\mathrm{H}\underline{\mathbb{F}}_p) \end{tikzcd}$$

vanishes since $1-p=1\in\pi_0^G(\mathrm{gl}_1(\mathrm{H}\underline{\mathbb{F}}_p)$ is the basepoint component. So the map $\tilde{f}$ factors through some map

$$S^{2\lambda+4}\longrightarrow\Sigma^{\lambda+2}\mathrm{gl}_1(\mathrm{H}\underline{\mathbb{F}}_p).$$

But

$$\pi^G_{\lambda+2}\mathrm{gl}_1(\mathrm{H}\underline{\mathbb{F}}_p)\simeq\pi^G_{\lambda+2}\mathrm{H}\underline{\mathbb{F}}_p=0$$

since $S^{\lambda+2}$ is 2-connective, whence the claim. □

Finally, we arrive at the proof of the main theorem of the section.

Proof of Theorem 5.1. Choose a dotted map as in Lemma 5.14 and let f be its adjoint,

$$f:\mathrm{B}_{\leqslant 2}S^{\lambda+1}_{(p)}\longrightarrow\mathrm{B}^{\lambda+2}\mathrm{GL}_1(S^0_{(p)}).$$

Then the map $1-p:S^{\lambda+1}_{(p)}\to\mathrm{B}^{\lambda+1}\mathrm{GL}_1(S^0_{(p)})$ factors as a composite:

$$\begin{tikzcd} S^{\lambda+1}_{(p)} \arrow[r] & \Omega\mathrm{B}_{\leqslant 2}S^{\lambda+1}_{(p)} \arrow[r, "\Omega f"] & \Omega\mathrm{B}^{\lambda+2}\mathrm{GL}_1(S^0_{(p)}) \arrow[r] & \mathrm{B}^{\lambda+1}\mathrm{GL}_1(S^0_{(p)}) \end{tikzcd} ,$$

each of which is an H-map. This proves part (i) of the theorem.

To prove part (ii), consider the diagram:

$$\begin{tikzcd} S^{\lambda+1}_{(p)} \arrow[r] & \Omega\mathrm{B}_{\leqslant 2}S^{\lambda+1}_{(p)} \arrow[r, "\Omega f"] & \Omega\mathrm{B}^{\lambda+2}\mathrm{GL}_1(S^0_{(p)}) \arrow[r] \arrow[d, "\Omega\mathrm{B}^{\lambda+2}\mathrm{GL}_1(\iota)"'] & \mathrm{B}^{\lambda+1}\mathrm{GL}_1(S^0_{(p)}) \arrow[d] \\ & & \Omega\mathrm{B}^{\lambda+2}\mathrm{GL}_1(\mathrm{H}\underline{\mathbb{F}}_p) \arrow[r] & \mathrm{B}^{\lambda+1}\mathrm{GL}_1(\mathrm{H}\underline{\mathbb{F}}_p) \end{tikzcd}$$

where $\iota:S^0_{(p)}\to\mathrm{H}\underline{\mathbb{F}}_p$ is the unit map.

The composite

$$\begin{tikzcd} \mathrm{B}_{\leqslant 2}S^{\lambda+1}_{(p)} \arrow[r, "f"] & \mathrm{B}^{\lambda+2}\mathrm{GL}_1(S^0_{(p)}) \arrow[r, "\mathrm{B}^{\lambda+2}\mathrm{GL}_1(\iota)"] & \mathrm{B}^{\lambda+2}\mathrm{GL}_1(\mathrm{H}\underline{\mathbb{F}}_p) \end{tikzcd}$$

is null by Lemma 5.15. The loop of this composite is then null through $\mathbb{A}_\infty$-maps and the result follows. □

6. Computing the zeroth homotopy Mackey functor

In this section, we establish that the zeroth homotopy Mackey functor of our Thom spectrum is as expected. That is to say, we give a proof that

$$\underline{\pi}_0\left(\Omega^\lambda S^{\lambda+1}\right)^\mu=\underline{\mathbb{F}}_p.$$

By construction, $\left(\Omega^\lambda S^{\lambda+1}\right)^\mu$ receives a map from the mod p Moore spectrum $M(p)=(S^1)^\mu$. This is enough to guarantee that $p=0$ in $\underline{\pi}_0\left(\Omega^\lambda S^{\lambda+1}\right)^\mu$. However, $\underline{\pi}_0S^0$ is the Burnside Mackey functor $\underline{A}$, and $\underline{A}/(p)$ is not $\underline{\mathbb{F}}_p$. For example, when $G=C_p$, we have

$$\underline{A}/(p)=\begin{tikzcd} \mathbb{F}_p\left\{[C_p]\right\} \arrow[d, bend right] \\ \mathbb{F}_p \arrow[u, bend right] \end{tikzcd} .$$

We will need to use some extra structure on $\left(\Omega^\lambda S^{\lambda+1}\right)^\mu$ to show that $[C_p]$ also vanishes. More generally, we must show that $[C_{p^n}/C_{p^k}]$ vanishes in the Hurewicz image for all k.

For the remainder of this section we write $G=C_{p^n}$ for a cyclic group of prime power order.

Definition 6.1. We say that a G-spectrum X is **equipped with norms** if it is equipped with a unit $S^0\to X$ and maps of H-spectra $\mathrm{N}^HX\to X$ extending the unit for all $H\subseteq G$.

The Thom spectrum $\left(\Omega^\lambda S^{\lambda+1}\right)^\mu$ is equipped with norms, as we now show. This result is well-known; compare, for example, [Hil17, Theorem 2.12].

Lemma 6.2. *If X is an $\mathbb{E}_\lambda$-algebra then it is canonically equipped with norms.*

Proof. Since the restriction of an $\mathbb{E}_\lambda$-algebra is still an $\mathbb{E}_\lambda$-algebra, it will suffice to construct the norm $\mathrm{N}^G X \to X$.

By definition, X comes equipped with a map

$$\widetilde{\mathrm{Conf}}_{p^n}(\lambda)_+ \wedge_{\Sigma_{p^n}} X^{\wedge p^n} \longrightarrow X,$$

where $\widetilde{\mathrm{Conf}}_{p^n}(\lambda)$ denotes the G-space of configurations of p^n ordered points in λ. Consider the inclusion $G \hookrightarrow \Sigma_{p^n}$ which sends a generator to the standard p^n-cycle $(1, 2, ..., p^n)$ and let Γ denote the graph of this inclusion. Let $\zeta = e^{2\pi i/p^n}$. Then the ordered tuple $(1, \zeta, \zeta^2, ..., \zeta^{p^n-1}) \in \widetilde{\mathrm{Conf}}_{p^n}(\lambda)$ produces a $G \times \Sigma_{p^n}$-equivariant inclusion:

$$\frac{G \times \Sigma_{p^n}}{\Gamma} \longrightarrow \widetilde{\mathrm{Conf}}_{p^n}(\lambda).$$

This, in turn, gives us a map

$$\left(\frac{G \times \Sigma_{p^n}}{\Gamma}\right)_+ \underset{\Sigma_{p^n}}{\wedge} X^{\wedge p^n} \longrightarrow X.$$

To complete the proof, we note that, for any G-spectrum Y, we have

$$\left(\frac{G \times \Sigma_{p^n}}{\Gamma}\right)_+ \underset{\Sigma_{p^n}}{\wedge} Y^{\wedge p^n} \simeq \mathrm{N}^G Y.$$

Indeed, the norm $\mathrm{N}^G : \mathsf{Sp} \longrightarrow \mathsf{Sp}^G$ is uniquely determined by the property that it is symmetric monoidal, commutes with sifted homotopy colimits, and satisfies the identity

$$\mathrm{N}^G(T_+) \simeq (\mathrm{map}(G, T))_+$$

for all finite sets T. One verifies readily that the functor $\left(\frac{G\times\Sigma_{p^n}}{\Gamma}\right)_+ \wedge_{\Sigma_{p^n}} (-)$ extends in an essentially unique way to a symmetric monoidal functor satisfying these properties. □

Remark 6.3. We have been vague in how one interprets the expression $\left(\frac{G\times\Sigma_{p^n}}{\Gamma}\right)_+ \wedge_{\Sigma_{p^n}} Y$. Either, one computes this expression literally in orthogonal $G \times \Sigma_{p^n}$-spectra when X is appropriately cofibrant, or else one defines this as a symmetric monoidal functor $\mathsf{Sp}^{G\times\Sigma_{p^n}} \to \mathsf{Sp}^G$ extending the appropriate map on spaces and commuting with homotopy colimits and suspension spectra.

Proposition 6.4. *Suppose X is equipped with norms. Suppose further that $p = 0 \in \pi_0^G X$. Then $[H/K] = 0 \in \pi_0^H X$ for all $K \subseteq H \subseteq G$.*

Corollary 6.5. *We have*

$$\underline{\pi}_0 \left(\Omega^\lambda S^{\lambda+1}\right)^\mu = \underline{\mathbb{F}}_p.$$

Proof of Proposition 6.4. Recall that $G = C_{p^n}$. If the result is proved for $C_{p^{n-1}} \subseteq G$, then the classes

$$p, \mathrm{tr}^G_{C_{p^{n-1}}}([C_{p^{n-1}}]) = [G], \mathrm{tr}^G_{C_{p^{n-1}}}([C_{p^{n-1}}/C_p]) = [G/C_p], ..., [G/C_{p^{n-2}}]$$

all vanish in $\pi_0^G X$. The result now follows from the next lemma. □

Lemma 6.6. *If X is equipped with norms, then*

$$\mathrm{N}^G(p) \equiv [G/C_{p^{n-1}}] \mod (p, [G/K] : K \subsetneq C_{p^{n-1}}).$$

Proof. It suffices to prove this formula when $X = S^0$, i.e. for the Burnside Mackey functor $\underline{A}$. By [HHR16, Lemma A.36], or by definition depending on how one sets up the theory, the norm of p is the class of the G-set $\mathrm{map}(G, \{1, ..., p\})$. By recording the size of the image of a map, we get an equality in $A(G)$:

$$[\mathrm{map}(G, \{1, ..., p\})] = \sum_{0<k\leqslant n} \binom{p}{k} [\mathrm{surj}(G, \{1, ..., k\})]$$

where $\mathrm{surj}(G, \{1, ..., k\})$ denotes the G-set of surjective maps $G \to \{1, ..., k\}$. So we have

$$\mathrm{N}^G(p) \equiv [\mathrm{surj}(G, \{1, ..., p\})] \mod p.$$

We are only concerned with the orbits in $\mathrm{surj}(G, \{1, ..., p\})$ with isotropy $C_{p^{n-1}}$ or G. There is only one orbit with isotropy $C_{p^{n-1}}$, namely the orbit of the quotient map $G \to G/C_{p^{n-1}} \simeq \{1, ..., p\}$. There are p orbits with isotropy G, namely the p constant maps. This completes the proof. □

7. Proof of the main theorem

We are now ready to prove the main theorem, which we recall here for convenience.

Theorem 7.1. *Let $G = C_{p^n}$ and let $S^1 \to \mathrm{BGL}_1(S^0_{(p)})$ be adjoint to $1-p \in \pi_0^G S^0_{(p)}$. Denote by $\mu : \Omega^\lambda S^{\lambda+1} \to \mathrm{BGL}_1(S^0_{(p)})$ the extension of this map over the λ-loop space. Then the Thom class*

$$\left(\Omega^\lambda S^{\lambda+1}\right)^\mu \longrightarrow \mathrm{H}\underline{\mathbb{F}}_p$$

is an equivalence of G-spectra.

Before the proof, we record a well-known computation (the $p = 2$ case is proven in [HHR16, Proposition 3.18] and the odd primary proof is much the same).

Lemma 7.2. *For $G = C_{p^n}$ and p odd, we have*

$$\pi_* \mathrm{H}\underline{\mathbb{Z}}^{\Phi G} = \mathbb{F}_p[t],\ |t| = 2,$$
$$\pi_* \mathrm{H}\underline{\mathbb{F}}_p^{\Phi G} = \mathbb{F}_p[t] \otimes \Lambda(s),\ s = \beta t$$

When $p = 2$, the second computation becomes

$$\pi_* \mathrm{H}\underline{\mathbb{F}}_2^{\Phi G} = \mathbb{F}_2[s],\ |s| = 1.$$

We also will need the corresponding result about our Thom spectrum.

Lemma 7.3. *Let X denote the Thom spectrum $(\Omega^\lambda S^\lambda)^\mu$. Then the homotopy group $\pi_*(X^{\Phi G})$ of the geometric fixed points is isomorphic to $\mathbb{F}_p$ for each $* \geqslant 0$.*

Proof. As we have equipped X with the structure of an $\mathbb{A}_2$-algebra in $\mathbb{E}_\lambda$-algebras, the norm map

$$\mathrm{N}^G(X) \longrightarrow X$$

is a map of $\mathbb{A}_2$-algebras. In particular, $X^{\Phi G}$ is a module over $(\mathrm{N}^G(X))^{\Phi G} \simeq (\mathrm{N}^G(\mathrm{H}\mathbb{F}_p))^{\Phi G} \simeq \mathrm{H}\mathbb{F}_p$. Since $\mathrm{H}\mathbb{F}_p$ is a field spectrum, $X^{\Phi G}$ splits as a wedge of suspensions of $\mathrm{H}\mathbb{F}_p$. The homotopy groups of $X^{\Phi G}$ are then determined by the homology groups of $X^{\Phi G}$. By the Thom isomorphism, $\mathrm{H}_*(X^{\Phi G}) \simeq \mathrm{H}_*(\Omega^2 S^3 \times \Omega S^2)$, and the result follows. □

Proof of the main theorem. We prove the theorem by induction on n. When $n = 0$, this is the non-equivariant result of Hopkins-Mahowald. For the induction hypothesis we assume that the map

$$\alpha : X := \left(\Omega^\lambda S^{\lambda+1}\right)^\mu \longrightarrow \mathrm{H}\underline{\mathbb{F}}_p$$

is an equivalence after restriction to $C_{p^{n-1}}$, and we assume that $n \geqslant 1$ from now on.

We need only show that the map on geometric fixed point spectra

$$\alpha^{\Phi G} : X^{\Phi G} = \left(\left(\Omega^\lambda S^{\lambda+1}\right)^\mu\right)^{\Phi G} \longrightarrow \mathrm{H}\underline{\mathbb{F}}_p^{\Phi G}$$

is an equivalence. By Lemma 7.3, we know that (additively)

$$\pi_* X^{\Phi G} = \mathbb{F}_p[x] \otimes \Lambda(y),\ |x| = 2, |y| = 1.$$

Thus, by Lemma 7.2, both the source and target have the same homotopy groups, additively.

By Theorem 5.1, when p is odd we can put an $\mathbb{A}_2$-structure on X such that α, and hence $\alpha^{\Phi G}$, is an $\mathbb{A}_2$-map. When $p = 2$, the Thom class is already an $\mathbb{E}_{\lambda+1}$-map. In either case, $\alpha^{\Phi G}$ induces a ring map on homotopy. It therefore suffices to show that s and t lie in the image of $\alpha^{\Phi G}$. This follows somewhat formally from the inductive hypothesis and the computation of (6.5), as we now explain.

First we show that s lies in the image. Define a G-spectrum $S^{1-\lambda/2}$ as the cofiber of the transfer map

$$S^0 \to G_+ \wedge S^0 \to S^{1-\lambda/2}.$$

By Corollary 6.5, we know $\underline{\pi}_0 X = \underline{\mathbb{F}}_p$, which has zero transfer map. From the diagram of exact sequences

$$\begin{tikzcd}
\pi_1 X^G \arrow[r] \arrow[d] & {[S^{1-\lambda/2}, X]} \arrow[r] \arrow[d] & \pi_0^u X \arrow[r, "0"] \arrow[d] & \pi_0 X^G \arrow[d] \\
0 \arrow[r] & {[S^{1-\lambda/2}, \mathrm{H}\underline{\mathbb{F}}_p]} \arrow[r] & \mathbb{F}_p \arrow[r, "0"] & \mathbb{F}_p
\end{tikzcd} \tag{2}$$

we learn that

$$[S^{1-\lambda/2}, X] \to [S^{1-\lambda/2}, \mathrm{H}\underline{\mathbb{F}}_p]$$

is surjective. On the other hand, passage to geometric fixed points produces an isomorphism

$$[S^{1-\lambda/2}, \mathrm{H}\underline{\mathbb{F}}_p] \to \pi_1 \mathrm{H}\underline{\mathbb{F}}_p^{\Phi G}.$$

It follows that s lies in the image of α^G.

Next, we argue that $\underline{\pi}_1 X = 0$. Indeed, let $\mathcal{P}$ denote the family of proper subgroups of G and let $X_{h\mathcal{P}}$ denote the spectrum $(X \wedge \mathrm{E}\mathcal{P}_+)^G$ (see, for example, [HHR16, 2.5.2]). Then there is a diagram of cofiber sequences of spectra

$$\begin{tikzcd}
X_{h\mathcal{P}} \arrow[r] \arrow[d] & X^G \arrow[r] \arrow[d] & X^{\Phi G} \arrow[d] \\
\left(\mathrm{H}\underline{\mathbb{F}}_p\right)_{h\mathcal{P}} \arrow[r] & \mathrm{H}\mathbb{F}_p \arrow[r] & \mathrm{H}\underline{\mathbb{F}}_p^{\Phi G}
\end{tikzcd}$$

By the induction hypothesis, the left vertical map is an equivalence. Since we have shown that s is hit, the right vertical map is an isomorphism on both π_0 and π_1. It follows that the middle vertical map is injective on π_1, and hence that $\pi_1 X^G = 0$, proving the claim.

Returning to the diagram (2) we learn that $[S^{1-\lambda/2}, X] \to [S^{1-\lambda/2}, \mathrm{H}\underline{\mathbb{F}}_p]$ is in fact an isomorphism.

Finally, we turn to the following cofiber sequence:

$$S^{1-\lambda/2} \to G_+ \wedge S^0 \to S^{2-\lambda}.$$

This leads to the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \pi_{2-\lambda} X \arrow[r] \arrow[d] & \pi_0^u X \arrow[r] \arrow[d, "\simeq"] & {[S^{1-\lambda/2}, X]} \arrow[d, "\simeq"] \\
0 \arrow[r] & \pi_{2-\lambda} \mathrm{H}\underline{\mathbb{F}}_p \arrow[r] & \pi_0^u \mathrm{H}\underline{\mathbb{F}}_p \arrow[r] & {[S^{1-\lambda/2}, \mathrm{H}\underline{\mathbb{F}}_p]}
\end{tikzcd}$$

Thus $\pi_{2-\lambda} X \to \pi_{2-\lambda} \mathrm{H}\underline{\mathbb{F}}_p$ is an isomorphism. Passage to geometric fixed points gives an isomorphism $\pi_{2-\lambda} \mathrm{H}\underline{\mathbb{F}}_p \to \pi_2 \mathrm{H}\underline{\mathbb{F}}_p^{\Phi G}$, and this completes the proof. ☐

8. CONCLUDING REMARKS

We end with some miscellaneous remarks.

The integral Eilenberg-MacLane spectrum. Recall that Mahowald also showed that $\mathrm{H}\mathbb{Z}_{(p)}$ can be obtained as the Thom spectrum of a bundle on $\Omega^2(S^3\langle 3\rangle)$. This result also holds in our context. Define $S^{\lambda+1}\langle\lambda+1\rangle$ as the fiber of the unit map

$$S^{\lambda+1} \to K(\underline{\mathbb{Z}}, \lambda+1) := \Omega^\infty \left(\Sigma^{\lambda+1}\mathrm{H}\underline{\mathbb{Z}}\right).$$

Then we have the following result.

Theorem 8.1. *There is an equivalence of* $\mathbb{E}_\lambda$*-algebras*

$$\left(\Omega^\lambda(S^{\lambda+1}\langle\lambda+1\rangle)\right)^\mu \simeq \mathrm{H}\underline{\mathbb{Z}}_{(p)}.$$

Proof. We argue as in Antolín-Camarena-Barthel [AB14, §5.2], though we need not develop all the technology present there. We have a fiber sequence

$$\Omega^\lambda S^{\lambda+1}\langle\lambda+1\rangle \to \Omega^\lambda S^{\lambda+1} \to S^1.$$

Decomposing S^1 into a 0-cell and a 1-cell, and trivializing the fibration on each cell, produces a decomposition of the Thom spectrum $\left(\Omega^\lambda S^{\lambda+1}\right)^\mu$ as a cofiber

$$\left(\Omega^\lambda S^{\lambda+1}\langle\lambda+1\rangle\right)^\mu \xrightarrow{x} \left(\Omega^\lambda S^{\lambda+1}\langle\lambda+1\rangle\right)^\mu \longrightarrow \left(\Omega^\lambda S^{\lambda+1}\right)^\mu \simeq \mathrm{H}\underline{\mathbb{F}}_p.$$

Each of these Thom spectra came from bundles classified by $\mathbb{A}_2$-maps, which is enough to ensure that the map x induces a map $\pi_* \left(\Omega^\lambda S^{\lambda+1}\langle\lambda+1\rangle\right)^\mu \to \pi_* \left(\Omega^\lambda S^{\lambda+1}\langle\lambda+1\rangle\right)^\mu$ of modules over $\pi_0 \left(\Omega^\lambda S^{\lambda+1}\langle\lambda+1\rangle\right)^\mu$. In particular, on homotopy the map corresponds to multiplication by some element $x \in \pi_0 \left(\Omega^\lambda S^{\lambda+1}\langle\lambda+1\rangle\right)^\mu$. Arguments similar to those in the proof of the main theorem show that

$$\underline{\pi}_0 \left(\Omega^\lambda S^{\lambda+1}\langle\lambda+1\rangle\right)^\mu \simeq \underline{\mathbb{Z}}_{(p)},$$

so we must have $x = p$. The result follows from Nakayama's lemma once one argues that the genuine fixed point spectra have finitely generated homotopy groups in each degree. (For example, isotropy separation reduces us to the corresponding statement on geometric fixed points, where it follows from the Thom isomorphism.) $\square$

Remark 8.2. Unlike the classical case, it is unclear whether the statement globalizes to a construction of $\mathrm{H}\underline{\mathbb{Z}}$. Our methods do not construct $\mathrm{H}\underline{\mathbb{F}}_\ell$ as a Thom spectrum when ℓ does not divide the order of G.

Remark 8.3. The main results of Antolín-Camarena-Barthel [AB14] and Beardsley [Bea16] carry over to the equivariant context once one identifies equivariant Thom spectra as parameterized homotopy colimits in the sense of [BDG$^+$16]. It would be nice, going forward, to have a reference that verifies this.

Brown-Gitler spectra. One of Mahowald's motivations for proving the equivalence $(\Omega^2S^3)^\mu \simeq \mathrm{H}\mathbb{F}_2$ is that the left hand side carries a natural filtration due to Milgram and May. This produces a filtration of $\mathrm{H}\mathbb{F}_2$ by spectra which turn out to be the Brown-Gitler spectra of [BG73] (see [BP78, Coh79, HK99]). The G-space $\Omega^\lambda S^{\lambda+1}$ also carries the arity filtration from the $\mathbb{E}_\lambda$-operad, so we could *define* equivariant Brown-Gitler spectra using this filtration. It would be interesting to know if these spectra could be of any use. In the case $G = C_2$, using the arity filtration of $\Omega^\rho S^{\rho+1}$, the second author has been able to establish some equivariant versions of classical results for these Brown-Gitler spectra. The odd primary case is less well-understood.

Milnor operations. The Milnor operations $Q_i : \mathrm{H}\mathbb{F}_p \to \Sigma^{|Q_i|}\mathrm{H}\mathbb{F}_p$ can be characterized as precisely those operations which arise as $\mathbb{E}_1$-maps

$$\mathrm{H}\mathbb{F}_p \to \mathrm{H}\mathbb{F}_p \oplus \Sigma^{|Q_i|}\mathrm{H}\mathbb{F}_p$$

augmented over $\mathrm{H}\mathbb{F}_p$, where the target denotes the trivial square-zero extension. By the Hopkins-Mahowald theorem and the universal properties of Thom spectra, such $\mathbb{E}_1$-maps are in canonical bijection with $\mathbb{E}_1$-maps

$$\Omega^2S^3 \longrightarrow K(\mathbb{F}_p, |Q_i|).$$

These, in turn, are in bijection with just ordinary maps

$$\Omega S^3 \longrightarrow K(\mathbb{F}_p, |Q_i| + 1).$$

In other words: the Milnor operations come from the cohomology of the space ΩS^3.

Denote by $\Omega^{\lambda/2}X$ the C_p-space of maps $\mathrm{map}_*(S^{\lambda/2}, X)$, where $S^{\lambda/2}$ is the cofiber of the map $C_{p+} \to S^0$. Using a generalization of the James splitting and an argument as above, the authors have constructed analogues of Milnor operations in C_p-equivariant homotopy theory. The properties of these operations and their possible relationship to an equivariant analogue of the Brown-Peterson spectrum is the subject of ongoing work.

Remark 8.4. It seems possible that the C_p-equivariant dual Steenrod algebra might be understood via its equivalence with $\pi_\star(\mathrm{H}\underline{\mathbb{F}}_p \wedge \Omega^\lambda S^{\lambda+1})$. This is also the subject of work in progress by the second author and Krishanu Sankar using entirely different methods. We note that, at odd primes, $\mathrm{H}\underline{\mathbb{F}}_p \wedge \mathrm{H}\underline{\mathbb{F}}_p$ does not split as a wedge of $RO(C_p)$-graded suspensions of $\mathrm{H}\underline{\mathbb{F}}_p$, so the relationship between the Steenrod algebra and its dual is slightly more mysterious. The second author and Krishanu Sankar have, however, obtained splittings involving more exotic objects than representation spheres.

Questions. We end with a few questions that we do not intend to investigate in the near future, but which others may find interesting.

Question 8.5. Calculations indicate that the spectrum $\left(\Omega^\lambda S^{\lambda+1}\right)^\mu$ is not $\mathrm{H}\underline{\mathbb{F}}_p$ for the groups C_n when n is not a power of p. What can be said about $\left(\Omega^\lambda S^{\lambda+1}\right)^\mu$ as an S^1 or even $\mathrm{SU}(2)/\pm 1 \simeq \mathrm{SO}(3)$-equivariant spectrum? At least, one expects an interesting C_{p^∞}-spectrum.

Question 8.6. Is $\mathrm{H}\underline{\mathbb{F}}_p$ a Thom spectrum for any group G that is not cyclic of p-power order? Perhaps an obstruction could be found. Alternatively, it would be very interesting if a different source of group actions on Ω^2S^3 were located.

Question 8.7. In the case $G = C_2$ there are two different operadic filtrations of $\Omega^{2\sigma}S^{2\sigma+1} \simeq \Omega^\rho S^{\rho+1}$. This leads to two different notions of Brown-Gitler spectra. How are they related?

Question 8.8. What are the Thom spectra obtained by killing other natural elements in the Burnside ring in a highly structured manner? What is the free $\mathbb{E}_\lambda$-algebra in C_p-spectra with $[C_p] = 0$?

Question 8.9. The original results of Hopkins and Mahowald offer a powerful nilpotence criterion: In the homotopy of $\mathbb{E}_2$-algebras, p-torsion classes are nilpotent if and only if their mod p Hurewicz images are nilpotent. Is any interesting analogue of that criterion afforded by the results here? In [Hah17], the first author more generally investigated nilpotence of p^k-torsion elements in $\mathbb{E}_n$-ring spectra. Are there interesting extensions of that work for equivariant operads like $\mathbb{E}_\lambda$?

Appendix A. Proof of Theorem 1.6

For convenience, we recall the statement of Theorem 1.6, which this appendix is devoted to proving. The result is entirely non-equivariant.

Theorem. *Let S^0_p denote the p-complete sphere spectrum, and suppose that $p > 2$. Then there is no triple loop map*

$$X \longrightarrow \mathrm{BGL}_1(S^0_p),$$

for any triple loop space X, that makes $\mathrm{H}\mathbb{F}_p$ as a Thom spectrum.

Proof. Suppose, for the sake of contradiction, that such a triple loop map

$$X \longrightarrow \mathrm{BGL}_1(S^0_p)$$

exists. The Thom isomorphism then implies that

$$\mathrm{H}\mathbb{F}_p \wedge \Sigma^\infty_+ X \simeq \mathrm{H}\mathbb{F}_p \wedge \mathrm{H}\mathbb{F}_p$$

as $\mathrm{H}\mathbb{F}_p$–$\mathbb{E}_3$-algebras.

In particular, by Theorem 1.5,

$$\mathrm{H}\mathbb{F}_p \wedge \Sigma^\infty_+ X \simeq \mathrm{H}\mathbb{F}_p \wedge \Sigma^\infty_+ \Omega^2 S^3$$

as $\mathrm{H}\mathbb{F}_p$–$\mathbb{E}_2$-algebras, and the latter object is the free $\mathrm{H}\mathbb{F}_p$–$\mathbb{E}_2$-algebra on a class in degree 1.

The Hurewicz theorem gives a map $S^1 \to X$, which extends to a double-loop map $\Omega^2 S^3 \to X$, and the above discussion implies that this double loop map is a homology isomorphism. Thus, the p-completion of X is the p-completion of $\Omega^2 S^3$, as a double loop space.

Transporting the $\mathbb{E}_3$-algebra structure on X yields an $\mathbb{E}_3$-algebra structure on the p-completion of $\Omega^2 S^3$, extending the usual $\mathbb{E}_2$-algebra structure. The theorems of Dwyer, Miller, and Wilkerson [DMW87] show that there is a unique such $\mathbb{E}_3$-algebra structure, and so the p-completion of $\mathrm{B}^3 X$ must be the p-completion of $\mathbb{H}P^\infty$.

Now, the composite

$$X \longrightarrow \mathrm{BGL}_1(S^0_p) \longrightarrow \mathrm{BGL}_1(\mathrm{H}\mathbb{F}_p)$$

is null, and it follows that there is a factorization through the fiber F of $\mathrm{BGL}_1(S^0_p) \longrightarrow \mathrm{BGL}_1(\mathrm{H}\mathbb{F}_p)$. The equivalence $\mathbb{Z}_p^\times \cong \mu_{p-1} \times \mathbb{Z}_p$ implies that the homotopy groups of F are p-complete. Thus, with $\mathbb{H}P^\infty_p$ denoting the p-completion of $\mathbb{H}P^\infty$, there is a commuting diagram

$$\begin{tikzcd}
 & \mathrm{B}^3 X \arrow[r] \arrow[d] & B^4 GL_1(S^0_p). \\
\mathbb{H}P^\infty \arrow[r] & \mathbb{H}P^\infty_p \arrow[ur] &
\end{tikzcd}$$

In particular, there is a triple-loop map

$$\Omega^2 S^3 \longrightarrow \Omega^3 \mathbb{H}P^\infty \longrightarrow \mathrm{BGL}_1(S^0_p)$$

with Thom spectrum equivalent (at least after p-completion) to $\mathrm{H}\mathbb{F}_p$.

The underlying double loop map is determined by a class in $1 + p\alpha \in \pi_3(\mathrm{B}^3\mathrm{GL}_1(S^0_p)) \cong \mathbb{Z}_p^\times$. Our original assumption, made for the sake of contradiction, is reduced to the assertion that a dashed arrow exists the diagram below:

$$\begin{tikzcd}
S^4 \arrow[r, "1+p\alpha"] \arrow[d] & \mathrm{B}^4\mathrm{GL}_1(S^0_p). \\
\mathbb{H}P^\infty \arrow[ur, dashed] &
\end{tikzcd}$$

We will show this to be impossible by proving the non-existence of a solution to the weaker lifting problem

$$\begin{tikzcd}
S^4 \arrow[r, "1+p\alpha"] \arrow[d] & \Sigma^\infty \mathrm{B}^4\mathrm{GL}_1(S^0_p) \arrow[r, "\ell"] & \Sigma^4 L_{K(1)}S^0, \\
\Sigma^\infty \mathbb{H}P^\infty \arrow[urr, dashed] & &
\end{tikzcd}$$

where $L_{K(1)}S^0$ is $K(1)$-local sphere spectrum and ℓ is the Rezk logarithm [Rez06]. We first calculate the composite

$$S^4 \xrightarrow{1+p\alpha} \mathrm{B}^4\mathrm{GL}_1(S^0_p) \xrightarrow{\ell} \Sigma^4 L_{K(1)}S^0,$$

using Rezk's formula [Rez06, Theorem 1.9] for the logarithm at odd primes:

$$\ell(1+p\alpha) = log(1+p\alpha) - \frac{1}{p} log(1+p\alpha).$$

If α were not a p-adic unit, then the composite $\Omega^2 S^3 \longrightarrow \mathrm{BGL}_1(S^0) \longrightarrow \mathrm{BGL}_1(\mathrm{H}\mathbb{Z}/p^2)$ would be null as a 2-fold loop map, providing a ring map $\mathrm{H}\mathbb{F}_p \longrightarrow \mathrm{H}\mathbb{Z}/p^2$. Since this is absurd, α must be a p-adic unit, and we learn that $\ell(1+p\alpha)$ is also a p-adic unit.

Without loss of generality, then, we are reduced to showing the impossibility of the following lifting problem:

$$\begin{tikzcd}
S^4 \arrow[r, "1"] \arrow[d] & \Sigma^4 L_{K(1)}S^0, \\
\Sigma^\infty \mathbb{H}P^\infty \arrow[ur, dashed] &
\end{tikzcd}$$

where 1 is the unit of the ring spectrum $L_{K(1)}S^0$.

Let KU_p denote p-complete complex K-theory. Recall that the composite

$$L_{K(1)}S^0 \longrightarrow \mathrm{KU}_p \xrightarrow{\psi^q - 1} \mathrm{KU}_p$$

is null for any Adams operation ψ^q with q relatively prime to p. Since p is odd, to finish the problem it will suffice for us to show that no element of $\mathrm{KU}_p^4(\mathbb{H}P^\infty)$ simultaneously:

(1) Restricts to the unit in $\mathrm{KU}^4_p(S^4)$.
(2) Is invariant under the action of ψ^2.

Now,

$$\mathrm{KU}^*_p(\mathbb{H}P^\infty) \cong \mathbb{Z}_p[\![e]\!][\beta^\pm],$$

where $|e| = 0$ and β is the Bott class in degree -2. Of course, $\psi^2(\beta) = 2\beta$, and it will be necessary also to understand $\psi^2(e)$.

Remembering that $\mathbb{H}P^\infty$ is $\mathrm{BSU}(2)$, we may calculate $\psi^2(e)$ by determining the restriction of e along the inclusion of the maximal torus $\mathrm{B}S^1 \longrightarrow \mathrm{BSU}(2)$. Indeed, $\mathrm{KU}^*_p(\mathrm{B}S^1) \cong \mathbb{Z}_p[\![x]\!][\beta^{\pm 1}]$, where $x = L - 1$. On the other hand, $e = V - 2$, where V is the standard representation of $\mathrm{SU}(2)$ on $\mathbb{C}^2$. The restriction of e is thus $L + L^{-1} - 2$, where

$$L^{-1} = (x+1)^{-1} = 1 - x + x^2 - x^3 + \cdots .$$

Since

$$\psi^2(L+L^{-1}-2) = L^2+L^{-2}-2 = (x+1)^2 + \frac{1}{(x+1)^2} - 2 = \left(x+1+\frac{1}{x+1}-2\right)^2 + 4\left(x+1+\frac{1}{x+1}-2\right),$$

we calculate that

$$\psi^2(e) = e^2 + 4e.$$

An element of $\mathrm{KU}^4(\mathbb{H}P^\infty)$ is of the form $\beta^{-2}P(e)$, where $P(e)$ is a power series in $\mathbb{Z}_p[\![e]\!]$. The lifting problem in question is equivalent to finding a power series $P(e) = e + c_2 e^2 + \cdots$ such that

$$P(e) = 2^{-2} P\left(\psi^2(e)\right).$$

Using the calculations above, this can be rewritten as the relation

$$4P(e) = P(e^2 + 4e).$$

The relation

$$4(e + c_2 e^2 + c_3 e^3 + \cdots) = (e^2 + 4e) + c_2(e^2+4e)^2 + c_3(e^2+4e)^3 + \cdots$$

inductively determines each c_i, given $c_2, \cdots, c_{i-1}$, according to the formula

$$c_i = \frac{2}{(2i)!} \prod_{j=2}^{i} \left(-(j-1)^2\right).$$

In particular, this formula does not yield a p-adic integer for $i = \frac{p+1}{2}$, implying that there is no lift through $\mathbb{H}P^{\frac{p+1}{2}}$. □

Remark A.1. The Adams conjecture provides a map from the connective cover of the $K(1)$-local sphere spectrum into $gl_1(S^0_{(p)})$. Using a variant of this due to Bhattacharya and Kitchloo [BK18], it is possible to construct maps $\mathbb{H}P^k \longrightarrow \mathrm{B}^4\mathrm{GL}_1(S^0_{(p)})$. Indeed, [BK18] employs arguments very similar to the ones above in order to produce multiplicative structures on Moore spectra. The authors believe, but have not verified, that it is possible to equip the map $S^3 \xrightarrow{1-p} \mathrm{B}^3\mathrm{GL}_1(S^0_{(p)})$ with an $\mathbb{A}_{\frac{p-1}{2}}$-algebra structure in this manner.

Remark A.2. It is well-known that the integral Eilenberg–Maclane spectrum $\mathrm{H}\mathbb{Z}_{(p)}$ is the Thom spectrum of a double loop composite

$$\Omega^2(S^3\langle 3\rangle) \longrightarrow \Omega^2 S^3 \longrightarrow \mathrm{BGL}_1(S^0_p).$$

One could attempt to refine this to a triple loop map, using the equivalence $\Omega\mathbb{H}P^\infty\langle 4\rangle \simeq S^3\langle 3\rangle$. The same obstruction as above proves that this strategy cannot work at odd primes, because the map

$$\mathbb{H}P^\infty\langle 4\rangle \longrightarrow \mathbb{H}P^\infty$$

is a $K(1)$-local equivalence.

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