

**CRYSTALLINITY FOR REDUCED SYNTOMIC COHOMOLOGY
AND THE MOD $(p, v_1^{p^{n-2}})$ K -THEORY OF \mathbb{Z}/p^n**

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ABSTRACT. We prove that the functor taking an animated ring R to its mod $(p, v_1^{p^n})$ derived syntomic cohomology factors through the functor $R \mapsto R/p^{n+2}$. We then use this to completely and explicitly compute the mod $(p, v_1^{p^n})$ syntomic cohomology of \mathbb{Z}/p^k whenever $k \geq n + 2$.

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1. INTRODUCTION

Let X denote a smooth qcqs p -adic formal \mathbb{Z}_p -scheme. The existence of crystalline cohomology implies, rather miraculously, that the (p -completed, derived) deRham cohomology of X depends functorially on the (derived) mod p reduction $X_{p=0}$. Here, we prove and apply similar crystallinity properties for prismatic and syntomic cohomology.

This paper achieves three main goals:

- (1) To determine new crystallinity properties of p -adic cohomology theories. We prove in particular that, for each $n \geq 0$, $\mathbb{F}_p(*)\langle X \rangle/v_1^{p^n}$ depends functorially on $X_{p^{n+2}=0}$. That is to say, while no p -power reduction of X completely recovers the syntomic complexes $\mathbb{F}_p(*)\langle X \rangle$, v_1 -power reductions of syntomic cohomology may be functorially recovered from p -power reductions of X .
- (2) To study the algebraic K -theory of \mathbb{Z}/p^n . Using [BMS19], this may be understood through syntomic cohomology. Our crystallinity result implies that, so long as $k \leq p^{n-2}$, the mod (p, v_1^k) syntomic cohomology of \mathbb{Z}/p^n agrees with the mod (p, v_1^k) syntomic cohomology of the trivial square-zero extension $\mathbb{Z}_p \oplus \mathbb{Z}_p[1]$. In particular, the answer is independent of n in this range!
- (3) To explicitly compute the mod p syntomic cohomology, with all Breuil–Kisin twists, of the trivial square-zero extension $\mathbb{Z}_p \oplus \mathbb{Z}_p[1]$. To make this calculation, we first prove that classic splitting results for TC of square-zero extensions, due to Lindenstrauss and McCarthy, respect the motivic filtration. Even given these splittings, the feasibility of the computation rests on several tricks and happy coincidences, and in particular the surprise that it suffices to compute both can and φ up to v_1 -adic associated graded.

One upshot of all this is the first closed form, complete calculation of

$$\pi_* \left(K(\mathbb{Z}/p^n)/(p, v_1^k) \right),$$

whenever $k < p^{n-2}$ and p is odd.¹ When $p = 2$ and $k < 2^{n-2}$ is divisible by 4, we obtain the associated graded of the motivic filtration on this algebraic K -theory.

1.1. Crystallinity of p -adic cohomology theories. The “crystalline miracle” may be formulated as follows: the p -complete derived de Rham cohomology of p -complete animated commutative rings

$$A \mapsto \mathrm{dR}_{A/\mathbb{Z}_p}$$

factors through the derived mod p reduction

$$A \mapsto A/p,$$

via the derived crystalline cohomology functor. This plays a foundational role in p -adic geometry and Hodge theory, as crystalline cohomology defines a good p -adic cohomology theory for characteristic p schemes.

In recent years, new p -adic cohomology theories have been introduced and used to study mixed characteristic phenomena: prismatic cohomology, which is a deformation of de Rham cohomology, and syntomic cohomology, which is closely related to algebraic K -theory and étale cohomology.

Our main result is an analogue of the “crystalline miracle” for certain reduced variants of these cohomology theories: they factor through reduction mod p^n . One novel aspect of this theorem is that it provides a concrete understanding of how the prismatic or syntomic cohomology of a p -adic formal scheme X is built up from its mod p^n reductions $X_{p^n=0}$.

Theorem 1.1. *Let $n \geq 0$. The following functors on $\mathrm{CAlg}_p^{\mathrm{Ani}}$ factor through the mod p^{n+2} reduction functor $\mathrm{CAlg}_p^{\mathrm{Ani}} \rightarrow \mathrm{CAlg}_{\mathbb{Z}/p^{n+2}}^{\mathrm{Ani}}$:*

(1) *mod $(p, v_1^{p^n})$ syntomic cohomology²*

$$R \mapsto \mathbb{F}_p(\ast)(R)/v_1^{p^n}$$

(2) *mod $(p, v_1^{p^n})$ Nygaard-filtered prismatic cohomology*

$$R \mapsto N^{\geq \ast} \Delta_R\{\ast\}/(p, v_1^{p^n})$$

(3) *mod $(p, v_1^{p^{n+1}})$ prismatic cohomology*

$$R \mapsto \Delta_R\{\ast\}/(p, v_1^{p^{n+1}})$$

(4) *mod $(p^k, (F^{n+1-k})^*I)$ prismatic cohomology, where $k \geq 1$, F is the Frobenius, and I is the Hodge-Tate ideal*

$$R \mapsto \Delta_R\{\ast\}/(p^k, (F^{n+1-k})^*I)$$

Remark 1.2. Parts (2) and (3) are consequences of the stronger Theorem 3.32 about mod $(p, a^{p^{n+1}} \mu^{p^n})$ Nygaard-filtered prismatic cohomology

$$N^{\geq \ast} \Delta_R\{\ast\}/(p, a^{p^{n+1}} \mu^{p^n}),$$

where $v_1 = a\mu$. The element a here is closely related to the class $a_\lambda : S^{-\lambda} \rightarrow S^0$ of S^1 -equivariant homotopy theory.

¹A powerful algorithm to compute the integral homotopy groups of $K(\mathbb{Z}/p^n)$ was recently developed in [AKN24]. We view this algorithm as complementary to our results, as we explain in Section 1.3.

²We recall the meaning of the class v_1 in Definition 2.10 and Remark 3.17. This class is the same as the v_1 of chromatic homotopy theory.

Theorem 1.1 came to us as a surprise. At the outset of this project, our goal was to study the mod (p, v_1) algebraic K -theory of \mathbb{Z}/p^n . We began to suspect a version of Theorem 1.1 after carrying out the computation and finding something unexpected: the answer does not depend on $n \geq 2$.

Remark 1.3. At least for syntomic cohomology, Theorem 1.1 is optimal in the following sense: for every $n \geq 2$, there is a pair of rings A_1, A_2 such that $A_1/p^n \cong A_2/p^n$ but $\mathbb{F}_p(*) (A_1)/v_1^{p^n} \not\cong \mathbb{F}_p(*) (A_2)/v_1^{p^n}$.

In fact, $A_1 = \mathbb{Z}/p^n$ and $A_2 = \mathbb{Z}/p^{n+1}$ work, as shown by Achim Krause and the third author in [KS24].

Our proof of Theorem 1.1, which was inspired by [BL22b, Example 5.15] and [Pet23, Section 6.3], makes essential use of the stacky approach to prismatic and syntomic cohomology [Dri20, BL22a, BL22b, Bha22]. As a consequence, we actually prove the following stacky refinement of Theorem 1.1.

Theorem 1.4. *The functors*

$$(1) \quad X \mapsto (X^{\text{Syn}})_{p=v_1^{p^n}=0}$$

$$(2) \quad X \mapsto (X^{\text{Nyg}})_{p=v_1^{p^n}=0}$$

$$(3) \quad X \mapsto (X^\Delta)_{p=v_1^{p^{n+1}}=0}$$

$$(4) \quad X \mapsto (X^\Delta)_{p^k=(F^{n+1-k})^*I=0}$$

factor through the functor $X \mapsto X_{p^{n+2}=0}$, where $X_{p^{n+2}=0}$ is regarded as derived \mathbb{Z}/p^{n+2} -scheme. Here, X is a derived p -adic formal scheme, and all of the vanishing loci are taken in a derived sense.

Remark 1.5. As in Remark 1.2, parts (2) and (3) are consequences of a stronger theorem about $(X^{\text{Nyg}})_{p=a^{p^{n+1}}\mu^{p^n}=0}$.

Remark 1.6. Theorem 1.4 implies that the map $(X^{\text{Syn}})_{p=v_1^{p^n}=0} \rightarrow (X^{\text{Syn}})_{p=0}$ factors through $((X_{p^{n+2}=0})^{\text{Syn}})_{p=0}$. On the other hand, Antieau–Krause–Nikolaus have shown that the map $((X_{p^n=0})^{\text{Syn}})_{p=0} \rightarrow (X^{\text{Syn}})_{p=0}$ factors through $(X^{\text{Syn}})_{p=v_1^{1+p+\dots+p^{n-1}}=0}$ [AKN24, Theorem 1.8]. These results may be interpreted as saying that, on $(X^{\text{Syn}})_{p=0}$, the v_1 -adic filtration and the filtration induced by the p -adic filtration on X are commensurate. We view this as a quantitative aspect of the chromatic redshift philosophy in algebraic K -theory.

As a sample application, we determine bounds on when the mod p^n reduction of a sufficiently nice p -adic formal scheme X determines the ranks of the mod p étale cohomology of its generic fiber. We introduce in Definition 4.1 the notion of affine cohomological dimension of a p -adic formal scheme X , an invariant of X in $\mathbb{N} \cup \{\infty\}$ that is bounded above by the Krull dimension of the underlying topological space of $X_{p=0}$ and that vanishes if X is affine.

Theorem 1.7. *Let X denote a p -torsionfree F -smooth qcqs p -adic formal scheme of affine cohomological dimension d . Set $b(d) = \lceil \log_p \left(\lceil \frac{d+1}{p-1} \rceil + 1 \right) \rceil + 2$. Then*

$$b_{i,j}(X; \mathbb{F}_p) := \dim_{\mathbb{F}_p} \mathrm{H}_{\text{ét}}^i(X_\eta; \mu_p^{\otimes j})$$

only depends on $X_{p^{b(d)=0}}$.³ In other words, if X_1 and X_2 as above satisfy $(X_1)_{p^{b(d)=0}} \cong (X_2)_{p^{b(d)=0}}$, then $b_{i,j}(X_1; \mathbb{F}_p) = b_{i,j}(X_2; \mathbb{F}_p)$.

For example, Theorem 1.7 implies that the $b_{i,j}(X; \mathbb{F}_p)$ are determined by $X_{p^3=0}$ when $d \leq p^2 - 2p$.

Remark 1.8. The definition of F -smoothness is somewhat technical—for this, we refer the reader to [BM23]. In the locally Noetherian case, F -smoothness is the same as regularity. Another source of F -smooth p -adic formal schemes are those which are smooth over $\mathrm{Spf}\mathcal{O}_K$ for a perfectoid field K .

Remark 1.9. Theorem 1.7 should be contrasted with the result of Fontaine–Messing [FM87, Remark 6.4], which states that if X is a smooth and proper \mathbb{Z}_p -scheme, then the isomorphism class of $H_{\mathrm{et}}^i(X_{\overline{\mathbb{Q}_p}}; \mathbb{Z}_p)$ for $i < p - 1$ may be recovered from $X_{p=0}$.

1.2. K -theory of \mathbb{Z}/p^n . The original purpose of this project was the explicit computation of the mod (p, v_1) K -theory of \mathbb{Z}/p^n , via mod (p, v_1) syntomic cohomology. We achieve this, and more, based on following corollary of Theorem 1.1:

Corollary 1.10. *Let $\mathbb{Z}_p\langle\epsilon\rangle$ denote the free animated \mathbb{Z}_p algebra on a class ϵ in degree 1. For $n \geq k + 2$, there are equivalences*

$$\mathbb{F}_p(*) (\mathbb{Z}/p^n) / v_1^{p^k} \simeq \mathbb{F}_p(*) (\mathbb{Z}_p\langle\epsilon\rangle) / v_1^{p^k}.$$

In particular, the answer is independent of n .

To compute the mod $(p, v_1^{p^k})$ syntomic cohomology of \mathbb{Z}/p^n , for any $n \geq k + 2$, it thus suffices to compute the mod p syntomic cohomology of $\mathbb{Z}_p\langle\epsilon\rangle$ as an $\mathbb{F}_p[v_1]$ -module. The animated ring $\mathbb{Z}_p\langle\epsilon\rangle$ is isomorphic to the trivial square-zero extension of \mathbb{Z}_p by $\mathbb{Z}_p[1]$. To compute its syntomic cohomology, we study the interaction of a formula for the topological cyclic homology of a square-zero extension with the Bhatt–Morrow–Scholze motivic filtration in Section 5. Using this, we are able to leverage our computational knowledge of $\mathrm{THH}(\mathbb{Z}_p)$ as a cyclotomic spectrum give a complete computation of the mod p syntomic cohomology of $\mathbb{Z}_p\langle\epsilon\rangle$.

Theorem 1.11. *There is an isomorphism of $\mathbb{F}_p[v_1]$ -modules*

$$\pi_* \mathbb{F}_p(*) (\mathbb{Z}_p\langle\epsilon\rangle) \cong (\pi_* \mathbb{F}_p(*) (\mathbb{Z}_p)) \oplus \bigoplus_{\substack{k>0 \\ p \nmid k}} \bigoplus_{n=0}^{\infty} \left(A_{n,k} \oplus B_{n,k} \oplus C_{n,k} \oplus \bigoplus_{r=1}^n (D_{n,k,r} \oplus E_{n,k,r}) \right),$$

where $A_{n,k}, B_{n,k}, C_{n,k}, D_{n,k,r}, E_{n,k,r}$ are explicit collections of elements described in Definition 6.18. More specifically, linearly independent $\mathbb{F}_p[v_1]$ -module generators for $A_{n,k}, B_{n,k}, C_{n,k}, D_{n,k,r}$, and $E_{n,k,r}$ are enumerated in Definition 6.17, and the v_1 -torsion order of each of these generators is enumerated as Lemma 6.19.

Remark 1.12. The retract

$$\mathbb{Z}_p \rightarrow \mathbb{Z}_p\langle\epsilon\rangle \rightarrow \mathbb{Z}_p$$

of rings leads to the $\pi_* \mathbb{F}_p(*) (\mathbb{Z}_p)$ summand in Theorem 1.11. Note that $\pi_* \mathbb{F}_p(*) (\mathbb{Z}_p)$ was computed by [LW22]. It is a free $\mathbb{F}_p[v_1]$ module on $p + 3$ explicit basis elements.

Remark 1.13. The computation constituting Theorem 1.11 is far from straightforward, even given the splitting results of Section 5. We perform several intermediate steps of the computation only at the level of v_1 -adic associated graded. It is then a striking fact that, for bidegree reasons alone, the final answer can be uniquely assembled from this partial information.

³Here, X_η is the adic generic fiber of X , i.e. $X \times_{\mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)} \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ in the category of (pre-)adic spaces.

From Theorem 1.11, one is easily able to deduce the mod (p, v_1^k) syntomic cohomology of \mathbb{Z}/p^n so long as $k \leq p^{n-2}$. In order to deduce results about algebraic K -theory, one first needs to understand both differentials and extension problems in the motivic spectral sequence converging to $\pi_*\mathrm{TC}(\mathbb{Z}/p^n)/(p, v_1^k)$. Controlling the behavior of this spectral sequence turns out to be somewhat more subtle than controlling the behavior of the motivic spectral sequence converging to the integral groups $\pi_*\mathrm{TC}(\mathbb{Z}/p^n)$, essentially because, when displayed in Adams grading, one has to worry about classes appearing below the x -axis. Nonetheless, in Section 7 we use a result of Achim Krause and the third author to prove that the following theorem:

Theorem 1.14. *Let p denote a prime, $n \geq 2$, and $k \leq p^{n-2}$.*

If p is odd, then there is an isomorphism of \mathbb{F}_p -vector spaces

$$\pi_*\mathrm{TC}(\mathbb{Z}/p^n)/(p, v_1^k) \cong \pi_* \bigoplus_{i \geq 0} \mathbb{F}_p(i)(\mathbb{Z}_p\langle \epsilon \rangle)[2i]/(p, v_1^k).$$

If $p = 2$ and k is divisible by 4, then the graded abelian group $\pi_\mathrm{TC}(\mathbb{Z}/2^n)/(2, v_1^k)$ admits a filtration with associated graded given by $\pi_* \bigoplus_{i \geq 0} \mathbb{F}_2(i)(\mathbb{Z}_2\langle \epsilon \rangle)[2i]/(2, v_1^k)$.*

Remark 1.15. At the prime 2, the filtration described at the end of Theorem 1.14 does not split, and indeed $\pi_*\mathrm{TC}(\mathbb{Z}/2^n)/(2, v_1^k)$ is not an \mathbb{F}_2 -vector space. Instead, there are motivic 2-extensions mapped forward from $\pi_*(\mathrm{TC}(\mathbb{Z}_2)/2)$ [LW22, Theorem 8.21]. We do not know whether this accounts for all of the 2-extensions.

Applying the Dundas–Goodwillie–McCarthy theorem and the known computation of the algebraic K -theory and topological cyclic homology of \mathbb{F}_p , we obtain the following theorem bridging the gap between TC and algebraic K -theory:

Theorem 1.16. *Let p denote a prime, $n \geq 2$, and let $k \leq p^{n-2} - 1$. If $p = 2$, then assume that k is divisible by 4. Then there is an exact sequence*

$$0 \rightarrow \mathbb{F}_p\{\partial\} \rightarrow \pi_*K(\mathbb{Z}/p^n)/(p, v_1^k) \rightarrow \pi_*\mathrm{TC}(\mathbb{Z}/p^n)/(p, v_1^k) \rightarrow \mathbb{F}_p\{\partial\} \rightarrow 0.$$

Remark 1.17. We have to exclude the case of $k = p^{n-2}$ because we do not know whether $\partial \in \pi_{-1}\mathrm{TC}(\mathbb{Z}/p^n)/p$ is $v_1^{p^{n-2}}$ -torsion.

Recall from [AKN24] that the even K -groups of \mathbb{Z}/p^n vanish in degrees large relative to n . In [KS24], Achim Krause and the third author use Theorem 1.1 and Theorem 1.11 to give precise conditions for when these K -groups vanish. As a final result, we prove the following consequence of Theorem 1.1:

Theorem 1.18. *For $n \geq 2$ and $i \geq 0$, the map*

$$K_{2i}(\mathbb{Z}_p) \rightarrow K_{2i}(\mathbb{Z}/p^n)$$

is surjective. In particular, $K_{2i}(\mathbb{Z}/p^n)$ is cyclic.

1.3. Comparison with the results of [AKN24]. In [AKN24], Antieau–Krause–Nikolaus combine the algebra of prismatic envelopes with a clever filtration argument to give a finitary algebraic model for the syntomic cohomology of \mathcal{O}_K/ϖ^n of a given weight. Using this, they give an algorithm which computes the algebraic K -groups of \mathcal{O}_K/ϖ^n , and have written a computer program which implements this algorithm. Results of their program appear in tables [AKN24, Figure 1 & Appendix A] [AKN22, Figures 1-4].

In contrast, we give a closed form description of the mod $(p, v_1^{p^{n-2}})$ syntomic cohomology of \mathbb{Z}/p^n in all weights. In particular:

- The methods of [AKN24] compute integral K -groups, while we are at best only able to get at mod p K -groups.

- As p and n get large, the compute time of [AKN24]’s algorithm also becomes large. On the other hand, as p and n get large, the degree of $v_1^{p^{n-2}}$ becomes large, so that mod $(p, v_1^{p^{n-2}})$ K -theory agrees with mod p K -theory in a larger and larger range.

For example, 48 hours on a high-performance cluster was enough to compute $K_*(\mathbb{Z}/7^7)$ through degree $* = 15$ using the program of [AKN24], but not $K_{16}(\mathbb{Z}/7^7)$ [AKN24, Appendix A.4]. Our results determine $\pi_*(K(\mathbb{Z}/7^7)/7)$ through degree $\approx |v_1^{7^5}| = 201684$.

- One may compare our results with the tables of Antieau–Krause–Nikolaus and see that they are consistent.
- The results of [AKN24] apply to \mathcal{O}_K/ϖ^n , where \mathcal{O}_K is a characteristic $(0, p)$ complete discrete valuation ring with finite residue field and uniformizer ϖ . Here, we have restricted our computations to the case of \mathbb{Z}/p^n .

Remark 1.19. In degrees i large enough that their even vanishing theorem applies, Antieau–Krause–Nikolaus apply work of Angeltveit to prove that the order of $K_{2i-1}(\mathbb{Z}/p^n)$ is exactly $(p^i - 1)p^{i(n-1)}$ [AKN24, Corollary 1.6]. The other most basic question one could ask about this abelian group is its rank, i.e. how many summands appear when it is written as a direct sum of cyclic groups of prime power order. Again applying even vanishing, computing this rank is equivalent to computing the mod p algebraic K -group $\pi_{2i-1}(K(\mathbb{Z}/p^n)/p)$. In fact, in low degrees where even vanishing does not apply, the combination of our Theorem 1.18 with [KS24] still reduces computing the rank of $K_{2i-1}(\mathbb{Z}/p^n)$ to the computation of mod p K -groups. Thus, for example, we now know the rank of $K_*(\mathbb{Z}/7^7)$ for all $* \leq 201682$.

In order to obtain a complete, closed form computation of $\pi_*(K(\mathbb{Z}/p^n)/p)$, the natural strategy is to use the v_1 -Bockstein spectral sequence

$$E_1 = \pi_*(\mathrm{TC}(\mathbb{Z}/p^n)/(p, v_1))[v_1] \implies \pi_*(\mathrm{TC}(\mathbb{Z}/p^n)/p).$$

This spectral sequence has finitely many pages, because v_1 acts nilpotently on $\pi_*(\mathrm{TC}(\mathbb{Z}/p^n)/p)$. We give in this paper explicit names to every element on the E_1 -page of the Bockstein spectral sequence. We also explain how to compute the first several differentials by determining the homotopy groups of $\mathrm{TC}(\mathbb{Z}/p^n)/(p, v_1^{p^{n-2}})$, but there remain a few undetermined differentials.

By our work here, elements on the E_1 -page appear in five infinite families. For low values of n , Bockstein differentials on the initial elements in each family may be understood using [AKN24] and high performance computing. Preliminary experiments in this direction give hope that the Bockstein spectral sequence follows a reasonably controlled pattern. After the v_1 -Bockstein spectral sequence is completely understood, one might turn to the p -Bockstein spectral sequence determining integral algebraic K -groups.

1.4. Questions and conjectures. Our work raises several additional questions that we have not attempted to address, but which we would like to highlight here.

To begin with, we have elected in this paper to stick to \mathbb{Z}/p^n , rather than more general rings of the form \mathcal{O}_K/ϖ^n . It would be interesting to extend our computations to the case of \mathcal{O}_K/ϖ^n . In the case that $p^2 \mid \varpi^n$, the mod (p, v_1) syntomic cohomology of \mathcal{O}_K/ϖ^n may be studied using Theorem 1.1. In the case that $\varpi^n \mid p$, \mathcal{O}_K/ϖ^n is a truncated polynomial algebra over a finite field, so its K -theory is known by [HM97].

Question 1.20. How does the mod (p, v_1) syntomic cohomology of \mathcal{O}_K/ϖ^n behave when $p^2 \nmid \varpi^n$ and $\varpi^n \nmid p$? Curiously, [AKN22, Figure 2] shows that the syntomic cohomology of $\mathbb{F}_2[z]/z^3$ and $\mathbb{Z}_2[2^{1/2}]/2^{3/2}$ agree in a range.

When A is a $\mathbb{Z}_p[\zeta_p]$ -algebra, v_1 admits a natural $(p-1)$ st root β in the mod p syntomic cohomology of A . We wonder if our theorem can be refined to study the mod (p, β^k) syntomic cohomology in this case.

Question 1.21. Is there a refined version of Theorem 1.1 for p -complete animated $\mathbb{Z}_p[\zeta_p]$ -algebras, where we quotient on the inside by powers of $(\zeta_p - 1)$ and on the outside by (p, β^k) , where $\beta \in H^0(\mathbb{F}_p(1)(\mathbb{Z}_p[\zeta_p]))$ is the Bott element, a $(p-1)$ st root of v_1 ?

In general, can Theorem 1.1 be improved in the ramified situation? Can modding out by p^2 be replaced by modding out by $p\varpi$?

Next, it is natural to ask the extent to which our theorems hold on the level of TP , TC^- and TC , as opposed to their motivic associated graded.

Question 1.22. Is there a version of Theorem 1.1 that holds for mod (p, v_1^k) reduced TP , TC^- and TC ?

Finally, one might ask whether there is an analogue of Theorem 1.1 that holds for the syntomic cohomology of higher chromatic ring spectra.

Question 1.23. Is there a version of Theorem 1.1 that holds at higher chromatic heights? For example, does the mod (p, v_1, \dots, v_{n+1}) syntomic cohomology of chromatic height n ring spectra factor through reduction by $(p^{i_0}, v_1^{i_1}, \dots, v_n^{i_n})$ for some sequence (i_1, \dots, i_n) ?

1.5. Outline. In Section 2, we carry out our crystallinity argument for prismatic cohomology, proving part (4) of Theorem 1.1 and Theorem 1.4. In Section 3, we extend this argument from the prismatic to the Nygaard-filtered prismatic and the syntomification, proving the remaining parts of Theorem 1.1 and Theorem 1.4. We deduce Theorem 4.3 in Section 4.

Next, we study the motivic filtration on the topological cyclic homology of $R\langle\epsilon\rangle$ in Section 5. In Section 6, we use this to explicitly compute the mod p syntomic cohomology of $\mathbb{Z}_p\langle\epsilon\rangle$. Finally, we study consequences for the K -theory and topological cyclic homology of \mathbb{Z}/p^n in Section 7.

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2. PRISMATIZATION AND THEOREM 1.1(4)

Our goal in this section is to prove Theorem 1.1(4) and Theorem 1.4(4). This will also serve as a warmup for the other parts of Theorem 1.1 and Theorem 1.4, since we will not need to engage here with Nygaard-filtered prismatic or syntomification.

2.1. Background on prismatic. We begin with some background on prismatic from [Dri20, BL22a, BL22b].

Definition 2.1. A *Cartier–Witt divisor* over a $(p$ -nilpotent) ring R consists of an invertible $W(R)$ -module I and a $W(R)$ -module map

$$I \rightarrow W(R),$$

such that $I \rightarrow W(R) \rightarrow R$ is nilpotent and $I \rightarrow W(R) \xrightarrow{\delta} W(R)$ generates the unit ideal.

A morphism $(I \xrightarrow{\alpha} W(R)) \rightarrow (J \xrightarrow{\beta} W(R))$ of Cartier–Witt divisors is a commuting diagram of $W(R)$ -modules

$$\begin{array}{ccc} I & \xrightarrow{\alpha} & W(R) \\ \downarrow & \nearrow \beta & \\ J & & \end{array}$$

By [Dri20, Lemma 4.2.5], every morphism of Cartier–Witt divisors is an isomorphism.

Definition 2.2. We define \mathbb{Z}_p^Δ to be the p -adic formal stack whose functor of points sends a p -nilpotent ring R to its category of Cartier–Witt divisors, so that

$$\mathbb{Z}_p^\Delta(R) = \{(I, I \rightarrow W(R))\}.$$

Definition 2.3. We say that a Cartier–Witt divisor $I \rightarrow W(R)$ is *Hodge–Tate* if the composite $I \rightarrow W(R) \rightarrow R$ is zero. We let $\mathbb{Z}_p^{\text{HT}} \subset \mathbb{Z}_p^\Delta$ denote the closed substack of Hodge–Tate divisors.

Definition 2.4. There is a Frobenius map $F : \mathbb{Z}_p^\Delta \rightarrow \mathbb{Z}_p^\Delta$ which sends a Cartier–Witt divisor $I \xrightarrow{\alpha} W(R)$ to $F^*I := I \otimes_{W(R), F} W(R) \xrightarrow{F^*\alpha} W(R)$.

Remark 2.5. Because the Frobenius map $F : W(R) \rightarrow W(R)$ agrees with the map induced by the Frobenius of R when R is an \mathbb{F}_p -algebra, the prismatic Frobenius

$$F : \mathbb{Z}_p^\Delta \rightarrow \mathbb{Z}_p^\Delta$$

is a lift of the mod p Frobenius.

Definition 2.6. There is an animated ring stack \mathbb{G}^Δ lying over $\text{Spf}\mathbb{Z}_p^\Delta$, which sends a Cartier–Witt divisor $I \rightarrow W(R)$ to its animated quotient $W(R)/I$.

Given a derived p -adic formal scheme X , we define its prismaticization $X^\Delta \rightarrow \text{Spf}\mathbb{Z}_p^\Delta$ via $X^\Delta(R) = X(\mathbb{G}^\Delta(R))$. We define its Hodge–Tate locus to be $X^{\text{HT}} := X^\Delta \times_{\mathbb{Z}_p^\Delta} \mathbb{Z}_p^{\text{HT}}$.

Example 2.7. Explicitly, we have $(\mathbb{Z}/p^n)^\Delta(R) = \{(I, \alpha : I \rightarrow W(R), x \in I) \mid \alpha(x) = p^n\}$.

Definition 2.8. There is a line bundle $\mathcal{O}\{1\}$ over \mathbb{Z}_p^Δ which corresponds to the Breuil–Kisin twist, see [BL22a, Section 2 & Example 3.3.8] and [Dri20, Section 4.9]. Heuristically, it is defined as follows. Let \mathcal{I} denote the line bundle on \mathbb{Z}_p^Δ which sends $I \rightarrow W(R)$ to $I \otimes_{W(R)} R$. The map $I \rightarrow W(R)$ induces a map $\mathcal{I} \rightarrow \mathcal{O}_{\mathbb{Z}_p^\Delta}$ cutting out the Hodge–Tate locus. Then $\mathcal{O}\{1\} := \bigotimes_{n=0}^\infty (F^n)^*\mathcal{I}$.

Remark 2.9. By definition, we have $F^*\mathcal{O}_{\mathbb{Z}_p^\Delta}\{1\} \cong \mathcal{O}_{\mathbb{Z}_p^\Delta}\{1\} \otimes \mathcal{I}^{-1}$. Reducing this mod p , we find that

$$\mathcal{O}_{(\mathbb{Z}_p)_{p=0}^\Delta}\{p\} \cong \mathcal{O}_{(\mathbb{Z}_p)_{p=0}^\Delta}\{1\} \otimes \mathcal{I}_{p=0}^{-1},$$

hence that

$$\mathcal{I}_{p=0} \cong \mathcal{O}_{(\mathbb{Z}_p)_{p=0}^\Delta}\{1-p\}.$$

Definition 2.10. We define v_1 to be the section of $\mathcal{O}_{(\mathbb{Z}_p)_{p=0}^\Delta}\{p-1\}$ corresponding to the canonical map $\mathcal{I}_{p=0} \rightarrow \mathcal{O}_{(\mathbb{Z}_p)_{p=0}^\Delta}$ under the isomorphism

$$\mathcal{I}_{p=0} \cong \mathcal{O}_{(\mathbb{Z}_p)_{p=0}^\Delta}\{1-p\}.$$

Remark 2.11. By definition, the closed substack $(\mathbb{Z}_p^{\text{HT}})_{p=0} \subset (\mathbb{Z}_p^\Delta)_{p=0}$ agrees with $(\mathbb{Z}_p^\Delta)_{p=v_1=0}$.

Finally, we recall the comparison between the cohomology of the prismaticization and prismatic cohomology.

Theorem 2.12 ([BL22b, Corollaries 8.13 & 8.17]). *Let X be a qcqs p -quasisyntomic p -adic formal scheme. Then there is a natural isomorphism*

$$\Delta_X \{i\} \cong R\Gamma(X^\Delta, \mathcal{O}_{X^\Delta} \{i\}).$$

2.2. Crystallinity of reduced prismatic cohomology.

Definition 2.13. Given a morphism of formal stacks $A \rightarrow \mathbb{Z}_p^\Delta$, and an integer $n \geq 0$, we denote by $F^{-n}A$ the pullback

$$\begin{array}{ccc} F^{-n}A & \longrightarrow & A \\ \downarrow & & \downarrow \\ \mathbb{Z}_p^\Delta & \xrightarrow{F^n} & \mathbb{Z}_p^\Delta, \end{array}$$

where F^n denotes the n -fold iterated composite of $F : \mathbb{Z}_p^\Delta \rightarrow \mathbb{Z}_p^\Delta$.

Our main result is as follows:

Theorem 2.14. *For integers $n \geq 0$ and $k \geq 1$, there exist dashed arrows making the following diagram commute:*

$$\begin{array}{ccc} F^{-n}((\mathbb{Z}_p^{\text{HT}})_{p^{k+1}=0}) & \xrightarrow{\quad\quad\quad} & (\mathbb{Z}_p)_{p^{k+1}=0}^\Delta \\ & \dashrightarrow & \nearrow \\ & & (\mathbb{Z}/p^{n+k+1})_{p^{k+1}=0}^\Delta, \end{array}$$

such that the arrows are compatible as k and n vary.

Corollary 2.15. *Taking $k = 1$ and reducing modulo p , we obtain the diagram*

$$\begin{array}{ccc} (\mathbb{Z}_p)_{p=v_1^{p^n}=0}^\Delta & \xrightarrow{\quad\quad\quad} & (\mathbb{Z}_p)_{p=0}^\Delta \\ & \searrow & \nearrow \\ & & (\mathbb{Z}/p^{n+2})_{p=0}^\Delta. \end{array}$$

Using Theorem 2.14, we are able to prove Theorem 1.1(4) and Theorem 1.4(4). We start with the latter.

Proof of Theorem 1.4(4). Form the diagram

$$\begin{array}{ccc} (X^\Delta)_{p=v_1^{p^n}=0} & \longrightarrow & (\mathbb{Z}_p^\Delta)_{p=v_1^{p^n}=0} \\ \downarrow & & \downarrow \\ (X_{p^{n+2}=0})^\Delta & \longrightarrow & (\mathbb{Z}/p^{n+2})^\Delta \\ \downarrow & & \downarrow \\ X^\Delta & \longrightarrow & \mathbb{Z}_p^\Delta. \end{array}$$

The outer square is a pullback by definition, and the bottom square is a pullback because prismaticization preserves limits of derived p -adic formal schemes. It follows that the top square is also a pullback, so that

$$(X^\Delta)_{p=v_1^{p^n}=0} \cong (X_{p^{n+2}=0})^\Delta \times_{(\mathbb{Z}/p^{n+2})^\Delta} (\mathbb{Z}_p^\Delta)_{p=v_1^{p^n}=0}$$

gives the desired factorization of

$$X \mapsto (X^\Delta)_{p=v_1^{p^n}=0}$$

through

$$X \mapsto X_{p^{n+2}=0}.$$

□

Proof of Theorem 1.1(4). Since derived prismatic cohomology is left Kan extended from p -completed polynomial algebras, it suffices to prove the theorem for p -completed polynomial algebras. But this follows from combining Theorem 1.4(4) with Theorem 2.12, since p -completed polynomial algebras and their mod p^n reductions are p -quasisyntomic. □

Our proof of Theorem 2.14 will be an induction on n . The base inductive case is provided by the following result of Bhatt–Lurie:

Proposition 2.16. *There is a commutative diagram:*

$$\begin{array}{ccc} \mathbb{Z}_p^{\text{HT}} & \longrightarrow & \mathbb{Z}_p^\Delta \\ \downarrow & & \downarrow F \\ \mathbb{F}_p^\Delta & \longrightarrow & \mathbb{Z}_p^\Delta \end{array}$$

In particular, the natural map $\mathbb{Z}_p^{\text{HT}} \subset \mathbb{Z}_p^\Delta$ factors through $F^{-1}(\mathbb{F}_p^\Delta)$.

Proof. This square is stated as [BL22a, Proposition 3.6.6], once one identifies

$$\rho_{dR} : \text{Spf} \mathbb{Z}_p \rightarrow \mathbb{Z}_p^\Delta$$

with the natural map $\mathbb{F}_p^\Delta \rightarrow \mathbb{Z}_p^\Delta$ as in [BL22b, Remark 3.13]. □

The inductive step is provided by the following lemma:

Lemma 2.17. *For each $n \geq 1$, there is a commuting diagram:*

$$\begin{array}{ccc} F^{-1}(\mathbb{Z}/p^n)_{p^{n+1}=0}^\Delta & \longrightarrow & (\mathbb{Z}/p^n)_{p^{n+1}=0}^\Delta \\ \vdots \downarrow & & \downarrow \\ (\mathbb{Z}/p^{n+1})_{p^{n+1}=0}^\Delta & & \\ \downarrow & & \downarrow \\ (\mathbb{Z}_p)_{p^{n+1}=0}^\Delta & \xrightarrow{F} & (\mathbb{Z}_p)_{p^{n+1}=0}^\Delta \end{array}$$

where the outer square is a pullback.

Proof. Let $(I \xrightarrow{\alpha} W(S), x) \in F^{-1}(\mathbb{Z}/p^n)^\Delta(S)$, where $x \in F^*I$ is a lift of $p^n \in W(S)$.

Now, we have a commutative square

$$\begin{array}{ccc} I & \xleftarrow{V} & F^*I \\ \downarrow \alpha & & \downarrow F^*\alpha \\ W(S) & \xleftarrow{V} & W(S), \end{array}$$

where V is induced by the Witt vector Verschiebung. Then $V(x) \in I$ is a lift of $V(p^n) \in W(S)$. By assumption $p^{n+1} = 0$ on S , so that $V(p^n) = p^{n+1} \in W(S)$ (by [Pet23, Lemma 6.14], following ideas of Devalapurkar), and $V(x)$ is the desired lift of $p^{n+1} \in W(S)$. □

We are now ready to prove Theorem 2.14:

Proof of Theorem 2.14. We will induct on $n \geq 0$ to construct a dashed arrow

$$\begin{array}{ccc}
 F^{-n}((\mathbb{Z}_p^{\text{HT}})_{p^{k+1}=0}) & \xrightarrow{\hspace{10em}} & (\mathbb{Z}_p)_{p^{k+1}=0}^{\Delta} \\
 \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} & \searrow \text{---} & \nearrow \text{---} \\
 & (\mathbb{Z}/p^{n+k+1})_{p^{k+1}=0}^{\Delta} &
 \end{array}$$

for fixed $k \geq 1$. To begin, we first construct the dashed arrow when $n = 0$. Reducing Proposition 2.16 modulo p^{k+1} , there is a diagram

$$\begin{array}{ccc}
 (\mathbb{Z}_p^{\text{HT}})_{p^{k+1}=0} & \xrightarrow{\hspace{2em}} & (\mathbb{Z}_p^{\Delta})_{p^{k+1}=0} \\
 \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} & \searrow \text{---} & \nearrow \text{---} \\
 F^{-1}((\mathbb{F}_p^{\Delta})_{p^{k+1}=0}) & &
 \end{array}$$

Since the map $F^{-1}(\mathbb{F}_p^{\Delta}) \rightarrow \mathbb{Z}_p^{\Delta}$ factors through $F^{-1}(\mathbb{Z}/p^k)^{\Delta}$, the above commutative diagram in particular induces a diagram

$$\begin{array}{ccc}
 (\mathbb{Z}_p^{\text{HT}})_{p^{k+1}=0} & \xrightarrow{\hspace{2em}} & (\mathbb{Z}_p^{\Delta})_{p^{k+1}=0} \\
 \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} & \searrow \text{---} & \nearrow \text{---} \\
 F^{-1}((\mathbb{Z}/p^k)_{p^{k+1}=0}^{\Delta}) & &
 \end{array}$$

Applying Lemma 2.17, we conclude the desired diagram when $n = 0$.

Now we proceed with the inductive step. In particular, suppose for the sake of induction that we have constructed a diagram

$$\begin{array}{ccc}
 F^{-n+1}((\mathbb{Z}_p^{\text{HT}})_{p^{k+1}=0}) & \xrightarrow{\hspace{10em}} & (\mathbb{Z}_p)_{p^{k+1}=0}^{\Delta} \\
 \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} & \searrow \text{---} & \nearrow \text{---} \\
 & (\mathbb{Z}/p^{n+k})_{p^{k+1}=0}^{\Delta} &
 \end{array}$$

Applying F^{-1} , we obtain a diagram

$$\begin{array}{ccc}
 F^{-n}((\mathbb{Z}_p^{\text{HT}})_{p^{k+1}=0}) & \xrightarrow{\hspace{10em}} & (\mathbb{Z}_p)_{p^{k+1}=0}^{\Delta} \\
 \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} & \searrow \text{---} & \nearrow \text{---} \\
 & F^{-1}(\mathbb{Z}/p^{n+k})_{p^{k+1}=0}^{\Delta} &
 \end{array}$$

The desired diagram is then constructed by applying Lemma 2.17. □

3. SYNTOMIFICATION AND THEOREM 1.1(1-3)

The argument we give for Theorem 1.1(1-3) here is very much analogous to the argument given for Theorem 1.1(4) in the previous section, though we must deal with more complicated formal stacks.

3.1. Background on Nygaardification and syntomification. Here, we recall some background on the Nygaard-filtered prismatization from [Dri20, Bha22]. Let W_S denote the Witt ring scheme over a base scheme S .

Definition 3.1. A (relatively) affine W_S -module M is said to be *admissible* if there exists an S -line bundle L_M , an invertible W_S -module scheme M' , and a short exact sequence

$$0 \rightarrow L_M^{\#} \rightarrow M \rightarrow F_* M' \rightarrow 0.$$

By [Dri20, Lemma 3.12.7], L_M , M' and the short exact sequence may be recovered functorially from M .

Example 3.2. Any invertible W_S -module I may be tensored with the exact sequence

$$0 \rightarrow \mathbb{G}_a^\# \rightarrow W_S \xrightarrow{F} F_*W_S \rightarrow 0$$

to obtain an exact sequence

$$0 \rightarrow I \otimes_{W_S} \mathbb{G}_a^\# \rightarrow I \rightarrow I \otimes_{W_S} F_*W_S \rightarrow 0.$$

Since $I \otimes_{W_S} F_*W_S \cong F_*(I \otimes_{W_S, F} W_S)$, I is an admissible W_S -module.

Example 3.3. Given a map $d : I \rightarrow W_S$, we may define an admissible W_S -module M as a pullback in the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{G}_a^\# & \longrightarrow & M & \longrightarrow & F_*I \longrightarrow 0 \\ & & \downarrow = & & \downarrow & & \downarrow F_*d \\ 0 & \longrightarrow & \mathbb{G}_a^\# & \longrightarrow & W_S & \xrightarrow{F} & F_*W_S \longrightarrow 0. \end{array}$$

Remark 3.4. It is not true in general that if M is an admissible W_S -module, then $F^*M = M \otimes_{W_S, F} W_S$ is an admissible W_S -module. Correspondingly, the Nygaard-filtered prismaticization of \mathbb{Z}_p will *not* admit a lift of the mod p Frobenius.

If S is an \mathbb{F}_p -scheme, then we may consider Fr_S^*M , which is an admissible W_S -module.

Lemma 3.5. *Admissible W_S -modules M are affine commutative flat group schemes over S .*

Proof. We just need to prove the flatness of M . Zariski locally, M lies in a short exact sequence

$$0 \rightarrow \mathbb{G}_a^\# \rightarrow M \rightarrow F_*W_S \rightarrow 0,$$

so this follows from flatness of $\mathbb{G}_a^\#$ and F_*W_S . \square

Applying [DG70, Section IV.3.4], we obtain the following corollary:

Corollary 3.6. *For \mathbb{F}_p -schemes S , admissible W_S -modules are equipped with natural Frobenius and Verschiebung homomorphisms*

$$F : M \rightarrow \mathrm{Fr}_S^*M, \quad V : \mathrm{Fr}_S^*M \rightarrow M$$

satisfying the relation $FV = VF = p$.

Definition 3.7. A *filtered Cartier–Witt divisor* over S consists of an admissible W_S -module M and a W_S -module map $d : M \rightarrow W_S$ which fits into a map of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & L_M^\# & \longrightarrow & M & \longrightarrow & F_*M' \longrightarrow 0 \\ & & \downarrow d^\# & & \downarrow d & & \downarrow F_*d' \\ 0 & \longrightarrow & \mathbb{G}_a^\# & \longrightarrow & W_S & \xrightarrow{F} & F_*W_S \longrightarrow 0, \end{array}$$

where $d' : M' \rightarrow W_S$ is a Cartier–Witt divisor.

Definition 3.8. We define $\mathbb{Z}_p^{\mathrm{Nyg}}$ to be the formal stack whose functor of points sends a p -nilpotent scheme S to its underlying groupoid of filtered Cartier–Witt divisors, so that

$$\mathbb{Z}_p^{\mathrm{Nyg}}(S) = \{(M, M \rightarrow W_S)\}.$$

Definition 3.9. The assignment $(d : M \rightarrow W_S) \mapsto (d' : M' \rightarrow W_S)$ in the above definition defines a morphism of stacks

$$F' : \mathbb{Z}_p^{\mathrm{Nyg}} \rightarrow \mathbb{Z}_p^\Delta.$$

Definition 3.10. By [Dri20, Lemma 3.12.4(ii)], the map $d^\# : L_M^\# \rightarrow \mathbb{G}_a^\#$ corresponds to a unique map $a(d) : L_M \rightarrow \mathbb{G}_a$. The assignment $(d : M \rightarrow W_S) \mapsto (a(d) : L_M \rightarrow \mathbb{G}_a)$ determines a morphism of stacks

$$a : \mathbb{Z}_p^{\text{Nyg}} \rightarrow \mathbb{A}^1 / \mathbb{G}_m.$$

Definition 3.11. We have the animated ring stack \mathbb{G}^{Nyg} over $\mathbb{Z}_p^{\text{Nyg}}$ given by the animated quotient W_S/M of fpqc sheaves.⁴ Given a derived p -adic formal scheme X , we define its Nygaard-filtered prismaticization $X^{\text{Nyg}} \rightarrow \text{Spf} \mathbb{Z}_p^{\text{Nyg}}$ via $X^{\text{Nyg}}(R) = X(\mathbb{G}^{\text{Nyg}}(R))$.

Example 3.12. For example, $(\mathbb{Z}/p^n)^{\text{Nyg}}$ is the fpqc sheafification of the functor $R \mapsto \{(M, d : M \rightarrow W_R, x \in M(R) | d(x) = p^n \in W(R))\}$.

Definition 3.13. By Example 3.2, any Cartier–Witt divisor is also a filtered Cartier–Witt divisor. This determines a morphism $j_{HT} : \mathbb{Z}_p^\Delta \rightarrow \mathbb{Z}_p^{\text{Nyg}}$, which turns out to be an open immersion, and whose composition with $F' : \mathbb{Z}_p^{\text{Nyg}} \rightarrow \mathbb{Z}_p^\Delta$ is the Frobenius map on \mathbb{Z}_p^Δ .

Definition 3.14. Given a Cartier–Witt divisor $d : I \rightarrow W_S$, we may construct a filtered Cartier–Witt divisor by taking M to be a pullback in the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{G}_a^\# & \longrightarrow & M & \longrightarrow & F_* I & \longrightarrow & 0 \\ & & \downarrow \text{id} & & \downarrow & & \downarrow F_* d & & \\ 0 & \longrightarrow & \mathbb{G}_a^\# & \longrightarrow & W_S & \xrightarrow{F} & F_* W_S & \longrightarrow & 0 \end{array}$$

By Example 3.3, this determines a morphism

$$j_{dR} : \mathbb{Z}_p^\Delta \rightarrow \mathbb{Z}_p^{\text{Nyg}}.$$

This identifies \mathbb{Z}_p^Δ with $(\mathbb{Z}_p^{\text{Nyg}})_{a \neq 0}$, the fiber of $a : \mathbb{Z}_p^{\text{Nyg}} \rightarrow \mathbb{A}^1 / \mathbb{G}_m$ over $\mathbb{G}_m / \mathbb{G}_m$.

Definition 3.15. Pulling back $\mathcal{O}_{\mathbb{Z}_p^\Delta}\{1\}$ along $F' : \mathbb{Z}_p^{\text{Nyg}} \rightarrow \mathbb{Z}_p^\Delta$ and $\mathcal{O}_{\mathbb{A}^1 / \mathbb{G}_m}(1)$ along $a : \mathbb{Z}_p^{\text{Nyg}} \rightarrow \mathbb{A}^1 / \mathbb{G}_m$, we obtain a bigraded family of line bundles on $\mathbb{Z}_p^{\text{Nyg}}$: $(F')^* \mathcal{O}_{\mathbb{Z}_p^\Delta}\{i\} \otimes a^* \mathcal{O}_{\mathbb{A}^1 / \mathbb{G}_m}(j)$.

We set

$$\mathcal{O}_{\mathbb{Z}_p^{\text{Nyg}}}\{1\} := (F')^* \mathcal{O}_{\mathbb{Z}_p^\Delta}\{1\} \otimes a^* \mathcal{O}_{\mathbb{A}^1 / \mathbb{G}_m}(-1).$$

Definition 3.16. Let S be an \mathbb{F}_p -scheme and M be an admissible W_S -module. Then the Frobenius induces a W_S -linear map $F : M \rightarrow F_*(\text{Fr}_S^* M)$. The map F kills $L_M^\#$, hence factors through a map $F_* M' \rightarrow F_*(\text{Fr}_S^* M)$, which gives rise to a map $f_M : M' \rightarrow \text{Fr}_S^* M$.

This determines a natural map $\phi_M : L_{M'} \rightarrow L_{\text{Fr}_S^* M} \cong L_M^{\otimes p}$, i.e. a map of line bundles

$$\mu : (F')^* \mathcal{I}_{p=0} \rightarrow a^* \mathcal{O}_{\mathbb{A}^1 / \mathbb{G}_m, p=0}(-p).$$

We define v_1 to be the section of $\mathcal{O}_{(\mathbb{Z}_p)^{\text{Nyg}}}\{p-1\} \cong (F')^* \mathcal{I}_{p=0}^{-1} \otimes a^* \mathcal{O}_{\mathbb{A}^1 / \mathbb{G}_m, p=0}(1-p)$ corresponding to the map

$$a\mu : (F')^* \mathcal{I}_{p=0} \rightarrow t^* \mathcal{O}_{\mathbb{A}^1 / \mathbb{G}_m, p=0}(1-p).$$

Remark 3.17. The open immersion $j_{HT, p=0} : (\mathbb{Z}_p)_{p=0}^\Delta \rightarrow (\mathbb{Z}_p)_{p=0}^{\text{Nyg}}$ identifies $(\mathbb{Z}_p)_{p=0}^\Delta$ with $(\mathbb{Z}_p^{\text{Nyg}})_{p=0, \mu \neq 0}$.

When pulled back along j_{HT} , a becomes identified with the canonical map $\mathcal{I} \rightarrow \mathcal{O}_{\mathbb{Z}_p^\Delta}$.

Similarly, when pulled back along $j_{dR, p=0}$, μ becomes identified with $\mathcal{I} \rightarrow \mathcal{O}_{\mathbb{Z}_p^\Delta}$.

It follows that $v_1 = a\mu$ pulls back along both $j_{HT, p=0}$ and $j_{dR, p=0}$ to the section v_1 of $\mathcal{O}_{(\mathbb{Z}_p)_{p=0}^{\text{Nyg}}}\{p-1\}$ identified in Definition 2.10.

⁴Any filtered Cartier–Witt divisor is a quasi-ideal by [Dri20, Lemma 3.12.12].

3.2. Drinfeld's description of $(\mathbb{Z}_p^{\text{Nyg}})_{p=a^p\mu=0}$. In this section, we recall a compact presentation of $(\mathbb{Z}_p^{\text{Nyg}})_{p=a^p\mu=0}$ due to Drinfeld [Dri20, Section 7.4].

Recollection 3.18. Recall the stack $\mathbb{A}^1/\mathbb{G}_m$, whose points are given by $\mathbb{A}^1/\mathbb{G}_m(S) = (\mathcal{L}, a : \mathcal{L} \rightarrow \mathcal{O}_S)$, where \mathcal{L} is a line bundle on S .

Definition 3.19. We define a stack T over $(\mathbb{A}^1/\mathbb{G}_m)_{p=0}$ by $T(S) = (\mathcal{L}, a : \mathcal{L} \rightarrow \mathcal{O}_S, u : \mathcal{O}_S \rightarrow \mathcal{L}^{\otimes p}, a^p u = 0)$.

Definition 3.20. We define a group scheme $\mathbb{G}_{m,a}^\#$ over $(\mathbb{A}^1/\mathbb{G}_m)_{p=0}$ by

$$\mathbb{G}_{m,a}^\#(S) = \{\alpha \in (\mathcal{L}^{\otimes p})^\#(S) \mid (1 - (a^p)^\#)(\alpha) \in \mathbb{G}_m^\#(S)\}$$

with group operation $\alpha_1 * \alpha_2 = \alpha_1 + \alpha_2 - [a^p](\alpha_1)\alpha_2$.

Definition 3.21. There is a map $T \rightarrow \mathbb{Z}_p^{\text{Nyg}}$ over $\mathbb{A}^1/\mathbb{G}_m$ given by sending (\mathcal{L}, a, u) to the admissible module M defined by the following pullback square:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{L}^\# & \longrightarrow & M & \longrightarrow & F_*W_S \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow & & \downarrow F_*[u] \\ 0 & \longrightarrow & \mathcal{L}^\# & \longrightarrow & [\mathcal{L}] & \xrightarrow{F} & [\mathcal{L}] \otimes_{W_S} F_*W_S \longrightarrow 0, \end{array}$$

which is made into a filtered Cartier–Witt divisor via the map $\xi : M \rightarrow [\mathcal{L}] \oplus F_*W_S \xrightarrow{[a], V} W_S$, which fits into the diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{L}^\# & \longrightarrow & M & \longrightarrow & F_*W_S \longrightarrow 0 \\ & & \downarrow a^\# & & \downarrow d & & \downarrow p \\ 0 & \longrightarrow & \mathbb{G}_a^\# & \longrightarrow & W_S & \longrightarrow & F_*W_S \longrightarrow 0. \end{array}$$

Here $[\mathcal{L}]$ is the Teichmüller lift of the line bundle \mathcal{L} to an invertible W_S -module.

Endow the morphism $T \rightarrow \mathbb{Z}_p^{\text{Nyg}}$ with the action of $\mathbb{G}_{m,t}^\#$ which is trivial on T and $\mathbb{Z}_p^{\text{Nyg}}$ and acts on the filtered Cartier–Witt divisor $d : M \rightarrow W_S$ via the action on $M \hookrightarrow [\mathcal{L}] \oplus F_*W_S$ given by

$$\alpha \cdot (x, y) = (x + V(\alpha y), (1 - [a^{\otimes p}](\alpha))y).$$

Proposition 3.22 ([Dri20, Corollary 7.4.6]). *The induced morphism $T/\mathbb{G}_{m,t}^\# \rightarrow \mathbb{Z}_p^{\text{Nyg}}$ identifies the source with $(\mathbb{Z}_p^{\text{Nyg}})_{p=a^p\mu=0}$.*

Remark 3.23. If we let $S \hookrightarrow T$ denote the closed substack cut out by the replacing the equation $a^p\mu = 0$ with $a\mu = 0$, then we find that $S/\mathbb{G}_{m,t}^\#$ identifies with $(\mathbb{Z}_p^{\text{Nyg}})_{p=v_1=0}$.

Additionally, $S \rightarrow \mathbb{Z}_p^{\text{Nyg}}$ may be identified with $(\mathbb{F}_p)_{p=0}^{\text{Nyg}} \rightarrow \mathbb{Z}_p^{\text{Nyg}}$, as follows from examining [Bha22, Section 5.4].

3.3. Syntomification.

Definition 3.24. We define the syntomification X^{Syn} of a derived p -adic formal scheme X via the pushout square:

$$\begin{array}{ccc} X^\Delta \amalg X^\Delta & \xrightarrow{(j_{HT}, j_{dR})} & X^{\text{Nyg}} \\ \downarrow \Delta & & \downarrow j^{\text{Nyg}} \\ X^\Delta & \xrightarrow{j^\Delta} & X^{\text{Syn}}. \end{array}$$

Definition 3.25. The line bundles $\mathcal{O}_{\mathbb{Z}_p^{\text{Nyg}}}\{1\}$ and $\mathcal{O}_{\mathbb{Z}_p^\Delta}\{1\}$ glue to a line bundle $\mathcal{O}_{\mathbb{Z}_p^{\text{Syn}}}$ on $\mathbb{Z}_p^{\text{Syn}}$. Moreover, the sections v_1 of $\mathcal{O}_{(\mathbb{Z}_p)_{p=0}^{\text{Nyg}}}\{p-1\}$ and $\mathcal{O}_{(\mathbb{Z}_p)_{p=0}^\Delta}\{p-1\}$ glue to give a section v_1 of $\mathcal{O}_{(\mathbb{Z}_p)_{p=0}^{\text{Syn}}}\{p-1\}$.

Finally, the following theorem compares Nygaard filtered prismatic cohomology and syntomic cohomology to the coherent cohomology of the Nygaardification and syntomifications, and will appear in forthcoming work of Bhatt–Lurie (see also [Bha22, Theorem 5.5.10 and Remark 5.5.18]).

Theorem 3.26 ([BL]). *Let X be a qcqs p -quasisyntomic p -adic formal scheme. Then there is a natural isomorphism*

$$\mathbb{Z}_p(i)(X) \cong R\Gamma(X^{\text{Syn}}, \mathcal{O}_{X^{\text{Syn}}}\{i\}).$$

and a natural isomorphism

$$\mathcal{N}^{\geq i} \Delta_X\{i\} \cong R\Gamma(X^{\text{Nyg}}, \mathcal{O}_{X^{\text{Nyg}}}\{i\}).$$

3.4. The base case. The main purpose of this subsection is to produce and study the following diagram, which serves as a base case to our later inductive proof of Theorem 1.1(1-3) and Theorem 1.4(1-3).

Construction 3.27. There is a commutative diagram

$$\begin{array}{ccccc} (\mathbb{Z}_p)_{p=v_1=0}^{\text{Nyg}} & \longrightarrow & (\mathbb{Z}_p)_{p=a^p \mu=0}^{\text{Nyg}} & \longrightarrow & (\mathbb{Z}_p)_{p=0}^{\text{Nyg}} \\ \downarrow F & & \downarrow F & & \downarrow F \\ & \swarrow & (\mathbb{F}_p)_{p=0}^{\text{Nyg}} & \swarrow & \\ & & & & \\ (\mathbb{Z}_p)_{p=v_1=0}^{\text{Nyg}} & \longrightarrow & (\mathbb{Z}_p)_{p=a^p \mu=0}^{\text{Nyg}} & \longrightarrow & (\mathbb{Z}_p)_{p=0}^{\text{Nyg}} \end{array}$$

such that the composite map $(\mathbb{F}_p)_{p=0}^{\text{Nyg}} \rightarrow (\mathbb{Z}_p)_{p=0}^{\text{Nyg}}$ is the mod p reduction of the functor $(-)^{\text{Nyg}}$ applied to the ring homomorphism $\mathbb{Z}_p \rightarrow \mathbb{F}_p$.

Proof. Suppose \mathcal{G} is a group scheme over \mathbb{F}_p on which the Frobenius map is constant at the identity. Then, if \mathcal{G} acts trivially on a stack \mathcal{T} , there is a natural diagram

$$\begin{array}{ccc} \mathcal{T} & \longleftarrow & \mathcal{T}/\mathcal{G} \\ \downarrow F & & \downarrow F \\ \mathcal{T} & \longrightarrow & \mathcal{T}/\mathcal{G}. \end{array}$$

In the case that $\mathcal{G} = \mathbb{G}_{m,t}^{\#}$ and $\mathcal{T} = T$ are as in Definition 3.20 and Definition 3.19, we obtain a diagram

$$\begin{array}{ccc} T & \longleftarrow & (\mathbb{Z}_p)_{p=a^p \mu=0}^{\text{Nyg}} \\ \downarrow F & & \downarrow F \\ T & \longrightarrow & (\mathbb{Z}_p)_{p=a^p \mu=0}^{\text{Nyg}}. \end{array}$$

Similarly, we have

$$\begin{array}{ccc} S & \longleftarrow & (\mathbb{Z}_p)_{p=v_1=0}^{\text{Nyg}} \\ \downarrow F & & \downarrow F \\ S & \longrightarrow & (\mathbb{Z}_p)_{p=v_1=0}^{\text{Nyg}}. \end{array}$$

Using the identification of $S \rightarrow \mathbb{Z}_p^{\text{Nyg}}$ with $(\mathbb{F}_p)_{p=0}^{\text{Nyg}} \rightarrow \mathbb{Z}_p^{\text{Nyg}}$, it therefore suffices to note that there is a further factorization,

$$\begin{array}{ccc} S & \dashleftarrow & T \\ & \searrow & \downarrow F \\ & & T \end{array}$$

coming from the factorization of graded rings

$$\begin{array}{ccc} \mathbb{F}_p[\mu, a]/(\mu a) & \dashrightarrow & \mathbb{F}_p[\mu, a]/(\mu a^p) \\ & \nwarrow & \uparrow F \\ & & \mathbb{F}_p[\mu, a]/(\mu a^p). \end{array}$$

□

In order to understand prismatization or syntomification, and not only Nygaardification, it is necessary to comment on the restrictions of the above diagram under the open immersions

$$j_{HT}, j_{dR} : \mathbb{Z}_p^\Delta \rightarrow \mathbb{Z}_p^{\text{Nyg}}.$$

We do so in the remainder of this section.

Proposition 3.28. *The restriction of the subdiagram*

$$\begin{array}{ccc} (\mathbb{Z}_p)_{p=a^p \mu=0}^{\text{Nyg}} & \longrightarrow & (\mathbb{Z}_p)_{p=0}^{\text{Nyg}} \\ \swarrow & \downarrow F & \downarrow F \\ (\mathbb{F}_p)_{p=0}^{\text{Nyg}} & & (\mathbb{Z}_p)_{p=0}^{\text{Nyg}} \\ \searrow & \downarrow F & \downarrow F \\ (\mathbb{Z}_p)_{p=a^p \mu=0}^{\text{Nyg}} & \longrightarrow & (\mathbb{Z}_p)_{p=0}^{\text{Nyg}} \end{array}$$

under j_{HT} is a diagram

$$\begin{array}{ccc} (\mathbb{Z}_p)_{p=v_1^p=0}^\Delta & \longrightarrow & (\mathbb{Z}_p)_{p=0}^\Delta \\ \swarrow & \downarrow F & \downarrow F \\ (\mathbb{F}_p)_{p=0}^\Delta & & (\mathbb{Z}_p)_{p=0}^\Delta \\ \searrow & \downarrow F & \downarrow F \\ (\mathbb{Z}_p)_{p=v_1^p=0}^\Delta & \longrightarrow & (\mathbb{Z}_p)_{p=0}^\Delta \end{array}$$

Proof. This follows from the facts that the maps in Construction 3.27 are compatible with inverting μ , and that the restriction of $(\mathbb{Z}_p)_{p=a^p \mu=0}^{\text{Nyg}}$ along j_{HT} is $(\mathbb{Z}_p)_{p=v_1^p=0}^\Delta$. □

Proposition 3.29. *The restrictions of the subdiagram*

$$\begin{array}{ccc} (\mathbb{Z}_p)_{p=v_1=0}^{\text{Nyg}} & \longrightarrow & (\mathbb{Z}_p)_{p=0}^{\text{Nyg}} \\ \downarrow F & & \downarrow F \\ (\mathbb{F}_p)_{p=0}^{\text{Nyg}} & \longrightarrow & (\mathbb{Z}_p)_{p=0}^{\text{Nyg}} \end{array}$$

along j_{HT} and j_{dR} are given by

$$\begin{array}{ccc} \mathrm{Spec}(\mathbb{F}_p)/(W[F] \times \mu_p) & \longrightarrow & (\mathbb{Z}_p)_{p=0}^\Delta \\ \downarrow & & \downarrow F \\ \mathrm{Spec}(\mathbb{F}_p) & \longrightarrow & (\mathbb{Z}_p)_{p=0}^\Delta, \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Spec}(\mathbb{F}_p)/((W^\times)[F]) & \longrightarrow & (\mathbb{Z}_p)_{p=0}^\Delta \\ \downarrow & & \downarrow F \\ \mathrm{Spec}(\mathbb{F}_p) & \longrightarrow & (\mathbb{Z}_p)_{p=0}^\Delta, \end{array}$$

respectively. These diagrams are identified under the isomorphism $W[F] \times \mu_p \cong (W^\times)[F]$ of [Dri20, Lemma 3.3.4]. Thus, there exists a diagram

$$\begin{array}{ccc} (\mathbb{Z}_p)_{p=v_1=0}^{\mathrm{Syn}} & \longrightarrow & (\mathbb{Z}_p)_{p=0}^{\mathrm{Syn}} \\ \downarrow & & \downarrow F \\ (\mathbb{F}_p)_{p=0}^{\mathrm{Syn}} & \longrightarrow & (\mathbb{Z}_p)_{p=0}^{\mathrm{Syn}} \end{array}$$

Proof. This is a consequence of [Dri20, Lemma 7.5.2 and Remark 7.5.3]. \square

3.5. A factorization via Verschiebung. Our goal in this subsection is to construct the diagram of the following proposition, which will be the key tool powering our inductive understanding of \mathbb{Z}/p^{n+1} in terms of \mathbb{Z}/p^n :

Proposition 3.30. *There is a commuting diagram:*

$$\begin{array}{ccc} F^{-1}(\mathbb{Z}/p^n)_{p=0}^{\mathrm{Nyg}} & \longrightarrow & (\mathbb{Z}/p^n)_{p=0}^{\mathrm{Nyg}} \\ \vdots & & \downarrow \\ (\mathbb{Z}/p^{n+1})_{p=0}^{\mathrm{Nyg}} & & \downarrow \\ \downarrow & & \downarrow F \\ (\mathbb{Z}_p)_{p=0}^{\mathrm{Nyg}} & \xrightarrow{F} & (\mathbb{Z}_p)_{p=0}^{\mathrm{Nyg}}, \end{array}$$

where the outer square is a pullback. Moreover, the restrictions of this diagram under the two open immersions

$$j_{HT}, j_{dR} : (\mathbb{Z}_p)_{p=0}^\Delta \hookrightarrow (\mathbb{Z}_p)_{p=0}^{\mathrm{Nyg}}$$

are isomorphic, so that there exists a diagram

$$\begin{array}{ccc} F^{-1}(\mathbb{Z}/p^n)_{p=0}^{\mathrm{Syn}} & \longrightarrow & (\mathbb{Z}/p^n)_{p=0}^{\mathrm{Syn}} \\ \downarrow & & \downarrow \\ (\mathbb{Z}/p^{n+1})_{p=0}^{\mathrm{Syn}} & & \downarrow \\ \downarrow & & \downarrow F \\ (\mathbb{Z}_p)_{p=0}^{\mathrm{Syn}} & \xrightarrow{F} & (\mathbb{Z}_p)_{p=0}^{\mathrm{Syn}}. \end{array}$$

Proof. We start by constructing the dashed morphism. For a test \mathbb{F}_p -algebra R , an object of $F^{-1}(\mathbb{Z}/p^n)(R)$ is a tuple $(M, d : M \rightarrow W_R, x \in \mathrm{Fr}_R^* M(R))$ such that $(\mathrm{Fr}_R^* d)(x) = p^n$.

Now, using Corollary 3.6, we have a commutative square

$$\begin{array}{ccc}
M(R) & \xleftarrow{V} & \mathrm{Fr}_R^* M(R) \\
\downarrow d & & \downarrow \mathrm{Fr}_R^* d \\
W(R) & \xleftarrow{V} & W(R).
\end{array}$$

In particular, $V(x) \in M(R)$ defines a lift of $V(p^n) \in W(R)$. By the assumption $p = 0$ on R , $V(p^n) = p^{n+1} \in W(R)$, and we may take $V(x)$ as the desired lift of $p^{n+1} \in W(R)$.

What remains is to check that the pullbacks under j_{HT} and j_{dR} agree.

Pulling back under j_{HT} , it is immediate that this agrees with the diagram constructed in Lemma 2.17. It therefore remains to check that we recover the same diagram if we pull back using j_{dR} .

Starting with a Cartier–Witt divisor $d : I \rightarrow W$, its image under j_{dR} is given by the pullback

$$\begin{array}{ccc}
M & \longrightarrow & F_* I \\
\downarrow & & \downarrow F_* d \\
W & \xrightarrow{F} & F_* W.
\end{array}$$

To understand how $j_{dR} : R^\Delta \rightarrow R^{\mathrm{Nyg}}$ works for general R , we need to recall the isomorphism $j_{dR}^* \mathbb{G}^{\mathrm{Nyg}} \cong \mathbb{G}^\Delta \cong j_{HT}^* \mathbb{G}^{\mathrm{Nyg}}$, which is induced via the ring isomorphism $W/M \cong F_* W / F_* I$ coming from the above pullback square.

For $R = \mathbb{Z}/p^n$, this may be understood as follows: using the above pullback square, lifts of p^n from $F_* W$ to $F_* I$ correspond to lifts of p^n from W to M . What we need to check is that the first part of this proof, which takes $x \in F^* M(R)$, a lift of p^n to $V(x) \in M(R)$, a lift of p^{n+1} , is compatible with this correspondence.

This holds by the naturality of V , which implies that we have a commutative cube:

$$\begin{array}{ccccc}
& & \mathrm{Fr}_R^* M & \longrightarrow & F_* \mathrm{Fr}_R^* I \\
& & \swarrow & & \swarrow \\
& & \mathrm{Fr}_R^* W & \longrightarrow & F_* \mathrm{Fr}_R^* W \\
& & \downarrow & & \downarrow \\
& & M & \longrightarrow & F_* I, \\
& & \swarrow & & \swarrow \\
& & W & \longrightarrow & F_* W
\end{array}$$

where the vertical maps are given by V .

□

3.6. Crystallinity.

Theorem 3.31. *For integers $n \geq 0$, there exist compatible commutative diagrams*

$$\begin{array}{ccc}
(\mathbb{Z}_p)_{p=v_1^{p^n}=0}^{\mathrm{Syn}} & \longrightarrow & (\mathbb{Z}_p)_{p=0}^{\mathrm{Syn}} \\
& \searrow & \nearrow \\
& (\mathbb{Z}/p^{n+2})_{p=0}^{\mathrm{Syn}} &
\end{array}$$

$$\begin{array}{ccc}
 (\mathbb{Z}_p)_{p=v_1^{p^n}=0}^{\text{Nyg}} & \xrightarrow{\quad\quad\quad} & (\mathbb{Z}_p)_{p=0}^{\text{Nyg}} \\
 & \searrow \quad \swarrow & \\
 & (\mathbb{Z}/p^{n+2})_{p=0}^{\text{Nyg}} &
 \end{array}$$

and

$$\begin{array}{ccc}
 (\mathbb{Z}_p)_{p=v_1^{p^n}=0}^{\Delta} & \xrightarrow{\quad\quad\quad} & (\mathbb{Z}_p)_{p=0}^{\Delta} \\
 & \searrow \quad \swarrow & \\
 & (\mathbb{Z}/p^{n+2})_{p=0}^{\Delta} &
 \end{array}$$

Proof. We present the proof for syntomification, noting that the arguments for prismaticization and for Nygaardification are perfectly analogous. Our goal is to induct on $n \geq 0$ to construct a dashed arrow

$$\begin{array}{ccc}
 F^{-n} \left((\mathbb{Z}_p)_{p=v_1=0}^{\text{Syn}} \right) & \xrightarrow{\quad\quad\quad} & (\mathbb{Z}_p)_{p=0}^{\text{Syn}} \\
 & \dashrightarrow \quad \swarrow & \\
 & (\mathbb{Z}/p^{n+2})_{p=0}^{\text{Syn}} &
 \end{array}$$

To begin, we first construct the dashed arrow when $n = 0$. By the last sentence of Proposition 3.29, there is a diagram

$$\begin{array}{ccc}
 (\mathbb{Z}_p)_{p=v_1=0}^{\text{Syn}} & \xrightarrow{\quad\quad\quad} & (\mathbb{Z}_p)_{p=0}^{\text{Syn}} \\
 \downarrow & \nearrow & \\
 F^{-1}((\mathbb{F}_p)_{p=0}^{\text{Syn}}) & &
 \end{array}$$

The case $n = 0$ thus follows from the last sentence of Proposition 3.30.

Now we proceed with the inductive step. In particular, suppose for the sake of induction that we have constructed a diagram

$$\begin{array}{ccc}
 F^{-n+1} \left((\mathbb{Z}_p)_{p=v_1=0}^{\text{Syn}} \right) & \xrightarrow{\quad\quad\quad} & (\mathbb{Z}_p)_{p=0}^{\text{Syn}} \\
 & \dashrightarrow \quad \swarrow & \\
 & (\mathbb{Z}/p^{n+1})_{p=0}^{\text{Syn}} &
 \end{array}$$

Applying F^{-1} , we obtain a diagram

$$\begin{array}{ccc}
 F^{-n} \left((\mathbb{Z}_p)_{p=v_1=0}^{\text{Syn}} \right) & \xrightarrow{\quad\quad\quad} & (\mathbb{Z}_p)_{p=0}^{\text{Syn}} \\
 & \dashrightarrow \quad \swarrow & \\
 & F^{-1}(\mathbb{Z}/p^{n+1})_{p=0}^{\text{Syn}} &
 \end{array}$$

The desired diagram is then constructed by applying the last sentence of Proposition 3.30. \square

For prismaticization and Nygaardification individually, but not syntomification, we can improve the above result:

Theorem 3.32. *For each $n \geq 0$, there exists a commutative diagram*

$$\begin{array}{ccc}
(\mathbb{Z}_p)_{p=(a^p\mu)^{p^n}=0}^{\text{Nyg}} & \xrightarrow{\quad\quad\quad} & (\mathbb{Z}_p)_{p=0}^{\text{Nyg}} \\
& \searrow & \nearrow \\
& (\mathbb{Z}/p^{n+2})_{p=0}^{\text{Nyg}} &
\end{array}$$

which restricts, under j_{HT} , to a commutative diagram

$$\begin{array}{ccc}
(\mathbb{Z}_p^\Delta)_{p=v_1^{p^{n+1}}=0} & \xrightarrow{\quad\quad\quad} & (\mathbb{Z}_p)_{p=0}^\Delta \\
& \searrow & \nearrow \\
& (\mathbb{Z}/p^{n+2})_{p=0}^\Delta &
\end{array}$$

Proof. We run an argument similar to that of Theorem 3.31. Our goal is to induct on $n \geq 0$ to construct a dashed arrow

$$\begin{array}{ccc}
F^{-n} \left((\mathbb{Z}_p)_{p=(a^p\mu)=0}^{\text{Nyg}} \right) & \xrightarrow{\quad\quad\quad} & (\mathbb{Z}_p)_{p=0}^{\text{Nyg}} \\
& \dashrightarrow & \nearrow \\
& (\mathbb{Z}/p^{n+2})_{p=0}^{\text{Nyg}} &
\end{array}$$

To begin, we first construct the dashed arrow when $n = 0$. By Proposition 3.28, there is a diagram

$$\begin{array}{ccc}
(\mathbb{Z}_p)_{p=a^p\mu=0}^{\text{Nyg}} & \xrightarrow{\quad\quad\quad} & (\mathbb{Z}_p)_{p=0}^{\text{Nyg}} \\
\downarrow & \nearrow & \\
F^{-1} \left((\mathbb{F}_p)_{p=0}^{\text{Nyg}} \right) & &
\end{array}$$

compatible with restriction to the prismaticization. Combining this with Proposition 3.30 gives the case $n = 0$.

Now we proceed with the inductive step. In particular, suppose for the sake of induction that we have constructed a diagram

$$\begin{array}{ccc}
F^{-n+1} \left((\mathbb{Z}_p)_{p=a^p\mu=0}^{\text{Nyg}} \right) & \xrightarrow{\quad\quad\quad} & (\mathbb{Z}_p)_{p=0}^{\text{Nyg}} \\
& \dashrightarrow & \nearrow \\
& (\mathbb{Z}/p^{n+1})_{p=0}^{\text{Nyg}} &
\end{array}$$

Applying F^{-1} , we obtain a diagram

$$\begin{array}{ccc}
F^{-n} \left((\mathbb{Z}_p)_{p=a^p\mu=0}^{\text{Nyg}} \right) & \xrightarrow{\quad\quad\quad} & (\mathbb{Z}_p)_{p=0}^{\text{Nyg}} \\
& \dashrightarrow & \nearrow \\
& F^{-1} \left((\mathbb{Z}/p^{n+1})_{p=0}^{\text{Nyg}} \right) &
\end{array}$$

The desired diagram is then constructed by applying Proposition 3.30. \square

Proof of Theorem 1.1(1)-(3) and Theorem 1.4(1)-(3). We claim that it is enough to prove Theorem 1.4. Indeed, if we assume Theorem 1.4, then it follows from Theorem 2.12 and Theorem 3.26 that Theorem 1.1 holds when restricted to the category of p -complete rings R such that both R and R/p^{n+2} are p -quasisyntomic. In particular, it holds for the category of p -completed polynomial \mathbb{Z}_p -algebras. Since all of the functors

under consideration are left Kan extended from this subcategory, we obtain the full statement of Theorem 1.1.

Now we prove Theorem 1.4(2). The proof of Theorem 1.4(3) is exactly analogous, except that we substitute Theorem 3.32 for Theorem 3.31.

Using Theorem 3.31, we may produce a diagram of stacks:

$$\begin{array}{ccc}
 (X^{\mathrm{Nyg}})_{p=v_1^{p^n}=0} & \longrightarrow & (\mathbb{Z}_p^{\mathrm{Nyg}})_{p=v_1^{p^n}=0} \\
 \downarrow & & \downarrow \\
 (X_{p^{n+2}=0})^{\mathrm{Nyg}} & \longrightarrow & (\mathbb{Z}/p^{n+2})^{\mathrm{Nyg}} \\
 \downarrow & & \downarrow \\
 (X^{\mathrm{Nyg}}) & \longrightarrow & (\mathbb{Z}_p^{\mathrm{Nyg}}).
 \end{array}$$

The outer square is a pullback by definition, and the bottom square is a pullback because $(-)^{\mathrm{Nyg}}$ preserves limits of derived p -adic formal schemes. It follows that the top square is also a pullback, so that the formula $(X_{p^{n+2}=0})^{\mathrm{Nyg}} \times_{(\mathbb{Z}/p^{n+2})^{\mathrm{Nyg}}} (\mathbb{Z}_p^{\mathrm{Nyg}})_{p=v_1^{p^n}=0}$ gives a functorial description of $(X^{\mathrm{Nyg}})_{p=v_1^{p^n}=0}$ only depending on $X_{p^{n+2}=0}$.

Finally, to prove Theorem 1.4(1), we note that by Theorem 3.31, the functorial descriptions of $(X^{\mathrm{Nyg}})_{p=v_1^{p^n}=0}$ and $(X^\Delta)_{p=v_1^{p^n}=0}$ in terms of $X_{p^{n+2}=0}$ are compatible with restriction along j_{HT} and j_{dR} . We may thus glue these loci together to obtain $(X^{\mathrm{Syn}})_{p=v_1^{p^n}=0}$ functorially. \square

4. RECOVERING THE BETTI NUMBERS OF THE GENERIC FIBER

As a sample application of our result, we give conditions for when the étale Betti numbers of the generic fiber of a p -adic formal scheme can be read off from its mod p^n reduction.

Definition 4.1. Let X be a p -adic formal scheme. Let C_X be the category of Zariski sheaves of abelian groups \mathfrak{F} such that $H^i(\mathfrak{F}|_U) = 0$ for $i > 0$ on all affine opens $U \subset X$. Then we say the *affine cohomological dimension* of X is the smallest $d \in \mathbb{N} \cup \{\infty\}$ such that $H^i(\mathfrak{F}) = 0$ for all $i > d$, $\mathfrak{F} \in C_X$.

Example 4.2. The affine cohomological dimension of X is always at most the Zariski cohomological dimension, which is given by the Krull dimension of the underlying topological space⁵ by [Sch92, Theorem 4.5]. However, it can be significantly smaller—for example, it vanishes when X is affine.

Theorem 4.3. Let X denote a p -torsionfree F -smooth qcqs p -adic formal scheme of affine cohomological dimension d . Set $b(d) = \lceil \log_p \left(\lceil \frac{d+1}{p-1} \rceil + 1 \right) \rceil + 2$. Then

$$b_{i,j}(X; \mathbb{F}_p) := \dim_{\mathbb{F}_p} H_{\mathrm{et}}^i(X_\eta; \mu_p^{\otimes j})$$

only depends on $X_{p^{b(d)}=0}$.⁶ In other words, if X_1 and X_2 as above satisfy $(X_1)_{p^{b(d)}=0} \cong (X_2)_{p^{b(d)}=0}$, then $b_{i,j}(X_1; \mathbb{F}_p) = b_{i,j}(X_2; \mathbb{F}_p)$.

Remark 4.4. For the technical definition of F -smooth, we refer the reader to [BM23]. For example, X could be a smooth p -adic formal scheme of relative dimension d over $\mathrm{Spf}\mathcal{O}_K$ for a local or perfectoid field K .

⁵This is equal to the topological space associated to $X_{p=0}$

⁶Here, X_η is the adic generic fiber of X , i.e. $X \times_{\mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)} \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$ in the category of (pre-)adic spaces.

Lemma 4.5. *Let X denote a p -torsion free F -smooth qcqs p -adic formal scheme of affine cohomological dimension d . Then the v_1 -power torsion part of $H^* \bigoplus_{i=0}^{\infty} \mathbb{F}_p(i)(X)$ is killed uniformly by $v_1^{\lceil \frac{d+1}{p-1} \rceil}$.*

Proof. By [AMMN22, Theorem 5.1(i)], we know that the sheaf $\mathbb{F}_p(i)_X$ lies in $[-i-1, 0]$ as a sheaf on the Zariski site of X . It follows since X is affine cohomological dimension d that the global sections $\mathbb{F}_p(i)(X)$ lie in $[-i-d-1, 0]$.

Now, it follows from [BM23, Proof of Proposition 5.2] that $v_1 : H^j \mathbb{F}_p(i)(X) \rightarrow H^j \mathbb{F}_p(i+p-1)(X)$ is injective for $j \leq i$. Combining this with the dimension bound of the previous paragraph, we conclude that all v_1 -power torsion in $H^* \bigoplus_{i=0}^{\infty} \mathbb{F}_p(i)(X)$ is killed by $v_1^{\lceil \frac{d+1}{p-1} \rceil}$. \square

Question 4.6. If $X = \mathrm{Spf} A$ is affine, then Lemma 4.5 shows the mod p syntomic cohomology of X can only have simple v_1 -torsion. Is there an affine F -smooth p -torsion free p -adic formal scheme X such that the mod p syntomic cohomology of X has v_1 -torsion?

Lemma 4.7. *Let $\mathbb{F}_p[x]$ denote the graded polynomial algebra over \mathbb{F}_p with $|x| = 1$, and let C denote a complex of nonnegatively graded $\mathbb{F}_p[x]$ -modules. Suppose that the x -power torsion in $H_* C$ is bounded by x^n .*

Let C/x^{n+1} denote the cofiber of C by x^{n+1} . Then we have

$$\dim_{\mathbb{F}_p} H_i C[x^{-1}] = \dim_{\mathbb{F}_p} x^n (H_i(C/x^{n+1})).$$

In particular, $\dim_{\mathbb{F}_p} H_i C[x^{-1}]$ may be read off from C/x^{n+1} .

Proof. Consider

$$0 \rightarrow (H_i C)_{\mathrm{tors}} \rightarrow H_i C \rightarrow (H_i C)_{\mathrm{tf}} \rightarrow 0.$$

Using the fact that C is nonnegatively graded, one sees that $(H_i C)_{\mathrm{tf}}$ is a free graded $\mathbb{F}_p[x]$ -module. In particular, we have

$$\dim_{\mathbb{F}_p} H_i C[x^{-1}] = \dim_{\mathbb{F}_p} (H_i C)_{\mathrm{tf}}/x.$$

Since x^n kills $(H_i C)_{\mathrm{tors}}$, we have further that

$$\dim_{\mathbb{F}_p} (H_i C)_{\mathrm{tf}}/x = \dim_{\mathbb{F}_p} x^n (H_i C)_{\mathrm{tf}}/x^{n+1} = \dim_{\mathbb{F}_p} x^n (H_i C)/x^{n+1}$$

Now, we have

$$H_i(C/x^{n+1}) \cong (H_i C)/x^{n+1} \oplus H_{i-1} C[x^{n+1}] = (H_i C)/x^{n+1} \oplus (H_{i-1} C)_{\mathrm{tors}},$$

so that

$$\dim_{\mathbb{F}_p} x^n H_i(C/x^{n+1}) = \dim_{\mathbb{F}_p} x^n (H_i C)/x^{n+1},$$

from which the lemma follows. \square

Proof of Theorem 4.3. By the étale comparison theorem [AMMN22, Theorem 5.1], $H_{\mathrm{et}}^i(X_\eta; \mu_p^{\otimes j})$ may be obtained as the colimit of

$$H^i \mathbb{F}_p(j)(X) \xrightarrow{v_1} H^i \mathbb{F}_p(j+p-1)(X) \xrightarrow{v_1} \dots$$

By Lemma 4.5, the v_1 -torsion in $H^j \mathbb{F}_p(*) (X)$ is bounded by $v_1^{\lceil \frac{d+1}{p-1} \rceil}$. Using Lemma 4.7 and the étale comparison theorem, we thus see that

$$\dim_{\mathbb{F}_p} H_{\mathrm{et}}^i(X_\eta; \mu_p^{\otimes j}) = \dim_{\mathbb{F}_p} v_1^{\lceil \frac{d+1}{p-1} \rceil} H^i \left(\bigoplus_{\substack{k \geq 0 \\ k \equiv j \pmod{p-1}}} \mathbb{F}_p(k)(X) / v_1^{\lceil \frac{d+1}{p-1} \rceil + 1} \right)$$

Now, it follows from Theorem 1.1 that $\mathbb{F}_p(j)(X) / v_1^{\lceil \frac{d+1}{p-1} \rceil + 1}$ may be recovered functorially from $X_{\log_p(\lceil \frac{d+1}{p-1} \rceil + 1) + 2} = 0$. \square

Remark 4.8. In the case $X = \mathrm{Spf} A$ is affine, we may recover the mod p Betti numbers of the generic fiber from $X_{p^3=0} = \mathrm{Spec} A/p^3$.

5. INVARIANTS OF $R\langle\epsilon\rangle$

By Theorem 1.1, we can compute the mod $(p, v_1^{p^n})$ syntomic cohomology of any animated ring R as a functor of the (derived) quotient R/p^{n+2} . If $R = \mathbb{Z}/p^m$ for $m \geq n+2$, then $R/p^{n+2} \cong \mathbb{Z}\langle\epsilon\rangle/p^{n+2}$, where $\mathbb{Z}\langle\epsilon\rangle$ is defined as follows:

Definition 5.1. We let $\mathbb{Z}\langle\epsilon\rangle$ denote $\mathbb{Z} \otimes_{\mathbb{Z}\langle x \rangle} \mathbb{Z}$, where $x \mapsto 0$ along both augmentations. In other words, $\mathbb{Z}\langle\epsilon\rangle$ is the free animated ring on a degree 1 class.

In particular, for $m \geq n+2$, the mod $(p, v_1^{p^n})$ syntomic cohomology of \mathbb{Z}/p^m is the same as the mod $(p, v_1^{p^n})$ syntomic cohomology of $\mathbb{Z}\langle\epsilon\rangle$. We will spend the rest of the paper computing the mod $(p, v_1^{p^n})$ syntomic cohomology of $\mathbb{Z}\langle\epsilon\rangle$ and drawing consequences for algebraic K -theory.

To calculate the syntomic cohomology of $\mathbb{Z}\langle\epsilon\rangle$ by quasi-syntomic descent, in this section we will understand the syntomic cohomology of $R\langle\epsilon\rangle = R \otimes_{\mathbb{Z}} \mathbb{Z}\langle\epsilon\rangle$ for quasiregular semiperfectoid rings R . We will first recall a classical decomposition of $\mathrm{THH}(R\langle\epsilon\rangle)$ in the category of cyclotomic spectra, which straightforwardly implies the following theorem:

Theorem 5.2. *For any animated ring R , there is an equivalence:*

$$\mathrm{TC}(R\langle\epsilon\rangle)_p^\wedge \simeq \mathrm{TC}(R)_p^\wedge \oplus \bigoplus_{p \nmid \ell} \mathrm{TR}(R; \Sigma^{2\ell} R)_p^\wedge.$$

Here, the sum ranges over all positive integers ℓ not divisible by p , and $\mathrm{TR}(R; \Sigma^{2\ell} R)$ refers to $\langle p \rangle$ -polygonic TR with coefficients in the bimodule $\Sigma^{2\ell} R$.

Remark 5.3. Since any animated ring R is connective, for each $\ell > 0$ the homotopy groups of $\mathrm{TR}(R; \Sigma^{2\ell} R)_p^\wedge$ are concentrated in degrees $2\ell - 1$ and above. In particular, the infinite sum appearing in Theorem 5.2 is also an infinite product, and therefore p -complete.

Next, we observe that the above decomposition interacts well with the Bhatt–Morrow–Scholze motivic filtration. In particular:

Theorem 5.4. *Suppose that R is a qrsp ring. Then, functorially in R , $\mathrm{fil}_{\mathrm{mot}}^* \mathrm{TC}(R\langle\epsilon\rangle)_p^\wedge$ is the equalizer of the diagram of filtered \mathbb{E}_∞ -rings*

$$\tau_{\geq 2*} \mathrm{TC}^-(R\langle\epsilon\rangle)_p^\wedge \begin{array}{c} \xrightarrow{\tau_{\geq 2*} \varphi^{hS^1}} \\ \xrightarrow{\tau_{\geq 2*} \mathrm{can}} \end{array} \tau_{\geq 2*} \mathrm{TP}(R\langle\epsilon\rangle)_p^\wedge.$$

Explicitly, defining

$$\mathrm{fil}_{\mathrm{mot}}^* \mathrm{TR}(R; \Sigma^{2\ell} R)_p^\wedge$$

to be the limit of the diagram

$$\begin{array}{ccccc} \tau_{\geq 2*} \left(\mathrm{THH}(R; (\Sigma^{2\ell} R)^{\otimes p})^{tC_p} \right)_p^\wedge & & \tau_{\geq 2*} \left(\mathrm{THH}(R; (\Sigma^{2\ell} R)^{\otimes p^2})^{tC_{p^2}} \right)_p^\wedge & & \dots \\ \varphi \nearrow & \leftarrow \mathrm{can} & \varphi^{hC_p} \nearrow & \leftarrow \mathrm{can} & \varphi^{hC_{p^2}} \nearrow \\ \tau_{\geq 2*} \mathrm{THH}(R; \Sigma^{2\ell} R)_p^\wedge & & \tau_{\geq 2*} \left(\mathrm{THH}(R; (\Sigma^{2\ell} R)^{\otimes p})^{hC_p} \right)_p^\wedge & & \tau_{\geq 2*} \left(\mathrm{THH}(R; (\Sigma^{2\ell} R)^{\otimes p^2})^{hC_{p^2}} \right)_p^\wedge \end{array}$$

of nonunital filtered \mathbb{E}_∞ -rings, there is a decomposition

$$\mathrm{fil}_{\mathrm{mot}}^* \mathrm{TC}(R\langle\epsilon\rangle)_p^\wedge \cong \mathrm{fil}_{\mathrm{mot}}^* \mathrm{TC}(R)_p^\wedge \oplus \bigoplus_{p \nmid \ell} \mathrm{fil}_{\mathrm{mot}}^* \mathrm{TR}(R; \Sigma^{2\ell} R)_p^\wedge.$$

The content of the above theorem is the identification of the motivic filtration on $\mathrm{TC}^-(R\langle\epsilon\rangle)_p^\wedge$ and $\mathrm{TP}(R\langle\epsilon\rangle)_p^\wedge$ with the double-speed Postnikov filtration when R is qrsp. This is not formal: while R is qrsp, $R\langle\epsilon\rangle$ is certainly not! This allows us to relate the motivic filtration on $\mathrm{TC}(R\langle\epsilon\rangle)_p^\wedge$, which is defined in terms of $R\langle\epsilon\rangle$, to a filtration on $\mathrm{TR}(R; \Sigma^{2\ell}R)_p^\wedge$ that is defined in terms of R . Descending from qrsp rings to p -quasisyntomic rings, and then left Kan extended from p -completed polynomial rings to p -complete animated rings, we obtain the following corollary:

Corollary 5.5. *Let R denote a p -complete animated ring. Define*

$$\mathrm{fil}_{\mathrm{mot}}^* \mathrm{TR}(R; \Sigma^{2\ell}R)_p^\wedge$$

to be the limit of the diagram

$$\begin{array}{ccccc} \mathrm{fil}_{\mathrm{mot}}^* \left(\mathrm{THH}(R; (\Sigma^{2\ell}R)^{\otimes p})^{tC_p} \right)_p^\wedge & & \mathrm{fil}_{\mathrm{mot}}^* \left(\mathrm{THH}(R; (\Sigma^{2\ell}R)^{\otimes p^2})^{tC_{p^2}} \right)_p^\wedge & & \dots \\ \uparrow \varphi & \swarrow \mathrm{can} & \uparrow \varphi^{hC_p} & \swarrow \mathrm{can} & \uparrow \varphi^{hC_{p^2}} \\ \mathrm{fil}_{\mathrm{mot}}^* \mathrm{THH}(R; \Sigma^{2\ell}R)_p^\wedge & & \mathrm{fil}_{\mathrm{mot}}^* \left(\mathrm{THH}(R; (\Sigma^{2\ell}R)^{\otimes p})^{hC_p} \right)_p^\wedge & & \mathrm{fil}_{\mathrm{mot}}^* \left(\mathrm{THH}(R; (\Sigma^{2\ell}R)^{\otimes p^2})^{hC_{p^2}} \right)_p^\wedge \end{array}$$

Then there is a decomposition

$$\mathrm{fil}_{\mathrm{mot}}^* \mathrm{TC}(R\langle\epsilon\rangle)_p^\wedge \cong \mathrm{fil}_{\mathrm{mot}}^* \mathrm{TC}(R)_p^\wedge \oplus \bigoplus_{p \nmid \ell} \mathrm{fil}_{\mathrm{mot}}^* \mathrm{TR}(R; \Sigma^{2\ell}R)_p^\wedge.$$

5.1. The cyclotomic spectrum $\mathrm{THH}(R\langle\epsilon\rangle)$. In order to identify $\mathrm{THH}(R\langle\epsilon\rangle)$ as a cyclotomic spectrum, it will be helpful to note the following alternative definition of the ring $R\langle\epsilon\rangle$:

Lemma 5.6. *For any animated ring R , $R\langle\epsilon\rangle$ is equivalent to the split square-zero extension of R by the bimodule ΣR .*

Proof. The homotopy groups of $R\langle\epsilon\rangle = R \otimes_{R[x]} R$ may be calculated by a Tor spectral sequences, which immediately collapses to give the formula $\pi_* R\langle\epsilon\rangle \cong \Lambda_{\pi_* R}(\epsilon)$, where $|\epsilon| = 1$. This homotopy ring is the same as that of the split square-zero extension $R \oplus \Sigma R$, so it remains only to find a map inducing an isomorphism. To produce such a map, we simply note that $R\langle\epsilon\rangle$ is the free animated R -algebra on a degree 1 class, so there is an R -algebra map that sends ϵ to the corresponding degree 1 class in the split square-zero extension. \square

In light of Lemma 5.6, we may understand $\mathrm{THH}(R\langle\epsilon\rangle)$ in terms of the literature on THH of square-zero extensions, which is a cornerstone of the proof of the Dundas–Goodwillie–McCarthy theorem. In particular, we recall from [Ras18, Proposition 4.5.1] (see also [LM12]) that there is a natural equivalence of S^1 -equivariant spectra:

$$\mathrm{THH}(R\langle\epsilon\rangle) \simeq \mathrm{THH}(R) \oplus \bigoplus_{k \geq 1} \mathrm{THH}(R) \otimes \mathrm{Hom}((S^1/C_k)_+, (\Sigma^2 R)^{\otimes k})$$

Remark 5.7. Since R is connective, for each $k \geq 1$ the spectrum underlying

$$\mathrm{THH}(R) \otimes \mathrm{Hom}((S^1/C_k)_+, (\Sigma^2 R)^{\otimes k})$$

is $(2k - 1)$ -connective. In particular, the infinite direct sum above is also an infinite product.

By the previous remark,

$$\mathrm{THH}(R\langle\epsilon\rangle)^{tC_p} \simeq \mathrm{THH}(R)^{tC_p} \times \prod_{k \geq 1} \left(\mathrm{THH}(R) \otimes \mathrm{Hom}((S^1/C_k)_+, (\Sigma^2 R)^{\otimes k}) \right)^{tC_p}.$$

In terms of the above direct product decomposition above, the cyclotomic Frobenius is given by a direct product of the following maps [Ras18, Lemma 4.8.1]:

- (1) The cyclotomic Frobenius

$$\mathrm{THH}(R) \rightarrow \mathrm{THH}(R)^{tC_p}$$

- (2) For each integer $k \geq 1$, the Frobenius

$$\mathrm{THH}(R) \otimes \mathrm{Hom}((S^1/C_k)_+, (\Sigma^2 R)^{\otimes k}) \rightarrow (\mathrm{THH}(R) \otimes \mathrm{Hom}((S^1/C_{pk})_+, (\Sigma^2 R)^{\otimes pk}))^{tC_p},$$

obtained by setting $M = \Sigma^2 R$ in the construction of [KMN23, Theorem 6.31].

We also have the following definitions:

Definition 5.8. The $\langle p \rangle$ -polygonic TR of R with coefficients in M , denoted by $\mathrm{TR}(R; M)_p^\wedge$, is the limit of the diagram

$$\begin{array}{ccccc} (\mathrm{THH}(R; (M)^{\otimes p})^{tC_p})_p^\wedge & & (\mathrm{THH}(R; (M)^{\otimes p^2})^{tC_{p^2}})_p^\wedge & & \dots \\ \nearrow \varphi & \swarrow \mathrm{can} & \nearrow \varphi^{hC_p} & \swarrow \mathrm{can} & \nearrow \varphi^{hC_{p^2}} \\ \mathrm{THH}(R; M)_p^\wedge & & (\mathrm{THH}(R; (M)^{\otimes p})^{hC_p})_p^\wedge & & (\mathrm{THH}(R; (M)^{\otimes p^2})^{hC_{p^2}})_p^\wedge \end{array}$$

Based on this definition, it is clear that

$$\mathrm{TC}(R\langle \epsilon \rangle)_p^\wedge \simeq \mathrm{TC}(R)_p^\wedge \times \prod_{p \nmid \ell} \mathrm{TR}(R; \Sigma^{2\ell} R)_p^\wedge \simeq \mathrm{TC}(R)_p^\wedge \oplus \bigoplus_{p \nmid \ell} \mathrm{TR}(R; \Sigma^{2\ell} R)_p^\wedge.$$

We also define truncated variants of TR (which we use in Section 6) $\mathrm{TR}^{[n]}(R; M)_p^\wedge$, as the fiber of the map

$$\prod_0^n (\mathrm{THH}(R; (M)^{\otimes p^i})^{hC_{p^i}}) \xrightarrow{\mathrm{can} - \varphi^{hC_{p^i}}} \prod_1^n (\mathrm{THH}(R; (M)^{\otimes p^i})^{tC_{p^i}})$$

Similarly, we can define motivic filtrations $\mathrm{fil}_{\mathrm{mot}}^* \mathrm{TR}^{[n]}(R; M)_p^\wedge$ and associated graded $\mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}^{[n]}(R; M)_p^\wedge$ as in Corollary 5.5 by truncating the diagram.

Note that $\lim_n \mathrm{fil}_{\mathrm{mot}}^* \mathrm{TR}^{[n]}(R; M)_p^\wedge = \mathrm{fil}_{\mathrm{mot}}^* \mathrm{TR}(R; M)_p^\wedge$.

5.2. The motivic filtration. If R is a qrsp ring, then the Bhatt–Morrow–Scholze motivic filtration on $\mathrm{THH}(R)_p^\wedge$ is given by the double speed Postnikov filtration $\tau_{\geq 2*} \mathrm{THH}(R)_p^\wedge$ [BMS19]. In this section, we observe that the same is true of $\mathrm{THH}(R\langle \epsilon \rangle)_p^\wedge$. Namely,

$$\mathrm{fil}_{\mathrm{mot}}^* \mathrm{THH}(R\langle \epsilon \rangle)_p^\wedge = \tau_{\geq 2*} \mathrm{THH}(R\langle \epsilon \rangle)_p^\wedge.$$

In this sense $R\langle \epsilon \rangle$ behaves much like a discrete quasi-regular semi-perfectoid ring, despite the fact that $R\langle \epsilon \rangle$ is not discrete. The main theorem is as follows:

Theorem 5.9. *Let R be a qrsp ring. Then*

$$\mathrm{fil}_{\mathrm{mot}}^* \mathrm{THH}(R\langle \epsilon \rangle)_p^\wedge \simeq \tau_{\geq 2*} \mathrm{THH}(R\langle \epsilon \rangle)_p^\wedge,$$

$$\mathrm{fil}_{\mathrm{mot}}^* \mathrm{TC}^-(R\langle \epsilon \rangle)_p^\wedge \simeq \tau_{\geq 2*} \mathrm{TC}^-(R\langle \epsilon \rangle)_p^\wedge,$$

and

$$\mathrm{fil}_{\mathrm{mot}}^* \mathrm{TP}(R\langle \epsilon \rangle) \simeq \tau_{\geq 2*} \mathrm{TP}(R\langle \epsilon \rangle).$$

Remark 5.10. The above theorem is stated for a single qrsp ring R , but in fact the identification of motivic filtrations with double-speed Postnikov filtrations is functorial in R . Indeed, if $A \rightarrow B$ is any map of \mathbb{E}_∞ -rings there is a unique (up to contractible choice) lift of such a map to a filtered \mathbb{E}_∞ -ring map $\tau_{\geq 2*} A \rightarrow \tau_{\geq 2*} B$. Thus, the above theorem identifies the motivic filtrations on $\mathrm{THH}(R\langle \epsilon \rangle)_p^\wedge$, $\mathrm{TC}^-(R\langle \epsilon \rangle)_p^\wedge$, and $\mathrm{TP}(R\langle \epsilon \rangle)_p^\wedge$

as sheaves on the quasisyntomic site. Similarly, this implies that the motivic filtrations on the maps

$$\varphi, \text{can} : \text{TC}^-(R)_p^\wedge \rightarrow \text{TP}(R)_p^\wedge$$

are $\tau_{\geq 2*}\varphi$ and $\tau_{\geq 2*}\text{can}$, respectively.

Our proof of Theorem 5.9 begins with a description of the motivic associated graded on $\text{THH}(R\langle\epsilon\rangle)$:

Proposition 5.11. *Suppose that R is a qrsp ring. Then*

$$\pi_* \text{gr}_{\text{mot}}^* \text{THH}(R\langle\epsilon\rangle) = (\pi_* \text{gr}_{\text{mot}}^* \text{THH}(R))\langle\epsilon\rangle \otimes \Gamma(\sigma\epsilon),$$

where $\|\epsilon\| = (1, -1)$ and $\|\sigma\epsilon^{(k)}\| = (2k, 0)$.

As a consequence,

$$\pi_* \text{gr}_{\text{mot}}^* \text{THH}(R\langle\epsilon\rangle)_p^\wedge = (\pi_* \text{gr}_{\text{mot}}^* \text{THH}(R)_p^\wedge)\langle\epsilon\rangle \otimes \Gamma(\sigma\epsilon),$$

where $\|\epsilon\| = (1, -1)$, $\|\sigma\epsilon^{(k)}\| = (2k, 0)$, and all classes in $\pi_* \text{gr}_{\text{mot}}^* \text{THH}(R)_p^\wedge$ have Adams filtration 0.

Proof. As an \mathbb{E}_∞ -ring spectrum,

$$R\langle\epsilon\rangle \simeq R \otimes_{R[x]} R.$$

which may be calculated as the geometric realization of the simplicial commutative ring

$$R \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} R[x_1] \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} R[x_1, x_2] \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} \cdots$$

Applying THH , we see that $\text{THH}(R\langle\epsilon\rangle)$ may be calculated as the geometric realization of the simplicial ring spectrum

$$\text{THH}(R) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} \text{THH}(R[x_1]) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} \text{THH}(R[x_1, x_2]) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} \cdots$$

The motivic filtration on $\text{THH}(R\langle\epsilon\rangle)$ is then computed as the geometric realization of

$$\text{fil}_{\text{mot}}^* \text{THH}(R)_p \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} \text{fil}_{\text{mot}}^* \text{THH}(R[x_1]) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} \text{fil}_{\text{mot}}^* \text{THH}(R[x_1, x_2]) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} \cdots$$

In particular, we may calculate $\pi_* \text{gr}_{\text{mot}}^* \text{THH}(R)$ using the spectral sequence beginning with the cohomology of the simplicial ring

$$\pi_* \text{gr}_{\text{mot}}^* \text{THH}(R) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} \pi_* \text{gr}_{\text{mot}}^* \text{THH}(R[x_1]) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} \pi_* \text{gr}_{\text{mot}}^* \text{THH}(R[x_1, x_2]) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} \cdots$$

We now note that, by the HKR theorem,

$$\pi_* \text{gr}_{\text{mot}}^* \text{THH}(R[x_1, x_2, \dots, x_k]) \cong \pi_* \text{gr}_{\text{mot}}^* \text{THH}_*(R)[x_1, x_2, \dots, x_k]\langle\sigma x_1, \sigma x_2, \dots, \sigma x_k\rangle.$$

The above cobar complex computes

$$\text{Tor}_{\pi_* \text{gr}_{\text{mot}}^* \text{THH}(R)[x]\langle\sigma x\rangle}(\pi_* \text{gr}_{\text{mot}}^* \text{THH}(R), \pi_* \text{gr}_{\text{mot}}^* \text{THH}(R)),$$

which is $\pi_* \text{gr}_{\text{mot}}^* \text{THH}(R)\langle\epsilon\rangle \otimes \Gamma(\sigma\epsilon)$. The class ϵ is represented by x_1 , and therefore has Adams filtration -1 . The class $\sigma\epsilon$ is represented by σx_1 , and therefore has Adams filtration 0. \square

To deduce Theorem 5.9, we will apply the following lemma:

Lemma 5.12. *Let $\text{fil}^* R$ be a complete filtered spectrum, with colimit spectrum R , such that the bigraded homotopy groups $\pi_* \text{gr}^* R$ are concentrated in Adams filtrations 0 and -1 . Then there is a canonical identification $\tau_{\geq 2*} R \simeq \text{fil}^* R$.*

Proof. By assumption, for each integer k , $\text{gr}^k R$ has homotopy groups concentrated in degrees $2k$ and $2k + 1$. Since the filtered spectrum is complete, it follows that the k -th filtered piece is $2k$ -connective. The canonical map from $\text{fil}^* R$ to R with the constant filtration factors through $\tau_{\geq 2*} R$, and the resulting map is easily seen to be an equivalence. \square

Proof of Theorem 5.9. By combining Proposition 5.11 with Lemma 5.12, we deduce immediately that

$$\text{fil}_{\text{mot}}^* \text{THH}(R\langle\epsilon\rangle)_p^\wedge \simeq \tau_{\geq 2*} \text{THH}(R\langle\epsilon\rangle)_p^\wedge.$$

There is an algebraic homotopy fixed point spectral sequence

$$\pi_*(\text{gr}_{\text{mot}}^* \text{THH}(R\langle\epsilon\rangle)_p^\wedge[t]) \implies \pi_* \text{gr}_{\text{mot}}^* \text{TC}^-(R)_p^\wedge,$$

$\|t\| = (-2, 0)$, from which one learns that the bigraded homotopy groups $\pi_* \text{gr}_{\text{mot}}^* \text{TC}^-(R)_p^\wedge$ are concentrated in Adams filtration 0 and -1 . Similarly, the algebraic Tate spectral sequence

$$\pi_*(\text{gr}_{\text{mot}}^* \text{THH}(R\langle\epsilon\rangle)_p^\wedge[t^{\pm 1}]) \implies \pi_* \text{gr}_{\text{mot}}^* \text{TP}(R)_p^\wedge$$

implies that $\pi_* \text{gr}_{\text{mot}}^* \text{TP}(R)_p^\wedge$ is concentrated in Adams filtrations 0 and -1 . Applying Lemma 5.12, we conclude the result. \square

6. THE MOD p SYNTOMIC COHOMOLOGY OF $\mathbb{Z}_p\langle\epsilon\rangle$

In this section, we compute the mod p syntomic cohomology of $\mathbb{Z}_p\langle\epsilon\rangle$. Our strategy will be to use the associated graded of Corollary 5.5 to reduce this to the computation of the homotopy groups of

$$\text{gr}_{\text{mot}}^* \left(\text{THH}(\mathbb{Z}_p; (\Sigma^{2\ell} \mathbb{Z}_p)^{\otimes_{\mathbb{Z}_p} p^n})^{hC_{p^n}} \right) / p$$

for $\ell \nmid p$ and $n \geq 0$, and the Frobenius and canonical maps between them. Before we start, we note that by tracing through the definitions, there is a natural equivalence of C_{p^n} -spectra:

$$\text{THH}(\mathbb{Z}_p; (\Sigma^{2\ell} \mathbb{Z}_p)^{\otimes_{\mathbb{Z}_p} p^n}) \simeq \text{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_n},$$

where ρ_n is the complex regular representation of C_{p^n} .

In Section 6.1, we use an algebraic version of homotopy fixed points spectral sequence to study the groups

$$\pi_* \text{gr}_{\text{mot}}^* \left(\text{THH}(\mathbb{Z}_p; (\Sigma^{2\ell} \mathbb{Z}_p)^{\otimes_{\mathbb{Z}_p} p^n})^{hC_{p^n}} \right) / p.$$

Then, in Section 6.2, we compute the Frobenius and canonical maps up to v_1 -adic filtration, where they admit simple formulas.

Finally, in Section 6.3, we use these two ingredients together with Corollary 5.5 to compute the mod p syntomic cohomology of $\mathbb{Z}_p\langle\epsilon\rangle$. Along the way, we will see that the up-to- v_1 -adic-filtration computations of Section 6.2 are sufficient to compute the full answer.

Notation 6.1. As notation, we will use a dot as in $\dot{+}, \dot{=}, \dot{\mapsto}$ to mean “up to a unit in \mathbb{F}_p^\times .” For example, $a \dot{=} b$ means $a = cb$ for $c \in \mathbb{F}_p^\times$.

6.1. Twisted spectral sequences. As above, we let ρ_n denote the complex regular representation of C_{p^n} . In this section, we compute

$$\mathrm{gr}_{\mathrm{mot}}^* \left((\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_n})^{hC_{p^n}} \right) / p.$$

using the spectral sequence coming from the Nygaard filtration.

To do this, let us first rewrite these groups as the cohomology of a quasicohherent sheaf on $\mathbb{Z}_p^{\mathrm{Nyg}}$.

Proposition 6.2. *There are natural equivalences*

$$\mathrm{gr}_{\mathrm{mot}}^i(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_n})^{hC_{p^n}} \simeq R\Gamma(\mathbb{Z}_p^{\mathrm{Nyg}}; (\mathcal{O}/a^*\mathcal{O}(1) \otimes \mathcal{I}_n) \{i\} \otimes a^*\mathcal{O}(\ell p^n) \otimes \bigotimes_{i=0}^{p^n-1} \mathcal{I}_{v_p(i)}^{-\ell})[2i].$$

Here, we let $\mathcal{I} \rightarrow \mathcal{O}$ denote the Hodge–Tate divisor on \mathbb{Z}_p^Δ , let $a^*\mathcal{O}(1) \rightarrow \mathcal{O}$ denote the Nygaard divisor on $\mathbb{Z}_p^{\mathrm{Nyg}}$, and let $\mathcal{I}_n \rightarrow \mathcal{O}$ on $\mathbb{Z}_p^{\mathrm{Nyg}}$ denote $(F')^*(\mathcal{I} \otimes \cdots \otimes (F^{n-1})^*\mathcal{I}) \rightarrow \mathcal{O}$.

Proof. To begin, we recall that we have

$$\mathrm{gr}_{\mathrm{mot}}^i \mathrm{THH}(\mathbb{Z}_p)^{hS^1} \simeq R\Gamma(\mathbb{Z}_p^{\mathrm{Nyg}}; \mathcal{O}\{i\})[2i].$$

A priori, the left hand side is the Nygaard completion of the right hand side, but the absolute prismatic cohomology of \mathbb{Z}_p is already Nygaard-complete.

Recall that the divisors $(F')^*\mathcal{I} \rightarrow \mathcal{O}$ and $a^*\mathcal{O}(1) \rightarrow \mathcal{O}$ on $\mathbb{Z}_p^{\mathrm{Nyg}}$, correspond on qrsps to the ideals generated by the prismatic element d and t , respectively. It therefore follows from [Rig22, Lemma 3.1] that there is an isomorphism:

$$\mathrm{gr}_{\mathrm{mot}}^i \mathrm{THH}(\mathbb{Z}_p)^{hC_{p^n}} \simeq R\Gamma(\mathbb{Z}_p^{\mathrm{Nyg}}; (\mathcal{O}/a^*\mathcal{O}(1) \otimes (F')^*\mathcal{I} \otimes \cdots \otimes (F')^*(F^{n-1})^*\mathcal{I}) \{i\})[2i],$$

i.e. an isomorphism

$$\mathrm{gr}_{\mathrm{mot}}^i \mathrm{THH}(\mathbb{Z}_p)^{hC_{p^n}} \simeq R\Gamma(\mathbb{Z}_p^{\mathrm{Nyg}}; (\mathcal{O}/a^*\mathcal{O}(1) \otimes \mathcal{I}_n) \{i\})[2i].$$

To deal with the twisting, we note that if we let λ denote the standard complex representation of S^1 , then there are fiber sequences

$$\mathbb{S}^{-\lambda^k} \rightarrow \mathbb{S}^0 \rightarrow \mathbb{D}(\Sigma_+^\infty S^1/C_k),$$

which give rise to fiber sequences

$$(\mathrm{THH}(R) \otimes \mathbb{S}^{-\lambda^k})^{hS^1} \rightarrow \mathrm{THH}(R)^{hS^1} \rightarrow \mathrm{THH}(R)^{hC_k} \simeq \mathrm{THH}(R)^{hC_{p^{v_p(k)}}}.$$

It follows that we have

$$\mathrm{gr}_{\mathrm{mot}}^i(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{-\lambda^k})^{hS^1} \simeq R\Gamma(\mathbb{Z}_p^{\mathrm{Nyg}}; \mathcal{O}\{i\} \otimes a^*\mathcal{O}(1) \otimes I_{v_p(k)})[2i],$$

from which we may deduce that (cf. [Rig22, Lemma 4.3])

$$\mathrm{gr}_{\mathrm{mot}}^i(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{-\lambda^k})^{hC_{p^n}} \simeq R\Gamma(\mathbb{Z}_p^{\mathrm{Nyg}}; (\mathcal{O}/a^*\mathcal{O}(1) \otimes \mathcal{I}_n) \{i\} \otimes a^*\mathcal{O}(1) \otimes I_{v_p(k)})[2i].$$

Finally, writing ρ_n as the restriction of $\bigoplus_{i=0}^{p^n-1} \lambda^i$ from S^1 to C_{p^n} , we conclude that

$$\mathrm{gr}_{\mathrm{mot}}^i(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_n})^{hC_{p^n}} \simeq R\Gamma(\mathbb{Z}_p^{\mathrm{Nyg}}; (\mathcal{O}/a^*\mathcal{O}(1) \otimes \mathcal{I}_n) \{i\} \otimes a^*\mathcal{O}(\ell p^n) \otimes \bigotimes_{i=0}^{p^n-1} \mathcal{I}_{v_p(i)}^{-\ell})[2i].$$

□

Now let us discuss spectral sequences. Given a complex of quasicohherent sheaves \mathcal{F} on $\mathbb{Z}_p^{\mathrm{Nyg}}$, there are spectral sequences associated to the Nygaard filtration of the form:

$$\pi_* R\Gamma((\mathbb{Z}_p^{\mathrm{Nyg}})_{a=0}; \mathcal{F}_{a=0} \otimes (\bigoplus_{i=0}^{\infty} \mathcal{O}(-i))) \Rightarrow \pi_* R\Gamma(\mathbb{Z}_p^{\mathrm{Nyg}}; \mathcal{F})_a^\wedge$$

and

$$\pi_* R\Gamma((\mathbb{Z}_p^{\text{Nyg}})_{a=0}; \mathcal{F}_{a=0} \otimes (\bigoplus_{i=-\infty}^{\infty} \mathcal{O}(-i))) \Rightarrow \pi_* R\Gamma(\mathbb{Z}_p^{\text{Nyg}}; \mathcal{F})_a^\wedge[a^{-1}].$$

(Indeed, to see this we push forward to A^1/\mathbb{G}_m and use the correspondence between complexes on this stack and filtered complexes.) These are algebraic versions of the S^1 -homotopy fixed points and S^1 -Tate spectral sequences on the level of THH. It follows from the definitions that the former is a truncation of the latter.

Next, we reduce modulo p and compute these spectral sequences in the cases relevant to $\text{gr}_{\text{mot}}^i(\text{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_n})^{hC_{p^n}}$.

We start with the untwisted case, when $\ell = 0$. By [BM23, Construction 2.4], we have the following starting point.

Proposition 6.3. *There is an isomorphism*

$$H^*((\mathbb{Z}_p^{\text{Nyg}})_{p=a=0}; \mathcal{O}\{*\}_{p=a=0}) \cong \mathbb{F}_p[\mu] \otimes \Lambda(\lambda_1),$$

where $|\mu| = (2p, 0)$ and $|\lambda_1| = (2p - 1, 1)$.

Lemma 6.4. *Let \mathcal{L} denote a line bundle over \mathbb{Z}_p^Δ . Then the pullback of \mathcal{L} along the composite*

$$(\mathbb{Z}_p^{\text{Nyg}})_{a=0} \hookrightarrow \mathbb{Z}_p^{\text{Nyg}} \xrightarrow{F'} \mathbb{Z}_p^\Delta$$

is trivial.

Proof. By the proof of [Bha22, Proposition 5.3.7], we have a commutative square

$$\begin{array}{ccc} (\mathbb{Z}_p^{\text{Nyg}})_{a=0} & \longrightarrow & \mathbb{Z}_p^{\text{Nyg}} \\ \downarrow & & \downarrow F' \\ \text{Spf}\mathbb{Z}_p & \xrightarrow{dR} & \mathbb{Z}_p^\Delta, \end{array}$$

where the bottom horizontal map corresponds to the Cartier–Witt divisor $p : W(\mathbb{Z}_p) \rightarrow W(\mathbb{Z}_p)$. The lemma therefore follows from the fact that any line bundle over $\text{Spf}\mathbb{Z}_p$ is trivializable. \square

Proposition 6.5. *The Nygaard spectral sequence for cofiber of $\mathcal{F} = (\mathcal{O}/a^*\mathcal{O}(1) \otimes \mathcal{I}_n)\{*\}$ by p takes the form*

$$\mathbb{F}_p[\mu, t^{\pm 1}] \otimes E(\lambda_1, u_n) \Rightarrow \pi_* \text{gr}_{\text{mot}} \text{THH}(\mathbb{Z}_p)^{tC_{p^n}}/p,$$

with differentials generated under the Leibniz rule by:

$$\begin{aligned} d_{2(p+\dots+p^{k+1})}(t^{p^k}) &\doteq v_1^{p+\dots+p^k} t^{p^{k+1}+p^k} \lambda_1 \\ d_{2(1+p+\dots+p^n)}(u_n) &\doteq v_1^{1+\dots+p^{n-1}} t^{p^n}. \end{aligned}$$

and the fact that $v_1 = t\mu$ and t^{p^n} are permanent cycles. In particular, the spectral sequence collapses at the $E_{2(1+\dots+p^n)+1}$ -page.

Proof. To start, we note that by computation on the level of qrsp's, the mod p reduction of the map $a^*\mathcal{O}(1) \otimes \mathcal{I}_n \rightarrow \mathcal{O}$ is detected in Nygaard filtration $1 + p + \dots + p^n$ by a unit multiple of the element $\mu^{1+p+\dots+p^{n-1}}$. Indeed, mod p we have $\mathcal{I}_n \cong (F')^*\mathcal{I}^{1+\dots+p^{n-1}}$ as divisors, so it suffices to note that μ detects the prismatic element $d \bmod p$ in Nygaard filtration p .

In particular, this divisor produces the zero map upon pullback to $(\mathbb{Z}_p^{\text{Nyg}})_{p=a=0}$, so that on this locus we have

$$\mathcal{O}/a^*\mathcal{O}(1) \otimes \mathcal{I}_n \cong \mathcal{O} \oplus (a^*\mathcal{O}(1) \otimes \mathcal{I}_n)[1]$$

Next, we note that the pullback of $(F')^*\mathcal{I}$ to $(\mathbb{Z}_p^{\text{Nyg}})_{a=0}$ is trivial by Lemma 6.4, so that on $(\mathbb{Z}_p^{\text{Nyg}})_{p=a=0}$ we further have

$$\mathcal{O}/a^*\mathcal{O}(1) \otimes \mathcal{I}_n \cong \mathcal{O} \oplus a^*\mathcal{O}(1)[1].$$

The computation of the E_2 -page now follows from Proposition 6.3, where u_n corresponds to the $a^*\mathcal{O}(1)[1]$ factor. The differential on u_n therefore follows from our discussion above of the attaching map corresponding to this factor.

The differentials on t are mapped forward from the spectral sequence for the mod p reduction of $\mathcal{F} = \mathcal{O}\{*\}$, which was done in [LW22, Propositions 6.32 & 6.34]. \square

Next we study how the twisting by a representation sphere affects this spectral sequence. We start by noting that on $(\mathbb{Z}_p^{\text{Nyg}})_{p=0}$, there is an isomorphism $(F')^*\mathcal{I} \simeq \mathcal{O}\{1-p\} \otimes a^*\mathcal{O}(1-p)$. Furthermore using the fact that $F^*\mathcal{L} \cong \mathcal{L}^p \bmod p$, we find that $\mathcal{I}_n \cong (F')^*\mathcal{I}^{1+\dots+p^{n-1}} \cong \mathcal{O}\{1-p^n\} \otimes a^*\mathcal{O}(1-p^n)$. Finally, we conclude that

$$\begin{aligned} a^*\mathcal{O}(\ell p^n) \otimes \bigotimes_{i=0}^{p^n-1} \mathcal{I}_{v_p(i)}^{-\ell} \\ \cong \mathcal{O}\{\ell \sum_{i=0}^{p^n-1} (p^{v_p(i)} - 1)\} \otimes a^*\mathcal{O}(\ell \sum_{i=0}^{p^n-1} p^{v_p(i)}) \\ \cong \mathcal{O}\{\ell n(p^n - p^{n-1}) - \ell p^n\} \otimes a^*\mathcal{O}(\ell n(p^n - p^{n-1})), \end{aligned}$$

since $\sum_{i=0}^{p^n-1} p^{v_p(i)} = n(p^n - p^{n-1})$.

From this we conclude that the twisted spectral sequence is as desired.

Theorem 6.6. *The Nygaard spectral sequence for $\text{gr}_{\text{mot}}^*(\text{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_n})^{hC_{p^n}}$ is of the form:*

$$\sigma\epsilon^{(\ell p^n)}(\mathbb{F}_p[v_1, t] \otimes E(\lambda_1, u_n)) \Rightarrow \pi_* \text{gr}_{\text{mot}} \text{THH}(\mathbb{Z}_p)^{tC_{p^n}}/p,$$

where $|\sigma\epsilon^{(\ell p^n)}| = (2\ell p^n, 0, 0)$. The differentials are determined by the following two facts:

- It is a module over the untwisted Nygaard spectral sequence, which was computed in Proposition 6.5.
- The differentials on $\sigma\epsilon^{(\ell p^n)}$ are determined by the differentials on $t^{-\ell n(p^n - p^{n-1})}$ in the spectral sequence of Proposition 6.5 via the formula

$$d_m(\sigma\epsilon^{(\ell p^n)}) = \frac{d_m(t^{-\ell n(p^n - p^{n-1})})}{t^{-\ell n(p^n - p^{n-1})}} \sigma\epsilon^{(\ell p^n)}.$$

Proof. This follows from combining the following two facts:

- Twisting a sheaf by $\mathcal{O}\{k\}$ simply shifts the Nygaard spectral sequence by $(-2k, 0, 0)$.
- Twisting a sheaf by $a^*\mathcal{O}(k)$ changes where where in the Nygaard filtration we truncate: we truncate in filtrations $\geq -k$ instead of ≥ 0 . This has the effect of taking the truncation of the t -inverted spectral sequence spanned by t^i where $i \geq -k$. This is isomorphic to the spectral sequence obtained by shifting the spectral sequence by $(2k, 0, 0)$ and twisting the differentials by the differential of t^{-k} .

\square

Now, we use Theorem 6.6 to explicitly compute the Nygaard spectral sequence for $\pi_* \text{gr}_{\text{mot}}^*(\text{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_n})^{hC_{p^n}}/p$. Our results are summarized in the following two propositions.

Notation 6.7. Given two nonnegative integers a and b , we let $a\%b$ denote the unique integer between 0 and $b-1$ which is equivalent to a modulo b .

Proposition 6.8. *Let $n, \ell \geq 0$. The $E_{2(p+\dots+p^n)+1}$ -page of the Nygaard spectral sequence*

$$\sigma\epsilon^{(\ell p^n)} \mathbb{F}_p[v_1, t] \otimes E(\lambda_1, u_n) \Rightarrow \pi_* \mathrm{gr}_{\mathrm{mot}}^*(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell p^n})^{hC_{p^n}}/p$$

is given as a $\mathbb{F}_p[v_1]$ -module as follows:

$$\begin{aligned} & \sigma\epsilon^{(\ell p^n)} E(u_n) \otimes ((t^{p^{n-1}(p-\ell\%p)}, u^{p^{n-1}(\ell\%p)}) \mathbb{F}_p[t^{p^n}, \mu^{p^n}, v_1]/(v_1^{p^n} = t^{p^n} \mu^{p^n}) \otimes E(\lambda_1)) \\ & \oplus \bigoplus_{k=1}^{n-2} \bigoplus_{i > p^{k+1} | v_p(i)=k} \mathbb{F}_p[v_1]/(v_1^{p^+\dots+p^k}) \{t^i \lambda_1\} \\ & \oplus \bigoplus_{k=0}^{n-2} \bigoplus_{i=1}^{p-1} \mathbb{F}_p[v_1]/(v_1^{p^+\dots+p^k+p^k(p-i)}) \{t^{p^k i} \lambda_1\} \oplus \bigoplus_{j>0 | v_p(j)=k} \mathbb{F}_p[v_1]/(v_1^{p^+\dots+p^{k+1}}) \{\mu^j \lambda_1\} \\ & \oplus \mathbb{F}_p[v_1]/(v_1^{p^+\dots+p^{n-1}}) \{t^i \lambda_1 | i \geq p^n, v_p(i + \ell n p^{n-1}) = n-1\} \\ & \oplus \mathbb{F}_p[v_1]/(v_1^{p^+\dots+p^{n-1}+p^{n-1}(p-i)}) \{t^{p^{n-1} i} \lambda_1 | 1 \leq i < p, v_p(p^{n-1} i + n \ell p^{n-1}) = n-1\} \\ & \oplus \mathbb{F}_p[v_1]/(v_1^{p^+\dots+p^n}) \{\mu^j \lambda_1 | j \geq 0, v_p(j - \ell n p^{n-1}) = n-1\} \end{aligned}$$

Here, $(t^{p^{n-1}(p-\ell\%p)}, u^{p^{n-1}(\ell\%p)})$ refers to the sub- $\mathbb{F}_p[t^{p^n}, \mu^{p^n}, v_1]/(v_1^{p^n} = t^{p^n} \mu^{p^n})$ -module of $\mathbb{F}_p[t, \mu]$ generated by $t^{p^{n-1}(p-\ell\%p)}$ and $u^{p^{n-1}(\ell\%p)}$.

Proof. The differentials of this spectral sequence are determined by Theorem 6.6.

Note that we may ignore the element u_n since it doesn't play a role in the differentials of the spectral sequence up to this page, so we omit it from the discussion. Next, up to the $E_{2(p+\dots+p^{n+1})}$ -page, the spectral sequence agrees with that converging to $\pi_* \mathrm{gr}_{\mathrm{mot}}^*(\mathrm{THH}(\mathbb{Z}_p))^{hC_{p^{n+1}}}/p$, so by induction on n , we may assume that the $E_{2(p+\dots+p^{n+1})}$ -page is given by $\sigma\epsilon^{(\ell p^n)} \mathbb{F}_p[t^{p^{n-1}}, \mu^{p^{n-1}}, v_1]/(v_1^{p^{n-1}} = t^{p^{n-1}} \mu^{p^{n-1}}) \otimes E(\lambda_1)$ plus the second and third lines of the answer stated in the proposition. This latter part plays no role in this page of the spectral sequence since it is t -torsion.

So what remains is to understand homology of the $d_{2(p+\dots+p^n)}$ -differential on $\sigma\epsilon^{(\ell p^n)} \mathbb{F}_p[t^{p^{n-1}}, \mu^{p^{n-1}}, v_1]/(v_1^{p^{n-1}} = t^{p^{n-1}} \mu^{p^{n-1}}) \otimes E(\lambda_1)$. This differential goes from non-multiples of λ_1 to multiples of λ_1 . The kernel is exactly the submodule $\sigma\epsilon^{(\ell p^n)} (t^{p^{n-1}(p-\ell\%p)}, u^{p^{n-1}(\ell\%p)}) \mathbb{F}_p[t^{p^n}, \mu^{p^n}, v_1]/(v_1^{p^n} = t^{p^n} \mu^{p^n})$, and the image is given by $t^{p^n} v_1^{p^+\dots+p^{n-1}} \lambda_1$ times the complementary summand in $\sigma\epsilon^{(\ell p^n)} \mathbb{F}_p[t^{p^{n-1}}, \mu^{p^{n-1}}, v_1]/(v_1^{p^{n-1}} = t^{p^{n-1}} \mu^{p^{n-1}}) = t^{p^{n-1}} \mu^{p^{n-1}}$. Note that $\sigma\epsilon^{(\ell p^n)} \lambda_1 (t^{p^{n-1}(p-\ell\%p)}, u^{p^{n-1}(\ell\%p)}) \mathbb{F}_p[t^{p^n}, \mu^{p^n}, v_1]/(v_1^{p^n} = t^{p^n} \mu^{p^n})$ is not in the image, so together we find that $(t^{p^{n-1}(p-\ell\%p)}, u^{p^{n-1}(\ell\%p)}) \mathbb{F}_p[t^{p^n}, \mu^{p^n}, v_1]/(v_1^{p^n} = t^{p^n} \mu^{p^n}) \otimes E(\lambda_1)$ lies in the homology.

The λ_1 -multiples in the complementary summand are isomorphic to the the free $\mathbb{F}_p[v_1]$ -module generated by terms of the form $\lambda_1 t^{p^{n-1} i}$ for $p \nmid i + \ell, i > 0$, and $\lambda_1 \mu^{p^{n-1} i}$ for $p \nmid i - k, i \geq 0$. When $i \geq p$, the terms of the form $\lambda_1 t^{p^{n-1} i}$ are $v_1^{p^+\dots+p^{n-1}}$ -torsion because of differentials from $t^{p^{n-1}(i-p)}$. It follows that we get a summand $\mathbb{F}_p[v_1]/(v_1^{p^+\dots+p^{n-1}}) \{t^i \lambda_1 | i \geq p^n, v_p(i + \ell n p^{n-1}) = n-1\}$ on the $E_{2(p+\dots+p^{n+1})+1}$ -page from these terms.

The terms $\lambda_1 t^{p^{n-1} i}$ for $p \nmid i + \ell, 0 < i < p$ are similarly $v_1^{p^+\dots+p^{n-1}+p^{n-1}(p-i)}$ -torsion due to differentials from $\mu^{p^{n-1}(p-i)}$, so we obtain a term as below on the $E_{2(p+\dots+p^{n+1})+1}$ -page from this.

$$\oplus \mathbb{F}_p[v_1]/(v_1^{p^+\dots+p^{n-1}+p^{n-1}(p-i)}) \{t^{p^{n-1} i} \lambda_1 | 1 \leq i < p, v_p(p^{n-1} i + n \ell p^{n-1}) = n-1\}$$

Finally, the terms of the form $\mu^{p^{n-1}i}\lambda_1$ for $p \nmid i - \ell$ are $v_1^{1+\dots+p^n}$ -torsion, because of differentials from $\mu^{p^{n-1}(i+p)}$. This contributes the last summand to the $E_{2(p+\dots+p^{n+1})+1}$ -page below, since we have accounted for all elements of the spectral sequence.

$$\oplus \mathbb{F}_p[v_1]/(v_1^{1+\dots+p^n})\{\mu^j \lambda_1 | j \geq 0, v_p(j - \ell n p^{n-1}) = n - 1\}$$

□

Finally, we run the last nontrivial differential of the Nygaard spectral sequence to compute $\pi_* \mathfrak{gr}_{\text{mot}}^*(\text{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell p^n})^{hC_{p^n}}/p$:

Proposition 6.9. *For $n, \ell \geq 0$, the E_∞ -page of the Nygaard spectral sequence for $\pi_* \mathfrak{gr}_{\text{mot}}^*(\text{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell p^n})^{hC_{p^n}}/p$ is given by:*

$$\begin{aligned} & \sigma \epsilon^{(\ell p^n)} E(\lambda_1) \otimes (\mathbb{F}_p[v_1]/(v_1^{1+\dots+p^{n-1}}))\{t^i | i \geq p^n, i \equiv -n\ell p^{n-1} \pmod{p^n}\} \\ & \oplus \mathbb{F}_p[v_1]/(v_1^{1+\dots+p^{n-1}+(p^n-i)})\{t^i | 0 < i < p^n, i \equiv -n\ell p^{n-1} \pmod{p^n}\} \\ & \oplus \mathbb{F}_p[v_1]/(v_1^{1+\dots+p^n})\{\mu^j | j \geq 0, j \equiv n\ell p^{n-1} \pmod{p^n}\} \\ & \oplus E(u_n) \otimes \left(\bigoplus_{k=1}^{n-2} \bigoplus_{i > p^{k+1} | v_p(i)=k} \mathbb{F}_p[v_1]/(v_1^{1+\dots+p^k})\{t^i \lambda_1\} \right. \\ & \left. \oplus \bigoplus_{k=0}^{n-2} \bigoplus_{i=1}^{p-1} \mathbb{F}_p[v_1]/(v_1^{1+\dots+p^k+p^k(p-i)})\{t^{p^k i} \lambda_1\} \oplus \bigoplus_{j > 0 | v_p(j)=k} \mathbb{F}_p[v_1]/(v_1^{1+\dots+p^{k+1}})\{\mu^j \lambda_1\} \right) \\ & \oplus E(u_n) \otimes (\mathbb{F}_p[v_1]/(v_1^{1+\dots+p^{n-1}}))\{t^i \lambda_1 | i \geq p^n, v_p(i + \ell n p^{n-1}) = n - 1\} \\ & \oplus \mathbb{F}_p[v_1]/(v_1^{1+\dots+p^{n-1}+(p^n-i)})\{t^{p^{n-1}i} \lambda_1 | 1 \leq i < p, v_p(p^{n-1}i + n\ell p^{n-1}) = n - 1\} \\ & \oplus \mathbb{F}_p[v_1]/(v_1^{1+\dots+p^n})\{\mu^j \lambda_1 | j \geq 0, v_p(j - \ell n p^{n-1}) = n - 1\} \end{aligned}$$

Furthermore there are no extension problems when computing $\pi_*(\text{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell p^n})^{hC_{p^n}}/p$ as an $\mathbb{F}_p[v_1]$ -module.

Proof. By Theorem 6.6, to understand the E_∞ -page of the spectral sequence, it suffices to compute the cohomology of the $d_{2(1+\dots+p^n)}$ -differential on the $E_{2(1+\dots+p^n)}$ -page, which is described by Proposition 6.8. By Theorem 6.6, this differential is determined by the following formulae: $d_{2(1+\dots+p^n)}(\sigma \epsilon^{(\ell p^n)} u_n) = v_1^{1+\dots+p^{n-1}} t^{p^n}$ and the fact that it is zero on any term not containing u_n . Since $v_1^{1+\dots+p^{n-1}} t^{p^n}$ acts by 0 on all but the first summand of the result of Proposition 6.8, this differential is 0 on all of these summands, which therefore survive to contribute all but the first three claimed summands.

On the remaining summand

$$(t^{p^{n-1}(p-\ell\%p)}, u^{p^{n-1}(\ell\%p)}) \mathbb{F}_p[t^{p^n}, \mu^{p^n}, v_1]/(v_1^{p^n} = t^{p^n} \mu^{p^n}) \otimes E(\lambda_1)$$

, multiplication by $v_1^{1+\dots+p^{n-1}} t^{p^n}$ is injective, so we get a summand

$$(t^{p^{n-1}(p-\ell\%p)}, u^{p^{n-1}(\ell\%p)}) \mathbb{F}_p[t^{p^n}, \mu^{p^n}, v_1]/(v_1^{p^n} = t^{p^n} \mu^{p^n}, v_1^{1+\dots+p^{n-1}} t^{p^n}) \otimes E(\lambda_1).$$

We can separate this into $\mathbb{F}_p[v_1]$ -module summands as the submodule generated by t^i for $i \geq p^n$, the submodule generated by t^i for $0 < i < p^n$, and the submodule generated by μ^j for $j \geq 0$. A generator of the form t^i for $i \geq p^n$ is $v_1^{1+\dots+p^{n-1}}$ -torsion because of the fact that $(v_1^{1+\dots+p^{n-1}} t^{p^n}) t^{p^n-i} = 0$, giving the first claimed summand of the proposition. A generator of the form t^i for $0 < i < p^n, i \equiv -n\ell p^{n-1} \pmod{p^n}$ is $v_1^{1+\dots+p^{n-1}+(p^n-i)}$ -torsion since $v_1^{1+\dots+p^{n-1}+(p^n-i)} t^i = (v_1^{1+\dots+p^{n-1}} t^{p^n}) \mu^{p^n-i}$, giving the second claimed summand. A generator of the form μ^i is $v_1^{1+\dots+p^n}$ -torsion since $\mu^i v_1^{1+\dots+p^n} = \mu^i (v_1^{1+\dots+p^{n-1}} t^{p^n}) \mu^{p^n}$, giving the third summand.

Next, we show that there are no v_1 -extensions. In motivic weights ± 1 , every class is of the form $t^i \mu^j \lambda_1$ or $t^i \mu^j u_n$ respectively, and we will argue identically in either case, so we focus on the former. In particular, there is at most one class detected in each topological degree and filtration. If there is a potential $v_1 = t\mu$ extension problem, it is because for a class x detected by some $(t^i \mu^j \lambda_1)$, some power of $(t\mu)^k (t^i \mu^j \lambda_1)$ is zero on the E_∞ -page, but $v_1^k x$ is detected in higher filtration by $(t^p \mu)^l (t\mu)^k (t^i \mu^j \lambda_1)$ for some l . But we can see that on the E_2 -page this is divisible by at least as many $t\mu$ s as $t^i \mu^j \lambda_1$. It follows that by changing x by a class detected in the associated graded by $(t^p \mu)^l (t^i \mu^j \lambda_1)$, $v_1^k x$ will be detected in higher filtration. Continuing this way, and using that the filtration is complete, there are no extension problems in these weights.

A similar argument shows that in motivic weight 0, where the classes are of the form $t^i \mu^j$ or $t^i \mu^j \lambda_1 u_n$, there cannot be extension problems within each type of class, and from the latter to the former.

Let's rule out the case of extensions from the former to the latter. Suppose there exists a class x detected by $t^i \mu^j$, such that $(t\mu)^k (t^i \mu^j)$ is zero, yet $v_1^k x$ is nonzero and detected by a class of the form $(t^p \mu)^l (t\mu)^{k-1} t^i \mu^j \lambda_1 u_n$. Note that $l > 0$, since an extension is otherwise impossible for filtration reasons. Then, by changing x by an element detected by $(t^p \mu)^{l-1} t^{p-1} t^i \mu^j \lambda_1 u_n$ to get an x' , $v_1^k x$ is detected in higher filtration. Note that $(t^p \mu)^{l-1} t^{p-1} t^i \mu^j \lambda_1 u_n$ must be a permanent cycle, because by inspection we see that if $v_1 y$ is a permanent cycle in this spectral sequence, so is y . \square

Proposition 6.10. *Let $n, \ell \geq 0$. The E_∞ -page of the Nygaard spectral sequence for*

$$\pi_* \mathrm{gr}_{\mathrm{mot}}^*(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell \rho_n})^{tC_{p^n}}/p$$

is given by:

$$\begin{aligned} & \sigma \epsilon^{(\ell p^n)} \mathbb{F}_p[v_1, t^{\pm p^n}] / (v_1^{1+\dots+p^{n-1}}) \{t^{-n\ell p^{n-1}}\} \otimes E(\lambda_1) \\ & \oplus \left(E(u_n) \otimes \left(\bigoplus_{k=1}^{n-2} \mathbb{F}_p[v_1] / (v_1^{p^1+\dots+p^k}) \{t^i \lambda_1 | v_p(i) = k\} \right) \right) \\ & \oplus E(u_n) \otimes \mathbb{F}_p[v_1] / (v_1^{p^1+\dots+p^{n-1}}) \{t^i \lambda_1 | v_p(i + \ell n p^{n-1}) = n - 1\}. \end{aligned}$$

There are no extension problems when computing $\pi_* \mathrm{gr}_{\mathrm{mot}}^*(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell \rho_n})^{tC_{p^n}}/p$ as an $\mathbb{F}_p[v_1]$ -module.

Proof. The E_∞ -page of the spectral sequence can be read off from inverting t in Proposition 6.9. To see that no extension problems happen for a particular class, it suffices to prove this after multiplying by a sufficiently large power of t , for which it follows again by Proposition 6.9, since the two spectral sequences agree in high enough filtrations. \square

««« HEAD Now we want to get a grip on the Frobenius maps. To understand these, we begin by noting that

$$\varphi : R\Gamma((\mathbb{Z}_p \langle \epsilon \rangle)^{\mathrm{Nyg}})_{p=0}, \mathcal{O}\{*\} \rightarrow R\Gamma((\mathbb{Z}_p \langle \epsilon \rangle^\Delta)_{p=0}, \mathcal{O}\{*\})$$

is equivalent to the map inverting μ on the source because j_{HT} , which induces φ , is the inclusion of the nonvanishing locus of μ .

It therefore follows that

$$\varphi : \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}^-(\mathbb{Z}_p \langle \epsilon \rangle) / p \rightarrow \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TP}^-(\mathbb{Z}_p \langle \epsilon \rangle) / p$$

may be identified with the map that inverts μ and Nygaard completes. The same is true of

$$\varphi : \mathrm{gr}_{\mathrm{mot}}^*(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{k \rho_{n-1}})^{hC_{p^{n-1}}} / p \rightarrow \mathrm{gr}_{\mathrm{mot}}^*(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{k \rho_n})^{tC_{p^n}} / p$$

as it is a graded summand of the former map. Since the corresponding prismatic complexes are already Nygaard complete, this last Frobenius map may be identified with μ -inversion.

As a consequence, we can also compute $\pi_* \mathrm{gr}_{\mathrm{mot}}^*(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_n})^{tC_{p^n}}/p$ using the μ -inverted Nygaard spectral sequence. The result is as follows:

Proposition 6.11. *Let $n, \ell \geq 0$. The E_∞ -page of the μ -inverted Nygaard spectral sequence for $\pi_* \mathrm{gr}_{\mathrm{mot}}^*(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_n})^{tC_{p^n}}/p$ is given by:*

$$\begin{aligned} & \sigma\epsilon^{(\ell p^{n-1})} \mathbb{F}_p[v_1, \mu^{\pm p^{n-1}}]/(v_1^{1+\dots+p^{n-1}})\{\mu^{(n-1)\ell p^{n-2}}\} \otimes E(\lambda_1) \\ & \oplus \left(E(u_{n-1}) \otimes \left(\bigoplus_{k=0}^{n-3} \mathbb{F}_p[v_1]/(v_1^{p+\dots+p^k+p^{k+1}})\{\mu^j \lambda_1 | v_p(j) = k\} \right) \right) \\ & \oplus E(u_{n-1}) \otimes \mathbb{F}_p[v_1]/(v_1^{p+\dots+p^{n-1}})\{\mu^j \lambda_1 | v_p(j - \ell(n-1)p^{n-2}) = n-2\}. \end{aligned}$$

There are no extension problems when computing $\pi_ \mathrm{gr}_{\mathrm{mot}}^*(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_n})^{tC_{p^n}}/p$ as an $\mathbb{F}_p[v_1]$ -module.*

Proof. The E_∞ -page of the spectral sequence can be read off from inverting μ in Proposition 6.9. To see that no extension problems happen for a particular class, it suffices to prove this after multiplying by a sufficiently large power of μ , for which it follows again by Proposition 6.9, since the two spectral sequences agree up to higher and higher filtrations as the power of μ gets larger. \square

6.2. Working up to v_1 -adic filtration. We begin with the following observation.

Lemma 6.12. *Let $n, k \geq 0$. The graded \mathbb{F}_p -vector space*

$$(\pi_* \mathrm{gr}_{\mathrm{mot}}^*(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_n})^{tC_{p^n}}/p)/v_1$$

is at most one-dimensional in each degree.

Furthermore, the graded \mathbb{F}_p -vector space

$$(\pi_* \mathrm{gr}_{\mathrm{mot}}^*(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_n})^{hC_{p^n}}/p)/v_1$$

is at most one-dimensional in each degree with the following exception: if n is divisible by p , then degree $2\ell p^n$ is two-dimensional with given by classes represented in the Nygaard spectral sequence by $\sigma\epsilon^{(\ell p^n)}$ and $u_n \lambda_1 t^{p-1} \sigma\epsilon^{(\ell p^n)}$.

Proof. The lemma is easy for $n = 0$, so we assume $n \geq 1$.

For the Tate fixed points, we use the μ -inverted Nygaard spectral sequence. Then modulo v_1 all basis elements are of the form $u_{n-1}^{e_1} \lambda_1^{e_2} \mu^j \sigma\epsilon^{(\ell p^n)}$, where j is an integer and $e_1, e_2 \in \{0, 1\}$. Moreover, we have $e_1 = 0$ if $e_2 = 0$.

Now, we have $|u_{n-1}^{e_1} \lambda_1^{e_2} \mu^j \sigma\epsilon^{(\ell p^n)}| \equiv -e_1 - e_2 \pmod{2p}$, so that we may recover e_1 and e_2 from the degree. But, given a fixed (e_1, e_2) , it is clear that there is at most one class of the form $u_{n-1}^{e_1} \lambda_1^{e_2} \mu^j \sigma\epsilon^{(\ell p^n)}$ in each degree, so we are done.

Now we do the homotopy fixed points. We start by assuming that $n > 1$. In this case, modulo v_1 in odd degrees we have basis elements of the form

$$\begin{aligned} & \lambda_1 t^i \sigma\epsilon^{(\ell p^n)} \\ & \lambda_1 \mu^j \sigma\epsilon^{(\ell p^n)}, \end{aligned}$$

and it is clear that there is at most one of these in each odd degree. In even degrees, we have basis elements of the form

$$\begin{aligned} & t^{pi} \sigma \epsilon^{(\ell p^n)} \\ & \mu^j \sigma \epsilon^{(\ell p^n)} \\ & u_n \lambda_1 t^{pi} \sigma \epsilon^{(\ell p^n)} \\ & u_n \lambda_1 \mu^j \sigma \epsilon^{(\ell p^n)} \\ & u_n \lambda_1 t^i \sigma \epsilon^{(\ell p^n)}, i \in \{1, \dots, p-1\}. \end{aligned}$$

(That we can put the p in t^{pi} is where we use $n > 1$.) Considering degrees modulo $2p$, we see that the degrees of the first two lines do not intersect those of the second two lines. It is also clear that the first two and last three lines do not contain more than one elements of the same degree. We therefore only need to worry about the first two and last line. Since the degrees of the last line are $2\ell p^n + 2p - 4, \dots, 2\ell p^n$, we see that the only possible point of intersection occurs in degree $2\ell p^n$, where we can have both $\sigma \epsilon^{(\ell p^n)}$ and $u_n \lambda_1 t^{p-1} \sigma \epsilon^{(\ell p^n)}$ if p divides n .

Finally, we do the case $n = 1$. Again, the case of odd degrees is easy. Our even degree basis elements are:

$$\begin{aligned} & t^i \sigma \epsilon^{(\ell p)} \\ & \mu^j \sigma \epsilon^{(\ell p)} \\ & u_1 \lambda_1 \mu^j \sigma \epsilon^{(\ell p)} \\ & u_1 \lambda_1 t^i \sigma \epsilon^{(\ell p)}, i \in \{1, \dots, p-1\}. \end{aligned}$$

Again, it is clear that the first two and last two lines do not have any repeated degrees. The first and third line don't because $|t^i|$ is non-positive and $|u_1 \lambda_1 \mu^j|$ is positive. The second and third line don't by considering the reduction modulo $2p$. The final line again lies in degrees $2\ell p + 2p - 4, \dots, 2\ell p$, which could only intersect the first two in $2\ell p$ in the same way as the $n > 1$ case. But this is not possible because p does not divide $n = 1$. \square

Notation 6.13. Let $n, k \geq 0$. We use $\mathbf{Fil}_{v_1}^*$ to denote the v_1 -adic filtration on π_* , and $\mathrm{gr}_{v_1}^*$ to denote the associated graded of this filtration. Then by Lemma 6.12, the generators (as an $\mathbb{F}_p[v_1]$ -module) of the E_∞ -page of the Nygaard spectral sequence lift uniquely to elements in $\mathrm{gr}_{v_1}^*(\pi_* \mathrm{gr}_{\mathrm{mot}}^*(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell \rho_n})^{hC_{p^n}}/p)$, with the exception of $\sigma \epsilon^{(\ell p^n)}$ when n is divisible by p , in which case the indeterminacy is generated by $u_n \lambda_1 t^{p-1} \sigma \epsilon^{(\ell p^n)}$. We use this to unambiguously regard these symbols as elements of $\mathrm{gr}_{v_1}^*(\pi_* \mathrm{gr}_{\mathrm{mot}}^*(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell \rho_n})^{hC_{p^n}}/p)$, with the exception of $\sigma \epsilon^{(\ell p^n)}$, for which we fix once and for all a choice of lift.

We do similarly for the Tate construction and its Nygaard spectral sequence, without any exceptions.

Next we compute the maps can and φ on the level of the v_1 -adic associated graded of their homotopy groups. Recall that can is induced by j_{dR} and φ is induced by j_{HT} .

Proposition 6.14. *Let $n, k \geq 0$. Equip $\pi_* \mathrm{gr}_{\mathrm{mot}}^*(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{2\ell \rho_n})^{hC_{p^n}}/p$ and $\pi_* \mathrm{gr}_{\mathrm{mot}}^*(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{2\ell \rho_n})^{tC_{p^n}}/p$ with the v_1 -adic filtration. Then can and φ induce the maps*

$$\begin{aligned} \mathrm{can} : \mathrm{gr}_{v_1}^*(\pi_* \mathrm{gr}_{\mathrm{mot}}^*(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{2\ell \rho_n})^{hC_{p^n}}/p) &\rightarrow \mathrm{gr}_{v_1}^*(\pi_* \mathrm{gr}_{\mathrm{mot}}^*(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{2\ell \rho_n})^{tC_{p^n}}/p) \\ v_1^a \sigma \epsilon^{(p^n \ell)} t^i \lambda_1^{\epsilon_1} u_n^{\epsilon_2} &\mapsto v_1^a \sigma \epsilon^{(p^n \ell)} t^i \lambda_1^{\epsilon_1} u_n^{\epsilon_2} \\ v_1^a \sigma \epsilon^{(p^n \ell)} \mu^j \lambda_1^{\epsilon_1} u_n^{\epsilon_2} &\mapsto 0, j > 0 \end{aligned}$$

and

$$\begin{aligned} \varphi : \mathrm{gr}_{v_1}^* (\pi_* \mathrm{gr}_{\mathrm{mot}}(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{2\ell\rho_n})^{hC_{p^n}}/p) &\rightarrow \mathrm{gr}_{v_1}^* (\pi_* \mathrm{gr}_{\mathrm{mot}}(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{2k\rho_{n+1}})^{tC_{p^{n+1}}}/p) \\ v_1^a \sigma \epsilon^{(p^n \ell)} t^i \lambda_1^{\epsilon_1} u_n^{\epsilon_2} &\mapsto 0, i > 0 \\ v_1^a \sigma \epsilon^{(p^n \ell)} \mu^j \lambda_1^{\epsilon_1} u_n^{\epsilon_2} &\mapsto v_1^a \sigma \epsilon^{(p^{n+1} \ell)} t^{p^n \ell (p-1) - pj} \lambda_1^{\epsilon_1} u_{n+1}^{\epsilon_2}. \end{aligned}$$

Proof. Using Lemma 6.12, it is enough to check that the maps are as claimed at the level of an associated graded, with the exception of the class $\sigma \epsilon^{(\ell p^n)}$ in the case $p|n$, for which there is a potential indeterminacy of $u_n \lambda_1 t^{p-1} \sigma \epsilon^{(\ell p^n)}$. On this class, both can and φ are zero, so φ and can of $\epsilon^{(\ell p^n)}$ are determined by the map at the level of an associated graded.

For can , the map of spectral sequences is as claimed since the map of spectral sequence is given by inverting t , so the map is given as claimed on the associated graded of the Nygaard spectral sequence.

If the classes in the Tate construction are named using the μ -inverted Nygaard spectral sequence, then the map φ is simply given by inverting μ . To obtain the proposition, we need to transition from μ -inverted names to Tate names, which can be done up to a unit in \mathbb{F}_p^\times using the fact that there is at most one class in each degree. \square

6.3. The computation. In this section, we tackle the computation.

Consider φ - $\mathrm{can} : \mathbf{Fil}_{v_1}^* \prod_{i=0}^n \pi_* (\mathrm{gr}_{\mathrm{mot}}^*(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_i})^{hC_{p^i}}/p) \rightarrow \mathbf{Fil}_{v_1}^* \prod_{i=1}^n \pi_* (\mathrm{gr}_{\mathrm{mot}}^*(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_i})^{tC_{p^i}}/p)$.

First, we note that the v_1 -adic filtrations here are finite, since each of the spectra involved has bounded v_1 -torsion.

Lemma 6.15. *Let $n, \ell \geq 0$. The map*

$$\mathbf{Fil}_{v_1}^* \prod_{i=0}^n \pi_* (\mathrm{gr}_{\mathrm{mot}}^*(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_i})^{hC_{p^i}}/p) \rightarrow \mathbf{Fil}_{v_1}^* \prod_{i=1}^n \pi_* (\mathrm{gr}_{\mathrm{mot}}^*(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_i})^{tC_{p^i}}/p)$$

is surjective.

Proof. Since the v_1 -adic filtration is finite, it will suffice to prove surjectivity on the v_1 -adic associated graded. We use the formulas in Proposition 6.14. The image of can in the (v_1 -associated graded of the) C_{p^i} -Tate construction is spanned by those basis elements which have a nonnegative power of t . The image of φ in the C_{p^i} -Tate construction in particular contains every basis element with a negative power of t . Note also that φ - can is either φ or $-\mathrm{can}$ unless the power of μ and t is 0. Let us write the source as the sum of three terms $\bigoplus_0^n (S_{+,i} \oplus S_{0,i} \oplus S_{-,i})$ by having $S_{+,i}$ be the summand of $\mathrm{gr}_{v_1}^* \pi_* \mathrm{gr}_{\mathrm{mot}}^*(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_i})^{hC_{p^i}}/p$ where the power of μ is positive, $S_{-,i}$ be the summand where the power of t is positive, and $S_{0,i}$ be the summand where both powers are 0. Similarly, let $T_{+,i}$ be the summand of the target where the power of t is negative, $T_{0,i}$ be the summand where the power of t is zero, and $T_{-,i}$ the summand where it is positive.

Then we can put a filtration on the source and target where we filter the target so that $T_{+,i}, T_{-,i}, T_{0,i}$ are in filtration i , and filter the source so that $S_{-,i}, S_{+,i-1}, S_{0,i-1}$ are in filtration i . Then φ - can preserves this filtration, and on the associated graded, $S_{-,i}$ surjects onto $T_{-,i}$ via $-\mathrm{can}$, and $S_{+,i-1} \oplus S_{0,i-1}$ surjects onto $(\pi_* \mathrm{gr}_{\mathrm{mot}}^*(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_i})^{tC_{p^i}}/p)/T_{-,i}$ via φ , so this shows surjectivity. \square

As a consequence of Lemma 6.15, only the kernel of φ - can contributes to $\pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}^{[n]}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p)/p$.

Construction 6.16. We define a filtration on $\pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}^{[n]}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p)/p$ via

$$\mathbf{Fil}^i \pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}^{[n]}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p)/p := \ker(\mathbf{Fil}_{v_1}^i(\varphi - \mathrm{can})).$$

and similarly for $\pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p)/p$.

We use $\mathrm{gr}^i \pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}^{[n]}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p)/p$ to denote the associated graded of this filtration, and similarly for TR .

Now, we do some legwork necessary to state the output of our main computation.

Definition 6.17. Below, we list some mutually linearly independent families of $\mathrm{gr}_{v_1}^* \prod_{i=0}^{\infty} \pi_* \mathrm{gr}_{\mathrm{mot}}(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell \rho_i})^{hC_{p^i}}/p$ that lie in the kernel of $\varphi - \mathrm{can}$. Eventually, we will show that they form a basis for $\mathrm{gr}^* \pi_* \mathrm{gr}_{\mathrm{mot}} \mathrm{TR}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p)/p$ in a certain sense.

$$\lambda_1^e (\sigma \epsilon^{(p^n \ell)} \mu^j + \sigma \epsilon^{(p^{n+1} \ell)} t^{p^n \ell (p-1) - pj}) \quad (1_{n,\ell})$$

$$\text{where } e \in \{0, 1\}, 0 < j \leq p^{n-1} \ell (p-1) - p^n, j \equiv n \ell p^{n-1} \pmod{p^n}$$

$$\lambda_1^e (\sigma \epsilon^{(p^n \ell)} \mu^j + \sigma \epsilon^{(p^{n+1} \ell)} t^{p^n \ell (p-1) - pj} + \delta_{p^{n-1} \ell (p-1)}^j \sigma \epsilon^{(p^{n+2} \ell)} t^{p^{n+1} \ell (p-1)}) \quad (2_{n,\ell})$$

$$\text{where } e \in \{0, 1\}, p^{n-1} \ell (p-1) - p^n < j \leq p^{n-1} \ell (p-1), j > 0, j \equiv n \ell p^{n-1} \pmod{p^n}$$

Here δ_i^j denotes the Kronecker delta function.

$$\sigma \epsilon^{(p^n \ell)} t^i \lambda_1 u_n^e \quad (3_{n,\ell})$$

$$\text{where } e \in \{0, 1\}, 0 < i < p, v_p(i + \ell n p^{n-1}) = 0$$

In the next two families, we let $1 \leq r \leq n$.

$$\sigma \epsilon^{(p^n \ell)} \mu^j \lambda_1 u_n^e + \sigma \epsilon^{(p^{n+1} \ell)} t^{p^n \ell (p-1) - pj} \lambda_1 u_{n+1}^e \quad (4_{n,\ell,r})$$

$$\text{where } e \in \{0, 1\}, j > 0, v_p(j - \ell n p^{n-1}) = r - 1, j \leq p^{n-1} \ell (p-1) - p^r$$

$$\sigma \epsilon^{(p^n \ell)} \mu^j \lambda_1 u_n^e + \sigma \epsilon^{(p^{n+1} \ell)} t^{p^n \ell (p-1) - pj} \lambda_1 u_{n+1}^e + \delta_{p^{n-1} \ell (p-1)}^j \sigma \epsilon^{(p^{n+2} \ell)} t^{p^{n+1} \ell (p-1)} \lambda_1 u_{n+2}^e \quad (5_{n,\ell,r})$$

$$\text{where } e \in \{0, 1\}, j > 0, v_p(j - \ell n p^{n-1}) = r - 1, p^{n-1} \ell (p-1) - p^r < j \leq p^{n-1} \ell (p-1)$$

The constants in the $\dot{+}$ s are uniquely determined by the condition that these classes are in the kernel of $\varphi - \mathrm{can}$ (see Lemma 6.19).

Definition 6.18. We let $A_{n,\ell}$, $B_{n,\ell}$, $C_{n,\ell}$, $D_{n,\ell,r}$ and $E_{n,\ell,r}$ denote the sub- $\mathbb{F}_p[v_1]$ -modules of $\prod_{i=0}^{\infty} \mathrm{gr}_{v_1}^i \pi_* \mathrm{gr}_{\mathrm{mot}}(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell \rho_i})^{hC_{p^i}}/p$ generated by the families $(1_{n,\ell})$, $(2_{n,\ell})$, $(3_{n,\ell})$, $(4_{n,\ell,r})$ and $(5_{n,\ell,r})$, respectively.

Lemma 6.19. $A_{n,\ell}$, $B_{n,\ell}$, $C_{n,\ell}$, $D_{n,\ell,r}$ and $E_{n,\ell,r}$ are in the kernel of $\varphi - \mathrm{can}$, and their elements are linearly independent in the sense that $A_{n,\ell} + B_{n,\ell} + C_{n,\ell} + D_{n,\ell,r} + E_{n,\ell,r}$ is the direct sum of the cyclic $\mathbb{F}_p[v_1]$ -submodules generated by each element of $(1_{n,\ell})$, $(2_{n,\ell})$, $(3_{n,\ell})$, $(4_{n,\ell,r})$ and $(5_{n,\ell,r})$. Moreover the v_1 -torsion order of the generators is listed below:

$$(1_{n,\ell}): \lambda_1^e (\sigma \epsilon^{(p^n \ell)} \mu^j + \sigma \epsilon^{(p^{n+1} \ell)} t^{p^n \ell (p-1) - pj}) \text{ is } v_1^{1+\dots+p^n} \text{-torsion.}$$

$$(2_{n,\ell}): \lambda_1^e (\sigma \epsilon^{(p^n \ell)} \mu^j + \sigma \epsilon^{(p^{n+1} \ell)} t^{p^n \ell (p-1) - pj} + \delta_{p^{n-1} \ell (p-1)}^j \sigma \epsilon^{(p^{n+2} \ell)} t^{p^{n+1} \ell (p-1)}) \text{ is } v_1^{1+\dots+p^n+(p^{n+1}-(p^n \ell (p-1)-pj))} \text{-torsion unless } j = p^n \ell (p-1) \text{ and } k = 1, \text{ in which case it is } v_1^{1+\dots+p^{n+1}+p^{n+1}} \text{-torsion.}$$

$$(3_{n,\ell}): \sigma \epsilon^{(p^n \ell)} t^i \lambda_1 u_n^e \text{ is } v_1^{p-i} \text{-torsion.}$$

$$(4_{n,\ell,r}): \sigma \epsilon^{(p^n \ell)} \mu^j \lambda_1 u_n^e + \sigma \epsilon^{(p^{n+1} \ell)} t^{p^n \ell (p-1) - pj} \lambda_1 u_{n+1}^e \text{ is } v_1^{1+\dots+p^r} \text{-torsion.}$$

$(5_{n,\ell,r})$: $\sigma\epsilon^{(p^n\ell)}\mu^j\lambda_1u_n^e + \sigma\epsilon^{(p^{n+1}\ell)}t^{p^n\ell(p-1)-pj}\lambda_1u_{n+1}^e + \delta_{p^{n-1}\ell(p-1)}^j\sigma\epsilon^{(p^{n+2}\ell)}t^{p^{n+1}k(p-1)}\lambda_1u_{n+2}^e$
 is $v_1^{1+\dots+p^r+(p^{r+1}-(p^n\ell(p-1)-pj))}$ -torsion unless $n = r, j = p^{n-1}\ell(p-1), k = 1$ in
 which case it is $v_1^{1+\dots+p^{n+1}+p^{n+1}}$ -torsion.

Proof. We first prove that the classes are in $\text{gr}_{\text{mot}}^*\text{TR}(\mathbb{Z}_p; \Sigma^{2\ell}\mathbb{Z}_p)/p$ and check their v_1 -torsion orders. In general, doing this is a straightforward application of Proposition 6.14 and Proposition 6.9, where the first term in each sum has can vanishing, the last term has φ vanishing, and φ of each term is can of the next one. The v_1 -torsion order can be read off from the maximum of the v_1 -torsion order of each term in Proposition 6.9. Therefore, we will just indicate where the each term of each class appears in the description of the E_∞ -page of the Nygaard spectral sequence of Proposition 6.9.

For family $(1_{n,\ell})$, the first term in the family appears in the third line of Proposition 6.9, and the second term appears in the first line.

For family $(2_{n,\ell})$, the first term in the family appears in the first line of Proposition 6.9, and the second term appears in the second line unless $p^{n-1}\ell(p-1) = j$. In this case, the second term appears on the third line, and the third term appears on the first line if $k \geq 2$ and the second line otherwise. Note that the case $k = 0$ doesn't happen because $j > 0$.

For family $(3_{n,\ell})$, the term appears on the fifth line of Proposition 6.9 in the first of the two summands.

For family $(4_{n,\ell,r})$, the first term appears in the second part of the fifth line of Proposition 6.9 if $n > r$ and the last line if $n = r$. The second term appears in the fourth line if $n > r$ and the sixth line if $n = r$.

For family $(5_{n,\ell,r})$, the first term appears in the second part of the fifth line of Proposition 6.9 if $n > r$ and the last line if $n = r$. The second term appears in the first part of the fifth line if $n > r$ and the second to last line if $n = r$. The third term appears in the third to last line if $k > 1$, and in the second to last line if $k = 1$ (this term appears only when $n = r$).

For the claim about linear independence, we note that it is straightforward to verify that each element of each term of each family $(1_{n,\ell}) \dots (5_{n,\ell,r})$ only appears once and is a basis element of the answer of Proposition 6.9 in terms of cyclic $\mathbb{F}_p[v_1]$ -modules, so there can be no linear relations. Note that this holds despite the fact that elements may appear to be of the same type: for example elements of the form $v_1^{1+\dots+p^{n+1}+p^{n+1}-1}\lambda_1\sigma\epsilon^{(p^{n+1})}t^{p^{n+1}(p-1)}$ appear to come from both family $(2_{n,\ell})$ and $(5_{n,\ell,r})$, but the former happens when $p|n+1$ and the latter when $p \nmid n+1$. \square

Now, we need to deal with $\text{gr}^*(\pi_*\text{gr}_{\text{mot}}^*\text{TR}^{[m]}(\mathbb{Z}_p; \Sigma^{2\ell}\mathbb{Z}_p)/p)$ instead of $\text{gr}^*(\pi_*\text{gr}_{\text{mot}}^*\text{TR}(\mathbb{Z}_p; \Sigma^{2\ell}\mathbb{Z}_p)/p)$. First, some of these families get truncated.

Definition 6.20. We let $A_{n,\ell}^{[m]}, B_{n,\ell}^{[m]}, C_{n,\ell}^{[m]}, D_{n,\ell,r}^{[m]}$ and $E_{n,\ell,r}^{[m]}$ denote the images of these in $\prod_{i=0}^m \text{gr}_*^{v_1} \pi_* \text{gr}_{\text{mot}}^*(\text{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_i})^{hC_{p^i}}/p$.

Lemma 6.21. \bullet If $m > n$, we have $A_{n,\ell}^{[m]} = B_{n,\ell}^{[m]} = C_{n,\ell}^{[m]} = D_{n,\ell,r}^{[m]} = E_{n,\ell,r}^{[m]} = 0$.

\bullet If $n \geq m$, we have $A_{n,\ell} \cong A_{n,\ell}^{[m]}$, $C_{n,\ell} \cong C_{n,\ell}^{[m]}$ and $D_{n,\ell,r} \cong D_{n,\ell,r}^{[m]}$.

\bullet If $n > m+1$ or $n = m+1$ and either $r < n$ or $k \neq 1$, we have $B_{n,\ell} \cong B_{n,\ell}^{[m]}$ and $E_{n,\ell,r} \cong E_{n,\ell,r}^{[m]}$.

\bullet The maps $B_{n,1} \rightarrow B_{n,1}^{[m+1]}$ and $E_{n,1,n} \rightarrow E_{n,1,n}^{[m+1]}$ are isomorphic to $B_{n,1} \rightarrow B_{n,1}/(v_1^{1+\dots+p^n+(p^{n+1}-(p^n(p-1)-pj))}(\lambda_1^e(\sigma\epsilon^{(p^n)}\mu^j + \sigma\epsilon^{(p^{n+1})}t^{p^n(p-1)-pj} + \delta_{p^{n-1}(p-1)}^j\sigma\epsilon^{(p^{n+2})}t^{p^{n+1}(p-1)}))$ where $j = p^n(p-1)$, and

$$E_{n,1,n} \rightarrow E_{n,1,n}/(v_1^{1+\dots+p^n+(p^{n+1}-(p^n(p-1)-pj))} \sigma \epsilon^{(p^n)} \mu^j \lambda_1 u_n^e + \sigma \epsilon^{(p^{n+1})} t^{p^n(p-1)-pj} \lambda_1 u_{n+1}^e + \delta_{p^{n-1}(p-1)}^j \sigma \epsilon^{(p^{n+2})} t^{p^{n+1}(p-1)} \lambda_1 u_{n+2}^e).$$

- The maps $B_{n,\ell} \rightarrow B_{n,\ell}^{[n]}$ and $E_{n,\ell,r} \rightarrow E_{n,\ell,r}^{[n]}$ are isomorphic to $B_{n,\ell} \rightarrow B_{n,\ell}/(v_1^{1+\dots+p^n})$ and $E_{n,\ell,r} \rightarrow E_{n,\ell,r}/(v_1^{p^1+\dots+p^r})$, respectively.

Proof. This is a straightforward consequence of Lemma 6.19, by looking at the v_1 -torsion orders of the elements in the sums making up the families $(1_{n,\ell}) \dots (5_{n,\ell,r})$, and seeing for which i they are detected in $\mathrm{gr}_*^{v_1} \pi_* \mathrm{gr}_{\mathrm{mot}}^*(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_i})^{hC_{p^i}}/p$. \square

There are also two additional families in $\mathrm{gr}^*(\pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}^{[n]}(\mathbb{Z}_p; \Sigma^{2\ell}\mathbb{Z}_p)/p)$:

Definition 6.22.

$$\lambda_1^e \sigma \epsilon^{(p^n\ell)} \mu^j \tag{6_{n,\ell}}$$

$$\text{where } e \in \{0, 1\}, j > p^{n-1}\ell(p-1), j \equiv n\ell p^{n-1} \pmod{p^n}$$

$$\sigma \epsilon^{(p^n\ell)} \mu^j \lambda_1 u_n^e \tag{7_{n,\ell,r}}$$

$$\text{where } e \in \{0, 1\}, j > p^{n-1}\ell(p-1), v_p(j - \ell n p^{n-1}) = r - 1$$

Definition 6.23. We let $F_{n,\ell}^{[n]}$ and $G_{n,\ell,r}^{[n]}$ denote the $\mathbb{F}_p[v_1]$ -modules generated by $(6_{n,\ell})$ and $(7_{n,\ell,r})$ in $\prod_{i=0}^m \mathrm{gr}_*^{v_1} \pi_* \mathrm{gr}_{\mathrm{mot}}^*(\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_i})^{hC_{p^i}}/p$.

Lemma 6.24. $F_{n,\ell}^{[n]}$ and $G_{n,\ell,r}^{[n]}$ lie in $\mathrm{gr}^*(\pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}^{[n]}(\mathbb{Z}_p; \Sigma^{2\ell}\mathbb{Z}_p)/p)$, and their elements are linearly independent in the sense that $A_{n,\ell}^{[n]} + B_{n,\ell}^{[n]} + C_{n,\ell}^{[n]} + D_{n,\ell,r}^{[n]} + E_{n,\ell,r}^{[n]} + F_{n,\ell}^{[n]} + G_{n,\ell,r}^{[n]}$ is the direct sum of the cyclic $\mathbb{F}_p[v_1]$ -submodules generated by each element of $(1_{n,\ell}), (2_{n,\ell}), (3_{n,\ell}), (4_{n,\ell,r}), (5_{n,\ell}), (6_{n,\ell})$ and $(7_{n,\ell,r})$. Moreover the v_1 -torsion order of the classes are listed in Lemma 6.21 and below:

$$(6_{n,\ell}) \lambda_1^e \sigma \epsilon^{(p^n\ell)} \mu^j \text{ is } v_1^{1+\dots+p^n}\text{-torsion}$$

$$(7_{n,\ell,r}) \sigma \epsilon^{(p^n\ell)} \mu^j \lambda_1 u_n^e \text{ is } v_1^{p^1+\dots+p^r}\text{-torsion.}$$

Proof. To see that the classes in $(6_{n,\ell})$ and $(7_{n,\ell,r})$ are in $\mathrm{TR}^{[n]}$, we just need to see they are in the kernel of the map can , which is true because they are a multiple of μ . The v_1 -torsion orders can be read off from the fact that $(6_{n,\ell})$ appears in the family of Proposition 6.9 on the third line and $(7_{n,\ell,r})$ appears on the second part of the fifth line. Linear independence of the classes follows as in Lemma 6.19, because basis elements are not repeated among elements. \square

Our main theorem computes $\mathrm{gr}^*(\pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}^{[m]}(\mathbb{Z}_p; \Sigma^{2\ell}\mathbb{Z}_p)/p)$ and $\mathrm{gr}^*(\pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}(\mathbb{Z}_p; \Sigma^{2\ell}\mathbb{Z}_p)/p)$ as $\mathbb{F}_p[v_1]$ -modules.

Theorem 6.25. *There are isomorphisms:*

$$\mathrm{gr}^* \pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}^{[m]}(\mathbb{Z}_p; \Sigma^{2\ell}\mathbb{Z}_p)/p \cong \bigoplus_{n=0}^m \left(A_{n,\ell}^{[m]} \oplus B_{n,\ell}^{[m]} \oplus C_{n,\ell}^{[m]} \oplus \bigoplus_{r=1}^n (D_{n,\ell,r}^{[m]} \oplus E_{n,\ell,r}^{[m]}) \right) \oplus F_{m,\ell}^{[m]} \oplus \bigoplus_{r=1}^m (G_{m,\ell,r}^{[m]})$$

for all $m \geq 0$ and

$$\mathrm{gr}^* \pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}(\mathbb{Z}_p; \Sigma^{2\ell}\mathbb{Z}_p)/p \cong \bigoplus_{n=0}^{\infty} \left(A_{n,\ell} \oplus B_{n,\ell} \oplus C_{n,\ell} \oplus \bigoplus_{r=1}^n (D_{n,\ell,r} \oplus E_{n,\ell,r}) \right)$$

Before we prove this theorem, we use it to show that we can get rid of the associated graded.

Proposition 6.26. *There is a (noncanonical) isomorphism of graded $\mathbb{F}_p[v_1]$ -modules*

$$\pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p)/p \cong \mathrm{gr}^* \pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p)/p.$$

Proof. In fact, we show that $\mathbf{Fil}^i \pi_* \mathrm{TR}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p)/p = \mathbf{Fil}_{v_1}^i \pi_* \mathrm{TR}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p)/p$, from which the claim follows immediately. In the remainder of this proof, we will write \mathbf{Fil}^i and $\mathbf{Fil}_{v_1}^i$ for these graded modules.

By definition, $\mathbf{Fil}_{v_1}^i \subseteq \mathbf{Fil}^i$. It therefore suffices to show that $\mathbf{Fil}^i \subseteq \mathbf{Fil}_{v_1}^i$. Inspecting the description of $\mathrm{gr}^* \pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p)/p$ given in Theorem 6.25, we see that $v_1 : \mathrm{gr}^* \rightarrow \mathrm{gr}^{*+1}$ is surjective. Since \mathbf{Fil}^* is a finite filtration in each degree⁷, this implies that $v_1 : \mathbf{Fil}^* \rightarrow \mathbf{Fil}^{*+1}$ is surjective, so that $\mathbf{Fil}^i \subseteq \mathbf{Fil}_{v_1}^i$ by induction on i , as desired. \square

Proof of Theorem 6.25. Our proof is by induction on m ; the base case $m = 0$ is easy since $\mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}^{[0]}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p) \simeq \mathrm{gr}_{\mathrm{mot}}^* \mathrm{THH}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p)$.

By definition, there are pullback squares:

$$\begin{array}{ccc} \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}^{[m]}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p) & \longrightarrow & \mathrm{gr}_{\mathrm{mot}}^* (\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_m})^{hC_p^m} \\ \downarrow & & \downarrow \mathrm{can} \\ \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}^{[m-1]}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p) & \xrightarrow{\varphi} & \mathrm{gr}_{\mathrm{mot}}^* (\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_m})^{tC_p^m}, \end{array}$$

where $\varphi : \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}^{[m-1]}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p) \rightarrow \mathrm{gr}_{\mathrm{mot}}^* (\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_m})^{tC_p^m}$ is the composite $\mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}^{[m-1]}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p) \rightarrow \mathrm{gr}_{\mathrm{mot}}^* (\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_{m-1}})^{hC_p^{m-1}} \xrightarrow{\varphi} \mathrm{gr}_{\mathrm{mot}}^* (\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_m})^{tC_p^m}$.

Modding out by p and using Lemma 6.15, we see that

$$\begin{array}{ccc} \mathrm{gr}^* \pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}^{[m]}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p)/p & \longrightarrow & \mathrm{gr}_{v_1}^* \pi_* \mathrm{gr}_{\mathrm{mot}}^* (\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_m})^{hC_p^m}/p \\ \downarrow & & \downarrow \mathrm{can} \\ \mathrm{gr}^* \pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}^{[m-1]}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p)/p & \xrightarrow{\varphi} & \mathrm{gr}_{v_1}^* \pi_* \mathrm{gr}_{\mathrm{mot}}^* (\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_m})^{tC_p^m}/p \end{array}$$

is a pullback of graded $\mathbb{F}_p[v_1]$ -modules.

First, we note that $F_{m-1,\ell}^{[m-1]} \oplus \bigoplus_{r=1}^{m-1} \left(G_{m-1,\ell,r}^{[m-1]} \right)$ maps isomorphically onto the basis elements of $\mathrm{gr}_{v_1}^* \pi_* \mathrm{gr}_{\mathrm{mot}}^* (\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_m})^{tC_p^m}/p$ which contain a negative power of t . More specifically, $F_{m-1,\ell}^{[m-1]}$ hits elements in the first summand listed in Proposition 6.10, and $G_{m-1,\ell,r}^{[m-1]}$ hits elements in the second line if $r < m - 1$ and the third if $r = m - 1$.

On the other hand, the map

$$\mathrm{can} : \mathrm{gr}_{v_1}^* \pi_* \mathrm{gr}_{\mathrm{mot}}^* (\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_m})^{hC_p^m} \rightarrow \mathrm{gr}_{v_1}^* \pi_* \mathrm{gr}_{\mathrm{mot}}^* (\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_m})^{tC_p^m}$$

surjects onto the basis elements which contain a nonnegative power of t . Moreover, we can easily compute the kernel of this map, to be generated by all elements with a positive power of μ (i.e in the third summand, the second part of the fifth line, and the last line of Proposition 6.9), and the v_1 -multiples of elements from the second, first part of the fifth, and the second to last lines which are sent to 0. Note that in the case $k = 0$ in the first part of the fifth line, the whole summand is in the kernel.

⁷To prove this, we note that $\pi_* \mathrm{gr}_{\mathrm{mot}}^* (\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_m})^{hC_p^m}/p$ has bounded v_1 -torsion and $\pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p)/p \rightarrow \pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}^{[m]}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p)/p$ is an isomorphism in an increasing range of degrees.

From Lemma 6.19 and Lemma 6.24 there is an inclusion

$$\bigoplus_{n=0}^m \left(A_{n,\ell}^{[m]} \oplus B_{n,\ell}^{[m]} \oplus C_{n,\ell}^{[m]} \oplus \bigoplus_{r=1}^n \left(D_{n,\ell,r}^{[m]} \oplus E_{n,\ell,r}^{[m]} \right) \right) \oplus F_{m,\ell}^{[m]} \oplus \bigoplus_{r=1}^m \left(G_{m,\ell,r}^{[m]} \right) \\ \subseteq \mathrm{gr}^* \pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}^{[m]}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p)/p.$$

To check that it is an isomorphism, it then suffices to check that the kernel of the map

$$\bigoplus_{n=0}^m \left(A_{n,\ell}^{[m]} \oplus B_{n,\ell}^{[m]} \oplus C_{n,\ell}^{[m]} \oplus \bigoplus_{r=1}^n \left(D_{n,\ell,r}^{[m]} \oplus E_{n,\ell,r}^{[m]} \right) \right) \oplus F_{m,\ell}^{[m]} \oplus \bigoplus_{r=1}^m \left(G_{m,\ell,r}^{[m]} \right) \rightarrow \\ \bigoplus_{n=0}^{m-1} \left(A_{n,\ell}^{[m-1]} \oplus B_{n,\ell}^{[m-1]} \oplus C_{n,\ell}^{[m-1]} \oplus \bigoplus_{r=1}^n \left(D_{n,\ell,r}^{[m-1]} \oplus E_{n,\ell,r}^{[m-1]} \right) \right) \oplus F_{m-1,\ell}^{[m-1]} \oplus \bigoplus_{r=1}^{m-1} \left(G_{m-1,\ell,r}^{[m-1]} \right)$$

maps isomorphically onto the kernel of

$$\mathrm{can} : \mathrm{gr}_{v_1}^* \pi_* \mathrm{gr}_{\mathrm{mot}}^* (\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_m})^{hC_p^m} \rightarrow \mathrm{gr}_{v_1}^* \pi_* \mathrm{gr}_{\mathrm{mot}}^* (\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_m})^{tC_p^m}.$$

We explain how to match up the terms, where the reader may refer to Lemma 6.21 for a description of the kernels between each summand as m varies.

To do this, the terms with positive power of μ in the kernel of the canonical map coming from the third summand of Proposition 6.9 correspond to the kernel of $A_{m,\ell}^{[m]} \rightarrow A_{m,\ell}^{[m-1]}$, the kernel of $B_{m,\ell}^{[m]} \rightarrow B_{m,\ell}^{[m-1]}$, and $F_{m,\ell}^{[m]} \rightarrow 0$. The terms with positive powers of μ in the kernel of the canonical map from the second part of the fifth line and the last line of Proposition 6.9 correspond to the kernel of $D_{m,\ell,r}^{[m]} \rightarrow D_{m,\ell,r}^{[m-1]}$ and the kernel of $E_{m,\ell,r}^{[m]} \rightarrow E_{m,\ell,r}^{[m-1]}$ in the cases $r < m$ and $r = m$ respectively, and $G_{m,\ell,r}^{[m]} \rightarrow 0$.

The v_1 -multiples of elements from the second line in the kernel of the canonical map correspond to the kernel of $B_{m,\ell}^{[m+1]} \rightarrow B_{m,\ell}^{[m]}$, elements from the first part of the fifth and second to last lines correspond to the kernels of $E_{m,\ell,r}^{[m+1]} \rightarrow E_{m,\ell,r}^{[m]}$ in the case $1 \leq r < m$ and $r = m$ respectively. In the case $k = 0$ in the first part of the fifth line, this corresponds to the kernel of $C_{m,\ell}^{[m]} \rightarrow C_{m,\ell}^{[m-1]}$.

Finally, to obtain the claim for $\mathrm{gr}^* \pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p)/p$ from $\mathrm{gr}^* \pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}^{[m]}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p)/p$, we use the equivalence $\mathrm{gr}^* \pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p)/p \cong \varprojlim_m \mathrm{gr}^* \pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}^{[m]}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p)/p$ along with the fact that $\mathrm{gr}^* \pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}^{[m]}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p)/p \rightarrow \mathrm{gr}^* \pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}^{[m-1]}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p)/p$ is an equivalence in a range increasing in m^8 to deduce that the \varprojlim^1 -terms vanish. \square

Corollary 6.27. *The bigraded $\mathbb{F}_p[v_1]$ -module $\pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(\mathbb{Z}_p\langle\epsilon\rangle)/p$ is isomorphic to*

$$\pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(\mathbb{Z}_p)/p \oplus \bigoplus_{p|\ell > 0} \pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p)/p,$$

where for each value of ℓ , the homotopy groups are given by Theorem 6.25.

If $k \leq p^{n-2}$, then the mod (p, v_1^k) -syntomic cohomology of \mathbb{Z}_p/p^n is isomorphic to $\ker_{v_1^k}(\pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(\mathbb{Z}_p\langle\epsilon\rangle)) \oplus \mathrm{coker}_{v_1^k} \pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(\mathbb{Z}_p\langle\epsilon\rangle)[(2p-2)k+1, 1]$, where $[(2p-2)k+1, 1]$ indicates shifting motivic weight by 1 and topological degree by $(2p-2)k+1$.

Proof. The first claim follows from combining Corollary 5.5 in the case $R = \mathbb{Z}_p$ with Theorem 6.25. The second claim follows from taking the quotient by v_1^k , and applying Theorem 1.1(1) to identify the result for $\mathbb{Z}_p\langle\epsilon\rangle$ with that of \mathbb{Z}/p^n . \square

Remark 6.28. We note that the homotopy groups of the other term $\mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(\mathbb{Z}_p)/p$ are given in [LW22, Theorem 1.5] as $\mathbb{F}_p[v_1] \otimes E(\lambda_1) \otimes E(\partial) \oplus \mathbb{F}_p[v_1]\{t\lambda_1, \dots, t^{p-1}\lambda_1\}$.

⁸Because $\mathrm{can} : \pi_* \mathrm{gr}_{\mathrm{mot}}^* (\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_m})^{hC_p^m}/p \rightarrow \pi_* \mathrm{gr}_{\mathrm{mot}}^* (\mathrm{THH}(\mathbb{Z}_p) \otimes \mathbb{S}^{\ell\rho_m})^{tC_p^m}/p$ is, by inspection.

7. CONSEQUENCES FOR ALGEBRAIC K -THEORY

In this section, we use our results on syntomic cohomology to draw consequences for topological cyclic homology and algebraic K -theory.

7.1. Consequences for even K -groups. First, we prove some results about the groups $K_{2i}(\mathbb{Z}/p^n)$, which vanish for $i \gg 0$ by the even vanishing theorem of [AKN24]. We note that, as explained in [AKN24], when displayed in Adams grading the motivic spectral sequence computing $\pi_*\mathrm{TC}(\mathbb{Z}/p^n)$ is concentrated on the 0-line, 1-line, and 2-line. Furthermore, the 0-line is trivial above degree zero, by [AKN24, Corollary 2.13(ii)]. In particular, the motivic spectral sequence degenerates on the $E_2 = E_\infty$ -page.

Theorem 7.1. *For $n \geq 2$, the maps*

$$H^2(\mathbb{F}_p(i)(\mathbb{Z}_p)/v_1) \rightarrow H^2(\mathbb{F}_p(i)(\mathbb{Z}/p^n)/v_1)$$

$$H^2(\mathbb{F}_p(i)(\mathbb{Z}_p)) \rightarrow H^2(\mathbb{F}_p(i)(\mathbb{Z}/p^n))$$

$$H^2(\mathbb{Z}_p(i)(\mathbb{Z}_p)) \rightarrow H^2(\mathbb{Z}_p(i)(\mathbb{Z}/p^n))$$

are surjective.

Proof. By Theorem 1.1, the map

$$H^2(\mathbb{F}_p(i)(\mathbb{Z}_p)/v_1) \rightarrow H^2(\mathbb{F}_p(i)(\mathbb{Z}/p^n)/v_1)$$

is isomorphic to the map

$$H^2(\mathbb{F}_p(i)(\mathbb{Z}_p)/v_1) \rightarrow H^2(\mathbb{F}_p(i)(\mathbb{Z}_p\langle\epsilon\rangle)/v_1).$$

That the first map is surjective now follows from Corollary 6.27, since each of the terms $\mathrm{gr}_{\mathrm{mot}}^* \mathrm{TR}(\mathbb{Z}_p; \Sigma^{2\ell}\mathbb{Z}_p)/(p, v_1)$ has homotopy concentrated between the -1 -line and the 1 -line.

Running the v_1 -Bockstein and then the p -Bockstein, it then follows that the second and third maps are also surjective. \square

Corollary 7.2. *For $n \geq 2$ and $i \geq 0$, the map*

$$K_{2i}(\mathbb{Z}_p) \rightarrow K_{2i}(\mathbb{Z}/p^n)$$

is surjective. In particular, $K_{2i}(\mathbb{Z}/p^n)$ is cyclic.

Proof. It is clear that on K_0 , the map is the isomorphism $\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}$.

From the Dundas–Goodwillie–McCarthy theorem [DGM12], it follows that the fiber of the map $K(\mathbb{Z}/p^n) \rightarrow \mathrm{TC}(\mathbb{Z}/p^n)$ is the fiber of the map $K(\mathbb{F}_p) \rightarrow \mathrm{TC}(\mathbb{F}_p)$. It follows that $K(\mathbb{Z}/p^n)[\frac{1}{p}] \rightarrow K(\mathbb{F}_p)[\frac{1}{p}]$ is an isomorphism, and since $K(\mathbb{F}_p)[\frac{1}{p}]$ vanishes in positive degrees, it suffices to prove surjectivity after p -localization. Moreover the p -local K -groups are finite in positive degrees (see [Ang11, Theorem B]), so it suffices to prove surjectivity after p -completion.

The fiber of $K(\mathbb{F}_p) \rightarrow \mathrm{TC}(\mathbb{F}_p)$ is p -adically isomorphic to \mathbb{Z}_p in degree -2 , so the same is true for the fiber of the map $K(\mathbb{Z}/p^n) \rightarrow \mathrm{TC}(\mathbb{Z}/p^n)$. It thus suffices to show that the p -completion of $\mathrm{TC}(\mathbb{Z}_p) \rightarrow \mathrm{TC}(\mathbb{Z}/p^n)$ is surjective in positive even degrees. We achieve this by looking at the map of motivic spectral sequences, which are concentrated on the 1-line and 2-line in positive degrees. Since the classes on the 1-line are in odd degrees, it suffices to prove surjectivity on the 2-line, which is Theorem 7.1. \square

7.2. Consequences for mod (p, v_1^k) K -groups.

Recollection 7.3. When $p \geq 3$, there is a v_1 -self map $v_1 : \Sigma^{2p-2} \mathbb{S}/p \rightarrow \mathbb{S}/p$ [Ada66]. On the other hand, when $p = 2$ we have a v_1^4 -self map $v_1^4 : \Sigma^8 \mathbb{S}/2 \rightarrow \mathbb{S}/2$ [Ada66].

As a consequence, the mod p homotopy groups of a spectrum are equipped with a self-map v_1 (resp v_1^4 when $p = 2$). Moreover, we may form the generalized Moore complexes $\mathbb{S}/(p, v_1^k)$ and consider the mod (p, v_1^k) homotopy of spectra (where k is assumed to be divisible by 4 if $p = 2$).

As input, we will require the following result of Achim Krause and the third named author:

Theorem 7.4 (Krause–S.). *In $\pi_* \mathrm{gr}_*^{\mathrm{mot}} \mathrm{TC}(\mathbb{Z}/p^n)/p$, we have $v_1^{p^n-2} \partial \lambda_1 = 0$.*

In the above result, $\partial \lambda_1$ refers to the $\mathbb{F}_p[v_1]$ -module generator of the 2-line of $\pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(\mathbb{Z}_p)/p$, or more precisely its image along the natural map $\pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(\mathbb{Z}_p)/p \rightarrow \pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(\mathbb{Z}/p^n)/p$. Note that, by our Theorem 7.1, the $\mathbb{F}_p[v_1]$ -module generated by $\partial \lambda_1$ contains all non-zero elements on the 2-line of $\pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(\mathbb{Z}/p^n)/p$.

Proposition 7.5. *Let p denote a prime and $R = \mathbb{Z}_p\langle \epsilon \rangle$ or $R = \mathbb{Z}/p^n$ with $n \geq 2$. Then the motivic spectral sequence*

$$\pi_* \mathrm{gr}_*^{\mathrm{mot}} \mathrm{TC}(R)/p \Rightarrow \pi_* \mathrm{TC}(R)/p$$

degenerates at the E_2 -page. It has no hidden v_1 -extensions (v_1^4 -extensions if $p = 2$), in the sense that any class $x \in \pi_ \mathrm{gr}_*^{\mathrm{mot}} \mathrm{TC}(R)/p$ with $v_1^j x = 0$ detects a class $\tilde{x} \in \pi_* \mathrm{TC}(R)/p$ with $v_1^j \tilde{x} = 0$ (with 4 dividing j if $p = 2$).*

Proof. Since motivic spectral sequences exhibit a checkerboard pattern, to prove their degeneration it suffices to show that they are concentrated on the 0-line, 1-line, and 2-line. In the case of \mathbb{Z}/p^n , we recall that $\pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(\mathbb{Z}/p^n)$ is concentrated on the 0-line, 1-line, and 2-line, and furthermore the 0-line is p -torsionfree [AKN24, Corollary 2.13] (indeed, the 0-line consists of a single copy of \mathbb{Z}_p in degree 0). It follows that $\pi_* \mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(\mathbb{Z}/p^n)/p$ is also concentrated on the 0-line, 1-line and 2-line. To prove degeneration of the motivic spectral sequence computing $\pi_*(\mathrm{TC}(\mathbb{Z}_p\langle \epsilon \rangle)/p)$, we also note that its E_2 -page is concentrated on the 0-line, 1-line, and 2-line, for example by our explicit calculation of the E_2 -page as Corollary 6.27.

It remains to prove that there are no hidden v_1 -extensions (resp., hidden v_1^4 -extensions). We begin with the case of $R = \mathbb{Z}/p^n$. Such an extension would have to go from the 0-line to the 2-line, and we recall that as an \mathbb{F}_p -vector space the 2-line is generated by classes of the form $v_1^k \partial \lambda_1$ (which has degree $(2p-2)(k+1)$). The source of the extension would therefore be on the 0-line in a degree divisible by $2p-2$ but less than $2p-2$. The only possible source of a hidden v_1 -extension is therefore 1. But the v_1 -torsion order of 1 is equal to $1 + \dots + p^{n-1}$ by [AKN24, Theorem 6.3], and $v_1^{p^n-2} \partial \lambda_1 = 0$ by Theorem 7.4, so there is no possible hidden v_1 -extension.

Next, we consider the case of $R = \mathbb{Z}_p\langle \epsilon \rangle$, where we have the splitting

$$\mathrm{fil}_{\mathrm{mot}}^* \mathrm{TC}(\mathbb{Z}_p)/p \oplus \bigoplus_{p \nmid \ell > 0} \mathrm{fil}_{\mathrm{mot}}^* \mathrm{TR}(\mathbb{Z}_p; \Sigma^{2\ell} \mathbb{Z}_p)/p$$

of Theorem 5.4. For each value of ℓ , the homotopy groups of the associated graded are given by Theorem 6.25.

The summand $\mathrm{gr}_{\mathrm{mot}}^* \mathrm{TC}(\mathbb{Z}_p)/p$ is v_1 -torsionfree [LW22, Theorem 1.5], so there are no possible v_1 -extensions associated to that summand. The summands indexed by ℓ are concentrated on the 0- and 1-lines, so again there are no possible v_1 -extensions for sparsity reasons. \square

The following is an immediate corollary of the fact that there are no hidden v_1 -extensions:

Corollary 7.6. *Let p denote a prime, $n \geq 2$, and $k \geq 1$ be divisible by 4 if $p = 2$.*

Then the spectral sequences

$$\pi_* \mathrm{gr}_*^{\mathrm{mot}} \mathrm{TC}(\mathbb{Z}/p^n)/(p, v_1^k) \Rightarrow \pi_* \mathrm{TC}(\mathbb{Z}/p^n)/(p, v_1^k)$$

and

$$\pi_* \mathrm{gr}_*^{\mathrm{mot}} \mathrm{TC}(\mathbb{Z}_p\langle\epsilon\rangle)/(p, v_1^k) \Rightarrow \pi_* \mathrm{TC}(\mathbb{Z}_p\langle\epsilon\rangle)/(p, v_1^k)$$

degenerate at the E_2 -page.

Combining this with Theorem 1.1, we obtain the following corollary:

Corollary 7.7. *Let p denote a prime, $n \geq 2$, and $k \leq p^{n-2}$.*

If p is odd, then there is an isomorphism of \mathbb{F}_p -vector spaces

$$\pi_* \mathrm{gr}_*^{\mathrm{mot}} \mathrm{TC}(\mathbb{Z}_p\langle\epsilon\rangle)/(p, v_1^k) \cong \pi_* \mathrm{TC}(\mathbb{Z}/p^n)/(p, v_1^k).$$

If $p = 2$ and k is divisible by 4, then the graded abelian group $\pi_ \mathrm{TC}(R)/(2, v_1^k)$ admits a filtration with associated graded given by $\pi_* \mathrm{gr}_*^{\mathrm{mot}} \mathrm{TC}(\mathbb{Z}_2\langle\epsilon\rangle)/(2, v_1^k)$.*

Remark 7.8. There are 2-extensions mapped forward from $\pi_*(\mathrm{TC}(\mathbb{Z}_2)/2)$ [LW22, Theorem 8.21], but it is not clear that these constitute all 2-extensions.

Finally, we conclude with a consequence for algebraic K -theory.

Corollary 7.9. *Let p denote a prime, $n \geq 2$, and let $k \leq p^{n-2} - 1$. If $p = 2$, then assume that k is divisible by 4. Then there is an exact sequence*

$$0 \rightarrow \mathbb{F}_p\{v_1^k\partial\} \rightarrow \pi_* K(\mathbb{Z}/p^n)/(p, v_1^k) \rightarrow \pi_* \mathrm{TC}(\mathbb{Z}/p^n)/(p, v_1^k) \rightarrow \mathbb{F}_p\{\partial\} \rightarrow 0.$$

Proof. By the Dundas-Goodwillie-McCarthy theorem, we have a pullback square:

$$\begin{array}{ccc} K(\mathbb{Z}/p^n) & \longrightarrow & K(\mathbb{F}_p) \\ \downarrow & & \downarrow \\ \mathrm{TC}(\mathbb{Z}/p^n) & \longrightarrow & \mathrm{TC}(\mathbb{F}_p). \end{array}$$

Using the identification of $\pi_* K(\mathbb{F}_p)/p \rightarrow \pi_* \mathrm{TC}(\mathbb{F}_p)/p$ with $\mathbb{F}_p \rightarrow \mathbb{F}_p\{1, \partial\}$, we find that $K(\mathbb{Z}/p^n)/p \rightarrow \mathrm{TC}(\mathbb{Z}/p^n)/p$ identifies the source with the connective cover of the target. Equivalently, $\pi_* K(\mathbb{Z}/p^n)/p$ obtained from $\pi_* \mathrm{TC}(\mathbb{Z}/p^n)/p$ by removing ∂ .

Now, it follows from Theorem 1.1 that ∂ is not $v_1^{p^{n-2}-1}$ -torsion. The stated exact sequence is an immediate consequence. \square

Remark 7.10. To determine what happens when $k = p^{n-2}$, one would need to figure out the precise v_1 -torsion degree of ∂ .

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