ON THE BOUSFIELD CLASSES OF $H_\infty$-RING SPECTRA

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Abstract. We prove that any $K(n)$-acyclic, $H_\infty$-ring spectrum is $K(n+1)$-acyclic, affirming an old conjecture of Mark Hovey.

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Throughout this paper, all spectra will be $p$-local for a fixed prime $p$.

1. Introduction

A bedrock result of chromatic homotopy theory is that any $K(n)$-acyclic, finite spectrum is $K(n-1)$-acyclic. Our goal here is to prove that $H_\infty$-ring spectra enjoy the opposite phenomenon:

Theorem 1.1. Suppose $R$ is a $K(n)$-acyclic, $H_\infty$-ring spectrum. Then $R$ is $K(n+1)$-acyclic.

Corollary 1.1.1. Suppose $R$ is a complex-orientable, $H_\infty$-ring spectrum that kills a finite complex. Then $R$ has the Bousfield class of $E(n)$ for some $n$.

These results settle ‘Miscellaneous Problem 2’ from Mark Hovey’s 1999 list of unsolved problems in algebraic topology [Hov99].

We will prove Theorem 1.1 for $n > 0$. The theorem is already known when $n = 0$, where it is a consequence of an old conjecture due to J.P. May:

May Nilpotence Conjecture [MNN15, Theorem 2.1]. If $R$ is an $H_\infty$-ring spectrum, and $R \otimes \mathbb{Q} \simeq 0$, then $R$ is $K(n)$-acyclic for all $n > 0$.

The first written proof of May’s conjecture is due to Mathew, Naumann, and Noel [MNN15], and these authors found spectacular applications in joint work with Clausen [NN16].

Let $E$ denote the height $n+1$ Morava $E$-theory with $\pi_0 E \cong \mathbb{Z}_p[[u_1, u_2, ..., u_n]]$. Standard techniques, which we review in Section 2, reduce Theorem 1.1 to the following Theorem 1.2.

Theorem 1.2. Suppose $R$ is a $K(n+1)$-local, $H_\infty$-E-algebra such that, in $\pi_0 R$, some power of $u_n$ is in the ideal $(p, u_1, ..., u_{n-1})$. Then 1 is in the ideal $(p, u_1, ..., u_{n-1})$. 
Our proof of Theorem 1.2 is by infinite descent: we use power operations to show that, if some power of $u_n$ lies in $(p, u_1, ..., u_{n-1})$, then so must some lower power. This is analogous to the technique featured in [MNN15].

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2. USEFUL REDUCTIONS

In this section we reduce Theorem 1.1 to Theorem 1.2. Since the May Nilpotence Conjecture is proved [MNN15], we need only prove Theorem 1.1 when $n > 0$.

Fix such an integer $n > 0$ for the remainder of the paper. Recall that, for us, $E$ denotes a height $n + 1$ variant of Morava $E$-theory with $\pi_0 E \cong \mathbb{Z}_p[[u_1, u_2, ..., u_n]]$. For more details on $E$, see Section 3.

**Lemma 2.1.** Suppose $R$ is a spectrum. Then $R$ is $K(n + 1)$-acyclic if and only if $R \wedge E$ is.

**Proof.** This is Proposition 3.4 of [HS99]. The argument is that $K(n + 1)$ is a field spectrum, and so $K(n + 1) \wedge E$ splits as a wedge of suspensions of $K(n + 1)$. It follows that $K(n + 1) \wedge E \wedge R$ is a wedge of suspensions of $K(n + 1) \wedge R$.

By an $H_\infty$-$E$-algebra we simply mean an $H_\infty$-ring spectrum $R$ equipped with a map of $H_\infty$-rings $E \to R$. A small piece of this structure is a ring map $\pi_0 E \to \pi_0 (R)$, which allows us to speak of $u_1, u_2, ..., u_n \in \pi_0 (R)$.

**Lemma 2.2.** Suppose $R$ is a $K(n)$-acyclic, $H_\infty$-$E$-algebra. Then, in $\pi_0 (R)$, some power of $u_n$ is in the ideal $(p, u_1, ..., u_{n-1})$.

**Proof.** Let $S/I$ denote a type $n$ Moore spectrum $S/(p^{i_0}, v_1^{i_1}, ..., v_{n-1}^{i_{n-1}})$, as in [HS99] 3.4. The spectrum $X = R \wedge S/I$ is $K(n)$-acyclic by assumption, but also $K(j)$-acyclic for $j < n$. Since $R$ is $L_{n+1}$-local, $X$ is $L_{n+1}$-local, but $L_0 X \simeq 0$. By [HS99] 7.10, $L_0^j X \simeq 0$. Also by [HS99] 7.10,

$$L_0^j (R \wedge S/I) \simeq R \wedge T(S/I),$$

where $T(S/I)$ is the telescope of a $v_n$-self map on $S/I$.

On $\pi_0$, the map $R \wedge S/I \to R \wedge T(S/I)$ inverts $u_n$. Since the image of this map is null, it follows that some power of $u_n$ is 0 in $\pi_0 (R \wedge S/I)$.

To finish the proof, I will show that any element in the kernel of $\pi_0 (R) \to \pi_0 (R \wedge S/I)$ is a member of the ideal $(p, u_1, ..., u_{n-1})$. Indeed, we can decompose this map as a composition

$$\pi_0 (R) \to \pi_0 (R \wedge S/p^{i_0}) \to \pi_0 (R \wedge S/(p^{i_0}, v_1^{i_1})) \to \cdots \to \pi_0 (R \wedge S/I).$$

The kernel of the map that kills $v_k^{i_k}$ consists of elements that are multiples of $v_k^{i_k}$, and the result follows.

For the moment assume Theorem 1.2 which the rest of the paper is devoted to proving. We will deduce Theorem 1.1 from this assumption.

**Proof of Theorem 1.1** If $R$ is any $K(n)$-acyclic, $H_\infty$-ring spectrum, then $R \wedge E$ will be a $K(n)$-acyclic, $H_\infty$-$E$-algebra. By Lemma 2.2 some power of $u_n$ is in the ideal $(p, u_1, ..., u_{n-1}) \subseteq \pi_0 (R \wedge E)$. The same fact must be true in $\pi_0 L_{K(n+1)} (R \wedge E)$. By Theorem 1.2 1 is in the ideal...
which is the cap product with the class \( \pi_0(L_{K(n+1)}(R \wedge E)) \). It follows that, upon smashing \( L_{K(n+1)}(R \wedge E) \) with any type \((n+1)\)-Moore spectrum \( M \), one obtains 0. In particular, \( L_{K(n+1)}(R \wedge E) \) is acyclic with respect to the telescope of \( M \), and hence \( K(n+1) \)-acyclic. This implies that \( R \wedge E \) is \( K(n+1) \)-acyclic. By Lemma 2.1, \( R \) is itself \( K(n+1) \)-acyclic.

\[ \text{Corollary 1.1.1} \]
Let \( R \) be a complex-orientable, \( H_\infty \)-ring spectrum that kills a finite complex.
Then \( R \) has the Bousfield class of \( E(n) \) for some \( n \).

\[ \text{Proof.} \]
This follows immediately from [Hov95 1.11], which states that \( R \) has the Bousfield class of some wedge of Morava \( K \)-theories.

\[ \text{\( \square \)} \]

3. Power operations for \( H_\infty \)-\( E \)-algebras

Recall that, given any height \( n+1 \) formal group \( G_0 \) over \( \text{Spec}(\mathbb{F}_p) \), there is a universal deformation \( G_E \) defined over \( \text{Spf} \mathbb{Z}_p[[u_1, u_2, \ldots, u_n]] \). By work of Goerss and Hopkins [GH04, GH05], there is an associated \( E_\infty \)-ring spectrum \( E \), a height \( n+1 \) Morava \( E \)-theory. The coefficient ring \( E_0 = \pi_0 E \cong \mathbb{Z}_p[[u_1, u_2, \ldots, u_n]] \).

Let \( BC_p \) denote the classifying space of the cyclic group with \( p \)-elements. As noted in [HKR00 §5],

\[ E^0(BC_p) \cong E_0[[a]]/[p](a). \]

There is a stable transfer map \( \Sigma^+ \mathcal{E}_p \to \Sigma^+ \mathcal{E} \cong S \). This yields a map \( E_0 \to E^0(BC_p) \), the image of which generates an ideal \( \tau \subset E^0(BC_p) \). A simple calculation with a Gysin sequence [HKR00 6.15] shows that \( \tau = \left( \frac{[p](a)}{a} \right) \).

The total power operation is a ring homomorphism

\[ P : E_0 \to E^0(BC_p)/\tau. \]

In [AHS04 §3], the power operation is described in terms of the moduli problem associated to the ring

\[ D = E^0(BC_p)/\tau \cong E_0[[a]]/[p](a). \]

To summarize their work, the \( E_0 \)-algebra morphism \( E_0 \to D \) specifies a formal group \( G_\text{source} \) over \( \text{Spf}(D) \). There is an isogeny of formal groups \( G_\text{source} \to G_\text{target} \) over \( \text{Spf}(D) \), and this latter formal group is specified by the ring homomorphism \( P : E_0 \to D \). The interested reader may consult [AHS04] or [Str97] to learn more.

Now, suppose that \( x \) is an element of \( E^0_0(BC_p) = \pi_0(L_{K(n+1)}E \wedge \Sigma^+_\mathcal{E} BC_p) \). For each element \( \alpha \in E^0(BC_p) \), we obtain a diagram

\[ \begin{array}{ccc}
E \wedge \Sigma^+_\mathcal{E} BC_p & \xrightarrow{1 \wedge \alpha} & E \wedge E \\
\downarrow & & \downarrow m \\
\mathbb{S} & \xrightarrow{x} & L_{K(n+1)}(E \wedge \Sigma^+_\mathcal{E} BC_p) \rightarrow & E,
\end{array} \]

giving an element in \( E_0 \). Assembling this construction over all \( \alpha \) gives an \( E_0 \)-module map

\[ \phi_x : E^0(BC_p) \to E_0, \]

which is the cap product with the class \( x \).

In the case that the transfer ideal is in the kernel of \( \phi_x \), we obtain an additive operation

\[ \bar{\phi}_x : E_0 \xrightarrow{P} E^0(BC_p)/\tau \xrightarrow{\bar{\phi}} E_0. \]
Suppose now that $R$ is a homotopy commutative $E$-algebra, with associated homomorphisms $\iota : E_0 \to \pi_0 R$ and $\tau : E^0(BC_p) \to R^0(BC_p)$. If $\alpha \in E^0(BC_p)$ is such that $\tau(\alpha) = 0$, then in the diagram

$$E \wedge \Sigma_+ \infty BC_p \xrightarrow{1 \wedge \alpha} E \wedge E \xrightarrow{1 \wedge 1} E \wedge R$$
$$L_{K(n+1)}(E \wedge \Sigma_+ \infty BC_p) \xrightarrow{\iota} E \xrightarrow{m} R,$$

the composite $E \wedge \Sigma_+ BC_p \to R$ is null. If $R$ is furthermore $K(n+1)$-local, the map

$$L_{K(n+1)}(E \wedge \Sigma_+ BC_p) \to R$$

must also be null. When $R$ is a $K(n+1)$-local, $H_\infty$-$E$-algebra, there is a commutative diagram of ring homomorphisms

$$\xymatrix{ \pi_0 R \ar[r] & R^0(BC_p)/\text{tor} \\ \pi_0 E \ar[u] \ar[r]^p & E^0(BC_p)/\text{tor} \ar[u] }.$$

The combined structure ensures that, if $\iota(\beta) = 0$ for some $\beta \in E_0$, then $\iota(\tilde{\phi}(\beta)) = 0$ as well.

4. AN EXPLICIT FORMULA FOR A REDUCED POWER OPERATION

Here we remark that, with careful choice of coordinate, one can explicitly describe the total power operation $P$ (modulo certain ideals). We follow [Str97, §15] to select a height $n+1$ formal group law over $\mathbb{F}_p$ and a coordinate on the resultant $G_E$. The multiplication on $G_E$ is then presented by a formal group law $F(x, y) \in E_0[[x, y]]$ with properties outlined in the following proposition:

**Proposition 4.1.** [Str97, 15.6] For any integer $m > 0$, let $C_p^m$ denote the polynomial in $\mathbb{Z}[x, y]$ defined by

$$C_p^m(x, y) = \frac{x^{p^m} + y^{p^m} - (x + y)^{p^m}}{p}.$$  

Then,

1. For any $0 < k \leq n$,

$$F(x, y) \equiv x + y + u_k C_p^k(x, y) \mod (u_1, u_2, \ldots, u_{k-1}) + (x, y)^{p^k + 1}.$$  

2. $F(x, y) \equiv x + y + C_p^{n+1}(x, y) \mod (u_1, u_2, \ldots, u_n) + (x, y)^{p^{n+1} + 1}.$

**Corollary 4.1.1.** For any integer $i$, we use $[i]_P(x)$ to denote the $i$-series of $x$. For $i \geq 0$, $1 \leq k \leq n$, let $\gamma_{i,k}$ denote $\frac{i}{p} \gamma_{p^k}$. Then,

$$[i]_P(x) \equiv ix + u_k\gamma_{i,k} x^{p^k} \mod (p, u_1, u_2, \ldots, u_{k-1}, x^{p^k + 1}).$$

In particular, $[p]_P(x) \equiv u_k x^{p^k}$. Furthermore,

$$[i]_P(x) \equiv ix + \gamma_{i,k} x^{p^{n+1}} \mod (p, u_1, u_2, \ldots, u_n, x^{p^{n+1} + 1}),$$

and so, modulo this ideal, $[p]_P(x) \equiv x^{p^{n+1}}$. 


Proof. See also [Rez98, 5.7]. This is a simple induction on $i$, the statement being true when $i = 0$. For larger $i$, setting $u_{n+1} = 1$,

$$\begin{align*}
[i]_F(x) &= F([i-1]_F(x), x) \\
&\equiv [i-1]_F(x) + x + u_k C_{p^k}(x, [i-1]_F(x)) \\
&\equiv (i-1)x + u_k \gamma_{i-1,k} x^{p^k} + x + u_k C_{p^k}(x, (i-1)x + u_k \gamma_{i-1,k} x^{p^k}) \\
&\equiv (i-1)x + u_k \gamma_{i-1,k} x^{p^k} + x + x^{p^k} u_k \left(\frac{(i-1)p^k + 1 - ip^k}{p}\right) \\
&= ix + u_k x^{p^k} \left(\frac{p\gamma_{i-1,k} + (i-1)p^k + 1 - ip^k}{p}\right) \\
&= ix + u_k x^{p^k} \left(\frac{(i-1) - (i-1)p^k + (i-1)p^k + 1 - ip^k}{p}\right) \\
&= ix + u_k x^{p^k} \left(\frac{i - ip^k}{p}\right) \\
&= ix + u_k \gamma_{i,k} x^{p^k},
\end{align*}$$

as desired. \qed

Recall that the total power operation is a ring map

$$P : E_0 \to D \cong E_0[[a]]/(\langle p \rangle(a)/a).$$

The ring homomorphism $P$ classifies a formal group law $F'$ on $D$. The natural $E_0$-algebra map $E_0 \to D$ classifies a second formal group law on $D$, which, by abuse of notation, we denote $F$.

Lemma 4.2. There is an equality of elements in $D[[x]]$,

$$\prod_{k=0}^{p-1} ([p]_F(x) - F[k] a) = [p]_{F'} \left(\prod_{k=0}^{p-1} (x - F[k] a)\right)$$

Proof. In the language of Section 3, we have a diagram of formal groups over $\text{Spf}(D)$

$$\begin{array}{ccc}
\mathbb{G}_{\text{source}} & \xrightarrow{p} & \mathbb{G}_{\text{target}} \\
\mathbb{G}_{\text{source}} & \xrightarrow{p} & \mathbb{G}_{\text{target}}
\end{array}$$

Applying global sections, we obtain a commuting diagram of $E_0$-algebra homomorphisms

$$\begin{array}{ccc}
D[[y]] & \xrightarrow{y \mapsto [p]_F(y)} & D[[x]] \\
\uparrow x \mapsto [p]_{F'}(x) & & \uparrow x \mapsto [p]_{F'}(x) \\
D[[y]] & \xrightarrow{y \mapsto [p]_F(y)} & D[[x]]
\end{array}$$

By [Str97 7.13], both horizontal arrows send $y$ to $\prod_{k=0}^{p-1} (x - F[k] a)$. \qed
Remark 4.3. As elements of $D$,
\[ \prod_{i=1}^{p-1}([-i]_F a) = \prod_{i=1}^{p-1}([i]_F a). \]
We denote their common value by $\Psi$.

Proposition 4.4. For $0 < k \leq n$,
\[ P(u_k)^k \equiv -u_k \Psi \mod (p, P(u_1), P(u_2), ..., P(u_{k-1}), u_1, u_2, ..., u_{k-1}). \]

Proof. Corollary 4.1.1 implies both of the following equations:
\[ [p]_{F'}(x) \equiv P(u_k)x^k \mod (p, P(u_1), ..., P(u_{k-1}), x^{p^k+1}), \]
\[ [p]_F(x) \equiv u_kx^k \mod (p, u_1, ..., u_{k-1}, x^{p^k+1}). \]
In particular, both of the equations hold modulo $(p, u_1, ..., u_{k-1}, P(u_1), ..., P(u_{k-1}), x^{p^k+1})$, which is also where we perform the following calculations:
\[ [p]_{F'} \left( \prod_{i=0}^{p-1} (x - [i]_F a) \right) \equiv [p]_{F'} \left( x \cdot \prod_{i=1}^{p-1} (x - [i]_F a) \right)^k \equiv P(u_k)x^k \left( \prod_{i=1}^{p-1} (x - [i]_F a) \right)^k \equiv P(u_k)x^k \Psi^k. \]

On the other hand,
\[ \prod_{i=0}^{p-1} ([p]_F(x) - [i]_F a) \equiv \prod_{i=0}^{p-1} (u_kx^k - [i]_F a) \equiv u_kx^k \prod_{i=1}^{p-1} (u_kx^k - [i]_F a) \equiv u_kx^k \prod_{i=1}^{p-1} (-[i]_F a) \equiv u_kx^k \Psi. \]
The result follows by Lemma 4.2.

In the remainder of this section, we attempt to reduce the complexity of the total power operation $P : E_0 \to D$ by modding out both the domain and codomain by $(p, u_1, u_2, ..., u_{n-1})$. It is not possible to do this directly because $P$ is not an $E_0$-algebra map, and indeed we will need to mod out more of the codomain than just $(p, u_1, ..., u_{n-1})$.

Proposition 4.5. In the ring $E_0[[a]]/(p, u_1, u_2, ..., u_{n-1}) \cong \mathbb{F}_p[[u_n, a]]$, the element $[p]_F(a)$ is a product
\[ [p]_F(a) = U a^{p^n} g(a), \]
where
(1) \( U \) is a unit in \( \mathbb{F}_p[[u_n, a]] \),

(2) \( g(a) \) is a monic polynomial in \( (\mathbb{F}_p[[u_n]])[a] \) of degree \( p^{n+1} - p^n \),

(3) \( g(a) \equiv a^{p^{n+1}-p^n} \) modulo \( u_n \), and

(4) The constant term of \( g(a) \) is divisible by \( u_n \) but not \( u_n^2 \).

**Proof.** By Corollary 4.1.1 \( [p] \mathbb{F}(a) \equiv u_n a^{p^n} \) modulo \( a^{p^n+1} \). This means that we may factor \( [p] \mathbb{F}(a) = a^{p^n} (u_n + aq(a)) \)

for some power series \( q(a) \in \mathbb{F}_p[[u_n, a]] \). Corollary 4.1.1 also states that \( [p] \mathbb{F}(a) \equiv a^{p^{n+1}} \) modulo \( (u_n, a^{p^{n+1}+1}) \), and so the Weierstrass preparation theorem [HKR00, 5.1] implies

\[ u_n + aq(a) = Ug(a) \]

for some unit \( U \) and some monic polynomial \( g(a) \) of degree \( a^{p^{n+1}+p^n} \). Modding out both sides by \( a \), we learn that the constant term of \( g(a) \) is a unit times \( u_n \). Modding out both sides by \( u_n \), we arrive at an equation \( \overline{Ug(a)} = a^{p^{n+1}+p^n} + O(a^{p^{n+1}+p^n+1}) \), where the right-hand-side has no terms of degree less than \( p^{n+1} - p^n \). By looking at each coefficient of \( \overline{g(a)} \) in turn, starting with the constant coefficient, we learn that \( \overline{g(a)} = a^{p^{n+1}+p^n} \). \( \square \)

**Corollary 4.5.1.** The polynomial \( g(a) \) is irreducible, and \( \mathbb{F}_p[[u_n]]/[a]/g(a) \) is a DVR valued by powers of its maximal ideal \( m = \langle a \rangle \). The element \( u_n \) is in \( m^{p^{n+1}+p^n} \) but no higher power of \( m \).

The element \( \Psi \) is not 0 inside \( \mathbb{F}_p[[u_n]]/[a]/g(a) \).

**Proof.** The ring \( \mathbb{F}_p[[u_n]]/[a] \) is a UFD, and so Eisenstein’s criterion applies to show that \( g(a) \) is irreducible. It follows that the quotient \( \mathbb{F}_p[[u_n]]/[a]/g(a) \) is a local domain. When we further mod out by \( a \), we get \( \mathbb{F}_p[[u_n]]/[\overline{g(a)}] \cong \mathbb{F}_p \), since \( \overline{g(a)} \) is a unit times \( u_n \). Thus, \( \mathbb{F}_p[[u_n]]/[a]/g(a) \) is a DVR with maximal ideal generated by \( a \). We have that

\[ u_n = \text{(some unit)} a^{p^{n+1}+p^n} + \text{(terms of strictly higher valuation than } u_n) \]

and so \( u_n \) must have valuation \( p^{n+1} - p^n \).

To see that \( \Psi \) is not zero, recall that

\[ \Psi = \prod_{k=1}^{p-1} [k] \mathbb{F}(a), \]

and so can only be 0 if one of its factors is 0. However, for each \( 1 \leq k < p \), \( [k] \mathbb{F}(a) = ka + \ldots \) has valuation 1. \( \square \)

By Proposition 4.5 we may compose with a quotient homomorphism to obtain a reduced power operation

\[ N : E_0 P \rightarrow D \rightarrow \mathbb{F}_p[[u_n]]/[a]/g(a). \]

**Proposition 4.6.** For \( 1 \leq i \leq n - 1 \), \( N(u_i) = 0 \). Also, \( N(p) = 0 \).

**Proof.** Since \( N \) is a ring homomorphism, \( N(p) = p \). That \( N(p) = 0 \) follows, since \( p = 0 \) in \( \mathbb{F}_p[[u_n]]/[a]/g(a) \). The rest we prove by induction on \( i \), assuming that \( N(u_1), \ldots, N(u_{i-1}) \) are all zero. Since

\[ P(u_i) \Psi^j \equiv u_i \Psi \text{ modulo } (p, P(u_1), P(u_2), \ldots, P(u_{i-1}), u_1, u_2, \ldots, u_{i-1}), \]

we may conclude that

\[ N(u_i) \Psi^j = 0. \]
By Corollary 4.5.1, $N(u_i) = 0$. □

**Corollary 4.6.1.** The ring homomorphism $N : E_0 \to \mathbb{D}_p[[u_n]][a]/g(a)$ factors through a ring homomorphism

\[ \overline{N} : E_0/(p, u_1, u_2, \ldots, u_{n-1}) \cong \mathbb{D}_p[[u_n]] \to \mathbb{D}_p[[u_n]][a]/g(a) \]

**Proposition 4.7.** $\overline{N}(u_n) \Psi^{p^{n-1}} = u_n$

**Proof.** We have that

\[ P(u_n) \Psi = u_n \Psi \text{ modulo } (p, P(u_1), P(u_2), \ldots, P(u_{n-1}), u_1, u_2, \ldots, u_{n-1}). \]

The result follows from Corollary 4.5.1. □

### 5. A Proof of Theorem 1.2

In the previous section, we learned that the total power operation $P : E_0 \to D$ induces a ring homomorphism $\overline{N} : \mathbb{D}_p[[u_n]] \to \mathbb{D}_p[[u_n]][a]/g(a)$ such that $\overline{N}(u_n) \Psi^{p^{n-1}} = u_n$.

By Corollary 4.5.1 the ring $\mathbb{D}_p[[u_n]][a]/g(a)$ is a valuation ring. We define the **weight** $\text{wt}(f)$ of $f \in \mathbb{D}_p[[u_n]][a]/g(a)$ such that $\text{wt}(u_n) = 1$ and $\text{wt}(a) = \frac{1}{p^n+1-p^n}$. In other words, $\text{wt}(f)$ is just a rescaling of the natural valuation of $f$ by powers of the maximal ideal $\mathfrak{m} = (a)$. We also use $\text{wt}(f)$ to refer to the $u_n$-valuation of any $f \in \mathbb{D}_p[[u_n]]$.

**Proposition 5.1.** $\text{wt}(\Psi) = \frac{p-1}{p^n+1-p^n}$.

**Proof.** We have that $[i]_p(a) = ia + O(a^2)$, and so for $0 < i < p$ this has weight $\frac{1}{p^n+1-p^n}$. By definition, $\Psi$ is the product of all of these elements and the result follows. □

**Proposition 5.2.** $\overline{N}(u_n)$ has weight $\frac{p-1}{p^n+1-p^n}$

**Proof.** We have that

\[ 1 = \text{wt}(u_n) = \text{wt}(\overline{N}(u_n) \Psi^{p^{n-1}}) = \text{wt}(\overline{N}(u_n)) + \frac{(p^n-1)(p-1)}{p^n+1-p^n}, \]

so

\[ \text{wt}(\overline{N}(u_n)) = \frac{p^n+1-p^n - (p^n-1)(p-1)}{p^n+1-p^n} = \frac{p-1}{p^n+1-p^n}. \]

□

**Corollary 5.2.1.** For any non-zero power series $z \in \mathbb{D}_p[[u_n]]$ of weight at least 1, the weight of $\overline{N}(z)$ is less than the weight of $z$.

**Proof.** This follows from the facts that $\text{wt}(u_n') = t \text{wt}(u_n)$ and $\text{wt}(f_1 + f_2) = \text{wt}(f_1) + \text{wt}(f_2)$ whenever $\text{wt}(f_1) \neq \text{wt}(f_2)$. □

Recall from the end of Section 3 that, for every element $x \in E_0^\Sigma(BC_p) = \pi_0(E_{\Sigma}^\infty(BC_p) \cap E_{\Sigma}^\infty(BC_p))$, there is an $E_0$-module homomorphism $\phi_x : E_0^\Sigma(BC_p) \to E_0$. We can tensor over $E_0$ with the module $E_0/(p, u_1, u_2, \ldots, u_{p-1})$ to obtain a module homomorphism

\[ \overline{\phi}_x : \mathbb{D}_p[[u_n, a]]/(\langle p \rangle(a)) \to \mathbb{D}_p[[u_n]]. \]

**Proposition 5.3.** For any non-zero $z \in \mathbb{D}_p[[u_n]]$ of weight at least 1, there exists an $x \in E_0^\Sigma(BC_p)$ such that:

- The $E_0$-module map $\overline{\phi}_x : \mathbb{D}_p[[u_n, a]]/(\langle p \rangle(a)) \to \mathbb{D}_p[[u_n]]$ kills $g(a)$.
The resultant additive operation
\[ \mathbb{F}_p[[u_n]] \xrightarrow{\nabla} \mathbb{F}_p[[u_n]]/a(a) \xrightarrow{\delta} \mathbb{F}_p[[u_n]] \]
sends \( z \) to a power series of strictly smaller weight.

Proof. The operation \( x \mapsto \phi_x \) gives a map \( E^\ast_0(BC) \to \mathrm{Hom}_{E_0}\text{-modules}(E^0(BC), E) \). By e.g. \cite{Str98} \S3, this map is in fact bijective. Now, \( \overline{N}(z) \in \mathbb{F}_p[[u_n]]/a(a) \), has a unique representative polynomial \( f(a) \in \mathbb{F}_p[[u_n]]/a \) of degree \( < p^{n+1} - p^n \). By Corollary 5.2.1 there is some \( i < p^{n+1} - p^n \) such that the coefficient of \( a^i \) in \( f \) (an element \( q \in \mathbb{F}_p[[u_n]] \)) has weight less than \( wt(z) \). I claim that there is a choice of \( x \) which will induce an additive operation sending \( z \) to \( q \), finishing the proof. Indeed, Proposition 4.5 implies that there is a unique \( x \in \mathrm{Hom}_{E_0}\text{-modules}(E^0(BC), E) \) sending \( \delta_0 \) to 1, sending all other \( a^i \) for \( 0 < j < p^{n+1} - p^n \) to 0, and killing \( g(a), ag(a), a^2g(a), \ldots \).

Theorem 1.2. Suppose \( R \) is a \( K(n+1) \)-local, \( H_\infty \)-\( E \)-algebra such that, in \( \pi_0R \), some power of \( u_n \) is in the ideal \( (p, u_1, \ldots, u_{n-1}) \). Then 1 is in the ideal \( (p, u_1, \ldots, u_{n-1}) \).

Proof. The natural \( E_0 \)-algebra morphism \( \iota : E_0 \to \pi_0R \) yields an \( E_0 \)-algebra morphism
\[ \mathcal{T} : \mathbb{F}_p[[u_n]] \to \pi_0R/(p, u_1, u_2, \ldots, u_{n-1}). \]
By assumption, there is some non-zero \( z \in \mathbb{F}_p[[u_n]] \) that is sent to 0 by \( \mathcal{T} \). Choose such a \( z \) of minimal weight \( \geq 1 \). The previous proposition provides an additive operation \( E_0 \to E_0 \) that sends \( z \) to a power series \( \hat{z} \) of smaller weight. As explained at the end of Section 3 the \( H_\infty \)-structure ensures \( \mathcal{T}(\hat{z}) = 0 \). By the minimality of \( wt(z) \), it must be that \( wt(\hat{z}) = 0 \), so \( \hat{z} \) is just a unit in \( \mathbb{F}_p \).

References


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