THE ORIENTED COBORDISM RING

ARAMINTA GWYNNE

Abstract. We give an exposition of the computation of the oriented cobordism ring Ω_*^{SO} using the Adam's spectral sequence. Our proof follows Pengelley [15]. The unoriented and complex cobordism rings are also computed in a similar fashion.

Contents

Introduction

In his now famous paper [19], Thom showed that cobordism rings are isomorphic to stable homotopy groups of Thom spectra, and used this isomorphism to compute certain cobordism rings. We'll use the notation $\mathfrak{N}_*$ for the unoriented cobordism ring, Ω_*^U for the complex cobordism ring, and Ω_*^{SO} for the oriented cobordism ring. Then Thom's theorem says that $\mathfrak{N}_* \cong \pi_*(MO)$, $\Omega_*^U \cong \pi_*(MU)$, and $\Omega_*^{SO} \cong \pi_*(MSO)$. Thom's theorem is one of the main examples of a general approach to problems in geometric topology: use classifying spaces to translate the problem into the world of algebraic topology, and then use algebraic tools to compute. For the majority of this paper, we will focus on the algebraic side of the computation. If the reader is unfamiliar with Thom's theorem, they may take it as motivation for computing the homotopy groups of Thom spectra. We will not

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make use of any geometry, including the definition of a cobordism, until §11. For the interested reader, proofs of Thom's theorem can be found in [19] or [7, Ch. 25] in the unoriented case. Definitions of the relevant cobordism theories can be found in [18, Ch. 4].

The rings $\pi_*(MU)$ and $\pi_*(MSO)$ were originally computed at around the same time spectra were being invented. The lack of a standard theory led to multiple notations and conventions being used in the literature. The computations also use the Adams spectral sequence, which we understand much better today than we did at the time of the first computations. Our goal is to give a clean, direct computation of π_*MSO, making full use of the stable world as well as spectral sequences. We assume knowledge of these things.

In the literature, oriented cobordism is the first difficult case. Unlike complex cobordism, oriented cobordism has torsion. This is the main source of difficulty. We attempt to place the unoriented, complex, and oriented cases in a framework that makes the computation of the oriented cobordism ring $\Omega_*^{SO} \cong \pi_*(MSO)$ seem natural.

Working in the stable world and making computations with spectral sequences often makes the geometry opaque. We will try to give geometric meaning to the things we say, most of which will come from Wall [21]. We'll also attempt to give a survey of the literature, pointing out changes in notation and places where the reader can find alternative proofs. Stong's book [18] collects all of these facts, but was again written around 1970, when notation was not yet standard. In [17], Ravenel uses modern language and notation, but does not give a full treatment of the oriented case. We'll assume the reader is familiar with computations of H^*BO, H^*BU, and H^*BSO, as well as with the stable Thom isomorphism and the stable Hurewicz map. The cohomology computations can be found in [13]. The passage of the Thom isomorphism and the Hurewicz map to stable maps is discussed in [7, Ch. 25].

The computation of π_*MSO splits into three parts: rational data, odd primary data, and even primary data.

In §3, we investigate the rational data. This part will be quick. The main point is that the rational stable Hurewicz map is an isomorphism, i.e. $\pi_*X \otimes \mathbb{Q} \cong H_*(X, \mathbb{Q})$ for any spectrum X. In the literature, it is shown that the rational Hurewicz map is an isomorphism in certain ranges, then passing to an isomorphism stably. This is done in [13] making use of Serre's Mod $\mathcal{C}$ theory (which requires the Serre spectral sequence to prove), and done in [5] directly. It turns out that proving the result in the stable world is essentialy a formality, and we reference the formal proof here.

The torsion computations require more work. In 1960, Milnor gave the first computation of the odd torsion in [10]. Milnor computed the structure of $H^*(MU; \mathbb{F}_p)$ and $H^*(MSO; \mathbb{F}_p)$ as modules over the Steenrod algebra $\mathcal{A}$ and used this knowledge to apply the Adams spectral sequence. Contrastingly, the original computation of 2-torsion (due to Wall [21], also in 1960) is done geometrically. Wall computes the $\mathcal{A}$-module structure of $H^*(MSO; \mathbb{F}_2)$ as a corollary of the Ω_*^{SO} computation. He later gave a direct proof of the module structure in [20]. An alternative proof of the module structure is given in [16, Thm. 1]. The computations show that $H^*(MSO; \mathbb{F}_2)$ is the direct sum of free modules over $\mathcal{A}$ and $\mathcal{A}/\mathcal{A}\mathrm{Sq}^1$. Around twenty years later, Pengelley [15] computed the 2-torsion of MSO using the $\mathcal{A}_*$-comodule structure and the Adams spectral sequence. In doing so, Pengelley gives an explicit description of the comodule structure on generators. In contrast with [10] and [21], Pengelley's computations take place over the dual Steenrod algebra $\mathcal{A}_*$.

Here, we attempt to give a unified computation of $\mathfrak{N}_*$, Ω_*^U, and Ω_*^{SO}. We will work almost exclusively over the dual Steenrod algebra as this simplifies many computations. For all three cobordism theories, we will compute the $\mathcal{A}_*$-comodule structure on homology and then use the Adams spectral sequence. The differences between the three theories can be seen in the comodule structure. Our main theorem is

Theorem. *Let p be any prime and q any odd prime. For each $k \geq 1$, there exist $x_k \in H_k(MO;\mathbb{F}_2)$, $v_k \in H_{2k}(MU;\mathbb{F}_p)$, $u_k \in H_{4k}(MSO;\mathbb{F}_q)$, and $y_k \in H_k(MSO;\mathbb{F}_2)$ so that as $\mathcal{A}_*$-comodules,*

$$\begin{cases} H_*(MO;\mathbb{F}_2) \cong \mathcal{A}_* \otimes \mathbb{F}_2[x_k \big| k \geq 2, k \neq 2^i - 1] \\ H_*(MU;\mathbb{F}_p) \cong \big(\mathcal{A}/(Q_0)\big)^\vee \otimes \mathbb{F}_p[v_k \big| k \geq 1, k \neq p^i - 1] \\ H_*(MSO;\mathbb{F}_q) \cong \big(\mathcal{A}/(Q_0)\big)^\vee \otimes \mathbb{F}_q[u_k \big| 2k \neq q^i - 1] \\ H_*(MSO;\mathbb{F}_2) \cong \big(\mathcal{A}/\mathcal{A}\mathrm{Sq}^1\big)^\vee \otimes \mathbb{F}_2[y_k \big| k \neq 2, k \neq 2^i - 1] \end{cases}$$

Here (Q_0) is the two-sided ideal generated by the Bockstein and the notation $\mathcal{S}^\vee$ denotes the $\mathbb{F}_p$-dual of $\mathcal{S}$. This is the essential theorem in the computation of all three cobordism theories. The proof will occupy most of this paper. Once we've proven the theorem, the rest of the computations will fall out of the Adams spectral sequence by some general theory given in [17].

The odd primary computation is due independently to Novikov [14] and Milnor [10]. The point is to show that there is no odd torsion, and this is done using the Adams spectral sequence. Following Milnor [10] and [18], we will prove this first for MU and then deduce the MSO case. Both Milnor and Stong make this computation using the Adams spectral sequence in cohomology. We'll instead use the spectral sequence in homology. We do this for two reasons. First, the computation of 2-torsion is more easily done in homology. Secondly, because $H_*(MSO,\mathbb{F}_2)$ is an algebra over the Steenrod algebra and we find it easier to work with an algebra co-module than with a co-algebra module. We note that Ravenel [17] computes the odd torsion for MU using the homology Adams spectral sequence, and we follow his computation closely.

The 2-torsion is where the hard work comes in. Unlike the other cobordism theories we consider here, MU and MO, oriented cobordism has both torsion and free parts. The torsion lives in $\mathbb{F}_2$, so the 2-torsion case is a posteriori harder. We hope to convince the reader that the $p = 2$ case is not that much harder. Like the odd prime case, we use the Adams spectral sequence. The only new part will be work describing how the E_2 page splits into things giving 2-torsion and free things. This step is due to Pengelley [15].

Finally, we would like to describe some alternative approaches to computing π_*MSO. In §7, we describe how the group structure of the oriented cobordism ring can be obtained from knowledge of the Steenrod algebra action on the module H^*MSO. We describe these computations for MO, MU, and MSO since they work similarly. In §11, we reincorporate some geometry. Since the 2-torsion part seems the most mysterious, we put our focus here. We describe Wall's generators for the free and 2-torsion part of $\pi_*MSO \otimes \mathbb{F}_2$. We conclude by comparing Wall's description in [21] of the 2-torsion with that of Pengelley in [15].

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1. Homology of classifying spaces

Here we recall the computations of polynomial bases for the homology of MO, MU, and MSO. Let Φ be the Thom isomorphism from the homology of the Thom spectrum to the homology of the classifying space. Let e_k be the dual of the kth power of the first Stiefel-Whitney class, $w_1^k \in H^*(BO;\mathbb{F}_2)$ and a_k be the dual of the kth power of the first Chern class, $c_1^k \in H^*(BU;R)$. Set $r_k = \Phi^{-1}(e_k) \in H_n(MO;\mathbb{F}_2)$ and $b_k = \Phi^{-1}(a_k) \in H_{2k}(MU;R)$.

Lemma 1.1. *Let R be any commutative ring and S a commutative ring in which 2 is invertible.*

$$H_*(MO;\mathbb{F}_2) = \mathbb{F}_2[r_i \big| i \geq 1]$$
$$H_*(MU;R) = R[b_i \big| i \geq 1]$$
$$H_*(MSO;S) \cong S[b_{2i} \big| i \geq 1]$$

where $H_*(MSO;S)$ is viewed in $H_*(MU;S)$ under the map induced by complexification.

For a proof of the MO and MU computations, see Kochman([6, Prop. 2.4.5, 2.4.7]. We'll prove the result for MSO.

Proof. Consider the forgetful map $f : BU \to BSO$ and the complexification map $c : BSO \to BU$. The composite $f \circ c : BSO \to BSO$ is homotopy equivalent to multiplication by 2. Hence

$$f_* \circ c_* : H_*(MSO;S) \to H_*(MU;S) \to H_*(MSO;S)$$

is multiplication by 2. Since $1/2 \in S$, the map $f_* \circ c_*$ is invertible. Thus c_* is injective. Identify $H_*(MSO;S)$ with it's image under c_*.

Composing in the other direction,

$$c_* \circ f_* : H_*(MU;S) \to H_*(MSO;S) \to H_*(MU;S)$$

is given by $1 + g$ where g is conjugation. Since $g^2 = 1$, the homology $H_*(MU;S)$ splits into two pieces: the part where g acts trivially and the part where g acts by a sign. Then $H_*(MSO;S)$ is isomorphic to the part of $H_*(MU;S)$ where g acts trivially. Since g is a ring map, it suffices to show that $g(b_{2i+1}) \neq b_{2i+1}$ and $g(b_{2i}) = b_{2i}$ for all i. Recall that $b_i = \Phi^{-1}(a_k)$ where a_k is the dual of $c_1^k \in H^*(BU(1);S) = H^*(\mathbb{CP}^\infty;S)$. Thus to understand $g(b_i)$ we need to understand the action of g on $H_*(BU;S)$. Dually, we need to understand the action on $H^*(\mathbb{CP}^\infty)$. By [13, Lem. 14.9], conjugation takes c_1 to $-c_1$. Thus $g(b_k) = (-1)^k b_k$. Thus

$$(c_* f_*)(b_k) = \begin{cases} 0 & k \text{ odd} \\ 2b_k & k\text{even} \end{cases}$$

The result follows. □

2. The stable Hurewicz map and rational data

Here we compute $\Omega_*^U \otimes \mathbb{Q}$ and $\Omega_*^{SO} \otimes \mathbb{Q}$. Since Ω_*^O is a $\mathbb{F}_2$-vector space, $\Omega_*^O \otimes \mathbb{Q} = 0$.

Theorem 2.1. *Let X be a spectrum. The rational stable Hurewicz map $h : \pi_*(X) \otimes \mathbb{Q} \to H_*(X; \mathbb{Q})$ is an isomorphism.*

This is a consequence of [1, Part III; Prop. 6.6(i)] (c.f. the example on the top of page 203).

Corollary 2.2. *We have*

$$\Omega_*^U \otimes \mathbb{Q} \cong \mathbb{Q}[v_{2i} \big| i \geq 1]$$
$$\Omega_*^{SO} \otimes \mathbb{Q} \cong \mathbb{Q}[u_{4i} \big| i \geq 1]$$

where the subscript denotes the degree of the generator.

Proof. Using the Hurewicz and Thom isomorphism,

$$\Omega_*^U \otimes \mathbb{Q} \cong \pi_*(MU) \otimes \mathbb{Q} \cong H_*(MU; \mathbb{Q}) \cong H_*(BU; \mathbb{Q}) = \mathbb{Q}[a_i \big| i \geq 1]$$

The proof for MSO is identical. □

3. Steenrod algebra structures

To compute the comodule structures of $H_*(MO), H_*(MU)$, and $H_*(MSO)$, we need a good basis for the dual Steenrod algebra $\mathcal{A}_*$ as well as the algebras $\big(\mathcal{A}/(Q_0)\big)^\vee$ and $\big(\mathcal{A}/\mathcal{A}\mathrm{Sq}^1\big)^\vee$ that appear in Theorem 4.1.

Let $E(x_1, x_2, \dots)$ denote the exterior algebra on generators $x_1, x_2, \dots$.

Theorem 3.1 (Milnor)**.** *For odd primes,*

$$\mathcal{A}_* = \mathbb{F}_p[\xi_1, \xi_2, \dots] \otimes E(\tau_0, \tau_1, \dots)$$

for certain $\xi_i \in (\mathcal{A}_*)_{2p^i-2}$ and $\tau_i \in (\mathcal{A}_*)_{2p^i-1}$. For $p = 2$,

$$\mathcal{A}_* = \mathbb{F}_2[\xi_1, \xi_2, \dots]$$

for $\xi_i \in (\mathcal{A}_*)_{2^i-1}$. The coproduct $\psi : \mathcal{A}_* \to \mathcal{A}_* \otimes \mathcal{A}_*$ is given on generators by

$$\psi(\xi_n) = \sum_{k=0}^{n} \xi_{n-k}^{p^k} \otimes \xi_k$$

$$\psi(\tau_n) = \tau_n \otimes 1 + \sum_{k=0}^{n} \xi_{n-k}^{p^k} \otimes \tau_k$$

A proof and definitions of the elements ξ_i and τ_i can be found in [6, Thm. 2.5.1] or in Milnor's original paper [9].

Let $Q_0 \in \mathcal{A}$ be the Bockstein. Below (α) denotes the two-sided ideal generated by α. Left-sided and right-sided ideals will be denoted $\mathcal{A}\alpha$ and $\alpha\mathcal{A}$, respectively.

Proposition 3.2. *Let $\big(\mathcal{A}/(Q_0)\big)^\vee$ denote the dual of $\mathcal{A}/(Q_0)$. Then*

$$\big(\mathcal{A}/(Q_0)\big)^\vee = \begin{cases} \mathbb{F}_2[\xi_1^2, \xi_2^2, \dots] & p = 2 \\ \mathbb{F}_p[\xi_1, \xi_2, \dots] & p \text{ odd} \end{cases}$$

For a proof, see [6, Prop. 2.5.2].

Let ζ_i denote the Hopf algebra conjugate of Milnor's basis element ξ_i.

Proposition 3.3. *At the prime 2,*

$$\left(\mathcal{A}/\mathcal{A}\mathrm{Sq}^1\right)^\vee \cong \mathbb{F}_2[\zeta_1^2, \zeta_2, \zeta_3, \dots]$$

Let $\mathcal{B}$ denote $\left(\mathcal{A}/\mathcal{A}\mathrm{Sq}^1\right)^\vee$.

Proof. Consider the composition

$$h : \mathcal{A} \otimes \Sigma\mathbb{F}_2 \to \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$$

where the first map sends $1 \otimes 1$ to $1 \otimes \mathrm{Sq}^1$ and the second map is multiplication. Then $coker(h) = \mathcal{A}/\mathcal{A}\mathrm{Sq}^1$. Dually, we want to compute the kernel of

$$h^\vee : \mathcal{A}_* \xrightarrow{\psi} \mathcal{A}_* \otimes \mathcal{A}_* \xrightarrow{1\otimes f} \mathcal{A}_* \otimes \Sigma\mathbb{F}_2$$

where the map $f : \mathcal{A} \to \Sigma H\mathbb{F}_2$ is the linear dual of $\xi_1 = \zeta_1$. We have

$$h^\vee(\zeta_1^2) = (1 \otimes f)(\psi(\zeta_1)^2) = (1 \otimes f)(1 \otimes \zeta_1^2) = 1 \otimes 0 = 0$$

since $\zeta_1^2 \neq \zeta_1$. Similarly, for any n we have

$$h^\vee(\zeta_n) = (1 \otimes f)(\psi(\zeta_n)) = (1 \otimes f)\Big(\sum_{i=0}^{n} \zeta_i \otimes \zeta_{n-i}^{2^i}\Big) = \sum_{i=0}^{n} \zeta_i \otimes f(\zeta_{n-i}^{2^i})$$

which is nonzero if and only if $2^i = 1$ and $n - i = 1$, i.e. if and only if $n = 1$. Thus

$$\mathbb{F}_2[\zeta_1^2, \zeta_2, \zeta_3, \dots] \subset \ker h^\vee = \mathcal{B}$$

A dimension count shows that this is the whole kernel. □

4. Main theorem: statement and remarks

This section contains the heart of the problem. The computations of the cobordism rings in §9 and §10 will be straightforward applications of the calculations here and the general algebraic manipulations of §8. The goal of this section is to prove the following table of results which was stated in the introduction.

Theorem 4.1. *Let p be any prime and q any odd prime. There exist $x_k \in H_k(MO; \mathbb{F}_2)$, $v_k \in H_{2k}(MU; \mathbb{F}_p)$, $u_k \in H_{4k}(MSO; \mathbb{F}_q)$, and $y_k \in H_k(MSO; \mathbb{F}_2)$ so that as $\mathcal{A}_*$-comodules,*

$$\begin{cases} H_*(MO; \mathbb{F}_2) \cong \mathcal{A}_* \otimes \mathbb{F}_2[x_k \big| k \geq 2, k \neq 2^i - 1] \\ H_*(MU; \mathbb{F}_p) \cong \big(\mathcal{A}/(Q_0)\big)^\vee \otimes \mathbb{F}_p[v_k \big| k \geq 1, k \neq p^i - 1] \\ H_*(MSO; \mathbb{F}_q) \cong \big(\mathcal{A}/(Q_0)\big)^\vee \otimes \mathbb{F}_q[u_k \big| 2k \neq q^i - 1] \\ H_*(MSO; \mathbb{F}_2) \cong \big(\mathcal{A}/\mathcal{A}\mathrm{Sq}^1\big)^\vee \otimes \mathbb{F}_2[y_k \big| k \neq 2, k \neq 2^i - 1] \end{cases}$$

where the comodule structure will be explained in the proof.

The proof of Theorem 4.1 relies on a result of Milnor and Moore. The relevant notation and definitions will be given below.

Theorem 4.2. *Let A be a (graded commutative, connected) Hopf algebra over a field k. Let N be a k-algebra and a left A-comodule. If there exists a surjective k-algebra, A-comodule map $f : N \to A$, then $N \cong A \otimes_k (k\square_A N)$ as left $k\square_A N$-modules and right A-comodules.*

For the original theorem see [12, Thm. 4.7]. The version stated here is [17, Cor. A1.1.18]. For another variation see [8, Thm. 21.2.2].

Definition 4.3. For any right A-comodule M and left A-comodule N, the cotensor product $M\Box_A N$ is defined to be the kernel

$$0 \to M\Box_A N \longrightarrow M \otimes_k N \xrightarrow{\psi\otimes 1 - 1\otimes\psi} M \otimes_k A \otimes_k N$$

where ψ is the appropriate comodule structure map.

Remark 4.4. As motivation for Theorem 4.2, consider the dual statement.

Theorem 4.5. *Let A be a (graded commutative, connected) Hopf algebra over a field k. Let M be a k-coalgebra and a left A-module. If there exists a injective k-algebra, A-module map $f : A \to M$, then $M \cong A \otimes_k k \otimes_A M$ as left $k \otimes_A M$-comodules and right A-modules.*

The conclusion that $M \cong A \otimes_k k \otimes_A M$ at first sight looks trivial. The subtle problem here is that $S \otimes_S T \cong T$ as S-modules. So $M \cong A \otimes_k k \otimes_A M$ as k-modules, but not necessarily as A-modules.

In the case $M = k$, we can give a more explicit description of $k\Box_A N$. We have isomorphisms $k \otimes_k N \cong N$ and $N \otimes_k A \otimes_k k \cong N \otimes_k A$. The cotensor product $k\Box_A N$ is thus the kernel of a map $N \to N \otimes_k A$. Tracing through the isomorphism, we see that

$$k\Box_A N = \{x \in N : \psi(x) = 1 \otimes x\}$$

In other words, $k\Box_A N$ is the set of primitive elements of N. In our applications, M will be the homology of the spectra of interest and $k = \mathbb{F}_p$. We want A to be

$$A = \begin{cases} \mathcal{A}_* & \text{for } MO \\ \left(\mathcal{A}/(Q_0)\right)^\vee & \text{for } MU \text{ at any prime and } MSO \text{ at odd primes} \\ \left(\mathcal{A}/\mathcal{A}\mathrm{Sq}^1\right)^\vee & \text{for } MSO \text{ at } p = 2 \end{cases}$$

These three cases get progressively harder. In fact, the third case is too much to hope for, see Remark 4.8 below.

To use Theorem 4.2, we need A to be a Hopf algebra. Clearly the Steenrod algebra is a Hopf algebra. By Propositions 3.2 and 3.3, we know that $\left(\mathcal{A}/(Q_0)\right)^\vee$ and $\left(\mathcal{A}/\mathcal{A}\mathrm{Sq}^1\right)^\vee$ are algebras. We get a compatible algebra structure on $\mathcal{A}/(Q_0)$ by observing that (Q_0) is a normal Hopf ideal and applying the following Proposition.

Proposition 4.6. *Let A be a Hopf algebra. If $I \subset A$ is a normal Hopf ideal, then A/I is a Hopf algebra.*

An ideal $I \subset A$ is a normal Hopf ideal if it is a two-sided ideal in the kernel of the counit so that $\psi(I) \subset I \otimes A + A \otimes I$.

Lemma 4.7. *The ideal $(Q_0) \subset \mathcal{A}$ is a normal Hopf ideal.*

Proof. We have $\psi(Q_0) = Q_0 \otimes 1 + 1 \otimes Q_0$. □

For a proof of Proposition 4.6, see [8, Rmk. 4.11].

Remark 4.8. Note that $\mathcal{A}\mathrm{Sq}^1$ has no chance of being a normal Hopf ideal since it is only one-sided. Although we cannot apply the above Proposition, one might still hope that $\mathcal{B} := \mathcal{A}/\mathcal{A}\mathrm{Sq}^1$ is a Hopf algebra. Unfortunately, it is not. There is no way to make $\mathcal{B}$ an algebra in a compatible way with its coalgebra structure. In particular, this means that we cannot consider comodules over $\mathcal{B}$. One might still have a glimmer of hope that we can get around this in the case we care about. We could ask if the comodule map $\psi : H_*(MSO; \mathbb{F}_2) \to \mathcal{A}_* \otimes H_*(MSO; \mathbb{F}_2)$ lands in $\mathcal{B} \otimes H_*(MSO; \mathbb{F}_2)$. This would allow us to mimic the construction of the cotensor product to get something we would want to call

$$\mathbb{F}_2 \square_{\mathcal{B}} H_*(MSO; \mathbb{F}_2)$$

Even this is hoping for too much. On the bright side, Pengelley [15] has constructed an explicit isomorphism between $H_*(MSO; \mathbb{F}_2)$ and $\mathcal{B} \otimes \mathbb{F}_2[y_k \big| k \neq 2, k \neq 2^i - 1]$. We will also see that $\{y_k : k \neq 2, k \neq 2^i - 1\}$ are primitive elements modulo elements not in $\mathcal{B}$.

Theorem 4.2 also requires a surjection $f : H_*(X) \to A$. Let $U \in H^*(X)$ the Thom class and $U : \mathcal{A} \to H^*(X)$ the map sending a to $a \cdot U$. We'll show that the following composition is surjective:

$$\begin{array}{ccc} H_*(X) & \overset{f}{\dashrightarrow} & A \\ \downarrow{\scriptstyle \psi} & & \uparrow{\scriptstyle m} \\ A \otimes H_*(X) & \xrightarrow{1 \otimes u} & A \otimes A \end{array}$$

Here ψ is the comodule structure map, m is multiplication, and u is the dual of the map U.

Thus to prove Theorem 4.1, we need to do two things:

(1) Compute the image of the map $f : H_*(X) \to \mathcal{A}_*$.
(2) Compute the primitive elements in homology.

Remark 4.9. It might seem strange to be working in dual land. We care about the *dual* of the Thom class map. Our answers involve *duals* of $\mathcal{A}/(Q_0)$ and $\mathcal{A}/\mathcal{A}\mathrm{Sq}^1$. However, working in cohomology is harder for two reasons. Firstly, cohomology is harder since in this case it is not an algebra. Secondly, the computations of the kernel of the Thom class map are all essentially repeating the computation of $\mathcal{A}_*$ done by Milnor. We find it easier to compute $\mathcal{A}_*$ once and for all and then use Milnor's basis to make the computations of u faster.

For the reader familiar with Eilenberg-MacLane spectra, we note the following:

Remark 4.10. The map u is induced from a map of spectra. Indeed, Let $U \in H^0(X; \mathbb{F}_p)$ be the Thom class. Then U is represented by a map $U : X \to H\mathbb{F}_p$. Identifying $H_*(H\mathbb{F}_p, \mathbb{F}_p)$ with $\mathcal{A}_*$, we see that the u is dual to U. Since u comes from a map of spectra, it is a map of $\mathcal{A}_*$-comodules. This gives a commutative diagram

$$\begin{array}{ccc} H_*(X) & \xrightarrow{u} & \mathcal{A}_* \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\ \mathcal{A}_* \otimes H_*(X) & \xrightarrow{1 \otimes u} & \mathcal{A}_* \otimes \mathcal{A}_* \end{array}$$

However, since $\psi \circ m \neq id$, this does not show that our computation of f is equivalent to the standard computation of u.

5. Proof of the main theorem

Let's start with the computations for MO.

Proof of Thm 4.1 for MO. We claim that $f : H_*(MO;\mathbb{F}_2) \to \mathcal{A}_*$ is surjective. Recall from Lemma 1.1 that

$$H_*(MO;\mathbb{F}_2) \cong \mathbb{F}_2[r_i \big| i \geq 1]$$

Claim: The comodule structure map $\psi : H_*(MO;\mathbb{F}_2) \to \mathcal{A}_* \otimes H_*(MO;\mathbb{F}_2)$ is given by

$$\psi(r_k) = \begin{cases} \xi_i \otimes \Phi^{-1}(1) + \text{decomposables} & k = 2^i - 1 \\ 1 \otimes r_k + \text{decomposables} & \text{otherwise} \end{cases}$$

Here Φ^{-1} is the inverse of the Thom isomorphism. For a proof see [6, Thm. 2.6.8(a)].

Recall that we've defined f to be the composition

$$\begin{array}{ccc} H_*(MO;\mathbb{F}_2) & \xrightarrow{f} & \mathcal{A}_* \\ \downarrow{\scriptstyle \psi} & & \uparrow{\scriptstyle m} \\ \mathcal{A}_* \otimes H_*(MO;\mathbb{F}_2) & \xrightarrow{1 \otimes u} & \mathcal{A}_* \otimes \mathcal{A}_* \end{array}$$

Note that $u(\Phi^{-1}(1)) = 1$ since $\Phi^{-1}(1)$ is the Thom class. When $k = 2^i - 1$, we therefore have

$$f(r_k) = m(1 \otimes u)(\psi(r_k))) = m(\xi_i \otimes 1 + \cdots) = \xi_i + \text{decomposables}$$

Since the ξ_i generate $\mathcal{A}_*$, the map f is surjective. The calculation of $\psi(r_k)$ also shows that $H_*(MO;\mathbb{F}_2)$ has a single primitive generator x_k for every $k \neq 2^i - 1$. By Theorem 4.2,

$$H_*(MO;\mathbb{F}_2) \cong \mathcal{A}_* \otimes \mathbb{F}_2[x_k \big| k \neq 2^i - 1]$$

□

Proof of Thm 4.1 for MU. We claim that $f : H_*(MU;\mathbb{F}_2) \to \mathcal{A}_*$ has image $(\mathcal{A}/(Q_0))^\vee$. By Lemma 1.1, we have

$$H_*(MU;\mathbb{F}_p) \cong \mathbb{F}_p[b_i \big| i \geq 1]$$

Claim: Let $\eta_i = \xi_i$ if p is odd and $\eta_i = \xi_i^2$ if $p = 2$. The comodule structure map $\psi : H_*(MU;\mathbb{F}_p) \to \mathcal{A}_* \otimes H_*(MU;\mathbb{F}_p)$ is given by

$$\psi(b_k) = \begin{cases} \eta_i \otimes \Phi^{-1}(1) + \text{decomposables} & k = p^i - 1 \\ 1 \otimes b_k + \text{decomposables} & \text{otherwise} \end{cases}$$

where Φ is the Thom isomorphism. For a proof, see [6, Thm. 2.6.8(b)].

Recall from Proposition 3.2, the η_i are in $\big(\mathcal{A}/(Q_0)\big)^\vee$. The claim therefore shows that $H_*(MU;\mathbb{F}_p)$ is a $\big(\mathcal{A}/(Q_0)\big)^\vee$-comodule.

When $k = p^i - 1$, we have

$$f(b_k) = m \circ (1 \otimes u) \circ \psi(b_k) = \eta_i + \text{decomposables}$$

Since the η_i generate $\big(\mathcal{A}/(Q_0)\big)^\vee$, the image of f contains $\big(\mathcal{A}/(Q_0)\big)^\vee$. The calculation of $\psi(\Phi^{-1}(a_k))$ also shows that $H_*(MU;\mathbb{F}_p)$ has a single primitive generator v_k for every $k \neq p^i - 1$. By Theorem 4.2,

$$H_*(MU;\mathbb{F}_p) \cong \big(\mathcal{A}/(Q_0)\big)^\vee \otimes \mathbb{F}_p[v_k \big| k \neq p^i - 1]$$

□

Proof of Thm 4.1 for MSO (odd). Let q be an odd prime. We claim that $f : H_*(MO;\mathbb{F}_2) \to \mathcal{A}_*$ has image $(\mathcal{A}/(Q_0))^\vee$. By Lemma 1.1,

$$H_*(MSO;\mathbb{F}_q) \cong \mathbb{F}_q[b_{2i} \big| i \geq 1]$$

Recall that

$$\psi(b_k) = \begin{cases} \eta_i \otimes \Phi^{-1}(1) + \text{decomposables} & k = p^i - 1 \\ 1 \otimes b_k + \text{decomposables} & \text{otherwise} \end{cases}$$

Thus $H_*(MSO;\mathbb{F}_q)$ is in fact a sub-$\big(\mathcal{A}/(Q_0)\big)^\vee$-comodule of $H_*(MU;\mathbb{F}_q)$. We also see that $H_*(MSO;\mathbb{F}_q)$ has a single primitive generator $u_k \in H_{2k}(MSO;\mathbb{F}_q)$ for each $2k \neq p^i - 1$. Thus

$$H_*(MSO;\mathbb{F}_q) \cong \big(\mathcal{A}/(Q_0)\big)^\vee \otimes \mathbb{F}_q[u_k \big| 2k \neq 2^i - 1]$$

□

Recall that we've defined $\mathcal{B} = \big(\mathcal{A}/\mathcal{A}\mathrm{Sq}^1\big)^\vee$.

Proof of Thm 4.1 for MSO (even). In Lemma 1.1, we described how the cohomology of MSO sat inside the cohomology of MU. The resulting polynomial generators allowed us to prove Theorem 4.1 in a way parallel to the proof for MU. Since the $\mathcal{A}_*$-comodule structures of the cohomology of MU and MSO diverge at the prime 2, we should not expect the generators b_{2i} to work well here. It turns out that we instead want generators coming from the map $H_*(MSO;\mathbb{F}_2) \to H_*(MO;\mathbb{F}_2)$ induced by inclusion. The motivation here comes from Wall's original computation of the 2-torsion of Ω_*^{SO} in [21].

For now, take the following Lemma on faith. We explain the motivation and give a proof in §11.

Lemma 5.1. *For $k = 2^j - 1$, define $z_k = r_k$ as in the proof for MO. For $k \neq 2^j - 1$, there exists z_k in the image of the Hurewicz map so that $H_*(MO;\mathbb{F}_2) = \mathbb{F}_2[z_k]$ and $H_*(MSO;\mathbb{F}_2)$ sits inside $H_*(MSO;\mathbb{F}_2)$ as $\mathbb{F}_2[y_k]$ where*

$$y_k = \begin{cases} z_{k/2}^2 & k = 2^j \\ z_k + z_1 z_{k-1} & n = 2i, i \neq 2^j \\ z_k & k = 2i - 1 \end{cases}$$

This is enough information for us to determine the comodule structure.

<u>Claim</u>: There exists $y_k \in H_{4k}(MSO;\mathbb{F}_2)$ so that $H_*(MSO;\mathbb{F}_2) \cong \mathbb{F}_2[y_k]$ and the comodule structure map $\psi : H_*(MSO;\mathbb{F}_2) \to \mathcal{A}_* \otimes H_*(MSO;\mathbb{F}_2)$ is given by

$$\psi(y_k) = \begin{cases} 1 \otimes y_1^2 & k = 2 \\ \sum_{n=0}^{i} \zeta_n \otimes y_{2^{i-n}-1}^{2^n} & k = 2^i - 1 \\ 1 \otimes y_k + \zeta_1 \otimes y_{k-1} & k \neq 2, k \neq 2^j - 1, k = 2i, i \neq 2^l \\ 1 \otimes y_k & \text{otherwise} \end{cases}$$

In particular, ψ does not land in $\mathcal{B} \otimes H_*(MSO;\mathbb{F}_2)$. The claim for $k = 2^i - 1$ follows from the computation of ψ on r_k and a little work with the conjugation map. For $k \neq 2^i - 1$, each z_k is primitive. Indeed, everything in the image of the Hurewicz map is primitive. Thus $\psi(y_k) = 1 \otimes y_k$ for k odd and $\neq 2^i - 1$. For $k = 2^i, i \neq 2^j$ we have

$$\begin{aligned}\psi(y_k) &= \psi(z_k + z_1 z_{k-1}) \\ &= 1 \otimes z_k + \psi(z_1)\psi(z_{k-1}) \\ &= 1 \otimes z_k + (1 \otimes z_1 + \zeta_1 \otimes 1)(1 \otimes z_{k-1}) \\ &= 1 \otimes y + \zeta_1 \otimes y_{n-1}\end{aligned}$$

For $k = 2$, we have $\psi(y_2) = \psi(z_1)\psi(z_1) = 1 \otimes y_1^2$. One can now compute $\psi(y_{2^j})$ inductively.

From our description of ψ, wee see that, module elements not in $\mathcal{B}$, y_k is primitive if and only if $k \neq 2$ and $k \neq 2^i-1$. Define $\gamma : H_*(MSO; \mathbb{F}_2) \to \mathcal{B} \otimes \mathbb{F}_2[y_k \big| k \neq 2, k \neq 2^i - 1]$ by

$$\gamma(y_k) = \begin{cases} \zeta_1^2 \otimes 1 & k = 2 \\ \zeta_j \otimes 1 & k = 2^j - 1 \\ 1 \otimes y_k & k \neq 2, k \neq 2^j - 1 \end{cases}$$

Note that γ is a map between polynomial algebras with the same number of generators in each degree. Since γ takes generators to generators, it is an isomorphism. From the description of the comodule structure on $H_*(MSO; \mathbb{F}_2)$, we see that γ is an isomorphism of $\mathcal{A}_*$-comodules. □

6. A spectrum level interpretation

To demonstrate the power of Theorem 4.1, we'll discuss a corollary which gives a "geometric" description of the spectra MO, MU, and MSO, and computes the group structure of the cobordism rings.

Ignoring explicit bases, we can restate Theorem 4.1 more concisely in cohomology.

Theorem 6.1.

$$\begin{cases} H^*(MO; \mathbb{F}_2) & \textit{is a free } \mathcal{A} \textit{ module} \\ H^*(MU; \mathbb{F}_p) & \textit{is a free } \mathcal{A}/(Q_0) \textit{ module for any } p \\ H^*(MSO; \mathbb{F}_q) & \textit{is a free } \mathcal{A}/(Q_0) \textit{ module for odd } q \\ H^*(MSO; \mathbb{F}_2) & \textit{is a direct sum of free } \mathcal{A} \textit{ modules and copies of } \mathcal{A}/\mathcal{A}\mathrm{Sq}^1 \end{cases}$$

Let's take a closer look at the result for MO. On a space level, we have $H^i(MO(n); \mathbb{F}_2) = [MO(n); K(\mathbb{F}_2, i)]$ for every i and n. One can define a spectrum $H\mathbb{F}_2$ with ith space $K(\mathbb{F}_2, i)$. In fact, we can make an Eilenberg-MacLane spectrum HG for any abelian group G. We then get a spectrum level statement $H^k(X; G) = [X, \Sigma^k HG]$ for any spectrum X. We have to take care to define homotopy classes of maps between spectra on the right-hand side. It turns out that the cohomology of the spectrum $H\mathbb{F}_2$ is isomorphic to the Steenrod Algebra. For details and rigorous definitions, see [7] page 185.

By Theorem 4.1, $H^*(MO; \mathbb{F}_2)$ is a free $\mathcal{A}$-module on (the dual of) the vector space $\mathbb{F}_2[x_k \big| k \neq 2^j - 1]$. Let $\{\lambda_\alpha\}$ be a $\mathcal{A}$-module basis for $H^*(MO; \mathbb{F}_2)$, where λ_α has degree $d(\alpha)$. Under the isomorphism

$$H^k(MO; \mathbb{F}_2) = [MO; \Sigma^k H\mathbb{F}_2]$$

each λ_α corresponds to a homotopy class of maps $MO \to \Sigma^{d(\alpha)} H\mathbb{F}_2$. Let

$$\lambda_\alpha : MO \to \Sigma^{d(\alpha)} H\mathbb{F}_2$$

also denote a representative of this homotopy class of maps.

Together, the λ_α define a map $\Lambda : MO \to \prod H\mathbb{F}_2 \simeq \bigvee H\mathbb{F}_2$. By construction, Λ induces an isomorphism on mod 2 cohomology. By consideration of the Bockstein operators and the universal coefficient theorem, Λ induces an isomorphism on integral cohomology and homology. We can then apply the Whitehead theorem to see that MO is homotopy equivalent to a wedge of Eilenberg-MacLane spectra.

A natural question to ask is if we can do something similar for MU and MSO. Do there exist spectra S and T so that MU and MSO split as wedges of S and T, respectively? Since we only have splitting in cohomology at primes, the best we can hope for is a spectrum level splitting of $MU_{(p)}$ and $MSO_{(p)}$ of the localizations. (For definitions spectra and of localizations of spaces, see [8] and [7, Ch. 25, §7].)

The solution for MSO at $p = 2$ again involves Eilenberg-MacLane spectra. The mysterious $\mathcal{A}/\mathcal{A}\mathrm{Sq}^1$ is the cohomology of $H\mathbb{Z}$ (with coefficients in $\mathbb{F}_2$). Just like in the MO case, we get a splitting of $MSO_{(2)}$ into wedges of $H\mathbb{Z}_{(2)}$ and $H\mathbb{F}_2$. For MU and MSO at an odd prime, we need a new spectrum called the Brown-Peterson spectrum, and denoted BP. The Brown-Peterson spectrum has cohomology $\mathcal{A}/(Q_0)$ and we again get analogous splittings of $MU_{(p)}$ and $MSO_{(q)}$. For constructions and in-depth discussion of BP see Ravenel [17], Chapter 4. The following consequence of Theorem 6.1 summarizes these results.

Theorem 6.2. *Let p be any prime and q any odd prime.*

$$\begin{cases} MO & \textit{splits as wedges of suspensions of } H\mathbb{F}_2 \\ MU_{(p)} & \textit{splits as wedges of suspensions of } BP \\ MSO_{(q)} & \textit{splits as wedges of suspensions of } BP \\ MSO_{(2)} & \textit{splits as wedges of suspensions of } H\mathbb{F}_2 \textit{ and of } H\mathbb{Z}_{(2)} \end{cases}$$

For a proof of the BP splittings, see [4, Thm. 1.3]. Brown and Peterson [4] also calculated the homotopy groups of BP. In particular, $\pi_*(BP)$ has no torsion. We can therefore read off the following results (cf. [4, Cor. 1.4]).

Corollary 6.3. *The ring $\pi_*(MU)$ has no torsion and the ring $\pi_*(MSO)$ has no odd torsion.*

Remark 6.4. Since we know generators for the homologies as $\mathcal{A}_*$-comodules, we get more information. As in the proof of the MO case, we can write down explicit homotopy equivalences between the spectra. The homotopy equivalences induce isomorphisms on homotopy groups. This allows us to read of the structures of $\pi_*(MO)$, $\pi_*(MU)$, and $\pi_*(MSO)$ from knowledge of $\pi_*(H\mathbb{F}_2)$, $\pi_*(BP)$, and $\pi_*(H\mathbb{Z}_{(2)})$.

Rather than give details on the construction of BP and the proof of Theorem 6.2, we compute the cobordism rings using the Adams spectral sequence.

7. Adams spectral sequence

We cite some computations of the E_2 page of the Adams spectral sequence. The results are coming word-for-word from Ravenel [17, Ch. 3], where he gives full proofs of everything we claim here.

Recall (cf. [17, Ch.2, §1]) that the E_2 page of the Adams spectral sequence for the prime p is

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathbb{F}_p, H_*(X))$$

To study the E_2 page of the Adams spectral sequence, we use the cobar complex. Let $\bar{\mathcal{A}}_*$ denote the kernel of the counit.

Definition 7.1. The cobar complex for $H_*(X)$ is the complex $C^*_{\mathcal{A}_*}(H_*(X))$ whose sth term is

$$C^s_{\mathcal{A}_*} = \Big(\bigotimes_s \bar{\mathcal{A}}_*\Big) \otimes H_*(X)$$

with coboundary operator $d_s : C^s_{\mathcal{A}_*} \to C^{s+1}_{\mathcal{A}_*}$ given by

$$d_s(a \otimes x) = 1 \otimes a \otimes x + \sum_{i=1}^{s} (-1)^i (a_1 \otimes \cdots a_{i-1} \otimes \Delta a_i \otimes a_{i+1} \cdots \otimes a_s) \otimes x + (-1)^{s+1} a \otimes \psi x$$

where $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is the coproduct and ψ is the comodule map.

Proposition 7.2 ([17], 3.1.2)**.** *The E_2-term for the classical Adams spectral sequence for $\pi_*(X)$ is the cohomology of $C^*_{\mathcal{A}_*}(H_*(X))$.*

Following Ravenel, we use the abbreviation $\mathrm{Ext}(H_*(X))$ for the E_2-term.

Lemma 7.3 ([17], 3.1.3)**.** *Let $a_0 \in Ext^{1,1}_{\mathcal{A}_*}(\mathbb{F}_p, \mathbb{F}_p)$ be the class represented by $[\tau_0]$ for p odd and $[\xi_1]$ for $p = 2$.*

(1) For $s \geq 0$, $Ext^{s,s}(H_*(S^0))$ is generated by a_0^s.
(2) If $x \in \mathrm{Ext}(H_*(X))$ is a permanent cycle represented by $\alpha \in \pi_*(X)$, then $a_0 x$ is a permanent cycle represented by $p\alpha$.

Recall that we've defined the cotensor product (4.3). We have the following change-of-rings result:

Proposition 7.4 ([17], A1.3.13)**.** *Let $f : \Gamma \to \Sigma$ a surjective map of Hopf algebras over $\mathbb{F}_p$. If N is a left Σ-comodule, then*

$$\mathrm{Ext}_\Gamma(\mathbb{F}_p, \Gamma \square_\Sigma N) \cong \mathrm{Ext}_\Sigma(\mathbb{F}_p, N)$$

Lemma 7.5. *Let N be a trivial $\mathcal{A}_*$-comodule. Then*

$$\mathrm{Ext}_{\mathcal{A}_*}(\mathbb{F}_p, N) \cong \mathrm{Ext}_{\mathcal{A}_*}(\mathbb{F}_p, \mathbb{F}_p) \otimes N$$

Lemma 7.6 ([17], 3.1.9)**.** *Let E be an exterior algebra over a field k on primitive generators $\{x_1, x_2 \ldots\}$. If $char(k) \neq 2$, assume each x_i has odd degree. Then*

$$\mathrm{Ext}_E(k, k) = k[g_1, g_2, \ldots]$$

where $g_i \in Ext^{1,|x_i|}$ is represented by $[x_i]$ in the cobar complex.

8. The easier torsion

We show that $\pi_*(MSO)$ has no odd torsion. Because the computation is parallel, we'll also show that $\pi_*(MU)$ has no torsion (odd or even). To be inclusive, we'll also compute torsion of $\pi_*(MO)$ using the same method. We begin with the MO case since it is the easiest and historically happened first.

8.1. **The MO case.**

Theorem 8.1. *We have $\pi_*(MO) \cong \mathbb{F}_2[x_i \big| i \neq 2^j - 1]$.*

Proof. By Theorem 4.1, $H_*(MO;\mathbb{F}_2) \cong P \otimes S$ where $P = \mathcal{A}_* = \mathbb{F}_2[\xi_i \big| i \geq 1]$ and $S = \mathbb{F}_2[x_i \big| i \neq 2^j - 1]$. Using the $H_*(MO;\mathbb{F}_2)$ $\mathcal{A}_*$-comodule structure, we compute

$$E_2 = \mathrm{Ext}_{\mathcal{A}_*}(\mathbb{F}_2, H_*(MO;\mathbb{F}_2)) \cong \mathrm{Ext}_{\mathcal{A}_*}(\mathbb{F}_2, \mathcal{A}_* \otimes S) \cong S$$

Since S lives on the 0th line of E_2, the whole spectral sequence collapses and we get $\pi_*(MO) \cong S$. □

8.2. **The MU and MSO cases.** Let

$$P = \big(\mathcal{A}/(Q_0)\big)^\vee = \begin{cases} \mathbb{F}_2[\xi_1^2, \xi_2^2, \dots] & p = 2 \\ \mathbb{F}_p[\xi_1, \xi_2, \dots] & p \text{ odd} \end{cases}$$

By Theorem 4.1, there exists $v_i \in H_{2i}(MU;\mathbb{F}_p)$ and $u_k \in H_{4k}(MSO;\mathbb{F}_q)$ so that,

$$H_*(MU;\mathbb{F}_p) \cong P \otimes C = P \otimes \mathbb{F}_p[v_i \big| i \neq p^k - 1]$$
$$H_*(MSO);\mathbb{F}_q) \cong P \otimes B = P \otimes \mathbb{F}_p[u_k \big| 2k \neq q^i - 1]$$

Lemma 8.2 (3.1.10)**.** *The E_2 page of the Adams spectral sequence for MU at any prime p is*

$$\mathrm{Ext}_{\mathcal{A}_*}(\mathbb{F}_p, H_*(MU;\mathbb{F}_p)) \cong \begin{cases} \mathbb{F}_p[a_1, a_2, \dots] \otimes C & p = 2 \\ \mathbb{F}_p[a_0, a_1, \dots] \otimes C & p \neq 2 \end{cases}$$

with $a_i \in Ext^{1,2p^i-1}$ represented by $[\tau_i]$ for $p > 2$ and by $[\xi_i]$ for $p = 2$.
The E_2 page for MSO at an odd prime q is

$$\mathrm{Ext}_{\mathcal{A}_*}(\mathbb{F}_q, H_*(MSO;\mathbb{F}_q)) \cong \mathbb{F}_q[a_0, a_1, \dots] \otimes B$$

Proof. For ease of notation, we'll prove this in the MSO case. The MU case is identical, (cf. [17, Ch. 2,§3]). Let $E = \mathcal{A}_* \otimes_P \mathbb{F}_p$. Then $P \otimes E \cong \mathcal{A}_*$ as $\mathbb{F}_p$-vector spaces and as E-comodules by the dual of Theorem 4.5. Note that

$$E = \begin{cases} E(\xi_1, \xi_2, \dots) & p = 2 \\ E(\tau_0, \tau_1, \dots) & p \neq 2 \end{cases}$$

For any let $\mathcal{A}_*$-comodule algebra N, we have

$$\mathcal{A}_* \square_E N \cong P \otimes E \square_E N \cong P \otimes N$$

In particular, this is true for $N = C$ or $N = B$. By Proposition 7.4, the E_2 page is

$$\mathrm{Ext}_{\mathcal{A}_*}(\mathbb{F}_q, P \otimes B) \cong \mathrm{Ext}_{\mathcal{A}_*}(\mathbb{F}_q, \mathcal{A}_* \square_E N) \cong \mathrm{Ext}_E(\mathbb{F}_q, B)$$

Since B is a trivial $\mathcal{A}_*$-comodule, Lemma 7.5,

$$\mathrm{Ext}_E(\mathbb{F}_p, C) \cong \mathrm{Ext}_E(\mathbb{F}_p, \mathbb{F}_p) \otimes C$$

By Lemma 7.6, we see that the E_2 page is $\mathbb{F}_q[g_1, g_2, \dots] \otimes B$ where the g_i are represented by generators for E as an exterior algebra. The result follows. □

Corollary 8.3. *The complex cobordism ring $\pi_*(MU)$ has no torsion and the oriented cobordism ring $\pi_*(MSO)$ has no odd torsion.*

Proof. For either MU at any prime or MSO at an odd prime, the above lemmas show that the E_2 page is generated by things in total degree $t-s=2p^i-1-1=2p^i-2$. Since all generators are in even degree, the differentials must all vanish. Thus $E_2=E_\infty$. By Lemma 7.3, all multiples of a_0^s are represented in $\pi_*(MU)$ by multiples of p^s. Hence $\pi_*(MU)\otimes\mathbb{Z}_{(p)}$ and $\pi_*(MSO)\otimes\mathbb{Z}_{(q)}$ have no torsion. □

Corollary 8.4. *The group $\pi_m(MU)$ is zero for m odd and free abelian for $m=2n$ of rank equal to the number of partitions of n. In other words, $\pi_*(MU)$ is isomorphic (as a group) to $\mathbb{F}_p[\hat{v}_{2i}\big|i\geq 1]$.*

Proof. This follows from Theorem 2.1 and the computation made in Lemma 1.1. □

One can show that $\pi_*(MU)$ is actually a polynomial ring on generators $\hat{v}_{2i}$ of degree $2i$ for every $i\geq 1$. For details on the ring structure, see the discussion following Corollary 3.1.10 in [17].

9. The harder torsion

We compute the two-torsion of MSO using the splitting of the E_2 page given by Pengelley [15].

Recall from Theorem 4.1 that $H_*(MSO;\mathbb{F}_2)\cong L\otimes C$ where

$$L=\mathbb{F}_2[\zeta_1^2,\zeta_2,\zeta_3,\dots]=\left(\mathcal{A}/\mathcal{A}\mathrm{Sq}^1\right)^\vee$$

and $C=\mathbb{F}_2[y_k\big|k\neq 2,k\neq 2^i-1]$.

We would like to mimic the argument for the odd-primary case using the results of §7. With slight modifications, this mostly works. The problem is in applying Lemma 7.5. As we noted in Remark 4.8, the polynomial generators of y_k are not all primitive. The work of Pengelley in [15] is to separate the primitive y_k.

Let $\beta: H_*(MSO;\mathbb{F}_2)\to H_*(MSO;\mathbb{F}_2)$ be the homology Bockstein.

Lemma 9.1. *For $k\neq 2$ and $k\neq 2^j-1$, the map β is given by*

$$\beta(y_k)=\begin{cases} y_{k-1} & k=2i, i\neq 2^l\\ 0 & \textit{otherwise}\end{cases}$$

Proof. By the claim following Lemma 5.1, we have

$$\psi(y_k)=\begin{cases}1\otimes y_1+\zeta_1\otimes 1 & k=1\\ 1\otimes y_k+\zeta_1\otimes y_{k-1} & k=2i, i\neq 2^l\\ 1\otimes y_k & k=2i-1, i\neq 2^l\end{cases}$$

Since y_k is primitive for k odd and $k\neq 2^i-1$, we have $\beta y_k=0$. For $k=2i$, $i\neq 2^l$, we have $y_k=z_k+z_1z_{k-1}$ (cf. Lemma 5.1). Since z_k and z_{k-1} are in the image of the Hurewicz map, they are both primitive. Thus $\beta(z_k)=\beta(z_{k-1})=0$. Since $z_1=y_1$ satisfies $\psi(z_1)=1\otimes z_1+\zeta_1\otimes 1$, we have $\beta(z_1)=1$. Thus for $k=2i$, $i\neq 2^l$ we have

$$\beta(y_k)=\beta(z_k+z_1z_{k-1})=\beta(z_1)z_{k-1}+z_1\beta(z_{k-1})=z_{k-1}$$

□

Thus the map β restricts to a map $C\to C$, which we continue to denote by β. Since $\beta^2=0$, we consider homology defined by β.

Claim 9.2. The E_2 page of the Adams spectral sequence for MSO at $p=2$ sits in a split short exact sequence

$$0 \to \operatorname{Im}\beta \to E_2 \to P(h_0) \otimes H_*(C,\beta) \to 0$$

We get this by placing $L \otimes C$ into a split short exact sequence, and applying Ext computations as in §8.

Lemma 9.3. *We have a split short exact sequence of $\mathcal{A}_*$-comodules*

$$0 \to \mathcal{A}_* \otimes \operatorname{Im}\beta \to L \otimes C \to L \otimes H_*(C,\beta) \to 0$$

Proof. As $\mathbb{F}_2$ vector spaces, we have a split short exact sequence

$$0 \to \operatorname{Im}\beta \to \ker\beta \to H_*(C,\beta) \to 0$$

This extends to a split short exact

$$0 \to E(\zeta_1) \otimes \operatorname{Im}\beta \to C \to H_*(C,\beta) \to 0$$

as $E(\zeta_1)$-comodules. Let $E = E(\zeta_1)$. Applying $\mathcal{A}_* \square_E -$, we get a split short exact

$$0 \to \mathcal{A}_* \otimes \operatorname{Im}\beta \to \mathcal{A}_* \square_E C \to \mathcal{A}_* \square_E H_*(C,\beta) \to 0$$

of $\mathcal{A}_*$-comodules. Now $\mathcal{A}_* \cong L \otimes E$ so that $\mathcal{A}_* \square_E H_*(C,\beta) \cong L \otimes H_*(C,\beta)$. □

To obtain the splitting of the E_2-page, we apply $\operatorname{Ext}_{\mathcal{A}_*}(\mathbb{F}_2, -)$ to the above exact sequence.

Proof of Claim 9.2. Since the exact sequence in Lemma 9.3 is split, we get a split short exact sequence

$$0 \to \operatorname{Ext}_{\mathcal{A}_*}(\mathbb{F}_2, \mathcal{A}_* \otimes \operatorname{Im}\beta) \to \operatorname{Ext}_{\mathcal{A}_*}(\mathbb{F}_2, L \otimes C) \to \operatorname{Ext}_{\mathcal{A}_*}(\mathbb{F}_2, L \otimes H_*(C,\beta)) \to 0$$
$$0 \to \operatorname{Im}\beta \to \operatorname{Ext}_{\mathcal{A}_*}(\mathbb{F}_2, H_*(MSO;\mathbb{F}_2)) \to \operatorname{Ext}_{\mathcal{A}_*}(\mathbb{F}_2, L \otimes H_*(C,\beta)) \to 0$$
$$0 \to \operatorname{Im}\beta \to E_2 \to \operatorname{Ext}_{\mathcal{A}_*}(\mathbb{F}_2, L \otimes H_*(C,\beta)) \to 0$$

We now want to apply the results of §7 to $\operatorname{Ext}_{\mathcal{A}_*}(\mathbb{F}_2, L \otimes H_*(C,\beta))$. To do so, we need an exterior algebra E so that $\mathcal{A}_* = L \otimes E$. We already have an exterior algebra in the works, namely what we have been calling $E = E(\zeta_1)$. As $\mathbb{F}_2$-vector spaces, $\mathcal{A}_* = L \otimes E$. By the dual of Theorem 4.5, we have $\mathcal{A}_* = L \otimes E$ as E-modules.

Now

$$\mathcal{A}_* \square_E H_*(C,\beta) \cong L \otimes E \square_E H_*(C,\beta) \cong L \otimes H_*(C,\beta)$$

By Proposition 7.4,

$$\operatorname{Ext}_{\mathcal{A}_*}(\mathbb{F}_2, L \otimes H_*(C,\beta)) \cong \operatorname{Ext}_E(\mathbb{F}_2, \mathbb{F}_2) \otimes H_*(C,\beta)$$

Now Lemma 7.6 gives,

$$\operatorname{Ext}_{\mathcal{A}_*}(\mathbb{F}_2, L \otimes H_*(C,\beta) \cong P(h_0) \otimes H_*(C,\beta))$$

where $P(h_0)$ is a polynomial algebra (over $\mathbb{F}_2$) on the single generator $g_i \in Ext^{1,|\zeta_1|}$ represented by $[\zeta_1]$. Since $|\zeta_1| = 1$, the generator h_0 lives in $E_2^{1,1}$ as claimed. □

We have proven the existence of a split exact

$$0 \to \operatorname{Im}\beta \to E_2 \to P(h_0) \to H_*(C,\beta) \to 0$$

On the 0th line, this becomes

$$0 \to \operatorname{Im}\beta \to E_2^{0,*} \to H_*(C,\beta) \to 0$$

Define

$$u_{4i} = \begin{cases} y_{4i} & i = 2^j \\ y_{2i}^2 & i \neq 2^j \end{cases}$$

Lemma 9.4. *As algebras,* $H_*(C, \beta) \cong \mathbb{F}_2[u_{4i}]_{i \geq 1}$*, and the inclusion* $\mathbb{F}_2[u_{4i}] \to \ker \beta$ *gives a splitting of*

$$0 \to \operatorname{Im} \beta \to E_2^{0,*} \cong \ker \beta \to H_*(C, \beta) \to 0$$

Let $Q : \pi_* MSO \to E_\infty^{0,*} \to E_2^{0,*}$ be projection onto the 0-line.

Theorem 9.5. *There exist* $\hat{u}_{4i} \in \pi_{4i} MSO$, $i \geq 1$ *so that*

$$\pi_* MSO \cong \mathbb{Z}[\hat{u}_{4i}] \oplus \operatorname{Im} \beta$$

We have $Q(\hat{u}_{4i}, 0) = \hat{u}_{4i}$ *and* $Q(0, b) = b$ *for* $b \in \operatorname{Im} \beta$*. The product structure is given by*

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2, Q(a_1, 0) b_2 + Q(a_2, 0) b_1 + b_1 b_2)$$

Proof. First note that all differentials vanish since the h_0 towers are only in dimensions $4i$. Thus $E_2 = E_\infty$. Now towers represent integral summands in $\pi_*(MSO)$. Since $\pi_* MSO$ has no odd torsion, the 2-torsion represented by $\operatorname{Im} \beta$ gets identified (via Q) with the whole torsion ideal in $\pi_* MSO$. As a consequence of Theorem 2.1, we have

$$\pi_*(MSO)/\operatorname{Im} \beta \cong H_*(MSO; \mathbb{Z})/(\text{torsion})$$

as rings. By the computations made in Lemma 1.1, $H_*(MSO; \mathbb{Z}) \cong \mathbb{Z}[\hat{u}_{4i}]$ with $\hat{u}_{4i} \in H_{4i}(MSO; \mathbb{Z})$. We therefore have an exact sequence

$$0 \to \operatorname{Im} \beta \to \pi_*(MSO) \to \mathbb{Z}[\hat{u}_{4i}] \to 0$$

We claim this splits. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Im} \beta & \longrightarrow & \pi_* MSO & \xrightarrow{\hat{\rho}} & \pi_* MSO/\operatorname{Im} \beta & \longrightarrow & 0 \\ & & \downarrow & & \downarrow Q & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Im} \beta & \longrightarrow & E_2^{0,*} & \xrightarrow{\rho} & H_*(C, \beta) & \longrightarrow & 0 \end{array}$$

of ring maps. To get a splitting of $\hat{\rho}$, we can lift any algebra map splitting ρ. But we gave a splitting of ρ in Lemma 9.4. Thus $\pi_*(MSO) \cong \mathbb{Z}[\hat{u}_{4i}] \oplus \operatorname{Im} \beta$. □

10. Relation to other proofs

We have identified the torsion in $\pi_*(MSO)$ with the image $\operatorname{Im} \beta$ of the homology Bockstein. The goal of this section is to give a more geometric description of the 2-torsion. The first description of the 2-torsion was given by Wall in [21]. Wall describes the torsion as the image of a map ∂_3 given on manifolds. This geometric interpretation allows us to explicitly describe the manifolds in Ω_*^{SO} creating the torsion. We will relate Pengelley's generators of $H_*(MSO; \mathbb{F}_2)$ to Wall's generators of $\mathfrak{N}_*$ and then describe how β and ∂_3 relate. To complete his description of Ω_*^{SO}, Wall shows how ∂_3 fits into an exact triangle

$$\begin{array}{ccc} \Omega_*^{SO} & \xrightarrow{\ 2\ } & \Omega_*^{SO} \\ & {\scriptstyle \partial_3} \nwarrow \quad \swarrow {\scriptstyle r} & \\ & \mathfrak{W} & \end{array}$$

We will end this section by giving a second, faster, description of Wall's exact triangle due to Atiyah in [3]. Atiyah's work gives a third description of torsion as the image of a map i^*. In an attempt to prevent confusion, we denote Pengelley's [15] map by β, Atiyah's [3] by i^*, and Wall's [21], by ∂_3 (this is consistent with Wall's notation).

We start by paying a debt owed to the reader. In the proof of Theorem 4.1 for MSO at $p = 2$, we claimed the existence of a certain basis $\{z_k\}$ for $H_*(MO;\mathbb{F}_2)$. Define $z_{2^j-1} = r_{2^j-1}$ as in Lemma 1.1. For z_k with $k \neq 2^j - 1$, define z_k to be the image under the Hurewicz map h of the class of the manifold X_k, where X_k is defined as follows.[1] More precisely, if $T : \mathfrak{N}_* \to \pi_*(MO)$ is Thom's map, then $z_k = h(T([X_k]))$. For what follows, we refer the reader to [21] for details.

Construction 10.1. For $k = 2^j$ let $X_k = \mathbb{RP}^k$.

For $k \neq 2^j$, X_{2k} will be a Dold manifold $P(m, n)$ and X_{2j-1} will be a manifold Wall calls $Q(m, n)$. Let C_2 act on $S^m \subset \mathbb{R}^{m+1}$ by $x \mapsto -x$ and on $\mathbb{CP}^n$ by conjugation. Let $P(m, n)$ be the quotient of $S^m \times \mathbb{CP}^n$ by the identification $(-x, z) \sim (x, \bar{z})$.

Give $\mathbb{R}^{m+1}$ coordinates $(x_1, \ldots, x_{m+1})$. Let $T : S^m \to S^m$ be reflection across the plane $x_m = 0$. Let $A : P(m, n) \to P(m, n)$ the map induced by $(x, z) \mapsto (Tx, z)$. Let $Q(m, n)$ be the mapping torus of A, i.e. $Q(m, n)$ is the quotient of $P(m, n)\times[0, 1]$ by the identification $(p, 0) \sim (Ap, 1)$.

Write $k = 2^{m-1}(2n + 1)$ with $s \neq 0$ and define

$$X_{2k} = P(2^r - 1, 2^r s) \text{ and } X_{2k-1} = Q(2^r - 1, 2^r s)$$

We determined the comodule structure of $z_{2^j-1} = r_{2^j-1}$ in Lemma 1.1. For details on the comodule structure of z_k for $k \neq 2^j - 1$, see [15].

(1) The manifolds X_k with $k \neq 2^j - 1$ are polynomial generators for $\mathfrak{N}_*$.
(2) If m is odd and n is even, then $P(m, n)$ is orientable and A reverses the orientation.
(3) The map $\beta : Q(m, n) \to S^1$ sending $\beta(p, s) = s$ is a fiber map with fiber $P(m, n)$.
(4) If $u \in H^1(S^1, \mathbb{F}_2)$ is a generator, then $\beta^*(u) = w_1(Q(m, n))$ is the first Stiefel-Whitney class.[2] Therefore the homotopy class
$$[\beta] \in [Q(m, n), S^1] = H^1(Q(m, n), \mathbb{Z})$$
maps to $w_1(Q(m, n))$ under the map $H^1(Q(m, n), \mathbb{Z}) \to H^1(Q(m, n), \mathbb{F}_2)$.

Wall's map ∂_3 is a generalization of the relation $Q(m, n) \to P(m, n)$, constructing an oriented manifold from a manifold with integral Stiefel-Whitney class.

Wall's Map ∂_3: Define $\mathfrak{W} \subset \mathfrak{N}_*$ to be the set of classes $[M]$ where $w_1(M)$ comes from an integral class. For any $M \in \mathfrak{N}_*$, let $f : M \to S^1$ represent

$$w_1(M) \in H^1(M; \mathbb{Z}) = [M, S^1]$$

Set $\partial_3(M) = f^{-1}\{0\}$.[3] We claim that $\partial_3(M)$ is always orientable so that $\partial_3 : \mathfrak{W} \to \Omega^{SO}_*$.

The following Lemma identifies $\operatorname{Im} \partial_3$ with the 2-torsion in Ω^{SO}_*.

[1] Pengelley [15] calls these M_k. The notation X_k is consistent with Wall [21].

[2] The proof of this claim uses the Serre spectral sequence. See the proof of Lemma 4 in [21] for details.

[3] This isn't quite true. We first need to approximate f by a smooth map.

Lemma 10.2. *An oriented manifold V^{n-1} can be obtained by the above construction from some M^n if and only if $2[V] = 0$ in Ω_*^{SO}.*

This is Lemma 1 in [21]. The proof is geometric and constructive.

Remark 10.3. The $(n-1)$-manifold $\partial_3(M)$ is dual to $w_1(M)$ in the following sense. Let $V = \partial_3(M) = f^{-1}\{0\}$ and $j : V \to M$ be the inclusion. Let $[V] \in H_{n-1}(V;\mathbb{F}_2$ and $[M] \in H_n(M;\mathbb{F}_2)$ be the fundamental classes. Let $\sigma_M : M \to BO$ classify the tangent bundle of M. Then $\sigma_M^*(w_1)$ is the Poincaré dual of $j_*([V])$.

There should be some relation between the facts that

(1) we have defined z_1 so that $\Phi(z_1)$ is the dual of w_1, and
(2) $\partial_3(M)$ is the dual of $w_1(M)$.

We now explain this relation and use it to motivate the definition of $y_{2i} = z_{2i} + z_1 z_{2i-1}$ for $k \neq 2^l$.

Claim 10.4. Let $i \neq 2^l$. In $H_*(BO;\mathbb{F}_2)$ we have $w_1 \cap \Phi(z_{2i}) = \Phi(z_{2i-1})$.

Proof. For any k, let $\sigma_k^* : X_k \to BO$ the map classifying the tangent bundle of X_k. For $i \neq 2^l$, let $j : X_{2k-1} \to X_{2k}$ be the inclusion and note that $\sigma_{2k} \circ i = \sigma_{2k-1}$. By Remark 10.3,

$$\begin{aligned} \sigma_{2i}^*(w_1) \cap [X_{2i}] &= j_*([X_{2i-1}]) \\ (\sigma_{2i})_*\big(\sigma_{2i}^*(w_1) \cap [X_{2i}]\big) &= (\sigma_{2i})_*\big(j_*([X_{2i-1}])\big) \\ w_1 \cap (\sigma_{2i})_*([X_{2i}]) &= (\sigma_{2i-1})_*([X_{2i-1}]) \end{aligned}$$

Let $\alpha : \mathfrak{N}_* \to H_*(BO)$ by $[M] \to (\sigma_M)_*([M])$. By [18, pg. 35], the diagram

$$\begin{array}{ccc} \mathfrak{N}_* & \xrightarrow{\ \alpha\ } & H_*(BO;\mathbb{F}_2) \\ \big\downarrow T & & \big\uparrow \Phi \\ \pi_*(MO) & \xrightarrow{\ h\ } & H_*(MO;\mathbb{F}_2) \end{array}$$

commutes.[4] Since $z_k = h(T(X_k))$,

$$w_1 \cap \Phi(z_{2i}) = w_1 \cap (\sigma_{2i})_*([X_{2i}]) = (\sigma_{2i-1})_*([X_{2i-1}]) = \Phi(z_{2i-1})$$

□

A similar argument shows that for every $a \in H^{2k-1}(BO;\mathbb{F}_2)$, we have

$$\langle aw_1, \Phi(z_{2k})\rangle = \langle a, \Phi(z_{2k-1})\rangle$$

where $\langle -, - \rangle$ denotes evaluation of a cohomology class on a homology class of the same degree. Since $\Phi(z_1)$ is the dual of w_1,

$$\begin{aligned} \langle aw_1, \Phi(y_{2i}) &= \langle aw_1, \Phi(z_{2i} + \Phi(z_1)\Phi(z_{2i-1}))\rangle \\ &= \langle a, \Phi(z_{2k-1})\rangle + \langle aw_1, \Phi(z_1)\Phi(z_{2i-1}) \\ &= 2\langle a, \Phi(z_{2k-1})\rangle \end{aligned}$$

is zero. Now $H^*(BSO;\mathbb{F}_2) \cong H^*(BO;\mathbb{F}_2)/(w_1)$ so that $\Phi(y_n) \in H_*(BSO;\mathbb{F}_2)$.

We can now prove the following Lemma from §4 (cf. Lemma 5.1).

---

[4] The statement in [18] uses the classifying map for the normal bundle. Since we're working over $\mathbb{F}_2$, the sign differences between the classifying map of the normal bundle and that of the tangent bundle do not matter.

Lemma. *The homology $H_*(MSO;\mathbb{F}_2)$ sits inside $H_*(MO;\mathbb{F}_2)$ as the polynomial ring $\mathbb{F}_2[y_k]$.*

Proof. We have just shown that for $k = 2i$ with $i \neq 2^l$, $y_k \in H_*(MSO;\mathbb{F}_2)$. For $k = 2^i - 1$, we have $y_k = z_k = \Phi^{-1}(e_k)$ coming from the Thom isomorphism. Since

$$\begin{CD} H_*(MSO) @>\Phi>> H_*(BSO;\mathbb{F}_2) \\ @VVV @VVV \\ H_*(MO) @>\Phi>> H_*(BO;\mathbb{F}_2) \end{CD}$$

commutes, $y_k \in H_*(MSO;\mathbb{F}_2)$.

For $k \neq 2^j$, we have $y_{2k-1} = z_{2k-1} = h(T([X_{2k-1}]))$ where $T : \mathfrak{N}_* \to \pi_*(MO)$ is Thom's map. Since X_{2k-1} is orientable, $T[X_{2k-1}]$ is in the image of $\pi_*MSO \to \pi_*MO$. Since

$$\begin{CD} \pi_*(MSO) @>h>> H_*(MSO;\mathbb{F}_2) \\ @VVV @VVV \\ \pi_*(MO) @>h>> H_*(MO;\mathbb{F}_2) \end{CD}$$

commutes, y_k is in the image of $H_*(MSO;\mathbb{F}_2)$ in $H_*(MO;\mathbb{F}_2)$.

For $k = 2^j$, we have $y_k = z_{k/2}^2 = h(T[\mathbb{RP}^{k/2}])^2$. By [21, Lemma 7], $(\mathbb{RP}^{k/2})^2$ is cobordant to $\mathbb{CP}^{k/2}$. Since $\mathbb{CP}^{k/2}$ is orientable, we again have $y_k \in H_*(MSO;\mathbb{F}_2)$.

The statement follows by a dimension count. □

Remark 10.5. Let $k \neq 2^j - 1$. For k odd, X_k is orientable and therefore in $\mathfrak{W}$. For k even, X_k is in $\mathfrak{W}$ by construction. The algebra $\mathfrak{W}$ is therefore generated by $\{X_k\}$. Analogous to Lemma 9.1, the map ∂_3 is given by

$$\partial_3(X_k) = \begin{cases} X_{k-1} & k = 2n, n \neq 2^j \\ 0 & \text{otherwise} \end{cases}$$

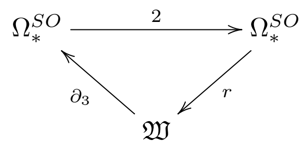
Define $f : H_*(MO;\mathbb{F}_2) \to H_*(MSO;\mathbb{F}_2)$ by $f(z_1) = 0$ and for $k \neq 1$,

$$f(z_k) = y_k = \begin{cases} z_{k/2}^2 & k = 2^i, i \neq 0 \\ z_k + z_1 z_{k-1} & k = 2i, i \neq 2^j \\ z_k & k = 2i - 1 \end{cases}$$

Following around generators, we see that the following commutes:

$$\begin{CD} \mathfrak{W} @>T>> \pi_*(MO) @>h>> H_*(MO;\mathbb{F}_2) \\ @. @. @VVfV \\ @V\partial_3VV @. H_*(MSO;\mathbb{F}_2) \\ @. @. @VV\beta V \\ \Omega_*^{SO} @>T>> \pi_*(MSO) @>h>> H_*(MSO;\mathbb{F}_2) \end{CD}$$

Wall goes on to show that ∂_3 fits into an exact triangle



where 2 is the multiplication by 2 map and r is the forgetful map.

Atiyah's Description: Right after Wall first computed the 2-torsion, Atiyah produced an alternative description of Wall's exact triangle. We now discuss Atiyah's approach.

We start by defining MSO-cohomology. Let X be a finite CW-complex and Y a subcomplex of X.

Definition 10.6. Define $MSO^k(X, Y) = [X/Y, MSO]_k$.

Recall that in Section §7, we defined Eilenberg-MacLane spectra HG and noted that for any spectrum T, we have $H^*(T, G) = [T, HG]$. This motivates calling $[X/Y, MSO]$ the MSO-cohomology of X/Y. One can formalize this by axiomatizing cohomology and defining "generalized cohomology theories." For details, see [7]. The group $[X/Y, MSO]$ is then the (relative) generalized cohomology theory defined by the spectrum MSO. It follows that MSO^k is a functor satisfying all the properties of H^* that we care about. In particular, MSO^k is a homotopy invariant and gives long exact sequences for cofibrations. Define $MSO^k(X) = MSO^k(X, *)$ and note that

$$MSO^{-k}(S^m) = \lim_{n\to\infty}[\Sigma^n\Sigma^k S^m, MSO(n)] = \pi_{m+k}(MSO) \cong \Omega^{SO}_{m+k}$$

The cofibration

$$S^1 \cong \mathbb{RP}^1 \to \mathbb{RP}^2 \to \mathbb{RP}^2/\mathbb{RP}^1 \cong S^2$$

gives a long exact sequence

$$\cdots \to MSO^n(\mathbb{RP}^2, \mathbb{RP}^1) \to MSO^n(\mathbb{RP}^2) \to MSO^n(\mathbb{RP}^1) \to MSO^{n+1}(\mathbb{RP}^2, \mathbb{RP}^1) \to \cdots$$

Defining $\mathfrak{W}_k = MSO^{2-k}(\mathbb{RP}^2)$, the long exact sequence becomes

$$\cdots \Omega_k^{SO} \to \mathfrak{W}_{k+1} \xrightarrow{i^*} \Omega_k^{SO} \to \Omega_k^{SO} \to \mathfrak{W}_k \xrightarrow{i^*} \Omega_{k-1}^{SO} \cdots$$

Define $\mathfrak{W} = \bigoplus \mathfrak{W}_k$. By [3], this notation is not contradictory. Wall's $\mathfrak{W}$ and Atiyah's $\mathfrak{W}$ agree. Moreover, Wall's exact triangle unwinds into the long exact sequence just described.

Let i^* denote the map $\mathfrak{W} \to \Omega_*^{SO}$. Consider $\mathbb{RP}^1$ and $\mathbb{RP}^2$ as based spaces with base point $\mathbb{RP}^0$. Unraveling all of this, $i^* : [\mathbb{RP}^2, MSO]_* \to [\mathbb{RP}^1, MSO]_*$ is just the map induced by the inclusion $i : \mathbb{RP}^1 \to \mathbb{RP}^2$. This gives a third description of the torsion in Ω_*^{SO}.

11. Consequences

We list some consequences of the structures of $\mathfrak{N}_*, \Omega_*^U$, and Ω_*^{SO}.

Lemma 11.1. *The ring Ω_*^{SO} has no 2^k-torsion for $k > 1$.*

This is a consequence of Proposition 6.2. For Wall's original proof, see [21, Thm. 2].

Proposition 11.2. *Two manifolds M and N are cobordant if and only if they have the same Stiefel-Whitney numbers. Equivalently, a manifold is a boundary if and only if its Stiefel-Whitney numbers vanish.*

For a proof, see [7, Ch. 25 §5].
We have similar results for Ω_*^U and Ω_*^{SO}.

Proposition 11.3. *Two stably almost complex manifolds M and N are complex cobordant if and only if they have the same Chern numbers.*

For a proof, see [18, pg. 117].

Proposition 11.4. *Two oriented manifolds M and N are oriented cobordant if and only if they have the same Stiefel-Whitney and Pontrjagin numbers.*

For a proof of necessity, see [13, pg. 186].[5] For a proof of sufficiency, see [21, Cor. 1]. Analogous statements for other cobordism theories are collected in [18, Ch. 4].

Corollary 11.5. *Let M be any manifold (not necessarily oriented). Then $M \times M$ is unoriented cobordant to an oriented manifold.*

Proof. We have shown that $\mathfrak{N}_*$ is a polynomial algebra on a single generator in each degree $k \neq 2^j - 1$. One can produce representatives $\{X_k\}_{k \neq 2^j - 1}$ for such generators where $X_{2k} = \mathbb{RP}^{2k}$ and X_{2k+1} (as above) is an oriented Dold manifold for all k. Since the square of an oriented manifold is oriented, it suffices to show that $(\mathbb{RP}^{2k})^2$ is cobordant to an oriented manifold for all k. We claim that $(\mathbb{RP}^{2k})^2$ and $\mathbb{CP}^{2k}$ have the same Stiefel-Whitney and Pontrjagin numbers. By Proposition 11.4, $(\mathbb{RP}^{2k})^2$ is cobordant to $\mathbb{CP}^{2k}$. For details, see [21, Prop. 3]. □

Remark 11.6. One also has the following: If M is any oriented manifold, then $M \times M$ is unoriented cobordant to a spin manifold (see [18] Ch. 4 for definitions). This was proved by P.G. Anderson in [2]. The proof mimics Wall's proof of Corollary 11.5 that we have just given. In [11], Milnor showed that the square of any complex projective space $\mathbb{CP}^n$ is (unoriented) cobordant to quaternionic projective space $\mathbb{HP}^n$ (a spin manifold). Anderson then constructs additional generators for Ω_*^{SO} and proves the conjecture on these generators.

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[5]This proves the Pontrjagin number part of necessity. The necessity of the Stiefel-Whitney numbers agreeing follows from the unoriented result, Proposition 11.2.

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