ON THE BOUSFIELD CLASSES OF H_∞-RING SPECTRA

JEREMY HAHN

Abstract. We prove that any $K(n)$-acyclic, D_p-ring spectrum is $K(n+1)$-acyclic, affirming an old conjecture of Mark Hovey.

Contents

1. Introduction

A bedrock result of chromatic homotopy theory is that any $K(h)$-acyclic, p-local finite spectrum is $K(h-1)$-acyclic. Our goal here is to prove that $\mathbb{E}_\infty$-ring spectra enjoy the opposite phenomenon:

Theorem 1.1. *Suppose that an $\mathbb{E}_\infty$-ring spectrum R is $K(h)$-acyclic at some prime p. Then R is also $K(h+1)$-acyclic at p.*

In fact, our arguments require much less than a full $\mathbb{E}_\infty$-structure:

Definition 1.2. For a fixed prime number p, a D_p-algebra R is a spectrum R equipped with:

(1) A unit map $\mathbb{S} \to R$ and a multiplication map $R \otimes R \to R$, making R into a homotopy commutative and associative ring spectrum.
(2) A factorization of the p-fold multiplication map $R^{\otimes p} \to R$ through the projection $R^{\otimes p} \to (R^{\otimes p})_{hC_p}$, such that the diagram

$$\begin{tikzcd} (\mathbb{S}^{\otimes p})_{hC_p} \arrow[r, "(1^{\otimes p})_{hC_p}"] \arrow[d] & (R^{\otimes p})_{hC_p} \arrow[d] \\ \mathbb{S} \arrow[r, "1"] & R \end{tikzcd}$$

commutes up to homotopy. Here, 1 is the unit map, and the left-hand vertical map is part of the natural $\mathbb{E}_\infty$-ring structure on the sphere $\mathbb{S}$.

Any $\mathbb{E}_\infty$-ring R is naturally a D_p-algebra, and any H_∞-ring R admits a D_p-algebra structure. In particular, Theorem 1.1 is a corollary of the following stronger result:

Theorem 1.3. *Suppose that a spectrum R admits a D_p-algebra structure. For any height $h \geq 0$ at the prime p, if R is $K(h)$-acyclic then R is also $K(h+1)$-acyclic.*

Remark 1.4. If a spectrum R admits a homotopy unital multiplication, then R is $K(h)$-acyclic if and only if it is $T(h)$-acyclic [Lan+22, Lemma 2.3], so Theorem 1.3 can also be read as a statement about telescopic localization.

Remark 1.5. The $h = 0$ case of Theorem 1.3 was proved by Mathew–Naumann–Noel [MNN15, Theorem 2.1][1]–it is known as the May nilpotence conjecture. Together with Clausen, these authors found spectacular applications of the May nilpotence conjecture to descent questions in algebraic K-theory [Cla+20b].

In order to prove Theorem 1.3, it suffices to consider not D_p-algebras in spectra, but rather $K(h+1)$-local D_p-E-algebras, where E is the height $h+1$ Morava E-theory with

$$\pi_* E \cong \mathbb{Z}_{p^{h+1}}[\![u_1, u_2, \cdots, u_h]\!][u^{\pm}].$$

We will study a $K(h+1)$-local D_p-E-algebra R by means of the weight p *power operation*

$$P : \pi_0(R) \to R^0(BC_p),$$

which takes $x : S^0 \to R$ to the homotopy class of the composite

$$BC_p \simeq (\mathbb{S}^{\otimes p})_{hC_p} \xrightarrow{(x^{\otimes p})_{hC_p}} (R^{\otimes p})_{hC_p} \to R.$$

The power operation P is multiplicative, but not additive. Nonetheless, the composition of P with the quotient map $R^0(BC_p) \to R^0(BC_p)/(\mathrm{tr})$ is additive. Here, (tr) denotes the *transfer ideal*, which is cut out by a certain element $\frac{[p](z)}{z} \in R^0(BC_p)$ that is related to the p-series of the formal group law on $\pi_0(E)$. In particular, for each $x \in \pi_0(R)$ the image of $P(x)$ in the quotient ring $R^0(BC_p)/(p, \mathrm{tr})$ depends only on x modulo p. One of our main technical results, which may be of substantial independent interest, is a higher chromatic analog of the preceding sentence. For each $1 \leq k \leq h$, we determine a quotient of R^{BC_p} in which $P(x)$ depends only on the value of x modulo $(p, u_1, \cdots, u_k)$:

Theorem 1.6. *Suppose that R is a $K(h+1)$-local D_p-E-algebra and $k \geq 1$. Then, for any $x \in \pi_0(R)$, the image of $P(x)$ in $\pi_0\left(R^{BC_p}/\left(p, u_1, \cdots, u_k, \frac{[p](z)}{z^{p^{k+1}}}\right)\right)$ depends only on the image of x in $\pi_0(R/(p, u_1, \cdots, u_k))$.*

For a more detailed version of this statement, see Section 3, where $\frac{[p](z)}{z^{p^{k+1}}}$ is denoted $g_{k+1}(z)$. Note that $[p](z)$ is divisible by $z^{p^{k+1}}$ only after modding out by $(p, u_1, \cdots, u_k)$.

Remark 1.7. In private communication, Nathaniel Stapleton has asked whether there exist power operations for the cohomology theory of spaces given by the spectrum $E/(p, u_1, \cdots, u_k)$. In other words, if X is a space, Stapleton asks whether there is a natural map

$$(E/(p, u_1, \cdots, u_k))^0 (X) \to \left(E^{BC_p}/\left(p, u_1, \cdots, u_k, \frac{[p](z)}{z^{p^{k+1}}}\right)\right)^0 (X).$$

The above result shows that such a power operation can be defined on any $x \in (E/(p, u_1, \cdots, u_k))^0 (X)$ that arises as the image of a class in $E^0(X)$.

Let us briefly describe how we prove Theorem 1.3. Suppose that R is a $K(h+1)$-local D_p-E-algebra. If R is $K(h)$-acyclic, then for some value of n it must be the case that $(u_h)^n \in \pi_0(E)$ maps to zero in $\pi_0(R/(p, u_1, \cdots, u_{h-1}))$. Using Theorem 1.6, we check that if $(u_h)^n$ maps to zero, then so does $(u_h)^{n-1}$. Iterating this argument, we eventually learn that one maps to zero in $\pi_0(R/(p, u_1, \cdots, u_{h-1}))$, which implies (since R is $K(h+1)$-local) that R is the zero ring. Such a proof is very much analogous to the one of the May nilpotence conjecture in [MNN15].

[1] While Mathew–Naumann–Noel state their theorem for H_∞-ring spectra, their proof uses only D_p-algebra structure.

Remark 1.8. In the long time since the first version of this paper was posted, Clausen–Mathew–Naumann–Noel [Cla+20a] and Land–Mathew–Meier–Tamme [Lan+22] found applications of Theorem 1.1 to foundational descent and purity results in algebraic K-theory. As part of their arguments, Clausen–Mathew–Naumann–Noel make the elegant observation that, for any $K(h+1)$-local $\mathbb{E}_\infty$-ring R, R is $K(h)$-acyclic if and only if R^{tC_p} is $K(h)$-acyclic, where R^{tC_p} denotes the Tate construction for the trivial C_p action. We record as Proposition 4.7 that this is also true of D_p-algebras. Thus, if a $K(h+1)$-local D_p-algebra R has $L_{K(h)}R^{tC_p} = 0$, then $R = 0$.

Remark 1.9. In their forthcoming work on the chromatic Nullstellensatz, Burklund–Schlank–Yuan prove that any $\mathbb{E}_\infty$-ring is either $K(h+1)$-acyclic or maps to some height $h+1$ Lubin–Tate theory; this in particular strengthens Theorem 1.1. In combination with work of Allen Yuan [Yua21], the Nullstellensatz implies chromatic redshift occurs in the algebraic K-theory of any $\mathbb{E}_\infty$-ring.

Remark 1.10. A version of Theorem 1.3 was first conjectured by Mark Hovey, where it appears as Miscellaneous Problem 2 in his 1999 list of unsolved problems in algebraic topology [Hov99].

Acknowledgements. I heartily thank Akhil Mathew for introducing me to this problem and pointing out its appearance on Mark Hovey's webpage. Thanks are due to Peter May, Denis Nardin, Eric Peterson, Andrew Senger, Nathaniel Stapleton, Dylan Wilson, Allen Yuan, and especially my advisor Mike Hopkins for helpful conversations. The author was supported by the NSF Graduate Fellowship under Grant DGE-1144152.

The original version of this paper was the first preprint I ever produced, and I hope the subsequent six years of experience have improved both my technical and expository powers. In particular, I thank Robert Burklund, Akhil Mathew, and Andrew Senger for bringing to my attention a mistake in Section 2 of the original version of this preprint.

2. Morava E theory and its weight p power operation

We fix throughout the remainder of this paper a prime p and an integer $h > 0$, and by default all spectra will be implicitly p-localized. By convention, we let $E = E_{h+1}$ denote the height $h+1$ Lubin–Tate theory associated to the Honda formal group over $\mathbb{F}_{p^{h+1}}$. We call this E_{h+1} *Morava E-theory*, and it has homotopy groups

$$\pi_* E \cong \mathbb{Z}_{p^{h+1}}[\![u_1, u_2, \cdots, u_h]\!][u^{\pm}],$$

where $|u_i| = 0$ and $|u| = -2$. By convention, we sometimes write $u_0 = p$. Any other Lubin–Tate theory would work just as well for our purposes, but we use this one for the sake of concreteness. The Goerss–Hopkins–Miller theorem equips E with a canonical $\mathbb{E}_\infty$-ring structure [GH04].

Convention 2.1. We fix a p-typical complex orientation of E

$$BP \to E,$$

and name our generators $u_i \in \pi_0 E$ such that indecomposable generators $v_i \in \pi_{2p^i-2}BP$ are carried to $u_i u^{-p^i+1}$ [DH04, p. 7]. Note that we do not in any way insist that this orientation be compatible with the E_∞-ring structure on Morava E-theory. Our fixed p-typical complex orientation is given by a class $t \in E^2(\mathbb{CP}^\infty)$, and it will be convenient to let $z \in E^0(\mathbb{CP}^\infty)$ denote the product of t with $u^{-1} \in \pi_2 E$. We can then speak of the p-series $[p](z)$ as a class in $E^0(\mathbb{CP}^\infty) \cong E_0[\![z]\!]$.

Remark 2.2. The fibration

$$S^1 \to \mathrm{B}C_p \to \mathbb{CP}^\infty$$

allows one to write $E^0(BC_p)$ as a quotient of $E^0(\mathbb{CP}^\infty) \cong E_0[\![z]\!]$ [HKR00, Lemma 5.7]. Specifically, one has

$$E^0(BC_p) \cong E_0[\![z]\!]/[p](z).$$

Definition 2.3. The central object of study of this paper is *the weight p power operation on Morava E-theory*, which is a multiplicative (but not additive) map

$$P : E^0 \to E^0(BC_p).$$

By definition, this map takes a class $x : S^0 \to E$ to the composite class

$$\Sigma^\infty_+ BC_p = (S^0)^{\otimes p}_{hC_p} \xrightarrow{(x^{\otimes p})_{hC_p}} (E^{\otimes p})_{hC_p} \to E,$$

where the last map arises from the D_p-algebra structure on Morava E-theory.

Definition 2.4. While P is not additive, and hence not a ring map, the composite

$$E^0 \xrightarrow{P} E^0(BC_p) \longrightarrow z^{-1}E^0(BC_p) = \pi_0 E^{tC_p}$$

is a ring map, which we will denote by φ. In fact, φ is π_0 of the Nikolaus–Scholze Frobenius [NS18, §IV.1], which is an $\mathbb{E}_\infty$ ring map $E \to E^{tC_p}$. By taking π_* of the Nikolaus–Scholze Frobenius, we extend the domain of φ to include elements of $\pi_* E$ that are not of degree 0; the output of φ will in general be a class in $z^{-1}E^*(BC_p)$.

Lemma 2.5. *For each $0 \le k \le h$, the following congruence holds in $z^{-1}E^0(BC_p)$:*

$$\varphi(u_k) \equiv \frac{\varphi\left(u^{p^k-1}\right) u_k}{u^{p^k-1}} \text{ modulo } p, \cdots, u_{k-1}$$

Proof. Recall that we have fixed a homotopy ring map $f : \mathrm{BP} \to E$, which on homotopy groups takes v_i to $u_i u^{-p^i+1}$. The lemma is thus implied by the statement that $\varphi(v_i) \equiv v_i$ modulo $(p, v_1, \cdots, v_{i-1})$ in $z^{-1}E^0(BC_p)$. To see this, consider the homotopy ring map given by the composite

$$\mathrm{BP} \otimes \mathrm{BP} \xrightarrow{f \otimes f} E \otimes E \xrightarrow{can \otimes \varphi} E^{tC_p},$$

where can denotes the canonical map that exists because we are considering E with trivial C_p-action. Taking homotopy groups, v_i will be the image of $\eta_L(v_i) \in \pi_*(\mathrm{BP} \otimes \mathrm{BP})$, while $\varphi(v_i)$ will be the image of $\eta_R(v_i)$. The result then follows from the fact that $\eta_L(v_i) \equiv \eta_R(v_i)$ modulo $(p, v_1, \cdots, v_{i-1})$ in $\pi_*(\mathrm{BP} \otimes \mathrm{BP})$. $\square$

Corollary 2.5.1. *For each $0 \le k \le h$, $\varphi(u_k) \equiv 0$ modulo $p, \cdots, u_k$ in $z^{-1}E^0(BC_p)$.*

It will be useful to formulate the above corollary in terms of P, instead of φ. To do so, recall (by, e.g., the formula at the top of [Goe+05, p.788]) that the p-series $[p](z) \in E_0[\![z]\!]$ satisfies the equation

$$[p](z) \equiv u_k z^{p^k} \text{ modulo } p, \cdots, u_{k-1}, z^{p^k+1},$$

for each $0 \le k \le h$. We may thus make the following definition:

Definition 2.6. For each $0 < k \le h$, let $g_k(z) \in E_0[\![z]\!]$ denote the power series such that $z^{p^k} g_k(z)$ is obtained from the p-series $[p](z)$ by setting $p, u_1, \cdots, u_{k-1} = 0$.

Remark 2.7. By the Weierstrass preparation theorem [HKR00, Lemma 5.1], z is not a zero-divisor in $E_0[\![z]\!]/(p, u_1, \cdots, u_{k-1}, g_k(z))$.

Proposition 2.8. *In $E^0(BC_p)$, the following congruence holds for each $0 \le k \le h-1$:*

$$P(u_k) \equiv 0 \text{ modulo } p, u_1, \cdots, u_k, g_{k+1}(z).$$

Proof. By Corollary 2.5.1 there exists some positive integer i such that $z^i P(u_k)$ is trivial in $E^0(BC_p)$ modulo $p, u_1, \cdots, u_k$. The result now follows from the fact that z is not a zero-divisor in $E^0(BC_p)/(p, u_1, \cdots, u_k, g_{k+1}(z))$. $\square$

3. The weight p power operation in a D_p-E-algebra

In the preceding section, we studied the weight p power operation $P : E^0 \to E^0(BC_p)$, where E is the height $h+1$ Morava E-theory. In this section, we study the analogous operation for any $K(h+1)$-local D_p-E-algebra R:

Definition 3.1. A $K(h+1)$-local D_p-E-algebra is a $K(h+1)$-local E-module R equipped with the data of:

- Multiplication and unit maps $R^{\otimes_E 2} \to R$ and $E \to R$, making R into a commutative and associative algebra object in the homotopy category of E-modules.
- A factorization of the p-fold multiplication map $R^{\otimes_E p} \to R$ through the projection $R^{\otimes_E p} \to (R^{\otimes_E p})_{hC_p}$.

Example 3.2. If A is a D_p-algebra in spectra, then $L_{K(h+1)}(E \otimes A)$ is naturally a $K(h+1)$-local D_p-E-algebra.

If R is a $K(h+1)$-local D_p-E-algebra, then R^{BC_p} is in particular an E^{BC_p}-module, so it makes sense to speak of the quotient of R^{BC_p} by any sequence of elements in $E^0(BC_p)$. With this in mind, the main result of this section is the following theorem:

Theorem 3.3. *Let R be a $K(h+1)$-local D_p-E-algebra, $k \geq 1$, and suppose that $x \in \pi_0 R$ has the property that x maps to 0 in $\pi_0(R/(p, u_1, \cdots, u_k))$. Then $P(x) \in \pi_0(R^{BC_p})$ maps to zero in $\pi_0\left(R^{BC_p}/(p, u_1, \cdots, u_k, g_{k+1}(z))\right)$.*

Let us explain for a moment the complications involved with proving Theorem 3.3. For each integer $0 \leq i \leq k$, the quotient E/u_i admits the structure of a homotopy E-algebra. The above theorem would be straightforward, even before coning off $g_{k+1}(z)$, if each E/u_i furthermore admitted a D_p-E-algebra structure. Unfortunately, the E/u_i fail to admit such structure: the first obstruction is the fact that $P(u_i) \in E^0(BC_p)$ does not map to 0 in $E^0(BC_p)/u_i$. However, $P(u_i)$ does project to 0 in $E^0(BC_p)/(p, u_1, \cdots, u_i, g_{i+1}(z))$, and it is this fact that will allow us to prove Theorem 3.3.

The remainder of this section will be entirely devoted to the proof of Theorem 3.3. Before continuing, it will be helpful to introduce some language from equivariant homotopy theory. Our use of equivariant language is entirely confined to the remainder of this section.

Definition 3.4. The category of E-modules with C_p-action, also known as the category of C_p-equivariant E-modules, is the category of functors from the groupoid BC_p to the category of E-modules. If M is an E-module, we will often write $M^{\otimes_E p}$ to denote the C_p-equivariant E-module $M \otimes_E M \otimes_E \cdots \otimes_E M$ where the C_p action permutes p tensor factors. For readability, when it is clear from context and in this section only, we will abbreviate $M^{\otimes_E p}$ by $M^{\otimes p}$.

Remark 3.5. The symmetric monoidal category of $K(h+1)$-local E-modules with C_p-action is equivalent to the category of $K(h+1)$-local modules over the $\mathbb{E}_\infty$-ring spectrum E^{BC_p} [HL13, Corollary 5.4.4]. Given a class $x \in E^0(BC_p)$, and an E^{BC_p}-module M, we may form the E^{BC_p}-module $M/x = E^{BC_p}/x \otimes_{E^{BC_p}} M$. In particular, if M is a $K(h+1)$-local E-module with C_p-action, and $x \in E^0(BC_p)$, it makes sense to speak of the C_p-equivariant E-module M/x.

Remark 3.6. Expanding on the above remark, given any sequence of elements $x_1, x_2, \cdots, x_m$ in $E^0(BC_p)$, and a $K(h+1)$-local C_p-equivariant E-module M, we may form the C_p-equivariant E-module $M/(x_1, \cdots, x_m)$. In fact, since $E^*(BC_p)$ is concentrated in even degrees, there exists an $\mathbb{E}_1$-E^{BC_p}-algebra structure on $E^{BC_p}/(x_1, \cdots, x_m)$, and a choice of such $\mathbb{E}_1$-algebra structure allows us to view C_p-equivariant E-module $M/(x_1, \cdots, x_m)$ as a module over $E^{BC_p}/(x_1, \cdots, x_m)$ [HW18].

The following lemma is the key technical fact powering our proof of Theorem 3.3. It concerns the C_p-equivariant pth tensor power of the non-equivariant u_i-Bockstein map $\Sigma^{-1}E/u_i \to E$.

Lemma 3.7. *For any $0 \le i \le h-1$, the C_p-equivariant E-module map*

$$\left(\Sigma^{-1} E/u_i\right)^{\otimes p} \to E^{\otimes p} = E$$

admits a C_p-equivariant section after modding out by $(p, u_1, \cdots, u_i, g_{i+1}(z))$. In other words, there is a section of the C_p-equivariant $E/(p, u_1, \cdots, u_i, g_{i+1}(z))$-module map

$$\left(\Sigma^{-1} E/u_i\right)^{\otimes p}/(p, u_1, \cdots, u_i, g_{i+1}(z)) \to E^{\otimes p}/(p, u_1, \cdots, u_i, g_{i+1}(z)) = E/(p, u_1, \cdots, u_i, g_{i+1}(z)).$$

Before proving the lemma, we recall the following useful fact:

Remark 3.8. The $\mathbb{E}_\infty$-ring spectrum E has an $\mathbb{F}_p^\times$ action preserving $\pi_0 E$. The fixed points of this action form an $\mathbb{E}_\infty$-ring spectrum that we denote by $\hat{E}$, which has homotopy groups concentrated in degrees that are multiples of $2p-2$. The inclusion of homotopy fixed points is an $\mathbb{E}_\infty$-ring map $\hat{E} \to E$, which is an isomorphism on π_0. We use the existence of $\hat{E}$ only to prove the lemma below; the key useful property is the triviality of homotopy groups in degrees that are not multiples of $2p-2$.

Proof of Lemma 3.7. To understand this point, it is (at least for the author) helpful to think of the C_p-equivariant E-module $(E/u_i)^{\otimes p}$ as a Thom spectrum. Indeed, the non-equivariant E-module E/u_i is the Thom spectrum of the map

$$S^1 \xrightarrow{1+u_i} \mathrm{BGL}_1(E).$$

The $\mathbb{E}_\infty$-ring structure on E induces a C_p-equivariant map $\mathrm{BGL}_1(E)^{\times p} \to \mathrm{BGL}_1(E)$, and we can use this to make a C_p-equivariant map

$$(S^1)^{\times p} \xrightarrow{(1+u_i)^{\times p}} (\mathrm{BGL}_1(E))^{\times p} \to \mathrm{BGL}_1(E)$$

that has Thom spectrum exactly the C_p-equivariant E-module $(E/u_i)^{\otimes p}$. Our goal will be accomplished upon proving that the composite map

$$(S^1)^{\times p} \to \mathrm{BGL}_1(E) \to \mathrm{BGL}_1(E/(p, u_1, \cdots, u_i, g_{i+1}(z)))$$

is C_p-equivariantly nullhomotopic. In fact, at primes $p > 2$ it is easier to prove the slightly stronger statement that the composite

$$(S^1)^{\times p} \to \mathrm{BGL}_1(\hat{E}) \to \mathrm{BGL}_1(\hat{E}/(p, u_1, \cdots, u_i, g_{i+1}(z)))$$

is C_p-equivariantly nullhomotopic.

We accomplish this by examining the natural equivariant cell decomposition of the pointed C_p space $(S^1)^{\times p}$. Since the non-equivariant space S^1 admits a cell decomposition with one 0-cell and one 1-cell, $(S^1)^{\times p}$ admits an equivariant cell decomposition with:

- One 0-cell.
- For each $0 < k < p$, $\frac{\binom{p}{k}}{p}$ induced cells with boundaries $(C_p)_+ \wedge S^{k-1}$.
- A ρ-cell (i.e., a cell with boundary $S^{\rho-1}$), where ρ is the real regular representation of C_p.

Noting that S^ρ is built from S^1 by attaching induced cells of dimension at least 2 and at most p, we see that it suffices to check the following three facts:

(1) The composite

$$S^1 \xrightarrow{\Delta} (S^1)^{\times p} \xrightarrow{(1+u_i)^{\times p}} \mathrm{BGL}_1(\hat{E}) \to \mathrm{BGL}_1(\hat{E}/(p, u_1, \cdots, u_i, g_{i+1}(z)))$$

is nullhomotopic.

(2) The composite

$$(C_p)_+ \wedge S^1 \to (S^1)^{\times p} \to \mathrm{BGL}_1(\hat{E}) \xrightarrow{(1+u_i)^{\times p}} \mathrm{BGL}_1(\hat{E}/(p, u_1, \cdots, u_i, g_{i+1}(z)))$$

is nullhomotopic.

(3) For $1 < k \leq p$, all maps

$$(C_p)_+ \wedge S^k \to \mathrm{BGL}_1(\hat{E}/(p, u_1, \cdots, u_i, g_{i+1}(z)))$$

are nullhomotopic.

Statement (1) is the claim that $P(1+u_i) = 1$ in $E^0(BC_p)/(p, u_1, \cdots, u_i, g_{i+1}(z))$. Since we have killed $g_{i+1}(z)$, we have killed the generator $[p](z)/z$ of the transfer ideal in $E^0(BC_p)$, and so we may write $P(1+u_i) = P(1) + P(u_i) = 1 + P(u_i)$. Now we conclude (1) from Proposition 2.8.

To see statements (2) and (3), we recall that a map

$$(C_p)_+ \wedge S^k \to \mathrm{BGL}_1(\hat{E}/(p, u_1, \cdots, u_i, g_{i+1}(z)))$$

is the data of a non-equivariant map from S^k to the non-equivariant ring underlying $\hat{E}/(p, u_1, \cdots, u_i, g_{i+1}(z))$. Because $g_{i+1}(z)$ is u_{i+1} plus a multiple of z, the non-equivariant ring underlying $\hat{E}/(p, u_1, \cdots, u_i, g_{i+1}(z))$ is the non-equivariant quotient $\hat{E}/(p, u_1, \cdots, u_i, u_{i+1})$. Since $(p, u_1, \cdots, u_{i+1})$ is a regular sequence in $\pi_0\hat{E} = \pi_0 E$, we may conclude (3) by the sparsity highlighted in Remark 3.8. To see statement (2), we need to check that $1 + u_i = 1$ in $E_0/(p, u_1, \cdots, u_i, u_{i+1})$, which follows from the fact that we have coned off u_i. $\square$

The next corollary studies the C_p-equivariant pth tensor power of the non-equivariant iterated Bockstein

$$\Sigma^{-k}E/(p, u_1, \cdots, u_k) \to \cdots \to \Sigma^{-2}E/(p, u_1) \to \Sigma^{-1}E/p \to E.$$

Note that this iterated Bockstein may also be described as the tensor product of the Bocksteins

$$\Sigma^{-1}E/u_i \to E$$

as i ranges from 0 to k.

Corollary 3.8.1. *For each $0 \leq k \leq h-1$, there is a C_p-equivariant section of the map*

$$(\Sigma^{-k}E/(p, \cdots, u_k))^{\otimes p} \to E^{\otimes p} = E$$

after modding out by $(p, u_1, \cdots, u_k, g_{k+1}(z))$.

Proof. We tensor together the sections provided by Lemma 3.7 for $i = k, k-1, \cdots, 0$. This gives a section of the map

$$(\Sigma^{-k}E/(p, \cdots, u_k))^{\otimes p} \to E$$

after first coning off by $(p, u_1, \cdots, u_k, g_{k+1}(z))$, then coning off $(p, u_1, \cdots, u_{k-1}, g_k(z))$, etc. Each of $p, u_1, \cdots, u_{k-1}, g_1(z), g_2(z), \cdots, g_k(z)$ is trivial modulo $(p, u_1, \cdots, u_k, g_{k+1}(z))$. So the above coning off process yields a direct sum of copies of $E/(p, u_1, \cdots, u_k, g_{k+1}(z))$, and we may project onto a single copy. $\square$

Proof of Theorem 3.3. By assumption, we are given a D_p-E-algebra R and a class $x \in \pi_0(R)$ such that x maps to 0 in $\pi_0(R/(p, \cdots, u_k))$. Consider now the following commutative diagram of C_p-equivariant E-modules

$$\begin{tikzcd}
(\Sigma^{-k}E/(p, \cdots, u_k))^{\otimes p} \arrow[r, "f"] & E \arrow[r, "x^{\otimes p}"] & R^{\otimes p} \\
\Sigma^{-1}(\Sigma^{-k}E/(p, \cdots, u_k))^{\otimes p}/g_{k+1}(z) \arrow[u, "g"] \arrow[r, "h"] & \Sigma^{-1}E/g_{k+1}(z) \arrow[u, "j"] &
\end{tikzcd}$$

Here, the map j is the $g_{k+1}(z)$ Bockstein $\Sigma^{-1}E/g_{k+1}(z) \to E$, where E is considered as a C_p-equivariant E-module with trivial action. The map g is obtained by tensoring j with the C_p-equivariant E-module $(\Sigma^{-k}E/(p, \cdots, u_k))^{\otimes p}$. The map f is the pth tensor power of the Bockstein $\Sigma^{-k}E/(p, \cdots, u_k) \to E$. Finally, the map h is obtained by tensoring f with $\Sigma^{-1}E/g_{k+1}(z)$.

The composite $x^{\otimes p} \circ f$ is trivial, because it is the pth tensor power of the non-equivariant composite

$$\Sigma^{-k} E/(p, \cdots, u_k) \to E \to R$$

that is trivial by assumption. In particular, this implies that $x^{\otimes p} \circ j \circ h$ is trivial. By Corollary 3.8.1, $x^{\otimes p} \circ j$ becomes trivial after further coning off $p, u_1, \cdots, u_k$.

Finally, we consider the C_p-equivariant map $R^{\otimes p} \to R$ into R with trivial action, which is given by the assumed D_p-E-algebra structure on R. From the above discussion, we learn in particular that the composite

$$\Sigma^{-1} E/g_{k+1}(z) \to E = (E^{\otimes p}) \xrightarrow{x^{\otimes p}} (R^{\otimes p}) \to R \to R/(p, u_1, \cdots, u_k)$$

is nullhomotopic. If we compose all but the initial map and final maps in this chain, we obtain $P(x) \in R^0(BC_p)$. The full composite precisely records the image of $P(x)$ in $\pi_0(R^{BC_p}/p, \cdots, u_k, g_{k+1}(z))$. □

4. The weight p power operation modulo $(p, u_1, \cdots, u_{h-1})$

In this final section, we study in greater detail the mod $(p, u_1, \cdots, u_{h-1})$ weight p power operation on a $K(h+1)$-local D_p-E-algebra R, beginning with the special case $R = E$. In particular, Theorem 3.3 ensures that the following definition is sensible.

Definition 4.1. Let

$$\overline{P} : \mathbb{F}_p[\![u_h]\!] \to \mathbb{F}_p[\![u_h, z]\!]/(g_h(z))$$

be the unique ring homomorphism fitting into the following diagram:

$$\begin{array}{ccc}
E_0 & \xrightarrow{\quad\quad P \quad\quad} & E^0(BC_p) \\
\downarrow & & \downarrow \\
E_0/(p, u_1, \cdots, u_{h-1}) & \xrightarrow{\overline{P}} & E^0(BC_p)/(p, u_1, \cdots, u_{h-1}, g_h(z)),
\end{array}$$

where the vertical maps are the natural quotient homomorphisms.

The codomain of $\overline{P}$ can be understood fairly explicitly, using the Weierstrass preparation theorem as in the following proposition:

Proposition 4.2. *There exists a polynomial $g(z) \in \mathbb{F}_p[\![u_h]\!][z]$ such that:*

- *In the ring $\mathbb{F}_p[\![u_h, z]\!]$, $g(z) = U g_h(z)$ where U is a unit.*
- *$g(z)$ is monic, of degree $p^{h+1} - p^h$, and $g(z) \equiv z^{p^{h+1}-p^h}$ modulo u_h.*
- *The constant term of $g(z)$ is divisible by u_h but not u_h^2.*

Proof. Recall that $[p](z) \equiv u_h z^{p^h}$ modulo $(p, u_1, \cdots, u_{h-1}, z^{p^h+1})$, and that $[p](z)$ is a unit multiple of $z^{p^{h+1}}$ modulo $(p, u_1, \cdots, u_h)$. The result then follows from the Weierstrass preparation theorem [HKR00, Lemma 5.1]. □

By the first of the above bullet points, we may describe the codomain $E^0(BC_p)/(p, u_1, \cdots, u_{h-1}, g_h(z))$ of $\overline{P}$ equally well as $E^0(BC_p)/(p, u_1, \cdots, u_{h-1}, g(z))$. We then have the following proposition:

Proposition 4.3. *The ring $E^0(BC_p)/(p, u_1, \cdots, u_{h-1}, g_h(z)) \cong \mathbb{F}_p[\![u_h]\!][z]/(g(z))$ is a discrete valuation ring, with uniformizer z.*

Proof. The ring $\mathbb{F}_p[\![u_n]\!][z]$ is a UFD, so Eisenstein's criterion applies to show that $g(z)$ is irreducible. It follows that the quotient $\mathbb{F}_p[\![u_n]\!][z]/g(z)$ is a local domain. Furthermore, $\mathbb{F}_p[\![u_n]\!][z]/(z, g(z)) \cong \mathbb{F}_p$, because the constant term of $g(z)$ is divisible by u_h but not u_h^2. It follows that z is a uniformizer. □

Convention 4.4. It will be convenient to scale the discrete valuations on $\mathbb{F}_p[\![u_h]\!]$ and $\mathbb{F}_p[\![u_h, z]\!]/(g(z))$ so that u_h has valuation 1 in both rings. This means that z has valuation $\frac{1}{p^{h+1}-p^h}$, and the valuation of any non-zero element in $\mathbb{F}_p[\![u_h, z]\!]/(g(z))$ is a multiple of $\frac{1}{p^{h+1}-p^h}$.

Proposition 4.5. *The class $\overline{P}(u_h) \in \mathbb{F}_p[\![u_h, z]\!]/g(z)$ has valuation $\frac{p-1}{p^{h+1}-p^h}$.*

Proof. The valuation on $\mathbb{F}_p[\![u_h, z]\!]/g(z)$ extends to a valuation on $z^{-1}\mathbb{F}_p[\![u_h, z]\!]/g(z)$. In this latter ring, we have the equation

$$\varphi(u_h) = u_h \frac{\varphi(u^{p^h-1})}{u^{p^h-1}} = u_h \left(\frac{\varphi(u)}{u}\right)^{p^h-1},$$

by Lemma 2.5. It therefore suffices to prove that the valuation of $\varphi(u)/u$ is $\frac{-(p-1)}{p^{h+1}-p^h}$, since this will imply that the valuation of $\varphi(u_h)$ is

$$1 - \frac{(p-1)(p^h-1)}{p^{h+1}-p^h} = \frac{p-1}{p^{h+1}-p^h}.$$

The valuation of $\frac{\varphi(u)}{u}$ does not depend on the choice of degree -2 unit u, so we may as well study a particularly convenient choice of degree -2 unit. By a theorem of Ando [And95] generalized by Zhu [Zhu20, Corollary 8.17], there exists an H_∞ ring homomorphism $\mathrm{MUP} \to E$, where MUP is the periodic complex bordism obtained as the Thom spectrum of the J-homomorphism $\mathrm{BU} \times \mathbb{Z} \to \mathrm{pic}(\mathbb{S})$. Assuming that u is the image of the degree -2 generator of $\pi_*(\mathrm{MUP})$ under such an H_∞-ring map, we have (as in [NS18, pg. 339]) that $\varphi(u)$ is u^p divided by the Euler class of the reduced real regular representation of C_p. In symbols, recalling that $z \in E^0(BC_p)$ denotes u^{-1} times the Euler class of the standard representation of C_p on $\mathbb{C}$, we have that

$$\varphi(u) = \frac{u}{z([2](z)) \cdots ([p-1](z))}.$$

For each integer k between 1 and $p-1$, $[k](z) = kz + \mathcal{O}(z^2)$ has valuation $\frac{1}{p^{h+1}-p^h}$, and it follows that $\varphi(u)/u$ has valuation $\frac{-(p-1)}{p^{h+1}-p^h}$. □

Corollary 4.5.1. *Let $x \in \mathbb{F}_p[\![u_h]\!]$ denote any non-zero element of positive valuation. Then $\overline{P}(x) \in \mathbb{F}_p[\![u_h, z]\!]/g_h(z)$ has strictly smaller valuation.*

Proof. Since we may write an arbitrary $x \in \mathbb{F}_p[\![u_h]\!]$ as a power of u_h times a unit, it suffices to check this for powers of u_h, where it follows from Proposition 4.5 since we have assumed $h > 0$. □

Corollary 4.5.2. *Suppose that R is a $K(h+1)$-local D_p-E-algebra, and that $x \in E_0$ is a non-zero element such that the image of x in $\pi_0(R/(p, \cdots, u_{h-1}))$ is trivial. Then, if x has positive valuation in $E_0/(p, \cdots, u_{h-1}) \cong \mathbb{F}_p[\![u_h]\!]$, there exists an element $y \in E_0$ which:*

(1) *Also maps to 0 in $\pi_0(R/(p, \cdots, u_{h-1}))$.*
(2) *Has valuation in $E_0/(p, \cdots, u_{h-1}) \cong \mathbb{F}_p[\![u_h]\!]$ strictly smaller than that of x.*

Proof. Let α denote the image of $P(x)$ in $E^0(BC_p)/(p, u_1, \cdots, u_{h-1}, g_h(z))$. By Theorem 3.3, the image of α in $R^0(BC_p)/(p, u_1, \cdots, u_{h-1}, g_h(z))$ is trivial.

By the isomorphism

$$E^0(BC_p)/(p, u_1, \cdots, u_{h-1}, g_h(z)) \cong \mathbb{F}_p[\![u_h]\!][z]/(g(z)),$$

where $g(z)$ is monic of degree $p^{h+1} - p^h$, we may write

$$\alpha = a_0 + a_1 z + a_2 z^2 + \cdots + a_{p^{h+1}-p^h-1} z^{p^{h+1}-p^h-1}$$

for some collection of coefficients $a_i \in \mathbb{F}_p[\![u_h]\!]$.

By Corollary 4.5.1, the valuation of α is strictly less than the valuation of x in $\mathbb{F}_p[\![u_h]\!] \cong E_0/(p, \cdots, u_{h-1})$. It follows that there exists some k, with $0 \leq k < p^{h+1} - p^h$, such that a_k has valuation strictly lower than that of x.

Noting that $E^0(BC_p)/(p, u_1, \cdots, u_{h-1}, g_h(z))$ is a free $E_0/(p, u_1, \cdots, u_{h-1})$ module, with basis dual to the functions that pick out the coefficients a_i, we may cap with the class that picks out a_k to learn that a_k maps to 0 in $\pi_0(R/(p, \cdots, u_{h-1}))$. We may then take y to be any lift of $a_k \in E_0/(p, \cdots, u_{h-1})$ to a class in E_0. □

Corollary 4.5.3. *Suppose that R is a $K(h+1)$-local D_p-E-algebra, and that $x \in E_0$ maps to zero in $\pi_0(R/(p, \cdots, u_{h-1}))$. Then, if x maps to a non-zero element of $E_0/(p, \cdots, u_{h-1}) \cong \mathbb{F}_p[\![u_h]\!]$, R is trivial.*

Proof. We learn from the above that $1 = 0$ in the homotopy groups of the ring spectrum $R/(p, \cdots, u_{h-1})$. Since R is $K(h+1)$-local, this implies $R \simeq 0$. □

We now turn to the proof of the main theorem of this article, which we restate for the reader's convenience:

Theorem 4.6. *Suppose that R is a D_p-algebra in spectra. For any height h at the prime p, if R is $K(h)$-acyclic then R is also $K(h+1)$-acyclic.*

Proof. To check that R is $K(h+1)$-acyclic, it suffices to check that $L_{K(h+1)}(R \otimes E)$ is trivial. Furthermore, since $K(h) \otimes L_{K(h+1)}(R \otimes E)$ is a module over the homotopy ring $K(h) \otimes R$, $L_{K(h+1)}(R \otimes E)$ will be $K(h)$-acyclic if R is $K(h)$-acyclic.

We may therefore assume without loss of generality that R is a $K(h+1)$-local D_p-E-algebra. If R is $K(h)$-acyclic, then it is $T(h)$-acyclic for some telescope $T(h) = v_h^{-1}\mathbb{S}/(p_0^i, v_1^{i_1}, \cdots, v_{h-1}^{i_{h-1}})$ [Lan+22, Lemma 2.3], so there must be some positive integer k such that $(u_h)^k \in \pi_0 E$ maps to zero in $\pi_0\left(R/(p, \cdots, u_{h-1})\right)$. In particular, we may apply Corollary 4.5.3 and learn that $R \simeq 0$. □

Following the arguments of [Cla+20a], we also have the following variant of the main theorem:

Corollary 4.6.1. *Suppose that R is a D_p-algebra in spectra, and let R^{tC_p} denote the Tate construction of R with trivial C_p action. For any height h at the prime p, if R^{tC_p} is $K(h)$-acyclic then R is also $K(h+1)$-acyclic.*

Proof. Note that $\left(L_{K(h+1)}(E \otimes R)\right)^{tC_p}$ is a module over R^{tC_p}, so the former is $K(h)$-acyclic whenever the latter is. The result therefore follows by the combination of the proposition below with Theorem 1.3. □

Proposition 4.7. *Suppose that R is a $K(h+1)$-local homotopy E-algebra, and let R^{tC_p} denote the Tate construction of R with trivial C_p action. If R^{tC_p} is $K(h)$-acyclic, then R is $K(h)$-acyclic.*

Proof. We follow the arguments of Clausen–Mathew–Naumann–Noel from [Cla+20a, §4]. First, one checks that E^{hC_p} is a free and finitely generated E-module, which follows as in many of our above arguments by the Weierstrass preparation theorem. It follows that the natural comparison map $E^{hC_p} \otimes_E M \to M^{hC_p}$ is an equivalence for any $K(h+1)$-local E-module M, considered with trivial C_p-action. This then implies that $E^{tC_p} \otimes_E M \simeq M^{tC_p}$.

It then remains to check that $K(h) \otimes_E E^{tC_p} \otimes_E R \simeq 0$ if and only if $K(h) \otimes_E R \simeq 0$, which follows from the explicit description $\pi_*(K(h) \otimes_E E^{tC_p}) \cong \mathbb{F}_p[\![u_h]\!][z^{\pm 1}][u_h^{-1}]/g(z)$. □

References

[And95] Matthew Ando. "Isogenies of formal group laws and power operations in the cohomology theories E_n". In: *Duke Math. J.* 79.2 (1995), pp. 423–485. ISSN: 0012-7094. DOI: 10.1215/S0012-7094-95-07911-3. URL: https://doi-org.libproxy.mit.edu/10.1215/S0012-709

[Cla+20a] Dustin Clausen et al. *Descent and vanishing in chromatic algebraic K-theory via group actions. arXiv preprint arXiv:2011.08233.* 2020.

[Cla+20b] Dustin Clausen et al. "Descent in algebraic K-theory and a conjecture of Ausoni-Rognes". In: *J. Eur. Math. Soc. (JEMS)* 22.4 (2020), pp. 1149–1200. ISSN: 1435-9855. DOI: `10.4171/JEMS/942`. URL: `https://doi-org.libproxy.mit.edu/10.4171/JEMS/942`.

[DH04] Ethan S. Devinatz and Michael J. Hopkins. "Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups". In: *Topology* 43.1 (2004), pp. 1–47. ISSN: 0040-9383. DOI: `10.1016/S0040-9383(03)00029-6`. URL: `https://doi-org.libproxy.mit.edu/10.1`

[GH04] P. G. Goerss and M. J. Hopkins. "Moduli spaces of commutative ring spectra". In: *Structured ring spectra*. Vol. 315. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2004, pp. 151–200. DOI: `10.1017/CBO9780511529955.009`. URL: `https://doi.org/10.1017/CBO9780511529955.009`.

[Goe+05] P. Goerss et al. "A resolution of the $K(2)$-local sphere at the prime 3". In: *Ann. of Math. (2)* 162.2 (2005), pp. 777–822. ISSN: 0003-486X. DOI: `10.4007/annals.2005.162.777`. URL: `https://doi-org.libproxy.mit.edu/10.4007/annals.2005.162.777`.

[HKR00] Michael J. Hopkins, Nicholas J. Kuhn, and Douglas C. Ravenel. "Generalized group characters and complex oriented cohomology theories". In: *J. Amer. Math. Soc.* 13.3 (2000), 553–594 (electronic). ISSN: 0894-0347. DOI: `10.1090/S0894-0347-00-00332-5`. URL: `http://dx.doi.org/10.1090/S0894-0347-00-00332-5`.

[HL13] Michael Hopkins and Jacob Lurie. *Ambidexterity in K(n)-Local Stable Homotopy Theory.* Available as of July 2022 at `https://www.math.ias.edu/~lurie/papers/Ambidexterity.pdf`. 2013.

[Hov99] Mark Hovey. *Algebraic Topology Problem List.* Available as of July 2022 at `https://www-users.cse.umn.edu/~tlawson/hovey/misc.html`. 1999.

[HW18] Jeremy Hahn and Dylan Wilson. *Quotients of even rings. arXiv preprint arXiv:1809.04723.* 2018.

[Lan+22] Markus Land et al. *Purity in chromatically localized algebraic K-theory. arXiv preprint arXiv:2001.10425v4.* 2022.

[MNN15] Akhil Mathew, Niko Naumann, and Justin Noel. "On a nilpotence conjecture of J. P. May". In: *J. Topol.* 8.4 (2015), pp. 917–932. ISSN: 1753-8416. DOI: `10.1112/jtopol/jtv021`. URL: `http://dx.doi.org/10.1112/jtopol/jtv021`.

[NS18] Thomas Nikolaus and Peter Scholze. "On topological cyclic homology". In: *Acta Math.* 221.2 (2018), pp. 203–409. ISSN: 0001-5962. DOI: `10.4310/ACTA.2018.v221.n2.a1`. URL: `https://doi-org.libproxy.mit.edu/10.4310/ACTA.2018.v221.n2.a1`.

[Yua21] Allen Yuan. *Examples of chromatic redshift in algebraic K-theory.* arXiv preprint arXiv:2111.10837. 2021.

[Zhu20] Yifei Zhu. "Norm coherence for descent of level structures on formal deformations". In: *J. Pure Appl. Algebra* 224.10 (2020), pp. 106382, 35. ISSN: 0022-4049. DOI: `10.1016/j.jpaa.2020.1063` URL: `https://doi-org.libproxy.mit.edu/10.1016/j.jpaa.2020.106382`.

Email address: `jhahn01@mit.edu`

Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Ave, Cambridge, MA 02139