MODELS OF G-SPECTRA AS PRESHEAVES OF SPECTRA

BERTRAND GUILLOU AND J.P. MAY

Abstract. Let G be a finite group. We give Quillen equivalent models for the category of G-spectra as categories of spectrally enriched functors from explicitly described domain categories to nonequivariant spectra. Our preferred model is based on equivariant infinite loop space theory applied to elementary categorical data. It recasts equivariant stable homotopy theory in terms of point-set level categories of G-spans and nonequivariant spectra. We also give a more topologically grounded model based on equivariant Atiyah duality.

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Introduction

The equivariant stable homotopy category is of fundamental importance in algebraic topology. It is the natural home in which to study equivariant stable homotopy theory, a subject that has powerful and unexpected nonequivariant applications. For recent examples, it plays a central role in the solution of the Kervaire invariant problem by Hill, Hopkins, and Ravenel, it is central to calculations of topological

Date: August 21, 2011.

cyclic homology and therefore to calculations in algebraic K-theory made by Angeltveit, Gerhardt, Hesselholt, Lindenstrauss, Madsen, and others, and it plays an interesting role by analogy and comparision in the work of Voevodsky and others in motivic stable homotopy theory. It is also of great intrinsic interest.

Setting up the equivariant stable homotopy category with its attendant model structures takes a fair amount of work. The original version was due to Lewis and May [11] and more modern versions that we shall start from are given in [12]. A result of Schwede and Shipley [20], reproven in [5], asserts that any stable model category $\mathscr{M}$ is equivalent to a category $\mathbf{Pre}(\mathscr{D},\mathscr{S})$ of spectrally enriched presheaves with values in a chosen category $\mathscr{S}$ of spectra. However, the domain $\mathscr{S}$-category $\mathscr{D}$ is a full $\mathscr{S}$-subcategory of $\mathscr{M}$ and typically is as inexplicit and mysterious as $\mathscr{M}$ itself. From the point of view of applications and calculations, this is therefore only a starting point. One wants a more concrete understanding of the category $\mathscr{D}$. We shall give explicit equivalents to the domain category $\mathscr{D}$ in the case when $\mathscr{M} = G\mathscr{S}$ is the category of G-spectra for a finite group G, and we fix a finite group G throughout.

We shall define an $\mathscr{S}$-category (or spectral category) $G\mathscr{B}$ by applying a suitable infinite loop space machine to simply defined categories of finite G-sets. The letter $\mathscr{B}$ stands for "Burnside", and $G\mathscr{B}$ is a spectrally enriched version of the Burnside category of G. We shall prove the following result.

Theorem 0.1 (Main theorem)**.** *There is a zig-zag of Quillen equivalences*

$$G\mathscr{S} \simeq \mathbf{Pre}(G\mathscr{B},\mathscr{S})$$

relating the category of G-spectra to the category of spectrally enriched contravariant functors $G\mathscr{B} \longrightarrow \mathscr{S}$.

As usual, we call such functors presheaves. We reemphasize the simplicity of our spectral category $G\mathscr{B}$: no prior knowledge of G-spectra is required to define it.

We give a precise description of the relevant categorical input and restate the main theorem more precisely in §1. The central point of the proof is to use equivariant infinite loop space theory to construct the spectral category $G\mathscr{B}$ from elementary categories of finite G-sets. We prove our main theorem in §2, using the equivariant Barratt-Priddy-Quillen (BPQ) theorem to compare $G\mathscr{B}$ to the spectral category $G\mathscr{D}$ given by the suspension G-spectra $\Sigma^\infty_G(A_+)$ of based finite G-sets A_+. It is crucial to our work that these G-spectra are self-dual. Our original proof (§3.2) took this as a special case of equivariant Atiyah duality, thinking of A as a trivial example of a smooth closed G-manifold. We later found a direct categorical proof (§2.3) of this duality based on equivariant infinite loop space theory and the equivariant BPQ theorem. This allows us to give an illuminating new proof of the required self-duality as we go along. We give an alternative model for the category of G-spectra in terms of classical Atiyah duality in §3.

We take what we need from equivariant infinite loop space theory as a black box in this paper, deferring the proofs of all but one detail to a sequel [7], with that detail deferred to another sequel [18].

We thank a diligent referee for demanding a reorganization of our original paper. We also thank Angelica Osorno and Inna Zakharevich for very helpful comments.

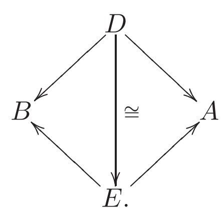
1. The $\mathscr{S}$-category $G\mathscr{B}$ and the $\mathscr{S}_G$-category $\mathscr{B}_G$

We first define the $\mathscr{S}$-category $G\mathscr{B}$ and restate our main theorem. We shall avoid categorical apparatus, but conceptually $G\mathscr{B}$ is obtained by applying a nonequivariant infinite loop space machine $\mathbb{K}$ to a category $G\mathscr{E}$ "enriched in permutative categories". The term in quotes can be made categorically precise [4, 9, 19], but we shall use it just as an informal slogan since no real categorical background is necessary to our work: we shall give direct elementary definitions of the examples we use, and they do satisfy the axioms specified in the cited sources. We then define a G-category $\mathscr{E}_G$ "enriched in permutative G-categories", from which $G\mathscr{E}$ is obtain by passage to G-fixed subcategories. Finally, we outline the proof of the main theorem, which is obtained by applying an equivariant infinite loop space machine $\mathbb{K}_G$ to $\mathscr{E}_G$.

1.1. **The bicategory $G\mathscr{E}$ of G-spans.** In any category $\mathscr{C}$ with pullbacks, the bicategory of spans in $\mathscr{C}$ has 0-cells the objects of $\mathscr{C}$. The 1-cells and 2-cells $A \longrightarrow B$ are the diagrams

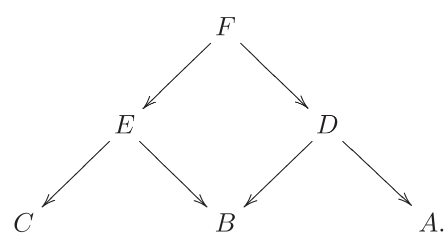
$$B \longleftarrow D \longrightarrow A$$

and



Composites of 1-cells are given by (chosen) pullbacks

(1.1)



The identity 1-cells are the diagrams $A \xleftarrow{=} A \xrightarrow{=} A$. The associativity and unit constraints are determined by the universal property of pullbacks. Observe that the 1-cells $A \longrightarrow B$ can just as well be viewed as objects over $B \times A$. Viewed this way, the identity 1-cells are given by the diagonal maps $A \longrightarrow A \times A$.

Our starting point is the bicategory of spans of finite G-sets. Here the disjoint union of G-sets over $B \times A$ gives us a symmetric monoidal structure on the category of 1-cells and 2-cells $A \longrightarrow B$ for each pair (A, B). We can think of the bicategory of spans as a category "enriched in the category of symmetric monoidal categories". Again, the notion in quotes does not make obvious mathematical sense since there is no obvious monoidal structure on the category of symmetric monoidal categories, but category theory due to the first author [4] (see also [9, 19]) explains what these objects are and how to rigidify them to categories enriched in permutative categories. We repeat that we have no need to go into such categorical detail. Rather than apply such category theory, we give a direct elementary construction of a strict structure that is equivalent to the intuitive notion of the category "enriched in symmetric monoidal categories" of spans of finite G-sets.

Definition 1.2. We first define a bipermutative category $G\mathscr{E}(1)$ equivalent to the symmmetric bimonoidal category of finite G-sets. Any finite G-set is isomorphic to a finite G-set of the form $A = (\mathbf{n}, \alpha)$, where $\mathbf{n} = \{1, \cdots, n\}$, α is a homomorphism $G \longrightarrow \Sigma_n$, and G acts on $\mathbf{n}$ by $g \cdot i = \alpha(g)(i)$ for $1 \leq i \leq n$. *We understand finite G-sets to be of this specific restricted form from now on.* A G-map $f\colon (\mathbf{m}, \alpha) \longrightarrow (\mathbf{n}, \beta)$ is a function $f\colon \mathbf{m} \longrightarrow \mathbf{n}$ such that $f \circ \alpha(g) = \beta(g) \circ f$ for $g \in G$. The morphisms of $G\mathscr{E}(1)$ are the isomorphisms $(\mathbf{n}, \alpha) \longrightarrow (\mathbf{n}, \beta)$ of G-sets. The disjoint union $D \amalg E$ of finite G-sets $D = (\mathbf{s}, \sigma)$ and $E = (\mathbf{t}, \tau)$ is $(\mathbf{s} + \mathbf{t}, \sigma + \tau)$, with $\sigma + \tau$ being the evident block sum $G \longrightarrow \Sigma_{s+t}$. With the evident commutativity isomorphism, this gives the permutative category $G\mathscr{E}(1)$ of finite G-sets; the empty finite G-set is the unit for $\amalg$. Similarly, the cartesian product $D \times E$ of D and E is $(\mathbf{st}, \sigma \times \tau)$ where the set $\mathbf{st}$ is identified with $\mathbf{s} \times \mathbf{t}$, ordered lexicographically, and $\sigma \times \tau$ is the evident block product. There is again an evident commutativity isomorphism, and $\amalg$ and $\times$ give $G\mathscr{E}(*)$ a structure of bipermutative category in the sense of [17]; the multiplicative unit is the trivial G-set $1 = (\mathbf{1}, \varepsilon)$, where $\varepsilon(g) = 1$ for $g \in G$.

We may view $G\mathscr{E}(1)$ as the category of finite G-sets over the one point G-set 1, and we generalize the definition as follows.

Definition 1.3. For a finite G-set A, we define a permutative category $G\mathscr{E}(A)$ of finite G-sets over A. The objects of $G\mathscr{E}(A)$ are the G-maps $p\colon D \longrightarrow A$. The morphisms $p \longrightarrow q$, $q\colon E \longrightarrow A$, are the G-isomorphisms $f\colon D \longrightarrow E$ such that $q \circ f = p$. Disjoint union of G-sets over A gives $G\mathscr{E}(A)$ a structure of permutative category; its unit is the empty set over A. When $A = 1$, $G\mathscr{E}(A)$ is the ("additive") permutative category of the previous definition.

Remark 1.4. There is also a product $\times\colon G\mathscr{E}(A) \times G\mathscr{E}(B) \longrightarrow G\mathscr{E}(A \times B)$. It takes (D, E) to $D \times E$, where D and E are finite G-sets over A and B, respectively. This product is also strictly associative and unital, with unit the unit of $G\mathscr{E}(1)$, and it has an evident commutativity isomorphism. Restriction to the object 1 gives the "multiplicative" permutative category of Definition 1.2. This product distributes over $\amalg$ and makes the enriched category $G\mathscr{E}$ of the next definition into a "strict symmetric monoidal category enriched in permutative categories" in a sense defined in [4].

Definition 1.5. We define a category $G\mathscr{E}$ "enriched in permutative categories" as follows. The 0-cells of $G\mathscr{E}$ are the finite G-sets, which may be thought of as the categories $G\mathscr{E}(A)$. The permutative category $G\mathscr{E}(A, B)$ of 1-cells and 2-cells $A \longrightarrow B$ is $G\mathscr{E}(B \times A)$, as defined in Definition 1.3. The composition

$$\circ\colon G\mathscr{E}(B, C) \times G\mathscr{E}(A, B) \longrightarrow G\mathscr{E}(A, C)$$

is defined via pullbacks, as in the diagram (1.1). Precisely, the pullback F is the sub G-set of $E \times D$ consisting of the elements (e, d) such that d and e map to the same element $b \in B$. This composition is strictly associative and unital.

Remark 1.6. We are suppressing some categorical details. The composition distributes over coproducts, and it should be defined on a "tensor product" rather than a cartesian product of permutative categories. Such a tensor product does in fact exist [9], but we shall not use the relevant category theory. Rather we will change notation to $\wedge$ since the composition is a pairing that gives rise to a pairing defined on the smash product of the spectra constructed from $G\mathscr{E}(B,C)$ and $G\mathscr{E}(A,B)$.

Remark 1.7. It is helpful to observe that the composition just defined can be viewed as a composite of maps of finite G-sets induced contravariantly and covariantly by the maps of finite G-sets

$$C \times B \times B \times A \xleftarrow{\mathrm{id} \times \Delta \times \mathrm{id}} C \times B \times A \xrightarrow{\quad \pi \quad} C \times A,$$

where $\pi : C \times B \times A \longrightarrow C \times A$ is the projection.

Before beginning work, we recall an old result that motivated this paper. The category $[G\mathscr{E}]$ of G-spans is obtained from the bicategory $G\mathscr{E}$ of G-spans by identifying spans from A to B if there is an isomorphism between them. Composition is again by pullbacks. We add spans from A to B by taking disjoint unions, and that gives the morphism set $[G\mathscr{E}](A,B)$ a structure of abelian monoid. We apply the Grothendieck construction to obtain an abelian group of morphisms $A \longrightarrow B$. This gives an additive category $\mathscr{A}b[G\mathscr{E}]$. The following result is [11, V.9.6]. Let Ho$G\mathscr{D}$ denote the full subcategory of the homotopy category Ho$G\mathscr{S}$ of G-spectra whose objects are the G-spectra $\Sigma^\infty_G(A_+)$, where A runs over the finite G-sets.

Theorem 1.8. *The categories Ho$G\mathscr{D}$ and $\mathscr{A}b[G\mathscr{E}]$ are isomorphic.*

1.2. **The precise statement of the main theorem.** Infinite loop space theory associates a spectrum $\mathbb{K}\mathscr{A}$ to a permutative category $\mathscr{A}$. There are several equivalent machines available. For definiteness, and because we have used it in working out the details, we use a modernized version of [14, 16] that lands in the category $\mathscr{S}$ of orthogonal spectra [13]. Precise details are given in [7]. With this choice, the zeroth space of $\mathbb{K}\mathscr{A}$ is the classifying space $B\mathscr{A}$. The objects $a \in \mathscr{A}$ are the vertices of the nerve of $\mathscr{A}$ and thus are points of $B\mathscr{A}$. Therefore each a determines a map $S \longrightarrow \mathbb{K}\mathscr{A}$, where S is the sphere spectrum. For any $\mathscr{A}$, $\mathbb{K}\mathscr{A}$ is a positive Ω-spectrum ([13, §14]) such that its structure map $B\mathscr{A} \longrightarrow \Omega(\mathbb{K}\mathscr{A})_1$ is a group completion.

Since $\mathscr{S}$ is closed symmetric monoidal under the smash product, it makes sense to enrich categories in $\mathscr{S}$. Our preferred version of spectral categories is categories enriched in $\mathscr{S}$, abbreviated $\mathscr{S}$-categories. Model theoretically, $\mathscr{S}$ is a particularly nice enriching category since its unit S is cofibrant in the stable model structure and $\mathscr{S}$ satisfies the monoid axiom [13, 12.5].

When a spectral category $\mathscr{D}$ is used as the domain category of a presheaf category, the objects and maps of the underlying category are unimportant. The important data are the morphism spectra $\mathscr{D}(A,B)$, the unit maps $S \longrightarrow \mathscr{D}(A,A)$, and the composition maps

$$\mathscr{D}(B,C) \wedge \mathscr{D}(A,B) \longrightarrow \mathscr{D}(A,C).$$

The presheaves $\mathscr{D}^{op} \longrightarrow \mathscr{S}$ can be thought of as (right) $\mathscr{D}$-modules.

Definition 1.9. We define a spectral category $G\mathscr{B}$. Its objects are the finite G-sets A, which may be viewed as the spectra $\mathbb{K}G\mathscr{E}(A)$. Its morphism spectra $G\mathscr{B}(A,B)$ are the spectra $\mathbb{K}G\mathscr{E}(B \times A)$. Its unit maps $S \longrightarrow G\mathscr{B}(A,A)$ are induced by the points $\mathrm{id}_A \in G\mathscr{E}(A,A)$ and its composition

$$G\mathscr{B}(B,C) \wedge G\mathscr{B}(A,B) \longrightarrow G\mathscr{B}(A,C)$$

is induced by composition in $G\mathscr{E}$.

As written, the definition makes little sense: to make the word "induced" meaningful requires properties of the infinite loop space machine $\mathbb{K}$ that we will spell out in §2.2. Once this is done, we will have the presheaf category $\mathbf{Pre}(G\mathscr{B},\mathscr{S})$ of $\mathscr{S}$-functors $(G\mathscr{B})^{op} \longrightarrow \mathscr{S}$ and and $\mathscr{S}$-natural transformations. As shown for example in [5], it is a cofibrantly generated model category enriched in $\mathscr{S}$, or $\mathscr{S}$-model category for short. As shown in [12], the category $G\mathscr{S}$ of (genuine) orthogonal G-spectra is also an $\mathscr{S}$-model category. Our main theorem can be restated as follows.

Theorem 1.10 (Main theorem)**.** *There is a zigzag of enriched Quillen equivalences connecting the $\mathscr{S}$-model categories $G\mathscr{S}$ and $\mathbf{Pre}(G\mathscr{B},\mathscr{S})$.*

Therefore G-spectra can be thought of as constructed from the very elementary category $G\mathscr{E}$ enriched in permutative categories, ordinary nonequivariant spectra, and the black box of infinite loop space theory. The following reassuring result falls out of the proof. Let $\mathscr{O}rb$ denote the orbit category of G. For a G-spectrum X, passage to H-fixed point spectra for $H \subset G$ defines a functor $X^{\bullet}\colon \mathscr{O}rb^{op} \longrightarrow \mathscr{S}$. Analogously, a presheaf $Y \in \mathbf{Pre}(G\mathscr{B},\mathscr{S})$ restricts to a functor $\mathscr{O}rb^{op} \longrightarrow \mathscr{S}$.

Corollary 1.11. *The zigzag of equivalences induces a natural zigzag of equivalences between the fixed point orbit functor on G-spectra and the restriction to orbits of presheaves; thus, if X corresponds to Y, then X^H is equivalent to $Y(G/H)$.*

Remark 1.12. There is an important missing ingredient needed for a fully satisfactory theory: we have not described the behavior of smash products under the equivalences of Theorem 1.10. This problem deserves study both in our work and in related work of others. The obvious guess is that $\mathbf{Pre}(G\mathscr{B},\mathscr{S})$ is symmetric monoidal and the zigzag connecting it to $G\mathscr{S}$ is a zigzag of symmetric monoidal Quillen equivalences. We see how the problem can be attacked, but we also have reason to believe that the obvious guess may be wrong. We intend to return to this question elsewhere.

Remark 1.13. Much of what we do applies to G-spectra indexed on an incomplete universe, provided that we restrict attention to those finite G-sets A that embed in that universe, so that Atiyah duality applies to the orbit G-spectra $\Sigma^{\infty}_G(A_+)$. By [10], duality fails for orbits that do not embed in the universe. Unfortunately, however, the cited restriction leads to the wrong weak equivalences, since we are then only entitled to see the homotopy groups of H-fixed point spectra for those H that embed in the given universe.

1.3. **The G-bicategory $\mathscr{E}_G$ of spans: intuitive definition.** Everything we do depends on first working equivariantly and then passing to fixed points. Following [6, §1.2], we fix some generic notations. For a category $\mathscr{C}$, let $G\mathscr{C}$ be the category of G-objects in $\mathscr{C}$ and G-maps between them. Let $\mathscr{C}_G$ be the G-category of G-objects and nonequivariant maps, with G acting by conjugation. The two categories are related conceptually by $G\mathscr{C} = (\mathscr{C}_G)^G$. The objects, being G-objects, are already G-fixed; we apply the G-fixed point functor to hom sets. More generally, we can start with a category $\mathscr{C}$ with actions by G on its objects and again define a category $G\mathscr{C}$ of G-maps and a G-category $\mathscr{C}_G$ with G-fixed category $G\mathscr{C}$.

We apply this framework to the category of finite G-sets. We have already defined the G-fixed bicategory $G\mathscr{E}$, and we shall give two definitions of G-bicategories $\mathscr{E}_G$ with fixed point bicategories equivalent to $G\mathscr{E}$. The first, given in this section, is more intuitive, but the second is more convenient for the proof of our main theorem.

Let U be a countable G-set that contains all orbit types G/H infinitely many times. Again let A, B, and C denote finite G-sets, but now let the D, E and F of §1.1 be finite subsets of the G-set U; these subsets need *not* be G-subsets. The action of G on U gives rise to an action of G on the finite subsets of U: for a finite subset D of U and $g \in G$, gD is another finite subset of U.

Definition 1.14. We define a G-category $\mathscr{E}_G(A)$. The objects of $\mathscr{E}_G(A)$ are the nonequivariant maps $p\colon D \longrightarrow A$, where A is a finite G-set and D is a finite subset of U. The morphisms $f\colon p \longrightarrow q$, $q\colon E \longrightarrow A$, are the bijections $f\colon D \longrightarrow E$ such that $q \circ f = p$. The group G acts on morphisms via the maps $g\colon D \longrightarrow gD$ and the formula $(gf)(gd) = gf(d)$.

Definition 1.15. We define a bicategory $\mathscr{E}_G$ with objects the finite G-sets and with G-categories of morphisms between objects specified by $\mathscr{E}_G(A,B) = \mathscr{E}_G(B \times A)$. Thinking of the objects of $\mathscr{E}_G(A,B)$ as nonequivariant spans $B \longleftarrow D \longrightarrow A$, composition and units are defined as in Definition 1.5.

Observe that taking disjoint unions of finite sets over A will not keep us in U and is thus not well-defined. Therefore the $\mathscr{E}_G(A)$ are not symmetric monoidal (let alone permutative) G-categories in the naive sense of symmetric monoidal categories with G acting compatibly on all data. In fact, the notion of a genuine permutative G-category, one that provides input for an equivariant infinite loop space machine, is subtle. We shall give two solutions to that categorical problem in [7]. In both, genuine permutative G-categories are described in terms of actions by an E_∞ operad of G-categories, to which equivariant infinite loop space theory applies. One solution gives each of the $\mathscr{E}_G(A)$ such a structure, but that is not the solution we shall use.

1.4. **The G-bicategory $\mathscr{E}_G$ of spans: working definition.** The other solution starts from a less intuitive definition of $\mathscr{E}_G$ and gives an equivalent way of solving that categorical problem. It uses a more convenient E_∞ operad of G-categories, denoted $\mathscr{O}_G$. We give details of this operad in [7], where we define a genuine permutative G-category to be an algebra over $\mathscr{O}_G$. To give the idea, we apply our general point of view on equivariant categories to the category $\mathscr{C}at$ of small categories. Thus, for G-categories $\mathscr{A}$ and $\mathscr{B}$, let $\mathscr{C}at_G(\mathscr{A},\mathscr{B})$ be the G-category of functors $\mathscr{A} \longrightarrow \mathscr{B}$ and natural transformations, with G acting by conjugation, and let $G\mathscr{C}at(\mathscr{A},\mathscr{B})$ be the category of G-functors and G-natural transformations.

Definition 1.16. Let $\tilde{G}$ (sometimes denoted EG in the literature[1]) be the groupoid with object set G and a unique morphism, denoted (h,k), from k to h for each pair of objects. Let G act from the right on $\tilde{G}$ by $h \cdot g = hg$ on objects and $(h,k)\cdot g = (hg,kg)$ on morphisms. The objects of $\mathscr{E}_G$ are the finite G-sets $A = (\mathbf{n},\alpha)$, regarded as discrete (identity morphisms only) G-categories. Define $\mathscr{O}(j) = \tilde{\Sigma}_j$; this is the jth category of an E_∞ operad of categories whose algebras are the permutative categories [16]. Define $\mathscr{O}_G(j)$ to be the G-category

$$\mathscr{C}at_G(\tilde{G},\tilde{\Sigma}_j) = \mathscr{C}at_G(\tilde{G},\mathscr{O}(j)).$$

Here G acts trivially on $\tilde{\Sigma}_j$. The left action of G on $\mathscr{O}_G(j)$ is induced by the right action of G on $\tilde{G}$, and the right action of Σ_j is induced by the right action of Σ_j on $\tilde{\Sigma}_j$. The functor $\mathscr{C}at_G(\tilde{G},-)$ is product preserving and the operad structure maps

[1]While $\tilde{G}$ is isomorphic as a G-category to the translation category of G, the action of G on that category is defined differently, as is explained in [8, Lemma 1.7].

are induced from those of $\mathscr{O}$. We interpret $\mathscr{O}(0)$ and $\mathscr{O}_G(0)$ to be trivial categories; $\mathscr{O}_G(1)$ is also trivial, with unique object denoted id.

Definition 1.17. Define the G-category $\mathscr{E}_G(A)$ by

$$\mathscr{E}_G(A) = \coprod_{n\geq 0} \mathscr{O}_G(n) \times_{\Sigma_n} A^n = (\coprod_{n\geq 1} \mathscr{O}_G(n) \times_{\Sigma_n} A^n)_+. \tag{1.18}$$

We interpret the term with $n = 0$ to be a trivial base category $*$, which explains the second equality, and we identify the term with $n = 1$ with A. An alternative formulation is $\mathscr{E}_G(A) = \mathbb{O}_G(A_+)$, where $\mathbb{O}_G$ denotes the monad in the category of based G-categories whose algebras are the same as the $\mathscr{O}_G$-algebras. Thus $\mathbb{O}_G(A_+)$ is the free $\mathscr{O}_G$-algebra (= genuine permutative G-category) generated by the based G-category A_+, with unit given by a disjoint trivial base category added to A.

The following result is neither obvious nor difficult. It is proven in [7].

Theorem 1.19. *The G-fixed permutative category $\mathscr{E}_G(A)^G$ is naturally isomorphic to the permutative category $G\mathscr{E}(A)$.*

The starting point of the proof is the observation that a functor $\tilde{G} \longrightarrow \tilde{\Sigma}_n$ is uniquely determined by its object function $G \longrightarrow \Sigma_n$. In particular, for a finite G-set $B = (\mathbf{n}, \beta)$ we may view the G-map $\beta\colon G \longrightarrow \Sigma_n$ as a G-fixed object of the category $\mathscr{O}_G(n)$, and all G-fixed objects of $\mathscr{O}_G(n)$ are of this form. With a little care, we see that a G-fixed object $(\beta; a_1, \cdots, a_n)$ of $\mathscr{O}_G(n) \times_{\Sigma_n} A^n$ can be interpreted as a G-map $B \longrightarrow A$ and that all finite G-sets over A are of this form.

The following is a sketch definition whose details will be fleshed out below.

Definition 1.20. The G-category $\mathscr{E}_G$ "enriched in permutative G-categories" has 0-cells the finite G-sets A, which may be thought of as the G-categories $\mathscr{E}_G(A)$. The permutative G-category $\mathscr{E}_G(A, B)$ of 1-cells and 2-cells $A \longrightarrow B$ is $\mathscr{E}_G(B \times A)$. The unit id_A of $A = (\mathbf{n}, \alpha)$ is the object $(\alpha; (1, 1), \cdots, (n, n))$ of $\mathscr{O}_G(n) \times_{\Sigma_n} (A \times A)^n$; it can be thought of as a G-map $1 \longrightarrow \mathscr{E}_G(A, A)$ of G-categories, where 1 is the trivial G-category. Composition is given by the following composite; its first map is a specialization of a pairing of free $\mathscr{O}_G$-algebras, and its second and third maps are specializations of contravariant functoriality of the free $\mathscr{O}_G$-algebra functor on inclusions and covariant functoriality on surjections that we shall shortly make precise.

$$\begin{array}{c} \mathscr{E}_G(C \times B) \wedge \mathscr{E}_G(B \times A) \\ \downarrow{\scriptstyle \omega} \\ \mathscr{E}_G(C \times B \times B \times A) \\ \downarrow{\scriptstyle (\mathrm{id} \times \Delta \times \mathrm{id})^*} \\ \mathscr{E}_G(C \times B \times A) \\ \downarrow{\scriptstyle \pi_!} \\ \mathscr{E}_G(C \times A). \end{array}$$

We shall place the following ad hoc definition of the required pairing ω in a suitable general context in [7], modernizing part of [15]. We first comment on its domain; compare Remark 1.6.

Remark 1.21. We can define the smash product of based G-categories in the same way as the smash product of based G-spaces. We are most interested in examples of the form $\mathscr{A}_+$ and $\mathscr{B}_+$ for unbased G-categories $\mathscr{A}$ and $\mathscr{B}$, and then $\mathscr{A}_+ \wedge \mathscr{B}_+$ can be identified with $(\mathscr{A} \times \mathscr{B})_+$. In particular,

$$(\coprod_{m\geq 1} \mathscr{O}_G(m) \times_{\Sigma_m} A^m)_+ \wedge (\coprod_{n\geq 1} \mathscr{O}_G(n) \times_{\Sigma_n} B^n)_+$$

is isomorphic to

$$(\coprod_{m\geq 1, n\geq 1} \mathscr{O}_G(m) \times \mathscr{O}_G(n) \times_{\Sigma_m\times\Sigma_n} A^m \times B^n)_+.$$

We do not claim that this is an $\mathscr{O}_G$-category, but an equivariant infinite loop space machine nevertheless constructs from it the smash product of the spectra constructed from $\mathscr{E}_G(A)$ and $\mathscr{E}_G(B)$.

Definition 1.22. Identify the ordered set $\mathbf{mn}$ with the set of pairs (i,j), $1 \leq i \leq m$ and $1 \leq j \leq n$, ordered lexicographically. This fixes a homomorphism $\Sigma_m \times \Sigma_n \longrightarrow \Sigma_{mn}$ and therefore a functor $\tilde{\Sigma}_m \times \tilde{\Sigma}_n \longrightarrow \tilde{\Sigma}_{mn}$. Applying the functor $\mathscr{C}at_G(\tilde{G}, -)$, we obtain pairings $\omega_{m,n}\colon \mathscr{O}_G(m) \times \mathscr{O}_G(n) \longrightarrow \mathscr{O}_G(mn)$. For finite G-sets A and B, we have the injection $A^m \times B^n \longrightarrow (A\times B)^{mn}$ that sends $(a_1, \cdots, a_m) \times (b_1, \cdots, b_n)$ to the set of pairs (a_i, b_j), ordered lexicographically. Combining, there result functors

$$\omega_{m,n}\colon (\mathscr{O}_G(m) \times_{\Sigma_m} A^m) \times (\mathscr{O}_G(n) \times_{\Sigma_n} B^n) \longrightarrow \mathscr{O}_G(mn) \times_{\Sigma_{mn}} (A\times B)^{mn}.$$

Distributing products over disjoint unions, these specify pairings of G-categories

$$\omega\colon \mathscr{E}_G(A) \wedge \mathscr{E}_G(B) \longrightarrow \mathscr{E}_G(A\times B).$$

The naturality maps in Definition 1.20 are both applications of the free $\mathscr{O}_G$-category functor to maps f of based finite G-sets. Conceptually, the definition (1.18) hides an extension of functors from $\mathscr{E}_G(A)$, which a priori appears to be a functor on *unbased* finite G-sets, to $\mathbb{O}_G(A_+)$, which is a functor on *based* finite G-sets.

Definition 1.23. For a map $f\colon A_+ \longrightarrow B_+$ of based finite G-sets, we obtain a functor $f_!\colon \mathscr{E}_G(A) \longrightarrow \mathscr{E}_G(B)$ by taking the disjoint union over n of the functors $\mathrm{id} \times_{\Sigma_n} f^n$. This is unproblematical if f is obtained from a map $A \longrightarrow B$ of unbased finite G-sets, so that $f^{-1}(*) = *$.[2] In general, however, the specification of $f_!$ depends on implicit basepoint identifications that are invisible to (1.18) but become visible when evaluating $\mathscr{E}_G f$. Because $\mathscr{O}_G(0)$ is the trivial category $*$, there is a degeneracy G-functor $\sigma_i\colon \mathscr{O}_G(n) \longrightarrow \mathscr{O}_G(n-1)$ associated to the ordered inclusion $\mathbf{n-1}\colon \longrightarrow \mathbf{n}$ that misses i. As in [14, 2.3], if γ is the structural map of the operad and $\nu \in \mathscr{O}_G(n)$,

$$\sigma_i(\nu) = \gamma(\nu; \mathrm{id}^{i-1}, *, \mathrm{id}^{n-i}).$$

If $a_i = *$, then $(\nu, a_1, \cdots, a_n)$ must be identified with $(\sigma_i(\nu), a_1, \cdots, \hat{a}_i, \cdots, a_n)$, where $\hat{a}_i$ means delete a_i. In particular, if $i\colon A \longrightarrow B$ is an inclusion of unbased finite G-sets, define an associated retraction $r\colon B_+ \longrightarrow A_+$ of based finite G-sets

[2] With the intuitive version of $\mathscr{E}_G$, $f_!\colon \mathscr{E}_G(A) \longrightarrow \mathscr{E}_G(B)$ is just the pushforward functor obtained by composing f with maps over A.

by setting $ri(a) = a$ and $r(b) = *$ if $b \notin \mathrm{im}(A)$. Then define $i^* = r_! \colon \mathscr{E}_G(B) \longrightarrow \mathscr{E}_G(A)$.[3] By Remark 2.21 below, we may think of i^* as the dual of i.

The associativity of the composition defined in Definition 1.20 is an easy diagram chase, starting from the associativity of the pairing on $\mathscr{O}_G$. The verification that composition with the prescribed unit objects id_A gives identity functors illustrates how Definition 1.23 works. Set $B = A$ and consider the composite

$$(\mu; (c_1, a_1), \cdots, (c_m, a_m)) \circ \mathrm{id}_A \,.$$

We are focusing on objects, and $\mu \in \mathscr{O}_G(m)$, $c_i \in C$, $a_i \in A$, and $A = (\mathbf{n}, \alpha)$. Applying the pairing we get the object

$$(\omega_{m,n}(\mu, \alpha); (c_i, a_i, j, j)) \in \mathscr{O}_G(mn) \times_{\Sigma_{mn}} (C \times A \times A \times A)^{mn}.$$

The four-tuple (c_i, a_i, j, j) is in the image of $\mathrm{id} \times \Delta \times \mathrm{id}$ if and only if $a_i = j$. The r corresponding to this inclusion maps all other (c_i, a_i, j, j) to the basepoint, and we have an accompanying iterated degeneracy $\sigma \colon \mathscr{O}_G(mn) \longrightarrow \mathscr{O}_G(m)$ such that $\sigma(\omega_{m,n}(\mu, \alpha)) = \mu$. Therefore our composite is $(\mu; (c_1, a_1), \cdots, (c_m, a_m))$, as required. The proof that composition on the left with id_A is the identity functor is similar.

Theorem 1.19 has the following corollary by direct comparison of definitions.

Corollary 1.24. *The G-fixed category $(\mathscr{E}_G)^G$ enriched in permutative categories is isomorphic to the category $G\mathscr{E}$ enriched in permutative categories.*

1.5. **The categorical duality maps.** Since various specializations are central to our work, we briefly recall how duality works categorically, following [11, III§1] for example. We then define maps of $\mathscr{O}_G$-algebras that will lead in §2.3 to the proof that the objects of $G\mathscr{B}$ are self-dual.

Let $\mathscr{V}$ be a closed symmetric monoidal category with product $\wedge$, unit S, and hom objects $F(X,Y)$; write $DX = F(X,S)$. A pair of objects (X,Y) in $\mathscr{V}$ is a dual pair if there are maps $\eta \colon S \longrightarrow X \wedge Y$ and $\varepsilon \colon Y \wedge X \longrightarrow S$ such that the composites

$$X \cong S \wedge X \xrightarrow{\eta \wedge \mathrm{id}} X \wedge Y \wedge X \xrightarrow{\mathrm{id} \wedge \varepsilon} X \wedge S \cong X$$

$$Y \cong Y \wedge S \xrightarrow{\mathrm{id} \wedge \eta} Y \wedge X \wedge Y \xrightarrow{\varepsilon \wedge \mathrm{id}} S \wedge Y \cong Y$$

are identity maps. For any such pair, the adjoint $\tilde{\varepsilon} \colon Y \longrightarrow DX$ of ε is an isomorphism. We have a natural map

$$\zeta \colon Y \wedge DX = Y \wedge F(X,S) \longrightarrow F(X,Y) \tag{1.25}$$

in $\mathscr{V}$, namely the adjoint of

$$\mathrm{id} \wedge \varepsilon \colon Y \wedge DX \wedge X \longrightarrow Y \wedge S \cong Y,$$

where ε is the evident evaluation map. The map ζ is an isomorphism when either X or Y is dualizable [11, III.1.3]. When X is dualizable and Y is arbitrary, we have the composite isomorphism

$$\delta = \zeta \circ (\mathrm{id} \wedge \tilde{\varepsilon}) \colon Y \wedge X \longrightarrow Y \wedge DX \longrightarrow F(X,Y). \tag{1.26}$$

[3] With the intuitive version of $\mathscr{E}_G$, $i^* \colon \mathscr{E}_G(B) \longrightarrow \mathscr{E}_G(A)$ is just the functor obtained by pulling back maps over B to maps over A.

This map in various categories will play a central role in our work. When (X, Y) and (X', Y') are dual pairs, the dual of a map $f\colon X \longrightarrow X'$ is the composite

$$Y' \cong Y' \wedge S_G \xrightarrow{\mathrm{id}\wedge\eta} Y' \wedge X \wedge Y \xrightarrow{\mathrm{id}\wedge f\wedge\mathrm{id}} Y' \wedge X' \wedge Y \xrightarrow{\varepsilon'\wedge\mathrm{id}} S_G \wedge Y \cong Y. \tag{1.27}$$

There are two maps of $\mathscr{O}_G$-algebras that are central to duality and therefore to everything we do. Let $S^0 = \{*, 1\}$, where $*$ is the basepoint and 1 is not. We think of S^0 as 1_+, where 1 is the one-point G-set. Remember that $\mathscr{E}_G(A) = \mathbb{O}_G(A_+)$ is the free $\mathscr{O}_G$-algebra generated by A_+, where we view finite G-sets as categories with only identity morphisms. We have already seen the first map implicitly.

Definition 1.28. For a finite G-set A, define based G-maps

$$\varepsilon\colon (A\times A)_+ \longrightarrow S^0, \quad r\colon (A\times A)_+ \longrightarrow A_+ \ \text{ and } \ \pi\colon A_+ \longrightarrow S^0$$

by $r(a,b) = *$ if $a \neq b$ and $r(a,a) = a$, $\pi(a) = 1$, and $\varepsilon = \pi\circ r$, so that $\varepsilon(a,b) = *$ if $a \neq b$ and $\varepsilon(a,a) = 1$. Note that $r\circ\Delta = \mathrm{id}$ and that ε is just an example of a Kronecker δ-function. We agree to again write ε for the induced map of $\mathscr{O}_G$-algebras

$$\varepsilon = \mathscr{E}_G\varepsilon\colon \mathscr{E}_G(A\times A) \longrightarrow \mathscr{E}_G(1).$$

Definition 1.29. For a finite G-set A, regard the object $\mathrm{id}_A \in \mathscr{E}_G(A)$ as the map of G-categories $i_A\colon 1 \longrightarrow \mathscr{E}_G(A)$ that sends the object 1 to the object id_A. By freeness, there results a map of $\mathscr{O}_G$-algebras

$$\eta\colon \mathscr{E}_G(1) \longrightarrow \mathscr{E}_G(A\times A).$$

If $A = (\mathbf{n}, \alpha)$, then η is the disjoint union of maps

$$\mathscr{O}_G(m)/\Sigma_m \cong \mathscr{O}_G(m)\times_{\Sigma_m} 1^m \longrightarrow \mathscr{O}_G(mn)\times_{\Sigma_{mn}} (A\times A)^{mn}.$$

These are obtained by composing $\mathscr{O}_G(m)\times i_A^m$ with the map induced on passage to orbits from the maps

$$\begin{array}{c} \mathscr{O}_G(m)\times(\mathscr{O}_G(n)\times(A\times A)^n)^m \cong (\mathscr{O}_G(m)\times\mathscr{O}_G(n)^m)\times((A\times A)^n)^m \\ \big\downarrow \\ \mathscr{O}_G(mn)\times(A\times A)^{mn} \end{array}$$

given by shuffling and applying the structure map $\gamma\colon \mathscr{O}_G(m)\times\mathscr{O}_G(n)^m \longrightarrow \mathscr{O}_G(mn)$.

The following categorical observation will lead to our proof in §2.3 that the G-spectra $\Sigma_G^\infty(A_+)$ are self-dual. Since care of basepoints is crucial, we use the alternative notation $\mathbb{O}_G(A_+)$. Remember that $(A\times A)_+$ can be identified with $A_+\wedge A_+$. We identify $1_+\wedge A_+$ and $A_+\wedge 1_+$ with A_+ at the bottom center of our diagrams.

Proposition 1.30. *The left and right squares commute in the following diagrams, and*

$$\mathbb{O}_G(\mathrm{id}\wedge\varepsilon)\circ\zeta_\ell = \mathrm{id} = \mathbb{O}_G(\varepsilon\wedge\mathrm{id})\circ\zeta_r. \tag{1.31}$$

Therefore the diagrams obtained by removing the maps ζ_ℓ and ζ_r commute.

$$\begin{array}{ccccc} \mathbb{O}_G(A_+\wedge A_+)\wedge\mathbb{O}_G(A_+) & \xrightarrow{\omega} & \mathbb{O}_G(A_+\wedge A_+\wedge A_+) & \xleftarrow{\omega} & \mathbb{O}_G(A_+)\wedge\mathbb{O}_G(A_+\wedge A_+) \\ {\scriptstyle\eta\wedge\mathrm{id}}\big\uparrow & & {\scriptstyle\zeta_\ell}\big\uparrow\ \big\downarrow{\scriptstyle\mathbb{O}_G(\mathrm{id}\wedge\varepsilon)} & & \big\downarrow{\scriptstyle\mathrm{id}\wedge\varepsilon} \\ \mathbb{O}_G(1_+)\wedge\mathbb{O}_G(A_+) & \xrightarrow[\omega]{} & \mathbb{O}_G(A_+) & \xleftarrow[\omega]{} & \mathbb{O}_G(A_+)\wedge\mathbb{O}_G(1_+) \end{array}$$

$$\begin{array}{ccccc}
\mathbb{O}_G(A_+) \wedge \mathbb{O}_G(A_+ \wedge A_+) & \xrightarrow{\omega} & \mathbb{O}_G(A_+ \wedge A_+ \wedge A_+) & \xleftarrow{\omega} & \mathbb{O}_G(A_+ \wedge A_+) \wedge \mathbb{O}_G(A_+) \\
\uparrow{\scriptstyle \mathrm{id} \wedge \eta} & & {\scriptstyle \zeta_r}\uparrow \ \downarrow{\scriptstyle \mathbb{O}_G(\varepsilon \wedge \mathrm{id})} & & \downarrow{\scriptstyle \varepsilon \wedge \mathrm{id}} \\
\mathbb{O}_G(A_+) \wedge \mathbb{O}_G(1_+) & \xrightarrow[\omega]{} & \mathbb{O}_G(A_+) & \xleftarrow[\omega]{} & \mathbb{O}_G(1_+) \wedge \mathbb{O}_G(A_+)
\end{array}$$

Proof. In the right vertical arrows, ε means $\mathbb{O}_G(\varepsilon)$. Since the right squares are just naturality diagrams, they clearly commute. For the rest, we must first define the maps ζ_ℓ and ζ_r. Remember that the elements of A are the elements of $\mathbf{n} = \{1, \cdots, n\}$, permuted according to $\alpha\colon G \longrightarrow \Sigma_n$. Define $j_\ell : A \longrightarrow (A \times A \times A)^n$ by $j_\ell(a) = \big((1, 1, a), \cdots, (n, n, a)\big)$. Then define $J_\ell\colon A_+ \longrightarrow \mathscr{O}_G(A_+ \wedge A_+ \wedge A_+)$ by

$$J_\ell(a) = (\alpha, j_\ell(a)) \in \mathscr{O}_G(n) \times_{\Sigma_n} (A \times A \times A)^n.$$

Define

$$\zeta_\ell\colon \mathbb{O}_G(A_+) \longrightarrow \mathscr{O}_G(A_+ \wedge A_+ \wedge A_+)$$

to be the map of $\mathscr{O}_G$-algebras induced by freeness. For $\mu \in \mathscr{O}_G(m)$ and $\nu \in \mathscr{O}_G(q)$,

$$(1.32) \qquad \zeta_\ell(\omega(\mu, \nu); (a_1, \cdots, a_q)^m) = (\gamma(\omega(\mu, \nu); \alpha^{mq}); (j_\ell(a_1), \cdots, j_\ell(a_q))^m)$$

where γ is the structural map of the operad $\mathscr{O}_G$. Define j_r, J_r, and ζ_r by symmetry.

Clearly $\mathbb{O}_G(\mathrm{id} \wedge \varepsilon)$ sends $J_\ell(a)$ to a. Indeed, a is one of the elements $j \in \mathbf{n}$ and $\mathrm{id} \wedge \varepsilon$ sends the coordinates (i, i, a) with $i \neq j$ to the basepoint and the coordinate (j, j, a) to a. Since $\mathbb{O}_G(\mathrm{id} \wedge \varepsilon) \circ \zeta_\ell$ is a map of $\mathscr{O}_G$-algebras with domain the free $\mathscr{O}_G$-algebra $\mathbb{O}_G(A_+)$, this implies the first equality in (1.31); the symmetric argument proves the second equality. It remains to prove that the left squares of our diagrams commute; by symmetry it suffices to consider the first diagram. Consider an element

$$x = ((\mu; 1^m), (\nu; a_1, \cdots, a_q)) \in (\mathscr{O}_G(m) \times_{\Sigma_m} 1^m) \times (\mathscr{O}_G(q) \times_{\Sigma_q} A^q),$$

where $m \geq 1$, $q \geq 1$, $\mu \in \mathscr{O}_G(m)$, $\nu \in \mathscr{O}_G(q)$, and $a_k \in A$ for $1 \leq k \leq q$. Write $[j, j, a_k]$ for the element of $(A^3)^{mnq}$ with $(i, j, k)th$ coordinate (j, j, a_k), $1 \leq i \leq m$, $1 \leq j \leq n$, and $1 \leq k \leq q$. Then

$$(1.33) \quad \omega \circ (\eta \wedge \mathrm{id})(x) = (\omega(\gamma(\mu; \alpha^m), \nu); [j, j, a_k]) \in \mathscr{O}_G(mnq) \times_{\Sigma_{mnq}} (A^3)^{mnq}.$$

On the other hand,

$$\omega(x) = (\omega(\mu, \nu); (a_1, \cdots, a_q)^m) \in \mathscr{O}_G(mq) \times_{\Sigma_{mq}} A^{mq}$$

and therefore

(1.34)

$$\zeta_\ell \omega(x) = (\gamma(\omega(\mu, \nu); \alpha^{mq}); (j_\ell(a_1), \cdots, j_\ell(a_q))^m) \in \mathscr{O}_G(mnq) \times_{\Sigma_{mnq}} (A^3)^{mnq}.$$

The coordinates in A^3 of the element on the right side of (1.34) differ from those of the right side of (1.33) by a permutation $\sigma \in \Sigma_{mnq}$, and it is a special case of the formula relating the pairing ω to the structure map γ of the operad $\mathscr{O}_G$ that

$$(1.35) \qquad \gamma(\omega(\mu, \nu); \alpha^{mq})\sigma = \omega(\gamma(\mu; \alpha^m), \nu).$$

Therefore the right sides of (1.33) and (1.34) are equal and $\omega \circ (\eta \wedge \mathrm{id}) = \zeta_\ell \circ \omega$. □

Remark 1.36. A more general form of (1.35) is the key defining property [15, 1.4(ii)] of a pairing of operads, such as ω. We have proven that the left and right squares of our diagrams are examples of maps of pairings of algebras over a permutative operad, as defined in [17, IX.1.3] and [15, 1.1]. Those sources are nonequivariant and outdated, but a modern treatment of equivariant pairings will be included in [18].

2. The proof of the main theorem

2.1. **The equivariant approach to Theorem 1.10.** As we will explain in [7], equivariant infinite loop space theory associates an orthogonal G-spectrum $\mathbb{K}_G\mathscr{A}_G$ to a (genuine) permutative G-category $\mathscr{A}_G$. The 0th space of $\mathbb{K}_G\mathscr{A}_G$ is the classifying G-space $B\mathscr{A}_G$. The 0th structure map $B\mathscr{A}_G \longrightarrow \Omega(\mathscr{B}_G\mathscr{A}_G)_1$ is an equivariant group completion.[4] The category $G\mathscr{S}$ of orthogonal G-spectra is the G-fixed category of a G-category $\mathscr{S}_G$ of G-spectra and non-equivariant maps with the same objects as $\mathscr{S}_G$ and with G acting by conjugation. Applying the functor $\mathbb{K}_G$ to $\mathscr{E}_G$, we obtain the following equivariant analogue of Definition 1.9.

Definition 2.1. We define a G-spectral category, or $\mathscr{S}_G$-category[5] $\mathscr{B}_G$. Its objects are the finite G-sets A, which may be viewed as the G-spectra $\mathbb{K}_G\mathscr{E}_G(A)$. Its morphism G-spectra $\mathscr{B}_G(A,B)$ are the G-spectra $\mathbb{K}_G\mathscr{E}_G(B\times A)$. Its unit G-maps $S_G \longrightarrow \mathscr{B}_G(A,A)$ are induced by the points $\mathrm{id}_A \in G\mathscr{E}(A,A)$ and its composition G-maps

$$\mathscr{B}_G(B,C)\wedge\mathscr{B}_G(A,B)\longrightarrow\mathscr{B}_G(A,C)$$

are induced by composition in $\mathscr{E}_G$.

Again, as written, the definition makes little sense: to make the word "induced" meaningful requires properties of the equivariant infinite loop space machine $\mathbb{K}_G$ that we will spell out in §2.2. This depends on having a functor that takes pairings of free $\mathscr{O}_G$-algebras to pairings of G-spectra.

The equivariant and non-equivariant infinite loop space functors are related by the following result.

Theorem 2.2 ([7])**.** *There is a natural equivalence of spectra*

$$\iota\colon \mathbb{K}(G\mathscr{A}) \longrightarrow (\mathbb{K}_G\mathscr{A}_G)^G$$

for permutative G-categories $\mathscr{A}_G$ with G-fixed permutative categories $G\mathscr{A}$.

In view of Corollary 1.24, there results an equivalence of $\mathscr{S}$-categories

$$G\mathscr{B}\xrightarrow{\ \simeq\ }(\mathscr{B}_G)^G.$$

The proof of Theorem 1.10 goes as follows. We start with the following specialization of a general result about stable model categories; it is discussed in [6, §3.1].

Theorem 2.3. *Let $G\mathscr{D}$ be the full $\mathscr{S}$-subcategory of $G\mathscr{S}$ whose objects are fibrant approximations of the suspension G-spectra $\Sigma^\infty_G(A_+)$, where A runs through the finite G-sets. Then there is an enriched Quillen adjunction*

$$\mathbf{Pre}(G\mathscr{D},\mathscr{S})\underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}}G\mathscr{S},$$

[4]The papers from around 1990, such as [2, 21] are not adequate for our purposes, in part because the target category of G-spectra was not yet well understood then. A full dress modern treatment of equivariant infinite loop space theory, complementing [7], is in progress [18].

[5]There is a slight abuse of language here since the notion of a category enriched in $\mathscr{S}_G$ (alias a G-spectral category) does not quite make sense in classical enriched category theory because the smash product of G-spectra is only functorial on G-maps, not on the more general maps in $\mathscr{S}_G$. The terminology is explained and justified in [6, 1.9].

and it is a Quillen equivalence.

Here $G\mathscr{D}$ is isomorphic to $(\mathscr{D}_G)^G$, where $\mathscr{D}_G$ is a full $\mathscr{S}_G$-subcategory $\mathscr{D}_G$ of $\mathscr{S}_G$.

Theorem 2.4 (Equivariant version of the main theorem)**.** *There is a zigzag of weak equivalences connecting the $\mathscr{S}_G$-categories $\mathscr{B}_G$ and $\mathscr{D}_G$.*

A weak equivalence between $\mathscr{S}_G$-categories with the same object sets is just an $\mathscr{S}_G$-functor that induces weak equivalences on morphism G-spectra.[6] On passage to G-fixed categories, this equivariant zigzag induces a zigzag of weak $\mathscr{S}$-equivalences connecting the $\mathscr{S}$-categories $G\mathscr{B}$ and $G\mathscr{D}$. In turn, by [5, 2.4], this zigzag induces a zigzag of Quillen equivalences between $\mathbf{Pre}(G\mathscr{B}, \mathscr{S})$ and $\mathbf{Pre}(G\mathscr{D}, \mathscr{S})$. Since $\mathbf{Pre}(G\mathscr{D}, \mathscr{S})$ is Quillen equivalent to $G\mathscr{S}$, it follows that Theorem 2.4 implies Theorem 1.10.

Remark 2.5. The functor $\mathbb{U}$ sends G/H to $F_G(\Sigma^\infty_G G/H_+, X)^G \cong X^H$. Keeping that fact in mind shows why Corollary 1.11 follows from the proof of Theorem 1.10.

To understand $G\mathscr{S}$ as an $\mathscr{S}$-category, we must first understand $\mathscr{S}_G$ as an $\mathscr{S}_G$-category. That is, to understand the G-fixed spectra $F_G(X,Y)^G$, we must first understand the function G-spectra $F_G(X,Y)$. Using infinite loop space theory to model function spectra implicitly raises a conceptual issue: there is no known infinite loop space machine that knows about function spectra. That is, given input data X and Y (permutative G-categories, E_∞-G-spaces, Γ-G-spaces, etc) for an infinite loop space machine $\mathbb{K}_G$, we do not know what input data will have as output the function G-spectra $F_G(\mathbb{K}_G X, \mathbb{K}_G Y)$. The problem does not even make sense as just stated because the output G-spectra $\mathbb{K}_G X$ are always connective, whereas $F_G(\mathbb{K}_G X, \mathbb{K}_G Y)$ is generally not. The most that one could hope for in general is to detect the connective cover of $F(\mathbb{K}_G X, \mathbb{K}_G Y)$. In our case, the relevant function G-spectra are connective since the suspension G-spectra $\Sigma^\infty_G(A_+)$ are self-dual, as we shall reprove in §2.3.

2.2. **Results from equivariant infinite loop space theory.** The proof of Theorem 2.4 is the heart of this paper, and of course it depends on equivariant infinite loop space theory and in particular on the relationship between the G-spectra $\mathscr{B}_G(A) = \mathbb{K}_G\mathscr{E}_G(A)$ and the suspension G-spectra $\Sigma^\infty_G(A_+)$. We collect the results that we need from [7] in this section, making Definitions 1.9 and 2.1 precise and expanding on Theorems 1.19 and 2.2. We warn the skeptical reader that the results of this paper depend on the two results just cited and on Theorems 2.6 and 2.8 below. The knowledgable expert will immediately accept the plausibility of these results, especially since those of the results which make sense when $G = e$ have been known for decades. However, their proofs require work that is far afield from the applications in this paper.

In fact, Theorem 2.4 is an application of a categorical version of the equivariant Barratt-Priddy-Quillen (BPQ) theorem for the identification of suspension G-spectra.[7] We state the theorem in full generality before restricting attention to finite G-sets. We shall find use for the full generality in §2.5.

Recall from Definition 1.17 that $\mathscr{E}_G(A)$ is the category $\mathbb{O}_G(A_+)$, where $\mathbb{O}_G$ is the free $\mathscr{O}_G$-category functor. We may view any based G-space X as a topological

[6] A more general definition is given in [5, 2.3].

[7] For $A = *$, Carlsson [1, p.6] mentions a space level version of the BPQ theorem. Shimakawa [21, p. 242] states and gives an incomplete sketch proof of a G-spectrum level version.

category[8] that is discrete in the categorical sense: its morphism and object spaces are both X, and its source, target, identity, and composition maps are all just the identity map of X. The functor $\mathbb{O}_G$ applies equally well to based topological G-categories, hence we have the topological $\mathscr{O}_G$-category $\mathbb{O}_G(X)$. The geometric realization of its nerve is the free E_∞ G-space generated by X.

Henceforward, we use the term stable equivalence, rather than weak equivalence, for the weak equivalences in our model categories of spectra and G-spectra.

Theorem 2.6 (Equivariant Barratt-Quillen Theorem, [7]). *For based G-spaces X, there is a natural stable equivalence*

$$\alpha\colon \Sigma^\infty_G X \longrightarrow \mathbb{K}_G\mathbb{O}_G(X).$$

Of course, the naturality statement says that the following diagram commutes for a map $f\colon X \longrightarrow Y$ of based G-spaces.

$$\begin{array}{ccc} \Sigma^\infty_G X & \xrightarrow{\alpha} & \mathbb{K}_G\mathbb{O}_G(X) \\ {\scriptstyle \Sigma^\infty_G f}\downarrow & & \downarrow{\scriptstyle \mathbb{K}_G\mathbb{O}_G(f)} \\ \Sigma^\infty_G(B_+) & \xrightarrow[\alpha]{} & \mathbb{K}_G\mathbb{O}_G(Y) \end{array} \tag{2.7}$$

There is a companion theorem that relates α to smash products. The pairing ω of Definition 1.22 generalizes to give a natural pairing

$$\omega\colon \mathbb{O}_G(X)\wedge\mathbb{O}_G(Y) \longrightarrow \mathbb{O}_G(X\wedge Y)$$

for based G-spaces X and Y.

Theorem 2.8. [7] *The pairing ω induces a natural stable equivalence*

$$\wedge\colon \mathbb{K}_G\mathbb{O}_G(X)\wedge\mathbb{K}_G\mathbb{O}_G(Y) \longrightarrow \mathbb{K}_G\mathbb{O}_G(X\wedge Y)$$

such that the following diagram commutes.

$$\begin{array}{ccc} \Sigma^\infty_G X\wedge\Sigma^\infty_G Y & \xrightarrow{\alpha\wedge\alpha} & \mathbb{K}_G\mathbb{O}_G(X)\wedge\mathbb{K}_G\mathbb{O}_G(Y) \\ {\scriptstyle \wedge}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \wedge} \\ \Sigma^\infty_G(X\wedge Y) & \xrightarrow[\alpha]{} & \mathbb{K}_G\mathbb{O}_G(X\wedge Y) \end{array} \tag{2.9}$$

The left map $\wedge$ in (2.9) is a canonical natural isomorphism, and this diagram says that the natural map α is lax monoidal. The result that we need to prove Theorem 2.4 is an immediate specialization.

Theorem 2.10. *For finite G-sets A, there is a lax monoidal natural stable equivalence*

$$\alpha\colon \Sigma^\infty_G(A_+) \longrightarrow \mathbb{K}_G\mathscr{E}_G(A).$$

Identifying $A_+\wedge B_+$ with $(A\times B)_+$, (2.9) specializes to the commutative diagram

$$\begin{array}{ccc} \Sigma^\infty_G(A_+)\wedge\Sigma^\infty_G(B_+) & \xrightarrow{\alpha\wedge\alpha} & \mathbb{K}_G\mathscr{E}_G(A)\wedge\mathbb{K}_G\mathscr{E}_G(B) \\ {\scriptstyle \wedge}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \wedge} \\ \Sigma^\infty_G(A\times B)_+ & \xrightarrow[\alpha]{} & \mathbb{K}_G\mathscr{E}_G(A\times B). \end{array} \tag{2.11}$$

[8]We understand a topological category to mean an internal category in the category of spaces, not just a category enriched in spaces.

We restate the naturality of α with respect to G-maps $f\colon A \longrightarrow B$ in the diagram

$$\begin{array}{ccc} \Sigma^\infty_G(A_+) & \xrightarrow{\alpha} & \mathbb{K}_G\mathscr{E}_G(A) \\ {\scriptstyle \Sigma^\infty_G f_+}\downarrow & & \downarrow{\scriptstyle \mathbb{K}_G f_!} \\ \Sigma^\infty_G(B_+) & \xrightarrow[\alpha]{} & \mathbb{K}_G\mathscr{E}_G(B). \end{array} \tag{2.12}$$

If $i\colon A \longrightarrow B$ is an inclusion with retraction $r\colon B_+ \longrightarrow A_+$, we have the induced map of G-spectra

$$\mathbb{K}_G i^* = \mathbb{K}_G r_! \colon \mathbb{K}_G\mathscr{E}_G(B) \longrightarrow \mathbb{K}_G\mathscr{E}_G(A),$$

and (2.12) specializes to

$$\begin{array}{ccc} \Sigma^\infty_G(B_+) & \xrightarrow{\alpha} & \mathbb{K}_G\mathscr{E}_G(B) \\ {\scriptstyle \Sigma^\infty_G r}\downarrow & & \downarrow{\scriptstyle \mathbb{K}_G i^*} \\ \Sigma^\infty_G(A_+) & \xrightarrow[\alpha]{} & \mathbb{K}_G\mathscr{E}_G(A) \end{array} \tag{2.13}$$

By Remark 2.21 below, we may identify $\mathbb{K}_G i^*$ as the dual of $\mathbb{K}_G i$ and thus $\Sigma^\infty_G r$ as the dual of $\Sigma^\infty_G i_+$.

We combine these diagrams to construct those that we need to prove Theorem 2.4. Let A, B, and C be finite G-sets and recall Definition 1.20.

Proposition 2.14. *The following diagram of G-spectra commutes.*

$$\begin{array}{ccc} \Sigma^\infty_G(C\times B)_+ \wedge \Sigma^\infty_G(B\times A)_+ & \xrightarrow{\alpha\wedge\alpha} & \mathbb{K}_G\mathscr{E}_G(C\times B) \wedge \mathbb{K}_G\mathscr{E}_G(B\times A) \\ {\scriptstyle \wedge}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \wedge} \\ \Sigma^\infty(C\times B\times B\times A)_+ & \xrightarrow{\alpha} & \mathbb{K}_G\mathscr{E}_G(C\times B\times B\times A) \\ {\scriptstyle \Sigma^\infty_G r}\downarrow & & \downarrow{\scriptstyle \mathbb{K}_G(\mathrm{id}\times\Delta\times\mathrm{id})^*} \\ \Sigma^\infty(C\times B\times A)_+ & \xrightarrow{\alpha} & \mathbb{K}_G\mathscr{E}_G(C\times B\times A) \\ {\scriptstyle \Sigma^\infty \pi}\downarrow & & \downarrow{\scriptstyle \mathbb{K}_G\pi_!} \\ \Sigma^\infty_G(C\times A)_+ & \xrightarrow[\alpha]{} & \mathbb{K}_G\mathscr{E}_G(C\times A) \end{array} \tag{2.15}$$

Here r is the retraction which sends the complement of the image of $\mathrm{id}\times\Delta\times\mathrm{id}$ *to the basepoint.*

Definition 2.16. To make Definition 2.1 and therefore Definition 1.9 precise, define the composition

$$\mathscr{B}_G(B,C)\wedge\mathscr{B}_G(A,B) \longrightarrow \mathscr{B}_G(A,C)$$

to be the right vertical composite in the diagram (2.15).

The diagram (2.15) relates the composition pairing of the $\mathscr{S}_G$-category $\mathscr{B}_G$ to remarkably simple and explicit maps between suspension G-spectra. In fact, recalling Definition 1.28 and again writing $\varepsilon = \Sigma^\infty_G\varepsilon$, we see that the left vertical composite in (2.15) can be identified with $\mathrm{id}\wedge\varepsilon\wedge\mathrm{id}$. We have proven the following result.

Theorem 2.17. *The following diagram of G-spectra commutes.*

$$\begin{array}{ccc}
\Sigma^\infty_G(C\times B)_+\wedge\Sigma^\infty_G(B\times A)_+ & \xrightarrow{\alpha\wedge\alpha} & \mathscr{B}_G(B,C)\wedge\mathscr{B}_G(A,B) \\
\cong\downarrow & & \\
\Sigma^\infty_G(C_+)\wedge\Sigma^\infty_G(B\times B)_+\wedge\Sigma^\infty_G(A_+) & & \\
\mathrm{id}\wedge\varepsilon\wedge\mathrm{id}\downarrow & & \downarrow\circ \\
\Sigma^\infty_G(C_+)\wedge S_G\wedge\Sigma^\infty_G(A_+) & & \\
\cong\downarrow & & \\
\Sigma^\infty_G(C\times A)_+ & \xrightarrow[\alpha]{} & \mathscr{B}_G(A,C)
\end{array}$$

2.3. **The self-duality of $\Sigma^\infty_G(A_+)$.** Let A be a finite G-set and write $\mathbb{A}=\Sigma^\infty_G(A_+)$ for brevity of notation. As recalled in §1.5, we must define maps $\eta\colon S_G\longrightarrow\mathbb{A}\wedge\mathbb{A}$ and $\varepsilon\colon\mathbb{A}\wedge\mathbb{A}\longrightarrow S_G$ in the stable homotopy category $HoG\mathscr{S}$ such that the composites

$$\mathbb{A}\xrightarrow{\eta\wedge\mathrm{id}}\mathbb{A}\wedge\mathbb{A}\wedge\mathbb{A}\xrightarrow{\mathrm{id}\wedge\varepsilon}\mathbb{A}\quad\text{and}\quad\mathbb{A}\xrightarrow{\mathrm{id}\wedge\eta}\mathbb{A}\wedge\mathbb{A}\wedge\mathbb{A}\xrightarrow{\varepsilon\wedge\mathrm{id}}\mathbb{A}\tag{2.18}$$

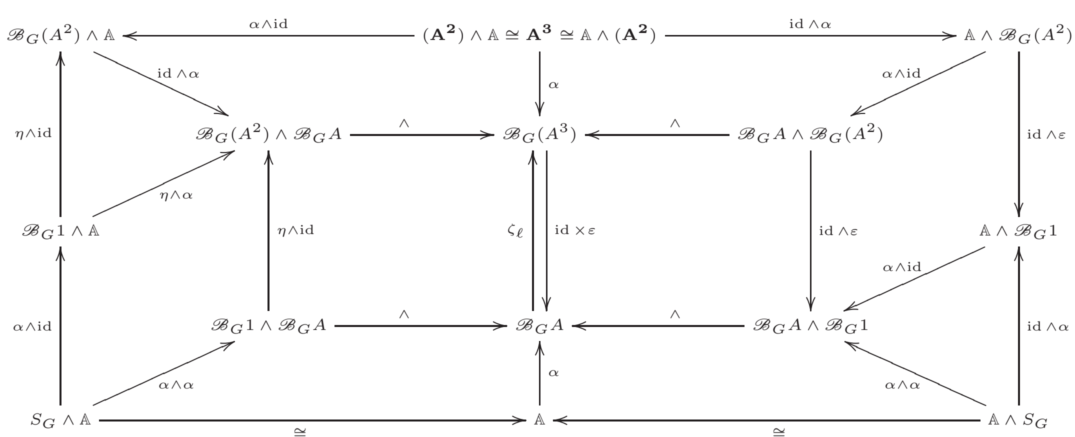
are the identity map in $HoG\mathscr{S}$. Using the stable equivalence α and the definitions of η and ε from Definitions 1.28 and 1.29, we let η and ε be the composites

$$S_G\xrightarrow{\alpha}\mathbb{K}_G\mathscr{E}_G(1)\xrightarrow{\mathbb{K}_G\eta}\mathbb{K}_G\mathscr{E}_G(A\times A)\xrightarrow{\alpha^{-1}}\Sigma^\infty_G(A\times A)_+\cong\mathbb{A}\wedge\mathbb{A}$$

and

$$\mathbb{A}\wedge\mathbb{A}\cong\Sigma^\infty_G(A\times A)_+\xrightarrow{\alpha}\mathbb{K}_G\mathscr{E}_G(A\times A)\xrightarrow{\mathbb{K}_G\varepsilon}\mathbb{K}_G\mathscr{E}_G(1)\xrightarrow{\alpha^{-1}}S_G.$$

The following commutative diagram proves that the first composite in (2.18) is the identity map in $HoG\mathscr{S}$; the second is dealt with similarly. We abbreviate notation by setting $\mathscr{B}_GA=\mathbb{K}_G\mathscr{E}_G(A)$. Remember that $\mathscr{E}_G(A)=\mathbb{O}_G(A_+)$. The center two squares are derived by use of the diagrams from Proposition 1.30.



Given Theorem 2.10, it is trivial that the outer parts of the diagram commute. We comment on the passage from the diagrams of Proposition 1.30 to the central squares of the diagram; compare Remark 1.36.

Remark 2.19. Nonequivariantly, the passage from pairings on the category level to pairings on the spectrum level is worked out in [15], implicitly using orthogonal spectra. The sequel [7] to this paper constructs the pairing $\wedge$ from the pairing ω of free $\mathscr{O}_G$-categories used here, but it does not treat its naturality with respect to maps of pairings that are not induced by maps of finite G-sets. Modernized generalizations and details will be supplied in [18].

Specializing general observations about duality recalled in §1.5, we have the following corollary. This homotopical input is the crux of the proof of Theorem 2.4.

Corollary 2.20. *For finite G-sets A and B, the canonical map*

$$\delta = \zeta \circ (\mathrm{id} \wedge \tilde{\varepsilon})\colon \mathbb{B} \wedge \mathbb{A} \longrightarrow \mathbb{B} \wedge D\mathbb{A} \longrightarrow F_G(\mathbb{A}, \mathbb{B})$$

of (1.26) is a stable equivalence.

We insert a mild digression concerning the identification of some of our maps.

Remark 2.21. For an inclusion $i\colon A \longrightarrow B$ of finite G-sets, (1.27) and the precise constructions of η and ε starting from Definitions 1.28 and 1.29 imply that the dual of i is the map $\mathbb{B} \longrightarrow \mathbb{A}$ induced by the evident retraction $r\colon B_+ \longrightarrow A_+$. A G-map $\pi\colon G/H \longrightarrow G/K$ is a bundle, and the dual of $\Sigma^\infty \pi_+$ is the associated transfer map (see e.g. [11, IV.pp 182 and 192]). It can be identified explicitly by a similar (but not especially illuminating) inspection of definitions.

2.4. **The proof that $\mathscr{B}_G$ is equivalent to $\mathscr{D}_G$.** We will have to chase large diagrams, and we again abbreviate notations by writing

$$\mathbb{A} = \Sigma^\infty_G(A_+), \quad \mathbb{B} = \Sigma^\infty_G(B_+), \quad \text{and} \quad \mathbb{C} = \Sigma^\infty_G(C_+)$$

for finite G-sets A, B, and C. We also abbreviate notation by writing

$$\mathscr{B}_G(A) = \mathscr{B}_G(*, A).$$

It is the G-spectrum $\mathscr{B}_G(A) = \mathbb{K}_G\mathscr{E}_G(A)$, which is equivalent to $\mathbb{A}$ by Theorem 2.10. Remember that we are free to choose any bifibrant equivalents of the G-spectra $\mathbb{A}$ as the objects of $\mathscr{D}_G$.

Proof of Theorem 2.4. We use model categorical arguments, and we work with the stable model structure on $G\mathscr{S}$. We use [5, §2.4] to obtain a model structure on the category $G\mathscr{S}\mathbb{O}$-$\mathscr{C}at$ of $G\mathscr{S}$-categories with the same object set $\mathbb{O}$ as $G\mathscr{E}$. Maps are weak equivalences or fibrations if they induce weak equivalences or fibrations on hom objects in $G\mathscr{S}$. Here the nature of the objects is irrelevant; we are concerned with $G\mathscr{S}$-categories with one object for each finite G-set A.

Let $\lambda\colon Q\mathscr{B}_G \longrightarrow \mathscr{B}_G$ be a cofibrant approximation of $\mathscr{B}_G$. By [5, 2.16], since S_G is cofibrant in the stable model structure each morphism G-spectrum $Q\mathscr{B}_G(A,B)$ is cofibrant in $G\mathscr{S}$. The maps $\lambda\colon Q\mathscr{B}_G(A,B) \longrightarrow \mathscr{B}_G(A,B)$ are stable acyclic fibrations. Digressively, since the $\mathscr{B}_G(A,B)$ are fibrant in the positive stable model structure, that is also true of the $Q\mathscr{B}_G(A,B)$; we will use this fact later, in §2.5.

Let $\rho\colon Q\mathscr{B}_G \longrightarrow RQ\mathscr{B}_G$ be a fibrant approximation of $Q\mathscr{B}_G$. The morphism G-spectra $RQ\mathscr{B}_G(A,B)$ are then bifibrant in the stable model structure. Therefore $RQ\mathscr{B}_G(A)$ is bifibrant for each A, and it is stably equivalent to $\mathbb{A}$. We take the $RQ\mathscr{B}_G(A)$ as the bifibrant approximations of the $\mathbb{A}$ that we use to define the full $G\mathscr{S}$-subcategory $\mathscr{D}_G$ of $G\mathscr{S}$.

We define $\mathscr{C}_G$ to be the full $G\mathscr{S}$-subcategory of $G\mathscr{S}$ with objects the $Q\mathscr{B}_G(A)$. To abbreviate notation, we agree to write

$$Q\mathscr{B}_G(*, A) = Q\mathscr{B}_G A \quad \text{and} \quad RQ\mathscr{B}_G(*, A) = RQ\mathscr{B}_G A.$$

With our notational conventions, it is consistent to write $Q\mathscr{B}_G(B\times A) = Q\mathscr{B}_G(A, B)$.

For finite G-sets A and B, let

$$\beta\colon Q\mathscr{B}_G(A, B) \longrightarrow \mathscr{C}_G(A, B) = F_G(Q\mathscr{B}_G A, Q\mathscr{B}_G B)$$

be the adjoint of the composition map

$$\circ\colon Q\mathscr{B}_G(A, B) \wedge Q\mathscr{B}_G A \longrightarrow Q\mathscr{B}_G B$$

and let

$$\gamma\colon RQ\mathscr{B}_G(A, B) \longrightarrow \mathscr{D}_G(A, B) = F_G(RQ\mathscr{B}_G A, RQ\mathscr{B}_G B)$$

be the adjoint of the composition map

$$\circ\colon RQ\mathscr{B}_G(A, B) \wedge RQ\mathscr{B}_G A \longrightarrow RQ\mathscr{B}_G B.$$

By [5, 5.6], these define $G\mathscr{S}$-functors

$$\beta\colon Q\mathscr{B}_G \longrightarrow \mathscr{C}_G \quad \text{and} \quad \gamma\colon RQ\mathscr{B}_G \longrightarrow \mathscr{D}_G.$$

It suffices to prove that each of the maps γ is a stable equivalence. For each finite G-set A, $\mathbb{A}$ is cofibrant and $\lambda\colon Q\mathscr{B}_G A \longrightarrow \mathscr{B}_G A$ is an acyclic fibration in the stable model structure. Therefore there is a map $\mu\colon \mathbb{A} \longrightarrow Q\mathscr{B}_G A$ such that the diagram

$$\begin{array}{ccc} & & Q\mathscr{B}_G A \\ & \overset{\mu}{\nearrow} & \big\downarrow{\scriptstyle \lambda} \\ \mathbb{A} & \xrightarrow[\alpha]{} & \mathscr{B}_G A \end{array}$$

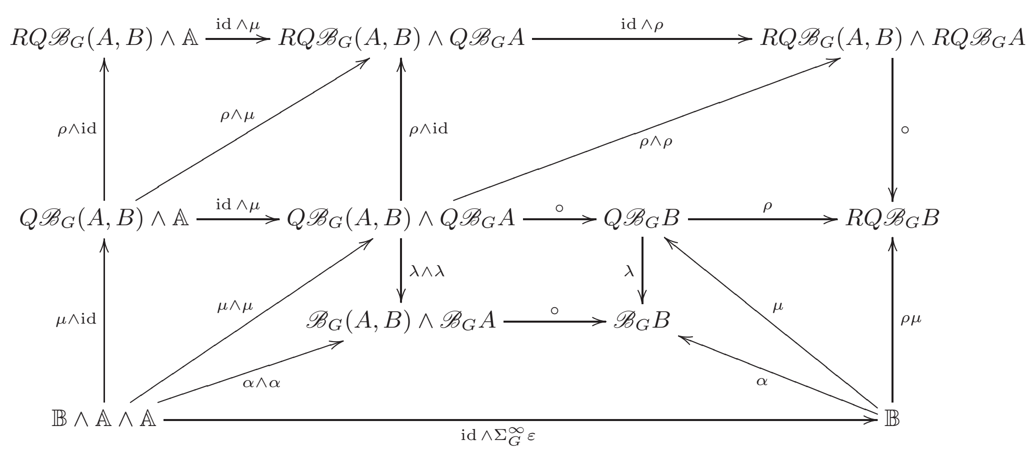
commutes. Since α and λ are stable equivalences, so is μ.

We claim that the following diagram of G-spectra commutes in $HoG\mathscr{S}$. Indeed, modulo inversion of maps which are stable equivalences, it commutes on the nose. As before, we identify $\mathbb{B} \wedge \mathbb{A} = \Sigma^\infty_G B_+ \wedge \Sigma^\infty_G A_+$ with $\Sigma^\infty_G (B \times A)_+$.

$$\begin{array}{ccccc}
RQ\mathscr{B}_G(A, B) & \xrightarrow{\gamma} & F_G(RQ\mathscr{B}_G A, RQ\mathscr{B}_G B) & \xrightarrow[\simeq]{F_G(\rho,\mathrm{id})} & F_G(Q\mathscr{B}_G A, RQ\mathscr{B}_G B) \\
{\scriptstyle \rho}\big\uparrow{\scriptstyle \simeq} & & & {\scriptstyle F_G(\mathrm{id},\rho)}\nearrow & {\scriptstyle \simeq}\big\downarrow{\scriptstyle F_G(\mu,\mathrm{id})} \\
Q\mathscr{B}_G(A, B) & \xrightarrow{\beta} & F_G(Q\mathscr{B}_G A, Q\mathscr{B}_G B) & & F_G(\mathbb{A}, RQ\mathscr{B}_G B) \\
{\scriptstyle \mu}\big\uparrow{\scriptstyle \simeq} & & & {\scriptstyle F_G(\mu,\mathrm{id})}\searrow & {\scriptstyle \simeq}\big\uparrow{\scriptstyle F_G(\mathrm{id},\rho)} \\
\mathbb{B} \wedge \mathbb{A} & \xrightarrow[\delta]{\simeq} & F_G(\mathbb{A}, \mathbb{B}) & \xrightarrow[F_G(\mathrm{id},\mu)]{\simeq} & F_G(\mathbb{A}, Q\mathscr{B}_G B)
\end{array}$$

The map δ is the stable equivalence of Corollary 2.20. The maps μ and ρ are also stable equivalences. The maps $F_G(\rho, \mathrm{id})$ and $F_G(\mu, \mathrm{id})$ that are labeled $\simeq$ are stable equivalences by [5, 1.22] since ρ and μ are maps between cofibrant objects and $RQ\mathscr{B}_G B$ is fibrant. The maps $F_G(\mathrm{id}, \mu)$ and $F_G(\mathrm{id}, \rho)$ that are labeled $\simeq$ are stable equivalences by [12, III.3.9], which shows that the functor $F_G(\mathbb{A}, -)$ preserves stable equivalences. Granting that the diagram commutes, it follows that γ is a stable equivalence since all of the other outer arrows of the diagram are stable equivalences.

To prove that the diagram commutes in $HoG\mathscr{S}$, we consider its adjoint. Remembering that $\lambda \circ \mu = \alpha$, we see that the adjoint can be written in the following expanded form. Here we have inserted the map $\circ\colon \mathscr{B}_G(A,B) \wedge \mathscr{B}_G A \longrightarrow \mathscr{B}_G B$ and wrong way arrows into its source and target for purposes of proof.



Since λ and ρ are maps of $G\mathscr{S}$-categories, it is apparent that all parts of the diagram commute except for the bottom trapezoid. Taking $(A,B,C) = (*,A,B)$ in Theorem 2.17, we see that the trapezoid commutes. Since the wrong way maps α and λ are stable equivalences and can be inverted upon passage to the homotopy category, this diagram and its adjoint commute there. □

2.5. **Identifications of suspension G-spectra and of tensors with spectra.** We expand the adjoint $\mathscr{S}$-equivalences in Theorem 1.10 more explicitly as follows.

$$
\begin{array}{ccccc}
G\mathscr{S} & \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} & \mathbf{Pre}(G\mathscr{D}, \mathscr{S}) & \underset{\gamma^*}{\overset{\gamma_!}{\leftrightarrows}} & \mathbf{Pre}((RQ\mathscr{B}_G)^G, \mathscr{S}) \\
 & & & & \rho^* \downarrow \uparrow \rho_! \\
\mathbf{Pre}(G\mathscr{B}, \mathscr{S}) & \underset{\iota^*}{\overset{\iota_!}{\rightleftarrows}} & \mathbf{Pre}((\mathscr{B}_G)^G, \mathscr{S}) & \underset{\lambda^*}{\overset{\lambda_!}{\leftrightarrows}} & \mathbf{Pre}((Q\mathscr{B}_G)^G, \mathscr{S})
\end{array}
\tag{2.22}
$$

The map $\iota : G\mathscr{B} \longrightarrow (\mathscr{B}_G)^G$ is the equivalence of Theorem 2.2. Before passage to G-fixed points, the proof in §2.4 gives stable equivalences of $\mathscr{S}_G$-categories

$$\rho\colon Q\mathscr{B}_G \longrightarrow RQ\mathscr{B}_G, \quad \gamma : RQ\mathscr{B}_G \longrightarrow \mathscr{D}_G, \text{ and } \lambda\colon Q\mathscr{B}_G \longrightarrow \mathscr{B}_G,$$

and these maps give stable equivalences of $\mathscr{S}$-categories after passage to fixed points.

For a finite G-set B, $\Sigma^\infty_G B_+$ corresponds under this zigzag to the presheaf $\mathbf{B}$ that sends A to $G\mathscr{B}(A,B)$. This is almost a tautology since, for $E \in G\mathscr{S}$, $\mathbb{U}(E)$ is the presheaf represented by E, while $G\mathscr{E}(-,B)$ is the functor represented by B. In the proof of Theorem 2.4, we chose the bifibrant approximation of $\Sigma^\infty_G B_+$ in $G\mathscr{D}_G$

to be $RQ\mathscr{B}_G(B)$. With B fixed, that proof shows that γ gives an equivalence of presheaves

$$RQ\mathscr{B}_G(-,B) \longrightarrow \gamma^*\mathbb{U}RQ\mathscr{B}_G(B)$$

(before passage to G-fixed points). The functors ρ^* and $\lambda_!$ and the isomorphism ι^* preserve representable functors, and therefore $\iota^*\lambda_!\rho^*RQ\mathscr{B}_G(-,B) \simeq K_G\mathscr{E}_G(-,B)$.

This observation can be generalized from finite based G-sets B_+ to arbitrary based G-spaces X. To see this, we mix general based G-spaces X with finite based G-sets A to obtain a functorial construction of a presheaf $\mathscr{P}_G(X)$.

Definition 2.23. Define a presheaf $\mathscr{P}_G(X)\colon (\mathscr{B}_G)^{op} \longrightarrow \mathscr{S}_G$ by letting

$$\mathscr{P}_G(X)(A) = \mathscr{K}_G\mathbb{O}_G(X \wedge A_+).$$

The contravariant functoriality map

$$\mathscr{P}_G(X)\colon \mathscr{B}_G(A,B) \longrightarrow F_G(\mathscr{B}_G(X)(B), \mathscr{B}_G(X)(A))$$

is the adjoint of the right vertical composite in the commutative diagram

(2.24)
$$\begin{array}{ccc}
\Sigma^\infty_G(X \wedge B_+) \wedge \Sigma^\infty_G(B_+ \wedge A_+) & \xrightarrow{\alpha\wedge\alpha} & \mathbb{K}_G\mathbb{O}_G(X \wedge B_+) \wedge \mathbb{K}_G\mathbb{O}_G(B_+ \wedge A_+) \\
{\scriptstyle\wedge}\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\wedge} \\
\Sigma^\infty(X \wedge B_+ \wedge B_+ \wedge A_+) & \xrightarrow{\alpha} & \mathbb{K}_G\mathbb{O}_G(X \wedge B_+ \wedge B_+ \wedge A_+) \\
{\scriptstyle\Sigma^\infty_G r}\downarrow & & \downarrow{\scriptstyle\mathbb{K}_G\mathbb{O}_G(r)} \\
\Sigma^\infty(X \wedge B_+ \wedge A_+) & \xrightarrow{\alpha} & \mathbb{K}_G\mathbb{O}_G(X \wedge B_+ \wedge A_+) \\
{\scriptstyle\Sigma^\infty\pi}\downarrow & & \downarrow{\scriptstyle\mathbb{K}_G\mathbb{O}_G\pi} \\
\Sigma^\infty_G(X \wedge A_+) & \xrightarrow[\alpha]{} & \mathbb{K}_G\mathbb{O}_G(X \wedge A_+).
\end{array}$$

Here r is the evident left inverse of $\mathrm{id}\wedge\Delta\wedge\mathrm{id}$ and π is the projection. The diagram commutes by the same concatenation of commutative diagrams as in Proposition 2.14.

Theorem 2.25. *Let X be a based G-space. Under our zigzag of equivalences, $\Sigma^\infty_G X$ corresponds naturally to the presheaf $(\mathscr{P}_G(X))^G$ that sends A to $\mathbb{K}\big(\mathbb{O}_G(X \wedge A_+)^G\big)$.*

Proof. Note that $\mathbb{K}_G\mathbb{O}_G(X \wedge -_+)$ is no longer a representable presheaf. We again work with G-spectra and obtain the conclusion after passage to G-fixed spectra. According to Theorem 2.6, we may replace $\Sigma^\infty_G X$ by the positive fibrant G-spectrum $\mathbb{K}_G\mathbb{O}_G(X)$, which we abbreviate to $\mathscr{B}_G(X)$ by a slight abuse of notation. After this replacement, the presheaf $\mathbb{U}(\Sigma^\infty_G X)$ may be computed as

$$\mathbb{U}(\Sigma^\infty_G X)(A) = F_G(RQ\mathscr{B}_G(A), \mathscr{B}_G(X)).$$

Therefore, following the chain of (2.22), we may compute $\rho^*\gamma^*\mathbb{U}(\Sigma^\infty_G X)$ as

$$\rho^*\gamma^*\mathbb{U}(\Sigma^\infty_G X) \simeq F_G(Q\mathscr{B}_G(-), \mathscr{B}_G(X)).$$

Thinking of (B,A) above replaced by $(A,*)$, the adjoint to the composite

$$\mathscr{P}_G(X)(A) \wedge Q\mathscr{B}_G(A) \xrightarrow{\mathrm{id}\wedge\lambda} \mathscr{P}_G(X)(A) \wedge \mathscr{B}_G(A) \xrightarrow{\circ} \mathscr{P}_G(X)(*) = \mathscr{B}_G(X) \tag{2.26}$$

defines a map of presheaves

$$\lambda^*\mathscr{P}_G(X) \longrightarrow F_G(Q\mathscr{B}_G(-), \mathscr{B}_G(X)) \tag{2.27}$$

with domain $Q\mathscr{B}_G$. It remains to show that this map is an equivalence. To compute the adjoint (2.27), observe that the composite (2.26) is the top horizontal composite in the commutative diagram

$$\begin{array}{ccccc}
\mathscr{P}_G(X)(A) \wedge Q\mathscr{B}_G(A) & \xrightarrow{\mathrm{id}\wedge\lambda} & \mathscr{P}_G(X)(A) \wedge \mathscr{B}_G(A) & \xrightarrow{\circ} & \mathscr{B}_G(X) \\
\uparrow{\scriptstyle \alpha\wedge\mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}\wedge\alpha} & & \\
\Sigma^\infty_G(X \wedge A_+) \wedge Q\mathscr{B}_G(A) & & \mathscr{B}_G(A,X) \wedge \Sigma^\infty_G A_+ & & \uparrow{\scriptstyle \alpha} \\
\uparrow{\scriptstyle \mathrm{id}\wedge\mu} & \nearrow{\scriptstyle \alpha\wedge\mathrm{id}} & & & \\
\Sigma^\infty_G(X \wedge A_+) \wedge \Sigma^\infty_G A_+ & \xrightarrow[\cong]{} & \Sigma^\infty_G X \wedge \Sigma^\infty_G (A_+ \wedge A_+) & \xrightarrow[\mathrm{id}\wedge\varepsilon]{} & \Sigma^\infty_G X.
\end{array}$$

We have used that $\lambda \circ \mu = \alpha$. The pentagon on the right is a special case of (2.24).

Therefore the map (2.27) is the top horizontal composite in the diagram

$$\begin{array}{ccccc}
\mathscr{P}_G(X)(A) & \longrightarrow & F_G(\mathscr{B}_G(A), \mathscr{B}_G(X)) & \xrightarrow{F_G(\lambda,\mathrm{id})} & F_G(Q\mathscr{B}_G(A), \mathscr{B}_G(X)) \\
\uparrow{\scriptstyle \alpha} & & & & \downarrow{\scriptstyle F_G(\mu,\mathrm{id})} \\
\Sigma^\infty_G(X \wedge A_+) & \xrightarrow[\delta]{} & F_G(\Sigma^\infty_G A_+, \Sigma^\infty_G X) & \xrightarrow[F_G(\mathrm{id},\alpha)]{} & F_G(\Sigma^\infty_G A_+, \mathscr{B}_G(X)).
\end{array}$$

The map α is a stable equivalence by Theorem 2.6. The map δ is the stable equivalence of (1.26). The map $F_G(\mathrm{id}, \alpha)$ is a stable equivalence by [12, III.3.9]. Finally, the map $F_G(\mu, \mathrm{id})$ is a stable equivalence by [5, 1.22]. □

There is another visible identification. The category $G\mathscr{S}$ and our presheaf categories are $\mathscr{S}$-complete, so that they have tensors and cotensors over $\mathscr{S}$ (see [5, §5.1]). It is formal that the left adjoint of an $\mathscr{S}$-adjunction preserves tensors and the right adjoint preserves cotensors. A quick chase of our zigzag of Quillen $\mathscr{S}$-equivalences gives the following conclusion.

Theorem 2.28. *For G-spectra Y and spectra X, if Y corresponds to a presheaf $\mathscr{P}Y$ under our zigzag of weak equivalences, then the tensor $Y \odot X$ corresponds to the tensor $\mathscr{P}Y \odot X$.*

3. Atiyah duality for finite G-sets

It is illuminating to see that we can come very close to constructing an alternative model for the spectrally enriched category $G\mathscr{D}$ just by applying the suspension G-spectrum functor Σ^∞_G to the category of based G-spaces and G-maps and then passing to G-fixed points. This is based on a close inspection of classical Atiyah duality specialized to finite G-sets. However, it depends on working in the alternative category $G\mathscr{Z}$ of S_G-modules [3, 12] rather than in the category $G\mathscr{S}$ of orthogonal G-spectra. Because every object of $G\mathscr{Z}$ is fibrant and its suspension G-spectra are easily understood, it is more convenient than $G\mathscr{S}$ for comparison with space level constructions. This leads us to a variant, Theorem 3.20, of Theorem 0.1 that does not invoke infinite loop space theory. It is more topological and less categorical. It is also more elementary.

3.1. **The categories $G\mathscr{Z}$, $G\mathscr{D}$, and $\mathscr{D}_G$.** Relevant background about $G\mathscr{Z}$ appears in [6, §3.4] and we just give a minimum of notation here. In analogy with Theorem 2.3, we have the following specialization of the same general result about stable model categories. It is discussed in [6, §3.1].

Theorem 3.1. *Let $G\mathscr{D}$ be the full $\mathscr{Z}$-subcategory of $G\mathscr{Z}$ whose objects are cofibrant approximations of the suspension G-spectra $\Sigma^\infty_G(A_+)$, where A runs through the finite G-sets. Then there is an enriched Quillen adjunction*

$$\mathbf{Pre}(G\mathscr{D},\mathscr{Z}) \underset{\mathbb{U}}{\overset{\mathbb{T}}{\rightleftarrows}} G\mathscr{Z},$$

and it is a Quillen equivalence.

Here $G\mathscr{D}$ is isomorphic to $(\mathscr{D}_G)^G$, where $\mathscr{D}_G$ is a full $\mathscr{Z}_G$-subcategory $\mathscr{D}_G$ of $\mathscr{S}_G$. All objects of $G\mathscr{Z}$ are fibrant, and we need to choose cofibrant approximations of the $\Sigma^\infty_G(A_+)$. The construction of $G\mathscr{Z}$ starts from the Lewis-May category $G\mathscr{S}p$ of G-spectra, and S_G-modules are G-spectra with additional structure. We have an elementary suspension G-spectrum functor $\Sigma^\infty_G\colon G\mathscr{T} \longrightarrow G\mathscr{S}p$. There is a left adjoint $\mathbb{F}\colon G\mathscr{S}p \longrightarrow G\mathscr{Z}$, which is a Quillen equivalence [3, 12]. Define $\mathbf{\Sigma}^\infty_G\colon G\mathscr{T} \longrightarrow G\mathscr{Z}$ to be the composite $\mathbb{F}\circ\Sigma^\infty_G$. Suspension G-spectra have natural structures as S_G-modules, and there is a natural stable equivalence of S_G-modules $\gamma\colon \mathbf{\Sigma}^\infty_{\mathbf{G}}X \longrightarrow \Sigma^\infty_G X$. Viewing Σ^∞_G as a functor $G\mathscr{T} \longrightarrow G\mathscr{Z}$, it is strong symmetric monoidal. However, the $\Sigma^\infty_G X$ are not cofibrant. The functor $\mathbf{\Sigma}^\infty_{\mathbf{G}}$ takes based G-CW complexes X, such as A_+ for a finite G-set A, to cofibrant S_G-modules. Therefore $\mathbf{\Sigma}^\infty_{\mathbf{G}}$ may be viewed as a cofibrant replacement functor for Σ^∞_G. In particular, we write $\mathbf{S_G} = \mathbf{\Sigma}^\infty_{\mathbf{G}}S^0$ and have a cofibrant approximation $\gamma\colon \mathbf{S_G} \longrightarrow S_G$ of the unit object S_G. Moreover, the cofibrant approximation $\mathbf{\Sigma}^\infty_{\mathbf{G}}(A_+)$ is isomorphic to $\mathbf{S_G}\wedge\Sigma^\infty_G(A_+)$ over $\Sigma^\infty_G(A_+)$.

As before, we consider finite G-sets A, B, and C, but we now agree to write

$$\mathbb{A} = \mathbf{\Sigma}^\infty_{\mathbf{G}}A_+, \quad \mathbb{B} = \mathbf{\Sigma}^\infty_{\mathbf{G}}B_+, \quad \text{and} \quad \mathbb{C} = \mathbf{\Sigma}^\infty_{\mathbf{G}}C_+.$$

The $\mathbb{A}$ are bifibrant objects of $G\mathscr{Z}$ and we let $G\mathscr{D}$ and $\mathscr{D}_G$ be the full subcategories of $G\mathscr{Z}$ and $\mathscr{Z}_G$ whose objects are the S_G-modules $\mathbb{A}$, where A runs over the finite G-sets. Then $\mathscr{D}_G$ is enriched in $G\mathscr{Z}$ and $G\mathscr{D} = (\mathscr{D}_G)^G$ is enriched in the category $\mathscr{Z}$ of S-modules. The functor $\mathbf{\Sigma}^\infty_{\mathbf{G}}$ is almost strong symmetric monoidal. Precisely, by [6, 3.9] there is a natural isomorphism

$$\mathbb{A}\wedge\mathbb{B} \cong \mathbf{S_G}\wedge\mathbf{\Sigma}^\infty_{\mathbf{G}}(A\times B)_+ \tag{3.2}$$

with appropriate coherence properties with respect to associativity and commutativity. Since S_G is the unit for the smash product, we can compose with

$$\gamma\wedge\mathrm{id}\colon \mathbf{S_G}\wedge\mathbf{\Sigma}^\infty_{\mathbf{G}}(A\times B)_+ \longrightarrow \mathbf{\Sigma}^\infty_{\mathbf{G}}(A\wedge B)_+$$

to give a pairing as if $\mathbf{\Sigma}^\infty_{\mathbf{G}}$ were a lax symmetric monoidal functor. However, the map $\gamma\colon \mathbf{S_G} \longrightarrow S_G$ points the wrong way for the unit map of such a functor.

3.2. **Space level Atiyah duality for finite G-sets.** To lift the self-duality of $Ho\mathscr{D}_G$ to obtain a new model for $\mathscr{D}_G$, we need representatives in $G\mathscr{Z}$ for the maps

$$\eta\colon S_G \longrightarrow \mathbb{A}\wedge\mathbb{A} \quad \text{and} \quad \varepsilon\colon \mathbb{A}\wedge\mathbb{A} \longrightarrow S_G$$

in Ho$G\mathscr{Z}$ that express the duality there. The map ε is induced from the elementary map ε of Definition 1.28. The observation that it plays a key role in Atiyah duality seems to be new. The definition of η requires desuspension by representation spheres.

Let A be a finite G-set and let $V = \mathbb{R}[A]$ be the real representation generated by A, with its standard inner product, so that $|a| = 1$ for $a \in A$. Since we are working on the space level, we may view $A_+ \wedge S^V$ as the wedge over $a \in A$ of the spaces (not G-spaces) $\{a\}_+ \wedge S^V$, with G acting by $g(a, v) = (ga, gv)$. There is no such wedge decomposition after passage to G-spectra.

Definition 3.3. Recall that $\varepsilon\colon (A\times A)_+ \longrightarrow S^0$ is the G-map defined by $\varepsilon(a, b) = *$ if $a \neq b$ and $\varepsilon(a, a) = 1$. Recall too that $(A \times B)_+$ can be identified with $A_+ \wedge B_+$ and that the functor $\mathbf{\Sigma^\infty_G}$ is almost strong symmetric monoidal. We shall also write ε for the composite map of S_G-modules

$$\mathbb{A} \wedge \mathbb{A} \cong \mathbf{S_G} \wedge \mathbf{\Sigma^\infty_G}(A \times A)_+ \xrightarrow{\mathrm{id} \wedge \mathbf{\Sigma^\infty_G}\varepsilon} \mathbf{S_G} \wedge \mathbf{S_G} \xrightarrow{\gamma\wedge\gamma} S_G \wedge S_G \cong S_G, \tag{3.4}$$

where the unlabeled isomorphisms are two instances of (3.2).

Definition 3.5. Embed A as the basis of the real representation $V = \mathbb{R}[A]$. The normal bundle of the embedding is just $A \times V$, and its Thom complex is $A_+ \wedge S^V$. We obtain an explicit tubular embedding $\nu\colon A \times V \longrightarrow V$ by setting

$$\nu(a, v) = a + (\rho(|v|)/|v|)v,$$

where $\rho\colon [0,\infty) \longrightarrow [0, d)$ is a homeomorphism for some $d < 1/2$; ν is a G-map since $|gv| = |v|$ for all g and v. Applying the Pontryagin-Thom construction, we obtain a G-map $t\colon S^V \longrightarrow A_+ \wedge S^V$, which is an equivariant pinch map

$$S^V \longrightarrow \vee_{a\in A} S^V \cong A_+ \wedge S^V.$$

To be more precise, after collapsing the complement of the tubular embedding to a point, we use ν^{-1} to expand each small homeomorphic copy of S^V to the canonical full-sized one; explicitly, if $|w| < d$, then

$$\nu^{-1}(a + w) = (a, (\rho^{-1}(|w|)/|w|)w).$$

The diagonal map on A induces the Thom diagonal $\Delta\colon A_+ \wedge S^V \longrightarrow A_+ \wedge A_+ \wedge S^V$, and we let

$$\eta\colon S^V \longrightarrow A_+ \wedge A_+ \wedge S^V \tag{3.6}$$

be the composite $\Delta \circ t$. Explicitly,

$$\eta(v) = \begin{cases} (a, a, (\rho^{-1}(|w|)/|w|)w) & \text{if } v = a + w \text{ where } a \in A \text{ and } |w| < d \\ * & \text{otherwise.} \end{cases} \tag{3.7}$$

The negative sphere G-spectrum S^{-V} in $G\mathscr{S}p$ is obtained by applying the left adjoint of the V^{th}-space functor to S^0, and S_G is isomorphic to $S^V \odot S^{-V}$ (see [11, I.4.2] and [12, IV.2.2]). Taking the tensor of η with S^{-V} we obtain a map of G-spectra

$$S_G \cong S^V \odot S^{-V} \longrightarrow (A_+ \wedge A_+ \wedge S^V) \odot S^{-V} \cong (A_+ \wedge A_+) \odot S_G \cong \Sigma^\infty_G(A_+ \wedge A_+).$$

Applying the functor $\mathbb{F}$ to this map and smashing with $\mathbf{S_G}$ we obtain the second map in the diagram

$$S_G \cong S_G \wedge S_G \xleftarrow{\gamma\wedge\gamma} \mathbf{S_G} \wedge \mathbf{S_G} \xrightarrow{\eta} \mathbf{S_G} \wedge \mathbf{\Sigma^\infty_G}(A \times A)_+ \cong \mathbb{A} \wedge \mathbb{A}. \tag{3.8}$$

The following result is a reminder about space level Atiyah duality. The notion of a V-duality was defined and explained for smooth G-manifolds in [11, §III.5].

Proposition 3.9. *The maps*

$$\eta\colon S^V \longrightarrow A_+ \wedge A_+ \wedge S^V \quad \textit{and} \quad \varepsilon \wedge \mathrm{id}\colon A_+ \wedge A_+ \wedge S^V \longrightarrow S^V$$

specify a V-duality between A_+ and itself.

Proof. This could be proven from scratch by proving the required triangle identities, but in fact it is a special case of equivariant Atiyah duality for smooth G-manifolds, A being a 0-dimensional example. Our specification of η is a specialization of the description of η for a general smooth G-manifold M given in [11, p. 152]. We claim that our $\varepsilon \wedge \mathrm{id}$ is a specialization of the definition of ε for a general smooth G-manifold given there. Indeed, letting s be the zero section of the normal bundle ν of the embedding $A \subset \mathbb{R}[A] = V$, we have the composite embedding

$$A \xrightarrow{\Delta} A \times A \xrightarrow{s\times\mathrm{id}} (A \times V) \times A \cong A \times A \times V.$$

The normal bundle of this embedding is $A \times V$, and we may view

$$\Delta \times \mathrm{id}\colon A \times V \longrightarrow A \times A \times V$$

as giving a big tubular neighborhood. The Pontryagin-Thom map here is obtained by smashing the map $r\colon (A \times A)_+ \longrightarrow A_+$ that sends (a, b) to a if $a = b$ and to $*$ if $a \neq b$ with the identity map of S^V. Composing with the map induced by the projection $\pi\colon A_+ \longrightarrow S^0$ that sends a to 1, this gives $\varepsilon \wedge \mathrm{id}$. We observed this factorization of ε in Definition 1.28 and we have used it before, in the proof of Theorem 2.17. □

Tensoring with S^{-V}, applying the functor $\mathbf{S_G} \wedge \mathbb{F}$, and composing with γ, we obtain the explicit duality maps in $G\mathscr{Z}$ displayed in (3.4) and (3.8).

3.3. **The weakly unital categories $G\mathscr{A}$ and $\mathscr{A}_G$.** Since the G-spectra $\mathbb{A}$ are self-dual, $F_G(\mathbb{A}, \mathbb{B})$ is naturally isomorphic to $\mathbb{B} \wedge \mathbb{A}$ in $\mathrm{Ho}G\mathscr{Z}$, and the composition and unit

$$F_G(\mathbb{B}, \mathbb{C}) \wedge F_G(\mathbb{A}, \mathbb{B}) \longrightarrow F_G(\mathbb{A}, \mathbb{C}) \quad \text{and} \quad S_G \longrightarrow F_G(\mathbb{B}, \mathbb{B}) \tag{3.10}$$

can be expressed as maps

$$\mathbb{C} \wedge \mathbb{B} \wedge \mathbb{B} \wedge \mathbb{A} \longrightarrow \mathbb{C} \wedge \mathbb{A} \quad \text{and} \quad S_G \longrightarrow \mathbb{A} \wedge \mathbb{A} \tag{3.11}$$

in $\mathrm{Ho}G\mathscr{Z}$. We want to understand these maps in terms of duality in $G\mathscr{Z}$, without use of infinite loop space theory. However, since we are working in $G\mathscr{Z}$, we must take the isomorphisms (3.2) and the cofibrant approximation $\gamma\colon \mathbf{S}_G \longrightarrow S_G$ into account, and we cannot expect to have strict units. The notion of a weakly unital enriched category was introduced in [5, §3.5] to formalize what we see here.

Thus we shall construct a weakly unital $G\mathscr{Z}$-category $\mathscr{A}_G$ and compare it with $\mathscr{D}_G$. The G-fixed category $G\mathscr{A}$ will be a weakly unital $\mathscr{Z}$-category.[9]

The objects of $\mathscr{A}_G$ and $G\mathscr{A}$ are the S_G-modules $\mathbb{A}$ for finite G-sets A. The morphism S_G-modules of $\mathscr{A}_G$ are $\mathscr{A}_G(\mathbb{A}, \mathbb{B}) = \mathbb{B} \wedge \mathbb{A}$. Composition is given by the maps

$$\mathrm{id} \wedge \varepsilon \wedge \mathrm{id}\colon \mathbb{C} \wedge \mathbb{B} \wedge \mathbb{B} \wedge \mathbb{A} \longrightarrow \mathbb{C} \wedge \mathbb{A}, \tag{3.12}$$

[9]Mnemonically, the $\mathscr{A}$ stands for Atiyah.

where ε is the map of (3.4); compare Theorem 2.17.

As recalled in §1.5, the adjoint $\tilde{\varepsilon}\colon \mathbb{A} \longrightarrow D\mathbb{A} = F_G(\mathbb{A}, S_G)$ of ε is a stable equivalence, and we have the composite stable equivalence

(3.13) $$\delta = \zeta \circ (\mathrm{id} \wedge \tilde{\varepsilon})\colon \mathbb{B} \wedge \mathbb{A} \longrightarrow \mathbb{B} \wedge D\mathbb{A} \longrightarrow F_G(\mathbb{A}, \mathbb{B}).$$

Formal properties of the adjunction $(\wedge, F_G)$ give the following commutative diagram in $G\mathscr{Z}$, which uses δ to compare composition in $\mathscr{A}_G$ with composition in $\mathscr{D}_G$.

(3.14) $$\begin{array}{ccc} \mathbb{C} \wedge \mathbb{B} \wedge \mathbb{B} \wedge \mathbb{A} & \xrightarrow{\mathrm{id} \wedge \varepsilon \wedge \mathrm{id}} & \mathbb{C} \wedge \mathbb{A} \\ {\scriptstyle \mathrm{id} \wedge \tilde{\varepsilon} \wedge \mathrm{id} \wedge \tilde{\varepsilon}} \downarrow & & \downarrow {\scriptstyle \mathrm{id} \wedge \tilde{\varepsilon}} \\ \mathbb{C} \wedge D\mathbb{B} \wedge \mathbb{B} \wedge D\mathbb{A} & \xrightarrow{\mathrm{id} \wedge \varepsilon \wedge \mathrm{id}} & \mathbb{C} \wedge D\mathbb{A} \\ {\scriptstyle \zeta \wedge \zeta} \downarrow & & \downarrow {\scriptstyle \zeta} \\ F_G(\mathbb{B}, \mathbb{C}) \wedge F_G(\mathbb{A}, \mathbb{B}) & \xrightarrow[\circ]{} & F_G(\mathbb{A}, \mathbb{C}) \end{array}$$

At the bottom, we do not know that the function S_G-modules or their smash product are cofibrant, but all objects at the top are cofibrant and thus bifibrant. In general, to compute the smash product of G-spectra X and Y in the homotopy category, we should take the smash product of cofibrant approximations QX and QY of X and Y. Since all objects of $G\mathscr{Z}$ are fibrant, to compute a map $X \wedge Y \longrightarrow Z$ in the homotopy category, we should represent it by a map $QX \wedge QY \longrightarrow QZ$ and take its homotopy class. The diagram displays such a cofibrant approximation of the composition in $\mathscr{D}_G$.

The unit $S_G \longrightarrow F_G(\mathbb{A}, \mathbb{A})$ of $\mathscr{A}_G$ is represented by the (formal) composite

(3.15) $$S_G \xrightarrow{\eta} \mathbb{A} \wedge \mathbb{A} \xrightarrow{\mathrm{id} \wedge \tilde{\varepsilon}} \mathbb{A} \wedge D\mathbb{A} \xrightarrow{\zeta} F_G(\mathbb{A}, \mathbb{A})$$

that is obtained by inverting the map $\gamma \wedge \gamma$ in (3.8) to obtain the map denoted η. The weak unital property is a way of expressing the unital property by maps in $\mathscr{Z}_G$, without use of inverses in $Ho\mathscr{Z}_G$. This is a bit tedious. Here are the details.

Definition 3.16. Let $V = \mathbb{R}[A]$. For $a \in A$, define $\xi_a\colon \{a\}_+ \wedge S^V \longrightarrow \{a\}_+ \wedge S^V$ by

(3.17) $$\xi_a(a, v) = \begin{cases} (a, (\rho^{-1}(|w|)/|w|)w) & \text{if } v = a + w \text{ and } |w| < d \\ * & \text{otherwise,} \end{cases}$$

where ρ is as in Definition 3.5. Then the wedge of the ξ_a is a G-map

(3.18) $$\xi\colon A_+ \wedge S^V \longrightarrow A_+ \wedge S^V;$$

ξ is G-homotopic to the identity map of $A_+ \wedge S^V$ via the explicit G-homotopy

$$h(a, v, t) = \begin{cases} (a, v) & \text{if } t = 0 \text{ or } v = a \\ (a, (1-t)v + t(\rho^{-1}(t|w|)/|w|)w) & \text{if } v = a + w \text{ and } t|w| < d \\ * & \text{otherwise.} \end{cases}$$

With η as specified in (3.6), easy and perhaps illuminating inspections show that the following unit diagrams already commute in $G\mathscr{T}$, before passage to homotopy. In both, A and B are finite G-sets. In the first, $V = \mathbb{R}[A]$. In the second, $V = \mathbb{R}[B]$

and we move S^V from the right to the left for clarity.

$$\begin{array}{ccc} B_+\wedge A_+\wedge S^V & \xrightarrow{\mathrm{id}\wedge\eta_A} & B_+\wedge A_+^3\wedge S^V \\ {\scriptstyle \mathrm{id}\wedge\xi_A}\downarrow & \swarrow{\scriptstyle \mathrm{id}\wedge\varepsilon\wedge\mathrm{id}} & \\ B_+\wedge A_+\wedge S^V & & \end{array} \quad \text{and} \quad \begin{array}{ccc} S^V\wedge B_+\wedge A_+ & \xrightarrow{\eta_B\wedge\mathrm{id}} & S^V\wedge B_+^3\wedge A_+ \\ {\scriptstyle \xi_B\wedge\mathrm{id}_A}\downarrow & \swarrow{\scriptstyle \mathrm{id}\wedge\varepsilon\wedge\mathrm{id}} & \\ S^V\wedge B_+\wedge A_+ & & \end{array}$$

Tensoring with S^{-V} and using the natural isomorphisms

$$(X\wedge S^V)\odot S^{-V}\cong X\odot S_G\cong \Sigma_G^\infty X$$

for based G-spaces X, we see that the space level G-equivalence ξ induces a spectrum level G-equivalence $\xi\colon \mathbb{A}\longrightarrow\mathbb{A}$.

Tensoring with S^{-V} and using (3.2) to pass to smash products of S_G-modules, a little diagram chase shows that the previous pair of diagrams in $G\mathscr{T}$ gives rise to the following pair of commutative diagrams in $G\mathscr{Z}$. These express the unit laws for a weakly unital $G\mathscr{Z}$-category $\mathscr{A}_G$ [5, §3.5] with objects the $\mathbb{A}$ and composition as specified in (3.12). The cited unit laws allow us to start with any chosen cofibrant approximation $\gamma\colon QS_G\longrightarrow S_G$ of the unit S_G, and we are led by (3.8) to choose our cofibrant approximation to be $\gamma\wedge\gamma\colon \mathbf{S_G}\wedge\mathbf{S_G}\longrightarrow S_G\wedge S_G\cong S_G$. Using the notation $\gamma\colon QS_G\longrightarrow S_G$ for this map, we obtain the required diagrams

$$\begin{array}{ccc} \mathbb{B}\wedge\mathbb{A}\wedge QS_G & \xrightarrow{\mathrm{id}\wedge\eta} & \mathbb{B}\wedge\mathbb{A}\wedge\mathbb{A}\wedge\mathbb{A} \\ {\scriptstyle \mathrm{id}\wedge\xi\wedge\gamma}\downarrow & & \downarrow{\scriptstyle \circ} \\ \mathbb{B}\wedge\mathbb{A}\wedge S_G & \xrightarrow[\cong]{} & \mathbb{B}\wedge\mathbb{A} \end{array} \quad \text{and} \quad \begin{array}{ccc} QS_G\wedge\mathbb{B}\wedge\mathbb{A} & \xrightarrow{\eta\wedge\mathrm{id}} & \mathbb{B}\wedge\mathbb{B}\wedge\mathbb{B}\wedge\mathbb{A} \\ {\scriptstyle \gamma\wedge\xi\wedge\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \circ} \\ S_G\wedge\mathbb{B}\wedge\mathbb{A} & \xrightarrow[\cong]{} & \mathbb{B}\wedge\mathbb{A}. \end{array}$$

Taking $A=S^0$ in our second space level diagram and changing B to A, we also obtain the following commutative diagrams in $G\mathscr{Z}$, where the second diagram is adjoint to the first.

$$(3.19)\qquad \begin{array}{ccc} QS_G\wedge\mathbb{A} & \xrightarrow{\eta\wedge\mathrm{id}} & \mathbb{A}\wedge\mathbb{A}\wedge\mathbb{A} \\ {\scriptstyle \gamma\wedge\xi}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\wedge\varepsilon} \\ S_G\wedge\mathbb{A} & \xrightarrow[\cong]{} & \mathbb{A} \end{array} \quad \text{and} \quad \begin{array}{ccccc} QS_G & \xrightarrow{\eta} & \mathbb{A}\wedge\mathbb{A} & \xrightarrow{\mathrm{id}\wedge\tilde{\varepsilon}} & bA\wedge D\mathbb{A} \\ {\scriptstyle \gamma}\downarrow & & & & \downarrow{\scriptstyle \zeta} \\ S_G & \xrightarrow[\eta]{} & F_G(\mathbb{A},\mathbb{A}) & \xrightarrow[F_G(\xi,\mathrm{id})]{} & F_G(\mathbb{A},\mathbb{A}) \end{array}$$

Here η at the bottom right is adjoint to the identity map of $\mathbb{A}$. In effect, this uses $\delta=\zeta\circ(\mathrm{id}\wedge\tilde{\varepsilon})$ to compare the actual unit η in $\mathscr{D}_G$ at the top with the weak unit in $\mathscr{A}_G$, which is given by the interrelated maps η, γ, and ξ.

3.4. **The category of presheaves with domain $G\mathscr{A}$.** The diagrams (3.14) and (3.19) show that the maps $\delta\colon \mathbb{A}\wedge\mathbb{B}\longrightarrow F_G(\mathbb{A},\mathbb{B})$ specify a map of weakly unital $\mathscr{Z}_G$-categories from the weakly unital $\mathscr{Z}_G$-category $\mathscr{A}_G$ to the (unital) $\mathscr{Z}_G$-category $\mathscr{D}_G$. Passing to G-fixed points, we obtain a weakly unital $\mathscr{Z}$-category $G\mathscr{A}$ and a map $\delta\colon G\mathscr{A}\longrightarrow G\mathscr{D}$ of weakly unital $\mathscr{Z}$-categories. Weakly unital presheaves and presheaf categories are defined in [5, 3.25]. By [5, 3.26], we obtain the same category of presheaves $\mathscr{Z}^{G\mathscr{D}}$ using unital or weakly unital presheaves. Since δ is an equivalence, we can adapt the methodology of [5, §2] to prove the following result. However, since we find the use of weakly unital categories unpleasant and our main result Theorem 1.10 more satisfactory, we shall leave the details to the interested reader. Nevertheless, it is this equivalence that best captures the geometric intuition behind our results.

Theorem 3.20. *The categories* $\mathbf{Pre}(G\mathscr{A}, \mathscr{X})$ *and* $\mathbf{Pre}(G\mathscr{D}, \mathscr{X})$ *are Quillen equivalent.*

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E-mail address: bertguillou@uky.edu

Department of Mathematics, University of Kentucky, Lexington, KY 40506 USA

E-mail address: may@math.uchicago.edu

Department of Mathematics, The University of Chicago, Chicago, IL 60637 USA