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# THE $(n+2)^{nd}$ HOMOTOPY GROUP OF THE $n$-SPHERE

By George W. Whitehead

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In this note we shall prove that $\pi_{n+2}(S^n)$ is cyclic of order two for all $n \geqq 2$[1]. This contradicts a result announced by Pontrjagin[2] in 1938.

The notation used here will follow that of my paper[3], hereafter referred to as GHI. The proof depends on a theorem recently proved by Blakers and Massey[4], together with a slight generalization of some of the results of GHI.

The following facts about $\pi_{n+2}(S^n)$ are known[5]:

(1) $\pi_4(S^2)$ is cyclic of order two;

(2) the suspension homomorphism $E: \pi_4(S^2) \to \pi_5(S^3)$ is onto;

(3) $E: \pi_{n+2}(S^n) \to \pi_{n+3}(S^{n+1})$ is an isomorphism onto for $n \geqq 3$. Hence it is sufficient to prove that $E: \pi_4(S^2) \to \pi_5(S^3)$ is an isomorphism into.

We first note that, as a consequenec of Theorem 3 of HGT, the homomorphism $\bar{\psi}_{r,r}: \pi_{n+1}(E^{2r}, \dot{E}^{2r}) \to \pi_{n+1}(S^r \times S^r, S^r \vee S^r)$ of GHI is an isomorphism into if $n \leqq 4r-4$, and therefore, if $h$ is an admissible mapping of $S^{2r-1}$ onto $\dot{E}^{2r}$, the homomorphism $\Psi_r = \psi_{r,r} \circ \partial^{-1} \circ \mathbf{h}: \pi_n(S^{2r-1}) \to \pi_{n+1}(E^{2r}, \dot{E}^{2r})$ is an isomorphism into for $n \leqq 4r-4$. Then if $\alpha \in \pi_n(S^r)$, $n \leqq 4r-4$, and if $Q\boldsymbol{\varphi}_r(\alpha) \in \text{Image } \psi_{r,r}$,

$$\pi_n(S^r) \xrightarrow{\boldsymbol{\varphi}_r} \pi_n(S^r \vee S^r) \xrightarrow{Q} \pi_{n+1}(S^r \times S^r, S^r \vee S^r) \xleftarrow{\Psi_r} \pi_n(S^{2r-1})$$

we may define $H(\alpha) = \Psi_r^{-1} Q\boldsymbol{\varphi}_r(\alpha)$, and $H$ is a homomorphism of a subgroup of $\pi_n(S^r)$ into $\pi_n(S^{2r-1})$.

Many of the results of §5 of GHI can be extended. We shall need the following extension of Theorem 5.40.

Lemma 1. *Let $h_r: S^{2r-1} \to S^r$ be a fibre map with Hopf invariant $\iota_{2r-1}$ $(r = 2, 4, or\ 8)$. Then if $n \leqq 4r-4$, $\alpha \in \pi_n(S^{2r-1})$, $\beta \in \pi_{n-1}(S^{r-1})$, then the generalized Hopf invariant of $\mathbf{h}_r(\alpha) + E(\beta)$ exists and is equal to $\alpha$.*

This lemma depends on the corresponding extensions of Theorems 5.15 and 5.19 of GHI, the proofs of which are unchanged.

Corollary. *If $\alpha_2$ is the non-zero element of $\pi_4(S^2)$, then $H(\alpha_2)$ exists and is the non-zero element $\nu_3$ of $\pi_4(S^3)$.*

We shall prove that $E(\alpha_2) \neq 0$ by extending Theorem 7.33 of GHI as follows.

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[1] I have been informed that this result has also been obtained by Pontrjagin. His proof is unknown to me. *Added in proof:* Pontrjagin's proof has been published in C.R. Acad. Sci. URSS, 70 (1950), pp. 957–959. His proof is entirely different from mine.

[2] C. R. Acad. Sci. URSS, 19 (1938), pp. 147–149, 361–363.

[3] Ann. of Math., 51 (1950), pp. 192–237.

[4] Proc. Nat. Acad. Sci. U. S. A., 35 (1949), pp. 322–328. A full exposition will appear in Ann. Math. This paper will be referred to as HGT.

[5] H. Freudenthal, Comp. Math. 5 (1937), pp. 299–314; W. Hurewicz and N. E. Steenrod, Proc. Nat. Acad. Sci. U. S. A., 27 (1941), pp. 60–64.

THEOREM. *If* $\alpha \in \pi_n(S^r)$ *and* $E(\alpha) = 0$, *and if* $H(\alpha)$ *exists, then*

$$\begin{cases} H(\alpha) = 0 & \text{if } r \text{ is odd and } n \leqq 4r - 4; \\ H(\alpha) \in 2\pi_n(S^{2r-1}) & \text{if } r \text{ is even and } n \leqq 3r - 2. \end{cases}$$

The proof of this theorem depends on an extension of Theorems 7.8 and 7.28 of GHI. Consider the diagram

$$\begin{array}{ccccc} \pi_{r+1}^{n+2} & \underset{\Lambda_0''}{\overset{\Lambda_0'}{\rightrightarrows}} & \pi_{n+3}(S^{r+1} \times S^{r+1}, S^{r+1} \vee S^{r+1}) & \xleftarrow{\Psi_{r+1}} & \pi_{n+2}(S^{2r+1}) \\ \Delta \downarrow & & \uparrow A & & \uparrow (-1)^r E \circ E \\ \pi_n(S^r) & \xrightarrow{Q_0 \varphi_r} & \pi_{n+1}(S^r \times S^r, S^r \vee S^r) & \xleftarrow{\Psi_r} & \pi_n(S^{2r-1}) \end{array}$$

The homomorphisms $\Delta$, $A$, $\Lambda_0'$, $\Lambda_0''$ were defined in §6 of GHI, and the proofs of the following facts are unchanged:

$$(1)\ A \circ \Psi_r = (-1)^r \Psi_{r+1} \circ (E \circ E);$$

$$(2)\ \Lambda_0' - \Lambda_0'' = A \circ Q \circ \varphi_r \circ \Delta.$$

In order to show that

$$(3)\ \Lambda_0'' = (-1)^{r+1} \Lambda_0'$$

we need the following extension of Theorem 4.22 of GHI.

LEMMA. *If* $\xi \in \text{Image } \Psi_{r+1}$, *and if* $n \leqq 4r - 2$, *then* $\boldsymbol{\delta}_{r+1}(\xi) = (-1)^{r+1}\xi$.

For if $n \leqq 4r - 2$ and $\gamma \in \pi_{n+3}(E^{r+1} \times E^{r+1}, (E^{r+1} \times E^{r+1})^{\cdot})$ then

$$((-1)^r \iota_{2r+1}) \circ \partial\gamma = (-1)^r \partial\gamma$$

because $\partial\gamma$ is a suspension and therefore (3.64) of GHI holds. Thus the proof of Theorem 4.22 applies without change.

The remainder of the proof of (3) duplicates that of Theorem 7.28 of GHI.

We now prove the theorem. Let $\alpha \in \pi_n(S^r)$ and suppose $E(\alpha) = 0$. Then there is an element $\xi \in \pi_{r+1}^{n+2}$ such that $\Delta(\xi) = \alpha$. Therefore we can combine (2) and (3) to conclude that

$$\begin{aligned} AQ\varphi_r(\alpha) &= AQ\varphi_r\Delta(\xi) = \Lambda_0'(\xi) - \Lambda_0''(\xi) \\ &= [1 + (-1)^r]\Lambda_0'(\xi). \end{aligned}$$

Now $Q\varphi_r(\alpha) = \Psi_r H(\alpha)$ because $H(\alpha)$ exists, and therefore

$$\begin{aligned} [1 + (-1)^r]\Lambda_0'(\xi) &= A\Psi_r H(\alpha) \\ &= (-1)^r \Psi_{r+1} EEH(\alpha). \end{aligned}$$

If $r$ is odd and $n \leqq 4r - 4$, then $\Psi_{r+1}$ and $E \circ E$ are isomorphisms into and we conclude that $H(\alpha) = 0$. If $r$ is even and $n \leqq 3r - 2$, then $\Psi_{r+1}$ is an isomorphism onto and $E \circ E$ is an isomorphism onto, and therefore

$$H(\alpha) = 2E^{-1}E^{-1}\Psi_{r+1}^{-1}\Lambda_0'(\xi) \in 2\pi_n(S^{2r-1}).$$

The proof is complete.

COROLLARY. *The suspension of the non-zero element $\alpha_2 \in \pi_4(S^2)$ is not zero.*

For $H(\alpha_2) = \nu_3 \notin 2\pi_4(S^3) = 0$.

COROLLARY. *The suspension homomorphism maps $\pi_4(S^2)$ isomorphically onto $\pi_5(S^3)$.*

COROLLARY. *The groups $\pi_{11}(S^5)$ and $\pi_{23}(S^9)$ are non-zero.*

For we may apply our theorem to the non-zero elements $\nu_4' \circ \nu_7' \in \pi_{10}(S^4)$ and $\nu_8'' \circ \nu_{15}'' \in \pi_{22}(S^8)$ constructed in §8 of GHI. By Lemma 1, we see that

$$H(\nu_4' \circ \nu_7') = \nu_7' \in \pi_{10}(S^7)$$

and $H(\nu_8'' \circ \nu_{15}'') = \nu_{15}'' \in \pi_{22}(S^{15})$. If $\nu_7' = 2\alpha$ with $\alpha \in \pi_{10}(S^7)$, then $\alpha = E^3\alpha'$ with $\alpha' \in \pi_7(S^4)$. Since $\nu_7' = E^3\nu_4'$ we have

$$E^3(2\alpha' - \nu_4') = 0.$$

Since $E^2 : \pi_8(S^5) \to \pi_{10}(S^7)$ is an isomorphism, we have

$$E(2\alpha' - \nu_4') = 0$$

and therefore $H(2\alpha' - \nu_4') = 2H(\alpha') - H(\nu_4')$ must belong to $2\pi_7(S^7)$. But $H(\nu_4') = \iota_7 \notin 2\pi_7(S^7)$, a contradiction. Similarly, $\nu_{15}'' \notin 2\pi_{22}(S^{15})$.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.