THE POWER OF MOD P BOREL HOMOLOGY

by

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TO PROFESSOR H. TODA ON HIS 60^{TH} BIRTHDAY

1. Setting the scene.

When seeking an analogue of ordinary cohomology for spaces with a specified finite group G of symmetries we may think first of Bredon's cohomology for obstruction theory [7], and its extention to the stable context [13], [14], [21]. However the difficulty of calculating with Bredon cohomology is a severe drawback. We shall consider instead a substitute which had already been developed by Borel and applied to give an alternative approach to Smith theory [6]. This is the theory defined for based G-spaces X by

$$b^*(X;A) = H^*(EG_+ \wedge_G X;A)$$

where EG_+ is a nonequivariantly contractible free G-space with a G-fixed basepoint added and A is an abelian group of coefficients. It is referred to as Borel cohomology (the term "ordinary equivariant cohomology" is also used but is liable to lead to confusion). Because X is made free before the quotient is taken it is much easier to work with than most Bredon theories.

One might expect this to make it a less powerful theory, but it turns out to be remarkably useful, especially for elementary abelian groups G and finite X. Indeed the remarkable recent theorem of Dwyer-Wilkerson [9] shows that in this case the localisation theorem can be refined to give a functorial description of the mod p cohomology of the fixed point set X^G in terms of the mod p Borel cohomology $b^*(X)$ of X. Indeed *the particular potency on finite complexes suggests that perhaps the power really comes from homology*: it is geometrically possible for homology to be strictly more powerful than cohomology since only infinite spaces may be essential but nonequivariantly contractible.

Various other known theorems give further indications of the status of the theory. For example a well known theorem of Quillen [18] considers more general groups G of equivariance. It states that the Borel cohomology of a finite space X is determined by the fixed point sets X^E for the various elementary abelian subgroups E of G. This shows that *Borel homology is somewhat limited to elementary abelian groups*. Moreover the general usefulness of the theory in both the elementary abelian and general cases has been established in the work of May [16] and Adams [4] on the Segal conjecture.

We shall attempt to put these various results in perspective by outlining the existence of a convergent and calculable Adams spectral sequence based on Borel homology for elementary abelian groups of equivariance and commenting on the extension of this result to more general groups. The convergence theorem should be contrasted with the weaker ones available for the Adams spectral sequences based on Borel and coBorel cohomology [10].

In Section 2 we recall how an Adams spectral sequence is constructed, and state our main theorem ((2.7)) . In Section 3 we outline the proof of the convergence theorem. In Section 4 we explain why the coalgebra of cooperations is flat over the coefficient ring for elementary abelian groups; this is the essential step in establishing the calculability of the spectral sequence. Further details and applications will appear elsewhere.

Henceforth G is a p-group and mod p coefficients are suppressed.

2. Constructing an Adams Spectral Sequence

We now move into the stable world in which suspension by every real representation has been forced to be an isomorphism [3], [12]. To be at ease in this world we need systematic methods of calculation and from the nonequivariant context we recall that Adams spectral sequences provide such methods. Indeed on Adams spectral sequence is a machine to perform obstruction theory formulated in the language of homological algebra; thus to construct an Adams spectral sequence one attempts to perform homological resolutions of geometric objects.

In most nonequivariant contexts one is either using mod p homology (in which case H_*X is always projective over $H_* = \mathbf{F}_p$) or one is simply interested in $\pi_*(Y) = [S^0,Y]_*$ when E_*S^0 is projective over E_*. Hence one can get away with just resolving Y by relative injectives, without the need to resolve X at all.

In the equivariant context Bredon homology with constant coefficients is very hard to use, and for other theories E the homology of a space will not always be projective. Nor can we be satisfied with the restriction to S^0 since at the very least we are interested in $\pi_*^H(Y) = [G/H_+,Y]_*^G$. Hence we are forced to resolve X by b_*-projectives.

Thus we must attempt to form a resolution.

(2.1)
$$\begin{array}{ccccccc} X = X^0 & \longrightarrow & X^1 & \longrightarrow & X^2 & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ P^0 & & P^1 & & P^2 & & \end{array}$$

where b_*P^i is b_*-protective and $P^i \longrightarrow X^i$ is onto in $b_*(\cdot)$. Hence

$0 \leftarrow b_*X \leftarrow b_*P^0 \leftarrow b_*S^{-1}P^1 \leftarrow \ldots$ is a resolution of b_*X by b_*-projectives in the usual sense.

This has been proved to be possible when G is elementary abelian, but more generally I do not know if it is possible even for $G = \mathbb{Z}/4$. If such a resolution (2.1) exists, all the usual UCT's exist and converge when G is a p-group and we use mod p coefficients .

Still, suppose that for G such a resolution exists and let $X^\infty = \underrightarrow{\text{holim}}_s X^s$ and let $\tilde{X}$ be defined by the cofibration

$$\tilde{X} \longrightarrow X \longrightarrow X^\infty.$$

Here $b_*X^\infty = 0$ so $b_* \tilde{X} = b_*X$.

We also form an Adams tower as usual

$$\begin{array}{ccccccc} Y = Y_0 & \leftarrow & Y_1 & \leftarrow & Y_2 & \leftarrow & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \\ Q_0 & & Q_1 & & Q_2 & & \end{array} \tag{2.2}$$

where $Q_s \simeq b \wedge Y_s$ and $Y_s \longrightarrow b \wedge Y_s$ is obtained via the unit of the ring spectrum b.

The way to construct an Adams spectral sequence with the right Adams filtration is well known. We combine these and form the filtration

$$W_s = \bigcup_{i+j=s} F(X^i, Y_j)$$

where the lattice of function spectra has been converted to one of inclusions.

This has

$$E_1^{s,*} = \bigoplus_{i+j=s} [P^i, Q_j]_*^G$$

and, by the Weak UCT,

$$[P^i, Q_j]_*^G = \text{Hom}^*_{b_*}(b_*P^i, b_*Q_j). \tag{2.3}$$

This of course follows from the UCT, but one can perhaps prove it directly. All we need for the E_2-term to work is the Weak Künneth Theorem

$$b_*(b \wedge Y_j) = b_*b \otimes_{b_*} b_*Y_j \tag{2.4}$$

Again this follows from the UCT or directly provided we check that b_*b is b_*-flat. We discuss this in §4.

In any case, given (2.4) b_*b is a b_*-comodule and $0 \longrightarrow b_*Y \longrightarrow b_*Q_0 \longrightarrow b_*SQ_1 \longrightarrow \ldots$ is a resolution by relative injective b_*-comodules. Hence by (2.3)

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{b_*b}(b_*X,\ b_*Y)$$

as usual.

Now there is the central matter of convergence.

By an argument based on Ravenel's [19] we reduce the problem to a more familiar one.

<u>Proposition (2.5)</u>: We have a cofibration

$$F(X,\ \underset{\overleftarrow{s}}{\mathrm{holim}}\ Y_s) \longrightarrow \underset{\overleftarrow{s}}{\mathrm{holim}}\ W_s \longrightarrow F(X^\infty, Y/\underset{\overleftarrow{s}}{\mathrm{holim}}\ Y_s)$$

Now the double filtration spectral sequence converges iff the term $\underset{\overleftarrow{s}}{\mathrm{holim}}\ W_s \simeq *$. However the usual type of convergence theorem says $\underset{\overleftarrow{s}}{\mathrm{holim}}\ Y_s \simeq *$. Fortunately in our case X^∞ is b_*-acyclic and $Y/\underset{\overleftarrow{s}}{\mathrm{holim}}\ Y_s$ is b-complete and so we have the following corollary telling us that the convergence problem is the same whether or not the resolution (2.1) of X exists.

<u>Corollary (2.6)</u>: $$\underset{\overleftarrow{s}}{\mathrm{holim}}\ W_s \simeq F(X,\ \underset{\overleftarrow{s}}{\mathrm{holim}}\ Y_s).$$

Finally we may state the main theorem.

<u>Theorem (2.7)</u>: If G is elementary abelian and Y is p-complete and bounded below then there is a convergent Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{b_*b}\ (b_*X, b_*Y) \Longrightarrow [X,Y]_*^G$$

<u>Remarks</u>: We have noted above that the main points in the proof are as follows

(1) Resolving X by b_*-projectives

(2) Proving weak Künneth and universal coefficient theorems.

(3) Proving b_*b is flat over b_*

and (4) Proving convergence.

We shall only comment here on (3) and (4). The prospects of proving (1) and (3) for more general p-groups are indistinct. On the other hand (2) is unlikely to give problems, and in (4) the only difficulty is in formulating palatable conditions.

One of the motivations for constructing the spectral sequence was a desire to understand relationship of the algebra to geometry in the proof of the Segal conjecture for elementary abelian groups. In the rank 1 case at least the Ext isomorphism of Lin [15] and Gunawardena [11] is the map of E_2-terms of the above spectral sequence induced by the relevant map of spaces. Thus the spectral sequence is a glass for the study of equivariant topology which automatically takes account of Segal invariance phenomena at the very outset.

3. Convergence of the spectral sequence

We have seen that whether or not we can construct the best behaved version of the spectral sequence, the convergence problem is that of proving $\underleftarrow{\text{holim}}_s Y_s \simeq *$ where the Y_s are terms in the Adams tower

$$\begin{array}{ccc} & \downarrow & \\ & Y_2 \longrightarrow Q_2 & \\ & \downarrow & \\ & Y_1 \longrightarrow Q_1 & \\ & \downarrow & \\ Y = & Y_0 \longrightarrow Q_0 & . \end{array}$$

In the standard tower $Q_s \simeq Y_s \wedge b$, but more generally we allow any b-injective resolution tower in the sense of Miller [17]. Since any two such resolutions are chain equivalent there is a map of spectral sequences inducing an iso of E_2-terms and hence the homotopy type of $\underleftarrow{\text{holim}}_s Y_s$ is independent of which tower we use.

Now in equivariant topology it is standard practice to reduce to less equivariant problems by passage to fixed points, and it is no loss of generality to restrict ourselves to normal subgroups N. This is true stably too but there are two ways of getting a G/N-spectrum from a G-spectrum. We shall use features of each of them, so we recall the relevant properties from [12].

The Lewis-May categorical fixed point functor $F^N(\cdot)$

We use this name since the functor's essential property is the isomorphism

$$\pi_*^N(X) = \pi_*(F^N X)$$

Indeed $F^N(\cdot)$ is a right adjoint.

$$[X, F^N Y]^{G/N} = [\epsilon^* X, Y]^G$$

where $\epsilon : G \to G/N$ is the quotient [12; II.4.4]. Thus $F^N(\cdot)$ has certain categorical properties of the fixed point functor, and in particular it preserves products. It does not have the geometric properties; for example it does not preserve smash products, and the fixed points of a free spectrum are rarely trivial - they have features of the quotient.

The geometric fixed point functor, $\Phi^N(\cdot)$

This fixed point functor extends the unstable one in the sense that for spaces X we have an equivalence $\Sigma^\infty X^N \simeq \Phi^N \Sigma^\infty X$. Accordingly it behaves in a more familar fashion. It preserves smash products and vanishes on suitably free spectra. It is to this that we refer with the name "geometric" and not to any means of definition. Finally it is represented in the sense that if $[\not\supseteq N]$ denotes the family of subgroups not containing N and $\tilde{E}[\not\supseteq]$ is the unreduced suspension of the classifying space for $[\not\supseteq]$ (i.e. "the part of S^0 over N") we have the following two facts which also combine to show Φ^N is a left adjoint.

(1) For any G-spectrum X $\quad \tilde{E}[\not\supseteq N] \wedge X \simeq \tilde{E}[\not\supseteq N] \bar{\wedge} \Phi^N X$

[The symbol $\bar{\wedge}$ allows for the fact that if we regard $\Phi^N X$ as a "G-spectrum" via the quotient it will not have structure maps for all representations of G. After smashing termwise with the space $\tilde{E}[\not\supseteq N]$ it is easy to build in the remaining representations to obtain the G-spectrum $\tilde{E}[\not\supseteq N] \bar{\wedge} \Phi^N X$ in (1).]
and

(2) For any G-spectra X and Y $\quad [X, \tilde{E}[\not\supseteq N] \wedge Y]^G = [\Phi^N X, \Phi^N Y]^{G/N}$.

Furthermore the relationship between the functors is close. The Lewis-May functor F^N is more general in that

$$\Phi^N(X) = F^N(\tilde{E}[\not\supseteq N] \wedge X)$$

i.e. $\Phi^N(X)$ is the F^N fixed points of the part of X over N.

Now we are in a position to state our theorem.

Theorem (3.1): If G is an elementary abelian p-group and Y is a bounded below spectrum which is p-complete then $\underleftarrow{\operatorname{holim}}_s Y_s \simeq *$.

The proof is by induction on the group order. The case G = 1 is covered by Adams' original theorem [1], [2]. Since b is H-equivariantly the Borel spectrum

$$[G/H_+, \underleftarrow{\operatorname{holim}}_s Y_s]^G_* = 0$$

if H ≠ G by induction.

Hence we are reduced to consideration of $F^G \underleftarrow{\operatorname{holim}}_s Y_s \simeq \underleftarrow{\operatorname{holim}}_s F^G Y_s$. Now it is very hard to analyse $F^G Y_s$, so we shall show in (3.3) that it is enough to consider spectra Y of various special forms. Indeed we may assume that Y is concentrated over N (it is H-contractible for any subgroup not containing N) and is as free as is consistent with this, i.e.

$$Y \simeq \tilde{E}[\not\supseteq N] \wedge EG/N_+ \wedge Y$$

for various normal subgroups N. Then we can say $F^G = F^{G/N} \circ F^N$ and

$$\begin{aligned} F^N(Y_s \wedge \tilde{E}[\not\supseteq N] \wedge EG/N_+) &\simeq \Phi^N(Y_s \wedge EG/N_+) \\ &\simeq \Phi^N(Y_s) \wedge EG/N_+ \end{aligned}$$

Now Φ^N applied to the original tower gives an Adams resolution by the ring spectrum $\Phi^N b$. So the proof amounts to the verification that this tower converges, and we shall do this by comparison with the G/N Borel Adams tower.

For this we may work in Miller's context and note that the condition we require for $\Phi^N(\cdot) \wedge EG/N_+$ of the original tower to be a $b_{G/N}$ tower is that

$$\begin{array}{ccc} b_{G/N} \wedge S^0 \wedge EG/N_+ & & \\ \downarrow & \overset{1}{\searrow} & \\ b_{G/N} \wedge \Phi^N b_G \wedge EG/N_+ & \dashrightarrow & b_{G/N} \wedge S^0 \wedge EG/N_+ \end{array}$$

have a solution. We remark that it is futile to try to prove that $b_{G/N} \wedge EG/N_+$ is a $\Phi^N b_G$ module spectrum; even though $(b_{G/N} \wedge EG/N_+)_*$ $= H_*(BG/N_+)$ is a module over H^*BG_+ it is not over $H^*BG_+[e_N^{-1}]$. It seems to be sufficient to find a good map $\Phi^N b_G \wedge EG/N_+ \longrightarrow b_{G/N} \wedge EG/N_+$ (maps in the other direction are easy to come by). We note that this is certainly impossible in general since $\Phi^G b_G$ is contractible when G is not elementary abelian [20].

In our case we find

$$G \cong N \times G/N$$

and it is quite easy to see that we have the following.

<u>Lemma 3.2</u>: $\Phi^N b_G$ is equivalent to a wedge of copies of $b_{G/N}$ (one for each element of an $\mathbb{F}_p$-basis of $H^*(BN_+)[e_N^{-1}]$).

Hence the resolution $\{ \Phi^N(Y_s) \wedge EG/N_+ \}_s$ is a highly nonminimal resolution of $\Phi^N Y \wedge EG/N_+$.

Hence

$$\underset{\overleftarrow{s}}{\text{holim}}\{\Phi^N(Y_s) \wedge EG/N_+\} \simeq \underset{\overleftarrow{s}}{\text{holim}}(\Phi^N(Y) \wedge EG/N_+)_s$$

Now if $N \neq 1$ this is contractible by induction.

If $N = 1$ we have to consider

$$\underset{\overleftarrow{s}}{\text{holim}}(Y_s \wedge EG_+).$$

To see this is contractible we first check that the result is G-free. Once we know this it is enough to check $\underset{\overleftarrow{s}}{\text{holim}}\ (Y_s \wedge EG_+)$ is nonequivariantly contractible and this follows from Adams' original result [1], [2].

Now whenever the $Y_s \wedge EG_+$ are uniformly bounded below, the natural map $(\prod_s Y_s) \wedge EG_+ \longrightarrow \prod_s (Y_s \wedge EG_+)$ is an equivalence. This is because in the diagram

$$\begin{array}{ccc} \left[T,\ \prod_s(Y_s \wedge EG_+)\right]^G & \overset{\cong}{\longleftarrow} & \left[T,\ \prod_s(Y_s \wedge EG_+^{(n)})\right]^G \\ \uparrow & & \uparrow \cong \\ \left[T,\ (\prod_s Y_s) \wedge EG_+\right]^G & \overset{\cong}{\longleftarrow} & \left[T,\ (\prod_s Y_s) \wedge EG_+^{(n)}\right]^G \end{array}$$

the horizontals are iso if $\dim T < c + n$ where c is the connectivity of the Y_s. It follows that $\coprod_s (Y_s \wedge EG_+)$ is free.

Finally to see $Y_s \wedge EG_+$ are uniformly bounded below we note that they are G-free and hence that we may appeal to the classical nonequivariant case.

It remains to justify the sufficiency of considering Y to be of the given special form. For this let G be a group with all subgroups normal.

Lemma (3.3): S^0 may be built from spaces $\tilde{E}[\not\supseteq N] \wedge EG/N_+$ for various normal subgroups N of G, using a finite number of cofibrations.

First we make some elementary observations in which $\mathcal{F}$, $\mathcal{G}$, are families of subgroups of G.

Observations (3.4): (a) $E\mathcal{F}_+ \wedge E\mathcal{G}_+ \simeq E(\mathcal{F} \cap \mathcal{G})_+$

(b) $\tilde{E}\mathcal{F} \wedge \tilde{E}\,\mathcal{G} \simeq \tilde{E}(\mathcal{F} \cup \mathcal{G})$

(c) $\tilde{E}\mathcal{F} \wedge E\mathcal{G}_+ \simeq *$ iff $\mathcal{G} \subseteq \mathcal{F}$.

The fundamental construction is as follows:

Suppose $\mathcal{G} \subseteq \mathcal{F}$ then we may pick $K \in \mathcal{G}\backslash\mathcal{F}$ adjacent to $\mathcal{F}$ and consider

$$(3.5)\qquad \begin{array}{ccccc} \tilde{E}\mathcal{F} \wedge E\mathcal{G}_+ \wedge E[\subseteq K]_+ & \longrightarrow & \tilde{E}\mathcal{F} \wedge E\mathcal{G}_+ & \longrightarrow & \tilde{E}\mathcal{F} \wedge \tilde{E}[\subseteq K] \wedge E\mathcal{G}_+ \\ \wr\! \| & & & & \wr\! \| \\ \tilde{E}\mathcal{F} \wedge E[\subseteq K]_+ & & & & \tilde{E}(\mathcal{F} \cup (K)) \wedge E\mathcal{G}_+ \end{array}$$

Hence we may keep adding conjugacy classes (K) to $\mathcal{F}$ until the right hand term is contractible. To deal with the first term we use

$$(3.6)\qquad \begin{array}{ccccc} \tilde{E}\mathcal{F} \wedge E[\subseteq K]_+ \wedge E[\not\supseteq K]_+ & \longrightarrow & \tilde{E}\mathcal{F} \wedge E[\subseteq K]_+ & \longrightarrow & \tilde{E}\mathcal{F} \wedge \tilde{E}[\not\supseteq K] \wedge E[\subseteq K]_+ \\ \wr\! \| & & & & \wr\! \| \\ * \text{ since } [\subseteq K] \cap [\supseteq K] \subseteq \mathcal{F} & & & & \tilde{E}[\not\supseteq K] \wedge E[\subseteq K]_+ \end{array}$$

This is already enough since all subgroup K are normal and $E[\subseteq N]_+ = EG/N_+$.

Remark: For other groups the obstruction to an analogue of (3.3) should lie in a suitable Grothendieck group. The statement of the lemma is false for the dihedral group of order 8.

4. The E_2-term

In this section we assume G is elementary abelian of rank r.

Recall that all we need to do is check that b_*b is b_*-flat: in fact we shall show b_*b is "just like b_*S^0".

First choose maximal subgroups $M_1, \ldots, M_r$ such that

$$G \cong G/M_1 \times G/M_2 \times \ldots \times G/M_r.$$

and let η_i be nontrivial simple real representation of G via G/M_i.

Then of course we have the cofibration

$$E(G/M_i)_+ \longrightarrow S^0 \longrightarrow S^{\infty\eta_i}$$

and this induces a short exact sequence

$$0 \to b_* \to b_*[e(\eta_i)^{-1}] \longrightarrow b_*(EG/M_{i+}) \longrightarrow 0$$

since $e_i = e(\eta_i)$ is not a zero divisor. Indeed we see that, since e_1, e_2, $\ldots, e_{r-1}$, e_r is a regular sequence in b_*

$$0 \to b_* \to b_*[e_1^{-1}] \to b_*(E(G/M_1)_+)[e_2^{-1}] \longrightarrow \ldots$$
$$\to b_*(E(G/M_1 \times G/M_2 \times \ldots \times G/M_{r-1})_+)[e_r^{-1}] \to b_*(EG_+) \to 0$$

is exact.

From this we easily see that

$$\text{weak dim } b_*(E(G/M_1 \times \ldots \times G/M_i)_+) \leq i$$

Now with rather more effort we can prove for certain spectra T that the analogous sequence

$$(4.1) \qquad 0 \to b_*(T) \to b_*(T)[e_1^{-1}] \to b_*(E(G/M_1)_+ \wedge T)[e_2^{-1}] \to \ldots$$
$$\to b_*(E(G/M_1 \times \ldots \times G/M_{r-1})_+ \wedge T)[e_r^{-1}] \to b_*(EG_+ \wedge T) \to 0$$

is exact. Specifically this is true for the pointed function space $T = X^{EG_+}$ for any G-fixed space X, and hence for $T = b$ by passing to limits.

Now we work the sequence in reverse. Since the projection $EG_+ \longrightarrow S^0$ gives an equivalence $EG_+ \wedge X^{EG_+} \simeq EG_+ \wedge X$ (cf (4.3)) we have an isomorphism

$$(4.2) \qquad b_*(EG_+ \wedge X^{EG_+}) = b_*EG_+ \otimes H_*X$$

and so $b_*(EG_+ \wedge X^{EG_+})$ is of weak dimension r.

We steadily work down to $b_*(X^{EG_+})$ and prove it has weak dimension 0 as required.

This is not quite so simple since we cannot calculate $b_*(E(G/M_1 \times G/M_2 \times \ldots \times G/M_i)_+ \wedge T)[e_{i+1}^{-1}]$ as it stands. We must use a shorter version of (4.1) to deduce facts about it in terms of more localised groups. Finally, those sufficiently localised can be calculated for geometric reasons:

<u>Lemma (4.3)</u>: If $A = B \times C$ then we have an equivalence

$$X^{BC_+} \wedge EB_+ \wedge \tilde{E}[\not\supset C] \xrightarrow{\simeq} X^{E(B\times C)_+} \wedge EB_+ \wedge \tilde{E}[\not\supset C]$$

The two simplest examples serve to illustrate how the argument is implemented without obscuring matters in notation.

Example 1: If $G = C_p$

then for $T = X^{EG_+}$ we have

$$0 \longrightarrow b_*T \longrightarrow b_*T\,[e^{-1}] \longrightarrow b_*(T \wedge EG_+) \longrightarrow 0.$$

Now (1) $X^{EG_+} \wedge EG_+ \simeq X \wedge EG_+$ so the final term is tractable

and (2) $X^{EG_+} \wedge S^{\infty\tilde{\rho}} \simeq (X^{EG_+})^G \wedge S^{\infty\tilde{\rho}}$

$\simeq X^{BG_+} \wedge S^{\infty\tilde{\rho}}$ so the middle term is tractable.

Example 2: If $G = C_2 \times C_2 = \{1,x,y,z\}$

take $M_1 = \langle x\rangle$, $M_2 = \langle y\rangle$ and write $e_{\hat{x}}$, $e_{\hat{y}}$ for the respective Euler classes.

Then for $T = X^{EG_+}$ we have

$$0 \longrightarrow b_*T \longrightarrow b_*T\,[e_{\hat{x}}^{-1}] \longrightarrow b_*(EG/x_+ \wedge T)\,[e_{\hat{y}}^{-1}] \longrightarrow b_*(EG_+ \wedge T) \longrightarrow 0$$

Now (1) $X^{EG_+} \wedge EG_+ \simeq X \wedge EG_+$ so the final term is tractable

(2) $X^{EG_+} \wedge EG/x_+ \wedge S^{\infty\hat{y}} \simeq X^{EG_+} \wedge EG/x_+ \wedge S^{\infty\hat{y}\oplus\hat{z}}$

$\simeq X^{EG/y_+} \wedge EG/x_+ \wedge S^{\infty\hat{y}\oplus\hat{z}}$

$\simeq (X^{EG/y_+})^x \wedge EG/x_+ \wedge S^{\infty\hat{y}\oplus\hat{z}}$

$\simeq X^{Bx_+} \wedge EG/x_+ \wedge S^{\infty\hat{y}\oplus\hat{z}}$

so the penultimate term is tractable.

(3) $0 \rightarrow b_*T[e_{\hat{x}}^{-1}] \rightarrow b_*T[e_{\hat{y}}^{-1}] \rightarrow b_*(EG/y_+ \wedge T)[e_{\hat{x}}^{-1}] \rightarrow 0.$

The final term is already dealt with in (2), and for the middle term we use

$$0 \rightarrow b_*T[e_{\hat{x}}^{-1},\, e_{\hat{y}}^{-1}] \rightarrow b_*T[e_{\hat{x}}^{-1}, e_{\hat{y}}^{-1}, e_{\hat{z}}^{-1}] \rightarrow b_*(EG/z_+ \wedge T)[e_{\hat{x}}^{-1}, e_{\hat{y}}^{-1}] \rightarrow 0.$$

The final term is dealt with in (2) and the middle term is

$X^{EG_+} \wedge S^{\infty\tilde{\rho}} \simeq (X^{EG_+})^G \wedge S^{\infty\tilde{\rho}}$

$\simeq X^{BG_+} \wedge S^{\infty\tilde{\rho}}$ and therefore tractable.

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