ON THE MOTIVIC SEGAL CONJECTURE
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Abstract. We establish motivic versions of the theorems of Lin and Gunawardena, thereby confirming the motivic Segal conjecture for the algebraic group \( \mu_\ell \) of \( \ell \)-th roots of unity, where \( \ell \) is any prime. To achieve this we develop motivic Singer constructions associated to the symmetric group \( S_{\ell} \) and to \( \mu_\ell \), and introduce a delayed limit Adams spectral sequence.

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1. Introduction

Let \( \gamma^1 \downarrow \mathbb{R}P^\infty \simeq BC_2 \) be the tautological line bundle over infinite-dimensional real projective space, let \( \mathbb{R}P^\infty_m = Th(-m\gamma^1) \) be the Thom spectrum of the negative of \( m \) times \( \gamma^1 \), and let \( \mathbb{R}P^\infty = \lim_m \mathbb{R}P^\infty_m \). Mahowald conjectured that there is a 2-adic equivalence \( \mathbb{R}P^\infty \simeq S^{-1} \), see [Ada74, p. 5]. More generally, Segal conjectured for finite groups \( G \) that there is an \( I(G) \) adjic equivalence \( (S_G)^G \simeq (S_G)^{hG} \) from the fixed points to the homotopy fixed points of the \( G \)-equivariant sphere spectrum. Here \( I(G) \) denotes the augmentation ideal in the Burnside ring of \( G \).

Mahowald’s conjecture, which is equivalent to Segal’s Burnside ring conjecture for \( C_2 \), was proved by Lin in [Lin80, Thm. 1.2]. For odd primes \( \ell \), Gunawardena [Gun81] proved Segal’s conjecture for \( C_\ell \), obtaining an \( \ell \)-adic equivalence \( L^\infty \simeq S^{-1} \). Here \( L^\infty \) denotes a homotopy limit of Thom spectra over the infinite-dimensional lens space \( L^\infty \simeq BC_\ell \). Segal’s conjecture was later affirmed for all finite groups by Carlsson [Car84], building on [MMS82], [AGMS85] and [CMP87].

In this paper we promote the classical theorems of Lin and Gunawardena to the motivic setting, obtaining \( \pi_* \)-isomorphisms \( S \simeq \Sigma^{1,0}L^\infty \), after \( (\ell, \eta) \)-adic completion, for all primes \( \ell \). Here \( S \) denotes the motivic sphere spectrum, and now

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Let $S$ be a finite dimensional Noetherian scheme, essentially smooth over a field or Dedekind domain containing $1/\ell$. There is a $\pi_{*,*}$-isomorphism
\[ e_{\ell,n}^*: S^\wedge_n / L^\wedge_n \longrightarrow (\Sigma^{1,0} L^\infty)^\wedge_{\ell,n} \]
in the stable motivic homotopy category $\mathcal{S}H(S)$. If $S = \text{Spec} k$ for $k$ a field, then $e_{\ell,n}^*$ is a motivic equivalence.

In other words, we prove the motivic Segal conjecture in its non-equivariant form, in the case of the algebraic group $\mu_{\ell}$, for any prime $\ell$. For $\ell = 2$ this is the motivic version of Mahowald’s conjecture and Lin’s theorem. For $\ell$ odd it is the motivic version of Gunawardena’s theorem.

Already for $S = \text{Spec} k$ in the algebraically closed case $k = \mathbb{C}$, the additional information about motivic weight has proved to be a valuable new tool for calculational purposes [Sa10], [WX20]. In the real case $k = \mathbb{R}$, many new phenomena arise [HI11], [DI7], [BI22]. Our results are valid even in the arithmetically most substantial cases of (rings of $\ell$-integers in) number fields. In particular, we have made an effort to not have to assume that the mod $\ell$ motivic cohomology groups $H^{*,*} = H^{*,*}(S; \mathbb{Z}/\ell)$ are finite in each bidegree. Our results enable an analysis of the motivic Mahowald root invariants, refining [Mah85], [MR99]. Such applications have already appeared in [Qui21a], [Qui21b]. We expect the interplay between the motivic cohomology of number fields and the Mahowald root invariants to be very rich.

In Section 2 we review from Voevodsky’s article [Voe03a] the Hopf algebroid structure of the motivic dual Steenrod algebra $\mathcal{A}_{*,*}$, and of its quotients $A(n)_{*,*} = \mathcal{A}_{*,*}/I(n)$. In Section 3 we generalize the approach of Adams–Gunawardena–Miller from [AGMS8] §2, and introduce the $A(n)_{*,*}$-bicomodule algebras $C(n)_{*,*} = \mathcal{A}_{*,*}/J(n)$ and their localizations $B(n)_{*,*}$ away from $\xi_1$. In Section 4 we dualize these constructions, following Boardman [Boa82] §3, obtaining the motivic Steenrod algebra $\mathcal{A}$, its finite subalgebras $A(n)$, and the $A(n)$-$A(n-1)$-bimodules $C(n)$ and $B(n)$. In Section 5 we generalize the (small) Singer construction of Sin[81] and LS[82], obtaining an $\mathcal{A}$-module $R_S(M) = \text{colim}_m B(n) \otimes_{A(n-1)} M$ and a natural homomorphism $\epsilon: R_S(M) \to M$ for each $\mathcal{A}$-module $M$.

We prove in Theorem 5.8 that $R_S(H^{*,*}) \cong \Sigma^{1,0} H^{*,*}(BS\ell)_\text{loc}$ is a shifted localization of the motivic cohomology of the geometric classifying space of the symmetric group $S_m$ on $m$ letters. In Section 6 we recast [AGMS8] §5 and construct a (large) Singer construction $R_\ell(M)$ and a natural $\mathcal{A}$-module homomorphism $\epsilon: R_\ell(M) \to M$. In Section 7 we show that $R_\ell(H^{*,*}) \cong \Sigma^{1,0} H^{*,*}(BS\ell)_\text{loc}$ is a shifted localization of the motivic cohomology of the infinite lens space $B\ell$. In Section 8 we prove that the evaluation homomorphisms $\epsilon$ are Ext-equivalences. Here we deviate from the Tor-equivalence approach of [AGMS8] §2, due to the two-sided nature of Hopf algebroids.

In Section 10 we construct the tower $\{L^\infty\}_m$ of motivic spectra, and the map $\epsilon: S \to \Sigma^{1,0} L^\infty$ to the suspension of their homotopy limit. We show in Proposition 10.10 that the continuous cohomology $H^{*,*}_c(L^\infty) = \text{colim}_m H^{*,*}(L^\infty)_m$ is isomorphic as an $\mathcal{A}$-module to $H^{*,*}(B\ell)_\text{loc}$, and that $\epsilon$ induces the Ext-equivalence $\epsilon$, via the identifications above. The plan is now to compare the motivic mod $\ell$ Adams spectral sequence for $S$ with the tower of Adams spectral sequences associated to
the $L_{2m}$. This works fine in the presence of sufficient finiteness to ensure that the algebraic limit of these Adams spectral sequences is again a spectral sequence, as is the case in the classical setting of [COPS77]. However, for base schemes $S$ such that $H_{t,u}^{*,*}$ is not finite in each bidegree, this approach can fail. Instead, we form a modified Adams spectral sequence, called the delayed limit Adams spectral sequence, where any $\lim^1$-classes arising from non-exactness are shifted up into the next filtration degree.

In Section 8 we prepare for this construction by introducing some terminology for motivic generalized Eilenberg–MacLane spectra, and formulate a finiteness condition, called bifinite type, which lets us identify the $E_1$- and $E_2$-terms of motivic Adams spectral sequences in algebraic terms. In Section 9 we introduce the delayed limit Adams spectral sequence in Definition 9.1 and identify its $E_2$-term as $\text{Ext}$ for a continuous cohomology $\mathcal{A}$-module in Proposition 9.2. In Proposition 9.3 we show that the delayed limit Adams spectral sequence converges conditionally, and in Proposition 9.7 we adapt a comparison theorem from [Boa99] for morphisms of conditionally convergent spectral sequences. In Section 11 the threads are brought together. See Theorem 11.1 for the proof of Theorem 1.1.

This article is based on the first author’s PhD thesis [Gre13], guided by the second author.

2. The motivic Steenrod algebra and its dual

Let $S$ be a Noetherian (separated) scheme of finite (Krull) dimension $d$, essentially smooth over a field or a Dedekind domain, and let $C$ be a prime that is invertible on $S$.

Let $\mathcal{SH}(S)$ be Voevodsky’s motivic stable homotopy category [Voe98, Def. 5.7], [Jar00] associated to smooth schemes over $S$. It is triangulated, and has a compatible closed symmetric monoidal structure given by the motivic sphere spectrum $S = \Sigma^\infty S_+$, the smash product pairing $- \wedge -$, the twist isomorphism $\gamma$ and the function spectrum $F(-,-)$. Let $H = HZ/C$ be the motivic Eilenberg–MacLane spectrum representing motivic cohomology with coefficients in $Z/C$. It is a commutative ring spectrum, with unit map $\eta: S \to H$ and product $\mu: H \wedge H \to H$. Moreover, $H$ is known to be cellular [Hov15, Prop. 8.1], [Spi18, Cor. 10.4], i.e., an iterated homotopy colimit of stable motivic spheres.

Let $H_{*,*} = \pi_{*,*}(H) = H_{*,*,*}$ denote the motivic homology and cohomology groups of the base scheme $S$. Then $H_{p,q} = 0$ unless $0 \leq p \leq \min\{q + d, 2q\}$, cf. [Gei04, Cor. 4.4], [Hov15, Cor. 4.26]. For $x \in \pi_{t,u}(X)$, where $X$ is any motivic spectrum, we refer to $t$ and $u$ as the topological degree and weight of $x$, respectively. We write $|x| = \deg(x) = t$, $\wt(x) = \mu$ and $||x|| = (t, u)$. The cup product induced by $\mu$ gives $H_{*,*} = H_{*,*,*}$ the structure of a bigraded commutative $Z/C$-$\alpha$-algebra. Only the parity of the topological degree plays a role in bigraded commutativity.

Let $\mathcal{A} = H_{*,*,*}(H) = \pi_{*,*,*}(F(H,H))$ denote the motivic Steenrod algebra, and let $\mathcal{A}_{*,*} = H_{*,*,*}(H \wedge H)$ denote its dual. Then $\mathcal{A}_{*,*}$ is free as a left $H_{*,*}$-module, cf. Lemma 2.1 so the pair $(H_{*,*}, \mathcal{A}_{*,*})$ admits the structure of a bigraded Hopf algebroid [Ada69, Lec. 3], [MR77, §1], [Rav86, Def. A1.1.1]. Its structure maps are the following $Z/C$-$\alpha$-algebra homomorphisms:

1. the left unit $\eta_L: H_{*,*} \to \mathcal{A}_{*,*}$ induced by $1 \wedge \eta: H = H \wedge S \to H \wedge H$;
2. the right unit $\eta_R: H_{*,*} \to \mathcal{A}_{*,*}$ induced by $\eta \wedge 1: H = S \wedge H \to H \wedge H$;
3. the product $\phi: \mathcal{A}_{*,*} \otimes \mathcal{A}_{*,*} \to \mathcal{A}_{*,*}$ induced by $(\mu \wedge \mu)(1 \wedge \gamma \wedge 1): H \wedge H \wedge H \to H \wedge H$;
4. the counit $\epsilon: \mathcal{A}_{*,*} \to H_{*,*}$ induced by $\mu: H \wedge H \to H$;
5. the coproduct $\psi: \mathcal{A}_{*,*} \to \mathcal{A}_{*,*} \otimes H_{*,*} \mathcal{A}_{*,*}$ induced by $1 \wedge \eta \wedge 1: H \wedge H = H \wedge S \wedge H \to H \wedge H \wedge H \cong (H \wedge H) \wedge H (H \wedge H)$;
sequences with $E$.

Lemma 2.1. They shall be interpreted to be zero for in \([Voe03a, Thm. 6.10]\). More explicitly,

$$\mathcal{A}_{+,\ast} = H_{+,\ast}[r_0, r_1, \ldots, E_1 ; \xi_1, \xi_2, \ldots] / (r_i^2 - T_i \mid i \geq 0)$$

is a bigraded commutative $H_{+,\ast}$-algebra generated by classes $r_i$ in bidegree $\|r_i\| = (2\ell^i - 1, \ell^i - 1)$ and $E_1$ in bidegree $\|E_1\| = (2\ell^0 - 2, \ell^0 - 1)$, where

$$T_i = \begin{cases} \tau r_i + \rho r_{i+1} + \rho_0 r_{i+1} & \text{for } \ell = 2, \\ 0 & \text{for } \ell = 0 \text{ odd.} \end{cases}$$

Here the elements $\rho \in H^{1,1} = H_{-1,-1}$ and $\tau \in H^{0,1} = H_{0,-1}$ are specified for $\ell = 2$ in \([Voe03a, Thm. 6.10]\). They shall be interpreted to be zero for $\ell$ odd. In these terms,

(1) the algebra unit is $\eta_L$;
(2) $\eta_R = \chi \eta_L$ satisfies $\eta_R(\rho) = \rho$ and $\eta_R(\tau) = \tau + \rho r_0$;
(3) the algebra product is $\phi$;
(4) the counit $\epsilon$ maps each $r_i$ and $E_1$ to 0;
(5) the coproduct $\psi$ satisfies

$$\psi(r_i) = r_i \otimes 1 + \sum_{i+j=k} \xi_i^{\ell^i} \otimes \tau_j \quad \text{and} \quad \psi(E_1) = \sum_{i+j=k} \xi_i^{\ell^i} \otimes \xi_j,$$

where $\xi_0 = 1$;
(6) the conjugation $\chi$ satisfies

$$\tau_i + \sum_{i+j=k} \xi_i^{\ell^i} \chi(\tau_j) = 0 \quad \text{and} \quad \sum_{i+j=k} \xi_i^{\ell^i} \chi(\xi_j) = 0,$$

and $\chi^2 = 1$.

See \([Voe03a, Thm. 12.6, Lem. 12.11, Rem. 12.12]\), \([Voe10, Thm. 3.49]\), \([HK017, Thm. 5.6]\) and \([Spi18, Thm. 10.26]\) for proofs.

Lemma 2.1. The monomials

$$\tau^E r^R = \tau_{r_0} r_{e_1} \cdots \tau_{r_1} \xi_1^{e_2} \cdots,$$

where $E = (e_0, e_1, \ldots)$ and $R = (r_1, r_2, \ldots)$ range through the finite length integer sequences with $e_s \in \{0, 1\}$ and $r_s \geq 0$, form a basis for

$$\mathcal{A}_{+,\ast} = H_{+,\ast}[r_0, r_1, \ldots, E_1 ; \xi_1, \xi_2, \ldots] / (r_i^2 - T_i \mid i \geq 0)$$

as a free left $H_{+,\ast}$-module.

Proof. For $\ell$ odd this is clear. The claim for $\ell = 2$ follows from the form of the relations $\tau_i^2 = T_i$, since $\xi_i, r_{i+1}$ and $\rho_0 r_{i+1}$ have higher weight than $\tau_i^2$.

Lemma 2.2. The same monomials $\tau^E r^R$ as in Lemma 2.1 form a basis for $\mathcal{A}_{+,\ast}$ as a free right $H_{+,\ast}$-module.

Proof. For $t \geq 0$ let

$$F^t \mathcal{A}_{+,\ast} = \langle \tau^E r^R \mid \deg(\tau^E r^R) \geq t \rangle \subset \mathcal{A}_{+,\ast}$$

be the left $H_{+,\ast}$-submodule generated by the monomials from Lemma 2.1 of topological degree $\geq t$. These are also right $H_{+,\ast}$-submodules, since $e_0 L = id = e_0 R$ implies $\eta_L \equiv \eta_R \mod F^1 \mathcal{A}_{+,\ast} = \ker(\epsilon)$, and $F^t \mathcal{A}_{+,\ast} \cdot F^{t+1} \mathcal{A}_{+,\ast} \subset F^{t+1} \mathcal{A}_{+,\ast} \subset F^t \mathcal{A}_{+,\ast}$. (The first inclusion uses that $\tau_i^2 = T_i$ has topological degree less than or equal to that.
of \( \xi_{i+1}, \tau_{i+1} \) and \( \tau_0\xi_{i+1} \). This defines a decreasing filtration of \( \mathcal{A}_{*,*} \) by \( H_{*,*}H_{*,*} \)-bimodules, such that the left and right \( H_{*,*} \)-module actions agree on each filtration quotient

\[
\mathcal{g}_t^\ell \mathcal{A}_{*,*} = \frac{F^\ell \mathcal{A}_{*,*}}{F^\ell+1 \mathcal{A}_{*,*}}.
\]

The (cosets of the) degree \( t \) monomials \( \tau^E \xi^R \) from Lemma 2.4 freely generate this quotient as a left \( H_{*,*} \)-module, hence also as a right \( H_{*,*} \)-module. It follows that the degree \( \geq 0 \) monomials \( \tau^E \xi^R \) freely generate \( \mathcal{A}_{*,*} \) as a right \( H_{*,*} \)-module, since in any given bidegree \( F^t \mathcal{A}_{*,*} = 0 \) for all sufficiently large \( t \).

The classical definitions of [Ste62, §II.3, §VI.4] generalize to the motivic setting.

**Definition 2.3.** For \( n \geq -1 \), let \( I(n) \subset \mathcal{A}_{*,*} \) be the ideal

\[
I(n) = \langle \tau_{n+1}, \tau_{n+2}, \ldots, \xi_1^n, \xi_2^n, \ldots, \xi_n, \xi_{n+1}, \xi_{n+2}, \ldots \rangle
\]

generated by \( \tau_k \) for \( k \geq n + 1 \) and by \( \xi_i^j \) for \( i \geq 1, j \geq 0 \) and \( i + j \geq n + 1 \). Note that \( T_i \in I(n) \) for \( i \geq n \). Let

\[
\mathcal{A}((n)_{*,*}) = \mathcal{A}_{*,*}/I(n) = \frac{H_{*,*}[\tau_0, \ldots, \tau_n, \xi_1, \xi_2, \ldots, \xi_n]}{(\tau_0^2 - T_0, \ldots, \tau_n^2, \tau_{n-1}, \xi_1, \xi_2, \ldots, \xi_n)}
\]

be the quotient algebra.

**Example 2.4.**

\[
\begin{align*}
I(-1) &= \langle \tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots \rangle \\
I(0) &= \langle \tau_1, \tau_2, \ldots, \xi_1, \xi_2, \ldots \rangle \\
I(1) &= \langle \tau_2, \tau_3, \ldots, \xi_1, \xi_2, \ldots \rangle
\end{align*}
\]

so

\[
\begin{align*}
A(-1)_{*,*} &= H_{*,*} \\
A(0)_{*,*} &= H_{*,*}[\tau_0]/(\tau_0^2) \\
A(1)_{*,*} &= H_{*,*}[\tau_0, \tau_1, \xi_1]/(\tau_0^2 - T_0, \tau_1^2, \xi_1^2).
\end{align*}
\]

**Lemma 2.5.** There is a unique Hopf algebroid structure on \( (A_{*,*}, A((n)_{*,*})) \) making the canonical projection \( \pi_n: \mathcal{A}_{*,*} \rightarrow \mathcal{A}_{*,*}/I(n) = A((n)_{*,*}) \) a Hopf algebroid homomorphism.

**Proof.** The Hopf algebroid structure maps of \( (H_{*,*}, A(n)_{*,*}) \) are \( \mathbb{Z}/t \)-algebra homomorphisms, determined as follows:

1. The left unit \( \eta_{L,n}: H_{*,*} \rightarrow A(n)_{*,*} \) is the composite \( \pi_n \circ \eta_L \).
2. The right unit \( \eta_{R,n}: H_{*,*} \rightarrow A(n)_{*,*} \) is the composite \( \pi_n \circ \eta_R \).
3. The algebra product \( \phi_n: A(n)_{*,*} \otimes A(n)_{*,*} \rightarrow A(n)_{*,*} \) is characterized by \( \phi_n \circ (\pi_n \otimes \pi_n) = \pi_n \otimes \phi \), and exists because \( I(n) \subset \mathcal{A}_{*,*} \) is an ideal.
4. The counit \( \epsilon_n: A(n)_{*,*} \rightarrow H_{*,*} \) is characterized by \( \epsilon_n \circ \pi_n = \epsilon \), and exists because \( \epsilon(x) = 0 \) for each generator \( x \) of \( I(n) \).
5. The coproduct \( \psi_n: A(n)_{*,*} \rightarrow A(n)_{*,*} \otimes H_{*,*} \) is characterized by \( \psi_n \circ \pi_n = (\pi_n \otimes \pi_n)\psi \), and exists because \( (\pi_n \otimes \pi_n)\psi(x) = 0 \) for each generator \( x \) of \( I(n) \).
6. The conjugation \( \chi_n: A(n)_{*,*} \rightarrow A(n)_{*,*} \) is characterized by \( \chi_n \circ \pi_n = \pi_n \circ \chi \), and exists because \( \chi(x) \in I(n) \) for each generator \( x \) of \( I(n) \).

In more detail, the explicit formulas for the coproduct show that \( \psi(\tau_k) \), for all \( k \geq n + 1 \), and \( \psi(\xi_i^j) = \psi(\xi_i)^j \), for all \( i \geq 1, j \geq 0 \) with \( i + j \geq n + 1 \), are in the image of

\[
I(n) \otimes H_{*,*} \otimes \mathcal{A}_{*,*} \otimes H_{*,*} \otimes I(n) \rightarrow \mathcal{A}_{*,*} \otimes H_{*,*} \otimes \mathcal{A}_{*,*}.
\]
so that \( \psi(I(n)) \) is contained in this image. Likewise, the recursive formulas for the conjunction show that \( \chi(\tau_k) \) and \( \chi(\xi_l^i) = \chi(\xi_l^j) \) are in \( I(n) \), for the same \( k, i \) and \( j \), so that \( \chi(I(n)) = I(n) \). The verification that these structure maps make \((H_{E^-, A(n), s}) \) a Hopf algebroid, with \((\text{id}, \pi_n) \) a Hopf algebroid homomorphism, follows formally from the fact that \((H_{E^-, A(n), s}) \) is a Hopf algebroid. \( \square \)

**Lemma 2.6.** The monomials \( \tau^E \xi^R \), where \( E = (e_0, \ldots, e_n) \) and \( R = (r_1, \ldots, r_n) \) range through the integer sequences with \( e_s \in \{0, 1\} \) and \( 0 \leq r_s < \ell^{n+1-s} \), form a basis for \( A(n)_{s,s} \) as a finitely generated free left \( H_{s,s} \)-module.

**Proof.** The ideal \( I(n) \) equals the free left \( H_{s,s} \)-submodule of \( \mathcal{A}_{s,s} \) generated by the monomials \( \tau^E \xi^R \) from Lemma 2.6 for which \( e_s = 1 \) for some \( s \geq n + 1 \) or \( r_s \geq \ell^{n+1-s} \) for some \( s \geq 1 \). This implies the claim. \( \square \)

**Lemma 2.7.** The same monomials \( \tau^E \xi^R \) as in Lemma 2.6 form a basis for \( A(n)_{s,s} \) as a free right \( H_{s,s} \)-module.

**Proof.** Replace \( \mathcal{A}_{s,s} \) and Lemma 2.1 in the proof of Lemma 2.2 by \( A(n)_{s,s} \) and Lemma 2.6. \( \square \)

The inclusions \( I(n) \subset I(n-1) \) induce a tower of surjective Hopf algebroid homomorphisms
\[
\mathcal{A}_{s,s} \longrightarrow \ldots \longrightarrow A(n)_{s,s} \longrightarrow A(n-1)_{s,s} \longrightarrow \ldots \longrightarrow H_{s,s}.
\]

The composites
\[
\lambda: \mathcal{A}_{s,s} \xrightarrow{\psi} \mathcal{A}_{s,s} \otimes_{H_{s,s}} \mathcal{A}_{s,s} \xrightarrow{\tau_n \otimes \text{id}} A(n)_{s,s} \otimes_{H_{s,s}} \mathcal{A}_{s,s}
\]
and
\[
\rho: \mathcal{A}_{s,s} \xrightarrow{\psi} \mathcal{A}_{s,s} \otimes_{H_{s,s}} \mathcal{A}_{s,s} \xrightarrow{\text{id} \otimes \pi_n^{-1}} \mathcal{A}_{s,s} \otimes_{H_{s,s}} A(n-1)_{s,s}
\]
are \( \mathbb{Z}/\ell \)-algebra homomorphisms giving \( \mathcal{A}_{s,s} \) the structure of an \( A(n)_{s,s} \)-\( A(n-1)_{s,s} \)-bicomodule algebra, and the projection \( \pi_n: \mathcal{A}_{s,s} \rightarrow A(n)_{s,s} \) is a morphism in the category of such bicomodule algebras.

**Definition 2.8.** Let
\[
X(n)_{s,s} = H_{s,s}\{\tau^E \xi^R \mid e_0 = \cdots = e_n = 0, \ell^n \mid r_1, \ldots, \ell \mid r_n\}
\]
be the free left \( H_{s,s} \)-module generated by the monomials \( \tau^E \xi^R \) with \( E = (e_0, e_1, \ldots) \) and \( R = (r_1, r_2, \ldots) \) satisfying \( e_s = 0 \) for \( 0 \leq s \leq n \) and \( \ell^{n+1-s} \mid r_s \) for \( 1 \leq s \leq n \).

Let \( \alpha_n: \mathcal{A}_{s,s} \rightarrow X(n)_{s,s} \) be the left \( H_{s,s} \)-module homomorphism mapping \( \tau^E \xi^R \) to the same monomial if \( e_0 = \cdots = e_n = 0 \) and \( \ell^n \mid r_1, \ldots, \ell \mid r_n \), and to 0 otherwise.

**Lemma 2.9.** The composite
\[
\mathcal{A}_{s,s} \xrightarrow{\lambda} A(n)_{s,s} \otimes_{H_{s,s}} \mathcal{A}_{s,s} \xrightarrow{\text{id} \otimes \alpha_n} A(n)_{s,s} \otimes_{H_{s,s}} X(n)_{s,s}
\]
is a left \( A(n)_{s,s} \)-co-module isomorphism.

**Proof.** Both \( \lambda \) and \( \text{id} \otimes \alpha_n \) respect the left \( A(n)_{s,s} \)-coactions, so it suffices to show that their composite is a left \( H_{s,s} \)-module isomorphism. Each monomial in the basis from Lemma 2.1 for \( \mathcal{A}_{s,s} \) factors uniquely as \( \tau^E \xi^R = \tau^{E'} \xi^{R'} \cdot \tau^{E''} \xi^{R''} \) with
\[
\begin{align*}
E' &= (e_0, \ldots, e_n, 0, \ldots) \\
R' &= (r_1, \ldots, r_n, 0, \ldots) \\
E'' &= (0, \ldots, 0, e_{n+1}, \ldots) \\
R'' &= (r_1, \ldots, r_n, r_{n+1}, \ldots)
\end{align*}
\]
where \( r_s < \ell^{n+1-s} \) for \( 1 \leq s \leq n \), and \( \ell^{n+1-s} \mid r_s \) for \( 1 \leq s \leq n \),
and $E = E' + E''$, $R = R' + R''$. Hence the restricted multiplication

$$A(n)_{*,*} \otimes_{H_{*,*}} X(n)_{*,*} \xrightarrow{\phi} \mathcal{A}_{*,*}$$

defines a left $H_{*,*}$-module isomorphism. We show that the composite

$$(\text{id} \otimes \alpha_n) \lambda \phi: A(n)_{*,*} \otimes_{H_{*,*}} X(n)_{*,*} \to A(n)_{*,*} \otimes_{H_{*,*}} X(n)_{*,*}$$

is bijective. For $t \geq 0$ let $F^t X(n)_{*,*}$ be the free left $H_{*,*}$-submodule generated by the monomials from Definition 2.8 that have topological degree $\geq t$. These define a decreasing filtration of $X(n)_{*,*}$, with associated graded modules $\text{gr}^i X(n)_{*,*} = F^i X(n)_{*,*}/F^{i+1} X(n)_{*,*}$. Direct calculation of $\lambda = (\pi_n \otimes \text{id}) \psi$ shows that

$$\lambda(\tau^{E'} \xi^{R'}) \equiv \tau^{E'} \xi^{R'} \otimes 1 \mod A(n)_{*,*} \otimes_{H_{*,*}} F^1 \mathcal{A}_{*,*},$$

where $F^1 \mathcal{A}_{*,*} = \ker(\epsilon)$ as before, and

$$\lambda(\tau^{E'} \xi^{R'}) = 1 \otimes \tau^{E''} \xi^{R''},$$

since each $\tau_k$ for $k \geq n + 1$ and each $\xi^i_j$ for $i + j \geq n + 1$ is left $A(n)_{*,*}$-comodule primitive. It follows that for $\tau^{E'} \xi^{R'} \in A(n)_{*,*}$ and $\tau^{E''} \xi^{R''} \in F^1 X(n)_{*,*}$ we have

$$(\text{id} \otimes \alpha_n) \lambda(\tau^{E'} \xi^{R'} \otimes \tau^{E''} \xi^{R''}) \equiv \tau^{E'} \xi^{R'} \otimes \tau^{E''} \xi^{R''} \mod A(n)_{*,*} \otimes_{H_{*,*}} F^1 X(n)_{*,*}.$$}

Hence $(\text{id} \otimes \alpha_n) \lambda \phi$ maps $A(n)_{*,*} \otimes_{H_{*,*}} F^i X(n)_{*,*}$ to itself, for each $t \geq 0$, and the induced homomorphism

$$A(n)_{*,*} \otimes_{H_{*,*}} \text{gr}^i X(n)_{*,*} \to A(n)_{*,*} \otimes_{H_{*,*}} \text{gr}^i X(n)_{*,*}$$

is the identity. The lemma follows, since $A(n)_{*,*} \otimes_{H_{*,*}} F^i X(n)_{*,*}$ is eventually zero in any given bidegree. \qed

3. Some bicomodule algebras

The classical definitions of [AGM85, §2] also generalize to the motivic setting.

**Definition 3.1.** For $n \geq 0$, let $J(n) \subset \mathcal{A}_{*,*}$ be the ideal

$$J(n) = (\tau_{n+1}, \tau_{n+2}, \ldots, \xi_2^{n-1}, \xi_3^{n-2}, \ldots, \xi_n, \xi_{n+1}, \xi_{n+2}, \ldots)$$

generated by $\tau_k$ for $k \geq n + 1$ and by $\xi^i_j$ for $i \geq 2$, $j \geq 0$ and $i + j \geq n + 1$. Note that $I(n) = J(n) + (\xi_1^i)$. Let

$$C(n)_{*,*} = \mathcal{A}_{*,*}/J(n) = \frac{H_{*,*}[\tau_0, \ldots, \tau_n, \xi_1, \ldots, \xi_n]}{(\tau_0^2 - T_0, \ldots, \tau_n, \xi_1^i, \ldots, \xi_i^n)}$$

be the quotient algebra. Let

$$B(n)_{*,*} = C(n)_{*,*}[1/\xi_1] = \frac{H_{*,*}[\tau_0, \ldots, \tau_n, \xi_1^{\pm 1}, \xi_2, \ldots, \xi_n]}{(\tau_0^2 - T_0, \ldots, \tau_n, \xi_1^{\pm 1}, \ldots, \xi_i^n)}$$

be the localization of $C(n)_{*,*}$ away from $\xi_1$.

**Example 3.2.**

$$J(0) = (\tau_1, \tau_2, \ldots, \xi_2, \xi_3, \ldots)$$

$$J(1) = (\tau_2, \tau_3, \ldots, \xi_2, \xi_3, \ldots)$$

$$J(2) = (\tau_3, \tau_4, \ldots, \xi_2, \xi_3, \ldots)$$
so
\[ C(0)_{*,*} = H_{*,*}[\tau_0, \xi_1]/(\tau_0^2 - T_0) \]
\[ C(1)_{*,*} = H_{*,*}[\tau_0, \tau_1, \xi_1]/(\tau_0^2 - T_0, \tau_1^2) \]
\[ C(2)_{*,*} = H_{*,*}[\tau_0, \tau_1, \tau_2, \xi_1, \xi_2]/(\tau_0^2 - T_0, \tau_1^2 - T_1, \tau_2^2, \xi_1^2) \]

and
\[ B(0)_{*,*} = H_{*,*}[\tau_0, \xi_1^{\pm 1}]/(\tau_0^2 - T_0) \]
\[ B(1)_{*,*} = H_{*,*}[\tau_0, \tau_1, \xi_1^{\pm 1}]/(\tau_0^2 - T_0, \tau_1^2) \]
\[ B(2)_{*,*} = H_{*,*}[\tau_0, \tau_1, \tau_2, \xi_1^{\pm 1}, \xi_2]/(\tau_0^2 - T_0, \tau_1^2 - T_1, \tau_2^2, \xi_1^2, \xi_2^2) \].

**Lemma 3.3.** (a) The monomials \( \tau^E \xi^R \), where \( E = (e_0, \ldots, e_n) \) and \( R = (r_1, \ldots, r_n) \) range through all sequences with \( e_s \in \{0, 1\} \) for \( 0 \leq s \leq n \), \( r_1 \geq 0 \) and \( 0 \leq r_s < \ell^{n+1-s} \) for \( 2 \leq s \leq n \), form a basis for \( C(n)_{*,*} \) as a free left \( H_{*,*} \)-module.

(b) The monomials
\[ \tau^E \xi^R = \tau_0^{e_0} \cdots \tau_n^{e_n} \xi_1^{r_1} \xi_2^{r_2} \cdots \xi_n^{r_n} \]
with \( (E, R) \) as in (a), except that \( r_1 \) can now be any integer, form a basis for \( B(n)_{*,*} \) as a free left \( H_{*,*} \)-module.

**Proof.** The ideal \( J(n) \) equals the free left \( H_{*,*} \)-submodule of \( \mathcal{A}_{*,*} \) generated by the monomials \( \tau^E \xi^R \) from Lemma 2.1 for which \( e_s = 1 \) for some \( s \geq n+1 \) or \( r_s \geq \ell^{n+1-s} \) for some \( s \geq 2 \). This implies part (a). Part (b) follows by inverting \( \xi_1 \). \( \square \)

**Lemma 3.4.** (a) The same monomials \( \tau^E \xi^R \) as in Lemma 3.3(a) form a basis for \( C(n)_{*,*} \) as a free right \( H_{*,*} \)-module.

(b) The same monomials \( \tau^E \xi^R \) as in Lemma 3.3(b) form a basis for \( B(n)_{*,*} \) as a free right \( H_{*,*} \)-module.

**Proof.** For part (a), replace \( \mathcal{A}_{*,*} \) and Lemma 2.1 in the proof of Lemma 2.2 by \( C(n)_{*,*} \) and Lemma 3.3(a).

For part (b), instead replace these by \( B(n)_{*,*} \) and Lemma 3.3(b), and allow the filtration index \( t \) in the proof of Lemma 2.2 to run over all integers, noting that in any given bidegree \( F^t B(n)_{*,*} = B(n)_{*,*} \) for all sufficiently negative \( t \). Alternatively, part (b) can be deduced from part (a) by inverting \( \xi_1 \), but the given proof also ensures that the left and right \( H_{*,*} \)-actions on \( gr^t B(n)_{*,*} \) agree, which will be needed in Lemma 4.10(b). \( \square \)

**Example 3.5.** (a) The monomials
\[ \{ \tau_0^r \xi_1^s | e \in \{0, 1\}, r \geq 0 \} \]
form a basis for \( C(0)_{*,*} \), both as a left \( H_{*,*} \)-module and as a right \( H_{*,*} \)-module.

(b) The monomials
\[ \{ \tau_0^r \xi_1^s | e \in \{0, 1\}, r \in \mathbb{Z} \} \]
form a basis for \( B(0)_{*,*} \), both as a left \( H_{*,*} \)-module and as a right \( H_{*,*} \)-module. The homological bidegree of \( \tau_0^r \xi_1^s \) is \( (e + (2 \ell - 2)r, (\ell - 1)r) \).
The inclusions $J(n) \subseteq I(n)$ and the localization homomorphisms yield a commutative diagram of $\mathbb{Z}/\ell$-algebras and algebra homomorphisms

\[
\begin{array}{ccccccccc}
\mathcal{A}_n & \xrightarrow{\pi_n} & A(n)_*, & \xrightarrow{=} & A(n-1)_*, & \xrightarrow{=} & A(0)_*, & \xrightarrow{=} & H_*, \\
\mathcal{A}_n & \xrightarrow{\pi_n} & C(n)_*, & \xrightarrow{=} & C(0)_*, \\
\mathcal{A}_n[1/\xi_1] & \xrightarrow{\beta_n} & B(n)_*, & \xrightarrow{=} & B(0)_*,
\end{array}
\]

Lemma 3.6. There is a unique $A(n)_*, A(n-1)_*,$-bicomodule algebra structure on $C(n)_*$ making the canonical projection $\pi_n: \mathcal{A}_n \xrightarrow{} J(n) = C(n)_*$ an $A(n)_*, A(n-1)_*,$-bicomodule algebra homomorphism.

Proof. The bicomodule structure maps are $\mathbb{Z}/\ell$-algebra homomorphisms, determined as follows:

1. The left coaction $\lambda_n: C(n)_* \rightarrow A(n)_* \otimes H_*, C(n)_*$ is characterized by $\lambda_n \circ \pi'_n = (\text{id} \otimes \pi'_n) \circ \lambda$, and exists because $(\pi_n \otimes \pi'_n)\psi(x) = 0$ for each generator $x$ of $J(n)$.

2. The right coaction $\rho_n: C(n)_* \rightarrow C(n)_* \otimes H_*, A(n-1)_*$ is characterized by $\rho_n \circ \pi'_n = (\pi'_n \otimes \text{id}) \circ \rho$, and exists because $(\pi'_n \otimes \pi_n)\psi(x) = 0$ for each generator $x$ of $J(n)$.

More explicitly, $\psi(\tau_k)$ and $\psi(\xi^\ell_1)$ are in the image of both

$I(n) \otimes H_*, \mathcal{A}_n \oplus \mathcal{A}_n \otimes H_*, J(n) \rightarrow \mathcal{A}_n \otimes H_*, \mathcal{A}_n$

and

$J(n) \otimes H_*, \mathcal{A}_n \oplus \mathcal{A}_n \otimes H_*, I(n-1) \rightarrow \mathcal{A}_n \otimes H_*, \mathcal{A}_n$

for each $k \geq n+1$ and each $i \geq 2, j \geq 0$ and $i+j \geq n+1$, respectively. The verification that the algebra homomorphisms $\lambda_n$ and $\rho_n$ define coactions, and that they commute, follows formally from the fact that $\mathcal{A}_n$ is an $A(n)_*, A(n-1)_*$-bicomodule.

Lemma 3.7. Let $\|\xi^\ell_1\| = ((2\ell-2)\ell^n, (\ell-1)\ell^n)$ denote the bidegree of $\xi^n_1$. There is a short exact sequence

$0 \rightarrow \Sigma^\ell_\mathcal{A}_n \|C(n)_*, A(n)_*, A(n-1)_* \rightarrow 0$

of $A(n)_*, A(n-1)_*$-bicomodules, where $\xi^n_1$ denotes $x \mapsto x \cdot \xi^n_1$.

Proof. From the definition of $I(n)$ and $J(n)$ it is clear that multiplication by $\xi^n_1$ acts injectively on $C(n)_*$ with cokernel $A(n)_*$. It remains to verify that $\xi^n_1$ is an $A(n)_*, A(n-1)_*$-bicomodule homomorphism, i.e., that it commutes with the left $A(n)_*$-coaction and the right $A(n-1)_*$-coaction. This is equivalent to $\xi^n_1$ being left $A(n)_*$-comodule primitive and right $A(n-1)_*$-comodule primitive, which follows from the observations that

$\psi(\xi^n_1) \equiv 1 \otimes \xi^n_1 \mod I(n) \otimes H_*, \mathcal{A}_n$

and

$\psi(\xi^n_1) \equiv \xi^n_1 \otimes 1 \mod \mathcal{A}_n \otimes H_*, I(n-1)$.
Definition 3.8. We assign to
\[ B(n)_{\ast,*} = C(n)_{\ast,*}[1/\xi_1^n] = \text{colim}(\Sigma^{-j}||\xi_1^n||C(n)_{\ast,*}) \]
the \( A(n)_{\ast,*}-A(n-1)_{\ast,*} \)-bicomodule structure given by the colimit of the diagram
\[ C(n)_{\ast,*} \xrightarrow{\xi_1^n} \Sigma^{-2||\xi_1^n||}C(n)_{\ast,*} \xrightarrow{\Sigma^{-j}||\xi_1^n||} C(n)_{\ast,*} \xrightarrow{\xi_1^n} \ldots \]

Lemma 3.9. \( B(n)_{\ast,*} \) is an \( A(n)_{\ast,*}-A(n-1)_{\ast,*} \)-bicomodule algebra, and the canonical morphism \( C(n)_{\ast,*} \to B(n)_{\ast,*} \) is an \( A(n)_{\ast,*}-A(n-1)_{\ast,*} \)-bicomodule algebra homomorphism.

Proof. The left \( A(n)_{\ast,*} \)-coaction
\[ \lambda_n : B(n)_{\ast,*} \to A(n)_{\ast,*} \otimes_{H_{\ast,*}} B(n)_{\ast,*} \]
is obtained from the left coaction
\[ \lambda_n : C(n)_{\ast,*} \to A(n)_{\ast,*} \otimes_{H_{\ast,*}} C(n)_{\ast,*} \]
by inverting (a positive power of) \( \xi_1 \). Since the latter coaction is an algebra homomorphism, so is the former. The case of right \( A(n-1)_{\ast,*} \)-coactions is entirely similar.

Definition 3.10. Let \( \gamma_n : C(n)_{\ast,*} \to C(0)_{\ast,*} \) and \( \beta_n : B(n)_{\ast,*} \to B(0)_{\ast,*} \) be the \( \mathbb{Z}/\ell \)-algebra homomorphisms shown in [\ref{fig:composites}]. Let \( \gamma_n' : C(n)_{\ast,*} \to H_{\ast,*}[\xi_1^{\ell^n}] \) be the composite of \( \gamma_n \) and the left \( H_{\ast,*} \)-module homomorphism
\[ C(0)_{\ast,*} = H_{\ast,*}[\tau_0, \xi_1]/(\tau_0^2 - T_0) \to H_{\ast,*}[\xi_1^{\ell^n}] \]
given for \( e \in \{0,1\} \) and \( r \geq 0 \) by
\[ \tau_0^e \xi_1^r \mapsto \begin{cases} \xi_1^r & \text{if } e = 0 \text{ and } \ell^n \mid r, \\ 0 & \text{otherwise}, \end{cases} \]
and let \( \beta_n' : B(n)_{\ast,*} \to H_{\ast,*}[\xi_1^{\ell^n}] \) be its localization.

Note that \( \tau_0 \cdot \eta_R(\tau) \) in \( C(0)_{\ast,*} \) maps by \( \gamma_0' \) to \( \rho \tau \xi_1 \). Hence \( \gamma_n' \) is sometimes not right \( H_{\ast,*} \)-linear.

Proposition 3.11. The composites
\[ C(n)_{\ast,*} \xrightarrow{\lambda_n} A(n)_{\ast,*} \otimes_{H_{\ast,*}} C(n)_{\ast,*} \xrightarrow{id \otimes \gamma_n'} A(n)_{\ast,*} \otimes_{H_{\ast,*}} H_{\ast,*}[\xi_1^{\ell^n}] \]
and
\[ B(n)_{\ast,*} \xrightarrow{\lambda_n} A(n)_{\ast,*} \otimes_{H_{\ast,*}} B(n)_{\ast,*} \xrightarrow{id \otimes \beta_n'} A(n)_{\ast,*} \otimes_{H_{\ast,*}} H_{\ast,*}[\xi_1^{\ell^n}] \]
are left \( A(n)_{\ast,*} \)-comodule isomorphisms.

Proof. The \( \mathbb{Z}/\ell \)-algebra homomorphism
\[ (id \otimes \gamma_n)\lambda_n : C(n)_{\ast,*} \to A(n)_{\ast,*} \otimes_{H_{\ast,*}} C(0)_{\ast,*} \]
is left \( H_{\ast,*} \)-linear and maps the remaining algebra generators by
\[ \begin{align*}
\tau_0 & \mapsto \tau_0 \otimes 1 + 1 \otimes \tau_0 \\
\tau_k & \mapsto \tau_k \otimes 1 + \xi_k \otimes \tau_0 \quad \text{for } 1 \leq k \leq n, \\
\xi_1 & \mapsto \xi_1 \otimes 1 + 1 \otimes \xi_1 \\
\xi_k & \mapsto \xi_k \otimes 1 + \xi_{k-1} \otimes \xi_1 \quad \text{for } 2 \leq k \leq n.
\end{align*} \]
In particular, it and \((\text{id} \otimes \gamma_n^r)\lambda_n\) respect the decreasing \((\xi^e_n)^r\)-adic filtrations defined (internally to this proof) for \(m \geq 0\) by

\[
F^m C(n)_{s,*} = C(n)_{s,*} \cdot (\xi^e_n)^m \\
F^m C(0)_{s,*} = C(0)_{s,*} \cdot (\xi^e_n)^m \\
F^m H_{s,*}[\xi^e_n] = H_{s,*}[\xi^e_n] \cdot (\xi^e_n)^m.
\]

The induced homomorphism

\[
\frac{F^m C(n)_{s,*}}{F^{m+1} C(n)_{s,*}} \to A(n)_{s,*} \otimes_{H_{s,*}} \frac{F^m H_{s,*}[\xi^e_n]}{F^{m+1} H_{s,*}[\xi^e_n]}
\]

of associated graded left \(A(n)_{s,*}\)-comodules is the isomorphism given by

\[
t^E \xi^R \cdot (\xi^e_n)^m \mapsto t^E \xi^R \otimes (\xi^e_n)^m,
\]

where \(c_s \in \{0, 1\}\) for \(0 \leq s \leq n\) and \(0 \leq r_s < \ell^{n+1-s}\) for \(1 \leq s \leq n\). Each filtration is eventually zero in each bidegree, so this implies that \((\text{id} \otimes \gamma_n^r)\lambda_n\) is an isomorphism. Inverting \(\xi^e_n\), it follows that

\[
(\text{id} \otimes \gamma_n^r)\lambda_n : B(n)_{s,*} \xrightarrow{\cong} A(n)_{s,*} \otimes_{H_{s,*}} H_{s,*}[\xi^e_n^r]
\]

is also an isomorphism. \(\square\)

**Proposition 3.12.** The composite

\[
B(n)_{s,*} \xrightarrow{\rho_n} B(n)_{s,*} \otimes_{H_{s,*}} A(n-1)_{s,*} \xrightarrow{\beta_n \otimes \text{id}} B(0)_{s,*} \otimes_{H_{s,*}} A(n-1)_{s,*}
\]

is a right \(A(n-1)_{s,*}\)-comodule algebra isomorphism.

**Proof.** The \(\mathbb{Z}/\ell\)-algebra homomorphism \((\beta_n \otimes \text{id})\rho_n\) is left \(H_{s,*}\)-linear and maps the remaining algebra generators by

\[
\begin{align*}
\tau_0 & \mapsto \tau_0 \otimes 1 + 1 \otimes \tau_0 \\
\tau_k & \mapsto \xi_1^{e_k-1} \otimes \tau_{k-1} + 1 \otimes \tau_k \quad \text{for } 1 \leq k \leq n, \\
\xi_1 & \mapsto \xi_1 \otimes 1 + 1 \otimes \xi_1 \\
\xi_k & \mapsto \xi_1^{e_k-1} \otimes \xi_{k-1} + 1 \otimes \xi_k \quad \text{for } 2 \leq k \leq n, \\
\xi_1^{-e_n} & \mapsto \xi_1^{-e_n} \otimes 1.
\end{align*}
\]

Letting

\[
\begin{align*}
\tilde{\tau}_k &= \tau_k \cdot \xi_1^{-e_k-1} \quad \text{for } 1 \leq k \leq n, \\
\tilde{\xi}_k &= \xi_k \cdot \xi_1^{-e_k-1} \quad \text{for } 2 \leq k \leq n,
\end{align*}
\]

we can rewrite the presentation in Definition 5.1 as

\[
B(n)_{s,*} = \frac{H_{s,*}[\tau_0, \tilde{\tau}_1, \ldots, \tilde{\tau}_n, \xi_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_n]}{(\tau_0^2 - T_0, \tilde{\tau}_1^2 - T_1, \ldots, \tilde{\tau}_n^2 - T_n, \xi_1^{e_{n+1} - e_1}, \ldots, \xi_1^{e_n}),}
\]

where for \(1 \leq i \leq n\) we use the notation

\[
\tilde{T}_i = \begin{cases} 
\tilde{\tau}_{i+1} \rho \tilde{\tau}_{i+1} + \rho \tilde{\tau}_0 \tilde{\tau}_{i+1} & \text{for } \ell = 2, \\
0 & \text{for } \ell \text{ odd.}
\end{cases}
\]

Note that \(\tilde{\tau}_k = \tau_k \cdot \xi_1^{e_k - e_{k-1}} \cdot \xi_1^{-e_{k-1}}\) and \(\tilde{\xi}_k = \xi_k \cdot \xi_1^{e_k - e_{k-1}} \cdot \xi_1^{-e_{k-1}}\). Hence \((\beta_n \otimes \text{id})\rho_n\) satisfies

\[
\tilde{\tau}_k \mapsto 1 \otimes \tau_{k-1} + \xi_1^{e_k-1} \otimes \tau_k + \ldots + \xi_1^{e_k-1} \otimes \tau_{k-1} \xi_1^{e_{k-1} - e_{k-1}} + \xi_1^{-e_{n-1}} \otimes \tau_{k-1} \xi_1^{e_{n-1} - e_{k-1}}
\]

for \(1 \leq k \leq n\), and
\[
\xi_k \mapsto 1 \otimes \xi_{k-1} + \xi_1^{\ell_k-1} \otimes \xi_k + \ldots + \xi_1^{\ell_k-1} \otimes \xi^n_{k-1} \otimes \xi_n \xi_k + \ldots
\]
for \(2 \leq k \leq n\). The omitted summands involve binomial coefficients, and each summand after the first has a negative power of \(\xi_1\) as its left hand tensor factor.

Hence \((\beta_n \otimes \text{id})\rho_n\) respects the increasing filtrations defined (internally to this proof) for \(m \in \mathbb{Z}\) by
\[
F_m B(n)_{*,*} = H_{*,*} \{ \tau_{0}^{e_{0}} \xi_{1} \cdots \tau_{n}^{e_{n}} \xi_{1} \cdots \xi_{n} | e_0 + 2r_1 \leq m \},
\]
where \(e_s \in \{0, 1\}\) for \(0 \leq s \leq n\), \(r_1 \in \mathbb{Z}\) and \(0 \leq r_s < \ell_{n+1-s}\) for \(2 \leq s \leq n\) as in Lemma 3.3\(b\). The induced homomorphism
\[
\frac{F_m B(n)_{*,*}}{F_{m-1} B(n)_{*,*}} \otimes H_{*,*} A(n-1)_{*,*}
\]
of associated graded right \(A(n-1)_{*,*}\)-comodules is the left \(H_{*,*}\)-module isomorphism given by
\[
\tau_{0}^{e_{0}} \xi_{1} \cdots \tau_{n}^{e_{n}} \xi_{1} \cdots \xi_{n} \mapsto \tau_{0}^{e_{0}} \xi_{1} \cdots \tau_{n}^{e_{n}} \xi_{1} \cdots \xi_{n-1}
\]
for \(e_0 + 2r_1 = m\). In particular, \(\hat{\tau}_k \mapsto \tau_{k-1}\) and \(\hat{\xi}_k \mapsto \xi_{k-1}\). Each filtration is exhaustive and eventually zero in each bidegree, so this implies that \((\beta_n \otimes \text{id})\rho_n\) is an isomorphism.

\[\square\]

4. . . AND THEIR DUAL BIMODULES

We now dualize the results of the previous section, following [Boa82].

**Definition 4.1** ([Boa82] Def. 3.2). Given a left \(H_{*,*}\)-module \(M\) we define the dual left \(H_{*,*}\)-module to be
\[
M^\vee = \text{Hom}_{H_{*,*}}(M, H_{*,*}).
\]
The left action of \(h \in H_{*,*}\) on \(f \in M^\vee\) is given by
\[
(hf)(m) = h f(m) = (-1)^{|h||f|} f(hm)
\]
for \(m \in M\), where \(|h|\) and \(|f|\) are the topological degrees of \(h\) and \(f\), respectively. If \(M\) is an \(H_{*,*}\)-\(H_{*,*}\)-bimodule then \(M^\vee\) is also a bimodule, with right action defined by
\[
(fh)(m) = (-1)^{|h||m|} f(mh).
\]

**Example 4.2.** The canonical isomorphism \(H^\vee_{*,*} \cong H_{-,*} = H_{-,*}^\vee\), taking \(f\) to \(f(1)\), is \(H_{*,*}\)-\(H_{*,*}\)-bilinear.

**Lemma 4.3** ([Boa82] Lem. 3.3]). Let \(M\) be an \(H_{*,*}\)-\(H_{*,*}\)-bimodule and let \(N\) be a left \(H_{*,*}\)-module.

(a) There is a natural homomorphism \(\theta : M^\vee \otimes_{H_{*,*}} N^\vee \to (M \otimes_{H_{*,*}} N)^\vee\) of left \(H_{*,*}\)-modules (or of \(H_{*,*}\)-\(H_{*,*}\)-bimodules, if \(N\) is a bimodule), given by
\[
\theta(f \otimes g)(m \otimes n) = (-1)^{|g||m|} f(m g(n))
\]
for \(f \in M^\vee\), \(g \in N^\vee\), \(m \in M\) and \(n \in N\).

(b) If \(L\) is another bimodule, the diagram
\[
\begin{array}{ccc}
L^\vee \otimes_{H_{*,*}} M^\vee \otimes_{H_{*,*}} N^\vee & \xrightarrow{\theta \otimes \text{id}} & (L \otimes_{H_{*,*}} M)^\vee \otimes_{H_{*,*}} N^\vee \\
\xrightarrow{\text{id} \otimes \theta} & & \\
L^\vee \otimes_{H_{*,*}} (M \otimes_{H_{*,*}} N)^\vee & \xrightarrow{\theta} & (L \otimes_{H_{*,*}} M \otimes_{H_{*,*}} N)^\vee
\end{array}
\]
commutes.

(c) Both composites $M^\vee \cong M^\vee \otimes_{H_{\ast \ast}} H_{\ast \ast} \xrightarrow{\theta} (M \otimes_{H_{\ast \ast}} H_{\ast \ast})^\vee = M^\vee$ and $M^\vee \cong H_{\ast \ast}^\vee \otimes_{H_{\ast \ast}} M^\vee \xrightarrow{\theta} (H_{\ast \ast} \otimes_{H_{\ast \ast}} M)^\vee = M^\vee$ are the identity homomorphism.

**Lemma 4.4 (Boa82 Lem. 3.4).** (a) Let $(H_{\ast \ast}, \Gamma)$ be a Hopf algebroid. The dual $\Gamma^\vee$ is a bigraded $\mathbb{Z}/t$-algebra, containing $H_{\ast \ast}^\vee$ as a subalgebra.

(b) Let $M$ be a left $\Gamma$-comodule. The dual $M^\vee$ is a left $\Gamma^\vee$-module.

(c) Let $(H_{\ast \ast}, \Sigma)$ be a second Hopf algebroid, and let $N$ be a $\Gamma$-$\Sigma$-bicomodule. The dual $N^\vee$ is a $\Gamma^\vee$-$\Sigma^\vee$-bimodule.

**Proof.** Let $\psi : \Gamma \rightarrow \Gamma \otimes_{H_{\ast \ast}} \Gamma$ be the coproduct, and let $\lambda : M \rightarrow \Gamma \otimes_{H_{\ast \ast}} M$ be the left coaction. Boardman uses Lemma 4.3 to define the multiplication on $\Gamma^\vee$ as the composite

$$\Gamma^\vee \otimes \Gamma^\vee \rightarrow \Gamma^\vee \otimes_{H_{\ast \ast}} \Gamma^\vee \xrightarrow{\theta} (\Gamma \otimes_{H_{\ast \ast}} \Gamma)^\vee \xrightarrow{\psi^\vee} \Gamma^\vee$$

and to define the left action on $M^\vee$ as the composite

$$\Gamma^\vee \otimes M^\vee \rightarrow \Gamma^\vee \otimes_{H_{\ast \ast}} M^\vee \xrightarrow{\theta} (\Gamma \otimes_{H_{\ast \ast}} M)^\vee \xrightarrow{\lambda^\vee} M^\vee.$$ 

Likewise, we define the bimodule action on $N^\vee$ as the now evident composite

$$\Gamma^\vee \otimes N^\vee \otimes \Sigma^\vee \rightarrow \Gamma^\vee \otimes_{H_{\ast \ast}} N^\vee \otimes_{H_{\ast \ast}} \Sigma^\vee \rightarrow (\Gamma \otimes_{H_{\ast \ast}} N \otimes_{H_{\ast \ast}} \Sigma)^\vee \rightarrow N^\vee.$$ 

The dual $\psi^\vee : H_{\ast \ast}^\vee \rightarrow \Gamma^\vee$ of the Hopf algebroid counit is split by $\eta^\vee_L$ (and by $\eta^\vee_R$), and exhibits $H_{\ast \ast}^\vee$ as a subalgebra of $\Gamma^\vee$.

The dual $\mathbb{Z}/t$-algebra $\Gamma^\vee$ is usually non-commutative. Switching to cohomological grading, we now refer to the duals of (left or right) $H_{\ast \ast}$-module actions as (left or right) $H^{\ast \ast}$-module actions.

**Notation 4.5.** The motivic Steenrod algebra $\mathcal{A} = \mathcal{A}_{\ast \ast}$ is the dual of the Hopf algebroid $(H_{\ast \ast}, \mathcal{A}_{\ast \ast})$, cf. [Voc03], §13, and contains $H^{\ast \ast}$ as a subalgebra. It is freely generated as a left $H^{\ast \ast}$-module by the Milnor basis $\{\rho(E, R)\}_{E,R}$ defined to be dual to the monomial basis $\{r^E\xi^R\}_{E,R}$ of Lemma 2.1. The cohomological bidegree of $\rho(E, R)$ is equal to the homological bidegree of $r^E\xi^R$. In particular, the Steenrod operation $P^r$ is dual to $\tau_0^r\xi^r$, for $E \in \{0, 1\}$ and $r \geq 0$, cf. [Voc03a Lem. 13.1, Lem. 13.5]. By [Voc03a Lem. 11.1, Cor. 12.5] and the Adem relations [Voc03a Thm. 10.3], [Rin12 Thm. 4.5.1] the operations $\beta, P^1, P^2, P^3, \ldots,$ together with the elements of $H^{\ast \ast}$, generate $\mathcal{A}$ as a $\mathbb{Z}/t$-algebra. When $\ell = 2$ we write $S^2$ for $P^r$ in cohomological bidegree $(2r, r)$ and $S^3$ for $P^{2r+1}$ in cohomological bidegree $(2r+1, 1)$.

**Lemma 4.6.** The operations $\rho(E, R)$, for $(E, R)$ as in Lemma 2.1, also form a basis for $\mathcal{A}$ as a right $H^{\ast \ast}$-module.

**Proof.** Recall the decreasing $H_{\ast \ast}$-$H_{\ast \ast}$-bimodule filtration $F^t\mathcal{A}_{\ast \ast}$ of $\mathcal{A}_{\ast \ast}$, from the proof of Lemma 2.2. For $t \geq 0$ let

$$F_{t-1}\mathcal{A} = \langle \rho(E, R) \mid \deg(\tau^E\xi^R) < t \rangle \subset \mathcal{A}$$

be the left $H^{\ast \ast}$-submodule generated by the operations $\rho(E, R)$ of cohomological topological degree $< t$. This is also a right $H^{\ast \ast}$-submodule, in view of the short exact sequence

$$0 \rightarrow F_{t-1}\mathcal{A} \rightarrow \mathcal{A}_{\ast \ast} \rightarrow (F_t\mathcal{A}_{\ast \ast})^\vee \rightarrow 0.$$ 

Hence $\{F_t\mathcal{A}\}_t$ is an increasing filtration of $\mathcal{A}$ by $H^{\ast \ast}$-$H^{\ast \ast}$-bimodules, with filtration quotients

$$\text{gr}_t\mathcal{A} = F_t\mathcal{A} / F_{t-1}\mathcal{A} \cong (\text{gr}_t\mathcal{A}_{\ast \ast})^\vee.$$ 

Since the left and right $H_{*,*}$-module actions agree on $\text{gr}^t\mathcal{A}$, the dual left and right $H^{*,*}$-module actions on $\text{gr}_t\mathcal{A}$ are also equal. Hence the (cosets of the) operations $\rho(E,R)$ of degree $= t$ freely generate $\text{gr}_t\mathcal{A}$ as a right $H^{*,*}$-module. Since the filtration is exhaustive, the set of degree $\geq 0$ operations is a right $H^{*,*}$-module basis for $\mathcal{A}$.

**Definition 4.7.** For $n \geq -1$ let the $\mathbb{Z}/\ell$-algebra $A(n) = A(n)^{\vee,*} \subset \mathcal{A}$ be the dual of the Hopf algebroid $(H_{*,*}, A(n)^{\vee,*})$.

**Lemma 4.8.** The operations $\rho(E,R)$, for $(E,R)$ as in Lemma 2.6, form a basis for $A(n)$ as a finitely generated free left $H^{*,*}$-module. In particular, there is an exhaustive sequence of $\mathbb{Z}/\ell$-algebra homomorphisms

$$H^{*,*} \subset \cdots \subset A(n-1) \subset A(n) \subset \cdots \subset \mathcal{A}.$$ 

**Proof.** This follows from (the proof of) Lemma 2.6 since $I(n) \subset \mathcal{A}$ is a monomial ideal. The sequence is dual to the tower (2.1). □

**Lemma 4.9.** The operations $\rho(E,R)$, for $(E,R)$ as in Lemma 2.6, also form a basis for $A(n)$ as a free right $H^{*,*}$-module.

**Proof.** Replace $\mathcal{A}$ and Lemma 2.6 in the proof of Lemma 4.8 by $A(n)^{\vee,*}$ and Lemma 2.7. □

**Example 4.10.** (a) $A(0) = H^*(\beta)/\langle \beta^2 \rangle$ with $[\beta,x] = \beta(x)$ for $x \in H^{*,*}$, where $[\beta,x] = \beta x - (-1)^{|x|} x \beta$ denotes the graded commutator.

(b) For $\ell = 2$,

$$A(1) = \frac{H^*(\beta,P^1)}{\langle \beta^2,P^1 P^1 = \tau \beta P^1 \beta, (\beta P^1)^2 = (P^1 \beta)^2 \rangle}$$

with $[\beta,x] = \beta(x)$ and $[P^1,x] = P^1(x)$ for $x \in H^{*,*}$. In the figure below, each bullet represents a copy of $H^{*,*}$, the operations $\beta = Sq^1$ and $P^1 = Sq^2$ map one and two columns to the right, respectively, and the dashed arrow indicates that $P^1 P^1 = Sq^2 Sq^2$ is $\tau$ times the generator $\beta P^1 \beta = Sq^3 Sq^1$.

(The following property is sometimes taken as the definition of $A(n)$.)

**Lemma 4.11.** For $n \geq 0$ the operations $\beta, P^1, P^\ell, \ldots, P^{\ell^{n-1}}$, together with the elements of $H^{*,*}$, generate $A(n)$ as a $\mathbb{Z}/\ell$-algebra.

**Proof.** For $\ell$ odd, the Adem relations [Voe03a, Thm. 10.3] show that the subalgebra of $A(n)$ generated by $\beta, P^1, P^\ell, \ldots, P^{\ell^{n-1}}$ is isomorphic to the classical finite subalgebra $A(n)^{\ell}$ of the classical Steenrod algebra. By [Mil58, Prop. 2] it has $\mathbb{Z}/\ell$-module basis equal to the $H^{*,*}$-module basis for $A(n)$ of Lemma 1.3.

For $\ell = 2$, the $\tau$- and $\rho$-coefficients in the Adem relations [Rio12, Thm. 4.5.1] (correcting [Voe03a, Thm. 10.2]) mean that Milnor’s product formula [Mil58, Thm. 4b] requires adjustment in the motivic setting. For $i \geq 0$ let $Q_i$ be the Milnor basis element dual to $\tau_i$, and for $i \geq 1$ and $j \geq 0$ let $P^t_j$ be the Milnor basis element dual...
to $\xi_i^{t_i}$. In particular, $Q_0 = \beta$ and $P_i^0 = P^{t_i}$. The arrays

\[
\begin{align*}
\xi^{n-1}_1 & \quad \xi^{n-2}_2 & \quad P_1^{n-1} \\
\xi^{n-2}_1 & \quad \xi^{n-3}_2 & \quad P_1^{n-2} & \quad P_2^{n-2} \\
\vdots & \quad \vdots & \quad \vdots & \quad \vdots \\
\xi_1 & \quad \xi_2 & \quad \ldots & \quad \xi_n & \quad P_0^1 & \quad P_0^2 & \quad \ldots & \quad P_0^n \\
\tau_0 & \quad \tau_1 & \quad \ldots & \quad \tau_{n-1} & \quad \tau_n & \quad Q_0 & \quad Q_1 & \quad \ldots & \quad Q_{n-1} & \quad Q_n
\end{align*}
\]

may be helpful, cf. [Mar83, p. 232]. Let $n \geq 1$ and suppose, by induction, that the lemma holds for $A(n-1)$. We show that the inclusions

\[
A(n-1) \langle P_1^{n-1} \rangle \subset A(n-1) \langle P_1^{n-1}, P_2^{n-2} \rangle \subset \ldots
\]

\[
\subset A(n-1) \langle P_1^{n-1}, \ldots, P_n^0, Q_n \rangle \subset A(n)
\]

are all equalities. Here we write $A(n-1) \langle P_1^{n-1}, \ldots, P_k^{n-k} \rangle$ to denote the subalgebra of $A(n)$ generated by $A(n-1)$ and the $P_i^{n-i}$ with $1 \leq t \leq k$, and similarly in the case with $Q_n$. This will complete the inductive step, since $A(n-1) \langle P_1^{n-1} \rangle$ is generated by $\beta, P_1^1, \ldots, P_1^{n-2}, P_2^{n-1}$ and the elements of $H^{*,*}$. Consider $2 \leq k \leq n$. We claim that

\[
(P_{k-1}^{n+k}, P_1^{n-k}) = P_k^{n-k} \mod A(n-1).
\]

The left hand commutator is an $H^{*,*}$-linear combination of Milnor basis elements $\rho(E,R)$ in $A(n)$, as in Lemma 4.8. The $H^{*,*}$-coefficient of $\rho(E,R)$ is the sum of the $H_{*,*}$-coefficients of

\[
\xi_{k-1}^{n+1-k} \otimes \xi_1^{n-k} \quad \text{and} \quad \xi_1^{n-k} \otimes \xi_{k-1}^{n+1-k}
\]

in $\psi(\tau E \xi R)$, where we can ignore signs since $\ell = 2$. The basis element $P_k^{n-k}$ appears with coefficient 1, due to the term $\xi_{k-1}^{n+1-k} \otimes \xi_1^{n-k}$ in $\psi(\xi_k^{n-k})$.

For other $\rho(E,R)$ not in $A(n-1)$, degree considerations show that exactly one of $\xi_1^{n-k}, \ldots, \xi_{k-1}^{n-k}$ must divide $\tau E \xi R$. When $1 \leq t \leq k-2$, no term of the coproduct

\[
\psi(\xi_t^{n-k}) = \sum_{i+j=t} \xi_i^{n-i} \otimes \xi_j^{n-j}
\]

divides either one of the tensor products in (4.2). Hence the $\rho(E,R)$ with these $\xi_i^{n-k}$ dividing $\tau E \xi R$ do not contribute to the commutator in (4.1). In the one remaining case, $t = k-1$, the coproduct $\psi(\xi_{k-1}^{n-k})$ contains two terms dividing those in (4.2), namely $1 \otimes \xi_{k-1}^{n+1-k}$ and $\xi_{k-1}^{n+1-k} \otimes 1$. The complementary factors $\xi_1^{n-k} \otimes 1$ and $1 \otimes \xi_{k-1}^{n-k}$ only appear in

\[
\psi(\xi_1^{n-k}) = 1 \otimes \xi_1^{n-k} + \xi_1^{n-k} \otimes 1,
\]

so the last possible contribution to (4.1) is $\rho(E,R)$ dual to $\xi_1^{n-k} : \xi_{k-1}^{n+1-k}$, with $H^{*,*}$-coefficient the sum of the $H_{*,*}$-coefficients in

\[
\psi(\xi_1^{n-k} : \xi_{k-1}^{n+1-k}) = \psi(\xi_1^{n-k}) \cdot \psi(\xi_{k-1}^{n+1-k}).
\]

Since each of $\xi_{k-1}^{n+1-k} \otimes \xi_1^{n-k}$ and $\xi_1^{n-k} \otimes \xi_{k-1}^{n+1-k}$ occurs twice in this product, this last contribution is $0 \mod \ell$. This establishes claim (4.1). The analogous formula

\[
(P_1^{t_1}, Q_0) = Q_n
\]

holds strictly in $A(n)$, and was already proved in [Voe03a, Prop. 13.6]. It follows by induction on $k$ that

\[
A(n-1) \langle P_1^{n-1} \rangle = A(n-1) \langle P_1^{n-1}, \ldots, P_n^0 \rangle = A(n-1) \langle P_1^{n-1}, \ldots, P_n^0, Q_n \rangle.
\]
Finally, the identity
\[ A(n-1)(P_1^{n-1}, \ldots, P_n^{n}, Q_n) = A(n) \]
follows by classical filtration-by-excess considerations, as in \cite[Prop. 15.8]{Mar83},
where the excess of \( \rho(E, R) \) is defined to be \( \sum_s e_s + 2\sum_s r_s \).

**Lemma 4.12.** The operations \( \rho(E, R) \) for \((E, R)\) as in Definition \( 2.5 \) form a basis for \( \mathcal{A} \) as a free left \( A(n) \)-module.

**Proof.** This follows by dualization from Lemma \( 2.9 \). □

**Lemma 4.13.** Let \( E_n = (1, \ldots, 1) \) and \( R_n = (\ell^n - 1, \ldots, \ell - 1) \), so that \( t_n = \deg(\tau E^n \xi R_n) \) is the highest topological degree of a monomial in \( A(n)_{\ast \ast} \). Then \( A(n)^{p,q} = 0 \) unless \( 0 \leq p \leq q + d + t_n \). Hence the subset
\[ \{(e, r) \mid A(n)^{p-c-(2t-2)r,q-(t-1)r} \neq 0 \} \subset \{0, 1\} \times \mathbb{Z} \]
is finite, for each given cohomological bidegree \( (p, q) \).

**Proof.** The \( H_{\ast \ast} \) module generators of \( A(n)_{\ast \ast} \) lie in homological bidegrees \((t, u)\) with \( 0 \leq t \leq t_n \) and \( u \geq 0 \). Hence the \( H^{\ast \ast} \) module generators of \( A(n) \) lie in homological bidegrees \((p, q)\) with \( 0 \leq p \leq t_n \) and \( q \geq 0 \). Since \( H^{\ast \ast} \) is concentrated in bidegrees with \( 0 \leq p \leq q + d \), it follows that \( A(n) \) is concentrated in the infinite triangular region where \( 0 \leq p \leq q + d + t_n \). Each line of slope 1/2 in the \((p, q)\) plane intersects this triangular region in a bounded interval, which implies the finiteness assertion. □

**Definition 4.14.** For \( n \geq 0 \) let the \( A(n)-A(n-1) \)-bimodules \( C(n) = C(n)_{\ast \ast} \subset \mathcal{A} \) and \( B(n) = B(n)_{\ast \ast} \) be the duals of the \( A(n)_{\ast \ast}-A(n-1)_{\ast \ast} \)-bicomodules \( C(n)_{\ast \ast} \) and \( B(n)_{\ast \ast} \), respectively. Let the symbol
\[ \rho(E, R) \in B(n) \]
be dual to \( \tau E^n \xi R \) in the monomial left \( H_{\ast \ast} \)-module basis for \( B(n)_{\ast \ast} \). The dual of the localization monomorphism \( C(n)_{\ast \ast} \to B(n)_{\ast \ast} \) is a canonical \( A(n)-A(n-1) \)-bimodule epimorphism \( B(n) \to C(n) \).

**Lemma 4.15.** (a) The operations \( \rho(E, R) \) for \((E, R)\) as in Lemma \( 3.3(b) \), form a basis for \( C(n) \) as a free left \( H^{\ast \ast} \)-module.

(b) The symbols \( \rho(E, R) \) for \((E, R)\) as in Lemma \( 3.3(b) \), form a basis for \( B(n) \) as a free left \( H^{\ast \ast} \)-module.

(c) The canonical epimorphism \( B(n) \to C(n) \) satisfies
\[ \rho(E, R) \mapsto \begin{cases} \rho(E, R) & \text{for } r_1 \geq 0, \\ 0 & \text{for } r_1 < 0. \end{cases} \]

**Proof.** Part (a) follows from (the proof of) Lemma \( 3.3(a) \), since \( J(n) \subset \mathcal{A}_{\ast \ast} \) is a monomial ideal. Part (b) likewise follows from Lemma \( 3.3(b) \). The restriction of \( \rho(E, R) \) to \( C(n)_{\ast \ast} \) is then dual to \( \tau E^n \xi R \) if \( r_1 \geq 0 \), and zero otherwise, proving (c). □

**Lemma 4.16.** (a) The operations \( \rho(E, R) \) for \((E, R)\) as in Lemma \( 3.3(a) \), also form a basis for \( C(n) \) as a free right \( H^{\ast \ast} \)-module.

(b) The symbols \( \rho(E, R) \) for \((E, R)\) as in Lemma \( 3.3(b) \), also form a basis for \( B(n) \) as a free right \( H^{\ast \ast} \)-module.

**Proof.** For part (a), replace \( \mathcal{A}_{\ast \ast} \) and Lemma \( 2.2 \) in the proof of Lemma \( 4.3 \) by \( C(n)_{\ast \ast} \) and Lemma \( 3.3(a) \).

For part (b), instead replace these by \( B(n)_{\ast \ast} \) and Lemma \( 5.4(b) \), and allow the filtration index \( t \) in the proof of Lemma \( 4.4 \) to run over all integers, noting that in any given bidegree \( F_t B(n) = 0 \) for all sufficiently negative \( t \). □
Example 4.17. (a) The Steenrod operations
\[ \{ \beta^e P^r \mid e \in \{0, 1\}, r \geq 0 \} \]
form a basis for \( C(0) \subset A^e \) as a left \( H^*\)-module, and as a right \( H^*\)-module. When \( \ell = 2 \), these are the Steenrod operations \( Sq^k \) for \( k \geq 0 \).

(b) The symbols
\[ \{ \beta^e P^r \mid e \in \{0, 1\}, r \in \mathbb{Z} \}, \]
with \( \beta^e P^r \) dual to \( \tau_0^e \xi_1^r \), form a basis for \( B(0) \) as a left \( H^*\)-module, and as a right \( H^*\)-module. When \( \ell = 2 \), these are the symbols \( Sq^k \) for \( k \in \mathbb{Z} \). The homomorphism \( B(0) \to C(0) \) maps \( \beta^e P^r \) to the corresponding Steenrod operation for \( r \geq 0 \), and to zero for \( r < 0 \).

Lemma 4.18. For each \( n \geq 0 \) there is a commutative diagram of \( A(n)\)-\( A(n-1)\)-bimodules
\[
\begin{array}{ccc}
A(n) & \longrightarrow & A(n+1) \\
\downarrow & & \downarrow \\
C(n) & \longrightarrow & C(n+1) \\
\downarrow & & \downarrow \\
B(n) & \longrightarrow & B(n+1),
\end{array}
\]
where the bimodule structures on the right hand side are obtained by restriction from the inherent \( A(n+1)\)-\( A(n)\)-bimodule structures.

Proof. This is readily obtained by comparing diagram \([3.1]\) to its analogue with \( n \) replaced by \( n+1 \), and dualizing. \( \square \)

Proposition 4.19. (a) The inclusion \( H_\ast, [\xi_1^{\ell n}]^\vee \subset C(n) \) extends to an isomorphism
\[ A(n) \otimes_{H_\ast} H_\ast, [\xi_1^{\ell n}]^\vee \stackrel{\cong}{\longrightarrow} C(n) \]
of left \( A(n)\)-modules. Hence the Steenrod operations
\[ \{ P^r \mid r \geq 0 \text{ with } \ell n \mid r \} \]
form a basis for \( C(n) \) as a free left \( A(n)\)-module.

(b) The inclusion \( H_\ast, [\xi_1^{\pm \ell n}]^\vee \subset B(n) \) extends to an isomorphism
\[ A(n) \otimes_{H_\ast} H_\ast, [\xi_1^{\pm \ell n}]^\vee \stackrel{\cong}{\longrightarrow} B(n) \]
of left \( A(n)\)-modules. Hence the symbols
\[ \{ P^r \mid r \in \mathbb{Z} \text{ with } \ell n \mid r \} \]
form a basis for \( B(n) \) as a free left \( A(n)\)-module. The cohomological bidegree of \( P^r \) is \( ((2\ell - 2)r, (\ell - 1)r) \).

Proof. The homomorphisms
\[ \theta : A(n)_\ast \otimes_{H_\ast} H_\ast, [\xi_1^{\ell n}]^\vee \longrightarrow (A(n)_\ast \otimes_{H_\ast} H_\ast, [\xi_1^{\ell n}]^\vee) \]
and
\[ \theta : A(n)_\ast \otimes_{H_\ast} H_\ast, [\xi_1^{\pm \ell n}]^\vee \longrightarrow (A(n)_\ast \otimes_{H_\ast} H_\ast, [\xi_1^{\pm \ell n}]^\vee) \]
are isomorphisms. This follows from Lemmas \([4.3]\)(c) and \([4.13]\) since in each case the source of \( \theta \) is a direct sum of shifted copies of \( A(n) \), the target of \( \theta \) is the corresponding product, and in each bidegree \((p, q)\) only finitely many of the factors in the product are nonzero.

The claims in (a) and (b) then follow by dualization from Proposition \([3.1]\) \( \square \)
Proposition 4.20. The inclusion \( B(0) \subset B(n) \) extends to an isomorphism
\[
B(0) \otimes_{H_{\ast \ast}} A(n-1) \cong B(n)
\]
of right \( A(n-1) \)-modules. Hence the symbols
\[
\{ \beta^{\varepsilon} P^r \mid \varepsilon \in \{0,1\}, r \in \mathbb{Z} \}
\]
form a basis for \( B(n) \) as a free right \( A(n-1) \)-module. The cohomological bidegree of \( \beta^{\varepsilon} P^r \) is \((e + (2\ell - 2)r, (\ell - 1)r)\).

Proof. The homomorphism
\[
\theta : B(0)_{\ast \ast} \otimes_{H_{\ast \ast}} A(n-1)_{\ast \ast} \rightarrow (B(0)_{\ast \ast} \otimes_{H_{\ast \ast}} A(n-1)_{\ast \ast})^\vee
\]
is an isomorphism, by Lemmas 4.3(c) and 4.13. Thus the claim follows by dualization from Proposition 3.12 and Example 4.17(b).

5. The small motivic Singer construction

In this section and the next, we generalize the classical Singer construction \( R_+(M) \) of [Sin81] and [LS82] to the motivic context, following the strategy of [AGM85]. We shall write \( R_+(M) \) for the (small) construction associated to the symmetric group \( C_\ell \), which is denoted \( R_+(M) \) in [LS82] and \( T_\ell(M) \) in [AGM85], and whose desuspension \( \Sigma^{-1}R_+(M) \) is denoted \( R_+(M) \) in [Sin80] and \( T_\ell(M) \) in [AGM85]. We shall write \( R_+(M) \) for the (large) construction associated to the cyclic group \( C_\ell \) and the algebraic group \( \mu_\ell \) of \( \ell \)-th roots of unity, which is denoted \( T(M) \) in [AGM85] and \( R_+(M) \) in [LNR12]. For \( \ell = 2 \) the two constructions agree.

Lemma 5.1. Let \( n \geq 0 \).

(a) For each left \( A(n-1) \)-module \( M \), the tensor product \( B(n) \otimes_{A(n-1)} M \) is a left \( A(n) \)-module. The inclusion \( B(0) \subset B(n) \) induces an isomorphism
\[
B(0) \otimes_{H_{\ast \ast}} M \cong B(n) \otimes_{A(n-1)} M .
\]

(b) If \( M \) is a left \( A(n) \)-module, then the inclusion \( B(n) \subset B(n+1) \) induces an isomorphism
\[
B(n) \otimes_{A(n-1)} M \cong B(n+1) \otimes_{A(n)} M
\]
of left \( A(n) \)-modules.

(c) If \( M \) is a left \( \mathcal{A} \)-module, then the composition \( B(n) \rightarrow C(n) \subset \mathcal{A} \) induces a left \( A(n) \)-module homomorphism
\[
\epsilon_n : B(n) \otimes_{A(n-1)} M \rightarrow \mathcal{A} \otimes_{A(n-1)} M \rightarrow M ,
\]
and these are compatible for varying \( n \).

Proof. (a) This is clear from the \( A(n) \)-\( A(n-1) \)-bimodule structure of \( B(n) \) and Proposition 4.20.

(b) The morphism exists because \( B(n) \subset B(n+1) \) is an \( A(n) \)-\( A(n-1) \)-bimodule homomorphism, with respect to the restricted bimodule structure on the target. It is an isomorphism by comparison with the isomorphisms of part (a) for \( n \) and \( n+1 \).

(c) This follows because the inclusions \( C(n) \subset C(n+1) \subset \mathcal{A} \) are \( A(n) \)-\( A(n-1) \)-bimodule homomorphisms. In each case the morphism \( \mathcal{A} \otimes_{A(n-1)} M \rightarrow M \) is induced by the left module action \( \mathcal{A} \otimes M \rightarrow M \).

Definition 5.2. Let \( M \) be any left \( \mathcal{A} \)-module.

(a) Let the small motivic Singer construction
\[
R_+(M) = \text{colim}_n (B(n) \otimes_{A(n-1)} M)
\]
be the colimit of the sequence of isomorphisms
\[ B(0) \otimes_{H_{-\cdot}} M \xrightarrow{\sim} \ldots \xrightarrow{\sim} B(n) \otimes_{A(n-1)} M \xrightarrow{\sim} B(n+1) \otimes_{A(n)} M \xrightarrow{\sim} \ldots, \]
equipped with the unique left \( \mathcal{A} \)-module structure for which the canonical map
\[ B(n) \otimes_{A(n-1)} M \to R_\mathcal{S}(M) \]
is an isomorphism of \( A(n) \)-modules, for each \( n \geq 0 \).

(b) Let the small evaluation homomorphism
\[ \epsilon : R_\mathcal{S}(M) \to M \]
be the left \( \mathcal{A} \)-module homomorphism such that its restriction to \( B(n) \otimes_{A(n-1)} M \) is equal to the \( A(n) \)-module homomorphism \( \epsilon_n \) of Lemma 5.1(c), for each \( n \geq 0 \).

Evidently, \( R_\mathcal{S} \) is an exact and colimit-preserving endofunctor of left \( \mathcal{A} \)-modules, and \( \epsilon : R_\mathcal{S} \to \text{id} \) is a natural transformation.

**Lemma 5.3.** As a left \( A(0) \)-module, the small motivic Singer construction is given by the tensor product
\[ R_\mathcal{S}(M) \cong B(0) \otimes_{H_{-\cdot}} M \]
with the \( A(0) \)-action from \( B(0) \). Each element of \( R_\mathcal{S}(M) \) is thus a finite sum of terms \( \beta^e P^r \otimes m \), with \( e \in \{0,1\} \), \( r \in \mathbb{Z} \) and \( m \in M \), where \( \beta(\beta^r \otimes m) = \beta P^r \otimes m \) and \( \beta(\beta^r \otimes m) = 0 \). The small evaluation homomorphism is given by
\[ \epsilon(\beta^e P^r \otimes m) = \begin{cases} \beta^e P^r(m) & \text{for } r \geq 0, \\ 0 & \text{for } r < 0. \end{cases} \]

**Proof.** Clear. \( \square \)

The following formulas generalize the one of Singer [Sin80 (2.1)] for \( \ell = 2 \) and a rewriting of the those of Li-Singer [LS82 §3] for \( \ell \) odd. By \( \tau^j \mod 2 \) we mean \( \tau^0 = 1 \) for \( j \) even and \( \tau^1 = \tau \) for \( j \) odd.

**Proposition 5.4.** For \( \ell = 2 \) and \( a \geq 0 \) even the action of \( Sq^a \) on \( R_\mathcal{S}(M) \) is given by
\[ Sq^a(Sq^b \otimes m) = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} \tau^j \mod 2 \cdot Sq^{a+b-j} \otimes Sq^j(m) \]
for \( b \) even, and
\[ Sq^a(Sq^b \otimes m) = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j} \otimes Sq^j(m) \]
\[ + \sum_{j=1}^{[a/2]} \binom{b-1-j}{a-2j} \rho \cdot Sq^{a+b-j-1} \otimes Sq^j(m) \]
for \( b \) odd. For \( \ell \) odd and \( a \geq 0 \) the action of \( P^a \) on \( R_\mathcal{S}(M) \) is given by
\[ P^a(P^b \otimes m) = \sum_{j=0}^{[a/\ell]} (-1)^{a+j} \binom{(\ell-1)(b-j)-1}{a-\ell j} P^{a+b-j} \otimes P^j(m) \]
and

\[ P^a(\beta P^b \otimes m) = \sum_{j=0}^{\lfloor a/\ell \rfloor} (-1)^{a+j} \binom{\ell-1}{a-\ell j} \beta P^{a+b-j} \otimes P^j(m) \]

\[ + \sum_{j=0}^{\lfloor (a-1)/\ell \rfloor} (-1)^{a+j-1} \binom{\ell-1}{a-\ell j - 1} P^{a+b-j} \otimes \beta P^j(m) \]

for all \( b \in \mathbb{Z} \).

**Proof.** For \( a = 0 \) the formulas confirm that \( Sq^0 \) and \( P^0 \) are the identity operations.

For \( \ell = 2 \) and \( a > 0 \) even, choose \( n \) so that \( Sq^n \in A(n) \). Then \( Sq^i \in A(n-1) \) for all \( 0 \leq j \leq \lfloor a/2 \rfloor \), and \( Sq^i \otimes m = Sq^i \otimes Sq^i(m) \) in \( B(n) \otimes A_{n-1} \). When \( a < 2b \) the formulas for \( Sq^a(Sq^b \otimes m) \) then follow from the Adem relations \[\text{Voe03a} \text{ Thm. 10.2] for } Sq^a Sq^b, \text{ as corrected in } \text{Rio12 Thm. 4.5.1].}\]

Similarly, for \( \ell \) odd and \( a > 0 \), choose \( n \) so that \( P^n \in A(n) \). Then \( P^i \beta P^j \in A(n-1) \) for all \( 0 \leq j \leq \lfloor a/\ell \rfloor \), and \( P^i P^j \otimes m = P^i \otimes P^j(m) \) and \( P^i \beta P^j \otimes m = P^i \otimes P^j(m) \) in \( B(n) \otimes A_{n-1} \). When \( a < \ell b \) the formulas for \( P^n(P^b \otimes m) \) and \( P^n(\beta P^b \otimes m) \) then follow from the Adem relations \[\text{Voe03a Thm. 10.3] for } P^n P^b \text{ and } P^n \beta P^b. \text{ The last Adem relation is valid for } 0 < a \leq \ell b, \text{ cf. } \text{Rio12 Thm. 4.5.2].}\]

For the rest of the argument, \( \ell \) can be even or odd. By Definition 5.5 the left \( A(n)_* \)-coaction on \( B(n)_* \) commutes with multiplication by \( \xi^\ell \), so the left \( A(n)_* \)-action on \( B(n)_* \) commutes with the operation \( \beta^\ell = P^{\ell} \mapsto P^{\ell} \beta^\ell t^\ell \). All odd \( \ell \) binomial coefficients in sight also repeat \( \ell \)-periodically in \( b \). Hence the formulas for \( a \geq \ell b \) follow from those for \( a < \ell b \).

**Corollary 5.5.** For \( \ell = 2 \) and \( a \geq 0 \) even the action of \( Sq^a \) on \( R_S(H^{*,*}) \cong B(0) \)

is given by

\[ Sq^a(Sq^b) = \binom{b-1}{a} Sq^{a+b} \]

for \( b \in \mathbb{Z} \).

For \( \ell \) odd and \( a \geq 0 \) even the action of \( P^n \) on \( R_S(H^{*,*}) \cong B(0) \)

is given by

\[ P^n(P^b) = (-1)^a \binom{\ell-1}{a} P^{a+b} \]

and

\[ P^n(\beta P^b) = (-1)^a \binom{\ell-1}{a} \beta P^{a+b} \]

for \( b \in \mathbb{Z} \).

**Proof.** This is the special case \( M = H^{*,*} \) of Proposition 5.4, where we identify \( R_S(H^{*,*}) \cong B(0) \otimes_{H^{*,*}} H^{*,*} = B(0) \) and note that \( Sq^j(1) = 0 \) and \( P^j(1) = 0 \) in \( H^{*,*} \) for all \( j > 0 \). When \( \ell = 2 \), the formulas for \( P^n(P^b) \) and \( P^n(\beta P^b) \) agree with the given formulas for \( Sq^{2b}(Sq^b) \) and \( Sq^{2b}(Sq^{2b+1}) \), since \( (b-1)_a \equiv (2b-1)_a \) and \( (b)_a \equiv (2b)_a \) mod 2.

**Notation 5.6.** Let \( B\mu_\ell \) and \( B\nu_\ell \) be the geometric classifying spaces of the linear algebraic groups \( \mu_\ell \) and \( \nu_\ell \), respectively. Recall from \[\text{Voe03a Thm. 6.10, Thm. 6.16] that } H^{*,*}(B\mu_\ell) = H^{*,*}[u,v]/(u^2 = \tau v + \mu u) \]

with \( \beta(u) = v, \) and

\[ H^{*,*}(B\nu_\ell) = H^{*,*}[v,d]/(c^2 = \tau d + \rho c) \]

with \( \beta(c) = d, \) as graded commutative \( \mathcal{A}\)-module \( H^{*,*}\)-algebras. The cohomological bidegrees of \( u, v, \) and \( c \) are \((1,1), (2,1), (2\ell-3, \ell-1) \) and \((2\ell-2, \ell-1) \), respectively.
The coefficients $\tau$ and $\rho$ are interpreted as 0 when $\ell$ is odd. Any choice of a primitive $\ell$-th root of unity $\zeta$ defines a map $p_{\zeta}: B_{\mu\ell} \to BS_{d}$ inducing
\[
p_{\zeta}^*: c \mapsto -uv^{\ell-2}
\]
\[
p_{\zeta}^*: d \mapsto -v^{\ell-1}.
\]
We suppress $p_{\zeta}^*$ from the notation, viewing $H^{*,*}(BS_{d})$ as an $\mathcal{A}$-module subalgebra of $H^{*,*}(B_{\mu\ell})$. The natural left $\mathcal{A}$-module structure on $H^{*,*}(B_{\mu\ell})$ is determined by the cases
\[
\beta^r P^r(u) = \begin{cases} 
  u & \text{for } (e, r) = (0, 0), \\
  v & \text{for } (e, r) = (1, 0), \\
  0 & \text{otherwise},
\end{cases}
\]
\[
\beta^r P^r(v) = \begin{cases} 
  v & \text{for } (e, r) = (0, 0), \\
  v^{\ell} & \text{for } (e, r) = (0, 1), \\
  0 & \text{otherwise},
\end{cases}
\]
and the Cartan formula [Voc03a] Prop. 9.7, leading to the expressions
\[
P^r(uv) = \binom{k}{r} uv^{(\ell-1)r+k}
\]
\[
\beta P^r(uv) = \binom{k}{r} uv^{(\ell-1)r+1+k}
\]
\[
P^r(v) = \binom{k}{r} v^{(\ell-1)r+k}
\]
\[
\beta P^r(v) = 0.
\]
The restricted $\mathcal{A}$-module action on $H^{*,*}(BS_{d})$ is given by
\[
P^r(cd) = (-1)^r \binom{(\ell-1)(k+1)-1}{r} cd^{r+k}
\]
\[
\beta P^r(cd) = (-1)^r \binom{(\ell-1)(k+1)-1}{r} d^{r+1+k}
\]
\[
P^r(d) = (-1)^r \binom{(\ell-1)k}{r} d^{r+k}
\]
\[
\beta P^r(d) = 0,
\]
for $r \geq 0$ and $k \geq 0$, cf. [Rio12] Prop. 4.4.6.

In particular, $\beta(v^r) = 0$ and $P^r(v^r) = 0$ for all $0 < r < \ell^n$, so multiplication by $v^r$ acts left $A(n)$-linearly on $H^{*,*}(B_{\mu\ell})$, cf. Lemma [4.11]. Likewise, multiplication by $d^n$ acts left $A(n)$-linearly on $H^{*,*}(BS_{d})$. Hence the following two localizations inherit compatible left $A(n)$-module structures for all $n \geq 0$. These combine to well-defined left $\mathcal{A}$-module structures, such that the localization homomorphisms are maps of $\mathcal{A}$-module $H^{*,*}$-algebras.

**Definition 5.7.** Let
\[
H^{*,*}(B_{\mu\ell})_{\text{loc}} = H^{*,*}(B_{\mu\ell})[1/v] = H^{*,*}[u, v^{\pm 1}]/(u^2 = \tau v + \rho u)
\]
\[
= H^{*,*}\{ u^i v^k \mid i \in \{0, 1\}, k \in \mathbb{Z} \}
\]
and
\[
H^{*,*}(BS_{d})_{\text{loc}} = H^{*,*}(BS_{d})[1/d] = H^{*,*}[c, d^{\pm 1}]/(c^2 = \tau d + \rho c)
\]
\[
= H^{*,*}\{ c^i d^k \mid i \in \{0, 1\}, k \in \mathbb{Z} \}.
\]
Lemma 6.2. The Frobenius forms, the associated Frobenius pairings, and the ad-
joint isomorphisms, are all left $\mathcal{A}$-module isomorphisms.

Theorem 5.8. Let $\Sigma = \Sigma^{1,0}$. There is a left $\mathcal{A}$-module isomorphism
\[
R_S(H^{*,*}) \xrightarrow{\sim} \Sigma H^{*,*}(BS_t)_{\text{loc}}
\]
defined by
\[
P^k \mapsto \Sigma cd^{-1} \quad \text{and} \quad \beta P^k \mapsto -\Sigma d^k
\]
for $k \in \mathbb{Z}$. The composite $\Sigma H^{*,*}(BS_t)_{\text{loc}} \cong R_S(H^{*,*}) \xrightarrow{\sim} H^{*,*}$ is the left $\mathcal{A}$-linear homomorphism given by
\[
\Sigma cd^{-1} \mapsto 1 \quad \text{and} \quad \Sigma c'd^k \mapsto 0
\]
for $(i, k) \neq (1, -1)$, where $i \in \{0, 1\}$ and $k \in \mathbb{Z}$.

Proof. By Corollary 5.5, Notation 5.6 and Definition 5.7, the indicated $H^{*,*}$-module isomorphism maps $P_r(P^k)$ and $P_r(\beta P^k)$ to $P_r(\Sigma cd^{-1})$ and $P_r(-\Sigma d^k)$, respectively, for all $r \geq 0$ and $k \in \mathbb{Z}$. Moreover, $\beta(\Sigma cd^{-1}) = -\Sigma d^k$, and $\beta(-\Sigma d^k) = 0$. Hence, the isomorphism is $\mathcal{A}$-linear. The calculation of the composite follows by noting that $\beta P_r(1) = 0$ in $H^{*,*}$ unless $(i, r) = (0, 0)$. \qed

6. The large motivic Singer construction

Our next aim, following [AGM85], §5, is to construct the large Singer construction $R_S(M)$ as an extension of $R_S(M)$, with $R_S(H^{*,*}) \cong \Sigma H^{*,*}(B\mu_{\ell})_{\text{loc}}$. We first note that $H^{*,*}(BS_t)_{\text{loc}} \subset H^{*,*}(B\mu_{\ell})_{\text{loc}}$ is a pair of graded Frobenius algebras.

Definition 6.1. Let the residue homomorphisms
\[
\text{res}: \Sigma H^{*,*}(B\mu_{\ell})_{\text{loc}} \longrightarrow H^{*,*}
\]
be the left $H^{*,*}$-linear Frobenius forms defined for $i \in \{0, 1\}$ and $k \in \mathbb{Z}$ by
\[
\text{res}(\Sigma a^i e^k) = \text{res}(\Sigma c'd^k) = \begin{cases} 1 & \text{for } (i, k) = (1, -1), \\ 0 & \text{otherwise.} \end{cases}
\]

The associated Frobenius pairings
\[
\Sigma H^{*,*}(B\mu_{\ell})_{\text{loc}} \otimes_{H^{*,*}} H^{*,*}(B\mu_{\ell})_{\text{loc}} \longrightarrow H^{*,*}
\]
\[
\Sigma H^{*,*}(BS_t)_{\text{loc}} \otimes_{H^{*,*}} H^{*,*}(BS_t)_{\text{loc}} \longrightarrow H^{*,*}
\]
map $\Sigma x \otimes y$ to $\text{res}(\Sigma xy)$, and the adjoint $H^{*,*}$-linear homomorphisms
\[
\Sigma H^{*,*}(B\mu_{\ell})_{\text{loc}} \xrightarrow{\sim} (H^{-*,*}(B\mu_{\ell})_{\text{loc}})^{\vee}
\]
\[
\Sigma H^{*,*}(BS_t)_{\text{loc}} \xrightarrow{\sim} (H^{-*,*}(BS_t)_{\text{loc}})^{\vee}
\]
are the isomorphisms given by
\[
\Sigma e^k \mapsto (uv^{-k-1})^{\vee}
\]
\[
\Sigma uv^{k-1} \mapsto (u^{-k})^{\vee} + \rho \cdot (uv^{-k})^{\vee}
\]
and
\[
\Sigma d^k \mapsto (cd^{-k-1})^{\vee}
\]
\[
\Sigma cd^{k-1} \mapsto (d^{-k})^{\vee} + \rho \cdot (cd^{-k})^{\vee}
\]
for $k \in \mathbb{Z}$.

Lemma 6.2. The Frobenius forms, the associated Frobenius pairings, and the adjoint isomorphisms, are all left $\mathcal{A}$-linear.
Proof. The residue homomorphism in the case of $B\mu_k$ is $\mathcal{A}$-linear, because for $r > 0$ we have $P^r(u^k) = 0$ whenever $(\ell - 1)r + k = -1$, since
$$
\left(\frac{(\ell - 1)(-r) - 1}{r}\right) = (-1)^r \left(\frac{\ell r}{r}\right) \equiv 0 \mod \ell.
$$
The case of $BS_t$ follows from this, or from the second part of Theorem 6.3. The $\mathcal{A}$-linearity of the remaining homomorphisms follows formally. □

Recall the cotensor product $\square$ of comodules, e.g. from [EM69 §2].

Definition 6.3. Let
$$
R^S(H_{n,*}) = \lim_n (B(n)_{n,*} \square_{A(n-1)_{n,*}} H_{n,*}) \cong B(0)_{n,*}
$$
be the (achieved) limit of the right $A(n-1)_{n,*}$-comodule primitives in $B(n)_{n,*}$. It is a left $A(n)_{n,*}$-comodule algebra for each $n \geq 0$, and these coactions combine to a completed left $\mathcal{A}_{n,*}$-comodule algebra structure. We write
$$
R^S(H_{n,*}) = H^{*,*}[\tilde{\tau}, \tilde{\xi}]^\perp / (\tilde{\tau}^2 = \tau \tilde{\xi} + \rho \tilde{\xi}),
$$
with $\tilde{\tau}$ and $\tilde{\xi}$ mapping to $\tau_0$ and $\xi_1$ in $B(0)_{n,*}$, respectively. Note that $R^S(H^{*,*}) \cong R^S(H_{n,*})$, with $\beta P^r$ dual to $\tilde{\tau}^r \tilde{\xi}$ in the monomial basis.

Lemma 6.4. The composite left $\mathcal{A}$-module isomorphism
$$
R^S(H^{*,*}) \xrightarrow{=} \Sigma H^{*,*}(BS_t)_{\text{loc}} \xrightarrow{=} (H^{-*,*-}(BS_t)_{\text{loc}})^\vee
$$
is the dual of the $H^{*,*}$-algebra isomorphism
$$
\Phi: H^{-*,*-}(BS_t)_{\text{loc}} \xrightarrow{=} R^S(H_{n,*})
$$
given by
$$
c \mapsto -\tilde{\tau}^{-1} + \rho \cdot 1 \quad \text{and} \quad d \mapsto -\tilde{\xi}^{-1}.
$$
Proof. The composite isomorphism maps $P^k$ to $(d^{-k})^\vee + \rho \cdot (cd^{-k})^\vee$ and maps $\beta P^k$ to $-(cd^{-k})^\vee$, hence is dual to the $H^{*,*}$-linear homomorphism mapping $d^{-k}$ to $(P^k)^\vee = \tilde{\tau}^k$ and mapping $cd^{-k}$ to $-(\beta P^k)^\vee + \rho \cdot (P^k)^\vee = -\tilde{\tau}^{k-1} + \rho \cdot \tilde{\xi}$. This is indeed an algebra isomorphism. □

For a left $\mathcal{A}_{n,*}$-comodule $M_{n,*}$, the $A(n)_{n,*}$-$A(n-1)_{n,*}$-bicomodule algebra product $\phi$ on $B(n)_{n,*}$ induces a pairing
$$
(B(n)_{n,*} \square_{A(n-1)_{n,*}} H_{n,*}) \otimes H_{n,*} \rightarrow (B(n)_{n,*} \square_{A(n-1)_{n,*}} M_{n,*})
$$
of left $A(n)_{n,*}$-comodules for each $n \geq 0$, making
$$
R^S(M_{n,*}) = \lim_n (B(n)_{n,*} \square_{A(n-1)_{n,*}} M_{n,*})
$$
an $R^S(H_{n,*})$-module in completed left $\mathcal{A}_{n,*}$-comodules. Viewing $R^S(M_{n,*})$ as an $H^{-*,*-}(BS_t)_{\text{loc}}$-module via the algebra isomorphism $\Phi$, we can form the induced $H^{-*,*-}(B\mu_k)_{\text{loc}}$-module
$$
R^\mu(M_{n,*}) = H^{-*,*-}(B\mu_k)_{\text{loc}} \otimes_{H^{-*,*-}(BS_t)_{\text{loc}}} R^S(M_{n,*}).
$$
As a left $A(n)_{n,*}$-comodule, it is isomorphic to a finite direct sum
$$
R^\mu(M_{n,*}) \cong H^{*,*} \{1, v, \ldots, v^{(\ell - 2)}\} \otimes H^{*,*} R^S(M_{n,*}),
$$
where each power of $v^m$ is $A(n)_{n,*}$-comodule primitive.

Dually, for a left $\mathcal{A}$-module $M$ the completed $A(n)$-$A(n-1)$-bimodule coproduct
$$
B(n) = \lim_n (B(n)_{n,*} \square_{H_{n,*}} B(n)_{n,*})^\vee = B(n) \hat{\otimes}_{H^{*,*}} B(n)
$$
induces a “copairing”

\[ B(n) \otimes A(n-1) M \rightarrow (B(n) \otimes A(n-1) H^* \otimes H^*). \]

of left \( A(n) \)-modules for each \( n \geq 0 \), making the small Singer construction

\[ R_S(M) = \text{colim}_n (B(n) \otimes A(n-1) M) \]

a completed \( R_S(H^\ast) \)-comodule in left \( \mathcal{A} \)-modules. Here \( R_S(H^\ast) \) has the completed \( H^\ast \)-coalgebra structure dual to the \( H^\ast \)-algebra structure on \( R_S(H^\ast) \) that appears in Lemma 6.4. It corresponds via the isomorphism in Theorem 5.8 to a completed \( H^\ast \)-coalgebra structure on \( \Sigma H^\ast(BS_\ell) \). Moreover, the algebra inclusion \( H^\ast(BS_\ell) \subset H^\ast(B\mu) \) in left \( \mathcal{A} \)-modules corresponds under duality and the Frobenius isomorphisms from Definition 6.1 to a completed \( H^\ast \)-coalgebra epimorphism

\[ \pi : \Sigma H^\ast(B\mu) \rightarrow \Sigma H^\ast(BS_\ell) \]

in left \( \mathcal{A} \)-modules, given by

\[ \Sigma u^{(\ell-1)k} \mapsto (-1)^k \Sigma d^k \quad \text{and} \quad \Sigma u^{(\ell-1)k-1} \mapsto (-1)^k \Sigma d^{k-1}, \]

while the remaining \( H^\ast \)-module generators \( \Sigma u_i v^k \) with \( i \in \{0,1\} \) and \( k \in \mathbb{Z} \) map to zero. This discussion motivates the following definition.

**Definition 6.5.** Let \( M \) be any left \( \mathcal{A} \)-module.

(a) Let the large motivic Singer construction

\[ R_\mu(M) = \Sigma H^\ast(B\mu) \] \[ \rightarrow \Sigma H^\ast(BS_\ell) \]

be the left \( \mathcal{A} \)-module coinduced from \( R_S(M) \) along the completed \( H^\ast \)-coalgebra epimorphism \( \pi \). As a left \( A(\ast) \)-module it is isomorphic to the finite direct sum

\[ R_\mu(M) \cong H^\ast \{1, v^\circ, \ldots, v^{\circ(\ell-2)}\} \otimes H^\ast \]

where \( A(\ast) \) acts trivially, i.e., \( \eta_{\mu, \ast} : A(\ast) \rightarrow H^\ast \), on each power of \( v^\circ \).

(b) Let the large evaluation homomorphism

\[ \epsilon : R_\mu(M) \rightarrow M \]

be the composite \( \epsilon(\pi \otimes 1) : R_\mu(M) \rightarrow R_S(M) \rightarrow M. \)

**Corollary 6.6.** There is a left \( \mathcal{A} \)-module isomorphism

\[ R_\mu(H^\ast) \cong \Sigma H^\ast(B\mu) \].

The composite \( \Sigma H^\ast(B\mu) \cong R_\mu(H^\ast) \rightarrow H^\ast \) equals the residue homomorphism for \( B\mu \).

**Proof.** This follows directly from Theorem 5.8.

**Lemma 6.7.** As a left \( A(0) \)-module, the large motivic Singer construction is given by the tensor product

\[ R_\mu(M) \cong H^\ast \{\Sigma u_i v^k \mid i \in \{0,1\}, k \in \mathbb{Z} \} \otimes H^\ast M \]

with the \( A(0) \)-action from \( \Sigma H^\ast(B\mu) \). Each element of \( R_\mu(M) \) is thus a finite sum of terms \( \Sigma u_i v^k \otimes m \), with \( i \in \{0,1\}, k \in \mathbb{Z} \) and \( m \in M \), where \( \beta(\Sigma u v^k \otimes m) = -\Sigma u^{k+1} \otimes m \) and \( \beta(\Sigma v^k \otimes m) = 0 \).

**Proof.** Clear.

The following formulas generalize the classical one of Singer [Sin81 (3.2)] for \( \ell = 2 \), and of Lunøe–Nielsen and the second author [LNR12 Def. 3.1] for \( \ell \) odd. The latter two formulas were surely known to the authors of [AGM85].
Proposition 6.8. For $\ell = 2$ and $r \geq 0$ the action of $Sq^{2r}$ on $R_{\mu}(M) \cong R_S(M)$ is given by
\[
Sq^{2r}(\Sigma uv^k \otimes m) = \sum_{j=0}^{[r/2]} \left(\sum_{k=j}^{\ell} \left(\begin{array}{c} k - j \\ r - 2j \end{array}\right) \Sigma uv^{r+k-j} \otimes Sq^{2j}(m) \right)
+ \sum_{j=0}^{[(r-1)/2]} \left(\sum_{k=j}^{\ell} \left(\begin{array}{c} k - j \\ r - 2j - 1 \end{array}\right) \Sigma uv^{r+k-j} \otimes Sq^{2j+1}(m) \right)
\]
and
\[
Sq^{2r}(\Sigma v^k \otimes m) = \sum_{j=0}^{[r/2]} \left(\sum_{k=j}^{\ell} \left(\begin{array}{c} k - j \\ r - 2j \end{array}\right) \Sigma uv^{r+k-j} \otimes Sq^{2j}(m) \right)
+ \sum_{j=0}^{[(r-1)/2]} \left(\sum_{k=j}^{\ell} \left(\begin{array}{c} k - j - 1 \\ r - 2j - 1 \end{array}\right) \Sigma (u + \rho)v^{r+k-j-1} \otimes Sq^{2j+1}(m) \right).
\]
Here $\Sigma (u + \rho)v^{r+k-j-1} = \Sigma uv^{r+k-j-1} + \rho \cdot \Sigma v^{r+k-j-1}$.
For $\ell$ odd and $r \geq 0$ the action of $P^r$ on $R_{\mu}(M)$ is given by
\[
P^r(\Sigma uv^{k-1} \otimes m) = \sum_{j=0}^{[r/\ell]} \left(\sum_{k=j}^{\ell} \left(\begin{array}{c} k - (\ell - 1)j - 1 \\ r - \ell j \end{array}\right) \Sigma uv^{k+(\ell-1)(r-j)-1} \otimes P^j(m) \right)
\]
and
\[
P^r(\Sigma v^k \otimes m) = \sum_{j=0}^{[r/\ell]} \left(\sum_{k=j}^{\ell} \left(\begin{array}{c} k - (\ell - 1)j \\ r - \ell j \end{array}\right) \Sigma uv^{k+(\ell-1)(r-j)-1} \otimes P^j(m) \right)
+ \sum_{j=0}^{[(r-1)/\ell]} \left(\sum_{k=j}^{\ell} \left(\begin{array}{c} k - (\ell - 1)j - 1 \\ r - \ell j - 1 \end{array}\right) \Sigma uv^{k+(\ell-1)(r-j)-1} \otimes \beta P^j(m) \right).
\]

Proof. For $\ell = 2$, the formulas are obtained from Proposition 5.4 by replacing $Sq^{2k}$ and $Sq^{2k+1}$ by $\Sigma cd^{k-1} = \Sigma uv^{k-1}$ and $-\Sigma d^{k} = \Sigma v^{k}$, respectively, as in Theorem 5.8. The summations over $0 \leq j \leq [a/2]$ split into two cases, according to the parity of $j$, and the resulting terms can be collected as shown.

For $\ell$ odd, we first rewrite $R_{\mu}(M)$ as

$$\Sigma H^* \langle BS_\infty \rangle \otimes H \cdots M = \Sigma H^* \{c^id^k \mid i \in \{0, 1\}, k \in \mathbb{Z} \} \otimes H \cdots M,$$

replacing $P^k$ and $\beta P^k$ by $\Sigma cd^{k-1}$ and $-\Sigma d^{k}$, respectively. For $r \geq 0$ the action of $P^r$ on $R_{\mu}(M)$ is then given by
\[
P^r(\Sigma (cd^{-1})(-d)^b \otimes m) = \sum_{j=0}^{[r/\ell]} \left(\sum_{k=j}^{\ell} \left(\begin{array}{c} (\ell - 1)(b - j) - 1 \\ r - \ell j \end{array}\right) \Sigma (cd^{-1})(-d)^{r+b-j} \otimes P^j(m) \right)
\]
and
\[
P^r(\Sigma (-d)^b \otimes m) = \sum_{j=0}^{[r/\ell]} \left(\sum_{k=j}^{\ell} \left(\begin{array}{c} (\ell - 1)(b - j) \\ r - \ell j \end{array}\right) \Sigma (-d)^{r+b-j} \otimes P^j(m) \right)
+ \sum_{j=0}^{[(r-1)/\ell]} \left(\sum_{k=j}^{\ell} \left(\begin{array}{c} (\ell - 1)(b - j - 1) \\ r - \ell j - 1 \end{array}\right) \Sigma (cd^{-1})(-d)^{r+b-j} \otimes \beta P^j(m) \right)
\]
for $b \in \mathbb{Z}$. Substituting $cd^{-1} = uv^{-1}$, $-d = v^{\ell-1}$ and $k = (\ell - 1)b$ we obtain the claimed formulas, in the cases where $k$ is a multiple of $\ell - 1$. The general cases follow, since for $n$ so large that $0 < r < \ell^n$ the action of $P^r$ on $R_{\mu}(M)$ commutes
with multiplication by $v^{\ell^n}$, and $\ell^n$ is relatively prime to $\ell - 1$. All mod $\ell$ binomial coefficients in sight are $\ell^n$-periodic as functions of $k \in \mathbb{Z}$. □

7. The evaluations are Ext-equivalences

We can now adapt [AGMS5] Lem. 2.2 to the motivic setting. We sidestep their use of $\mathbb{F}_p \otimes_{\mathcal{A}} (-)$ and Tor-equivalences, since $H^{\ast,\ast}$ is not naturally a right $\mathcal{A}$-module, and pass directly to $\text{Hom}_{\mathcal{A}}(-, H^{\ast,\ast})$ and Ext-equivalences.

**Lemma 7.1.** Let $M$ be a free left $\mathcal{A}$-module. The Singer constructions $R_S(M)$ and $R_\mu(M)$ are free as left $A(n)$-modules, for each $n$, and flat as left $\mathcal{A}$-modules.

**Proof.** Since $M$ is left $\mathcal{A}$-free, it is left $A(n-1)$-free by Lemma 4.12. Hence the left $A(n)$-module $R_S(M) = B(n) \otimes_{A(n-1)} M$ is a direct sum of (suitably suspended) copies of $B(n)$, each of which is left $A(n)$-free by Proposition 4.19(b). Therefore $R_S(M)$ is left $A(n)$-free for each $n$, so that

$$\text{Tor}_{\mathcal{A}}(K, R_S(M)) \cong \text{colim}_n \text{Tor}_s^A(K, R_S(M)) = 0$$

for each right $\mathcal{A}$-module $K$ and $s \geq 1$. Equivalently, $R_S(M)$ is left $\mathcal{A}$-flat.

Moreover, $R_\mu(M)$ is a direct sum, as a left $A(n)$-module, of copies of $R_S(M)$. Hence it is also left $A(n)$-free for each $n$, and therefore left $\mathcal{A}$-flat, by the same argument as before. □

**Proposition 7.2.** Let $M$ be a free left $\mathcal{A}$-module. The evaluation homomorphisms induce isomorphisms

$$\text{Hom}(\epsilon, \text{id}): \text{Hom}_{\mathcal{A}}(M, H^{\ast,\ast}) \cong \text{Hom}_{\mathcal{A}}(R_S(M), H^{\ast,\ast})$$

**Proof.** It suffices to consider the case $M = \mathcal{A} = \text{colim}_n A(n-1)$. Then

$$\text{Hom}_{A(n)}(R_S(A(n-1)), H^{\ast,\ast}) = \text{Hom}_{A(n)}(B(n) \otimes_{A(n-1)} A(n-1), H^{\ast,\ast})$$

$$= \text{Hom}_{A(n)}(B(n), H^{\ast,\ast})$$

$$\cong \text{Hom}_{H^{\ast,\ast}}(H^{\ast,\ast}\{P^r \mid r \in \mathbb{Z} \text{ with } \ell^n \mid r\}, H^{\ast,\ast})$$

$$\cong \prod_{\ell^n \mid r} H^{\ast,\ast}\{\zeta_1^r\}$$

by Proposition 4.19(b), where we identify $\zeta_1^r$ with the dual of the left $A(n)$-module generator $P^r \in B(n)$. The bidegrees of the classes $\zeta_1^r$ with $\ell^n \mid r$ are integer multiples of $(2\ell - 2)\ell^n, (\ell - 1)\ell^n$. In any fixed bidegree, only the factor generated by $\zeta_1^0$ can make a nonzero contribution to this product when $n$ is sufficiently large. Hence

$$\text{Hom}_{\mathcal{A}}(R_S(\mathcal{A}), H^{\ast,\ast}) \cong \lim_n \text{Hom}_{A(n)}(R_S(A(n-1)), H^{\ast,\ast})$$

$$\cong \lim_n \prod_{\ell^n \mid r} H^{\ast,\ast}\{\zeta_1^r\} \cong H^{\ast,\ast}\{\zeta_1^0\}.$$ 

Since $\epsilon$ maps $P^0 \otimes 1$ to $P^0(1) = 1$, it follows that

$$H^{\ast,\ast} \cong \text{Hom}_{\mathcal{A}}(\mathcal{A}, H^{\ast,\ast}) \xrightarrow{\text{Hom}(\epsilon, 1)} \text{Hom}_{\mathcal{A}}(R_S(\mathcal{A}), H^{\ast,\ast}) \cong H^{\ast,\ast}\{\zeta_1^0\}$$

is an isomorphism.

Similarly,

$$\text{Hom}_{A(n)}(R_\mu(A(n-1)), H^{\ast,\ast})$$

$$\cong H^{\ast,\ast}\{1, v^{\ell^n}, \ldots, v^{\ell^n(\ell-2)}\} \otimes_{H^{\ast,\ast}} \prod_{\ell^n \mid r} H^{\ast,\ast}\{\zeta_1^r\}.$$
The bidegrees of the classes $v^i j \otimes \xi^j_1$ with $0 \leq j \leq \ell - 2$ and $\ell^n \mid r$ are integer multiples of $(2^\ell, \ell^n)$. Again, in any fixed bidegree, only the factor generated by $v^i j \otimes \xi^j_1$ can make a nonzero contribution for $n$ sufficiently large. Hence

$$\text{Hom}_{\mathcal{A}}(R_n(\mathcal{A}), H^{*,*}) \cong \lim_n \text{Hom}_{A(n)}(R_n(A(n-1)), H^{*,*})$$

$$\cong H^{*,*} \{v^0 \otimes \xi^0_1\}.$$ 

Since $\epsilon(\pi \square 1)$ maps $v^0 \otimes P^0$ to $P^0(1) = 1$, it follows that

$$H^{*,*} \cong \text{Hom}_{\mathcal{A}}(\mathcal{A}, H^{*,*}) \xrightarrow{\text{Hom}(s,1)} \text{Hom}_{\mathcal{A}}(R_n(\mathcal{A}), H^{*,*}) \cong H^{*,*} \{v^0 \otimes \xi^0_1\}$$

is an isomorphism.

The Ext-groups for modules over $\mathcal{A}$ or $A(n)$ are trigraded. In the case of an $\mathcal{A}$-module $M$ we write $\text{Ext}_{\mathcal{A}}^{s,t,u}(M, H^{*,*})$ for the group in tridegree $(s, t, u)$, where $s$ is the cohomological degree and $(t, u)$ is the internal bidegree.

**Definition 7.3.** An $\mathcal{A}$-module homomorphism $\theta: L \rightarrow M$ will be said to be an Ext-equivalence if the induced homomorphism

$$\theta^*: \text{Ext}_{\mathcal{A}}^{s,t,u}(M, H^{*,*}) \rightarrow \text{Ext}_{\mathcal{A}}^{s,t,u}(L, H^{*,*})$$

is an isomorphism.

We can now generalize part of [AGMS85, Prop. 1.2, Thm. 1.3].

**Theorem 7.4.** Let $M$ be any left $\mathcal{A}$-module. The (small and large) evaluation homomorphisms

$$\epsilon: R_S(M) \rightarrow M \quad \text{and} \quad \epsilon: R^*_\mu(M) \rightarrow M$$

are Ext-equivalences.

**Proof.** Let

$$\ldots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

be a free $\mathcal{A}$-module resolution of $M$. Then

$$\ldots \rightarrow R_S(F_1) \rightarrow R_S(F_0) \rightarrow R_S(M) \rightarrow 0$$

is a flat $\mathcal{A}$-module resolution, and a free $A(n)$-module resolution for each $n$, by Lemma [7.1]. By Proposition [7.2] the evaluation homomorphism induces an isomorphism

$$\text{Hom}(\epsilon, 1): \text{Hom}_{\mathcal{A}}(F_s, H^{*,*}) \xrightarrow{\cong} \text{Hom}_{\mathcal{A}}(R_S(F_s), H^{*,*})$$

for each $s \geq 0$. Passing to cohomology, it also induces isomorphisms

$$\epsilon^*: \text{Ext}_{\mathcal{A}}^{s,t,u}(M, H^{*,*}) \xrightarrow{\cong} H^s(\text{Hom}_{\mathcal{A}}(R_S(F_s), H^{*,*}))$$

for all $s \geq 0$. Let

$$\ldots \rightarrow E_1 \rightarrow E_0 \rightarrow R_S(M) \rightarrow 0$$

be a free $\mathcal{A}$-module resolution of $R_S(M)$, and choose an $\mathcal{A}$-module chain map $\zeta: E_* \rightarrow R_S(F_*)$ over $R_S(M)$. There is then an induced map $\zeta^*$ of lim-lim homomorphisms, from

$$0 \xrightarrow{\text{lim}_n H^{*,-1}(\text{Hom}_{A(n)}(R_S(F_*), H^{*,*}))} H^s(\text{Hom}_{\mathcal{A}}(R_S(F_*), H^{*,*}))$$

$$\rightarrow \lim_n H^s(\text{Hom}_{A(n)}(R_S(F_*), H^{*,*})) \rightarrow 0$$

to

$$0 \xrightarrow{\text{lim}_n \text{Ext}^{s,-1}_{A(n)}(R_S(M), H^{*,*})} \text{Ext}^s_{\mathcal{A}}(R_S(M), H^{*,*})$$

$$\rightarrow \lim_n \text{Ext}^s_{A(n)}(R_S(M), H^{*,*}) \rightarrow 0,$$
for all $n$ and $s$. Applying $\lim_0 = \lim_{-1}$, we deduce that

$$\zeta^* : H^s(\text{Hom}_{A(n)}(R_S(F_s), H^{••}^*)) \to \text{Ext}^s_{A(n)}(R_S(M), H^{••})$$

for all $n$ and $s$. Applying $\lim_0$ and $\lim_{-1}$, we deduce that

$$\zeta^* : H^s(\text{Hom}_{A}(R_S(F_s), H^{••}^*)) \to \text{Ext}^s_{A}(R_S(M), H^{••})$$

is an isomorphism. Hence the composite $\epsilon^* = \zeta^* \epsilon'$ is also an isomorphism, as claimed.

The proof for $R_\mu$ in place of $R_S$ is identical. \hfill \Box

**Corollary 7.5.** The residue homomorphisms

$$\text{res} : \Sigma H^{••^*}(BSt)_{\text{loc}} \to H^{••^*} \quad \text{and} \quad \text{res} : \Sigma H^{••^*}(B\mu)_{\text{loc}} \to H^{••^*}$$

are Ext-equivalences.

**Proof.** In view of Theorem 5.8 and Corollary 6.6 this is the case $M = H^{••^*}$ of Theorem 7.4. \hfill \Box

8. Generalized Eilenberg–MacLane spectra

Since $H$ is cellular, the monomial basis $\{ \tau^{E \xi R}_\ell \}_{\ell \in \mathbb{Z}}$ for $A_{••}$ as a right (or left) $H_{••}$-module determines an equivalence $H \wedge H \simeq \bigvee\{E \wedge R\}_{\ell \in \mathbb{Z}} \wedge H$ of right (or left) $H$-module spectra. Here $(E, R)$ ranges over the sequences in Lemma 2.4, and $\|E, R\| = \|\tau^{E \xi R}\|$. It follows that the natural homomorphism

$$A_{••} \otimes H_{••}, H_{••}(X) \xrightarrow{\zeta_\ast} \pi_{••}(H \wedge H, H \wedge X)$$

is an isomorphism for any motivic spectrum $X$, and that $1 \wedge \eta \wedge 1 : H \wedge X = H \wedge S \wedge X \to H \wedge H \wedge X$ induces a natural left $A_{••}$-coaction on $H_{••}(X)$.

If $M \simeq H \wedge X$ for some motivic spectrum $X$, then the fork

$$\pi_{••} M \xrightarrow{\eta} \pi_{••}(H \wedge M) \xrightarrow{\eta \wedge 1} \pi_{••}(H \wedge H \wedge M)$$

is split by $\mu : \pi_{••}(H \wedge M) \simeq \pi_{••}(H \wedge H \wedge X) \to \pi_{••}(H \wedge X) \simeq \pi_{••} M$ and $\mu : \pi_{••}(H \wedge H \wedge M) \to \pi_{••}(H \wedge M)$, hence exhibits $\pi_{••} M$ as a split equalizer [MH98 §VI.6]. Under the identifications $\pi_{••}(H \wedge M) = H_{••}(M)$ and $\pi_{••}(H \wedge H \wedge M) \simeq A_{••} \otimes H_{••}, H_{••}(M)$, this provides an isomorphism

$$(8.1) \quad \pi_{••} M \cong H_{••} \boxtimes_{A_{••}} H_{••}(M) \cong \text{Hom}_{A_{••}}(H_{••}, H_{••}(M))$$

of $\pi_{••} M$ with the left $A_{••}$-comodule primitives in $H_{••}(M)$.

**Definition 8.1.** By a motivic GEM (short for generalized Eilenberg–MacLane spectrum) we shall mean a left $H$-module spectrum

$$M \simeq \bigvee_{\alpha \in J} \Sigma^{p_{\alpha} \cdot q_{\alpha}} H$$

that is equivalent to a wedge sum of bigraded suspensions of $H$.

These are precisely the $H$-cellular module spectra $M$, in the sense of [D10, §7.9], with the property that $\pi_{••} M$ is free as a left $H_{••}$-module. This generalizes the split Tate objects of [Voe10 §2.4], in that we allow arbitrary bigraded suspensions.

If $M$ is a motivic GEM, we can write $M \simeq H \wedge X$ with $X = \bigvee_{\alpha \in J} \Sigma^{p_{\alpha} \cdot q_{\alpha}}$. Then

$$H_{••}(X) \cong \bigoplus_{\alpha \in J} \Sigma^{p_{\alpha} \cdot q_{\alpha}} H_{••}$$
as left $\mathcal{A}_{*,*}$-comodules, and the natural homomorphism
\[
\pi_{*,*}F(X, H) = \pi_{*,*}F_H(H \wedge X, H) \xrightarrow{\sim} \text{Hom}_{\mathcal{A}_{*,*}}(H_{*,*}(X), H_{*,*}) = H_{*,*}(X)^{\vee}
\]
is an isomorphism, so that
\[
H_{*,*}(X) \cong \prod_{\alpha \in J} \Sigma^{p_\alpha \cdot q_\alpha} H_{*,*}
\]
as left $\mathcal{A}$-modules.

**Definition 8.2.** With these notations we say that $H \wedge X$ (or $H_{*,*}(X)$, or $H_{*,*}(X)$) has \textit{bifinite type} if for each bidegree $(p, q)$ there are only finitely many $\alpha \in J$ for which
\[
p_\alpha \leq p \quad \text{or} \quad p_\alpha - q_\alpha \leq p - q.
\]

This condition is more restrictive than the notions of motivically finite type from [DI10, Def. 2.11, Def. 2.12] and of finite type from [HKO11, §2]. It ensures that both inclusions
\[
\bigoplus_{\alpha \in J} \Sigma^{p_\alpha \cdot q_\alpha} H_{*,*} \xrightarrow{\sim} \prod_{\alpha \in J} \Sigma^{p_\alpha \cdot q_\alpha} H_{*,*}
\]
are isomorphisms, so that the canonical homomorphism
\[
H_{*,*}(X) \xrightarrow{\sim} \text{Hom}_{\mathcal{A}_{*,*}}(H_{*,*}(X), H_{*,*}) = H_{*,*}(X)^{\vee}
\]
is an isomorphism. Moreover, if $H \wedge X$ has bifinite type, then so does $H \wedge H \wedge X \simeq \bigvee_{(E, R)} \Sigma^{E, R} H \wedge X$. Hence $f \mapsto f^\vee$ defines an isomorphism
\[
\text{Hom}_{\mathcal{A}_{*,*}}(H_{*,*}(X), H_{*,*}) \xrightarrow{\sim} \text{Hom}_{\mathcal{A}}(H_{*,*}(X), H_{*,*})
\]
from the $\mathcal{A}_{*,*}$-comodule homomorphisms $H_{*,*} \to H_{*,*}(X)$ to the $\mathcal{A}$-module homomorphisms $H_{*,*}(X) \to H_{*,*}$.

9. A delayed limit Adams spectral sequence

Let
\[
\cdots \to Y(m + 1) \overset{f_{m+1}}{\to} Y(m) \overset{f_m}{\to} Y(m - 1) \to \cdots
\]
be any tower of motivic spectra. Its homotopy limit $Y = \text{holim}_m Y(m)$ sits in a homotopy cofiber sequence
\[
\Sigma^{-1} \prod_m Y(m) \overset{i}{\to} Y \overset{j}{\to} \prod_m Y(m) \overset{k}{\to} \prod_m Y(m),
\]
where $k$ is the difference between the identity map and the product of the maps $f_m$.

Let
\[
\cdots \to S_2 \overset{\alpha}{\to} S_1 \overset{\alpha}{\to} S_0 \leftarrow \cdots
\]
be the canonical mod $\ell$ Adams resolution of the motivic sphere spectrum $S = S_0$, inductively defined by the homotopy cofiber sequences
\[
S_{s+1} \overset{\alpha}{\to} S_s \overset{\beta}{\to} H \wedge S_s \overset{\gamma}{\to} \Sigma S_{s+1}
\]
where $\beta = \eta \wedge 1$. (Dashed arrows indicate morphisms of degree $-1$.) Form the smash products

\[
Y_s(m) = S_s \wedge Y(m)
\]
\[
K_s(m) = H \wedge S_s \wedge Y(m)
\]

so as to obtain a tower of canonical Adams resolutions

(9.2)

\[
\vdots \rightarrow Y_2(m+1) \rightarrow Y_2(m) \rightarrow Y_2(m-1) \rightarrow \ldots
\]
\[
\vdots \rightarrow K_1(m+1) \rightarrow K_1(m) \rightarrow K_1(m-1) \rightarrow \ldots
\]
\[
\vdots \rightarrow Y_1(m+1) \rightarrow Y_1(m) \rightarrow Y_1(m-1) \rightarrow \ldots
\]
\[
\vdots \rightarrow K_0(m+1) \rightarrow K_0(m) \rightarrow K_0(m-1) \rightarrow \ldots
\]
\[
\vdots \rightarrow Y_0(m+1) \rightarrow Y_0(m) \rightarrow Y_0(m-1) \rightarrow \ldots
\]

Let

\[
Y_s = \text{holim}_m Y_s(m)
\]
\[
K_s = \text{holim}_m K_s(m)
\]

be the homotopy limits of the terms in these Adams resolutions. These fit in a commutative diagram

(9.3)

\[
\begin{array}{ccc}
\Sigma^{-1} \prod_m Y_2(m) & \xrightarrow{i} & Y_2 & \xrightarrow{j} & \prod_m Y_2(m) \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^{-1} \prod_m K_1(m) & \xrightarrow{i} & K_1 & \xrightarrow{j} & \prod_m K_1(m) \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^{-1} \prod_m Y_1(m) & \xrightarrow{i} & Y_1 & \xrightarrow{j} & \prod_m Y_1(m) \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^{-1} \prod_m K_0(m) & \xrightarrow{i} & K_0 & \xrightarrow{j} & \prod_m K_0(m) \\
\end{array}
\]

with horizontal homotopy cofiber sequences extending to

\[
\Sigma^{-1} \prod_m Y_s(m) \xrightarrow{i} Y_s \xrightarrow{j} \prod_m Y_s(m) \xrightarrow{k} \prod_m Y_s(m)
\]
\[
\Sigma^{-1} \prod_m K_s(m) \xrightarrow{i} K_s \xrightarrow{j} \prod_m K_s(m) \xrightarrow{k} \prod_m K_s(m)
\]

The subdiagram

(9.4)

\[
\begin{array}{ccc}
\vdots \rightarrow Y_2 & \xrightarrow{\alpha} & Y_1 & \xrightarrow{\beta} & Y_0 \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma^{-1} \prod_m Y_2(m) & \xrightarrow{i} & \Sigma^{-1} \prod_m K_1(m) & \xrightarrow{i} & \Sigma^{-1} \prod_m Y_1(m) \\
\end{array}
\]
is not generally an Adams resolution. Nonetheless, one may consider its associated homotopy spectral sequence, with $E_1$-term
\[ \lim E_1^{s,*} = \pi_{s,*}(K_s) \]
and abutment $\pi_{s,*}(Y_0) = \pi_{s,*}(Y)$. Under finiteness hypotheses which ensure that all limits in sight are exact, the $E_2$-term of this limit Adams spectral sequence was described in \cite{CPS} Prop. 7.1 and \cite{LN12} Prop. 2.2. In our motivic context, these finiteness hypotheses are only realistic if $H^{*,*}$ is finite in each bidegree, which excludes some very interesting base schemes $S$, such as Spec of a global field.

To avoid this restrictive hypothesis, we shall instead show that there is a modified Adams spectral sequence, in the style of \cite{M17} Lem. 5.3.1 and \cite{R82}, with the same abutment as before, whose $E_2$-term is recognizable under more flexible finiteness conditions. This kind of modification is referred to in \cite{BR21} §12.6 as a delayed Adams spectral sequence, to distinguish it from another kind of modified Adams spectral sequence in current usage (the hastened one).

To construct the delayed limit Adams spectral sequence we may assume that the maps $i$ and $\alpha$ in (9.3) are all cofibrations, let $W_0 = Y_0$, and form the pushouts
\[ W_s = Y_s \cup \Sigma^{-1} \prod_m Y_{s-1}(m) \]
along $\Sigma^{-1} \prod_m Y_{s}(m)$, for all $s \geq 1$. There are then homotopy cofiber sequences
\[ W_1 \rightarrow W_0 \xrightarrow{\beta_1} L_0 = \prod_m K_0(m) \rightarrow \Sigma W_1 \]
and
\[ W_{s+1} \rightarrow W_s \xrightarrow{\beta_{s+1}} L_s = \prod_m K_s(m) \vee \Sigma^{-1} \prod_m K_{s-1}(m) \rightarrow \Sigma W_{s+1} \]
for all $s \geq 1$, defining the spectra $L_s$. This produces a delayed resolution
\[ \ldots \rightarrow W_2 \xrightarrow{\kappa} W_1 \xrightarrow{\alpha} W_0 \]
\[ \downarrow \gamma \quad \downarrow \gamma \quad \downarrow \beta \]
\[ L_1 \quad \downarrow \gamma \quad \downarrow \beta \]
\[ L_0 \]
of $W_0 = Y_0 \simeq Y$. The inclusions $Y_s \subset W_s$ induce a map of diagrams from (9.3) to (9.5).

**Definition 9.1.** The delayed limit Adams spectral sequence of the tower (9.1) is the homotopy spectral sequence associated to the resolution (9.5), with $E_1$-term
\[ \lim E_1^{s,*} = \pi_{s,*}(L_s) \]
and $d_1$-differential induced by the composite
\[ \beta \gamma: L_s \rightarrow \Sigma W_{s+1} \rightarrow \Sigma L_{s+1}. \]

We now make the assumption that each $H \wedge Y(m)$ is a motivic GEM of bifinite type. For example, this is the case if each $Y(m)$ is cellular with $H_{s,*}(Y(m))$ free of bifinite type as a left $H_{s,*}$-module. It follows by induction on $s$ that each $K_s(m)$ is a motivic GEM of bifinite type. Hence the isomorphisms (8.1) and (8.2) identify the $(E_1, d_1)$-term
\[ \ldots \rightarrow \pi_{s,*}K_2(m) \leftarrow \pi_{s,*}K_1(m) \leftarrow \pi_{s,*}K_0(m) \leftarrow 0 \]
of the Adams spectral sequence for $Y(m)$ with $\text{Hom}_{\mathcal{A}}(-, H^{*,*})$ applied to the free $\mathcal{A}$-module resolution
\[ \ldots \rightarrow H^{*,*}(K_2(m)) \rightarrow H^{*,*}(K_1(m)) \rightarrow H^{*,*}(K_0(m)) \rightarrow 0 \]
of $H_{\ast}(Y(m))$. In view of (12), these resolutions are compatible for varying $m$. Passing to colimits over $m$, we obtain a flat $A$-module resolution

$$\ldots \longrightarrow H_{\ast}(K_2) \xrightarrow{\partial} H_{\ast}(K_1) \xrightarrow{\partial} H_{\ast}(K_0) \longrightarrow 0$$

of $H_{\ast}(Y)$, where we write

$$H_{\ast}(Y) = \text{colim}_m H_{\ast}(Y(m))$$

$$H_{\ast}(K_s) = \text{colim}_m H_{\ast}(K_s(m))$$

for the “continuous” cohomology groups of the towers $\{Y(m)\}_m$ and $\{K_s(m)\}_m$, respectively. The $A$-module $H_{\ast}(K_s)$ might not be free, but remains flat, since such modules are preserved under filtered colimits. These colimits can also be written as cokernels, as in the following diagram with exact rows and columns.

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \bigoplus_m H_{\ast}(K_1(m)) & \xrightarrow{k} & \bigoplus_m H_{\ast}(K_1(m)) & \xrightarrow{j} & H_{\ast}(K_1) & \rightarrow & 0 \\
\downarrow{\partial} & & \downarrow{\partial} & & \downarrow{\partial} & & \downarrow{\partial} & & \downarrow{\partial} \\
0 & \rightarrow & \bigoplus_m H_{\ast}(K_0(m)) & \xrightarrow{k} & \bigoplus_m H_{\ast}(K_0(m)) & \xrightarrow{j} & H_{\ast}(K_0) & \rightarrow & 0 \\
\downarrow{\epsilon} & & \downarrow{\epsilon} & & \downarrow{\epsilon} & & \downarrow{\epsilon} & & \downarrow{\epsilon} \\
0 & \rightarrow & \bigoplus_m H_{\ast}(Y(m)) & \xrightarrow{k} & \bigoplus_m H_{\ast}(Y(m)) & \xrightarrow{j} & H_{\ast}(Y) & \rightarrow & 0 \\
\downarrow{0} & & \downarrow{0} & & \downarrow{0} & & \downarrow{0} & & \downarrow{0}
\end{array}
\]

Omitting the bottom row and the right hand column, we have a bicomplex of free $A$-modules, whose total complex $(F_\ast, \partial)$ is a free resolution of $H_{\ast}(Y)$. Here

$$F_0 = \bigoplus_m H_{\ast}(K_0(m))$$

and

$$F_s = \bigoplus_m H_{\ast}(K_s(m)) \oplus \bigoplus_m H_{\ast}(\Sigma^{-1}K_{s-1}(m))$$

for $s \geq 1$. Hence we can recognize

$$\text{del}E_1^{0,\ast} = \pi_{\ast}L_0 = \prod_m \pi_{\ast}K_0(m) \cong \prod_m \text{Hom}_A(H^{\ast}(K_0(m)), H^{\ast})$$

$$\cong \text{Hom}_A(\bigoplus_m H^{\ast}(K_0(m)), H^{\ast}) = \text{Hom}_A(F_0, H^{\ast})$$

and

$$\text{del}E_1^{s,\ast} = \pi_{\ast}L_s \cong \prod_m \pi_{\ast}K_s(m) \oplus \prod_m \pi_{\ast}\Sigma^{-1}K_{s-1}(m)$$

$$\cong \prod_m \text{Hom}_A(H^{\ast}(K_s(m)), H^{\ast}) \oplus \prod_m \text{Hom}_A(H^{\ast}(\Sigma^{-1}K_{s-1}(m)), H^{\ast})$$

$$\cong \text{Hom}_A(\bigoplus_m H^{\ast}(K_s(m)) \oplus \bigoplus_m H^{\ast}(\Sigma^{-1}K_{s-1}(m)), H^{\ast})$$

$$= \text{Hom}_A(F_s, H^{\ast})$$
Lemma 9.4. For each $s \geq 1$. Moreover, $d_1: \del E^{s,*,*}_1 \to \del E^{s+1,*,*}_1$ is induced by the boundary operator $\partial: F_{s+1} \to F_s$ in the total complex, as can be verified by tracing through the definitions. Since $(F_s, \partial)$ is a free $\mathcal A$-module resolution of $H_*^{\mathbb Z}(Y)$, we obtain the desired isomorphism

$$\del E^{s,*,*}_2 \cong \Ext^{s,*}_A(H_*^{\mathbb Z}(Y), H^{*,*})$$

for each $s \geq 0$.

**Proposition 9.2.** Let $\cdots \to Y(m+1) \to Y(m) \to \cdots$ be a tower of motivic spectra, with each $H \wedge Y(m)$ a motivic GEM of bifinite type. Let $Y = \holim_m Y(m)$ and $H_*^{\mathbb Z}(Y) = \colim_m H_*^{*,*}(Y(m))$. The delayed limit Adams spectral sequence

$$\del E^{s,*,*}_1 \Longrightarrow \pi_{s,*}(Y)$$

has $E_2$-term

$$\del E^{s,*,*}_2 = \Ext^{s,*}_A(H_*^{\mathbb Z}(Y), H^{*,*})$$,

with $\Ext$ calculated in the category of $\mathcal A$-modules.

**Proof.** This summarizes the discussion so far in this section. \(\square\)

Let $Y_\infty(m) = \holim_m Y(m)$. The Adams spectral sequence for $Y(m)$ is conditionally convergent [Boa99, Def. 5.10] to $\pi_{s,*} Y(m)$ if and only if $\pi_{s,*} Y_\infty(m) = 0$. This fails in many interesting examples, but often becomes true after $(\ell, \eta)$-adic completion in the following sense.

**Definition 9.3.** Let $\ell \in \pi_{0,0}(S)$ be $\ell$ times the class of the identity map $S \to S$, and let $\eta \in \pi_{1,1}(S)$ be the class of the Hopf fibration $S^{3,2} \simeq \mathbb A^2 - \{0\} \to \mathbb P^1 \simeq S^{2,1}$. For any motivic spectrum $X$ let

$$X_{\ell}^\wedge = \holim_n X/\ell^n$$

$$X_{\ell, \eta}^\wedge = \holim_n (X_{\ell}^\wedge)/\eta^n \simeq \holim_n X/(\ell^n, \eta^n)$$

be the $\ell$- and $(\ell, \eta)$-adic completions of $X$, respectively, as in [HKO11, p. 574] and [Man21, Def. 3.2.9]. There are canonical completion maps $X \to X_{\ell}^\wedge \to X_{\ell, \eta}^\wedge$.

**Lemma 9.4.** For each $H$-module spectrum $M$ the completion map $M \to M_{\ell, \eta}^\wedge$ is a $\pi_{s,*}$-isomorphism.

**Proof.** Multiplication by $\ell$ and by $\eta$ act trivially on $H_{s,*}$, hence also on $\pi_{s,*} M$. The tower of short exact sequences

\[
\begin{array}{ccccccccc}
\vdots & \vdots & \vdots & \vdots & & & & & & \\
0 & \pi_{s,*}(M) & \pi_{s,*}(M/\ell^n) & \pi_{s,*}(\Sigma M) & 0 & & & & & \\
\downarrow{\text{id}} & \downarrow{\text{id}} & \downarrow{\text{id}} & \downarrow{\text{id}} & & & & & & \\
0 & \pi_{s,*}(M) & \pi_{s,*}(M/\ell^{n+1}) & \pi_{s,*}(\Sigma M) & 0 & & & & & \\
\end{array}
\]
and the lim-lim\(^1\) sequence imply that \(M \to M_{\ell}^\wedge\) is a \(\pi_{\ast,\ast}\)-isomorphism. Likewise, the tower of short exact sequences

\[
\cdots \to \pi_{\ast,\ast}(M_{\ell}^\wedge / \eta^{n+1}) \to \pi_{\ast,\ast}(\Sigma^{n+2, n+1}M_{\ell}^\wedge) \to 0 \to \cdots
\]

\[
\cdots \to \pi_{\ast,\ast}(M_{\ell}^\wedge / \eta^{n}) \to \pi_{\ast,\ast}(\Sigma^{n+1, n}M_{\ell}^\wedge) \to 0 \to \cdots
\]

and the lim-lim\(^1\) sequence imply that \(M_{\ell}^\wedge \to M_{\ell, \eta}^\wedge\) is a \(\pi_{\ast,\ast}\)-isomorphism. \(\square\)

Applying \((\ell, \eta)\)-adic completion to the tower of Adams resolutions \([9.2]\) yields another diagram of the same shape. At its lower edge, the tower of spectra

\[
\cdots \to Y(m+1)_{\ell, \eta}^\wedge \to Y(m)_{\ell, \eta}^\wedge \to Y(m-1)_{\ell, \eta}^\wedge \to \cdots
\]

has homotopy limit \(Y_{\ell, \eta}^\wedge\). For each \(m\) the Adams resolution of \(Y(m)\) maps to the diagram

\[
\cdots \to Y_2(m)_{\ell, \eta}^\wedge \xrightarrow{\alpha} Y_1(m)_{\ell, \eta}^\wedge \xrightarrow{\alpha} Y_0(m)_{\ell, \eta}^\wedge
\]

of homotopy cofiber sequences, and by Lemma \([9.4]\) the induced map of homotopy spectral sequences is an isomorphism from the \(E_1\)-term and onward. The new spectral sequence has abutment \(\pi_{\ast,\ast}(Y(m)_{\ell, \eta}^\wedge)\), and is conditionally convergent to this target if and only if \(\pi_{\ast,\ast}(Y_{\infty}(m)_{\ell, \eta}^\wedge) = 0\).

We now make the additional assumption, for each \(m\), that the Adams spectral sequence for \(Y(m)\) converges conditionally to \(\pi_{\ast,\ast}(Y(m)_{\ell, \eta}^\wedge)\). The following theorem was proved by Hu–Kriz–Ormsby \([HKO11\text{ Thm. }1]\) in the case of a cellular spectrum \(X\) of finite type over \(\text{Spec} k\) for \(k\) a field of characteristic 0. It was generalized to bounded below spectra \(X\) over \(S\), in our generality, by Mantovani \([Man21]\).

The homotopy t-structure on \(SH(S)\) is defined as in \([Hoy15\text{ §2.1}]\), and a motivic spectrum is bounded below if it lies in \(SH(S)_{\geq -m}\) for some finite \(m\). (When \(S = \text{Spec} k\) for a field \(k\), Morel’s stable \(k\)-connectivity theorem \([Mor05\text{ Thm. }3]\) shows that \(X\) lies in \(SH(S)_{\geq -m}\) if and only if the homotopy sheaves \(\pi_{t, u}(X)\) vanish whenever \(t - u < -m\).)

**Theorem 9.5.** Suppose that \(X\) in \(SH(S)\) is bounded below in the homotopy t-structure. Then the mod \(\ell\) Adams spectral sequence for \(X\) is conditionally convergent to \(\pi_{\ast,\ast}(X_{\ell, \eta}^\wedge)\).

**Proof.** As reviewed in \([Man21\text{ §5}]\), this is an application of \([Man21\text{ Thm. }1.0.2,\text{ Thm. }1.0.4]\) in the case \(E = H\), which satisfies Mantovani’s hypotheses because of \([Hoy15\text{ Thm. }3.8,\text{ Thm. }7.12]\) and \([Spi18\text{ Thm. }10.3]\). \(\square\)

**Proposition 9.6.** Let \(Y = \text{holim}_m Y(m)\) be the homotopy limit of a tower of motivic spectra. Suppose, for each \(m\), that the mod \(\ell\) Adams spectral sequence for
Y(m) converges conditionally to \( \pi_{\ast,\ast}(Y(m)_{\ell,\eta}) \). Then the limit and delayed limit Adams spectral sequences

\[
\lim E_{1}^{s,\ast,\ast} = \pi_{\ast,\ast}(K_{s}) \implies \pi_{\ast,\ast}(Y_{\ell,\eta}) \\
\text{del} E_{1}^{s,\ast,\ast} = \pi_{\ast,\ast}(L_{s}) \implies \pi_{\ast,\ast}(Y_{\ell,\eta})
\]

are both conditionally convergent to the bigraded homotopy groups of the \((\ell, \eta)\)-adic completion of \( Y \).

**Proof.** Let \( Y_{\infty} = \text{holim}_{s} Y_{s} \) and \( W_{\infty} = \text{holim}_{s} W_{s} \). The inclusions

\[
W_{s+1} \subset Y_{s} \subset W_{s} \subset Y_{s-1}
\]

imply that \( Y_{\infty} \simeq W_{\infty} \). Granting that \( \pi_{\ast,\ast}(Y_{\infty}(m)_{\ell,\eta}) = 0 \) for all \( m \), the short exact \( \lim\text{-}\lim \) sequence shows that \( \pi_{\ast,\ast}(Y_{\infty}(m)_{\ell,\eta}) \cong \pi_{\ast,\ast}(W_{\infty}(m)_{\ell,\eta}) = 0 \), so that both the limit Adams spectral sequence and the delayed limit Adams spectral sequence are conditionally convergent to \( \pi_{\ast,\ast}(Y_{\ell,\eta}) \). \( \square \)

Let \( g: X \to Y = \text{holim}_{m} Y(m) \) be a map of motivic spectra. The resulting compatible maps from the canonical Adams resolution of \( X \) to the canonical Adams resolutions of the \( Y(m) \) induce a map from the former to the diagram \([9,4]\), which can be naturally continued to map to the delayed limit Adams resolution \([9,3]\).

Applying \((\ell, \eta)\)-adic completion, and passing to the associated homotopy spectral sequences, we obtain morphisms of spectral sequences

\[
g: E_{1}^{s,\ast,\ast}(X) \longrightarrow \lim E_{1}^{s,\ast,\ast}(Y) \longrightarrow \text{del} E_{1}^{s,\ast,\ast}(Y)
\]

with abutment

\[
g: \pi_{\ast,\ast}(X_{\ell,\eta}) \longrightarrow \pi_{\ast,\ast}(Y_{\ell,\eta}) \longrightarrow \pi_{\ast,\ast}(Y_{\ell,\eta})
\]

We can now appeal to a special case of Boardman’s comparison theorem \([Boa99\ Thm. 7.2]\) for conditionally convergent spectral sequences. This version of the comparison theorem is particularly convenient, in view of the failure of strong convergence for the motivic Adams spectral sequence for the sphere spectrum over a number field, demonstrated by Kuyling- Wilson in \([KW19 Cor. 7.8]\).

**Proposition 9.7.** Let \( g: X \to Y = \text{holim}_{m} Y(m) \), with \( H \land X \) and each \( H \land Y(m) \) a motivic GEM of bifinite type. Suppose that the mod \( \ell \) Adams spectral sequences for \( X \) and the \( Y(m) \) are conditionally convergent to \( \pi_{\ast,\ast}(X_{\ell,\eta}) \) and \( \pi_{\ast,\ast}(Y(m)_{\ell,\eta}) \), respectively. If the \( \mathcal{A} \)-module homomorphism

\[
g^{\ast}: H_{\ast,\ast}^{\ast}(Y) \longrightarrow H_{\ast,\ast}^{\ast}(X)
\]

is an Ext-isomorphism, so that

\[
g: E_{2}^{s,\ast,\ast}(X) = \text{Ext}_{\mathcal{A}}^{s,\ast,\ast}(H_{\ast,\ast}^{\ast}(X), H_{\ast,\ast}^{\ast})
\]

\[
\text{del} E_{2}^{s,\ast,\ast}(Y) = \text{Ext}_{\mathcal{A}}^{s,\ast,\ast}(H_{\ast,\ast}^{\ast}(Y), H_{\ast,\ast}^{\ast})
\]

is an isomorphism, then \( g \) induces an isomorphism

\[
g: \pi_{\ast,\ast}(X_{\ell,\eta}) \longrightarrow \pi_{\ast,\ast}(Y_{\ell,\eta})
\]

**Proof.** The identification of the \( E_{1} \)- and \( E_{2} \)-term for \( X \) follows as usual from \([8,1]\) and \([8,2]\), and the delayed limit \( E_{2} \)-term for \( Y \) is given by Proposition \([9,2]\). We now apply \([Boa99\ Thm. 7.2[ln]]\). If \( g \) induces an isomorphism of \( E_{2} \)-terms, then it certainly also induces isomorphisms of \( E_{\infty} \) and \( RE_{\infty} \)-terms. Hence \( g \) induces the stated isomorphism of (filtered) abutments. \( \square \)
of motivic Thom spectra, with $H^*_c(L^\infty_{\infty})$ realizing $H^*_c(B\mu_\infty)_{\text{loc}}$. The left hand square will commute up to a generalized sign, and replacing i by $i^4$ will give a strictly commuting diagram in the stable homotopy category $SH(S)$.

For an algebraic vector bundle $\alpha \downarrow X$ over a smooth scheme $X$ (over our base scheme $S$) we let $E(\alpha)$ denote its total space, let $E_0(\alpha) = E(\alpha) \setminus z(X)$ be the complement of its zero section, and let the Thom space $Th(\alpha) = E(\alpha)/E_0(\alpha)$ be the motivic quotient space, formed as in [Voe03a, §4].

Our next aim is to construct a diagram

$$
\begin{array}{c}
\begin{tikzpicture}
  \node (A1) at (0,0) {$Th(-(m+1)\gamma^*_n)$};
  \node (A2) at (3,0) {$Th(-(m+1)\gamma_{n+1}^*)$};
  \node (A3) at (0,-2) {$Th(-m\gamma^*_n)$};
  \node (A4) at (3,-2) {$Th(-m\gamma_{n+1}^*)$};
  \node (B1) at (0,-4) {$L^\infty_{2m}$};
  \node (B2) at (3,-4) {$L^\infty_{2m-2}$};

  \draw[->] (A1) -- node[above] {$i$} (A2);
  \draw[->] (A1) -- node[left] {$j$} (A3);
  \draw[->] (A2) -- node[right] {$j$} (A4);
  \draw[->] (A3) -- node[below] {$i$} (A4);
  \draw[->] (A2) -- (B2);
  \draw[->] (A1) -- (B1);
  \draw[->] (A3) -- (B1);
  \draw[->] (A4) -- (B2);
\end{tikzpicture}
\end{array}
$$

where $\gamma_n$ is an equivalence of Nisnevich sheaves, and by [MV99, Ex. 3.2.2] the projection $(\mathbb{A}^n \times \mathbb{A}^k)/\mu_l \to \mathbb{A}^k/\mu_l = L^{2k-1}$ is an $\mathbb{A}^1$-homotopy equivalence, so there is a homotopy cofiber sequence

$$
L^{2k-1} \xrightarrow{i_n} L^{2n+2k-1} \to Th(k\gamma^*_n)
$$

and bundle projection mapping $[x_1, \ldots, x_n; y]$ to $[x_1, \ldots, x_n]$. Let $\epsilon_n = \epsilon_n^0 \downarrow L^{2n-1}$ be the trivial rank $n$ bundle over $L^{2n-1}$, with total space $E(\epsilon_n) = L^{2n-1} \times \mathbb{A}^n$ and bundle projection to the first factor. There is a canonical embedding $\gamma_n \to \epsilon_n$, given in coordinates by

$$
[x_1, \ldots, x_n; y] \mapsto ([x_1, \ldots, x_n], x_1y, \ldots, x_ny).
$$

Let $\zeta_n = \epsilon_n^0 \downarrow L^{2n-1}$ be its cokernel, so that there is a short exact sequence

$$
0 \to \gamma_n \to \epsilon_n \to \zeta_n \to 0
$$

of algebraic vector bundles over $L^{2n-1}$. Let $\gamma^*_n$, $\epsilon^*_n$ and $\zeta^*_n$ denote the dual bundles, fitting in a short exact sequence

$$(10.1) \quad 0 \to \zeta^*_n \xrightarrow{j} \epsilon^*_n \xrightarrow{i} \gamma^*_n \to 0.$$}

Here the total space of $\gamma^*_n$ is given by the orbit space $E(\gamma^*_n) = (\mathbb{A}^n \times \mathbb{A}^1)/\mu_k$, where $\mu_k$ acts diagonally.

More generally, the total space of the $k$-fold direct sum $k\gamma^*_n = \gamma^*_n \oplus \cdots \oplus \gamma^*_n$, where $k \geq 0$, is

$$
E(k\gamma^*_n) = (\mathbb{A}^n \times \mathbb{A}^k)/\mu_k,
$$

which comes with a canonical map to $\mathbb{A}^{n+k}/\mu_k = L^{2n+2k-1}$. By [MV99] Lem. 3.1.6], the inclusion

$$
Th(k\gamma^*_n) = \frac{E(k\gamma^*_n)}{E_0(k\gamma^*_n)} = \frac{(\mathbb{A}^n \times \mathbb{A}^k)/\mu_k}{(\mathbb{A}^n \times \mathbb{A}^k)/\mu_k} \to \frac{(\mathbb{A}^n \times \mathbb{A}^k)/\mu_k}{(\mathbb{A}^n \times \mathbb{A}^k)/\mu_k}
$$

is an equivalence of Nisnevich sheaves, and by [MV99] Ex. 3.2.2] the projection $(\mathbb{A}^n \times \mathbb{A}^k)/\mu_k \to \mathbb{A}^k/\mu_k = L^{2k-1}$ is an $\mathbb{A}^1$-homotopy equivalence, so there is a homotopy cofiber sequence

$$(10.2) \quad L^{2k-1} \xrightarrow{i^n} L^{2n+2k-1} \to Th(k\gamma^*_n)$$
of motivic spaces. Following James [Jam59], Atiyah [Ati61] Prop. 4.3 and Kambe–Matsunaga–Toda [KMT16] Thm. 1 we may therefore write $Th(k\gamma_n^\ast) = L^{2n-2m-1}_{2k}$ for the Thom space of $k\gamma_n^\ast$ over $L^{2n-1}$, and refer to it as a motivic stunted lens space.

Following Mahowald and Adams [Ada74] p. 4, we are, however, more interested in the cases where $k = -m$ is negative, corresponding to Thom spectra $Th(-m\gamma_n^\ast) = L^{2n-2m-1}m$ of virtual bundles $-m\gamma_n^\ast$. In view of (10.1), $-m\gamma_n^\ast \cong \kappa^\ast_n - m\epsilon_n^\ast$ as virtual bundles over $L^{2n-1}$, where $m\epsilon_n^\ast$ is trivial of rank $mn$, which leads to the following definition.

**Definition 10.1.** For $m \geq 0$ let

$$L^{2n-2m-1}_{-2m} = Th(-m\gamma_n^\ast) = \Sigma^{-2mn,-mn}Th(m\epsilon_n^\ast)$$

denote a (finite, motivic) stunted lens spectrum.

Consider the inclusion $i : L^{2n-1}_{-2m} \rightarrow L^{2n+1}_{-2m}$. There are natural isomorphisms $i^\ast (\gamma_n+1) \cong \gamma_n$, $i^\ast (\epsilon_n+1) \cong \epsilon_n \oplus \epsilon_n^1$ and $i^\ast (\kappa_n+1) \cong \kappa_n \oplus \epsilon_n^1$, where $\epsilon_n^1$ is trivial of rank 1. Dually, $i^\ast (\gamma_n+1) \cong \gamma_n^\ast$, $i^\ast (\epsilon_n+1) \cong \epsilon_n^\ast \oplus (\epsilon_n^1)^\ast$ and $i^\ast (\kappa_n+1) \cong \kappa_n^\ast \oplus (\epsilon_n^1)^\ast$.

Consider also the inclusion $j : \kappa_n \rightarrow \epsilon_n$ of bundles over $L^{2n-1}_{-2m}$. Let $\kappa_n^\ast \rightarrow \epsilon_n^\ast$ be the maps obtained by applying $\Sigma^{-2m(n+1),-m(n+1)}$ and $\Sigma^{-2(n+1),-(n+1)n}$ to $\Sigma^{2m,m}Th(m\kappa_n^\ast) \cong Th(m(\kappa_n^\ast \oplus (\epsilon_n^1)^\ast)) \rightarrow Th(m\epsilon_n^\ast)$

and

$$Th((m+1)\kappa_n^\ast) \rightarrow Th(m\epsilon_n^\ast) = \Sigma^{2mn}Th(m\epsilon_n^\ast),$$

respectively. Here $sh$ denotes the isomorphism of spectra induced by the shuffle $m\epsilon_n^\ast \oplus m(\epsilon_n^1)^\ast \cong m(\kappa_n^\ast \oplus (\epsilon_n^1)^\ast)$.

**Definition 10.3.** Let $-\epsilon \in \pi_0(S)$ be the class of the symmetry isomorphism $\gamma : S^{2,1} \wedge S^{2,1} \cong S^{2,1} \wedge S^{2,1}$. It satisfies $(-\epsilon)^2 = 1$, since $\gamma^2 = id$.

**Lemma 10.4.** The rectangle

$$
\begin{array}{ccc}
L^{2n-2m-3}_{-2m-2} = Th(-(m+1)\gamma_n^\ast) & \xrightarrow{i} & Th(-(m+1)\gamma_n^\ast) = L^{2n-2m-1}_{-2m-2} \\
\downarrow j & & \downarrow j \\
L^{2n-2m-1}_{-2m} = Th(-m\gamma_n^\ast) & & L^{2n-2m-1}_{-2m} \\
\downarrow (m\epsilon_n) \cong & & \\
L^{2n-2m-1}_{-2m} = Th(-m\gamma_n^\ast) & \xrightarrow{i} & Th(-m\gamma_n^\ast) = L^{2n-2m+1}_{-2m-2}
\end{array}
$$

commutes up to homotopy.

**Proof.** The diagrams

$$
\begin{array}{ccc}
Th((m+1)(\kappa_n^\ast \oplus (\epsilon_n^1)^\ast)) & \xrightarrow{(m+1)i} & Th((m+1)\kappa_n^\ast) \\
\downarrow m \text{id} \oplus j \oplus \text{id} & & \downarrow m \text{id} \oplus j \\
Th(m(\kappa_n^\ast \oplus (\epsilon_n^1)^\ast) \oplus \epsilon_n \oplus (\epsilon_n^1)^\ast) & \xrightarrow{m \text{id} \oplus 1} & Th(m\kappa_n^\ast \oplus \epsilon_n^\ast)
\end{array}
$$
and

\[
\begin{array}{c}
\text{Th}((m+1)\zeta_n^* \oplus (m+1)(\epsilon_n^1)^*) \xrightarrow{\text{sh}} \text{Th}((m+1)(\zeta_n^* \oplus (\epsilon_n^1)^*)) \\
\downarrow \text{id} \oplus \theta \oplus \text{id} \\
\text{Th}(m\zeta_n^* \oplus \epsilon_n^* \oplus (m+1)(\epsilon_n^1)^*) \\
\downarrow \text{id} \oplus \chi_{m,n} \oplus \text{id} \\
\text{Th}(m\zeta_n^* \oplus m(\epsilon_n^1)^* \oplus \epsilon_n^* \oplus (\epsilon_n^1)^*)
\end{array}
\]

commute strictly, where \(\chi_{m,n}\) is induced by the symmetry isomorphism

\[
\epsilon_n^* \oplus m(\epsilon_n^1)^* \cong m(\epsilon_n^1)^* \oplus \epsilon_n^*,
\]

hence is homotopic to multiplication by \((-e)^{mn}\). Applying

\[
\sum_{-(m+1)(n+1),-(m+1)(n+1)}^{-(2m+7)n(n+1)}
\]
yields the stated homotopy commutative rectangle.

\[\square\]

**Corollary 10.5.** The square

\[
\begin{array}{c}
L_{-2m}^{2n-2m-3} = \text{Th}(-(m+1)\gamma_n^*) \xrightarrow{\ell^i} \text{Th}(-(m+1)\gamma_{n+4}) = L_{-2m}^{2n-2m+5} \\
\downarrow j \\
L_{-2m}^{2n-2m-1} = \text{Th}(-m\gamma_n^*) \xrightarrow{\ell^i} \text{Th}(-m\gamma_{n+4}) = L_{-2m}^{2n-2m+7}
\end{array}
\]

commutes up to homotopy.

**Proof.** This follows from Lemma [10.4] since \(mn + m(n+1) + m(n+2) + m(n+3)\) is always even.

\[\square\]

**Definition 10.6.** Let the (infinite, motivic) stunted lens spectrum

\[
L_{-2m}^\infty = \text{holim}_n L_{-2m}^{2n-2m-1} = \text{holim}_n \text{Th}(-m\gamma_n^*)
\]

be the homotopy colimit of the maps

\[
\ldots \rightarrow L_{-2m}^{2n-2m-1} \xrightarrow{i} L_{-2m}^{2n-2m+1} \rightarrow \ldots.
\]

For a fixed choice of commuting homotopies in Corollary [10.5] let \(j: L_{-2m}^\infty \rightarrow L_{-2m}^\infty\) be the induced map. Let

\[
L_{\infty} = \text{holim}_m L_{-2m}^\infty
\]

be the homotopy limit of the resulting tower

\[
\ldots \rightarrow L_{-2m}^\infty \xrightarrow{j} L_{-2m}^\infty \rightarrow \ldots.
\]

Recall Notation [5.6]

**Lemma 10.7.**

\[
H^*(L_{-2m}^{2n+1}) = H^*[u,v]/(u^2 = \tau v + \rho u, v^n),
\]

where \(v\) is the mod \(l\) Euler class of \(\gamma_n^* \downarrow L_{-2m}^{2n-1}\) and \(\beta(u) = v\).

**Proof.** This follows by the same argument as for [Voe03, Thm. 6.10], working with \(L_{-2m}^{2n-1} \rightarrow \mathbb{P}^{n-1}\) in place of \(BP_l \rightarrow \mathbb{P}^{\infty}\).  

\[\square\]
Definition 10.8. Let $U_{m\zeta^n} \in H^{2m(n-1), m(n-1)}(\text{Th}(m\zeta_n^*))$ be the mod $\ell$ Thom class of $m\zeta_n^* \downarrow L^{2n-1}$. Let
\[ U_{-m\gamma_n^*} \in H^{-2m, -m}(L^{2n-2m-1}) = H^{-2m, -m}(\text{Th}(-m\gamma_n^*)) \]
be its image under the (de-)suspension isomorphism. We write $x \mapsto x \cdot U_{m\zeta^n}$ and $x \mapsto x \cdot U_{-m\gamma_n^*}$ for the Thom isomorphisms
\[ H^{*,*}(L^{2n-1}) \cong H^{*,*+2m(n-1),*+m(n-1)}(\text{Th}(m\zeta_n^*)) \]
\[ H^{*,*}(L^{2n-1}) \cong H^{*,*-2m,*,*}(\text{Th}(-m\gamma_n^*)) , \]
cf. [Voe03a, Prop. 4.3].

Lemma 10.9. The homomorphisms
\[ i^*: H^{*,*}(\text{Th}(-m\gamma_{n+1}^*)) \to H^{*,*}(\text{Th}(-m\gamma_n^*)) \]
\[ j^*: H^{*,*}(\text{Th}(-m\gamma_n^*)) \to H^{*,*}(\text{Th}(-(m+1)\gamma_n^*)) \]
are given by
\[ i^*(x \cdot U_{-m\gamma_{n+1}^*}) = i^*(x) \cdot U_{-m\gamma_n^*} \]
\[ j^*(x \cdot U_{-m\gamma_n^*}) = xv \cdot U_{-(m+1)\gamma_n^*} . \]

Proof. The Thom class of $m\zeta_{n+1}^*$ maps under $i^*$ to the Thom class of $i^*(m\zeta_{n+1}^*) = m(\zeta_{n+1}^* \oplus (\epsilon_n^*)^\gamma)$, which corresponds under the suspension isomorphism to the Thom class of $m\zeta_n^*$. This proves the first formula, where $x \in H^{*,*}(L^{2n+1})$.

The Thom class of $m\zeta_n^*$ corresponds under the suspension isomorphism to the Thom class of $m\zeta_n^* \oplus \epsilon_n^*$. By the Jouanolou trick [Voe03a, Lem. 4.7] it maps under $j^*$ to the Euler class $v$ of $\gamma_n^*$ times the Thom class of $(m+1)\zeta_n^*$. This proves the second formula, where $x \in H^{*,*}(L^{2n-1})$. \qed

Proposition 10.10. The structure maps $L_{-2m-1} \to L_{-2m}$ and $L_{-\infty} \to L_{-2m}$ induce $\mathcal{A}$-module isomorphisms
\[ H^{*,*}(L_{-2m}) \cong H^{*,*}(B\mu)(U_{-m\gamma^*}) \cong H^{*,*}(B\mu)(v^{-m}) \]
and
\[ H^{*,*}(L_{-\infty}) = \lim_{m} H^{*,*}(L_{-2m}) \cong H^{*,*}(B\mu)_{\text{loc}} . \]

Proof. Since each $i^*: H^{*,*}(\text{Th}(-m\gamma_{n+1}^*)) \to H^{*,*}(\text{Th}(-m\gamma_n^*))$ is surjective, the lim-$\lim^1$ sequence gives an isomorphism
\[ H^{*,*}(L_{-2m}) \cong \lim_{n} H^{*,*}(\text{Th}(-m\gamma_n^*)) \cong \lim_{n} H^{*,*}(L^{2n-1})(U_{-m\gamma_n^*}) . \]
Letting $U_{-m\gamma^*}$ correspond to the compatible sequence $(U_{-m\gamma_n^*})_n$ gives the first isomorphism. The second isomorphism sends $U_{-m\gamma^*}$ to $v^{-m}$. The induced homomorphism $j^*: H^{*,*}(B\mu)(U_{-m\gamma^*}) \to H^{*,*}(B\mu)(U_{-(m+1)\gamma^*})$ maps $U_{-m\gamma^*}$ to $v \cdot U_{-(m+1)\gamma^*}$, hence corresponds to the homomorphism
\[ H^{*,*}(B\mu)(v^{-m}) \to H^{*,*}(B\mu)(v^{-m-1}) \]
\[ \text{sending } v^{-m} \text{ to } v \cdot v^{-m-1} . \]
It follows that $\lim_{m} H^{*,*}(L_{-\infty})$, i.e., the continuous cohomology $H^{*,*}_{c}(L_{-\infty})$, is isomorphic to the localization $H^{*,*}(B\mu)[1/v] = H^{*,*}_{c}(B\mu)_{\text{loc}}$. It remains to justify that these isomorphisms are compatible with the Steenrod operations. The short exact sequence
\[ 0 \to H^{*,*}(\text{Th}(m\gamma_n^*)) \to H^{*,*}(L^{2n+2m-1}) \xrightarrow{i^n} H^{*,*}(L^{2n-1}) \to 0 \]
induced from [10.2] shows that the Steenrod action on $U_{m\zeta_n^*}$ matches that on $v^m$ in $H^{*,*}(L^{2n-1})\{v^m\}$. By another application of the Jouanolou trick, and the Cartan formula, it follows that the Steenrod action on $U_{m\zeta_n^*}$ is compatible with that on
The next two lemmas confirm the assumptions required (for recognition of the $E_2$-term and conditional convergence) of the delayed limit Adams spectral sequence for $\Sigma L^\infty$.  

**Lemma 10.11.** The spectra $L_{2m-2m-1}^\infty$ are cellular and bounded below.  

**Proof.** The Zariski cover of $L_{2n-1}^{\infty}$ by the affines $(A^{i-1} \times \mathbb{A}^n) / \mu_\ell$, with $1 \leq i \leq n$, is completely stably cellular in the sense of [DI05, Def. 3.7]. It trivializes $\gamma_n$, hence also $m\gamma_n^\ast$. It follows as in [DI05, Cor. 3.10] that $L_{2n-2m-1}^{\infty} = \Sigma_{2n-2m-1} \otimes m\gamma_n^\ast$ is cellular. Inspection of the argument shows that it admits a cell structure with finitely many cells, all in bidegrees $(p,n)$ satisfying $p - n \geq -m$.  

Since this bound is uniform, it follows by passage to the homotopy colimit that $L_{2m}^{\infty}$ is also cellular, with cells in the same range of bidegrees. Hence $L_{2m}^{\infty}$ lies in $SH(S)_{\geq -m}$ of the homotopy $t$-structure.  

Recall Definitions 8.1 and 8.2.  

**Lemma 10.12.** The $H$-module spectra $H \wedge L_{2m}^{2n} - 2m-1$ and $H \wedge L_{2m}^{\infty} - 2m$ are motivic GEMs of bifinite type.  

**Proof.** These spectra are $H$-cellular by Lemma 10.11. The homology version of Lemma 10.7 shows that $H_{*,*}(L_{2n-1}^{\infty})$ is finitely generated and free over $H_{*,*}$ on generators in bidegrees $(i + 2k, i + k)$ for $i \in \{0, 1\}$ and $0 \leq k < n$. It then follows from the Thom isomorphism in motivic homology that $H_{*,*}(L_{2m-2m-1}^{\infty})$ is finitely generated and free on similar generators for $i \in \{0, 1\}$ and $-m \leq k < n - m$, and that $H_{*,*}(L_{2m}^{\infty})$ is free on one generator in each bidegree $(i + 2k, i + k)$ for $i \in \{0, 1\}$ and $k \geq -m$. In particular, $H_{*,*}(L_{2m}^{\infty})$ is of bifinite type.  

Finally, we construct maps  

\[(10.3) \quad J \xrightarrow{\epsilon} \Sigma^{2,1} \otimes_{\infty} \xrightarrow{d} \Sigma L_{-1}^{\infty} \xrightarrow{} \Sigma^{2,1} \otimes_{\infty} \Sigma L_{-1}^{\infty}, \]

whose composite induces the (large) residue homomorphism from Definition 6.1 in motivic cohomology.  

To define $\mathbb{P}^\infty_{-m}$, let the (finite, motivic) stunted projective spectrum $\mathbb{P}^{n-m-1} = Th(-m\gamma_n^\ast \downarrow \mathbb{P}^{n-1})$ be the Thom spectrum of the negative of $m\gamma_n^\ast = \partial(1)^m$ over $\mathbb{P}^{n-1}$. We have maps $i: \mathbb{P}^{n-m-1} \rightarrow \mathbb{P}^{n-m}$ and $j: \mathbb{P}^{n-m-1} \rightarrow \mathbb{P}^{n-m-2}$ as in Definition 10.2 and let $\mathbb{P}^\infty_{-m} = \text{holim}_{m} \mathbb{P}^{n-m-1}$ and $\mathbb{P}^{\infty}_{-m} = \text{holim}_{m} \mathbb{P}^{n-m}$ as in Definition 10.6. We obtain $H^\ast_{\ast}(\mathbb{P}^{\infty}_{-m}) \cong H^\ast_{\ast}(v^{\pm 1})$, by the same arguments as for lens spectra.  

**Proposition 10.13.** There is a map $c: J \rightarrow \Sigma^{2,1} \otimes_{\infty} 1$ in cohomology. The $H^\ast_{\ast}$-module generators $\Sigma^{2,1} \nu^k$ for $k \leq -2$ map to zero.  

**Proof.** We use that $\mathbb{P}^{n-1}$ is a smooth projective variety, with stable normal bundle $\nu = \epsilon_n^\ast - n\gamma_n^\ast = \partial - \partial(1)^n$.  

$\Sigma^{2mn,mn}v^{-m}$. Hence, by stability, the Steenrod action on $U_{-m\gamma_n^\ast}$ matches that on $v^{-m}$ in $H^\ast_{\ast}(L_{2n-1}^{\infty}) \{v^{-m}\}$. Passing to the limit over $n$ and the colimit over $m$ completes the argument.  

$\square$
By the construction leading to algebraic Atiyah duality, see [Voe03b Prop. 2.7], [Hat05 Cl. 2] and [Hoy17 §5.3], there is a Pontryagin–Thom collapse map
\[ c_n : \Sigma \xrightarrow{\alpha, \beta} Th(\nu \downarrow \mathbb{P}^{n-1}) = \Sigma^{2,1} \mathbb{P}^{-1}_{-n} \]
inducing the homomorphism \( \Sigma^{2,1} v^{-1} \) for \(-n \leq k \leq -2\) map to zero for bidegree reasons. When combined with the Thom diagonal and an adjunction, this leads to the Atiyah duality equivalence \( Th(\nu \downarrow \mathbb{P}^{n-1}) \cong D(\mathbb{P}^{n-1}) = F(\mathbb{P}^{n-1}, \mathbb{S}) \), under which \( c_n \) is functionally dual to the collapse map \( \mathbb{P}^{n-1} \to S^0 \). In particular, these maps are compatible up to homotopy for varying \( n \), and combine to define a map \( e : \Sigma \to \Sigma^{2,1} \mathbb{P}^{-1}_{-n} \cong D(\mathbb{P}^{n+1}) \), as required.

Since \( L^{2n} \) is only quasi-projective, we need a different method to obtain the second map of (10.3).

**Proposition 10.14.** There is a map \( d : \Sigma^{2,1} \mathbb{P}^{-1}_{-\infty} \to \Sigma L^{-1}_{-\infty} \) inducing
\[ \Sigma_{uv}^{-1} \xrightarrow{\alpha, \beta} \Sigma^{2,1} v^{-1} \]
in cohomology, modulo \( H^{*,*} \)-multiples of \( \Sigma^{2,1} v^k \) for \( k \leq -2 \).

**Proof.** For algebraic vector bundles \( \alpha, \beta \downarrow X \) over the same smooth scheme \( X \), where \( \beta \) may be virtual, there is a homotopy cofiber sequence
\[ Th(p^* \beta \downarrow E_0(\alpha)) \to Th(\beta \downarrow X) \to Th(\alpha \oplus \beta \downarrow X) \]
of motivic spectra. Here \( p : E_0(\alpha) \to X \) denotes the projection, and \( p^* \beta \) is the pullback of \( \beta \) along \( p \). We apply this with \( X = \mathbb{P}^{n-1} \), \( \alpha = \gamma_n \otimes \ell = \Theta(-\ell) \) and \( \alpha \oplus \beta = \nu \), the stable normal bundle of \( \mathbb{P}^{n-1} \). We identify \( E_0(\gamma_n) \cong L^{2n-1} \), as in [Voe03b Lem. 6.3]. Moreover, \( p^* \gamma_n \cong c_1 \) over \( L^{2n-1} \), i.e., \( \Theta(-\ell) \) and \( \Theta \) pull back to the same bundle, which implies that \( p^* \beta \cong -n\gamma_n^* \) as a stable bundle over \( L^{2n-1} \). This leads to the homotopy cofiber sequence
\[ L^{-1}_{-2n} \to Th(\beta \downarrow \mathbb{P}^{n-1}) \to \Sigma^{2,1} \mathbb{P}^{-1}_{-n} d_n \to \Sigma L^{-1}_{-2n} \]
The long exact sequence in cohomology shows that the connecting map \( d_n \) induces a homomorphism mapping \( \Sigma_{uv}^{-1} \) to \( \Sigma^{2,1} v^{-1} \), modulo \( H^{*,*} \)-multiples of \( \Sigma^{2,1} v_k \) for \(-n \leq k \leq -2 \). Again, these maps are compatible up to homotopy for varying \( n \), and combine to define the required map \( d \).

**Proposition 10.15.** There is a map \( e : \Sigma L^\infty_{-\infty} \) of motivic spectra, inducing the residue homomorphism
\[ e^* = \text{res} : \Sigma H^{*,*}(B\mu)_{\text{loc}} \to H^{*,*} \]
in cohomology.

**Proof.** We take \( e \) to be \( dc \) followed by the inclusion \( \Sigma L^{-1}_{-\infty} \to \Sigma L^\infty_{-\infty} \) (of the homotopy fiber of \( \Sigma L^\infty_{-\infty} \to \Sigma L^\infty_0 \)). To check that \( e \) induces the residue homomorphism, we use Corollary 10.6 in cohomological degree 0, giving an isomorphism
\[ \text{res} : \text{Hom}_\mathcal{M}(H^{*,*}, H^{*,*}) \xrightarrow{\cong} \text{Hom}_\mathcal{M}(\Sigma H^{*,*}(B\mu)_{\text{loc}}, H^{*,*}) \]
In other words, any \( \mathcal{M} \)-module homomorphism \( \Sigma H^{*,*}(B\mu)_{\text{loc}} \to H^{*,*} \) is characterized by its value on \( \Sigma_{uv}^{-1} \). Since \( e^* \) and \( \text{res} \) agree on this element, they are equal.
11. The motivic Lin and Gunawardena theorems

We can now prove a motivic refinement of the classical theorems of Lin [Lin80] (for \( \ell = 2 \)) and Gunawardena [Gun81] (for \( \ell \) an odd prime).

Recall that \( \mu_\ell \) denotes the algebraic group of \( \ell \)-th roots of unity, \( L^{2n-1} = (k^n \setminus \{0\})/\mu_\ell \) is an algebraic lens space, \( \gamma_n^* \downarrow L^{2n-1} \) is the dual of the tautological line bundle, \( L^{2n-2m-1} = \text{Th}(-m\gamma_n^*) \) is a stunted lens spectrum, and \( L^{2m}_\infty = \text{holim}_m L^{2m-2m-1} \) and \( L^{\infty}_\infty = \text{holim}_m L^{2m-2m-1}_\infty \) are infinite lens spectra. The continuous mod \( \ell \) cohomology \( H^*_c(L^{\infty}_\infty) = \text{colim}_m H^*_c(L^{2m-2m-1}) \) is isomorphic to the localization \( H^*_c(B\mu_\ell)_{\text{loc}} = H^*_c[u,v^{1/2}]/(u^2 = tv + pu) \). The map \( e: S \to \Sigma L^{\infty}_\infty \) induces the Ext-equivalence \( \text{res}: \Sigma H^*_c(B\mu_\ell)_{\text{loc}} \to H^*_c \).

**Theorem 11.1.** Let \( S \) be a Noetherian scheme of finite dimension \( d \), essentially smooth over a field or Dedekind domain, and let \( \ell \) be a prime that is invertible on \( S \). The \((\ell, \eta)\)-completed map

\[
e^*_{\ell,\eta}: S^\wedge_{\ell,\eta} \to (\Sigma L^{\infty}_\infty)^{\wedge}_{\ell,\eta}
\]

is a \( \pi_{*,*} \)-isomorphism. If \( S = \text{Spec} \ k \) for a field, then \( e^*_{\ell,\eta} \) is a motivic equivalence.

**Proof.** We apply Proposition 9.7 with \( X = S, Y = \Sigma L^{\infty}_\infty, Y(m) = \Sigma L^{2m}_\infty \) and \( g = e \).

The \( H \)-modules \( H \wedge S \) and \( H \wedge \Sigma L^{\infty}_\infty \) are motivic GEMs of bifinite type by Lemma 10.12. Moreover, \( S \) and each \( \Sigma L^{\infty}_\infty \) is bounded below in the homotopy \( t \)-structure on \( SH(S) \) by Lemma 10.11. By Theorem 9.3 the Adams spectral sequences for \( S \) and the \( \Sigma L^{\infty}_\infty \) are conditionally convergent to the \((\ell, \eta)\)-adic completions. By Proposition 10.10 the delayed limit Adams spectral sequence for \( \Sigma L^{\infty}_\infty \) is also conditionally convergent to the \((\ell, \eta)\)-adic completion. The \( \mathcal{A} \)-module homomorphism

\[
e^*: H^*_c((\Sigma L^{\infty}_\infty) \to (S)
\]

agrees with

\[
\text{res}: \Sigma H^*_c(B\mu_\ell)_{\text{loc}} \to H^*_c
\]

via the isomorphism of Proposition 10.10 by Proposition 10.15. Finally, \( e \) is an Ext-equivalence by Corollary 10.5. Hence the induced map of spectral sequences

\[
e^*: E^2_{\ell,\eta}(S) = \text{Ext}^*_c(H^*_c, H^*_c) \to \text{Ext}^*_c(\Sigma H^*_c(B\mu_\ell)_{\text{loc}}, H^*_c)
\]

is an isomorphism, from the \( E_2 \)-term and onward, and the map of abutments

\[
e: \pi_{*,*}(S^\wedge_{\ell,\eta}) \to \pi_{*,*}(\Sigma L^{\infty}_\infty)^{\wedge}_{\ell,\eta}
\]

is an isomorphism of (filtered) bigraded abelian groups.

Let \( C_{\ell,\eta} \) denote the homotopy cofiber of \( e^*_{\ell,\eta} \). If \( S = \text{Spec} \ k \) for a field \( k \), then the homotopy sheaves \( \pi_{*,*}(C_{\ell,\eta}) \) are pure in the sense of [Mor12] Def. 6.4.9], hence unramified in the sense of [Mor05] Def. 2.1], by [Mor05] Lem. 6.4.11] and [Mor12] Thm. 1.9. For any (irreducible) smooth \( k \)-scheme \( U \) with function field \( k(U) \), the vanishing of \( \pi_{*,*}(C_{\ell,\eta}) \) over \( \text{Spec} \ k(U) \) implies the vanishing of \( \pi_{*,*}(C_{\ell,\eta}) \) over \( U \), so that \( \pi_{*,*}(C_{\ell,\eta}) = 0 \) and \( e^*_{\ell,\eta} \) is a motivic equivalence. This application of Morel’s theorems also appears in [Hem22] Prop. 4. □

**Remark 11.2.** As an alternative to our fairly explicit construction of \( e: S \to \Sigma L^{\infty}_\infty \) in Proposition 10.15 one might appeal to the weight 0 part of the delayed limit Adams spectral sequence

\[
\text{del} E^4_{2,0} = \text{Ext}^4_{\mathcal{A}}(\Sigma H^*_c(B\mu_\ell)_{\text{loc}}, H^*_c) \to \pi_{*,0}(\Sigma L^{\infty}_\infty)^{\wedge}_{\ell,\eta}
\]
for $\Sigma L_{\infty}^\infty$ to show the existence of a homotopy class $e' \in \pi_{0,0}(\Sigma L_{\infty}^\infty)^{1,1} \eta$ detected in Adams filtration $s = 0$ by res: $\Sigma H^{\ast,\ast}(B\mu)_\text{loc} \to H^{\ast,\ast}$ in $\text{del}_E^{1,0,0} \subseteq \text{del}_E^{1,0,0} = \text{Hom}_s(\Sigma H^{\ast,\ast}(B\mu)_\text{loc}, H^{\ast,\ast})$.

By Corollary [Ada99, Thm. 7.3], see also [HR19, Thm. 3.9], it suffices to show the existence of a homotopy class $e'$ detected by a map $e': \Sigma S \to (\Sigma L_{\infty}^\infty)^{1,1} \eta$, which is always the case for $S = \text{Spec} k$, the canonical (or normalized cobr) $\mathcal{A}$-module resolution of $H^{\ast,\ast}$, and we don't know about $R^{\ast,\ast}$. New developments in topology (Proc. Sympos. Algebraic Topology, Oxford, 1972), London Math. Soc. Lecture Note Ser., No. 11, Cambridge Univ. Press, London, 1974, pp. 1–9. MR0339178

The class $e'$ is then represented by a map $e': S \to (\Sigma L_{\infty}^\infty)^{1,1} \eta$, which can be used in place of $e$.

References


ON THE MOTIVIC SEGAL CONJECTURE


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