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Tate cohomology in commutative algebra

J.P.C. Greenlees *

Department of Pure Mathematics, University of Sheffield, The Hicks Building, Sheffield S3 7RH, United Kingdom

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Abstract

We construct local Tate cohomology groups $\widehat{H}_I^*(A;M)$ of an A-module M at a finitely generated ideal I by splicing together the Grothendieck local cohomology groups [16] with the local homology groups of [14]. We give two quite different means of calculating them, show they vanish on I-free modules and prove that many elements of A act invertibly. These local Tate groups have applications in the study of completion theorems and their duals in equivariant topology.

0. Introduction

Tate introduced the group cohomology that bears his name for use in class field theory, soon after the Artin–Tate seminar of 1951–52 [3]; details first appeared in [7]. Its analogues are now widely used in number theory, group theory, transformation groups and equivariant topology [2,3,6,7,9,13,20]; there is a similar theory for modules over Frobenius algebras [18], which also has applications in number theory [19] and equivariant topology. In all cases the construction and purpose is broadly similar. The theory is constructed by splicing together a homology theory and a corresponding cohomology theory. This minimises "edge effects" of both and hence *highlights periodicity* and provides *systematic blindness* to "free" or "induced" structures. In the present paper we introduce a Tate cohomology $\widehat{H}_I^*(A;M)$ for modules M over a commutative ring A for each finitely generated ideal I, by splicing Grothendieck's local cohomology groups [16] with the local homology groups introduced in [14]: we call it local Tate cohomology. We show that it enjoys the analogue of the

* Email: j.greenlees@sheffield.ac.uk.

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blindness in that it vanishes on complexes exact when localised at primes not containing I, and we identify the relevant type of periodicity; we also show that $\hat{H}_I^*(A)$ is a ring over which the theory is module valued. More interesting perhaps is the "Warwick duality" (Theorem 4.1) which is a far reaching generalisation of the curious isomorphism

$$\mathbb{Z}_p^\wedge[1/p] \cong \varprojlim(\mathbb{Z}/p^\infty, p),$$

where $\mathbb{Z}/p^\infty$ denotes the quasi-cyclic group of p-power roots of unity and the inverse limit is over repeated multiplication by p. The proof is an easy formal manipulation which substitutes for the commutation of direct and inverse limits, but the author believes the fundamental importance of the result merits the distinction of a special name.

If we note that each side is isomorphic to the field of p-adic rationals we have the antecedent of our second main result (Theorem 7.1), that for a ring A which is Noetherian and of dimension 1 the Tate cohomology is concentrated in degree zero where it has the remarkable property that all regular elements of A act invertibly. The naive analogue is false for rings of higher dimension. The rationality theorem lies behind the striking fact that for finite groups the Tate theory for equivariant topological K-theory is a rational vector space [13]. This allows us in Appendix A to give a proof of the rationality of Tate spectra for arbitrary families of subgroups which does not depend on our knowledge of the coefficient ring of K-theory.

The author was led to the definition of local Tate cohomology directly from the Tate cohomology associated with equivariant topological K-theory. Bott periodicity moves the usual graded construction into a single degree, where it acquires precisely the present character with A being the representation ring and I being its augmentation ideal. The idea that an infinite-dimensional object needed for a graded version of a theory can be replaced by a finite-dimensional local one in the presence of suitable periodicity is the essence of the Atiyah–Segal completion theorem and the confirmed Segal conjecture [10,12]. The author intends to substantiate this assertion elsewhere, in a study of the algebraic counterpart to these completion theorems.

Many of the proofs that follow are direct translations of suitable proofs in equivariant topology, which topologists may find useful for motivation. Nonetheless, no topology is used until we draw topological conclusions in the appendices.

The outline of contents is as follows. *Section* 1: Recollections about local homology and cohomology. *Section* 2: Čech cohomology and the definition of a corresponding homology theory. *Section* 3: The definition of local Tate cohomology. *Section* 4: The Warwick duality: a second complex for calculating local Tate cohomology. *Section* 5: Vanishing on modules over I: the notion of I-freedom, and the relevance of local Tate cohomology. *Section* 6: Rationality of local Tate cohomology: the basic lemma for showing a ring element acts invertibly. *Section* 7: Rings of dimension 1: specialization of our results to this case and proof of the Rationality Theorem. *Section* 8: Ideals generated by a regular sequence: another specially good case. *Section* 9: Multiplicative structure. *Appendix* A: Rationality of Tate spectra and the local cohomology theorem. *Appendix* B: Axiomatic topological Tate theory.

The author is grateful to J.D.S. Jones for the discussion which revealed the topological Warwick duality [13, (2.6)], to J. Morava for expecting certain Tate theories to be rational, to R.Y. Sharp for various conversations about local cohomology (and the present simple proof of Lemma 7.4), and to J.P. May for many exchanges on related topological questions.

1. Local cohomology and local homology

In this section we recall suitable definitions of Grothendieck's local cohomology groups [16] and certain related local homology groups [14].

We suppose given a commutative ring A, and a finitely generated ideal $I = (\alpha_1, \ldots, \alpha_n)$. We shall use the specified generators in our construction, but the construction is independent of them up to quasi-isomorphism (i.e. the equivalence relation on chain complexes generated by homology isomorphisms) [10].

For a single element α we may form the Koszul cochain complex

$$K^\bullet(\alpha) = (\alpha : A \to A)$$

where the nonzero groups are in codegrees 0 and 1. When defining local cohomology it is usual to take the stabilised Koszul complex $\varinjlim_k K^\bullet(\alpha^k) = (A \longrightarrow A[1/\alpha])$, which, being flat, is adequate for this purpose. However we shall need a complex of projective A modules and accordingly we take a projective approximation to it; at one or two points it is useful to have a particular model and we define the projective stabilized Koszul complex by

$$K^\bullet_\infty(\alpha) = \mathrm{tel}_k\, K^\bullet(\alpha^k).$$

Here "tel" denotes the infinite mapping telescope or homotopy direct limit. Exact details can be found in [14], but for the moment we only need to know that $K^\bullet_\infty(\alpha)$ is a complex of infinitely generated free A-modules and there is a map $K^\bullet_\infty(\alpha) \to (A \to A[1/\alpha])$ which induces an isomorphism in homology. The Koszul cochain complex for a sequence $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ is obtained by tensoring together the complexes for the elements so that

$$K^\bullet_\infty(\boldsymbol{\alpha}) = K^\bullet_\infty(\alpha_1, \ldots, \alpha_n) = K^\bullet_\infty(\alpha_1) \otimes \cdots \otimes K^\bullet_\infty(\alpha_n).$$

The local cohomology and homology of an A-module M are then defined by

$$H_I^*(A;M) = H^*(K_\infty^\bullet(\boldsymbol{\alpha}) \otimes M),$$
$$H_*^I(A;M) = H_*(\mathrm{Hom}(K_\infty^\bullet(\boldsymbol{\alpha}), M)).$$

Remark 1.1. (i) From the classical universal coefficient theorems we see that any quasi-isomorphic flat complex gives isomorphic local cohomology groups, and any quasi-isomorphic projective complex gives isomorphic local homology groups, hence that the groups agree with the classical definitions and are independent of the choice of generators up to isomorphism.

(ii) We note that $K_\infty^\bullet(\alpha)$ is nonzero in codegrees -1, 0 and 1, and hence $K_\infty^\bullet(\boldsymbol{\alpha})$ is nonzero in codegrees from $-n$ to n. However it is easy to check directly that there is a homology isomorphism (vertically)

$$\begin{array}{ccc} A \oplus A[x] & \xrightarrow{\langle 1, \alpha x - 1\rangle} & A[x] \\ \downarrow & & \downarrow \\ A & \longrightarrow & A[1/\alpha] \end{array}$$

Hence $K_\infty^\bullet(\boldsymbol{\alpha})$ is quasi-isomorphic to a projective complex nonzero only in codegrees between 0 and n. This gives helpful vanishing results.

(iii) Here and elsewhere the definitions make sense if M is replaced by a chain complex.

It is clear from the definitions that there is a third quadrant universal coefficient spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_A^s(H_I^{-t}(A), M) \Longrightarrow H_{-t-s}^I(A;M),$$

with cohomological differentials (i.e. $d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$). This generalises Grothendieck's local duality spectral sequence [16]; see [14] for more details.

The above definitions in terms of elements are in most ways very unsatisfactory. The meaning of the constructions appears from the following two theorems.

Theorem 1.2 (Grothendieck [16]). *If A is Noetherian the local cohomology groups calculate the right derived functors of the left exact functor $M \mapsto \Gamma_I(M) = \{x \in M \mid I^k x = 0$ for some positive integer $k\}$. In symbols we have*

$$H_I^n(A;M) = (R^n\Gamma_I)(M). \qquad \square$$

Theorem 1.3 (Greenlees and May [14]). *If A is Noetherian then the local homology groups calculate the left derived funtors of the (not usually right exact) I-adic completion functor $M \mapsto M_I^\wedge = \varprojlim_k M/I^k M$. In symbols we have*

$$H_n^I(A;M) = (L_n(-)_I^\wedge)(M). \qquad \square$$

Remark. The conclusion of Theorem 1.3 is proved in [14] under much weaker hypotheses; these are stated using the notion of pro-regularity of the sequence $\boldsymbol{\alpha}$ for a module M [14, (1.8)]. Now [14, (2.5)] states that local homology calculates the left derived functors of completion provided A has bounded α_i torsion for all i and that $\boldsymbol{\alpha}$ is pro-regular for A. Furthermore if this is the case and $\boldsymbol{\alpha}$ is also pro-regular for M we have $H_0^I(A;M) = M_I^\wedge$ and $H_i^I(A;M) = 0$ for $i > 0$. We shall refer to any module for which the local homology is its completion concentrated in degree zero as *tame*. By the Artin–Rees lemma any finitely generated module over a Noetherian ring is tame.

The conclusion of Theorem 1.2 is also true under similar weakened hypotheses [11].

Remark. We also note that by Grothendieck vanishing, $H_I^n(A;M)$ is zero when $n > \dim A$; from the universal coefficient theorem we see that $H_n^I(A;M)$ is also zero in this range.

2. The definition of Čech homology

With the notations of the previous section we now define the projective Čech cochain complex $\check{C}^\bullet(\boldsymbol{\alpha})$. Indeed we have an obvious map $\pi : K^\bullet(\alpha) \to A$, and we may pass to telescopes and replace $\mathrm{tel}_k A$ by A to obtain a map $\pi : K^\bullet_\infty(\alpha) \to A$. Finally we take tensor products and define $\check{C}^\bullet(\boldsymbol{\alpha})$ as the mapping cone of the resulting map: we therefore have a cofibre sequence

$$K^\bullet_\infty(\boldsymbol{\alpha}) \longrightarrow A \longrightarrow \check{C}^\bullet(\boldsymbol{\alpha})$$

of cochain complexes, which induces a long exact sequence in cohomology.

From the construction it is clear that $\check{C}^\bullet(\boldsymbol{\alpha})$ is a projective approximation to the usual Čech complex for calculating the cohomology of $\mathrm{Spec}(A) \setminus V(I)$. This makes it legitimate to define the Čech cohomology by

$$\check{C}H_I^*(A;M) = H^*(\check{C}^\bullet(\boldsymbol{\alpha}) \otimes M),$$

and to introduce the homology theory

$$\check{C}H_*^I(A;M) = H_*(\mathrm{Hom}(\check{C}^\bullet(\boldsymbol{\alpha}), M)).$$

We have chosen this non-standard notation to minimise confusion between the myriad decorated H's; more standard notations would be either $\check{H}_I^*(A;M)$ or $H^*(\mathrm{Spec}(A) \setminus V(I); \tilde{M})$. The bottom Čech cohomology $\check{C}H_I^0(A;M)$ is sometimes known as the "ideal transform" of M.

There is a third quadrant universal coefficient spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^s(\check{C}H_I^{-t}(A), M) \Rightarrow \check{C}H^I_{-t-s}(A;M),$$

precisely as for the local theories; it too has cohomological differentials.

Warnings. (i) The chain complex $\check{C}^\bullet(\boldsymbol{\alpha})$ is *not* usually quasi-isomorphic to a tensor product of the corresponding complexes for principal ideals.

(ii) Although the usual Čech complex is concentrated in codegrees from 0 to $n-1$, the given projective approximation is concentrated in codegrees from $-n-1$ to n; from Remark 1.1(ii) of Section 1, it is quasi-isomorphic to a complex concentrated in codegrees -1 to n. We shall see that it is usual for the Čech homology group to be nonzero in degree -1. This explains the fact that there is no obvious dual of the notion of ideal transform, partly answering a question of P. Schenzel (private communication).

We do however have one useful principle for calculating Čech homology, based on the fact that $\check{C}^\bullet(\boldsymbol{\alpha})$ is the telescope of the mapping cones of the maps

$$\pi_k : K^\bullet(\alpha_1^k) \otimes \cdots \otimes K^\bullet(\alpha_d^k) \longrightarrow A. \tag{1}$$

This is immediate from the fact that telescopes commute with tensor products up to homology isomorphism [14, (0.1)]. The point is that the finite Koszul complexes are self-dual, in the sense that

$$\mathrm{Hom}(K^\bullet(\boldsymbol{\alpha}) \otimes C, D) = \mathrm{Hom}(C, D) \otimes K_\bullet(\boldsymbol{\alpha}). \tag{2}$$

According to the usual convention $C_i = C^{-i}$ for changing between chain and cochain complexes, $K_\bullet(\alpha)$ is still the complex $(A \xrightarrow{\alpha} A)$, but now thought of as a chain complex concentrated in degrees 0 and -1. Accordingly the dual of the map π_k is the tensor product of the inclusions

$$i_k : A \longrightarrow K_\bullet(\boldsymbol{\alpha}^k).$$

Finally, it is immediate from the definitions that if we apply $\mathrm{Hom}(-, M)$ to the mapping cone of π_k we obtain the mapping fibre of $M \otimes i_k$. In particular $\mathrm{Hom}(-, M)$ converts mapping telescopes (homotopy direct limits) into microscopes (homotopy inverse limits) [14]. Therefore we have an exact sequence

$$\begin{aligned} 0 \longrightarrow\ & \varprojlim{}_k^1 H_{n+1}(F(i_k) \otimes M) \longrightarrow \check{C}H_n^I(A; M) \\ & \longrightarrow \varprojlim{}_k H_n(F(i_k) \otimes M) \longrightarrow 0. \end{aligned} \tag{3}$$

This is apparently most useful in the case of a principal ideal.

Example. Suppose $I = (\alpha)$. Then we see that $\check{C}^\bullet(\boldsymbol{\alpha})$ is quasi-isomorphic to the complex $\alpha x - 1 : A[x] \to A[x]$ concentrated in codegrees -1 and 0. We then have

$$\check{C}H^0_{(\alpha)}(M) = M[1/\alpha],$$

$$\check{C}H_i^{(\alpha)}(M) = \begin{cases} \varprojlim\,(M, \alpha) & \text{if } i = 0, \\ \varprojlim{}^1(M, \alpha) & \text{if } i = -1, \end{cases}$$

and all other Čech groups vanish.

Remark. From the cofibre sequence of chain complexes defining the Čech complex we see that

$$0 \longrightarrow H_1^I(A;M) \longrightarrow \check{C}H_0^I(A;M) \longrightarrow M$$
$$\longrightarrow H_0^I(A;M) \longrightarrow \check{C}H_{-1}^I(A;M) \longrightarrow 0$$

is exact, so that if M is tame $\check{C}H_{-1}^I(A;M) = M_I^\wedge/M$ and $\check{C}H_0^I(A;M) = \bigcap_{n\geq 0} I^n M$. Unless M is quasi-complete we therefore find Čech homology is nonzero in dimension -1. On the other hand if local homology calculates left derived functors of completion (for instance if the ring A is Noetherian) and $\bigcap_{n\geq 0} I^n A = 0$ we see that the Čech homology groups calculate the left derived functors of the right exact functor $\check{C}H_{-1}^I$ (and of the functor $M \mapsto M_I^\wedge/M$ which is not right exact) in the sense that $L_n\check{C}H_{-1}^I(A;\bullet) = \check{C}H_{n-1}^I(A;\bullet)$.

3. The definition of local Tate cohomology

We have defined a cofibre sequence of cochain complexes

$$K_\infty^\bullet(\alpha) \xrightarrow{\pi} A \xrightarrow{j} \check{C}^\bullet(\alpha). \tag{4}$$

Equivariant topologists should think of this as the precise analogue of the fundamental cofibering $EG_+ \to S^0 \to \tilde{E}G$. The following is an exact translation of the topological definition in [9].

Definition 3.1. The *local Tate cohomology* $\hat{H}_I^*(A;M)$ of M at the ideal I is the cohomology of the cochain complex

$$T^\bullet(M) = \mathrm{Hom}(K_\infty^\bullet(\alpha), M) \otimes \check{C}^\bullet(\alpha).$$

Remark. From our remarks on efficient choices of complexes we see that $\hat{H}_I^i(A;M)$ is only nonzero in the range $-n \leq i \leq n-1$.

The definition immediately gives us a means of calculating the local Tate cohomology groups.

Proposition 3.2. *There is a third quadrant spectral sequence*

$$E_2^{s,t} = \check{C}H_I^s(A; H_{-t}^I(A;M)) \Longrightarrow \hat{H}_I^{s+t}(A;M)$$

with cohomological differentials. □

Corollary 3.3. *If $H_I^*(A)$ vanishes above dimension $d > 0$ (for example if A is Noetherian of dimension d) then $\hat{H}_I^i(A;M) = 0$ unless $-d \leq i \leq d-1$.* □

Corollary 3.4. *If M is tame the spectral sequence of Proposition 3.2 collapses to give an isomorphism*

$$\widehat{H}_I^s(A;M) \cong \check{C}H_I^s(A;M_I^\wedge),$$

so that in particular the local Tate cohomology is zero in negative codegrees. □

4. The Warwick duality

The subject of this section is the fact that there is an apparently very different construction of local Tate cohomology. The second construction gives a complex $TT^\bullet(M)$ quasi-isomorphic to $T^\bullet(M)$, and a second spectral sequence (Proposition 4.2) analogous to that of Proposition 3.2. Since the two spectral sequences converge to the same groups we obtain isomorphisms generalizing the curiosity mentioned in the introduction. The analogue for Tate cohomology of finite groups is the rather well-known phenomenon that the dual of a complete resolution is itself the suspension of a complete resolution. For generalised cohomology theories the phenomenon takes on the present character; special cases of it show that the spectra considered by Mahowald and others in relation to Lin's theorem are instances of Tate spectra (see [13, Sections 13 and 16] for further discussion and detailed references). The original topological formality [13, (2.6)] was isolated in conversation with J.D.S. Jones in the course of understanding why two apparently quite different explanations for a property of Tate cohomology were in fact the same. It was a desire to understand this algebraically that led to the present work.

Now recall that for any map $f : X^\bullet \longrightarrow Y^\bullet$ of cochain complexes there is an injective map $j_f : Y^\bullet \to C(f)$ to the mapping cone and the quotient gives a map $C(f) \to \Sigma^{-1}X^\bullet$ to the (cohomology) desuspension of $X^\bullet$ (i.e. $(\Sigma^{-1}X)^n = X^{n+1}$). Using the notation of (4) we let $d : \check{C}^\bullet(\boldsymbol{\alpha}) \to \Sigma^{-1}K_\infty^\bullet(\boldsymbol{\alpha})$ denote the quotient and we may state the main theorem of the section.

Theorem 4.1 (The algebraic Warwick duality). *The local Tate complex $T^\bullet(M)$ is quasi-isomorphic to the complex*

$$TT^\bullet(M) = \mathrm{Hom}(\check{C}^\bullet(\boldsymbol{\alpha}), M \otimes \Sigma^{-1}K_\infty^\bullet(\boldsymbol{\alpha})).$$

Indeed, the maps

$$\begin{array}{rc} T^\bullet(M) = & \mathrm{Hom}(A, \mathrm{Hom}(K^\bullet_\infty(\boldsymbol{\alpha}), M) \otimes \check{C}^\bullet(\boldsymbol{\alpha})) \\ & \Big\uparrow j^* \\ & \mathrm{Hom}(\check{C}^\bullet(\boldsymbol{\alpha}), \mathrm{Hom}(K^\bullet_\infty(\boldsymbol{\alpha}), M) \otimes \check{C}^\bullet(\boldsymbol{\alpha})) \\ & \Big\downarrow d_* \\ & \mathrm{Hom}(\check{C}^\bullet(\boldsymbol{\alpha}), \mathrm{Hom}(K^\bullet_\infty(\boldsymbol{\alpha}), M) \otimes \Sigma^{-1}K^\bullet_\infty(\boldsymbol{\alpha})) \\ & \Big\uparrow \pi^* \\ TT^\bullet(M) = & \mathrm{Hom}(\check{C}^\bullet(\boldsymbol{\alpha}), M \otimes \Sigma^{-1}K^\bullet_\infty(\boldsymbol{\alpha})) \end{array}$$

are homology isomorphisms.

Before turning to a proof we deduce certain consequences. The point of the theorem is that the spectral sequence for calculating the cohomology of the second form of the local Tate complex is quite different from that of Proposition 3.2.

Proposition 4.2. *There is a spectral sequence*

$$E_2^{s,t} = \check{C}H^I_{-s}(A; H^t_I(A; M)) \Rightarrow \widehat{H}^{s+t-1}_I(A; M)$$

with cohomological differentials which essentially lies in the second quadrant (more precisely $E_r^{s,t} \neq 0$ only if $s \leq 1$ and $t \geq 0$). □

Note. If M is tame this spectral sequence provides a reverse universal coefficient theorem. It leads from the Čech homology of the local cohomology of M to the Čech cohomology of the completion, $\check{C}H^{s+t-1}_I(A; M^\wedge_I)$.

By comparing the spectral sequences of Propositions 3.2 and 4.2 we obtain some interesting facts. We begin with the rather simple case when I is principal, recalling the example following (3).

Corollary 4.3. *If $I = (\alpha)$ is principal we have an exact sequence*

$$0 \longrightarrow \varprojlim{}^1 (H^0_{(\alpha)}(A; M), \alpha) \longrightarrow H^{(\alpha)}_0(A; M)[1/\alpha] \longrightarrow \varprojlim (H^1_{(\alpha)}(A; M), \alpha) \longrightarrow 0$$

and an isomorphism

$$H^{(\alpha)}_1(A; M)[1/\alpha] \cong \varprojlim (H^0_{(\alpha)}(A; M), \alpha). \qquad \square$$

Remark. If A is Noetherian and M is finitely generated the exact sequence of Corollary 4.3 gives

$$M^\wedge_\alpha[1/\alpha] = \varprojlim (H^1_{(\alpha)}(A; M), \alpha),$$

and the isomorphism is the identity $0 = 0$. The particular case $A = \mathbb{Z} = M$ and $\alpha = p$ was the curiosity of the Introduction.

Corollary 4.4. *Quite generally, if $H_I^i(A) = 0$ for $i > d$, by comparing the two methods of calculating local Tate cohomology in dimensions d, $d-1$ and $-d$, we find*

(d) $\check{C}H^I_{-1}(A; H^d_I(A;M)) = 0$ (*i.e.* $H^d_I(A;M)$ *is quasi-complete*).

($d-1$) *There is an exact sequence*

$$\begin{aligned}&\check{C}H^I_1(A; H^d_I(A;M))\\ &\longrightarrow \check{C}H^I_{-1}(A; H^{d-1}_I(A;M)) \longrightarrow \check{C}H^{d-1}_I(A; H^I_0(A;M))\\ &\longrightarrow \check{C}H^I_0(A; H^d_I(A;M)) \longrightarrow 0.\end{aligned}$$

($-d$) *There is an isomorphism*

$$\check{C}H^0_I(A; H^I_d(A;M) \cong \check{C}H^I_{d-1}(A; H^0_I(A;M)). \qquad \square$$

Proof of Theorem 4.1. The proof of Theorem 4.1 reduces to that of the following single lemma.

Lemma 4.5. *For any positive integer k, the map $\pi : K^\bullet_\infty(\boldsymbol{\alpha}) \to A$ induces a homology isomorphism $\pi \otimes K(\alpha_1^k) \otimes \cdots \otimes K(\alpha_n^k)$.*

Proof. It is enough to deal with the principal case, which is clear by replacing $K^\bullet_\infty(\alpha)$ with the quasi-isomorphic complex $A \longrightarrow A[1/\alpha]$. $\square$

Corollary 4.6. *The following five complexes are exact for any chain complex X.*

(i) $\check{C}^\bullet(\boldsymbol{\alpha}) \otimes K(\alpha_1^k) \otimes \cdots \otimes K(\alpha_n^k) \otimes X$.
(ii) $\check{C}^\bullet(\boldsymbol{\alpha}) \otimes K^\bullet_\infty(\boldsymbol{\alpha}) \otimes X$.
(iii) $\mathrm{Hom}(K(\alpha_1^k) \otimes \cdots \otimes K(\alpha_n^k), \check{C}^\bullet(\boldsymbol{\alpha}) \otimes X)$.
(iv) $\mathrm{Hom}(K^\bullet_\infty(\boldsymbol{\alpha}), \check{C}^\bullet(\boldsymbol{\alpha}) \otimes X)$.
(v) $\mathrm{Hom}(\check{C}^\bullet(\boldsymbol{\alpha}), X) \otimes K^\bullet_\infty(\boldsymbol{\alpha})$.

Proof. Part (i) follows from Lemma 4.5 if $X = A$ by the exact sequence of (4). The general case follows by the universal coefficient theorem since the complex for $X = A$ is projective.

Part (ii) follows from (i) by passage to mapping telescopes (which commute with tensor products and become direct limits in homology).

Part (iii) also follows from (i) by the self-duality (2) of the finite Koszul complex.

Part (iv) follows from (iii) since Hom converts telescopes in the first variable into microscopes.

Part (v) follows from (i) with $X = A$ by applying $\mathrm{Hom}(-, X)$, using self-duality of the finite Koszul complex and passing to telescopes. $\square$

We may now complete the proof of Theorem 4.1. Indeed j^* is a homology isomorphism by Corollary 4.6(iv), d_* is a homology isomorphism by Corollary 4.6(ii) and π^* is a homology isomorphism by Corollary 4.6(v). □

5. Vanishing on modules over I and periodicity

To explain the relevant property we relate the present theory to its antecedents. In group cohomology one takes $H^*(G;X) = \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, X)$, and to calculate it one takes a resolution of the ground ring $\mathbb{Z}$. By contrast what we have defined so far is analogous to $\mathrm{Ext}^*_{\mathbb{Z}G}(M, \mathbb{Z})$: we have worked from a resolution of M. Thus the analogue of the complete resolution in group theory is $T^\bullet(A)$, and $\widehat{H}^*_I(A;M)$ is really the analogue of the coefficient ring of a generalised cohomology theory. According to this analogy the value $\widehat{H}_I(A;M)_*(X)$ of the homology theory on a module or chain complex X is the homology of $T^\bullet(M) \otimes X$ and the cohomology $\widehat{H}_I(A;M)^*(X)$ is that of $\mathrm{Hom}(X, T^\bullet(M))$.

With this notation it is easy to give the vanishing property.

Theorem 5.1. *If A is Noetherian and the supporting primes of the homology of X all contain I then $\widehat{H}_I(A;M)^*(X) = 0$ and $\widehat{H}_I(A;M)_*(X) = 0$.*

Proof. Using $T^\bullet(M)$ for homology and $TT^\bullet(M)$ for cohomology we see that it is enough to check that $\check{C} \otimes X$ is exact. Now this happens precisely when $K^\bullet_\infty(\boldsymbol{\alpha}) \otimes X \to A \otimes X$ is a homology isomorphism; the advantage here being that it suffices to consider the principal case. However, $(A \to A[1/\alpha]) \otimes X \to A \otimes X$ is a homology isomorphism for $\alpha \in I$ because $X[1/\alpha]$ is exact by the hypothesis on X. □

By analogy with the topological theory we may refer to a complex of projectives with support in $V(I) = \{\wp \mid \wp \supseteq I\}$ as *I-free*. This is the start of a dictionary between equivariant topology and commutative algebra. In particular one may study algebraic analogues of topological completion theorems and their duals. I intend to return to this elsewhere.

The analogue for the present theory of the periodicity enjoyed by cyclic groups in group cohomology is that the ideal I behaves like a principal ideal.

6. Rationality of local Tate cohomology

In this section we begin investigating the very striking fact that many elements f of A induce isomorphisms of local Tate cohomology. The most powerful illustration of this comes for rings of dimension 1, for which we find in Theorem 7.1 that every element which is not a zero divisor is invertible. The philosophy comes from the note following Proposition 4.2: the completion tends to make elements not in I act invertibly whilst Čech cohomology tends to make elements of I act invertibly.

Our method is to combine the spectral sequence of Proposition 4.2 with the universal coefficient theorem for Čech homology. This means that the groups

$$E^{r,s,t} = \mathrm{Ext}^r_A(\check{C}H_I^{-s}(A), H_I^t(A;M))$$

lead via two spectral sequences to the groups $\widehat{H}_I^n(A;M)$. More precisely the displayed group makes a contribution to codegree $n = r + s + t - 1$. The differentials d^k of the first spectral sequence map from a section of $E^{r,s,t}$ to a section of $E^{r+k,s-k+1,t}$; the resulting section of $E^{r,s,t}$ at the E_∞ term of this spectral sequence contributes a section to $\check{C}H^I_{-r-s}(A; H_I^t(A;M))$. The differentials d^l of the second spectral sequence thus relate sections of $E^{r,s,t}$ to sections of extensions involving $E^{a,b,t-l+1}$ with $r + s + l = a + b$. In any case *if multiplication by f is an isomorphism for the groups $E^{r,s,t}$ with $r + s + t = n - 2, n - 1$ and n then multiplication by f is an isomorphism of $\widehat{H}_I^n(A;M)$.*

We have now established the relevance of the groups $\mathrm{Ext}^*_A(M,N)$ to our discussion and we consider multiplication by f.

Lemma 6.1. *If A is Noetherian and*
(i) *M is f torsion free,*
(ii) *M/f is finitely generated and*
(iii) *N is f divisible,*
then the kernel and cokernel of $f : \mathrm{Ext}^*_A(M,N) \to \mathrm{Ext}^*_A(M,N)$ have support in $\mathrm{Supp}(M/f) \cap \mathrm{Supp}(\mathrm{ann}(f,N))$.

Proof. By (ii) we have $\mathrm{Ext}^*_A(M/f,N)_\wp \cong \mathrm{Ext}^*_A((M/f)_\wp, N_\wp)$. If this is zero we see by considering the exact sequence $0 \to M \xrightarrow{f} M \to M/f \to 0$ guaranteed by (i) that f is an isomorphism at $\wp$. If $\wp \notin \mathrm{Supp}(M/f)$ the group is clearly zero. On the other hand if $\wp \notin \mathrm{Supp}(\mathrm{ann}(f,N))$, $f : N_\wp \to N_\wp$ is an isomorphism and so multiplication by f is a nilpotent isomorphism of $\mathrm{Ext}^*_A((M/f)_\wp, N_\wp)$, so again the group is zero. □

Remark. The Noetherian hypothesis is only used to deduce from (ii) that M/f has a resolution by finitely generated free modules. With a suitable interpretation of the statement about supports the lemma therefore holds for arbitrary A if (ii) is strengthened to include this assumption.

This rather trivial lemma gains force when we establish (i) and (ii) for relevant Čech groups and (iii) for relevant local cohomology groups.

7. Rings of dimension 1

Because of topological applications it is well worth making explicit the very degenerate case when A has Krull dimension 1. In this case the local homology and cohomology are concentrated in degree 0 and 1, and the universal coefficient theorem takes the form of an exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Ext}^1(H_I^1(A),M) \longrightarrow H_0^I(A;M) \\ &\longrightarrow \mathrm{Hom}(H_I^0(A),M) \longrightarrow \mathrm{Ext}^2(H_I^1(A),M) \longrightarrow 0 \end{aligned}$$

and the isomorphism

$$H_1^I(A;M) = \mathrm{Hom}(H_I^1(A),M).$$

We also obtain the periodicity $\mathrm{Ext}^s(H_I^0(A),M) \cong \mathrm{Ext}^{s+2}(H_I^1(A),M)$ for $s \geq 1$. Rather more simply, the Čech cohomology is concentrated in degree zero, the Čech homology is concentrated in degrees -1 and 0, and the universal coefficient theorem gives the isomorphisms

$$\check{C}H_0^I(A;M) = \mathrm{Hom}(\check{C}H_I^0(A),M)$$

and

$$\check{C}H_{-1}^I(A;M) = \mathrm{Ext}^1(\check{C}H_I^0(A),M).$$

Also $\mathrm{Ext}^s(\check{C}H_I^0(A),M) = 0$ for $s \geq 2$. Similarly the local Tate cohomology is concentrated in degrees -1 and 0. The spectral sequence of Proposition 3.2 collapses to give isomorphisms

$$\hat{H}_I^t(A;M) \cong \check{C}H_I^0(A;H_{-t}^I(A;M)),$$

and the spectral sequence of Proposition 4.2 gives the short exact sequence

$$0 \longrightarrow \check{C}H_{-1}^I(A;H_I^0(A;M)) \longrightarrow \hat{H}_I^0(A;M) \longrightarrow \check{C}H_0^I(A;H_I^1(A;M)) \longrightarrow 0$$

and the isomorphism

$$\hat{H}_I^{-1}(A;M) \cong \check{C}H_0^I(A;H_I^0(A;M)).$$

If M has bounded I-torsion then these degenerate considerably; indeed the higher local homology vanishes and so $\hat{H}_I^{-1}(A;M) = 0$. Also $H_I^0(A;M)$ has bounded I-torsion and so $\check{C}H_{-1}^I(A;H_I^0(A;M)) = 0$. In short, the local Tate cohomology is concentrated in degree zero where we have

$$\hat{H}_I^0(A;M) \cong \mathrm{Hom}(\check{C}H_I^0(A),H_I^1(A;M)).$$

This shows that under extremely weak hypotheses, any ideal in a ring of dimension 1 behaves in local Tate cohomology as though it contains a regular element.

One most unexpected property of the local Tate groups is that they are rational vector spaces in cases of interest. Indeed, assuming for the moment the multiplicative properties to be proved in Section 9, if A is the ring of integers in a number field and $I = \wp$ is a maximal ideal, the local Tate cohomology is the $\wp$-adic field, concentrated in degree zero.

Theorem 7.1 (Rationality theorem). *If A is a Noetherian ring of dimension 1 then $\widehat{H}_I^*(A)$ is concentrated in degree zero, and every regular element of A is invertible in $\widehat{H}_I^0(A)$. Furthermore, if I contains a regular element of A then $\widehat{H}_I^0(A)$ is the ring of fractions of the I-completion of A in which the set S of regular elements of A is inverted:*

$$\widehat{H}_I^0(A) = S^{-1}(A_I^\wedge).$$

Corollary 7.2. *If A is Noetherian of dimension 1 and torsion free as an abelian group the local Tate cohomology is a rational vector space.* □

Remark. Because the Burnside ring is one-dimensional it is easy to deduce from Corollary 7.2 that for finite groups of equivariance the Tate theory corresponding to equivariant K-theory is rational for any family of subgroups. This generalizes the proof given in [13] for the trivial family and provided the motivation for the Rationality Theorem. This deduction is explained in Appendix A.

Corollary 7.3. *If A is the ring of integers in the number field K and I is a maximal prime then the local Tate cohomology is the associated I-adic field.* □

Proof of Theorem 7.1. We have already proved that the local Tate cohomology is concentrated in degree zero, and that the Koszul–Čech exact sequence reads

$$0 \longrightarrow H_I^0(A_I^\wedge) \longrightarrow A_I^\wedge \longrightarrow \check{C}H_I^0(A_I^\wedge) \longrightarrow H_I^1(A_I^\wedge) \longrightarrow 0.$$

The final clause of Theorem 7.1 will follow by localising this exact sequence at S provided we show that every regular element f of A is invertible in $\widehat{H}_I^0(A)$.

We investigate the kernel and cokernel of the map $f : \widehat{H}_I^0(A) \to \widehat{H}_I^0(A)$, using the results of Section 6. Indeed we have seen that it is enough to verify the hypotheses

(i) $\check{C}H_I^0(A)$ has no f torsion
(ii) $\check{C}H_I^0(A)/f$ is finitely generated and
(iii) $H_I^1(A)$ is f divisible

of Lemma 6.1 and to show that

$$\mathrm{Supp}(\check{C}H_I^0(A)/f) \cap \mathrm{Supp}(\mathrm{ann}(f, H_I^1(A))) = \emptyset.$$

Proof of (i). Since $\check{C}H^0_I(A)$ is a submodule of a sum of modules $A[1/x]$ it is enough to show that if f is a regular element of A there is no f torsion in $A[1/x]$. But $f \cdot a/x^i = 0$ in $A[1/x]$ means that in A we have $x^n f a = 0$ for some n, and hence since f is regular, $x^n a = 0$.

Lemma 7.4. *If A is Noetherian of dimension* 1 *and f-torsion free then condition* (iii) *holds.*

I am grateful to R.Y. Sharp for suggesting the present proof of (iii), which is much simpler than my previous one.

Proof. To prove condition (iii) we take local cohomology of the short exact sequence

$$0 \longrightarrow A \xrightarrow{f} A \longrightarrow A/f \longrightarrow 0;$$

since A/f is zero-dimensional $H^1_I(A; A/f) = 0$, and hence multiplication by f is surjective on $H^1_I(A)$. □

Proof of (ii). The Koszul–Čech exact sequence gives the exact sequence

$$0 \longrightarrow H^0_I(A) \longrightarrow A \longrightarrow \check{C}H^0_I(A) \longrightarrow H^1_I(A) \longrightarrow 0.$$

By condition (iii) it follows that $\check{C}H^0_I(A)/f$ is a quotient of A/f; indeed the kernel K of the projection lies in an exact sequence $0 \to H^0_I(A)/f \to K \to \operatorname{ann}(f, H^1_I(A)) \to 0$. □

Lemma 7.5. *If A is Noetherian of dimension* 1 *and f-torsion free then*

$$\operatorname{Supp}(\check{C}H^0_I(A)/f) \cap \operatorname{Supp}(\operatorname{ann}(f, H^1_I(A))) = \emptyset.$$

Proof. We note first that any prime $\wp$ in either support must contain f and hence be maximal; also any prime in the support of $\operatorname{ann}(f, H^1_I(A))$ must contain I. Next, Čech and local cohomology commute with localisation, so we may suppose A is local with maximal ideal $\wp$. Since $\wp$ is then the only prime over f, f is $\wp$-primary and $(f) \supseteq \wp^e$ for some e. Since $\wp \supseteq I$ we have $I^e \subseteq (f)$ and the Čech cohomology may be calculated from a complex in which every term has f inverted, so $\check{C}H^0_I(A)/f$ is zero. □

We note that the argument shows that for rings of higher dimension the intersection of the supports in Lemma 7.5 consists of primes of codimension 2 or more. Also, since

$$\operatorname{ann}(f, H^1_I(A)) \subseteq H^0_{(f)}(A; H^1_I(A)) \subseteq H^1_{I+(f)}(A)$$

we see that $\operatorname{Supp}(\operatorname{ann}(f, H^1_I(A)))$ is empty if $I + (f)$ has depth at least two. Now if $I = (\chi)$ is principal and A is Noetherian the local Tate cohomology is again concentrated in codegree zero and the same proof will show that every regular element f for which $H^1_I(A/f) = 0$ is invertible if the ring is smooth in codimension 2.

It also seems worth illustrating the failure of the analogue of Theorem 7.1 for rings of larger dimension. For instance if $A = \mathbb{Z}[z, 1/z]$ and $I = (\chi)$, where $\chi = 1 - z$ the local Tate cohomology is $\mathbb{Z}[\![\chi]\!][1/\chi]$ concentrated in degree zero. This example arises in the calculation of S^1-equivariant Tate K-theory [13]. Evidently conditions (ii) and (iii) fail when f is an integer other than 1, 0 or -1. In this instance we see that it is really only integer primes that are at fault since the ring $\mathbb{Q}[z, 1/z]$ is a Noetherian domain of dimension 1, to which we may apply Theorem 7.1.

8. The regular case

To give some feel for these functors we consider another case with particularly simple behaviour: the case when I is generated by a regular sequence. Suppose then that I is generated by a regular sequence of length $d \geq 2$, so that $H^*_I(A)$ is concentrated in degree d. The universal coefficient theorem for local homology shows that in this case

$$H^I_j(A; M) = \mathrm{Ext}^{d-j}(H^d_I(A), M),$$

and $\mathrm{Ext}^m(H^d_I(A), M) = 0$ for $m > d$. The long exact sequence connecting Čech cohomology with local cohomology shows that Čech cohomology is concentrated in degrees 0 and $d-1$ where we have $\check{C}H^0_I(A) = A$ and $\check{C}H^{d-1}_I(A) = H^d_I(A)$. The universal coefficient theorem for Čech homology shows that for $1 \leq j \leq d-1$

$$\check{C}H^I_j(A; M) = \mathrm{Ext}^{d-1-j}(H^d_I(A), M),$$

whilst in low degrees we have the exact sequence

$$0 \longrightarrow \mathrm{Ext}^{d-1}(H^d_I(A), M) \longrightarrow \check{C}H^I_0(A; M) \longrightarrow M$$
$$\longrightarrow \mathrm{Ext}^d(H^d_I(A), M) \longrightarrow \check{C}H^I_{-1}(A; M) \longrightarrow 0.$$

These give rise to spectral sequences for local Tate cohomology as usual. We consider the still more special situation when I is also generated by an M-regular sequence (for instance if $M = A$). In this case the local cohomology of M is also concentrated in degree d, and Proposition 4.2 gives what must be regarded as the ideal situation:

$$\hat{H}^i_I(A; M) = \check{C}H^I_{d-1-i}(A; H^d_I(A; M)).$$

Therefore it follows that for $0 \leq i \leq d-2$ we have $\hat{H}^i_I(A) = \mathrm{Ext}^i(H^d_I(A), H^d_I(A; M))$, that there is a short exact sequence

$$0 \longrightarrow \mathrm{Ext}^{d-1}(H_I^d(A), H_I^d(A)) \longrightarrow \widehat{H}_I^{d-1}(A) \longrightarrow H_I^d(A)$$
$$\longrightarrow \mathrm{Ext}^d(H_I^d(A), H_I^d(A)) \longrightarrow 0,$$

and that the other local Tate groups are zero. In particular we want to emphasize the analogue of the display before Theorem 7.1:

$$\widehat{H}_I^0(A) \cong \mathrm{Hom}(\check{C}H_I^{d-1}(A), H_I^d(A)). \tag{5}$$

Since $H_I^d(A)$ has a tendency to be injective (for instance if I is the maximal ideal in a regular local ring) it is therefore typical for the local Tate cohomology to be concentrated in codegree 0 where we have (5) and codegree $d-1$ where we have

$$\widehat{H}_I^{d-1}(A) = H_I^d(A). \tag{6}$$

Local Tate cohomology tends to emphasize this sort of behaviour for any ideal and under weak hypotheses it appears rather as if the ideal defines a complete intersection with "orientable local normal bundle". Recall that if in addition M is tame $\widehat{H}_I^j(A;M) = \check{C}H_I^j(A;M_I^\wedge)$.

9. Products

In this section we run through the formal verification that local homology, Čech cohomology and local Tate cohomology of A are rings, and that the corresponding theory is module valued over this ring. Indeed the essence is that we have maps

$$\mu : \check{C}^\bullet(\boldsymbol{\alpha}) \otimes \check{C}^\bullet(\boldsymbol{\alpha}) \longrightarrow \check{C}^\bullet(\boldsymbol{\alpha})$$

and

$$\Delta : K_\infty^\bullet(\boldsymbol{\alpha}) \longrightarrow K_\infty^\bullet(\boldsymbol{\alpha}) \otimes K_\infty^\bullet(\boldsymbol{\alpha}),$$

which are compatible with the maps $A \to \check{C}^\bullet(\boldsymbol{\alpha})$ and $K_\infty^\bullet(\boldsymbol{\alpha}) \to A$. In fact both μ and Δ are homology isomorphisms; since the complexes concerned are complexes of free A-modules they are determined up to chain homotopy by specifying an inverse homology isomorphism, which we shall characterise up to chain homotopy in favourable cases.

For the inverse of Δ we recall that the augmentation $K_\infty^\bullet(\alpha) \to A$ gives various maps

$$m : K_\infty^\bullet(\alpha_1, \ldots, \alpha_n) \otimes K_\infty^\bullet(\alpha_1, \ldots, \alpha_n) \longrightarrow K_\infty^\bullet(\alpha_1, \ldots, \alpha_n)$$

each of which is a homology isomorphism.

Lemma 9.1. *There is a homotopy class of maps*

$$K^{\bullet}_{\infty}(\boldsymbol{\alpha}) \otimes K^{\bullet}_{\infty}(\boldsymbol{\alpha}) \longrightarrow K^{\bullet}_{\infty}(\boldsymbol{\alpha})$$

lying over the natural map $A \otimes A \longrightarrow A$; *it is unique if* $\check{C}H^I_0(A) = 0$.

Proof. Consider the total complex of $\mathrm{Hom}(K^{\bullet}_{\infty}(\boldsymbol{\alpha}), K^{\bullet}_{\infty}(\boldsymbol{\alpha}))$; by Lemma 4.6(iv) its homology is equal to that of $\mathrm{Hom}(K^{\bullet}_{\infty}(\boldsymbol{\alpha}), A)$, namely the local homology of A. □

We choose one of these homology isomorphisms m. Since $K^{\bullet}_{\infty}(\boldsymbol{\alpha})$ is projective we may choose a homotopy inverse to m and refer to it as Δ.

To construct μ we observe by Lemma 4.6(ii) that the augmentation induces a homology isomorphism

$$d : \check{C}^{\bullet}(\boldsymbol{\alpha}) \otimes A \longrightarrow \check{C}^{\bullet}(\boldsymbol{\alpha}) \otimes \check{C}^{\bullet}(\boldsymbol{\alpha}).$$

Lemma 9.2. *There is a homotopy class of maps*

$$\check{C}^{\bullet}(\boldsymbol{\alpha}) \longrightarrow \check{C}^{\bullet}(\boldsymbol{\alpha}) \otimes \check{C}^{\bullet}(\boldsymbol{\alpha})$$

lying under the natural map $A \longrightarrow A \otimes A$; *it is unique if* $H^0_I(A) = 0$.

Proof. By Lemma 4.6(iv) the homology of $\mathrm{Hom}(\check{C}^{\bullet}(\boldsymbol{\alpha}), \check{C}^{\bullet}(\boldsymbol{\alpha}))$ is equal to that of $\mathrm{Hom}(A, \check{C}^{\bullet}(\boldsymbol{\alpha})) = \check{C}^{\bullet}(\boldsymbol{\alpha})$. □

Since $\check{C}^{\bullet}(\boldsymbol{\alpha}) \otimes \check{C}^{\bullet}(\boldsymbol{\alpha})$ is projective we may choose a homotopy inverse to d and refer to it as μ.

In order to feel confident that the multiplication we define is independent of arbitrary choice we must assume that $\check{C}H^I_0(A) = H^0_I(A) = 0$, as happens for instance if A is Noetherian and I contains a regular element. Nonetheless, if we are prepared to make arbitrary choices we do obtain ring structures quite generally.

Corollary 9.3. *For any ring A and any finitely generated ideal I*

(i) $H^I_*(A)$ *is a graded ring and* $H^I_*(A; M)$ *is naturally a module over it. The natural map* $A \to H^I_*(A)$ *is a map of rings.*

(ii) $\check{C}H^*_I(A)$ *is a graded ring and* $\check{C}H^*_I(A; M)$ *is naturally a module over it. The natural map* $A \to \check{C}H^*_I(A)$ *is a map of rings.*

(iii) $\widehat{H}^*_I(A)$ *is a graded ring and* $\widehat{H}^*_I(A; M)$ *is naturally a module over it. The natural map* $A \to \widehat{H}^*_I(A)$ *is a map of rings.* □

Remark. It is worth noting that whilst (iii) is an immediate consequence of the above constructions when we use the first form $T^{\bullet}(A)$ it appears almost implausible when we use $TT^{\bullet}(A)$. For this reason the multiplicativity of certain forms of local Tate cohomology [13] was an unexpected surprise for those topologists who came across it via the analogue of $TT^\bullet(A)$.

A typical noncalculational consequence of Corollary 9.3 is immediate from Theorem 7.1.

Corollary 9.4. *Under the hypotheses of Theorem* 7.1 *the regular elements of A act invertibly on $\widehat{H}_I^*(A;M)$ for any module M.* □

Appendix A. Rationality of Tate spectra and the local cohomology theorem

In this and the following appendix we apply the above algebra to G-equivariant cohomology for a finite group G. It is necessary to work in a category where all such theories are represented, so we work in the Lewis–May stable homotopy category of G-spectra [17]. For a brief introduction and much relevant background see [13].

Recall that a family $\mathcal{F}$ is a collection of subgroups of G closed under passage to conjugates and subgroups, and that a universal space $E\mathcal{F}$ is characterised by the condition that $(E\mathcal{F})^H$ is contractible if $H \in \mathcal{F}$ and empty otherwise. In particular EG is the universal space for the family $\{1\}$. Given an equivariant homology theory $k_*^G(-)$ we may attempt to evaluate it on the universal space $E\mathcal{F}_+$. It turns out that in certain well behaved cases it is essentially the local cohomology of the coefficient module k_*^G at the ideal $I\mathcal{F}$ of the Burnside ring $A(G) = [S^0, S^0]^G$ defined by

$$I\mathcal{F} = \bigcap_{H \in \mathcal{F}} \ker\{A(G) \longrightarrow A(H)\}.$$

This may be proved by constructing a G-spectrum $M(I\mathcal{F})$ whose homology is obviously the local cohomology and a map

$$c : E\mathcal{F}_+ \longrightarrow M(I\mathcal{F}),$$

and then proving c becomes an equivalence of G-spectra when smashed with k. If this holds we say the local cohomology theorem is true for equivariant k-homology.

On the other hand we may construct a new theory $t_{\mathcal{F}}(k)_*^G(-)$ from $k_*^G(-)$ by a geometric version of the Tate construction [13]: the representing G-spectrum is defined by $t_{\mathcal{F}}(k) = F(E\mathcal{F}_+, k) \wedge \tilde{E}\mathcal{F}$, where $\tilde{E}\mathcal{F}$ is the unreduced suspension of $E\mathcal{F}_+$. It is observed in [13] that the truth of the local cohomology theorem is closely related to the rationality of the Tate theory: indeed this relationship was proved in [13] if $\mathcal{F} = \{1\}$ or for arbitrary $\mathcal{F}$ if k is periodic equivariant K-theory. The purpose of the algebra presented in the body of the paper was to establish the close relationship for arbitrary families and cohomology theories.

These constructions only work when the Burnside ring can be used as a good approximation to the coefficient ring k_*^G, which only happens if k_*^G is essentially one-dimensional. The algebra for rings of dimension more than one is fundamental to understanding what the appropriate version of the local cohomology theorem implies about the Tate spectrum when k_*^G is of dimension more than one. This is relevant to equivariant K-theory for positive dimensional groups (as in the Appendix to [10]) and to derivatives of equivariant bordism. For example the author presumes the fact that the Tate spectrum of a v_n-periodic complex oriented theory is v_{n-1}-periodic [15] is an instance of the higher-dimensional rationality theorem, although our ignorance of the coefficient rings of equivariant theories prevents us justifying this. The full generality of the algebra is used since there are Eilenberg–MacLane ring spectra HA for any ring A. The relevant topological constructions (directly analogous to those below) would require the use of highly structured equivariant ring spectra with the expected properties, so applications of the algebra depend on the development of a suitable theory. Recent work of Elmendorf, Kriz and May [8] in the nonequivariant case appear to generalize directly and to give the required properties, but details have yet to be worked out.

We may now give a little more detail by defining the local cohomology spectrum $M(I)$. For this we geometrically realise the stable Koszul complex: for a principal ideal $I = (\alpha)$ we take $M(I)$ to be the fibre of $S^0 \longrightarrow S^0[1/\alpha]$ and if $I = (\alpha_1, \ldots, \alpha_n)$ we take $M(I) = M(\alpha_1) \wedge \ldots \wedge M(\alpha_n)$. By construction we have a map $M(I) \longrightarrow S^0$, and we denote the cofibre by $\tilde{M}(I)$. The existence of the map c is then immediate because the map $M(I\mathcal{F}) \longrightarrow S^0$ is an H-equivalence for $H \in \mathcal{F}$ since $I\downarrow_H = (0)$ for $H \in \mathcal{F}$.

After the algebra above it is impossible to resist defining the topological local Tate spectra by

$$lt_I(k) = F(M(I), k) \wedge \tilde{M}(I).$$

The direct parallel between the algebraic and geometric constructions leads inevitably to a calculation of homotopy groups.

Lemma A.1. *If X is a finite spectrum we have a short exact sequence*

$$0 \longrightarrow \hat{H}_I^{-1}(k_G^{n-1}X) \longrightarrow lt_I(k)_G^n(X) \longrightarrow \hat{H}_I^0(k_G^n X) \longrightarrow 0.$$

Proof. Since X is finite we are calculating the homotopy groups of $F(M(I), F(X,k)) \wedge \tilde{M}(I)$. For this we use skeletal filtrations of $M(I)$ and $\tilde{M}(I)$ which correspond to the Koszul and Čech complexes of Section 1. Forming the total filtration of the function spectrum in the usual way, the resulting spectral sequence has E_1 term precisely corresponding to the chain complex used in the definition of algebraic local Tate cohomology. □

Corollary A.2. *The spectra $lt_I(k)$ are rational.*

Proof. This is immediate from the exact sequence above (Lemma A.1) and the Rationality Theorem (Theorem 7.1, Corollary 9.4) since the Burnside ring is one-dimensional. □

Since G is finite all rational G-spectra split [13, A.1].

Corollary A.3. *The spectra $lt_I(k)$ split as a product of Eilenberg–Mac Lane spectra:*

$$lt_I(k) \simeq \prod_{n\in\mathbb{Z}} \Sigma^n HT_n$$

where the rational Mackey functor $T_n = \widehat{H}_I^{-1}(k^\bullet_{n-1}) \oplus \widehat{H}_I^0(k^\bullet_n)$. □

The reason for the framework of this paper is the following lemma, whose proof is completely formal.

Lemma A.4. *If k is a ring spectrum and the local cohomology theorem holds for k-homology then*

$$t_{\mathcal{F}}(k) \simeq lt_{I\mathcal{F}}(k).$$

Proof. The local cohomology theorem states that the map c is an isomorphism of k-homology, so its cofibre is $k_*^G(-)$-acyclic. However if Y is any k-acyclic spectrum and m is a k-module then $m_*^G Y = 0$ since $m \wedge Y$ is a retract of $m \wedge k \wedge Y$ and $m_G^* Y = 0$ since any map $Y \to m$ factors as $Y \to Y \wedge k \to m \wedge k \to m$. Since k and any spectrum $F(Y,k)$ are k-modules, all four maps in the diagram

$$\begin{array}{ccccc} & F(M(I\mathcal{F}),k)\wedge \tilde{E}\mathcal{F} & \longrightarrow & F(E\mathcal{F}_+,k)\wedge \tilde{E}\mathcal{F} & = t_{\mathcal{F}}(k) \\ & \downarrow & & \downarrow & \\ lt_{I\mathcal{F}}(k) = & F(M(I\mathcal{F}),k)\wedge \tilde{M}(I\mathcal{F}) & \longrightarrow & F(E\mathcal{F}_+,k)\wedge \tilde{M}(I\mathcal{F}) & \end{array} \tag{A.1}$$

are equivalences. □

From Corollary A.2 we then have immediately the main conclusion.

Corollary A.5. *If k is a ring spectrum and the local cohomology theorem holds for the homology theory it represents then $t_{\mathcal{F}}(k)$ is a rational spectrum.* □

This happens for instance with equivariant K-theory [10,13] and with p-local and p-adic periodic Morava K-theories [15] and the resulting rational Tate spectra are far from trivial. The corresponding fact for mod p theories says they have trivial Tate spectra.

We pause here to insert an example showing the ring hypothesis is necessary.

Example A.6. It is obvious that the local cohomology theorem holds for $k = G/H_+$ with $H \in \mathcal{F}$. The class of spectra for which the theorem holds is also clearly closed under cofibre sequences and arbitrary wedges. It therefore holds for $k = E\mathcal{F}_+$. However $E\mathcal{F}_+$ is $I\mathcal{F}$-complete [13, 18.2] whilst $F(E\mathcal{F}_+, E\mathcal{F}_+) \simeq F(E\mathcal{F}_+, S^0)$ is the $I\mathcal{F}$-completion of S^0 by the generalized Segal conjecture [1,12]. It is therefore only in the rare event that $E\mathcal{F}_+ \simeq (S^0)^{\wedge}_{I\mathcal{F}}$ that the completion theorem holds for $k = E\mathcal{F}_+$.

For completeness we recall [13, 18.10] for the elementary partial converse.

Lemma A.7. *If k is a ring spectrum, the completion theorem holds for k and $t_{\mathcal{F}}(k)$ is a rational spectrum then the local cohomology theorem holds for $c_{\mathcal{F}}(k) = F(E\mathcal{F}_+, k)$.*

Proof. Since $I\mathcal{F}$ is rationally generated by idempotents, c is a rational equivalence [13, (A.14)] and so the vertical maps in Corollary A.5 are rational equivalences. Since k is a ring spectrum, the bottom horizontal map is a map of ring spectra with rational domain, and so the codomain is also rational. If $t_{\mathcal{F}}(k)$ is rational the right-hand vertical is an equivalence. If the completion theorem holds the horizontals are also equivalences. $\square$

Appendix B. The local Warwick duality and axiomatic Tate theory

In Appendix A we used the construction of the Tate spectra with the role of the universal space $E\mathcal{F}_+$ played by the local cohomology spectrum $M(I)$ for some ideal I in the Burnside ring. It seems worth making a couple of elementary observations about this type of construction.

We suppose given a G-spectrum E and a map $\varepsilon : E \to S^0$ and we let C denote the cofibre of ε. We may consider the E-completion spectrum $c_E(k) = F(E, k)$, the E-free spectrum $f_E'(k) = b_E(k) \wedge E$ and the E-Tate spectrum $t_E(k) = c_E(k) \wedge C$. These fit into a diagram

$$\begin{array}{ccccc}
k \wedge E & \longrightarrow & k & \longrightarrow & k \wedge C \\
\downarrow & & \downarrow & & \downarrow \\
F(E,k) \wedge E & \longrightarrow & F(E,k) & \longrightarrow & F(E,k) \wedge C \\
\| & & \| & & \| \\
f_E'(k) & & c_E(k) & & t_E(k)
\end{array}$$

just like (C) of [13]. To proceed further we consider a condition on the augmented spectrum E, stating that E and C are suitably complementary.

Condition B.1. (i) $C \wedge E \simeq *$,
(ii) $F(C,T) \wedge E \simeq *$ for all T and
(iii) $F(E, T \wedge C) \simeq *$ for all T.

We shall show that the familiar spectra $E\mathcal{F}_+$ and $M(I)$ satisfy these conditions using a lemma for suitably approximated E.

Lemma B.2. *Any augmented spectrum E which admits an inductive filtration with subquotients which are wedges of finite spectra Q so that $C \wedge Q \simeq *$ and $C \wedge DQ \simeq *$ satisfies Condition* B.1.

Proof. Clearly (i) holds by cofibre sequences and direct limits. For finite Q we find $F(C,T) \wedge Q \simeq F(C \wedge DQ, T)$ so (ii) holds for each Q; it therefore holds for E by cofibre sequences and passage to direct limits. Finally for finite Q we again have $F(Q, T \wedge C) \simeq T \wedge C \wedge DQ$ so (iii) also holds for Q and hence for E itself by cofibre sequences and passage to inverse limits. □

Example B.3. (i) The subquotients of the spectra $E\mathcal{F}_+$ in the skeletal filtration are wedges of suspensions of the self-dual spectra G/H_+ for $H \in \mathcal{F}$ and $\tilde{E}\mathcal{F}$ is H-contractible. Therefore $E\mathcal{F}_+$ satisfies the hypotheses of Lemma B.2 and hence Condition B.1. For compact Lie groups of positive dimension the dual of G/H_+ is the extended spectrum $G \ltimes_H S^{L(H)}$ where $L(H)$ is the tangent H-representation to G/H at eH so $E\mathcal{F}_+$ still satisfies the hypotheses of Lemma B.2.

(ii) The spectrum $M(I)$ is a smash product of spectra $S^{-1}/\alpha^{\infty} = \underrightarrow{\mathrm{holim}}_k S^{-1}/\alpha^k$ and therefore admits a filtration with subquotients which are suspensions of $S^{-1}/\alpha_1 \wedge \cdots \wedge S^{-1}/\alpha_n$ (which is self-dual up to suspension) where $I = (\alpha_1, \ldots, \alpha_n)$. On the other hand $\tilde{M}(I)$admits a finite filtration with subquotients of form $S^0[1/\alpha] \wedge Y$ for $\alpha \in I$ and α is nilpotent on $S^{-1}/\alpha_1 \wedge \ldots \wedge S^{-1}/\alpha_n$. Therefore $M(I)$ satisfies the hypotheses of the first example and hence Condition B.1.

Theorem B.4 (Warwick duality). *For any augmented spectrum E satisfying Condition* B.1 *and any spectrum X there is an equivalence*

$$F(E,X) \wedge C \simeq F(C, X \wedge \Sigma E).$$

The point is that the original definition on the left is most suitable for use in homology and the second form on the right is best for cohomology.

Proof. As familiar from [13, 2.6] and the body of the paper the proof is obtained from a string of more elementary equivalences:

$$F(E,X)\wedge C \overset{(a)}{\simeq} F(S^0, F(E,X)\wedge C) \overset{(b)}{\simeq} F(C, F(E,X)\wedge C)$$
$$\overset{(c)}{\simeq} F(C, F(E,X)\wedge E) \overset{(d)}{\simeq} F(C, X\wedge \Sigma E).$$

Of course the equivalence (a) is a basic property of S^0. The equivalence (b) follows since $F(E, F(E,X)\wedge C) \simeq *$ by B.1(iii) with $T = F(E,X)$. (c) follows since $F(C, F(E,X)) \simeq F(C\wedge E, X)$ and $C\wedge E \simeq *$ by B.1(i) and (d) is immediate from B.1(ii). □

Remark. We note that the form of proof via Lemma B.2, Example B.3 and Theorem B.4 was precisely that used in Section 4.

Combining Example B.3 and Theorem B.4 we have the desired conclusion.

Corollary B.5. *We have equivalences*
(i) $t_{\mathcal{F}}(k) \simeq F(\tilde{E}\mathcal{F}, k\wedge \Sigma E\mathcal{F}_+)$ *and*
(ii) $lt_I(k) \simeq F(\tilde{M}(I), k\wedge \Sigma M(I))$. □

It is also worth noting that B.1(iii) implies in particular that $c_E(t_E(k)) \simeq *$ for any k.

Corollary B.6. *For any spectrum k and any ideal I*

$$(lt_I(k))^{\wedge}_I \simeq *. \qquad \square$$

This leads to a great contrast between bounded and unbounded spectra for which the local cohomology theorem holds. In the case of the periodic K-theory spectra Tate spectra are highly nontrivial. On the other hand it is observed in [13, 18.2] that the Tate spectrum of a bounded below spectrum was $I\mathcal{F}$-complete, so by Corollary B.6 and Lemma A.4 we deduce these phenomena degenerate.

Corollary B.7. *If k is a bounded below ring spectrum for which the local cohomology theorem holds then*

$$t_{\mathcal{F}}(k) \simeq lt_{I\mathcal{F}}(k) \simeq *. \qquad \square$$

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