MEMOIRS
of the
American Mathematical Society

Volume 246 • Number 1162 (first of 6 numbers) • March 2017

Abelian Properties of Anick Spaces

Brayton Gray

ISSN 0065-9266 (print) ISSN 1947-6221 (online)

American Mathematical Society

MEMOIRS
of the
American Mathematical Society

Volume 246 • Number 1162 (first of 6 numbers) • March 2017

Abelian Properties of Anick Spaces

Brayton Gray

ISSN 0065-9266 (print) ISSN 1947-6221 (online)

American Mathematical Society
Providence, Rhode Island

Library of Congress Cataloging-in-Publication Data

Names: Gray, Brayton, 1940–

Title: Abelian properties of Anick spaces / Brayton Gray.

Description: Providence, Rhode Island : American Mathematical Society, 2017. | Series: Memoirs of the American Mathematical Society, ISSN 0065-9266 ; volume 246, number 1162 | Includes bibliographical references and index.

Identifiers: LCCN 2016055094 | ISBN 9781470423087 (alk. paper)

Subjects: LCSH: Abelian groups. | Topological groups. |Topological spaces. | Loop spaces. | H-spaces.

Classification: LCC QA387 .G7245 2017 | DDC 512/.55–dc23 LC record available at https://lccn.loc.gov/2016055094

DOI: http://dx.doi.org/10.1090/memo/1162

Memoirs of the American Mathematical Society

This journal is devoted entirely to research in pure and applied mathematics.

Subscription information. Beginning with the January 2010 issue, *Memoirs* is accessible from www.ams.org/journals. The 2016 subscription begins with volume 239 and consists of six mailings, each containing one or more numbers. Subscription prices for 2016 are as follows: for paper delivery, US$890 list, US$712.00 institutional member; for electronic delivery, US$784 list, US$627.20 institutional member. Upon request, subscribers to paper delivery of this journal are also entitled to receive electronic delivery. If ordering the paper version, add US$10 for delivery within the United States; US$69 for outside the United States. Subscription renewals are subject to late fees. See www.ams.org/help-faq for more journal subscription information. Each number may be ordered separately; *please specify number* when ordering an individual number.

Back number information. For back issues see www.ams.org/backvols.

Subscriptions and orders should be addressed to the American Mathematical Society, P. O. Box 845904, Boston, MA 02284-5904 USA. *All orders must be accompanied by payment.* Other correspondence should be addressed to 201 Charles Street, Providence, RI 02904-2294 USA.

Memoirs of the American Mathematical Society (ISSN 0065-9266 (print); 1947-6221 (online)) is published bimonthly (each volume consisting usually of more than one number) by the American Mathematical Society at 201 Charles Street, Providence, RI 02904-2294 USA. Periodicals postage paid at Providence, RI. Postmaster: Send address changes to Memoirs, American Mathematical Society, 201 Charles Street, Providence, RI 02904-2294 USA.

This publication is indexed in *Mathematical Reviews*®, *Zentralblatt MATH*, *Science Citation Index*®, *Science Citation Index*™*-Expanded*, *ISI Alerting Services*℠, *SciSearch*®, *Research Alert*®, *CompuMath Citation Index*®, *Current Contents*®*/Physical, Chemical & Earth Sciences.* This publication is archived in *Portico* and *CLOCKSS*.

Printed in the United States of America.

∞ The paper used in this book is acid-free and falls within the guidelines established to ensure permanence and durability.

Visit the AMS home page at http://www.ams.org/

10 9 8 7 6 5 4 3 2 1 21 20 19 18 17 16

Contents

Abstract

Anick spaces are closely connected with both EHP sequences and the study of torsion exponents. In addition they refine the secondary suspension and enter unstable periodicity. In this work we describe their H-space properties as well as universal properties. Techniques include a new kind on Whitehead product defined for maps out of co-H spaces, calculations in an additive category that lies between the unstable category and the stable category, and a controlled version of the extension theorem of Gray and Theriault (Geom. Topol. **14** (2010), no. 1, 243–275).

Received by the editor August 1, 2013 and, in revised form, August 22, 2013, September 25, 2014, January 10, 2015, and March 9, 2015.

Article electronically published on September 29, 2016.

DOI: `http://dx.doi.org/10.1090/memo/1162`

2010 *Mathematics Subject Classification.* Primary 55Q15, 55Q20, 55Q51; Secondary 55Q40, 55Q52, 55R99.

The author is affiliated with the Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago.

CHAPTER 1

Introduction

1.1. Statement of Results

By an Anick space we mean a homotopy CW complex T_{2n-1} which occurs in a fibration sequence.

$$\Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} \longrightarrow T_{2n-1} \longrightarrow \Omega S^{2n+1} \tag{1.1}$$

where the composition

$$\Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$$

is homotopic to the p^rth power map on $\Omega^2 S^{2n+1}$.

We will say that a space is a homotopy-Abelian H-space if it has a homotopy associative and homotopy commutative H-space structure.

Throughout this work we will assume that all spaces are localized at $p \geqslant 5$ unless otherwise indicated.

Theorem A. *There is a homotopy-Abelian Anick space for any $n \geqslant 1$ and $r \geqslant 1$.*

We will write $P^{2n}(p^r)$ for the $2n$ dimensional Moore space $S^{2n-1} \cup_{p^r} e^{2n}$. From (1.1) we see that T_{2n-1} is $2n-2$ connected and there is a $2n$-equivalence

$$i\colon P^{2n}(p^r) \longrightarrow T_{2n-1}$$

For any homotopy-Abelian H-space Z, let $[T_{2n-1}, Z]_H$ be the Abelian group of homotopy classes of based H-maps from T_{2n-1} to Z. Let

$$p_k(Z) = p^{r+k-1}\pi_{2np^k-1}(Z; Z/p^{r+k}).$$

Theorem B. *$[T_{2n-1}, Z]_H \cong \varprojlim G_k(Z)$ where $G_0(Z) = [P^{2n}(p^r), Z] = \pi_{2n}(Z; Z/p^r)$ and there are exact sequences:*

$$0 \longrightarrow p_k(\Omega Z) \xrightarrow{e} G_k(Z) \xrightarrow{r} G_{k-1}(Z) \xrightarrow{\beta} p_k(Z).$$

In particular, if $p^r\pi_*(Z) = 0$, there is an isomorphism

$$[P^{2n}(p^r), Z] \approx [T_{2n-1}, Z]_H$$

given by the restriction

$$P^{2n}(p^r) \longrightarrow T_{2n-1}.$$

Several examples are given in Chapter 7. In particular

Corollary C. *Given two homotopy-Abelian Anick spaces for the same values of n, r and $p > 3$, there is an H map between them which is a homotopy equivalence.*

COROLLARY D. *In any homotopy-Abelian H-space structure on an Anick space, the identity map has order p^r.*

Let $T_{2n} = S^{2n+1}\{p^r\}$. Then analogous results[1] to theorems A and B are well known ([**Nei83**], [**Gra93a**]) when $p > 3$, and in particular, any map

$$\alpha\colon P^{n+1}(p^r) \longrightarrow P^{m+1}(p^r)$$

corresponds to a unique H-map $\widehat{\alpha}$ such that the diagram

$$\begin{array}{ccc} P^{n+1}(p^r) & \xrightarrow{\alpha} & P^{m+1}(p^r) \\ \downarrow & & \downarrow \\ T_n & \xrightarrow{\widehat{\alpha}} & T_m \end{array}$$

commutes up to homotopy, for any n and m. This result was the object of the conjectures in [**Gra93a**].

In developing these results, several new techniques of geometric homotopy theory are introduced. These may be of some use in other problems. A summary of some of these techniques can be found in section 1.4.

In an appendix, we show that these results do not generally hold if $p = 3$, and we treat the special case when $n = 1$.

The author would like to thank Joseph Neisendorfer for many helpful conversations during this work.

1.2. History

A map $\pi_n\colon \Omega^2 S^{2n+1} \longrightarrow S^{2n-1}$ with the property that the composition

$$\Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$$

is homotopic to the p^rth power map was first discovered when $p \geqslant 3$ by Cohen, Moore and Neisendorfer ([**CMN79c**],[**Nei81**]) and played a crucial role in determining the maximal exponent for the torsion in the homotopy groups of spheres.

In [**CMN79a**], the authors raised the question of whether a fibration such as 1.1 could exist. The feasibility of constructing a secondary EHP sequence refining the secondary suspension ([**Mah75**], [**Coh83**]) together with a theory of compositions ([**Tod56**]) was studied in [**Gra93a**], [**Gra93b**]. This led the author to conjecture the existence of Anick spaces with theorems A and B and corollaries C and D.

At about the same time, David Anick was studying the decomposition of the loop space on a finite complex ([**Ani92**]). His intention was to find a list of indecomposable spaces which, away from a few small primes, could be used for decomposition. This led to the construction of a sequence of spaces for $p \geqslant 5$. The limit of this sequence is the space sought after in [**Gra93a**]. This work of Anick was published in a 270-page book ([**Ani93**]). In [**AG95**], the authors showed that the Anick space so constructed admitted an H-space structure when $p \geqslant 5$. They also proved a weaker version of Theorem B. They showed that if $p_k(Z) = 0$ for all k, an extension to an Anick space existed, but there was no indication that the extension would be an H-map or that it would be unique. At that time it was thought that the torsion condition was a peculiarity of the approach and, it was conjectured that, as in the case of $T_{2n} = S^{2n+1}\{p^r\}$, this requirement was unnecessary.

[1] In this case no torsion requirements are needed for Theorem B.

In [**The01**], the author asserted theorem A and corollary C and that for each homotopy Abelian H-space Z, there is an equivalence $[P^{2n}(p^r), Z] \simeq [T_{2n-1}, Z]_H$. This, however, is not consistent with the results in [**Gra12**], where counter examples are provided with Z is as Eilenberg–MacLane space. The assertions in [**The01**] depend on the author's Theorem 2.1 which is quoted as "to appear in Topology" in the author's bibliography. This, however, did not appear. Many of the results in the author's section 5 are inconsistent with results we obtain here. The main result of the author's section 4 is valid and we give a much simplified proof of it here (2.9).

In [**GT10**] a much simpler construction of the Anick spaces was obtained which worked for all $p \geqslant 3$. This result replicated the results of [**Ani93**] and [**AG95**] and extended them to the case $p = 3$. Furthermore they showed that the homotopy type of an Anick space that supports an H-space structure is uniquely characterized by n, p and r.

In [**Gra12**], a proof was given that if $p_k(\Omega Z) = 0$ for all k, there is at most one extension of a map $\alpha\colon P^{2n} \to Z$ to an H-map $\widehat{\alpha}\colon T \to Z$ and examples were presented to show that this torsion condition is necessary. The proof we give here is entirely different.

In [**GT10**], the authors constructed the EHP sequences conjectured in [**Gra93a**].

$$T_{2n-1} \xrightarrow{E} \Omega T_{2n} \xrightarrow{H} BW_n$$

$$T_{2n} \xrightarrow{E} \Omega T_{2n+1} \xrightarrow{H} BW_{n+1}$$

(where $T_{2n} = S^{2n+1}\{p^r\}$), and BW_n lies in a fibration sequence:

$$S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n.$$

More details appear in the next section. The only remaining unsettled conjecture in [**Gra93a**] for the Anick spaces is the following:

Conjecture E. There is a homotopy equivalence

$$BW_n \simeq \Omega T_{2np-1}(p)$$

where $T_{2np-1}(p)$ is the Anick space with $r = 1$.

In [**The11**], Theriault constructed $T_{2n-1}(2^r)$ for $r \geqslant 3$, but there is no H-space structure in this case.

1.3. Methods and Modifications

Throughout this work we will fix n and abbreviate T_{2n-1} as T if this will not lead to confusion. The construction in [**GT10**] begins with a fibration sequence:

$$W_n \longrightarrow S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n \tag{1.2}$$

where E^2 is the double suspension map ([**Gra88**]). The authors construct a factorization of the map ν:

$$\Omega^2 S^{2n+1} \xrightarrow{\Omega\partial} \Omega S^{2n+1}\{p^r\} \xrightarrow{H} BW_n$$

where ∂ occurs in the fibration sequence defining $S^{2n+1}\{p^r\}$, the fiber of the degree p^r map on S^{2n+1}. For any choice of H, there is a homotopy commutative diagram:

$$
\begin{CD}
\Omega^2 S^{2n+1} @= \Omega^2 S^{2n+1} \\
@V{\pi_n}VV @V{p^r}VV \\
S^{2n-1} @>>> \Omega^2 S^{2n+1} @>{\nu}>> BW_n \\
@VVV @V{\Omega\partial}VV @| \\
T @>{E}>> \Omega S^{2n+1}\{p^r\} @>{H}>> BW_n \\
@VVV @VVV \\
\Omega S^{2n+1} @= \Omega S^{2n+1}
\end{CD}
\tag{1.3}
$$

and consequently for each choice of H, the fiber of H is an Anick space by (1.1).

In [**GT10**], the authors proceed to show that for any such choice, the Anick space admits an H-space structure such that the fibration 1.1 is an H-fibration. Furthermore, they prove

PROPOSITION 1.4 ([**GT10**, 4.9]). *Fix n, r and $p \geqslant 3$. Then any two Anick spaces which admit an H-space structure are homotopy equivalent.*

The H-space structure is constructed in [**GT10**] as follows. The authors construct a splitting:

$$\Sigma T \simeq G \vee W \tag{1.5}$$

where G is atomic and W is a $4n-1$ connected wedge of Moore spaces. A map $\varphi\colon G \to S^{2n+1}\{p^r\}$ is constructed from (1.5) and the adjoint of the map E in (1.3)

$$G \longrightarrow \Sigma T \xrightarrow{\widetilde{E}} S^{2n+1}\{p^r\}$$

and a space E is constructed as the fiber of φ:

$$\Omega S^{2n+1}\{p^r\} \xrightarrow{\partial} E \longrightarrow G \xrightarrow{\varphi} S^{2n+1}\{p^r\}.$$

The H-space structure arises from a factorization of H through the space E:

$$
\begin{CD}
\Omega G @= \Omega G \\
@V{h}VV @VVV \\
T @>{E}>> \Omega S^{2n+1}\{p^r\} @>{H}>> BW_n \\
@VVV @V{\partial'}VV @| \\
R @>>> E @>{\nu_\infty}>> BW_n \\
@VVV @V{\pi}VV \\
G @= G
\end{CD}
\tag{1.6}
$$

h has a right homotopy inverse $g\colon T \to \Omega G$ defined by the splitting (1.5) since T is atomic. Thus T inherits an H space structure as a retract of ΩG. The H-space structure depends on h and consequently on the choice of ν_∞. The splitting (1.5) and map ν_∞ are defined inductively. The space G is filtered by subspaces G_k where

$$G_k = G_{k-1} \cup_{\alpha_k} CP^{2np^k}(p^{r+k}) \tag{1.7}$$

and the fibration $E \xrightarrow{\pi} G$ is the union of the induced fibrations

$$\Omega S^{2n+1}\{p^r\} \to E_k \to G_k.$$

The splitting (1.5) is approximated by a sequence of splittings:

$$\Sigma T^{2np^k} \simeq G_k \vee W_k$$

and ν_∞ is simultaneously constructed by induction over the restriction of (1.6) to G_k. The map ν_∞ is then the limit of maps

$$\nu_k \colon E_k \to BW_n.$$

The extension theorem [**GT10**, 21] is applied which guarantees that any map $\nu_{k-1} \colon E_{k-1} \to BW_n$ extends to E_k. An arbitrary choice is made for each $k > 0$. It seems likely that the number of choices for ν_∞ is uncountable.

1.4. Outline of Modifications

The basis of this paper is to modify and sharpen the construction in [**GT10**] as described in section 1.3. As explained there, the H-space properties that the Anick space inherits depend on a choice of a map

$$\nu_k \colon E_k \to BW_n.$$

In Chapter 2, we introduce maps

$$\Gamma_k \colon \Omega G_k * \Omega G_k \to E_k$$

and prove that if we choose ν_k such that $\nu_k\Gamma_k$ is null homotopic for each k, the induced H-space structure will be homotopy-Abelian. We also recall, at this point, various facts about the Anick spaces that were developed in [**GT10**] which will be needed in the sequel.

In Chapter 3, we recall from [**Gra11**] the construction of a Whitehead product pairing

$$[G, X] \times [H, X] \to [G \circ H, X]$$

where G and H are simply connected co-H spaces and $G \circ H$ is a new simply connected co-H space. This generalizes the classical pairing:

$$[\Sigma A, X] \times [\Sigma B, X] \to [\Sigma A \wedge B, X].$$

We also generalize Neisendorfer's theory of relative Whitehead products and H-space based Whitehead products in the mod p^r homotopy of a principal fibration [**Nei10a**] by replacing Moore spaces with arbitrary co-H spaces. We then reduce the question of whether $\nu_k\Gamma_k$ is null homotopic to whether a sequence of iterated H-space based and relative Whitehead products $G_k \circ (G_k \circ \ldots G_k) \to E_k$ are annihilated by ν_k for all $i \geqslant 2$.

In Chapter 4 we construct mod p^{r+i-1} homotopy classes $a(i)$ and $c(i)$ in E_k for $i \leqslant k$ and a mod p^{r+k} homotopy class β_k which will play a key role. We also introduce "index p approximation" and show that the iterated Whitehead product under investigation can be approximated by iterated Whitehead products in homotopy groups with coefficients in Z/p^s for $r \leqslant s \leqslant r+k$. This approximation excludes the case $n = 1$ which is handled in the appendix. These classes are the obstructions and we seek to choose ν_k which annihilates them.

When $k > 0$, the obstructions actually belong to two classes, A and C depending on whether they involve the $a(i)$ or the $c(i)$. (It turns out that any obstruction involving both $a(i)$ and $c(j)$ is automatically in the kernel of ν_k.) In 5.1 we simplify the procedure by defining a quotient space J_k of E_k which is universal for annihilating the classes as C. J_k is a principal fibration over a space D_k which is a quotient[2] of G_k and we seek a factorization of ν_k:

$$E_k \xrightarrow{\tau_k} J_k \xrightarrow{\gamma_k} BW_n$$

where γ_k annihilates the obstructions in A. In section 5.2 we introduce a congruence relation among homotopy classes and the relative Whitehead products and H-space based Whitehead products have better properties in the congruence homotopy category. This allows for a further reduction in obstructions to a collection of mod p^r homotopy classes.

In Chapter 6 we introduce the controlled extension theorem (6.1). This is a modification of the extension theorem in [**GT10**, 2.2] which allows maps defined on the total space of an induced fibration of a principal fibration over a subspace to be extended over the total space under certain conditions. In the controlled extension theorem, conditions are given for the extension to annihilate certain maps maps $u\colon P \to E$. This is immediately applied to the case $k = 0$ where we construct ν_o by induction over the skeleta of a space F_0.

A complexity arises because for each $k > 0$, there are level k obstructions in infinitely many dimensions. When we modify ν_k to eliminate these obstructions, we can't assume that it will be an extension of ν_{k-1} and consequently the level $k - 1$ obstructions may reappear. A separate argument (6.40) dispenses with this issue. In section 6.2 we introduce the inductive hypothesis (6.7) and a space F_k is analyzed to prepare for the inductive step. This is accomplished in section 6.3.

In Chapter 7, we discuss the universal properties of the Anick spaces. From the fibration sequence (1.6) we extract the following fibration sequence

$$\xrightarrow{*} \Omega R \longrightarrow \Omega G \xrightarrow{h} T \xrightarrow{*}$$

which we think of as a presentation of T.

The proof of Theorem B depends on an understanding of the map $R \to G$. From [**GT10**, 4.8] we know that for any choice of ν_∞, R is a wedge of Moore spaces. Certain of these Moore spaces are needed to resolve the relationship between $H_*(\Omega G; Z/p)$, which has infinitely many generators and $H_*(T; Z/p)$ which has two generators. These are the classes $a(i)$ and $c(i)$. The others are necessary to enforce the homotopy commutativity in $H_*(T; Z/p^r)$. These are either Whitehead products or generalized Whitehead products[3] defined by co-H-spaces in [**Gra11**].

[2]The space D_k, defined differently, occurs in the original construction of Anick ([**Ani93**]). The results of [**AG95**] are obtained by replacing D_k by G_k.

[3]In particular, the first element in $\pi_*(G)$ of order p^{r+k} could not be a classical Whitehead product for $k > 0$. It is defined as a composition

$$P^{4np^k}(p^{r+k}) \longrightarrow G_k \circ G_k \xrightarrow{W} G_k$$

where W is a generalized Whitehead product.

The obstruction to extensions depend on certain homotopy classes

$$\widetilde{\beta_k}\colon P^{2np^k-1}(p^{r+k}) \longrightarrow \Omega G_{k-1}$$

which must be annihilated in order for an extension to proceed.

In the appendix we discuss the case $n = 1$ and the case $p = 3$.

1.5. Conventions and Notation

All spaces will be localized at a prime $p \geqslant 3$ and usually we will assume $p \geqslant 5$. $H_*(X)$ and $H^*(X)$ will designate the mod p homology and cohomology. If other coefficients are used (usually $Z_{(p)}$) they will be specified in the usual way.

We write $P^m(p^s) = S^{m-1} \cup_{p^s} e^m$ for the Moore space. Throughout we will fix $r \geqslant 1$ and we will always have $s \geqslant r$. We will abbreviate $P^m(p^r)$ simply as P^m. We will write ι_{m-1} and π_m for the usual maps

$$S^{m-1} \xrightarrow{\iota_{m-1}} P^m(p^s) \xrightarrow{\pi_m} S^m.$$

We designate the symbols β, σ, ρ for the maps

$$\beta\colon P^m(p^s) \longrightarrow P^{m+1}(p^s)$$
$$\rho\colon P^m(p^s) \longrightarrow P^m(p^{s+1})$$
$$\sigma\colon P^m(p^s) \longrightarrow P^m(p^{s-1})$$

with $\beta = \iota_m\pi_m$, $\pi_m\rho = \pi_m$ and $\sigma\iota_{m-1} = \iota_{m-1}$. These symbols will not be indexed by the dimension and can be composed, so that we have formulas

$$\beta = \sigma\beta\rho$$
$$p = \sigma\rho = \rho\sigma$$
$$\beta\sigma^t = p^t\sigma^t\beta$$

where p is the degree p self map. We write

$$\delta_t = \beta\rho^t$$

and will frequently use the cofibration sequence

$$(1.8) \qquad P^{m-1}(p^s) \vee P^m(p^s) \xrightarrow{-\delta_t \vee \rho^t} P^m(p^{s+t})$$
$$\xrightarrow{p^s} P^m(p^{s+t}) \xrightarrow{\sigma^t \vee \sigma^t\beta} P^m(p^s) \vee P^{m+1}(p^s)$$

especially when $s = r + k - 1$ and $t = 1$.

We will write $\nu_p(m)$for the largest exponent of p that divides m and often set $s = \nu_p(m)$. For any map x we will write $\widetilde{x}$ for either its left or right adjoint, if there is no possibility of confusion.

By a diagram of fibration sequences, we will mean a diagram in which any sequences, either vertical or horizontal are fibration sequences up to homotopy.

CHAPTER 2

Abelian Structures

We begin by reviewing some material about principal fibrations. In section 2.1, we recall the construction of BW_n ([**Gra88**]) and the extension theorem ([**GT10**]), and for certain principal fibrations we construct a natural map $\Gamma\colon \Omega B * \Omega B \longrightarrow E$ in section 2.2. Γ is the lynchpin for generalizing Neisendorfer's H-space based Whitehead products. In 2.3 we give a short proof of a result of Theriault giving a criterion for an H-space structure to be homotopy-Abelian. We use this to relate the map Γ to the obstructions for the induced H-space structure on the fiber being homotopy Abelian. We conclude with Proposition 2.12 which presents the conditions we will establish in the next 4 chapters. Finally, we recall some results from [**GT10**] that will be used in the sequel.

2.1. Preliminaries

In [**Gra88**], a clutching construction was described for Hurewicz fibrations in case that the base is a mapping cone. This construction is particularly simple in the case of a principal fibration.

Suppose $\varphi\colon B \longrightarrow X$. We describe a principal fibration

$$\Omega X \xrightarrow{\ i\ } E \xrightarrow{\ \pi\ } B$$

where $E = \{(b, \omega) \in B \times PX \mid \omega(1) = \varphi(b)\}$, where PX is the space of paths $\omega\colon I \longrightarrow X$ with $\omega(0) = *$.

In case $B = B_0 \cup_\theta CA$, we have a pair of principal fibrations:

$$\begin{array}{ccccc} & & \Omega X & = & \Omega X \\ & & \downarrow & & \downarrow \\ & & E_0 & \longrightarrow & E \\ & \nearrow & \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi} \\ A & \xrightarrow{\ \theta\ } & B_0 & \longrightarrow & B \end{array}$$

Clearly θ lifts to a map $\theta'\colon A \longrightarrow E_0$. We assert that there is a lifting $\overline{\theta}\colon A \longrightarrow E_0$ such that the composition:

$$A \xrightarrow{\ \overline{\theta}\ } E_0 \longrightarrow E$$

is null homotopic. For if the composition

$$A \xrightarrow{\ \theta'\ } E_0 \longrightarrow E$$

is essential, it factors through ΩX up to homotopy, and we can use the principal action

$$\Omega X \times (E, E_0) \xrightarrow{a} (E, E_0)$$

to define a different lifting $\overline{\theta}\colon A \to E_0$ of θ in which the composition into E is null homotopic. In particular, the composition

$$(CA, A) \xrightarrow{\overline{\theta}} (E, E_0) \xrightarrow{\pi} (B, B_0)$$

induces an isomorphism in homology.

PROPOSITION 2.1 ([**Gra88**]) (Clutching Construction). *Suppose*

$$(E, E_0) \xrightarrow{\pi} (B, B_0)$$

is a Hurewicz fibration with fiber F where $B = B_0 \cup_\theta CA$. Then there is a map $F \times (CA, A) \xrightarrow{\varphi} (E, E_0)$, and a pushout diagram

$$\begin{array}{ccc} F \times CA & \xrightarrow{\varphi} & E \\ \uparrow & & \uparrow \\ F \times A & \xrightarrow{\varphi} & E_0 \end{array}$$

where $\pi\varphi\colon F \times CA \to B$ is the projection onto $CA \subset B$. In particular

$$E/E_0 \cong F \ltimes \Sigma A. \tag{2.2}$$

In case that π is a principal fibration with fiber $F = \Omega X$, we can take φ to be the composition:

$$\Omega X \times (CA, A) \xrightarrow{1 \times \overline{\theta}} \Omega X \times (E, E_0) \xrightarrow{a} (E, E_0)$$

where a is the principal action map.

The following result ([**Gra88**]) is a simple application of 2.1.

PROPOSITION 2.3. *Localized at a prime $p > 2$, there is a fibration sequence*

$$\Omega^2 S^{2n+1} \xrightarrow{\partial} BW_n \times S^{4n-1} \longrightarrow S^{2n} \xrightarrow{E} \Omega S^{2n+1}$$

where ∂ factors $\Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n \xrightarrow{i_1} BW_n \times S^{4n-1}$ and the homotopy fiber of ν is S^{2n-1}

$$S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n.$$

Furthermore if $p \geqslant 5$, BW_n has a homotopy-Abelian H-space structure and ν is an H-map.

Using the work of Cohen, Moore and Neisendorfer, Theriault has proved

PROPOSITION 2.4 ([**The08**]). *BW_n has H-space exponent*[1] *p.*

An important application of 2.1 is the extension theorem

[1]We say that an H-space has H-space exponent q if the q^{th} power map is null homotopic in some association.

Extension Theorem 2.5 ([**GT10**, 2.1]). *Suppose A is a co-H space such that the map $\overline{\theta}\colon A \to E_0$ is divisible by q in the co-H-space structure, and Z is a connected H-space with H-space exponent q. Then the restriction*

$$[E, Z] \to [E_0, Z]$$

is onto.

This is a powerful tool and is the key to showing that each map $\nu_{k-1}\colon E_{k-1} \to BW_n$ extends to a map $\nu_k\colon E_k \to BW_n$ in [**GT10**]. In section 6.1 we will enhance this to the controlled extension theorem, which will allow for a good choice of extension.

2.2. Whitehead Products

Whitehead and Samelson products in homotopy groups with coefficients were introduced by Neisendorfer ([**Nei80**]). These included relative Whitehead products and in [**Nei10a**], he introduced H-space based Whitehead products. These will be useful in Chapter 3 where we will generalize these constructions to classes defined on co-H spaces. At this point we will introduce a basic construction and show how it is related to the question of homotopy-Abelian H-space structures.

For any space W, we define a homotopy equivalence of pairs

$$\xi\colon (C(\Omega W), \Omega W) \to (PW, \Omega W)$$

by $\xi(\omega, s)(t) = \omega(st)$.

Now construct a map

$$\omega\colon \Omega U * \Omega V \to U \vee V.$$

We use the decomposition $\Omega U * \Omega V = \Omega U \times C(\Omega V) \cup C(\Omega U) \times \Omega V$ where C is the cone functor with vertex at 0 and define ω by

$$\Omega U \times C(\Omega V) \xrightarrow{\pi_2} C(\Omega V) \xrightarrow{\epsilon} V \longrightarrow U \vee V$$

$$C(\Omega U) \times \Omega V \xrightarrow{\pi_1} C(\Omega U) \xrightarrow{\epsilon} U \longrightarrow U \vee V$$

where ϵ is an evaluation.

Proposition 2.6 ([**Gan70**]). *ω lifts to a homotopy equivalence with the homotopy fiber of the inclusion $U \vee V \to U \times V$.*

Proof. The homotopy fiber is the union of the parts over U and V; i.e.,

$$F = \Omega U \times PV \cup PU \times \Omega V.$$

According to [**Str72**], the pair $(PW, \Omega W)$ is an NDR pair, so the induced map

$$\widehat{\xi}\colon \Omega U \times C(\Omega V) \cup C(\Omega U) \times \Omega V \to \Omega U \times PV \cup PU \times \Omega V$$

is a homotopy equivalence. □

Now suppose that

$$\Omega X \xrightarrow{i} E \xrightarrow{\pi} B$$

is a principal fibration induced by a map $\varphi\colon B \to X$ where X is an H-space. We can and will assume that the multiplication on X has a strict unit ([**Nei10a**, 11.1.11]).

PROPOSITION 2.7. *There is a strictly commutative diagram,*

$$\begin{array}{ccc} \Omega B * \Omega B & \xrightarrow{\Gamma} & E \\ {\scriptstyle\omega}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ B \vee B & \xrightarrow{\nabla} & B \end{array}$$

where ∇ is the folding map. Furthermore, Γ is natural with respect to the data.

PROOF. The square

$$\begin{array}{ccccc} B \vee B & & \xrightarrow{\quad\nabla\quad} & & B \\ \big\downarrow & & & & \big\downarrow{\scriptstyle\varphi} \\ B \times B & \xrightarrow{\varphi\times\varphi} & X \times X & \xrightarrow{\mu} & X \end{array}$$

is strictly commutative and induces a map of the induced principal fibrations. Γ is the composition of $\widehat{\xi}$ with the map

$$\Omega B \times PB \cup PB \times \Omega B \longrightarrow E = \{(b, \omega) \in B \times PX \mid \omega(1) = \varphi(b)\}$$

given by the formula

$$(\omega_1, \omega_2) \longrightarrow \begin{cases} (\omega_1(1), \overline{\omega}(t)) & \text{if } \omega_2(1) = * \\ (\omega_2(1), \overline{\omega}(t)) & \text{if } \omega_1(1) = * \end{cases}$$

where $\overline{\omega}(t) = \mu(\varphi(\omega_1(t)), \varphi(\omega_2(t)))$. □

Recall (for example [**Gra11**, 3.4]) that there is a natural homotopy equivalence

$$\Sigma(X \wedge Y) \simeq X * Y$$

such that the diagram

$$\begin{array}{ccc} \Sigma(X \wedge Y) & \simeq & X * Y \\ {\scriptstyle -\tau}\big\downarrow & & \big\downarrow{\scriptstyle\tau} \\ \Sigma(Y \wedge X) & \simeq & Y * X \end{array}$$

commutes up to homotopy, where the maps labeled τ are the transposition maps.

PROPOSITION 2.8. *Suppose that X is homotopy commutative. Then there is a homotopy commutative diagram:*

$$\begin{array}{ccccc} \Sigma(\Omega B \wedge \Omega B) & \simeq & \Omega B * \Omega B & & \\ & & & \searrow{\scriptstyle\Gamma} & \\ {\scriptstyle -\tau}\big\downarrow & & \big\downarrow{\scriptstyle\tau} & & E \\ & & & \nearrow{\scriptstyle\Gamma} & \\ \Sigma(\Omega B \wedge \Omega B) & \simeq & \Omega B * \Omega B & & \end{array}$$

PROOF. By [**Nei10a**, 11.1.11], we can assume that the homotopy of commutation $\mu_t(x_1, x_2)$ is stationary on the axes, where $\mu_0(x_1, x_2) = \mu(x_1, x_2)$ and $\mu_1(x_1, x_2) = \mu(x_2, x_1)$. We then define

$$\overline{\omega}_t(s) = \mu_t\left(\varphi(\omega_1(s)), \varphi(\omega_2(s))\right)$$

and use this to define $\Gamma_t\colon \Omega B * \Omega B \longrightarrow E$, a homotopy between Γ and $\Gamma\tau$. $\square$

2.3. Theriault's Criterion

The map $\omega\colon \Omega B*\Omega B \longrightarrow B\vee B$ plays a role in a useful condition for an H-space to have a homotopy-Abelian structure.

PROPOSITION 2.9 ([**The01**, 4.12]). *Suppose that*

$$\Omega B \xrightarrow{h} F \xrightarrow{i} E \xrightarrow{\pi} B$$

is a fibration sequence in which i is null homotopic. Suppose that there is a lifting $\overline{\omega}$ of $\nabla\omega$ in the diagram:

$$\begin{array}{ccc} \Omega B * \Omega B & \xrightarrow{\overline{\omega}} & E \\ \downarrow{\scriptstyle\omega} & & \downarrow{\scriptstyle\pi} \\ B \vee B & \xrightarrow{\nabla} & B \end{array}$$

Then the H-space structure defined on F by any right inverse $g\colon F \longrightarrow \Omega B$ of h defines a homotopy-Abelian H-space structure.

PROOF. For any pointed space Z let $G = [Z, \Omega B]$ and $X = [Z, F]$. Then G is a group which acts on X via the action map

$$\Omega B \times F \xrightarrow{a} F.$$

Since h has a right homotopy inverse, the orbit of $* \in X$ is all of X. The adjoint of the composition

$$\Sigma(\Omega B \wedge \Omega B) \simeq \Omega B \times \Omega B \xrightarrow{\omega} B \vee B \xrightarrow{\nabla} B$$

is well known to be homotopic to the commuter map

$$\Omega B \wedge \Omega B \xrightarrow{c} \Omega B.$$

(See, for example, [**Gra11**, 3.4]). Consequently, the existence of $\overline{\omega}$ implies that every commutator in G acts trivially on $* \in X$; i.e., $g(h*) = h(g*)$. Let $N = \{g \mid g* = *\}$ be the stabilizer of $*$. Then N is a normal subgroup since if $g \in N$, $(hgh^{-1})(*) = (hg)(h^{-1}*) = h^{-1}(hg*) = g* = *$. Consequently $X = G/N$ is a quotient group of G. It is Abelian since $g(h*) = h(g*)$. This group structure on $X = [Z, F]$ is natural for maps in Z. Apply this in case $Z = F \times F$ and $Z = F \times F \times F$ to construct a homotopy-Abelian H-space structure on F. $\square$

Now recall that we have fixed G_k and $\varphi_k\colon G_k \longrightarrow S^{2n+1}\{p^r\}$ and we set $G = \bigcup G_k$. Let $E = \bigcup E_k$. Consider the commutative diagram:

$$\begin{array}{ccc} & & \Omega S^{2n+1}\{p^r\} \\ & & \downarrow \\ \Omega G * \Omega G & \xrightarrow{\Gamma} & E \\ \downarrow{\scriptstyle\omega} & & \downarrow \\ G \vee G & \xrightarrow{\nabla} & G \end{array} \tag{2.10}$$

PROPOSITION 2.11. *If the composition:*

$$\Omega G * \Omega G \xrightarrow{\Gamma} E \xrightarrow{\nu_\infty} BW_n$$

is null homotopic, the induced H-space structure on T is homotopy Abelian.

PROOF. Compare (2.10) with (1.6) and apply (2.9). □

2.4. Compatibility of Modifications

In the sequel we will construct maps $\nu_k \colon E_k \to BW_n$ such that the composition:

$$\Omega G_k * \Omega G_k \xrightarrow{\Gamma_k} E_k \xrightarrow{\nu_k} BW_n$$

is null homotopic. We begin with an arbitrary choice of ν_k as in [**GT10**] and modify it using the controlled extension theorem (6.1). Having done this, we have no reason to assume that the composition:

$$E_{k-1} \xrightarrow{e_k} E_k \xrightarrow{\nu_k} BW_n$$

is homotopic to ν_{k-1}. However, all modifications occur in dimensions $2np^k$ and larger, so we can assume these maps agree up to dimension $2np^k - 2$. The proof of Theorem A will then follow from

PROPOSITION 2.12. *Suppose we can construct maps $\nu_k \colon E_k \to BW_n$ for each $k \geqslant 0$ such that:*

(a) $\Omega^2 S^{2n+1} \xrightarrow{\Omega\partial} \Omega S^{2n+1}\{p^r\} \xrightarrow{\partial'} E_k \xrightarrow{\nu_k} BW_n$
induces an isomorphism in H_{2np-2}

(b) *The composition:*

$$\Omega G_k * \Omega G_k \xrightarrow{\Gamma_k} E_k \xrightarrow{\nu_k} BW_n$$

is null homotopic for each $k \geqslant 0$.

(c) *The restrictions of $\nu_k e_k$ and ν_{k-1} to the $2np^k - 2$ skeleton of E_{k-1} are homotopic.*

Then there is a map

$$\nu_\infty \colon E \to BW_n$$

*in (*1.6*) such that $\nu_\infty \Gamma$ is null homotopic and thus the induced H-space structure on T is homotopy Abelian.*

PROOF. Since the inclusion:

$$\bigcup_{k \geqslant 0} E_k^{2np^{k+1}-2} \to E$$

is a homotopy equivalence, we can define $\nu_\infty \colon E \to BW_n$ which restricts to ν_k on $E_k^{2np^{k+1}-2}$. Since $\partial' \colon \Omega S^{2n+1}\{p^r\} \to E$ factors through E_k for each k, we have a homotopy commutative diagram:

$$\begin{array}{ccccc}
\Omega S^{2n+1}\{p^r\} & = & \Omega S^{2n+1}\{p^r\} & & \\
\downarrow & & \downarrow & \searrow^{H} & \\
 & & & & BW_n \\
 & & & \nearrow_{\nu_k} & \\
E_k & \longrightarrow & E & &
\end{array}$$

By (a), the composition:

$$\Omega^2 S^{1n+1} \xrightarrow{\Omega\partial} \Omega S^{2n+1}\{p^r\} \xrightarrow{H} BW_n$$

induces an isomorphism in H_{2np-2}. Using cup product and the Bockstein, we can conclude that this composition is an epimorphism in all dimensions and consequently the fiber of this composition is S^{2n-1}. Now compare with (1.3) to see that the fiber of H is an Anick space.

In the diagram

$$\begin{array}{ccccc}
(\Omega G_k * \Omega G_k)^{2np^{k+1}-2} & \longrightarrow & E_k^{2np^{k+1}-2} & \xrightarrow{\nu_k} & BW_n \\
\downarrow & & \downarrow & & \| \\
\Omega G * \Omega G & \xrightarrow{\Gamma} & E & \xrightarrow{\nu_\infty} & BW_n,
\end{array}$$

the upper composition is null homotopic for each k, so the lower composition is null homotopic as well, thus the result follows from 2.11. □

2.5. Properties of G and T

We now recall, for future use, the properties of G and T that we will be using in the sequel.

THEOREM 2.13 ([**GT10**]). *For $p \geqslant 3$, $r \geqslant 1$ and $n \geqslant 1$, there is an Anick space T; i.e., there is a fibration sequence*

$$\Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} \longrightarrow T \longrightarrow \Omega S^{2n+1}$$

such that the composition

$$S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n+1}$$

has degree p^r. Furthermore we have the following properties:

(a) *There exists a space G and maps f, g, h such that the compositions*

$$G \xrightarrow{f} \Sigma T \xrightarrow{\widetilde{g}} G$$

$$T \xrightarrow{g} \Omega G \xrightarrow{h} T$$

are homotopic to the identity, where $\widetilde{g}$ is the adjoint of g.

(b) *Both T and G are atomic and p-complete.*

(c) *The homotopy type of T and G are unique satisfying these conditions.*

(d) *$\Sigma T \wedge T$ is a wedge of Moore spaces.*

(e) *For any H-space structure defined by the maps g, h in* (a), *there is an H-map*

$$E\colon T \longrightarrow \Omega S^{2n+1}\{p^r\}$$

such that $Eh \sim \Omega\varphi$.

(f) *$H^*(T;Z)$ is generated by classes $v_i \in H^{2np^i}(T;Z)$ for each $i \geqslant 0$ subject to the relations $v_i^p = pv_{i+1}$ and $p^r v_0 = 0$.*

(g) *$H_*(T) \simeq Z/p[v] \otimes \bigwedge(u)$ where $|v| = 2n$ and $|u| = 2n-1$. Furthermore $\beta^{(r+i)}(v^{p^i}) = uv^{p^i-1}$.*

(h) $H_m(G;Z) = \begin{cases} Z/p^{r+i} & \textit{if } m = 2np^i \\ 0 & \textit{otherwise.} \end{cases}$

(i) *ΣG and $\Sigma^2 T$ are each homotopy equivalent to a wedge of Moore spaces.*

(j) *Any choice of $\nu_k\colon E_k \longrightarrow BW_n$ in (1.6) has a right homotopy inverse.*

(k) *R is homotopy equivalent to a wedge of* mod p^s *Moore spaces for $s \geqslant r$.*

(l) *$\Sigma G \wedge G$ is homotopy equivalent to a wedge of Moore spaces*

Proof. Most of these are restatements of results in [**GT10**]. Properties (a), (b), (c), (d), (e), and (f) are respectively 4.4, 4.7, 4.9, 4.3(m), 4.6, and 4.1 of [**GT10**]. Property (g) follows immediately by applying the Serre spectral sequence to 1.1. Properties (h) and (i) are 4.3(c) and 4.5 respectively. For (j), a right homotopy inverse is given by the composition

$$BW_n \longrightarrow BW_n \times S^{4n-1} \simeq E_{(1)} \longrightarrow E_0 \longrightarrow E_k$$

constructed from the proof of 3.5. Property (k) is 4.8. For (l) note that $\Omega G \simeq T \times \Sigma R$ since $hg \sim 1$. Thus

$$\Sigma\Omega G \simeq \Sigma T \vee \Sigma\Omega R \vee \Sigma T \wedge \Omega R \simeq W_1 \vee \Sigma T$$

where W_1 is a wedge of Moore spaces by (i) and (k). Thus $\Sigma^2\Omega G \in \mathcal{W}$ and

$$\begin{aligned} \Sigma\Omega G \wedge \Omega G &\simeq W_1 \wedge \Omega G \vee \Sigma T \wedge \Omega G \simeq W_2 \vee T \wedge \Sigma\Omega G \\ &\simeq W_3 \vee \Sigma T \wedge T \end{aligned}$$

which is in a wedge of Moore space by property (d). □

CHAPTER 3

Whitehead Products

In this chapter we will review and extend some results in [**Gra11**] which generalize the notion of Whitehead products. In particular[1] given two simply connected co-H spaces G and H, we will construct a new co-H space $G \circ H$ and a cofibration sequence

$$G \circ H \xrightarrow{W} G \vee H \longrightarrow G \times H.$$

Composition with W defines a more general notion of Whitehead products. In section 3.1 we will review the material in [**Gra11**]. In section 3.2 we will discuss relative Whitehead products and H-space based Whitehead products which have been developed for homotopy groups with coefficients in Z/p^r by Neisendorfer [**Nei10a**]. We will define these in the total space of a principal fibration using co-H spaces in place of Moore spaces. In section 3.3 we will use these products to decompose $\Omega G * \Omega H$ and $\Omega G \ltimes H$ when G and H are co-H spaces and decompose the map Γ from section 2.2 as a wedge of iterated Whitehead products when the base is a co-H space. In section 3.4 we recall and generalize slightly the results of Neisendorfer in the case that G and H are Moore spaces.

3.1. Defining Whitehead Products Using co-H Spaces

Given two simply connected co-H spaces G, H, we introduce a new co-H space $G \circ H$ together with a cofibration sequence:

$$G \circ H \xrightarrow{W} G \vee H \longrightarrow G \times H.$$

To do this, suppose that G and H are given co-H space structures by constructing right inverses to the respective evaluation maps:

$$G \xrightarrow{\nu_1} \Sigma\Omega G \xrightarrow{\epsilon_1} G$$

$$H \xrightarrow{\nu_2} \Sigma\Omega H \xrightarrow{\epsilon_2} H.$$

We define a self map $e\colon \Sigma(\Omega G \wedge \Omega H) \to \Sigma(\Omega G \wedge \Omega H)$ as the composition

$$\Sigma(\Omega G \wedge \Omega H) \xrightarrow{\epsilon_1 \wedge 1} G \wedge \Omega H \xrightarrow{\nu_1 \wedge 1} \Sigma(\Omega G \wedge \Omega H) \xrightarrow{1 \wedge \epsilon_2} \Omega G \wedge H \xrightarrow{1 \wedge \nu_2} \Sigma(\Omega G \wedge \Omega H);$$

$G \circ H$ is then defined as the telescopic direct limit of e. We then have:

PROPOSITION 3.1 ([**Gra11**, 2.1,2.3]). *The identity map of $G \circ H$ factors:*

$$G \circ H \xrightarrow{\psi} \Sigma(\Omega G \wedge \Omega H) \xrightarrow{\theta} G \circ H.$$

[1]Throughout Chapter 3, G and H will designate an arbitrary co-H space. Then we will return G as to designating the co-H space corresponding to T in future chapters.

Furthermore, if $f\colon G \to G'$ and $g\colon H \to H'$ are co-H maps, there are induced co-H maps so that the diagram

$$\begin{array}{ccccc} G\circ H & \xrightarrow{\psi} & \Sigma(\Omega G\wedge \Omega H) & \xrightarrow{\theta} & G\circ H \\ \downarrow{\scriptstyle f\circ g} & & \downarrow{\scriptstyle \Sigma(\Omega f\wedge\Omega g)} & & \downarrow{\scriptstyle f\circ g} \\ G'\circ H' & \xrightarrow{\psi'} & \Sigma(\Omega G'\wedge \Omega H') & \xrightarrow{\theta'} & G'\circ H' \end{array}$$

commutes up to homotopy.

Since $G\circ H$ is the limit of the telescope defined by e, $\theta e \sim \theta$, so the composition

$$G\circ H \xrightarrow{\psi} \Sigma(\Omega G\wedge\Omega H) \xrightarrow{e} \Sigma(\Omega G\wedge\Omega H) \xrightarrow{\theta} G\circ H$$

is homotopic to the identity. The map e, however, is a composition of 4 maps between co-H spaces, and thus $G\circ H$ is a retract of 3 different co-H spaces and one of them, $\Sigma(\Omega G\wedge\Omega H)$, in two potentially distinct ways. This provides 4 potentially distinct co-H space structures on $G\circ H$. We choose the structure defined by ψ and θ; viz.,

$$G\circ H \xrightarrow{\psi} \Sigma(\Omega G\wedge\Omega H) \xrightarrow{\Sigma\widetilde{\theta}} \Sigma\Omega(G\circ H)$$

or equivalently

$$G\circ H \xrightarrow{\psi} \Sigma(\Omega G\wedge\Omega H) \to \Sigma(\Omega G\wedge\Omega H)\vee\Sigma(\Omega G\wedge\Omega H) \xrightarrow{\theta\vee\theta} G\circ H\vee G\circ H$$

where $\widetilde{\theta}$ is the adjoint of θ.

PROPOSITION 3.2 ([**Gra11**, 2.3,2.5]). *There are co-H equivalences $G\circ\Sigma X\simeq G\wedge X$, $\Sigma(G\circ H)\simeq G\wedge H$ which are natural for co-H maps in G and H and continuous maps in X.*

PROPOSITION 3.3 ([**Gra12**, 3.3]). *There is a natural cofibration sequence*

$$G\circ H \xrightarrow{W} G\vee H \longrightarrow G\times H$$

where W is the composition:

$$G\circ H \xrightarrow{\psi} \Sigma(\Omega G\wedge\Omega H) \xrightarrow{\omega} G\vee H.$$

DEFINITION 3.4. Let $\alpha\colon G\to X$ and $\beta\colon H\to X$. We define the Whitehead product[2]

$$\{\alpha,\beta\}\colon G\circ H\to X$$

as the composition

$$G\circ H \xrightarrow{W} G\vee H \xrightarrow{\alpha\vee\beta} X.$$

[2]We use the notation $\{\alpha,\beta\}$ rather than the usual $[\alpha,\beta]$ since in an important application we need to make a distinction. That is the case when G and H are both Moore spaces. In this case $G\circ H$ is a wedge of two Moore spaces. By choosing the higher dimensional one, Neisendorfer [**Nei80**] defines internal Whitehead products in homotopy with coefficients in Z/p^r. This is denoted $[\alpha,\beta]$, while $\{\alpha,\beta\}$ is the "external" Whitehead product.

PROPOSITION 3.5. *Each Whitehead product* $\{\alpha, \beta\}\colon G \circ H \to X$ *factors through the "universal Whitehead product"*

$$w = \nabla\omega\colon \Sigma(\Omega X \wedge \Omega X) \to X \vee X \to X.$$

PROOF. $\{\alpha, \beta\}$ is the upper composition in the commutative diagram:

$$\begin{array}{ccccccc} G \circ H & \xrightarrow{\psi} & \Sigma(\Omega G \wedge \Omega H) & \xrightarrow{\omega} & G \vee H & & \\ & & \downarrow & & \downarrow & & \\ & & \Sigma(\Omega X \wedge \Omega X) & \xrightarrow{\omega} & X \vee X & \xrightarrow{\nabla} & X \end{array}$$

The result follows since the bottom composition is $w = \nabla\omega$ □

3.2. H-space Based and Relative Whitehead Products

In this section we will discuss H-space based Whitehead products and relative Whitehead products. In the case that G and H are Moore spaces, this material is covered in [**Nei10a**], and what we present is a mild generalization. We need to consider Whitehead products instead of their adjoints—the Samelson products (which Neisendorfer considered) since the domains are not necessarily suspensions. We also consider principal fibrations, so these products occur in the total space rather than the fiber of a fibration as in Neisendorfer's version. We wish to thank Joe Neisendorfer for several interesting conversations during the development of this material.

We begin with a principal fibration

$$\Omega X \xrightarrow{i} E \xrightarrow{\pi} B$$

induced by a map $\varphi\colon B \to X$. The (external) relative Whitehead product then is a pairing

$$[G, B] \times [H, E] \to [G \circ H, E].$$

In the case that X is a homotopy commutative H-space with strict unit, we also define the H-space based Whitehead product. It is a pairing

$$[G, B] \times [H, B] \to [G \circ H, E].$$

Suppose we are given maps:

$$G \xrightarrow{\alpha} B,\ H \xrightarrow{\beta} B,\ G \xrightarrow{\gamma} E,\ H \xrightarrow{\delta} E.$$

We will use the notation

$$\{\alpha, \gamma\}_r \in [G \circ H, E]$$

for the relative Whitehead product and

$$\{\alpha, \beta\}_\times \in [G \circ H, E]$$

for the H-space based Whitehead product. These products and the absolute Whitehead product are related by the following formulas to be proved:

(3.6c) $$\pi\{\alpha,\beta\}_\times \sim \{\alpha,\beta\}\colon G\circ H \to B;$$

(3.6e) $$\{\pi\gamma,\pi\delta\}_\times \sim \{\gamma,\delta\}\colon G\circ H \to E;$$

(3.11c) $$\pi\{\alpha,\delta\}_r \sim \{\alpha,\pi\delta\}\colon G\circ H \to B;$$

(3.11e) $$\{\pi\gamma,\delta\}_r \sim \{\gamma,\delta\}\colon G\circ H \to E.$$

(3.12) $$\{\alpha,\delta\}_r \sim \{\alpha,\pi\delta\}_\times\colon G\circ H \to E;$$

We begin with the H-space based Whitehead product. These are defined using the map Γ from (2.7). The product $\{\alpha,\beta\}_\times$ is defined as the homotopy class of the upper composition in the diagram:

$$\begin{array}{ccccccc}
G\circ H & \xrightarrow{\psi} & \Sigma(\Omega G\wedge\Omega H)\simeq\Omega G*\Omega H & \longrightarrow & \Omega B*\Omega B & \xrightarrow{\Gamma} & E \\
& \searrow^{W} & \downarrow{\scriptstyle\omega} & & \downarrow{\scriptstyle\omega} & & \downarrow{\scriptstyle\pi} \\
& & G\vee H & \xrightarrow{\alpha\vee\beta} & B\vee B & \xrightarrow{\nabla} & B.
\end{array}$$

Proposition 3.6. *Given $\alpha\colon G\to B$ and $\beta\colon H\to B$, the homotopy class of the H-space based Whitehead product*

$$\{\alpha,\beta\}_\times\colon G\circ H\to E$$

depends only on the homotopy classes of α and β. Furthermore

(a) *If $f\colon G'\to G$ and $g\colon H'\to H$ are co-H maps,*

$$\{\alpha,\beta\}_\times(f\circ g)\sim\{\alpha f,\beta g\}_\times.$$

(b) *Given an induced fibration*

$$\begin{array}{ccccc}
E' & \xrightarrow{\widetilde{\xi}} & E & & \\
\downarrow & & \downarrow & & \\
B' & \xrightarrow{\xi} & B & \xrightarrow{\varphi} & X
\end{array}$$

and $\alpha'\colon G\to B'$, $\beta'\colon H\to B'$, we have

$$\widetilde{\xi}\,\{\alpha',\beta'\}_\times\sim\{\xi\alpha',\xi\beta'\}_\times\colon G\circ H\to E.$$

(c) $\pi\{\alpha,\beta\}_\times\sim\{\alpha,\beta\}\colon G\circ H\to B.$

(d) *Suppose $\eta\colon X\to X'$ is a strict H-map and we have a pointwise commutative diagram*

$$\begin{array}{ccc}
B & \xrightarrow{\xi} & B' \\
{\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\
X & \xrightarrow{\eta} & X'
\end{array}$$

which defines a map of principal fibrations:

$$\begin{array}{ccc} E & \xrightarrow{\widetilde{\xi}} & E' \\ \downarrow & & \downarrow \\ B & \xrightarrow{\xi} & B'. \end{array}$$

Then

$$\widetilde{\xi}\{\alpha, \beta\}_\times \sim \{\xi\alpha, \xi\beta\}_\times \colon G \circ H \to E'.$$

(e) $\{\pi\gamma, \pi\delta\}_\times \sim \{\gamma, \delta\} \colon G \circ H \to E$.

Proof. These all follow directly from the definition except for (e). To prove this we apply (d) to the diagram

$$\begin{array}{ccc} E & \xrightarrow{\pi} & B \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle \varphi} \\ PX & \xrightarrow{\epsilon} & X \end{array}$$

where $k(b, \omega) = \omega$. Give PX the H-space structure of pointwise multiplication of paths in X. Then ϵ is a strict H-map. This gives a map of principal fibrations:

$$\begin{array}{ccc} \widetilde{E} & \xrightarrow{\widetilde{\pi}} & E \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle \pi} \\ E & \xrightarrow{\pi} & B. \end{array}$$

By (d) we have

$$\widetilde{\pi}\{\gamma, \delta\}_\times \sim \{\pi\gamma, \pi\delta\}_\times.$$

It suffices to show that $\widetilde{\pi} \sim e$ by applying part (c) to the left hand fibration. The space $\widetilde{E} \subset (B \times PX) \times PPX$ can be described as follows

$$\widetilde{E} = \{(b, \sigma) \in B \times PPX \mid \varphi(b) = \sigma(1, 1), \sigma(s, 0) = \sigma(0, t) = *\}$$

with $\widetilde{\pi}(b, \sigma) = (b, \omega)$ where $\omega(t) = \sigma(t, 1)$ and $e(b, \sigma) = (b, \omega')$ where $\omega'(t) = \sigma(1, t)$. Define $F \colon \widetilde{E} \times I \times I \to X$ by σ. The result then follows from:

Homotopy Rotation Lemma 3.7. *Suppose* $F \colon A \times I \times I \to B$ *and* $F(a, 0, t) = F(a, s, 0) = F(*, s, t) = *$. *Then there is a homotopy*

$$H \colon A \times I \times I \to B$$

such that

$$\begin{aligned} H(a, 0, t) &= F(a, 1, t) \\ H(a, 1, t) &= F(a, t, 1) \\ H(a, s, 1) &= F(a, 1, 1) \\ H(a, s, 0) &= H(*, s, t) = * \end{aligned}$$

Proof. The left side and the bottom of the square are mapped to the base-point. By rotating from the top to the right hand side pivoting at the point $(1, 1)$, we obtain the required homotopy. □ □

We now describe the relative Whitehead product. We assume a principal fibration

$$\Omega X \xrightarrow{\ i\ } E \xrightarrow{\ \pi\ } B$$

induced by a map $\varphi\colon B \to X$ (as in section 2.1), but we won't assume an H-space structure on X. Define $k\colon E \to PX$ by the second component, so $k(e)(1) = \varphi\pi(e)$. The principal action

$$a\colon \Omega X \times\ E \longrightarrow E$$

is defined by the formula

$$a(\omega, e) = (\pi(e), \omega')$$

where ω' is given by

$$\omega'(t) = \begin{cases} \omega(2t) & 0 \leqslant t \leqslant 1/2 \\ k(e)(2t-1) & 1/2 \leqslant t \leqslant 1. \end{cases}$$

We then describe a strictly commutative diagram of vertical fibration sequences

$$\begin{array}{ccc}
\Omega B & \xrightarrow{\quad d \quad} & \Omega X \\
\downarrow & & \downarrow{\scriptstyle i} \\
\Omega B \times E \cup PB & \xrightarrow{\ \Gamma'\ } & E \\
{\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \pi} \\
B \vee E & \xrightarrow{\ 1\vee\pi\ } & B
\end{array}$$

where $\Omega B \times E \cup PB$ is to be considered as a subspace of $PB \times E$, and the map φ is given as follows: $\varphi|_{\Omega B \times E}$ is the projection onto E and $\varphi|_{PB}$ is end point evaluation. The map Γ' is defind by the formula

$$\begin{aligned} \Gamma'(\omega, e) &= a(\varphi\omega, e) && \text{for } (\omega, e) \in \Omega B \times E \\ \Gamma'(\omega) &= (\omega(1), \varphi\widetilde{\omega}) && \text{for } \omega \in PB \end{aligned}$$

where[3] $\widetilde{\omega}(t) = \omega(2t)$, and the map $d\colon \Omega B \to \Omega X$ is given by $d(\omega) = \varphi\widetilde{\omega}$. The left hand fibration is the principal fibration induced by the projection $\pi_1\colon B \vee E \to B$.

Observe that

$$\Omega B \times E \cup PB \simeq \Omega B \times E \cup C(\Omega B) \simeq \Omega B \ltimes E.$$

We record an important commutative diagram

$$\begin{array}{ccc}
\Omega B \times E & \xrightarrow{\quad \Omega\varphi \times 1 \quad} & \Omega X \times E \\
\downarrow & & \downarrow{\scriptstyle a} \\
\Omega B \ltimes E \simeq \Omega B \times E \cup PB & \xrightarrow{\ \Gamma'\ } & E
\end{array} \tag{3.8}$$

which will be useful in evaluating the relative Whitehead products in homology.

[3] For convenience, we extend maps $[0,1] \xrightarrow{\ f\ } X$ to the real line by $f(x) = f(0)$ for $x < 0$ and $f(x) = f(1)$ for $x > 1$.

Consider the strictly commutative square

$$\begin{array}{ccc} B \vee E & = & B \vee E \\ \downarrow & & \downarrow {\scriptstyle \pi_1} \\ B \times E & \xrightarrow{\pi_1} & B. \end{array}$$

Taking homotopy fibers vertically, we obtain a diagram of principal fibrations

$$\begin{array}{ccc} \Omega(B \times E) & \xrightarrow{\Omega\pi_1} & \Omega B \\ \downarrow & & \downarrow \\ \Omega B \times PE \cup PB \times \Omega E & \xrightarrow{\zeta} & \Omega B \times E \cup PB \\ \downarrow & & \downarrow \\ B \vee E & = & B \vee E \end{array}$$

The map ζ is defined by the formula

$$\Omega B \times PE \xrightarrow{1 \times \epsilon} \Omega B \times E$$

$$PB \times \Omega E \xrightarrow{\pi_1} PB$$

Combining these diagrams, we obtain a strictly commutative diagram

$$\begin{array}{ccccccc} \Omega B * \Omega E & \simeq & \Omega B \times PE \cup PB \times \Omega E & \xrightarrow{\zeta} & \Omega B \times E \cup PB & \xrightarrow{\Gamma'} & E \\ & {\scriptstyle W} \searrow & \downarrow & & \downarrow & & \downarrow {\scriptstyle \pi} \\ & & B \vee E & = & B \vee E & \xrightarrow{1 \vee \pi} & B. \end{array} \tag{3.9}$$

For $\alpha\colon G \to B$ and $\delta\colon H \to E$, we define the relative Whitehead product

$$\{\alpha, \delta\}_r\colon G \circ H \to E$$

as the composition

$$G \circ H \xrightarrow{\psi} \Omega G * \Omega H \to \Omega B * \Omega E \xrightarrow{\zeta} \Omega B \times E \cup PB \xrightarrow{\Gamma'} E \tag{3.10}$$

and, analogous to 3.6, we have

Proposition 3.11. *The homotopy class of the relative Whitehead product* $\{\alpha, \delta\}_r$ *depends only on the homotopy classes of* α *and* δ. *Furthermore*

(a) *If* $f\colon G' \to G$ *and* $g\colon H' \to H$ *are co-H maps, then*

$$\{\alpha, \delta\}_r \cdot (f \circ g) \sim \{\alpha f, \delta g\}_r.$$

(b) *Given an induced fibration*

$$\begin{array}{ccccc} E' & \xrightarrow{\widetilde{\xi}} & E & & \\ \downarrow & & \downarrow & & \\ B' & \xrightarrow{\xi} & B & \xrightarrow{\varphi} & X \end{array}$$

and classes $\alpha'\colon G \to B'$, $\delta'\colon H \to E'$, *we have*

$$\widetilde{\xi}\{\alpha', \delta'\}_r \sim \{\xi\alpha', \widetilde{\xi}\delta'\}_r.$$

(c) $\pi\{\alpha, \delta\}_r \sim \{\alpha, \pi\delta\}$.

(d) *Suppose we have a strictly commutative diagram*

$$\begin{array}{ccc} B & \xrightarrow{\xi'} & B' \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ X & \xrightarrow{\eta} & X' \end{array}$$

inducing a map between principal fibrations:

$$\begin{array}{ccc} E & \xrightarrow{\xi'} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ B & \xrightarrow{\xi} & B' \end{array}$$

Then $\xi'\{\alpha, \delta\}_r \sim \{\xi\alpha, \xi'\delta\}_r$.

(e) $\{\pi\gamma, \delta\}_r \sim \{\gamma, \delta\}$.

Proof. All parts except (e) follow directly from the definitions. For part (e) we construct a map of principal fibrations exactly as in 3.6(e):

$$\begin{array}{ccc} \widetilde{E} & \xrightarrow{\widetilde{\pi}} & E \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle\pi} \\ E & \xrightarrow{\pi} & B \end{array}$$

Recall that $e \sim \widetilde{\pi}$, and since PX is contractible, both e and $\widetilde{\pi}$ are homotopy equivalences. Choose $\widetilde{\delta}\colon H \to \widetilde{E}$ such that $\widetilde{\pi}\widetilde{\delta} \sim \delta$. Then by part (d) we have

$$\{\pi\gamma, \delta\}_r \sim \widetilde{\pi}\{\gamma, \widetilde{\delta}\}_r \sim e\{\gamma, \widetilde{\delta}\}_r \sim \{\gamma, e\widetilde{\delta}\}$$

by part (c). However this is homotopic to $\{\gamma, \widetilde{\pi}\widetilde{\delta}\} \sim \{\gamma, \delta\}$. □

At this point we will discuss the compatibility of the H-space based Whitehead product and the relative Whitehead product.

Theorem 3.12. *Suppose X is an H-space with strict unit and we are given* $\alpha\colon G \to B$, $\delta\colon H \to E$. *Then*

$$\{\alpha, \delta\}_r \sim \{\alpha, \pi\delta\}_\times$$

Proof. To prove this we will combine two homotopies and 3.12 is a consequence of Proposition 3.15. The first homotopy will replace the sequential composition of paths in the definition of the action map a and Γ' with a blending of the homotopies using the H-space structure in X. The second homotopy will apply the homotopy rotation lemma (3.7). Recall the map $k\colon B \to PX$ with the property that $\epsilon k \sim \varphi\pi$

$$\begin{array}{ccc} E & \xrightarrow{\pi} & B \\ {\scriptstyle k}\downarrow & & \downarrow \\ PX & \xrightarrow{\epsilon} & X \end{array}$$

LEMMA 3.13. *There is a homotopy* $a_s\colon \Omega X \times E \to E$ *with* $a_1 = a$ *and* a_0 *given by the formula*

$$a_0(\omega, e) = (\pi(e), \mu(\omega(t), k(e)(t)))$$

and a compatible homotopy $\Gamma'_s\colon \Omega B \times E \cup PB \to E$ *with* $\Gamma'_1 = \Gamma'$ *and* Γ'_0 *given by the formula*

$$\Gamma'_0(\omega, e) = (\pi(e), \mu(\varphi\omega(t), k(e)(t))$$
$$\Gamma'_0(\omega) = (\omega(1), \omega\varphi) \quad \text{for } \omega \in PB$$

PROOF. Recall that any map $\omega\colon [0,1] \to X$ is to be extended to a map $\omega\colon R \to X$ by defining $\omega(x) = \omega(0)$ if $x < 0$ and $\omega(x) = \omega(1)$ if $x > 1$. Then, for example,

$$\mu(\omega(2t), k(e)(2t-1)) = \begin{cases} \omega(2t) & \text{if } 0 \leqslant t \leqslant 1/2 \\ k(e)(2t-1) & \text{if } 1/2 \leqslant t \leqslant 1 \end{cases}$$

since $\omega(1) = k(e)(0) = *$, the unit for μ. We define

$$a_s(\omega, e) = (\pi(e), \omega_s)$$

where

$$\omega_s(t) = \mu\left(\omega\left(\frac{2t}{2-s}\right), k(e)\left(\frac{2t-s}{2-s}\right)\right).$$

We define $\Gamma'_s(\omega, e) = a_s(\varphi\omega, e)$ and $\Gamma'_s(\omega) = \left(\omega(1), \varphi\omega\left(\frac{2t}{2-s}\right)\right)$ in case $\omega \in PB$. □

Using Γ'_0 we consider the composition

$$\Omega B * \Omega E \xrightarrow{\zeta} \Omega B \times E \cup PB \xrightarrow{\Gamma'_0} E.$$

Using the identification $\Omega B * \Omega E \simeq PB \times \Omega E \cup \Omega E \times PB$, we have the following formula for this composition

$$(3.14) \qquad (\omega_1, \omega_2) \longrightarrow (\omega_1, \omega_2(1)) \longrightarrow (\pi\omega_2(1), \mu(\varphi\omega_1(t), k(\omega_2(1))(t))).$$

We now apply 3.7 to the homotopy

$$F\colon PE \times I \times I \to X$$

given by $F(\omega, s, t) = k(\omega(t))(s)$ to obtain a homotopy

$$H\colon PE \times I \to X$$

with

$$H(\omega, 1, t) = k(\omega(1))(t)$$
$$H(\omega, 0, t) = \varphi\pi\omega(t)$$
$$H(\omega, s, 0) = *$$
$$H(\omega, s, 1) = \varphi\pi\omega(1).$$

From this we construct a homotopy

$$\Gamma_s\colon \Omega B \times PE \cup PB \times \Omega E \to E$$

given by

$$\Gamma_s(\omega_1, \omega_2) = \begin{cases} \omega_1(1) \quad \mu(\varphi\omega_1(t), H(\omega_2, s, t)) & \text{if } \omega_2(1) = * \\ \pi\omega_2(1) \quad \mu(\varphi\omega_1(t), H(\omega_2, s, t)) & \text{if } \omega_1(1) = *. \end{cases}$$

Then $\Gamma_1 = \Gamma(1 * \Omega\pi)$ and $\Gamma_0 = \Gamma_0'\zeta$. We have proved

PROPOSITION 3.15. *If X is an H space with a strict unit, there is a homotopy commutative diagram:*

$$\begin{array}{ccc} \Omega B \times E \cup PB & \xrightarrow{\Gamma'} & E \\ \zeta \uparrow & & \uparrow \Gamma \\ \Omega B * \Omega E & \xrightarrow{1 * \Omega\pi} & \Omega B * \Omega B \end{array}$$

□

Clearly 3.12 follows from 3.15. □

We next describe a simplification of the relative Whitehead product in case $G = \Sigma A$.

PROPOSITION 3.16. *Suppose $\alpha\colon \Sigma A \to B$ and $\delta\colon H \to E$. Then the relative Whitehead product*

$$A \wedge H \simeq \Sigma A \circ H \xrightarrow{\{\alpha, \delta\}_r} E$$

is represented by the composition

$$A \wedge H \xrightarrow{\theta} A \ltimes H \xrightarrow{\widetilde{\alpha} \ltimes \delta} \Omega B \ltimes E \xrightarrow{\Gamma'} E$$

where θ is a right homotopy inverse to the projection[4] *which pinches H to a point.*

PROOF. To construct the map θ we need to generalize the context in which the map ζ was defined in 3.9. The homotopy fiber of the map

$$CX \cup X \times Y \xrightarrow{\pi_2} Y$$

which pinches CX to the basepoint is of the form

$$CX \times \Omega Y \cup X \times PY \subset CX \times PY.$$

Using the homotopy equivalence $\xi\colon (C(\Omega Y), \Omega Y) \to (PY, \Omega Y)$ (see 2.6), we get a homotopy equivalent fibration sequence

(3.17) [5]
$$X * \Omega Y \xrightarrow{\zeta} \begin{array}{c} CX \cup X \times Y \\ \wr| \\ X \ltimes Y \end{array} \xrightarrow{\pi_2} Y$$

Furthermore, the composition

$$X * \Omega Y \xrightarrow{\zeta} X \ltimes Y \longrightarrow X \wedge Y$$

[4] θ will depend on the co-H structure of H.

[5] Curiously there is also a cofibration sequence

$$X * Y \xrightarrow{\zeta'} X \ltimes \Sigma Y \xrightarrow{\pi_2} \Sigma Y$$

where ζ' is the composition

$$X * Y \longrightarrow X * \Omega\Sigma Y \xrightarrow{\zeta} X \ltimes \Sigma Y.$$

collapses $X \cup CX \times \Omega Y$ to a point, so there is a commutative square

$$\begin{array}{ccc} X * \Omega Y & \xrightarrow{\zeta} & X \ltimes Y \\ \simeq \downarrow & & \downarrow \\ X \wedge \Sigma\Omega Y & \xrightarrow{1 \wedge \epsilon} & X \wedge Y. \end{array}$$

The relative Whitehead product in 3.16 is given by the upper composition in the homotopy commutative diagram

$$\begin{array}{ccccccccc} & & & & \Omega B * \Omega E & \xrightarrow{\zeta} & \Omega B \times E \cup PB & \xrightarrow{\Gamma'} & E \\ & & & & \Omega\alpha * \Omega\delta \uparrow & & \uparrow & & \\ (\Sigma A) \circ H & \to & \Sigma(\Omega\Sigma A \wedge \Omega H) & \simeq & \Omega\Sigma A * \Omega H & \xrightarrow{\zeta} & \Omega\Sigma A \times H \cup P\Sigma A & & \\ \simeq \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ A \wedge H & \xrightarrow{1 \wedge \nu} & \Sigma(A \wedge \Omega H) & \simeq & A * \Omega H & \xrightarrow{\zeta} & A \times H \cup CA & \simeq & A \ltimes H \end{array}$$

where the lower composition is the map θ. The right hand vertical map is the composition

$$A \ltimes H \xrightarrow{\widetilde{\alpha} \ltimes \delta} \Omega B \ltimes E \simeq \Omega B \times E \cup PB.$$

By the homotopy commutative square above, θ has a right homotopy inverse since $\epsilon\nu \sim 1$, and θ projects trivially to H since $\pi_2 \zeta$ is null homotopic. ☐

COROLLARY 3.18. *Suppose $\alpha \colon \Sigma A \to B$ and $\delta \colon H \to E$. Then for any ring R, the homomorphism*

$$(\{\alpha, \delta\}_r)_* \colon H_*(A \wedge H; R) \to H_*(E; R)$$

is given by the composition

$$H_*(A \wedge H; R) \subset H_*(A \times H; R) \xrightarrow{(\widetilde{\varphi\alpha} \times \delta)_*} H_*(\Omega X \times E; R) \xrightarrow{a_*} H_*(E; R)$$

PROOF. Apply 3.8 and 3.16. ☐

3.3. Iterated Whitehead Products and the Decomposition of $\Omega G * \Omega H$

We need, also, to discuss iterated Whitehead products. Suppose

$$\alpha_i \colon G_i \to X$$

for $1 \leqslant i \leqslant n$. We define the iterated Whitehead product

$$\{\alpha_n, \alpha_{n-1}, \dots, \alpha_1\} \colon G_n \circ (G_{n-1} \circ \cdots \circ G_1) \dots) \to X$$

as $\{\alpha_n, \{\alpha_{n-1}, \dots, \alpha_1\}\}$. In case $G_i = G$ for each i we define

$$G^{[n]} = G \circ G^{[n-1]}.$$

We also define $G^{[i]}H^{[j]}$ as $G \circ (G^{[i-1]}H^{[j]}_*)$ when $i > 1$ and as $G \circ H^{[j]}$ when $i = 1$.

Suppose now that $\alpha \colon G \to G \vee H$ and $\beta \colon H \to G \vee H$ are the inclusions. We then consider

$$ad^i(\alpha)(\{\alpha, \beta\}) = \{\alpha, \dots, \alpha, \beta\} \colon G^{[i+1]}H \to G \vee H.$$

Given a principal fibration

$$\Omega X \longrightarrow E \longrightarrow B$$

and maps $\alpha_i\colon G \to B$, $\beta\colon H \to E$, we define

$$\{\alpha_n, \ldots, \alpha_1, \beta\}_r\colon G_n \circ (G_{n-1} \circ \cdots \circ (G_1 \circ H) \ldots) \to E$$

as

$$\{\alpha_n, \{\alpha_{n-1}, \ldots, \alpha_1, \beta\}_r\}_r.$$

By an iterated application of 3.11(c), we have

$$\pi\{\alpha_n, \ldots, \alpha_1, \beta\}_r = \{\alpha_n, \ldots, \alpha_1, \pi\beta\}. \tag{3.19}$$

Now consider the principal fibration

$$\Omega G \xrightarrow{i} \Omega G \ltimes H \xrightarrow{\pi} G \vee H.$$

Let $K = \bigvee_{i \geqslant 1} G^{[i]}H$. Let $\beta\colon H \to \Omega G \ltimes H$ be the inclusion of the second factor.

Proposition 3.20. *The maps $ad_r^i(\alpha)(\beta)\colon G^{[i]}H \to \Omega G \ltimes H$ define a homotopy equivalence*

$$K \vee H \simeq \bigvee_{i \geqslant 1} G^{[i]}H \vee H \to \Omega G \ltimes H.$$

Proof. According to [**Gra11**, 3a], such a homotopy equivalence exists where the maps

$$\xi_i\colon G^{[i]}H \to \Omega G \ltimes H$$

are chosen so that $\pi\xi_i \sim \{\alpha, \ldots, \alpha, \pi\beta\}$. However by equation (3.19), $\pi\{\alpha, \ldots, \alpha, \beta\}_r \sim \{\alpha, \ldots, \alpha, \pi\beta\}$; since the map $i\colon \Omega G \to \Omega G \ltimes H$ is null homotopic, $\xi_i \sim \{\alpha, \ldots, \alpha, \beta\}_r$. □

We now consider the principal fibration

$$\Omega G \times \Omega H \longrightarrow \Omega G * \Omega H \longrightarrow G \vee H.$$

Using the map $\psi\colon G \circ H \to \Omega G * \Omega H$ we define iterated relative Whitehead products:

$$ad_r^{i,j} = ad_r^i(\beta) ad_r^j(\alpha)(\psi)\colon H^{[i]}G^{[j]}(G \circ H) \to \Omega G * \Omega H$$

Proposition 3.21. *The maps $ad_r^{i,j}$ for $i \geqslant 0$, $j \geqslant 0$ define a homotopy equivalence*

$$\bigvee_{\substack{i \geqslant 0 \\ j \geqslant 0}} H^{[i]}G^{[j]}(G \circ H) \to \Omega G * \Omega H.$$

Proof. Consider the diagram of principal fibrations

$$\begin{array}{ccc}
\Omega H & \xrightarrow{\iota_2} & \Omega G \times \Omega H \\
\downarrow & & \downarrow \\
\Omega G * \Omega H & = & \Omega G * \Omega H \\
\zeta \downarrow & & \pi \downarrow \\
\Omega G \ltimes H & \longrightarrow & G \vee H
\end{array}$$

The maps $ad_r^j\colon G^{[j]}(G \circ H) \to \Omega G \ltimes H$ defined by 3.20 lift to $\Omega G * \Omega H$ since ζ has a right homotopy inverse. Since the liftings project by π onto the maps:

$$ad^j : G^{[j]}(G \circ H) \to G \vee H,$$

these liftings are homotopic to the relative Whitehead product defined by π. However $\Omega G \ltimes H$ is homotopy equivalent to $H \vee K$ so the maps

$$ad_r^i H^{[i]} K \to \Omega G * \Omega H$$

define a homotopy equivalence

$$\bigvee_{i \geqslant 0} H^{[i]} K \to \Omega G * \Omega H.$$

Furthermore

$$K = \bigvee_{j \geqslant 0} G^{[j]}(G \circ H)$$

and the relative Whitehead products defined by the left hand fibration are mapped to the corresponding relative Whitehead products in the right hand fibration. Thus we have

$$\bigvee_{\substack{i \geqslant 0 \\ j \geqslant 0}} H^{[i]} G^{[j]}(G \circ H) \xrightarrow{\simeq} \bigvee_{i \geqslant 0} H^{[i]} K \xrightarrow{\simeq} \Omega G * \Omega H$$ □

THEOREM 3.22. *Suppose*

$$\Omega X \xrightarrow{i} E \xrightarrow{\pi} G$$

is a principal fibration induced by a map $\varphi\colon G \to X$ where X is an H-space with strict unit. Suppose $\nu\colon E \to Z$. Then the composition

$$\Omega G * \Omega G \xrightarrow{\Gamma} E \xrightarrow{\nu} Z$$

is null homotopic iff the compositions

$$\nu ad_r^i(\alpha)(\{\alpha, \alpha\}_\times)\colon G^{[[i+2]} \to E \to Z$$

are null homotopic for each $i \geqslant 0$, where $\alpha\colon G \to G$ is the identity map.

PROOF. In this case $G = H$ and the map of principal fibrations

$$\begin{array}{ccc} \Omega G \times \Omega G & \longrightarrow & \Omega X \\ \downarrow & & \downarrow \\ \Omega G * \Omega G & \longrightarrow & E \\ \downarrow & & \downarrow \\ G \vee G & \xrightarrow{\nabla} & G \end{array}$$

maps $H^{[i]} G^{[j]}(G \circ H)$ to $G^{[i+j]}(G \circ G)$ which only depends on $i + j$. □

3.4. Neisendorfer's Theory for Homotopy with Coefficients

In the case that the co-H spaces are Moore spaces, the resulting Whitehead products occur in the homotopy groups with coefficients. The adjoint theory of Samelson products is due to Neisendorfer [**Nei80**], and was crucial in the work of [**CMN79b**, **CMN79c**, **CMN79a**]. This theory has been further developed in [**Nei10a**] where H-space based Whitehead products were introduced.

We need to make a mild generalization of this in that we must consider the case where

$$G = \Sigma P^m(p^r) \qquad H = \Sigma P^n(p^s) \qquad s \geqslant r.$$

In this case

$$G \circ H = \Sigma P^{m+n}(p^r) \vee \Sigma P^{m+n-1}(p^r).$$

This splitting is not unique and we must choose a splitting.

Choose a map

$$\Delta\colon P^{m+n}(p^s) \longrightarrow P^m(p^s) \wedge P^n(p^s)$$

so that the diagram

$$\begin{array}{ccc} P^{m+n}(p^s) & \xrightarrow{\Delta} & P^m(p^s) \wedge P^n(p^s) \\ \downarrow{\scriptstyle \pi_{m+n}} & & \downarrow \\ S^{m+n} & \xrightarrow{\simeq} & S^m \wedge S^n \end{array} \tag{3.23}$$

commutes up to homotopy. Such a choice is possible when $m, n \geqslant 2$ for p odd and is unique up to homotopy.

Neisendorfer [**Nei80**] has produced internal Whitehead and Samelson products for homotopy with Z/p^s coefficients. The Whitehead product of $x \in \pi_{m+1}(X; Z/p^s)$ and $y \in \pi_{n+1}(X; Z/p^s)$ is an element

$$[x, y] \in \pi_{m+n+1}(X; Z/p^s)$$

defined as the homotopy class of the composition:

$$\begin{aligned} P^{m+n+1}(p^s) = {} & \Sigma P^{m+n}(p^s) \xrightarrow{\Sigma\Delta} \Sigma P^m(p^s) \wedge P^n(p^s) \\ = {} & P^{m+1}(p^s) \circ P^{n+1}(p^s) \xrightarrow{\{x,y\}} X \end{aligned} \tag{3.24}$$

As we will need to consider such pairings with different coefficients, suppose $x \in \pi_{m+1}(X; Z/p^r)$ and $y \in \pi_{n+1}(X; Z/p^{r+t})$. We can still form the external Whitehead product:

$$\Sigma P^m(p^r) \wedge P^n(p^{r+t}) = P^{m+1}(p^r) \circ P^{n+1}(p^{r+t}) \xrightarrow{\{x,y\}} X.$$

Since the map of degree p^{r+t} on $P^m(p^r)$ is null homotopic, there is a splitting:

$$P^m(p^r) \wedge P^n(p^{r+t}) \simeq P^{m+n}(p^r) \vee P^{m+n-1}(p^r).$$

We now choose an explicit splitting. Recall (1.5) $\delta_t = \beta\rho^t$.

PROPOSITION 3.25. *There is a splitting of* $P^m(p^r) \wedge P^n(p^{r+t})$ *defined by the two compositions:*

$$P^{m+n}(p^r) \xrightarrow{\Delta} P^m(p^r) \wedge P^n(p^r) \xrightarrow{1 \wedge \rho^t} P^m(p^r) \wedge P^n(p^{r+t})$$

$$P^{m+n-1}(p^r) \xrightarrow{\Delta} P^m(p^r) \wedge P^{n-1}(p^r) \xrightarrow{1 \wedge \delta_t} P^m(p^r) \wedge P^n(p^{r+t})$$

PROOF. $(1 \wedge \pi_n)(1 \wedge \rho^t)\Delta = (1 \wedge \pi_n)\Delta$ induces a mod p homology isomorphism, so $(1 \wedge \rho^t)\Delta$ induces a homology monomorphism. The second composition factors

$$P^{m+n-1}(p^r) \xrightarrow{\Delta} P^m(p^r) \wedge P^{n-1}(p^r) \xrightarrow{1 \wedge \pi_{n-1}} P^m(p^r) \wedge S^{n-1}$$
$$\xrightarrow{1 \wedge \iota_{n-1}} P^m(p^r) \wedge P^n(p^{r+t})$$

and the composition of the first two maps is a homotopy equivalence. Since the third map induces a mod p homology monomorphism, this composition does as well. Counting ranks, we see that the two maps together define a homotopy equivalence:

$$e\colon P^{m+n}(p^r) \vee P^{m+n-1}(p^r) \xrightarrow{\simeq} P^m(p^r) \wedge P^n(p^{r+t}) \qquad \square$$

We apply this to the internal Whitehead product (3.24) to get

PROPOSITION 3.26.

$$\{x, y\}e = [x, y\rho^t] \vee [x, y\delta_t]\colon P^{m+n}(p^r) \vee P^{m+n-1}(p^r) \to X. \qquad \square$$

3.26 resolves the external Whitehead product with different coefficients into internal Whitehead products with coefficients in Z/p^r as considered by Neisendorfer.

Suppose now that we are given a principal fibration

$$\Omega X \to E \to B$$

classified by a map $\varphi\colon B \to X$ where X is a homotopy commutative H-space with strict unit and we are given classes $u \in \pi_m(B; Z/p^r)$ and $v \in \pi_n(B; Z/p^{r+t})$. Then we have

PROPOSITION 3.27.

$$\{u, v\}_\times e = [u, v\rho^t]_\times \vee [u, v\delta_t]_\times\colon P^{m+n}(p^r) \vee P^{m+n-1}(p^{r+t}) \to E.$$

PROOF. Both $\{u, v\}_\times e$ and $[u, v\rho^t]_\times \vee [u, v\delta_t]_\times$ are the images under Γ of maps:

$$P^{m+n}(p^r) \vee P^{m+n-1}(p^r) \to \Sigma(\Omega B \wedge \Omega B)$$

which are homotopic after projection

$$\Sigma(\Omega B \wedge \Omega B) \xrightarrow{\omega} B \vee B$$

by 3.26. Since $\Omega\omega$ has a left homotopy universe, these maps are homotopic. Composing with $\Gamma\colon \Sigma(\Omega B \wedge \Omega B) \to E$ finishes the proof. $\square$

Similar to 3.27, we have

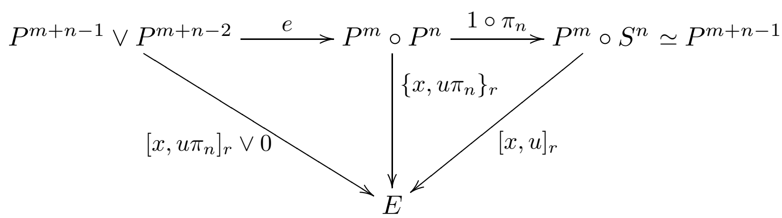
PROPOSITION 3.28. $\{x, u\}_r e = [x, u\rho^t]_r \vee [x, u\delta_t]_r$ *where* $x \in \pi_m(B; Z/p^r)$ *and* $u \in \pi_n(E; Z/p^{r+t})$. $\square$

There is one special case of this that we will need in section 6.3. This involves relative Whitehead products $[x, u]_r$ when $u\colon S^n \to E$ and $x\colon P^m \to B$. In this case

$$[x, u]_r = \{x, u\}_r\colon\ P^m \circ S^n \longrightarrow E\ .$$

PROPOSITION 3.29. $[x, u\pi_n]_r = [x, u]_r\colon\ P^{m+n-1} \longrightarrow E$.

PROOF. $\{x, u\pi_n\}_r e = [x, u\pi_n]_r \vee 0$ since $\pi_n \delta_t = 0$. Consequently we have a homotopy commutative diagram



where the upper composition is homotopic to projection onto the first factor. □

Suppose then we are given a principal fibration

$$\Omega X \xrightarrow{\ i\ } E \xrightarrow{\ \pi\ } B$$

induced by a map $\varphi\colon B \to X$ where X is a homotopy commutative H-space with strict unit. Suppose we are given classes

$$\alpha \in \pi_m(B; Z/p^r),\ \beta \in \pi_n(B; Z/p^r),\ \gamma \in \pi_k(E; Z/p^r),\ \delta \in \pi_\ell(E; Z/p^r).$$

Recall that by using the map Δ we define the internal H-space based Whitehead product

$$[\alpha, \beta]_\times = \{\alpha, \beta\}_\times \Delta \in \pi_{m+n-1}(E; Z/p^r)$$

and internal relative Whitehead product

$$[\alpha, \gamma]_r = \{\alpha, \gamma\}_r \Delta \in \pi_{m+k-1}(E; Z/p^r).$$

These are related as in 3.6, 3.11 and 3.12.

PROPOSITION 3.30. (a) $\pi_*[\alpha, \beta]_\times = [\alpha, \beta] \in \pi_{m+n-1}(B; Z/p^r)$
(b) $[\pi_*\gamma, \pi_*\delta]_\times = [\gamma, \delta] \in \pi_{k+\ell-1}(E; Z/p^r)$
(c) $[\alpha, \delta]_r = [\alpha, \pi_*\delta]_\times \in \pi_{m+\ell-1}(E; Z/p^r)$
(d) $\pi_*[\alpha, \delta]_r = [\alpha, \pi_*\delta] \in \pi_{m+\ell-1}(B; Z/p^r)$
(e) $[\pi_*\gamma, \delta]_r = [\gamma, \delta] \in \pi_{k+\ell-1}(E; Z/p^r)$

According to Neisendorfer [**Nei10a**], we also have standard Whitehead product formulas:

PROPOSITION 3.31. *The following identities hold:*
(a) $[\alpha, \beta]_\times = -(-1)^{(m+1)(n+1)}[\beta, \alpha]_\times$
(b) $[\alpha_1 + \alpha_2, \beta]_\times = [\alpha_1, \beta]_\times + [\alpha_2, \beta]_\times$
(c) $[\alpha, [\beta, \eta]]_\times = [[\alpha, \beta], \eta]_\times + (-1)^{(m+1)(n+1)}[\beta, [\alpha, \eta]]_\times$
for $\eta \in \pi_j(B; Z/p^r)$
(c′) $[\alpha, [\beta, \gamma]_r] = [[\alpha, \beta], \gamma]_r + (-1)^{(m+1)(n+1)}[\beta, [\alpha, \gamma]_r]_r$
(d) $\beta^{(r)}[\alpha, \beta]_\times = [\beta^{(r)}\alpha, \beta]_\times + (-1)^{m+1}[\alpha, \beta^{(r)}\beta]_\times$ *where* $\beta^{(r)}$ *is the Bockstein associated with the composition* $P^k(p^r) \to P^{k+1}(p^r)$ *for appropriate* k.

PROOF. See [**Nei10a**]. Neisendorfer considers the adjoint Samelson products, so there is a dimension shift. □

CHAPTER 4

Index p approximation

The goal of this chapter is to replace the co-H spaces $G_k^{[i]}$ from 3.22 by a finite wedge of Moore spaces in case $n > 1$. The iterated Whitehead products involving G_k are then replaced by iterated Whitehead products in $\bmod p^s$ homotopy, which are more manageable. In 4.1, we construct certain $\bmod p^{r+i-1}$ homotopy classes $a(i)$ and $c(i)$ for $i \leqslant k$. This is a refinement of a similar construction in [**GT10**], and leads to a ladder of cofibration sequences. In 4.2, we construct new co-H spaces L_k when $n > 1$, and introduce index p approximation. Using this we exploit the fact ([**The08**]) that the identity map of BW_n has order p to reduce the size of the set of obstructions. This allows for the replacement of the iterated relative and H-space based Whitehead products based on G_k with iterated relative and H-space based Whitehead products in the $\bmod p^s$ homotopy groups for $r \leqslant s \leqslant r+k$. The case $n = 1$ is simpler and we show that T is homotopy-Abelian in the appendix. Nevertheless, the constructions in Chapters 4, 5 and 6 will be used in Chapter 7 in case $n = 1$ as well.

4.1. Construction of the co-H Ladder

In this section we will assume an arbitrary H-space structure on the Anick space as given in [**GT10**] and use its existence to develop certain maps $a(k)$ and $c(k)$ for $k \geqslant 1$. We begin with a strengthening[1] of [**GT10**, 4.3(d)].

PROPOSITION 4.1. *There is a map*

$$e\colon P^{2np^k}(p^{r+k-1}) \vee P^{2np^k+1}(p^{r+k-1}) \longrightarrow \Sigma T$$

which induces an epimorphism in mod p *homology in dimensions* $2np^k$ *and* $2np^k+1$. *Furthermore the composition of* e *with the map*

$$\Sigma T \xrightarrow{\widetilde{E}} S^{2n+1}\{p^r\}$$

is null homotopic, where $\widetilde{E}$ *is adjoint to the map* E *of* 2.13*(e).*

PROOF. Recall by 2.13(g)

$$H_*(T) \simeq Z/p[v] \otimes \Lambda(u)$$

where $|v| = 2n$, $|u| = 2n-1$ and $\beta^{(r+i)}(v^{p^i}) = uv^{p^i-1}$.

Using some H-space structure map μ we consider the Hopf construction:

$$H(\mu)\colon \Sigma(T \wedge T) \longrightarrow \Sigma T.$$

Note that in homology

$$(H(\mu))_*\,(\sigma \otimes x \otimes y) = \sigma \otimes \mu_*(x \otimes y)$$

[1]The additional property that $\widetilde{E}e$ is null homotopic is included here and will be needed in 4.3

if $|x| > 0$ and $|y| > 0$. We now define homology classes

$$\alpha \in H_{2np^k+1}(\Sigma(T \wedge T); Z/p)$$
$$\beta \in H_{2np^k}(\Sigma(T \wedge T); Z/p)$$

by the formulas

$$\alpha = -\sigma \otimes v^{p^{k-1}} \otimes v^{p^{k-1}(p-1)}$$
$$\beta = \sigma \otimes v^{p^{k-1}} \otimes uv^{p^{k-1}(p-1)-1}$$

so we have

$$(H(\mu))_* \, \alpha = -\sigma \otimes v^{p^k}$$
$$(H(\mu))_* \, \beta = \sigma \otimes uv^{p^k-1}.$$

Also $\beta^{(r+k-1)}(\alpha)$ and $\beta^{(r+k-1)}(\beta)$ are both nonzero. By 2.13(d), $\Sigma(T \wedge T)$ is a wedge of Moore spaces; consequently there are maps

$$a\colon P^{2np^k+1}(p^{r+k-1}) \longrightarrow \Sigma(T \wedge T)$$
$$b\colon P^{2np^k}(p^{r+k-1}) \longrightarrow \Sigma(T \wedge T)$$

such that α is in the image of a_* and β is in the image of b_*. Combining these we get a map e

$$P^{2np^k+1}(p^{r+k-1}) \vee P^{2np^k}(p^{r+k-1}) \xrightarrow{a \vee b} \Sigma(\Gamma \wedge T) \xrightarrow{H(\mu)} \Sigma T$$

such that $\sigma \otimes v^{p^k}$ and $\sigma \otimes uv^{p^k-1}$ are in the image of e_*. From this we see that there is a homotopy commutative diagram

$$\begin{array}{ccc} P^{2np^k}(p^{r+k-1}) \vee P^{2np^k+1}(p^{r+k-1}) & \xrightarrow{e} & \Sigma T^{2np^k} \\ \Big\downarrow{\scriptstyle b \vee a} & & \Big\downarrow \\ \Sigma(T \wedge T) & \xrightarrow{H(\mu)} & \Sigma T \end{array} \tag{4.2}$$

where e induces an epimorphism in $\operatorname{mod} p$ homology in dimensions $2np^k$ and $2np^k + 1$. It remains to show that the composition

$$\Sigma(T \wedge T) \xrightarrow{H(\mu)} \Sigma T \xrightarrow{\widetilde{E}} S^{2n+1}\{p^r\}$$

is null homotopic.

However, since $E\colon T \longrightarrow \Omega S^{2n+1}\{p^r\}$ is an H map by 2.13(e), there is a homotopy commutative diagram:

$$\begin{array}{ccc} \Sigma(T \wedge T) & \xrightarrow{\Sigma(E \wedge E)} & \Sigma(\Omega S^{2n+1}\{p^r\} \wedge \Omega S^{2n+1}\{p^r\}) \\ {\scriptstyle H(\mu)}\Big\downarrow & & \Big\downarrow{\scriptstyle H(\mu')} \\ \Sigma T & \xrightarrow{\Sigma E} & \Sigma\Omega S^{2n+1}\{p^r\}. \end{array}$$

where μ' is the loop space structure map on $\Omega S^{2n+1}\{p^r\}$. Since $\Omega S^{2n+1}\{p^r\}$ is a loop space, the right hand map is part of the classifying space structure

$$\begin{array}{ccccc}
\Sigma(\Omega S^{2n+1}\{p^r\} \wedge \Omega S^{2n+1}\{p^r\}) & \longrightarrow & \dots & \longrightarrow & E^{\infty} \\
\downarrow{\scriptstyle H(\mu')} & & & & \downarrow \\
\Sigma\Omega S^{2n+1}\{p^r\} & \longrightarrow & \dots & \longrightarrow & S^{2n+1}\{p^r\}
\end{array}$$

where E_∞ is contractible and the bottom horizontal map is the evaluation map. The result follows since $\widetilde{E}$ is the composition:

$$\Sigma T \longrightarrow \Sigma\Omega S^{2n+1}\{p^r\} \xrightarrow{ev} S^{2n+1}\{p^r\}. \qquad \square$$

Now recall from 2.13(a) the maps

$$T \xrightarrow{g} \Omega G \xrightarrow{h} T$$

with $hg \sim 1$. Restricting we get $T^{2np^k} \xrightarrow{g_k} \Omega G_k$. Let φ_k be the restriction of φ to G_k. Then we have

$$\begin{array}{ccccc}
 & & T & \longleftarrow & T^{2np^k} \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g_*} \\
\Omega G & = & \Omega G & \longleftarrow & \Omega G_k \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Omega\varphi} & & \downarrow{\scriptstyle \Omega\varphi_k} \\
T & \xrightarrow{E} & \Omega S^{2n+1}\{p^r\} & = & \Omega S^{2n+1}\{p^r\}
\end{array}$$

where the left hand square commutes up to homotopy by 2.13(e). Since $hg = 1$, we get the homotopy commutative square:

$$\begin{array}{ccc}
T^{2np^k} & \xrightarrow{g_k} & \Omega G_k \\
\downarrow & & \downarrow{\scriptstyle \Omega\varphi_k} \\
T & \xrightarrow{E} & \Omega S^{2n+1}\{p^r\}.
\end{array}$$

Since e factors through ΣT^{2np^k}, we combine this with 4.1 to see that the central composition

$$\begin{array}{ccccc}
 & & & \overset{}{\dashrightarrow} & E_k \\
 & & & & \downarrow \\
P^{2np^k}(p^{r+k-1}) \vee P^{2np^k+1}(p^{r+k-1}) & \longrightarrow & \Sigma T^{2np^k} & \longrightarrow & G_k \\
 & \searrow{\scriptstyle e} & \downarrow & & \downarrow{\scriptstyle \varphi_k} \\
 & & \Sigma T & \xrightarrow{\widetilde{E}} & S^{2n+1}\{p^r\}
\end{array}$$

factors through E_k. We state this as

PROPOSITION 4.3. *For any H-space structure on T with corresponding maps h and g, there is a lifting of $\widetilde{g}_k e$ to E_k*

$$\begin{CD}
P^{2np^k}(p^{r+k-1}) \vee P^{2np^k+1}(p^{r+k-1}) @>{a(k)\vee c(k)}>> E_k \\
@V{e}VV @VV{\pi_k}V \\
\Sigma T^{2np^k} @>{\widetilde{g}_k}>> G_k.
\end{CD}$$

In the diagram below, the left column is a standard cofibration sequence (1.8) and the right column is a fibration sequence defined by the pinch map $\pi\colon G_k \longrightarrow G_k/G_{k-1} \simeq P^{2np^k+1}(p^{r+k})$

$$\begin{CD}
P^{2np^k}(p^{r+k}) @>{\theta_1}>> @. \Omega P^{2np^k+1}(p^{r+k}) \\
@V{p^{r+k-1}}VV @. @VVV \\
P^{2np^k}(p^{r+k}) @>{\theta_2}>> @. J \\
@V{\sigma\vee\sigma\beta}VV @. @VVV \\
P^{2np^k}(p^{r+k-1})\vee P^{2np^k+1}(p^{r+k-1}) @>{a(k)\vee c(k)}>> E_k @>{\pi_k}>> G_k \\
@V{-\delta_1\vee\rho}VV @. @VV{\pi}V \\
P^{2np^k+1}(p^{r+k}) @= @. P^{2np^k+1}(p^{r+k}).
\end{CD}$$

The homological properties of e and 4.3 imply that the bottom region commutes up to homotopy since $\widetilde{g}_k$ has a right homotopy inverse. The maps θ_1 and θ_2 are induced from this region in the standard way. For dimensional reasons θ_2 factors through $G_{k-1} \subset J$ and since

$$\begin{CD}
E_{k-1} @>>> G_{k-1} \\
@VVV @VVV \\
E_k @>>> G_k
\end{CD}$$

is a pullback diagram, θ_2 factors through E_{k-1}. θ_1 factors through

$$P^{2np^k-1}(p^{r+k})$$

also for dimensional reasons. We obtain

THEOREM 4.4. *There is a homotopy commutative ladder of cofibrations:*[2]

$$\begin{array}{ccccc}
P^{2np^k}(p^{r+k}) & = & = & = & P^{2np^k}(p^{r+k}) \\
\downarrow{\scriptstyle p^{r+k-1}} & & & & \downarrow{\scriptstyle \alpha_k} \\
P^{2np^k}(p^{r+k}) & \xrightarrow{\beta_k} & E_{k-1} & \xrightarrow{\pi_{k-1}} & G_{k-1} \\
\downarrow{\scriptstyle \sigma \vee \sigma\beta} & & \downarrow & & \downarrow \\
P^{2np^k}(p^{r+k-1}) \vee P^{2np^k+1}(p^{r+k-1}) & \xrightarrow{a(k) \vee c(k)} & E_k & \xrightarrow{\pi_k} & G_k \\
\downarrow{\scriptstyle -\delta_1 \vee \rho} & & & & \downarrow{\scriptstyle \pi} \\
P^{2np^k+1}(p^{r+k}) & = & = & = & P^{2np^k+1}(p^{r+k}).
\end{array}$$

Furthermore, for any choice of retraction $\nu_{k-1}\colon E_{k-1} \to BW_n$, we can construct $a(k)$ and $c(k)$ so that $\nu_{k-1}\beta_k \sim *$.

PROOF. We need only demonstrate the last statement. Suppose we are given a map $\overline{\beta}_k\colon P^{2np^k}(p^{r+k}) \to E_{k-1}$ so that the diagram commutes up to homotopy. Given a retraction $\nu_{k-1}\colon\ E_{k-1} \longrightarrow BW_n$, we get a splitting

$$\Omega E_{k-1} \simeq \Omega \overline{R}_{k-1} \times W_n$$

by 2.13(j) where $\overline{R}_{k-1}$ is the fiber of ν_{k-1}. We can then write $\beta_k = \overline{\beta}_k - \epsilon$ where ϵ is the component of $\overline{\beta}_k$ that factors through W_n and β_k factors through R_{k-1}. Since each map $P^{2np^k}(p^{r+k}) \to W_n$ has order p, ϵ has order p and thus $p^{r+k-1}\overline{\beta}_k = p^{r+k-1}\beta_k$ as $r+k-1 \geqslant 1$. Thus the upper region commutes up to homotopy when $\overline{\beta}_k$ is replaced by β_k. Since $p^{r+k-1}\epsilon = 0$, ϵ factors

$$P^{2np^k}(p^{r+k}) \xrightarrow{\sigma \vee \sigma\beta} P^{2np^k}(p^{r+k-1}) \vee P^{2np^k+1}(p^{r+k-1}) \xrightarrow{\epsilon'} E_{k-1}$$

as the left hand column is a cofibration sequence. We now redefine $a(k)$ and $c(k)$ by subtracting off the appropriate components of ϵ' and the middle region now commutes up to homotopy. Since this alteration of $a(k)$ and $c(k)$ factors through E_{k-1}, the projections to G_k vanish when projected to $P^{2np^k}(p^{r+k})$ so the bottom region also commutes up to homotopy. □

During the inductive construction of ν_k we will be assuming that ν_i is defined for $i < k$ and the alterations in 4.4 have been made so that β_k is in the kernel of ν_{k-1}.

4.2. Index p Approximation

The goal of this section is to replace the co-H space G_k by a sequence of approximations. The end result will be to replace $G_k^{[i]}$ by a wedge of mod p^s Moore spaces for $r \leqslant s \leqslant r+k$. We begin with a cofibration sequence based on the ladder 4.4. Throughout this section we will exclude the case $n = 1$. The case $n = 1$ is dealt with in the appendix.

[2]In order to keep the notation from being too cumbersome we will sometimes write β_k, $a(k)$ and $c(k)$ for the composition $\pi_{k-1}\beta_k$, $\pi_k a(k)$, and $\pi_k c(k)$ if it will not lead to confusion.

PROPOSITION 4.5. *For $k \geqslant 1$ there is a cofibration sequence*

$$P^{2np^k}(p^{r+k}) \xrightarrow{\xi_k} L_k \xrightarrow{\zeta_k} G_k \xrightarrow{\pi'} P^{2np^k+1}(p^{r+k})$$

where $L_k = G_{k-1} \vee P^{2np^k}(p^{r+k-1}) \vee P^{2np^k+1}(p^{r+k-1})$, ζ_k is induced by the inclusion of G_{k-1} and the maps $\pi_k a(k)$ and $\pi_k c(k)$, and $\pi' = p^{r+k-1}\pi$.

PROOF. This is a standard consequence of a ladder in which each third rail is an equivalence (as in the usual proof of the Mayer–Vietoris sequence). □

PROPOSITION 4.6. *If $n > 1$, there is a unique co-H space structure on L_k so that the cofibration in 4.5 is a cofibration of co-H maps.*

PROOF. Let P be the pullback in the diagram:

$$\begin{array}{ccc} P & \longrightarrow & G_k \vee G_k \\ \downarrow & & \downarrow \\ L_k \times L_k & \longrightarrow & G_k \times G_k. \end{array}$$

There is a map $\eta\colon L_k \to P$ which projects to the diagonal map on L_k and the composition:

$$L_k \xrightarrow{\zeta_k} G_k \longrightarrow G_k \vee G_k.$$

We first assert that η is unique up to homotopy. Since L_k is a wedge of Moore spaces, it suffices to show that if $\epsilon\colon L_k \to P$ projects trivially to $G_k \vee G_k$ and $L_k \times L_k$, it is itself trivial. Now the homotopy fiber of the map $P \to L_k \times L_k$ is the same as the homotopy fiber of $G_k \vee G_k \to G_k \times G_k$, i.e., $\Sigma(\Omega G_k \wedge \Omega G_k)$; we conclude that ϵ must factor through $\Sigma(\Omega G_k \wedge \Omega G_k)$, and that the composition

$$L_k \xrightarrow{\epsilon'} \Sigma(\Omega G_k \wedge \Omega G_k) \longrightarrow G_k \vee G_k$$

is null homotopic. This implies that ϵ' is null homotopic and hence ϵ is as well since $\Omega(G_k \vee G_k) \to \Omega(G_k \times G_k)$ has a right homotopy inverse.

The map ζ_k is a $2np^k - 1$ equivalence since $P^{2np^k+1}(p^{r+k})$ is $2np^k - 1$ connected. Since L_k and G_k are both $2n - 1$ connected, this implies that the composition

$$\Sigma(\Omega L_k \wedge \Omega L_k) \longrightarrow \Sigma(\Omega L_k \wedge \Omega G_k) \longrightarrow \Sigma(\Omega G_k \wedge \Omega G_k)$$

is a $2np^k + 2n - 2$ equivalence. Now consider the diagram of vertical fibrations:

$$\begin{array}{ccccc} \Sigma(\Omega L_k \wedge \Omega L_k) & \longrightarrow & \Sigma(\Omega G_k \wedge \Omega G_k) & \xrightarrow{\simeq} & \Sigma(\Omega G_k \wedge \Omega G_k) \\ \downarrow & & \downarrow & & \downarrow \\ L_k \vee L_k & \longrightarrow & P & \longrightarrow & G_k \vee G_k \\ \downarrow & & \downarrow & & \downarrow \\ L_k \times L_k & = & L_k \times L_k & \longrightarrow & G_k \times G_k. \end{array}$$

From this we see that the map $L_k \vee L_k \to P$ is a $2np^k + 2n - 2$ equivalence. Since $n > 1$, that implies that there is a unique lifting of $\eta\colon L_k \to P$ to $L_k \vee L_k$, which defines a co-H space structure on L_k such that ζ_k is a co-H map.

Similarly, we observe that the $2np^k + 2n - 3$ skeleton of the fiber of the map $L_k \vee L_k \longrightarrow G_k \vee G_k$ is $P^{2np^k}(p^{r+k}) \vee P^{2np^k}(p^{r+k})$, so the composition

$$P^{2np^k}(p^{r+k}) \xrightarrow{\xi_k} L_k \longrightarrow L_k \vee L_k$$

factors through $P^{2np^k}(p^{r+k}) \vee P^{2np^k}(p^{r+k})$ and such a factorization defines a co-H space structure on $P^{2np^k}(p^{r+k})$. That structure, of course, is unique. So this ξ_k is a co-H map with the suspension structure. □

Warning: L_k does not split as a co-product of co-H spaces. In particular, the inclusion $P^{2np^k+1}(p^{r+k-1}) \longrightarrow L_k$ is not a co-H map. If it were, the map

$$P^{2np^k+1}(p^{r+k-1}) \xrightarrow{c(k)} E_k \xrightarrow{\pi_k} G_k$$

would be a co-H map, contradicting [**AG95**, 2.2].

Write $[k]\colon \Sigma X \longrightarrow \Sigma X$ for the k-fold sum of the identity map.

Definition 4.7. Suppose $L \xrightarrow{f} G \xrightarrow{g} \Sigma K$ is a cofibration sequence of co-H spaces and co-H maps. We will say that f is an index p approximation if there is a co-H map $g'\colon \Sigma G \longrightarrow \Sigma^2 K$ such that Σg factors

$$\Sigma G \xrightarrow{g'} \Sigma^2 K \xrightarrow{[p]} \Sigma^2 K$$

up to homotopy. $f\colon L \longrightarrow G$ will be called an iterated index p approximation if f is homotopic to a composition

$$L = L_0 \longrightarrow L_1 \longrightarrow \dots \longrightarrow L_m = G$$

where each map $L_i \longrightarrow L_{i+1}$ is an index p approximation.

Thus, for example, $\zeta_k\colon L_k \longrightarrow G_k$ is an index p approximation.

Proposition 4.8. *Suppose that $f\colon L \longrightarrow G$ is an iterated index p approximation and $\nu\colon G \longrightarrow BW_n$. Then ν is null homotopic iff νf is null homotopic.*

Proof. We will only consider the case when f is an index p approximation, as the general result follows by an easy induction. Suppose then that $f\colon L \longrightarrow G$ is an index p approximation and Σg factors up to homotopy:

$$\Sigma G \xrightarrow{g'} \Sigma^2 K \xrightarrow{[p]} \Sigma^2 K.$$

Assume that νf is null homotopic, so we can factor ν as

$$G \xrightarrow{g} \Sigma K \xrightarrow{\nu'} BW_n.$$

Consider the diagram:

$$\begin{array}{ccccccc}
\Omega\Sigma^2 K & \xrightarrow{\Omega[p]} & \Omega\Sigma^2 K & \xrightarrow{\Omega\Sigma\nu'} & \Omega\Sigma BW_n & \longrightarrow & BW_n \\
\uparrow{\scriptstyle \widetilde{g}'} & & \uparrow & & \uparrow & \diagup\!\!\diagup & \\
G & \xrightarrow{g} & \Sigma K & \xrightarrow{\nu'} & BW_n & &
\end{array}$$

Since BW_n is homotopy associative ([**Gra88**]), the upper composition is an H-map. This composition is thus inessential if its restriction to ΣK is inessential. However this restriction factors through $[p]\colon \Sigma K \to \Sigma K$. Since BW_n has H-space exponent p ([**The08**]), we conclude that the upper composition is inessential. This $\nu \sim \nu' g$ is inessential as well. $\square$

LEMMA 4.9. *There is a homotopy commutative diagram*

$$\begin{array}{ccc} (\Sigma^2 K)\circ G & \xrightarrow{[p]\circ 1} & (\Sigma^2 K)\circ G \\ \wr\| & & \wr\| \\ (\Sigma K)\wedge G & \xrightarrow{[p]\wedge 1} & (\Sigma K)\wedge G \\ \wr\| & & \wr\| \\ \Sigma^2(K\circ G) & \xrightarrow{[p]} & \Sigma^2(K\circ G) \end{array}$$

where the equivalences are co-H equivalences.

PROOF. The vertical equivalences follow from 3.2. These equivalences are natural for co-H maps. However $[p]\colon \Sigma H \to \Sigma H$ is a co-H map since H is a co-H space. $\square$

LEMMA 4.10. *Suppose* $G_1 \xrightarrow{\alpha} G_2 \xrightarrow{\beta} G_3$ *is a cofibration sequence of co-H spaces and co-H maps. Then for each co-H space H,*

$$G_1\circ H \xrightarrow{\alpha\circ 1} G_2\circ H \xrightarrow{\beta\circ 1} G_3\circ H$$

$$H\circ G_1 \xrightarrow{1\circ\alpha} H\circ G_2 \xrightarrow{1\circ\beta} H\circ G_3$$

are both cofibration sequences.

PROOF. In the extended cofibration sequence, the composition of two adjacent maps is null homotopic

$$G_1 \xrightarrow{\alpha} G_2 \xrightarrow{\beta} G_3 \xrightarrow{\gamma} \Sigma G_1 \xrightarrow{\Sigma\alpha} \Sigma G_2 \xrightarrow{\Sigma\beta} \Sigma G_3 \longrightarrow \dots$$

and all maps are co-H maps. It follows that the same is true for the sequence:

$$G_1\circ H \xrightarrow{\alpha\circ 1} G_2\circ H \xrightarrow{\beta\circ 1} G_3\circ H \xrightarrow{\gamma\circ 1} (\Sigma G_1)\circ H \longrightarrow \dots$$

where $\Sigma(G_1\circ H) \simeq G_1\wedge H \simeq (\Sigma G_1)\circ H$. Since this sequence also induces an exact sequence in homology it is a cofibration sequence. The other case is similar. $\square$

PROPOSITION 4.11. *If $f\colon L \to G$ is an index p approximation, the maps $f\circ 1\colon L\circ H \to G\circ H$ and $1\circ f\colon H\circ L \to H\circ G$ are index p approximations as well.*

PROOF. Factor Σg as

$$\Sigma G \xrightarrow{g'} \Sigma^2 K \xrightarrow{[p]} \Sigma^2 K$$

and consider the diagram

$$\begin{array}{ccccc} (\Sigma G)\circ H & \xrightarrow{\Sigma g\circ 1} & (\Sigma^2 K)\circ H & \simeq & \Sigma^2(K\circ H) \\ =\Big\| & & \Big\uparrow{[p]\circ 1} & & \Big\uparrow{[p]} \\ (\Sigma G)\circ H & \xrightarrow{g'\circ 1} & (\Sigma^2 K)\circ H & \simeq & \Sigma^2(K\circ H) \end{array}$$

where the right hand square commutes by 4.9. The map

$$\Sigma g \circ 1 \colon (\Sigma G) \circ H \longrightarrow (\Sigma^2 K) \circ H$$

is the cofiber of $f \circ 1$, and $g' \circ 1$ is a co-H map since g' is a co-H map. Thus $f \circ 1$ is an index p approximation. The other case is similar. □

COROLLARY 4.12. *Suppose* $L \xrightarrow{f} G$ *is an index p approximation. Then*

$$f^{[i]} \colon L^{[i]} \longrightarrow G^{[i]}$$

is an iterated index p approximation.

PROOF. We first observe that

$$L \circ G^{[j]} \xrightarrow{f \circ 1} G \circ G^{[j]} = G^{[j+1]}$$

is an index p approximation by 4.11. We then see by induction that

$$L^{[i]}G^{[j]} = L \circ (L^{[i-1]}G^{[j]}) \longrightarrow L \circ (L^{[i-2]}G^{[i+1]}) = L^{[i-1]}G^{[j+1]}$$

is an index p approximation. Finally

$$L^{[i]} \longrightarrow L^{[i-1]}G$$

is an iterated index p approximation by induction since it factors as

$$L^{[i]} = L \circ (L^{[i-1]}) \longrightarrow L \circ (L^{[i-2]}G) = L^{[i-1]}G.$$

Consequently

$$L^{[i]} \longrightarrow L^{[i-1]}G \longrightarrow L^{[i-2]}G^{[2]} \longrightarrow \ldots \longrightarrow L \circ G^{[i-1]} \longrightarrow G^{[i]}$$

is an iterated index p approximation. □

THEOREM 4.13. *Suppose*

$$\Omega X \longrightarrow E \longrightarrow G$$

is a principal fibration classified by a map $\varphi \colon G \longrightarrow X$ *where* X *is an* H*-space with strict unit. Suppose* $f \colon L \longrightarrow G$ *is an index p approximation. Then, for any map* $\nu \colon E \longrightarrow BW_n$ *the compositions*

$$\Sigma(\Omega G \wedge \Omega G) \xrightarrow{\Gamma} E \xrightarrow{\nu} BW_n$$

is null homotopic iff the composition

$$\Sigma(\Omega L \wedge \Omega L) \xrightarrow{\Sigma(\Omega f \wedge \Omega f)} \Sigma(\Omega G \wedge \Omega G) \xrightarrow{\Gamma} E \xrightarrow{\nu} BW_n$$

is null homotopic.

PROOF. Suppose the composition

$$\Sigma(\Omega L \wedge \Omega L) \longrightarrow \Sigma(\Omega G \wedge \Omega G) \xrightarrow{\Gamma} E \xrightarrow{\nu} BW_n$$

is null homotopic. Let $\alpha\colon G \to G$ be the identity map and $\beta = \alpha f$. Since $f\colon H \to G$ is a co-H map, there is a homotopy commutative diagram:

$$\begin{array}{ccc} L^{[i]} & \xrightarrow{ad_r^{i-2}(\beta)(\{\beta,\beta\}_\times)} & \Sigma(\Omega L \wedge \Omega L) \\ {\scriptstyle f^{[i]}}\downarrow & & \downarrow \\ G^{[i]} & \xrightarrow{ad^{i-2}(\alpha)(\{\alpha,\alpha\}_\times)} & \Sigma(\Omega G \wedge \Omega G) \end{array}$$

by 3.6(d) and 3.11(d). Since f is an index p approximation, $f^{[i]}$ is an iterated index p approximation by 4.12; thus the compositions

$$G^{[i]} \xrightarrow{ad^{i-2}(\alpha)(\{\alpha,\alpha\}_\times)} \Sigma(\Omega G \wedge \Omega G) \xrightarrow{\quad\Gamma\quad} E \xrightarrow{\quad\nu\quad} BW_n$$

are null homotopic for all $i \geqslant 2$. The result then follows from 3.22. □

We will use this result to transfer conditions on ν_k to the composition:

$$\Sigma(\Omega L_k \wedge \Omega L_k) \longrightarrow \Sigma(\Omega G_k \wedge \Omega G_k) \xrightarrow{\Gamma_k} E_k.$$

We need to iterate this. We have to consider the issue that for $\zeta_k\colon L_k \to G_k$ to be a co-H map, we need to use an exotic co-H space structure on L_k. We will show that the triviality of the composition above does not depend on the co-H space structure of L_k. To see this, recall that the map $\Gamma_k\colon \Sigma(\Omega G_k \wedge G_k) \to E_k$ was defined in section 2.2 based on the fact that E_k was defined by a principal fibration

$$\Omega S^{2n+1}\{p^r\} \to E_k \to G_k$$

classified by a map $\varphi_k\colon G_k \to S^{2n+1}\{p^r\}$ where $S^{2n+1}\{p^r\}$ is an H-space with H-space structure map chosen to have a strict unit. The fact that G_k is a co-H space was not used.

For any space X and map $\zeta\colon X \to G_k$, we can construct the pullback

$$\begin{array}{ccc} \Omega S^{2n+1}\{p^r\} & = & \Omega S^{2n+1}\{p^r\} \\ \downarrow & & \downarrow \\ E(X) & \longrightarrow & E_k \\ \downarrow & & \downarrow \\ X & \xrightarrow{\quad\zeta\quad} & G_k \end{array}$$

which is induced by the composition $\varphi_k\zeta$. Consequently there is a map $\Gamma(X)\colon \Sigma(\Omega X \wedge \Omega X) \to E(X)$ and a strictly commutative diagram:

$$\begin{array}{ccc} \Sigma(\Omega X \wedge \Omega X) & \longrightarrow & \Sigma(\Omega G_k \wedge \Omega G_k) \\ {\scriptstyle \Gamma(X)}\downarrow & & \downarrow{\scriptstyle \Gamma_k} \\ E(X) & \longrightarrow & E_k. \end{array}$$

Consider the homotopy equivalence

$$X_k = G_{k-1} \vee P^{2np^k}(p^{r+k-1}) \vee P^{2np^k+1}(p^{r+k-1}) \simeq L_k$$

where we give X_k the split co-H space structure, so this map is not a co-H map. Nevertheless, we have a strictly commutative diagram

$$\begin{array}{ccccc}
\Sigma(\Omega X_k \wedge \Omega X_k) & \xrightarrow{\simeq} & \Sigma(\Omega L_k \wedge \Omega L_k) & \longrightarrow & \Sigma(\Omega G_k \wedge \Omega G_k) \\
\Big\downarrow{\scriptstyle \Gamma(X_k)} & & \Big\downarrow{\scriptstyle \Gamma(L_k)} & & \Big\downarrow{\scriptstyle \Gamma_k} \\
E(X_k) & \xrightarrow{\simeq} & E(L_k) & \longrightarrow & E_k \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow{\scriptstyle \pi_k} \\
X_k & \xrightarrow{\simeq} & L_k & \xrightarrow{\zeta_k} & G_k
\end{array}$$

since ζ_k is an iterated index p approximation, we have

PROPOSITION 4.14. *For any map* $\nu\colon E_k \to BW_n$, $\nu\Gamma_k$ *is null homotopic iff the composition*

$$\Sigma(\Omega X_k \wedge \Omega X_k) \xrightarrow{\Gamma(X_k)} E(X_k) \simeq E(L_k) \longrightarrow E_k \xrightarrow{\nu} BW_n$$

is null homotopic. □

We now define spaces with split co-H space structures (coproducts in the category of co-H spaces):

$$\begin{aligned}
W(j,k) &= \bigvee_{i=j}^{k} P^{2np^i}(p^{r+i-1}) \vee P^{2np^i+1}(p^{r+i-1}) \\
G(j,k) &= G_j \vee W(j+1,k) \\
L(j,k) &= L_j \vee W(j+1,k).
\end{aligned}$$

Consequently we have homotopy equivalences

$$G(j,k) \simeq L(j+1,k)$$

which are not co-H equivalences, and index p approximations

$$L(j,k) \xrightarrow{\zeta_j \vee 1} G(j,k).$$

This leads to a chain:

$$f\colon G(0,k) \simeq L(1,k) \to G(1,k) \simeq L(2,k) \to \dots \to G(k-1,k) \simeq L_k \to G_k.$$

THEOREM 4.15. *Suppose* $n > 1$. *Then for any given map* $\nu\colon E_k \to BW_n$ *the composition*

$$\Sigma(\Omega G_k \wedge \Omega G_k) \xrightarrow{\Gamma_k} E_k \xrightarrow{\nu} BW_n$$

is null homotopic iff the composition

$$\Sigma(\Omega G(0,k) \wedge \Omega G(0,k)) \xrightarrow{\Sigma(\Omega f \wedge \Omega f)} \Sigma(\Omega G_k \wedge \Omega G_k) \xrightarrow{\Gamma_k} E_k \xrightarrow{\nu} BW_n$$

is null homotopic, where

$$G(0,k) = P^{2n+1} \vee \bigvee_{i=1}^{k} P^{2np^i}(p^{r+i-1}) \vee P^{2np^i+1}(p^{r+i-1})$$

and the map $f\colon G(0,k) \to G_k$ *is defined by the inclusion of*

$$P^{2n+1} = G_0 \to G_k$$

and the maps $\pi_k c(i)$ *and* $\pi_k a(i)$ *for* $1 \leqslant i \leqslant k$*:*

$$P^{2np^i}(p^{r+i-1}) \xrightarrow{a(i)} E_i \longrightarrow E_k \xrightarrow{\pi_k} G_k$$

$$P^{2np^i+1}(p^{r+i-1}) \xrightarrow{c(i)} E_i \longrightarrow E_k \xrightarrow{\pi_k} G_k. \qquad \square$$

Let $E(0,k)$ be the induced fibration over $G(0,k)$. Define a lifting Γ:

$$\begin{array}{ccccc} & & E(0,k) & \longrightarrow & E_k \\ & \overset{\Gamma}{\nearrow} & \downarrow & & \downarrow \\ \Sigma(\Omega G(0,k) \wedge \Omega G(0,k)) & \longrightarrow & G(0,k) & \xrightarrow{f} & G_k \end{array}$$

obtained by pulling back the composition of Γ_k with $\Sigma(\Omega f \wedge \Omega f)$. Since $G(0,k)$ is a wedge of Moore spaces, the components of Γ are H-space based Whitehead products as defined by Neisendorfer [**Nei10a**]. This will be studied in the next chapter.

CHAPTER 5

Simplification

In this chapter we work with the obstructions obtained in Chapter 4. These are mod p^s homotopy classes for $r \leqslant s \leqslant r+k$. In section 5.1, we define a quotient space D_k of G_k and a corresponding principal fibration J_k over D_k. This has the property that roughly half of the obstructions vanish in J_k; we then seek a factorization of ν_k through J_k. In section 5.2 we introduce a congruence relation on homotopy classes, and show that we need only consider the obstructions up to congruence homotopy. This leads to a shorter list of obstructions. Congruence homotopy will also be useful in Chapter 6, since the properties of relative Whitehead products are simpler up to congruence homotopy.

5.1. Reduction

The inductive hypothesis (6.7) in the next chapter is a strengthening of Proposition 2.12, so in particular, we will be assuming that we have constructed a retraction $\nu_{k-1}\colon E_{k-1} \longrightarrow BW_n$ such that $\nu_{k-1}\Gamma_{k-1}$ is null homotopic. In section 4.1 we constructed classes $a(i)$, $c(i)$, and β_i for $i \leqslant k$ and in section 4.2 we reduced the constraints on the construction of ν_k to a condition involving the maps $a(i)$ and $c(i)$. Some of the material in this section and section 6.2 can be found at `arXiv:0804.1896`.

We now make a further simplification by burying the classes $c(i)$ in the base space. Specifically, we define a map

$$c\colon C_k = \bigvee_{i=1}^{k} P^{2np^i+1}(p^{r+i-1}) \longrightarrow E_k$$

by the compositions

$$P^{2np^i+1}(p^{r+i-1}) \xrightarrow{c(i)} E_i \longrightarrow E_k,$$

and define[1] D_k by a cofibration

$$C_k \xrightarrow{\pi_k c} G_k \longrightarrow D_k.$$

[1] The spaces D_k were first defined in [**Ani93**], but were abandoned in [**GT10**] as the related spaces G_k have better properties. As we will see, the spaces D_k have smaller homotopy which is useful in our analysis.

PROPOSITION 5.1. *There is a homotopy commutative diagram of cofibration sequences*

$$\begin{array}{ccccc}
P^{2np^k+1}(p^{r+k-1}) & \xrightarrow{\rho} & P^{2np^k+1}(p^{r+k}) & \xrightarrow{\sigma^{r+k-1}} & P^{2np^k+1}(p) \\
\uparrow & & \uparrow{\scriptstyle \pi} & & \uparrow \\
C_k & \xrightarrow{\pi_k c} & G_k & \longrightarrow & D_k \\
\uparrow & & \uparrow & & \uparrow \\
C_{k-1} & \xrightarrow{\pi_{k-1} c} & G_{k-1} & \longrightarrow & D_{k-1}
\end{array}$$

and

$$H_i(D_k; Z_{(p)}) = \begin{cases} Z/p^r & \text{if } i = 2n \\ Z/p & \text{if } i = 2np^j,\ 1 \leqslant j \leqslant k \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. The composition

$$P^{2np^k+1}(p^{r+k-1}) \xrightarrow{c(k)} E_k \xrightarrow{\pi_k} G_k \xrightarrow{\pi} P^{2np^k+1}(p^{r+k})$$

is ρ by 4.4. The homology calculation is immediate. □

Since $\varphi_k \pi_k$ is null homotopic, we can extend φ_k to a map

$$\varphi'_k \colon D_k \longrightarrow S^{2n+1}\{p^r\}.$$

Any such extension defines a diagram of vertical fibration sequences:

(5.2)
$$\begin{array}{ccccc}
E_k & \xrightarrow{\tau_k} & J_k & \xrightarrow{\eta_k} & F_k \\
{\scriptstyle \pi_k}\downarrow & & {\scriptstyle \xi_k}\downarrow & & {\scriptstyle \sigma_k}\downarrow \\
G_k & \longrightarrow & D_k & = & D_k \\
{\scriptstyle \varphi_k}\downarrow & & {\scriptstyle \varphi'_k}\downarrow & & \downarrow \\
S^{2n+1}\{p^r\} & \longrightarrow & S^{2n+1}\{p^r\} & \longrightarrow & S^{2n+1}
\end{array}$$

PROPOSITION 5.3. *We can choose an extension φ'_k of φ'_{k-1} in such a way that the composition*

$$P^{2np^k+1}(p^{r+k-1}) \xrightarrow{c(k)} E_k \xrightarrow{\tau_k} J_k$$

is null homotopic.

PROOF. We begin by defining D by a pushout square:

$$\begin{array}{ccc}
G_k & \longrightarrow & D \\
\uparrow & & \uparrow \\
G_{k-1} & \longrightarrow & D_{k-1}
\end{array}$$

Using the lower right hand square in 5.1 and the pushout property, we see that there is a cofibration

$$P^{2np^k+1}(p^{r+k-1}) \xrightarrow{\alpha} D \longrightarrow D_k$$

where α is the composition:

$$P^{2np^k+1}(p^{r+k-1}) \xrightarrow{c(k)} E_k \xrightarrow{\pi_k} G_k \longrightarrow D.$$

We use the pushout property to construct $\varphi\colon D \to S^{2n+1}\{p^r\}$ by φ'_{k-1} on D_{k-1} and φ_k on G_k. We seek a map φ'_k in the diagram

$$\begin{array}{ccccc}
E_k & \longrightarrow & J & \longrightarrow & J_k \\
{\scriptstyle \pi_k}\downarrow & & \downarrow & & \downarrow \\
G_k & \longrightarrow & D & \longrightarrow & D_k = D \cup_\alpha CP^{2np^k+1}(p^{r+k-1}) \\
{\scriptstyle \varphi_k}\downarrow & & {\scriptstyle \varphi}\downarrow & & {\scriptstyle \varphi'_k}\downarrow \\
S^{2n+1}\{p^r\} & = & S^{2n+1}\{p^r\} & = & S^{2n+1}\{p^r\}
\end{array}$$

We assert that we can choose φ'_k so that the composition

$$P^{2np^k+1}(p^{r+k-1}) \xrightarrow{c(k)} E_k \longrightarrow J \longrightarrow J_k$$

is null homotopic. Note that α is homotopic to the composition

$$P^{2np^k+1}(p^{r+k-1}) \xrightarrow{c(k)} E_k \longrightarrow J \longrightarrow D.$$

The assertion then follows from

LEMMA 5.4. *Suppose* $J \xrightarrow{\pi} D$ *is a principal fibration induced by a map* $\varphi\colon D \to S$. *Suppose* $c\colon Q \to J$ *and* D' *is the mapping cone of* πc. *Then there is a map* $\varphi'\colon D' \to S$ *with homotopy fiber* J' *such that the composition* $Q \to J \to J'$ *is null homotopic in the diagram:*

$$\begin{array}{ccccc}
Q & \xrightarrow{c} & J & \longrightarrow & J' \\
 & & {\scriptstyle \pi}\downarrow & & \downarrow \\
 & & D & \longrightarrow & D' \\
 & & {\scriptstyle \varphi}\downarrow & & {\scriptstyle \varphi'}\downarrow \\
 & & S & = & S.
\end{array}$$

PROOF. $J = \{(d,\omega) \in D \times PS \mid \omega(1) = \varphi(d)\}$ so $c(q)$ has components $(c_1(q), c_2(q))$ where $c_1(q) \in D$, $c_2(q) \in PS$ with $c_2(q)(0) = *$ and $c_2(q)(1) = \varphi(c_1(q))$. Write $D' = D \cup_\alpha CQ$ with 0 at the vertex of the cone and $\alpha(q) = c_1(q)$. Now define $\varphi'\colon D' \to S$ by $\varphi'(d) = \varphi(d)$ for $d \in D$ and $\varphi'(q,t) = c_2(q)(t)$. This is well defined and we can define a homotopy

$$H\colon Q \times I \to J' \subset D' \times PS$$

by the formula

$$H(q,t) = ((q,t), c_2(q)_t)$$

where $c_2(q)_t$ is the path defined as $c_2(q)_t(s) = c_1(q)(st)$. □

This proves the lemma and hence the proposition. □

Now assume $n > 1$ and define

$$U_k = P^{2n+1} \vee \bigvee_{i=1}^{k} P^{2np^i}(p^{r+i-1}), \tag{5.5}$$

so $G(0,k) = U_k \vee C_k$, and we have a homotopy commutative square

$$\begin{array}{ccc} G(0,k) & \longrightarrow & G_k \\ \downarrow & & \downarrow \\ U_k & \xrightarrow{a} & D_k \end{array}$$

where the left hand vertical map is the projection and a is defined on $P^{2np^i}(p^{r+i-1})$ as the composition:

$$p^{2np^i}(p^{r+i-1}) \xrightarrow{a(i)} E_k \xrightarrow{\tau_k} J_k \xrightarrow{\xi_k} D_k.$$

From this we construct homotopy commutative diagram:

$$\begin{array}{ccccc} \Sigma(\Omega G(0,k) \wedge \Omega G(0,k)) & \longrightarrow & \Sigma(\Omega G_k \wedge \Omega G_k) & \xrightarrow{\Gamma_k} & E_k \\ \downarrow & & \downarrow & & \downarrow \tau_k \\ \Sigma(\Omega U_k \wedge \Omega U_k) & \longrightarrow & \Sigma(\Omega D_k \wedge \Omega D_k) & \xrightarrow{\overline{\Gamma}_k} & J_k \end{array} \tag{5.6}$$

Proposition 5.7. *Suppose that $n > 1$ and there is a retraction*

$$\gamma_k \colon J_k \longrightarrow BW_n$$

such that the compositions

$$U_k^{[j]} \xrightarrow{\{a,\dots,a,\{a,a\}_\times\}_r} J_k \xrightarrow{\gamma_k} BW_n$$

are null homotopic for each $j \geqslant 2$. Then the composition

$$\Sigma(\Omega G_k \wedge \Omega G_k) \xrightarrow{\Gamma_k} E_k \xrightarrow{\tau_k} J_k \xrightarrow{\gamma_k} BW_n$$

is null homotopic.

Proof. By (5.6) and 4.15, it suffices to show that the composition

$$\Sigma(\Omega U_k \wedge \Omega U_k) \longrightarrow \Sigma(\Omega D_k \wedge \Omega D_k) \xrightarrow{\overline{\Gamma}_k} J_k \xrightarrow{\gamma_k} BW_n$$

is null homotopic. Define $E(U_k)_k$ as a pullback:

$$\begin{array}{ccc} E(U_k) & \longrightarrow & J_k \\ \downarrow & & \downarrow \\ U_k & \xrightarrow{a} & D_k \end{array}$$

Then by naturality, it suffices to show that the composition

$$\Sigma(\Omega U_k \wedge \Omega U_k) \xrightarrow{\Gamma(U_k)} E(U_k) \longrightarrow J_k \xrightarrow{\gamma_k} BW_n$$

is null homotopic. But since U_k is a co-H space, we can apply 3.22 to finish the proof. □

Now write $U_k = \Sigma P_k$ where

$$P_k = P^{2n} \vee \bigvee_{i=1}^{k} P^{2np^i-1}(p^{r+i-1})$$

so $U_k^{[j]} = \Sigma P_k \wedge \cdots \wedge P_k$ by 3.2.

Using the splitting of P_k into $k+1$ Moore spaces, we obtain

PROPOSITION 5.8. *Suppose $n > 1$. Then the map*

$$\{a, \ldots, a, \{a, a\}_\times\}_r \colon U_k^{[j]} = \Sigma P_k^{(j)} \longrightarrow J_k$$

when restricted to one of the $(k+1)^j$ iterated smash products of Moore spaces is an iterated external Whitehead product of the form

$$\{x_1, \ldots, x_{j-2}, \{x_{j-1}, x_j\}_\times\}_r$$

where each x_i is either $\xi_k \tau_k a(i) \colon P^{2np^i-1}(p^{r+i-1}) \longrightarrow D_k$ for $1 \leqslant i \leqslant k$ or the inclusion $P^{2n+1} \longrightarrow D_k$. □

By applying 3.26, we resolve these external Whitehead products into internal Whitehead products.

THEOREM 5.9. *Suppose $n > 1$. Then the restriction of the map*

$$\{a, \ldots, a, \{a, a\}_\times\}_r \colon \Sigma P_k \wedge \cdots \wedge P_k \longrightarrow J_k$$

to any Moore space in any decomposition of $\Sigma P_k \wedge \cdots \wedge P_k$ is homotopic to a linear combination of weight j iterated internal H-space based Whitehead products

$$[x_1, \ldots, x_{j-2}, [x_{j-1}, x_j]_\times]_r$$

where each x_i is one of the following: $\xi_k \tau_k a(i) \rho^t$, $\xi_k \tau_k a(i) \delta_t$, μ, ν for $1 \leqslant i \leqslant k$ and for appropriate values of t.

PROOF. This is an easy induction on j using 3.26. □

5.2. Congruence Homotopy Theory

The results of section 5.1, and in particular 5.9, indicate that the obstructions to constructing a suitable retraction $\nu_k = \gamma_k \tau_k$ are $\bmod p^s$ homotopy classes in J_k for $r \leqslant s \leqslant r+k$. These obstructions are iterated compositions of relative and H-space based Whitehead products. However, they are not Whitehead products of maps into J_k in general. Since BW_n is an H-space, any Whitehead products of classes in J_k will be annihilated by any such map γ_k. We are led to a coarser classification. We introduce a congruence relation among homotopy classes so that it is only necessary to annihilate a representative of each congruence class. A remarkable and useful feature is that the congruence homotopy of J_k is a module over the symmetric algebra on $\pi_k(\Omega D_k)$.

DEFINITION 5.10. Two maps $f, g \colon X \longrightarrow Y$ will be called congruent (written $f \equiv g$) if Σf and Σg are homotopic in $[\Sigma X, \Sigma Y]$. We write $e[X, Y]$ for the set of congruence classes of pointed maps: $X \longrightarrow Y$ and

$$e\pi_k(Y; Z/p^s) = e[P^k(p^s), Y].$$

Clearly congruence is an equivalence relation and composition is well defined on congruence classes. This defines the congruence homotopy category. It is easy to prove

PROPOSITION 5.11. *Suppose* $f \equiv g\colon X \to Y$ *and* $h\colon Y \to Z$ *where* Z *is an* H*-space. Then* hf *and* hg *are homotopic.*

Consequently, it is sufficient to classify the iterated H-space based Whitehead products of 5.9 up to congruence. We will also need to consider congruence in a different way in section 6.3. In constructing γ_k we will make alterations in dimensions where obstructions of level $k-1$ may resurface. This is a delicate point which has needed much attention. For this reason we need to develop some deeper properties of congruence homotopy theory.

PROPOSITION 5.12. *The inclusion* $Y_1 \vee Y_2 \xrightarrow{i} Y_1 \times Y_2$ *defines a* 1–1 *map* $\iota_*\colon e[X, Y_1 \vee Y_2] \to e[X, Y_1 \times Y_2]$. *Furthermore, if* G *is a co-*H *space* $e[G, X]$ *is a Abelian group and*

$$e[G, Y_1 \vee Y_2] \cong e[G, Y_1 \times Y_2] \cong e[G, Y_1] \oplus e[G, Y_2].$$

PROOF. Suppose $f, g\colon X \to Y_1 \vee Y_2$ and the compositions:

$$\Sigma X \xrightarrow{\Sigma f} \Sigma(Y_1 \vee Y_2) \xrightarrow{\Sigma i} \Sigma(Y_1 \times Y_2)$$

$$\Sigma X \xrightarrow{\Sigma g} \Sigma(Y_1 \vee Y_2) \xrightarrow{\Sigma i} \Sigma(Y_1 \times Y_2)$$

are homotopic. Since Σi has a left homotopy inverse, Σf and Σg are homotopic. The co-H space structure on G defines a multiplication on $[G, X]$ and the map

$$[G, X] \to [G, \Omega\Sigma X]$$

is multiplicative. However $[G, \Omega\Sigma X]$ is a Abelian group by a standard argument. Since $e[G, X]$ is a subgroup of $[G, \Omega\Sigma X] \cong [\Sigma G, \Sigma X]$, it also is Abelian. Finally observe that the composition

$$e[G, Y_1] \oplus e[G, Y_2] \to e[G, Y_1 \vee Y_2] \to e[G, Y_1 \times Y_2] \to e[G, Y_1] \oplus e[G, Y_2]$$

is the identity where the first and last maps are defined by naturality. Thus the composition of the first two is 1–1. But this composition is also onto since any element of $e[G, Y_1 \times Y_2]$ is represented by a map $G \to Y_1 \times Y_2$ so all these maps are isomorphisms. $\square$

PROPOSITION 5.13. *Suppose* G *and* H *are co-*H *spaces. Then composition defines a homomorphism:*

$$e[G, H] \otimes e[H, X] \to e[G, X].$$

PROOF. The only issue is the distributive law

$$(\beta_1 + \beta_2)\alpha \equiv \beta_1\alpha + \beta_2\alpha$$

for $\alpha\colon G \to H$ and $\beta_1, \beta_2\colon H \to X$. To prove this we show that the following diagram commutes up to congruence:

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & H \\ \downarrow & \equiv & \downarrow \\ G \vee G & \xrightarrow{\alpha \vee \alpha} & H \vee H. \end{array}$$

This certainly commutes after the inclusion of $H \vee H \to H \times H$. Thus it commutes up to congruence by 5.12. □

This will be useful when G and H are Moore spaces.

COROLLARY 5.14. *The category of co-H spaces and congruence classes of contiuous maps is an additive category.*

PROOF. $G \vee H$ is both a product and co-product by 5.12 and composition is bilinear by 5.13. □

The following result will be needed in 5.19 and in section 6.3.

THEOREM 5.15. *Suppose $\varphi\colon \Sigma^2 X \to P^{2m}(p^{r+t})$ has order p^r. Then there are maps $\varphi_1\colon \Sigma^2 X \to P^{2m}(p^r)$ and $\varphi_2\colon \Sigma^2 X \to S^{2m-1}$ such that*

$$\varphi \equiv \rho^t \varphi_1 + \iota_{2m-1}\varphi_2.$$

PROOF. According to [**CMN79b**, 11.1] or [**Gra99**, 1.2], there is a fibration sequence

$$\Omega P^{2m}(p^{r+t}) \xrightarrow{\partial} S^{2m-1}\{p^{r+t}\} \longrightarrow W \xrightarrow{\pi} P^{2m}(p^{r+t})$$

where W is a $4m-3$ connected wedge of Moore spaces and π is an iterated Whitehead product on each factor. In particular, π is null congruent. A right homotopy inverse for ∂ is constructed as follows. Given any map $\theta\colon U \to V$, there is a natural map from the fiber of θ to the loop space on the cofiber:

$$\Phi\colon F_\theta \to \Omega(V \cup_\theta CU).$$

This defines a map $\Phi\colon S^{2m-1}\{p^{r+t}\} \to \Omega P^{2m}(p^{r+t})$ and $\partial\Phi$ is a homology equivalence since $S^{2m-1}\{p^{r+t}\}$ is atomic. This defines a splitting of $\Omega P^{2m}(p^{r+t})$ and we have a direct sum decomposition

$$[\Sigma^2 X, W] \oplus [\Sigma X, S^{2m-1}\{p^{r+t}\}] \longrightarrow [\Sigma^2 X, P^{2m}(p^{r+t})]$$

$$(\alpha, \beta) \longleftrightarrow \varphi = \pi\alpha + \widetilde{\Phi\beta}$$

where $\widetilde{\Phi\beta}$ is the adjoint of $\Phi\beta\colon \Sigma X \to \Omega P^{2m}(p^{r+t})$. Since φ has order p^r, both α and β have order p^r. Since π is null congruent, we have

$$\varphi \equiv \widetilde{\Phi\beta}.$$

Now consider the diagram of fibration sequences:

$$\begin{array}{ccccc}
F & \xrightarrow{f} & S^{2m-1}\{p^{r+t}\} & \xrightarrow{p^r} & S^{2m-1}\{p^{r+t}\} \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
S^{2m-1}\{p^r\} & \longrightarrow & S^{2m-1} & \xrightarrow{p^r} & S^{2m-1} \\
{\scriptstyle * \sim p^{r+t}}\Big\downarrow & & {\scriptstyle p^{r+t}}\Big\downarrow & & {\scriptstyle p^{r+t}}\Big\downarrow \\
S^{2m-1}\{p^r\} & \longrightarrow & S^{2m-1} & \xrightarrow{p^r} & S^{2m-1}.
\end{array}$$

From this we see that $F \simeq S^{2m-1}\{p^r\} \times \Omega S^{2m-1}\{p^r\}$. We choose a splitting of F as follows: define a map ρ^t by the diagram of vertical fibration sequences:

$$\begin{array}{ccc}
S^{2m-1}\{p^r\} & \xrightarrow{\rho^t} & S^{2m-1}\{p^{r+t}\} \\
\Big\downarrow & & \Big\downarrow \\
S^{2m-1} & = & S^{2m-1} \\
{\scriptstyle p^r}\Big\downarrow & & {\scriptstyle p^{r+t}}\Big\downarrow \\
S^{2m-1} & \xrightarrow{p^t} & S^{2m-1}.
\end{array}$$

The map ρ^t factors through f and defines a splitting. Thus the composition

$$S^{2m-1}\{p^r\} \times \Omega S^{2m-1}\{p^r\} \xrightarrow{\simeq} F \xrightarrow{f} S^{2m-1}\{p^{r+t}\}$$

is homotopic to the map

$$S^{2m-1}\{p^r\} \times \Omega S^{2m-1}\{p^r\}$$
$$\xrightarrow{\rho^t \times \delta_t} S^{2m-1}\{p^{r+t}\} \times S^{2m-1}\{p^{r+t}\} \xrightarrow{\mu} S^{2m-1}\{p^{r+t}\}$$

where δ_t is the composition

$$\Omega S^{2m-1}\{p^r\} \xrightarrow{\Omega\pi} \Omega S^{2m-1} \xrightarrow{\iota} S^{2m-1}\{p^{r+t}\}.$$

Since β has order p^r, it factors through F and we conclude that β is homotopic to a composition:

$$\Sigma X \longrightarrow S^{2m-1}\{p^r\} \times \Omega S^{2m-1}\{p^r\}$$
$$\xrightarrow{\rho^t \times \delta_t} S^{2m-1}\{p^{r+t}\} \times S^{2m-1}\{p^{r+t}\} \longrightarrow S^{2m-1}\{p^{r+t}\}.$$

This map is homotopic to the sum of the two compositions

$$\Sigma X \longrightarrow S^{2m-1}\{p^r\} \xrightarrow{\rho^t} S^{2m-1}\{p^{r+t}\}$$
$$\Sigma X \longrightarrow \Omega S^{2m-1}\{p^r\} \longrightarrow \Omega S^{2m-1} \longrightarrow S^{2m-1}\{p^{r+t}\}.$$

By the naturality of Φ, $\Phi\beta$ is homotopic to the sum of the maps

$$\Sigma X \xrightarrow{\widetilde{\varphi}_1} \Omega P^{2m}(p^r) \xrightarrow{\Omega\rho^t} \Omega P^{2m}(p^{r+t})$$

$$\Sigma X \xrightarrow{\widetilde{\varphi}_2} \Omega S^{2m-1} \xrightarrow{\Omega\iota_{2m-1}} \Omega P^{2m}(p^{r+t}).$$

Thus $\varphi \equiv \widetilde{\phi\beta}$ which is homotopic to $\rho^t\varphi_1 + \iota_{2m-1}\varphi_2$. □

At this point we will examine relative Whitehead products in congruence homotopy. This will be applied to determining the congruence homotopy classes of the obstruction in Theorem 5.9.

Recall that relative Whitehead products are defined in a principal fibration:

$$\Omega X \longrightarrow E \longrightarrow B$$

classified by a map $B \to X$.

We define a Z/p^r module $M_*(B)$ by

$$M_m = \pi_{m+1}(B; Z/p^r) \approx \pi_m(\Omega B; Z/p^r).$$

Proposition 5.16. *The relative Whitehead product* $[\alpha, \delta]_r$ *induces a homomorphism:*

$$M_m \otimes e\pi_k(E; Z/p^r) \to e\pi_{m+k}(E; Z/p^r)$$

which commutes with the Hurewicz homomorphism; i.e., the diagram

$$\begin{array}{ccc} M_m \otimes e\pi_k(E; Z/p^r) & \longrightarrow & e\pi_{m+k}(E; Z/p^r) \\ \downarrow & & \downarrow \\ H_m(\Omega B; Z/p^r) \otimes H_k(E; Z/p^r) & \xrightarrow{a_*} & H_{m+k}(E; Z/p^r) \end{array}$$

commutes where a_* *is induced by the principal action.*

Proof. To see that the pairing is well defined, it suffices to show that if $\Sigma\delta \sim *$, then $\Sigma[\alpha, \delta]_r \sim *$. Recall that $[\alpha, \delta]_r$ is given in 3.16 by the composition:

$$P^{m+k} \xrightarrow{\Delta} P^m \wedge P^k \xrightarrow{\theta} P^m \ltimes P^k \xrightarrow{\widetilde{\alpha} \ltimes \delta} \Omega B \ltimes E \xrightarrow{\Gamma'} E.$$

However, since there is a natural homeomorphism

$$\Sigma(X \ltimes Y) \cong X \ltimes \Sigma Y,$$

$\Sigma[\alpha, \delta]_r$ factors through the map

$$P^m \ltimes P^{k+1} \xrightarrow{\widetilde{\alpha} \ltimes \Sigma\delta} \Omega B \ltimes \Sigma E.$$

This map is null homotopic since it factors through $\Omega B \ltimes *$ up to homotopy. Clearly the Hurewicz map factors through congruence homotopy and the homology calculation follows from 3.18. □

Definition 5.17. $A_*(B)$ is the graded symmetric algebra generated by $M_*(B)$.

Theorem 5.18. *The relative Whitehead product induces the structure of a graded differential* $A_*(B)$ *module on* $e\pi_*(E; Z/p^r)$.

Proof. By 5.16, there is an action of $M_*(A)$ on $e\pi_*(E; Z/p^r)$ and hence an action of the tensor algebra. It suffices to show that

$$[\alpha, [\beta, \gamma]_r]_r \equiv (-1)^{mn} [\beta, [\alpha, \gamma]_r]_r$$

where $\alpha \in M_m$ and $\beta \in M_n$. However we have

$$\begin{aligned} [\alpha, [\beta, \gamma]_r]_r &= [\alpha, [\beta, \pi_*(\gamma)]]_\times \\ &= [[\alpha, \beta], \pi_*(\gamma)]_\times + (-1)^{mn} [\beta, [\alpha, \pi_*(\gamma)]]_\times \end{aligned}$$

by 3.30(c), (d) and 3.31(c). But $[\alpha, \beta] = \pi_*[\alpha, \beta]_\times$ by 3.30(a), so we have

$$\begin{aligned} [[\alpha, \beta], \pi_*(\gamma)]_\times &= [\pi_*([\alpha, \beta]_\times), \pi_*(\gamma)]_\times \\ &= [[\alpha, \beta]_\times, \gamma] \end{aligned}$$

by 3.30(b). Since $[[\alpha, \beta]_\times, \gamma]$ is a Whitehead product of classes in $\pi_*(E; Z/p^r)$, $[[\alpha, \beta]_\times, \gamma] \equiv 0$. Thus

$$\begin{aligned} [\alpha, [\beta, \gamma]_r]_r &\equiv (-1)^{mn} [\beta, [\alpha, \pi_*(\gamma)]]_\times \\ &= (-1)^{mn} [\beta, [\alpha, \gamma]_r]_r. \end{aligned}$$ □

Since the action of $A_*(B)$ is associative, we will also use the notation

$$\alpha \cdot \gamma = [\alpha, \gamma]_r$$

to simplify the notation and distinguish this action from composition.

Proposition 5.19. *Suppose* $\xi\colon H' \to H$, $\delta\colon H \to E$ *and* $\alpha\colon \Sigma P \to B$. *Then*

(a) $\{\alpha, \delta\xi\}_r \equiv \{\alpha, \delta\}_r(1 \wedge \xi)\colon A \wedge H' \to E$.

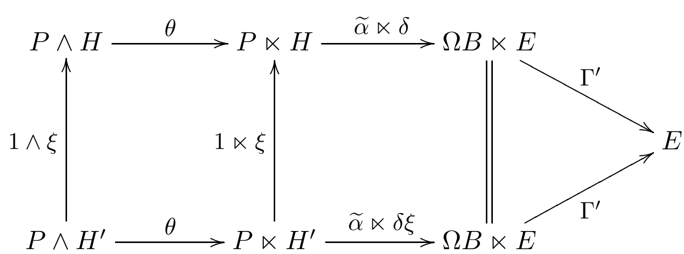
(b) *Suppose* $P = P^m$, $H = P^k$ *and* $H' = P^\ell$. *Then there is a splitting* $P \wedge H \simeq P^{m+k} \vee P^{m+k-1}$ *such that*

$$[\alpha, \delta\xi]_r \equiv [\alpha, \delta]_r(\Sigma^m \xi) \vee [\beta(\alpha), \delta]_r(\xi')$$

where $\xi'\colon P^{m+\ell} \to P^{m+k-1}$.

Remark. These relative Whitehead products are not homotopic in general unless ξ is a co-H map.

Proof. Using 3.16, we construct the following diagram:



The two right hand regions are commutative. We claim that the left hand region commutes up to congruence. Since $P \ltimes H$ is a co-H space, there is a homotopy equivalence

$$P \ltimes H \simeq P \wedge H \vee H$$

given by the sum of the map pinching H to a point and the projection onto H. Since the composition

$$P \wedge H \xrightarrow{\theta} P \ltimes H \longrightarrow P \wedge H$$

is a homotopy equivalence while the composition

$$P \wedge H \xrightarrow{\theta} P \ltimes H \longrightarrow H$$

is null homotopic, both compositions in the left hand square become homotopic when composed with the map

$$P \ltimes H \xrightarrow{\simeq} P \wedge H \vee H \longrightarrow (P \wedge H) \times H.$$

The result then follows from 5.12.

For part b, we split $P^m \wedge P^k$ by the composition

$$P^{m+k} \vee P^{m+k-1} \xrightarrow{\Delta \vee \Delta} P^m \wedge P^k \vee P^{m-1} \wedge P^k \xrightarrow{1 \vee \beta \wedge 1} P^m \wedge P^k$$

(compare to 3.25) $[\alpha, \delta\xi]_r$ is given by precomposing $\{\alpha, \delta\xi\}_r$ with Δ. The composition

$$P^{m+\ell} \xrightarrow{\Delta} P^m \wedge P^\ell \xrightarrow{1 \wedge \xi} P^m \wedge P^k \simeq P^{m+k} \vee P^{m+k-1}$$

is congruent to a map with components $\Sigma^m \xi$ and $\xi' \colon P^{m+k} \longrightarrow P^{m+k-1}$. The result then follows from the splitting. ☐

In case $n > 1$, we now apply the $A_*(D_k)$ module structure to the study of the congruence classes of the obstructions in Theorem 5.9. We will actually only consider the subalgebra of $A_*(D_k)$ generated by $\nu \in M_{2n} = \pi_{2n+1}(D_k Z/p_r)$ and $\mu = \beta\nu \in M_{2n-1}$. These elements generate a subalgebra

$$Z/p[\nu] \otimes \wedge(\mu) \subset A_*(D_k)$$

and $e\pi_*(J_k; Z/p^r)$ is a module over this algebra. We define classes

$$\begin{aligned}
\overline{a(i)} &= \tau_k a(i)\rho^{i-1} \colon P^{2np^i} \longrightarrow J_k\\
\overline{b(i)} &= \tau_k a(i)\delta_{i-1} \colon P^{2np^i-1} \longrightarrow J_k\\
\overline{a(0)} &= [\nu, \mu]_\times\\
\overline{b(0)} &= [\mu, \mu]_\times
\end{aligned}$$

for $1 \leqslant i \leqslant k$.

THEOREM 5.20. *In case $n > 1$, the collection of congruence classes of the set of obstructions listed in Theorem 5.9 is spanned, as a module over $Z/p[\nu] \otimes \wedge(\mu)$ by the classes $\overline{a(i)}$ and $\overline{b(i)}$ of weight $j \geqslant 2$ for $0 \leqslant i \leqslant k$, where the weight of $\overline{a(i)}$ and $\overline{b(i)}$ are both one, except when $i = 0$ where the weight is two.*

PROOF. We first consider internal H-space based Whitehead products of weight 2. Recall that by 3.30(b) $[\xi_k\gamma, \xi_k\delta]_\times = [\gamma, \delta]$ which is null congruent, so we need only consider weight 2 products $[x_1, x_2]_\times$ in which at least one of x_1, x_2 is not in the image of ξ_k. By 3.31(a), we will assume that $x_1 = \mu$ or ν. This gives the following possibilities for weight 2.

$$\overline{a(0)},\ \overline{b(0)},\ [\nu, \xi_k\overline{a(i)}]_\times,\ [\mu, \xi_k\overline{a(i)}]_\times,\ [\nu, \xi_k\overline{b(i)}]_\times,\ [\mu, \xi_k\overline{b(i)}]_\times$$

for $1 \leqslant i \leqslant k$. Applying 3.30(e) again, we see that for $j > 2$ the class of

$$[x_1, \ldots, x_{j-2}, [x_{j-1}, x_j]_\times]_r$$

is null congruent if x_1 is in the image ov ξ_k. Thus each of $x_1, \dots, x_{j-2}$ must be either μ or ν, and this class is in the $Z/p[\nu] \otimes \wedge(\mu)$ submodule generated by $[x_{j-1}, x_j]_\times$. □

We will refer to the submodule generated by $\overline{a(k)}$ and $\overline{b(k)}$ as the level k obstructions. In case $k = 0$ we have some simple relations:

PROPOSITION 5.21. *$\mu \cdot \overline{b(0)} \equiv 0$ and $\nu \cdot \overline{b(0)} \equiv 2\mu \cdot \overline{a(0)}$. Consequently the submodule generated by $\overline{a(0)}$ and $\overline{b(0)}$ has a basis consisting of $\nu^k \cdot \overline{b(0)}$ and $\nu^k \cdot \overline{a(0)}$ for $k \geqslant 0$.*

PROOF. By 3.30(c) and 3.31(c), we have

$$\mu \cdot [\mu, \mu]_\times = [\mu, [\mu, \mu]_\times]_r = [\mu, [\mu, \mu]]_\times = 0 \text{ since } p > 3.$$

Likewise, by 3.30(c),

$$\mu \cdot [\nu, \mu]_\times = [\mu, [\nu, \mu]_\times]_r = [\mu, [\nu, \mu]]_\times.$$

Using 3.31(a) and (c) we get

$$[\mu, [\nu, \mu]]_\times = [[\mu, \nu], \mu]_\times + [\nu, [\mu, \mu]]_\times = -[\mu, [\nu, \mu]]_\times + [\nu, [\mu, \mu]]_\times,$$

so $2\mu \cdot [\nu, \mu]_\times = [\nu, [\mu, u]]_\times = \nu \cdot [\mu, \mu]_\times$. □

Because of these relations we define[2] $x_2 = [\nu, \mu]_\times$ and $y_2 = \frac{1}{2}[\mu, \mu]_\times$. Then $x_k = \nu \cdot x_{k-1} = \nu^{k-2} \cdot \overline{a(0)}$ and $y_k = \mu \cdot x_{k-1}$.

PROPOSITION 5.22. *In case $n > 1$, the level 0 congruence classes are generated by x_j and y_j for $j \geqslant 2$ with the relations $\mu \cdot x_k \equiv \nu \cdot y_k$ and $\nu \cdot y_k \equiv 0$. Furthermore $\beta x_j \equiv j y_j$ and $\beta y_j \equiv 0$.*

PROOF. We have

$$\mu \cdot x_k \equiv \nu^{k-2} \mu \cdot x_2 \equiv \frac{1}{2} \nu^{k-1} \cdot [\mu, \mu]_\times \equiv \nu^{k-1} \cdot y_2 = \nu \cdot y_k$$

and

$$\mu \cdot y_k \equiv \mu \nu^{k-2} \cdot y_2 \equiv \frac{1}{2} \nu^{k-2} \mu \cdot [\mu, \mu]_\times \equiv 0$$

by 5.21. These relations imply that the x_k and y_k are linear generators. We will see in section 6.1 that they are actually linearly independent.

Likewise,

$$\beta x_k = (k-2)\nu^{k-3} \cdot x_2 + \nu^{k-2} \cdot [\mu, \mu]_\times \equiv (k-2)\mu \cdot x_{k-1} + 2y_k \equiv k y_k$$

and

$$\beta y_k \equiv (k-2)\nu^{k-3} \mu \cdot [\mu, \mu]_\times \equiv 0.$$ □

[2]The class $x_k \colon P^{2nk} \longrightarrow D_0 = P^{2n+1}$ is the adjoint of the similarly named class in [**CMN79b**].

CHAPTER 6

Constructing γ_k

In this chapter we assemble the material in the previous chapters and construct a proof of Theorem A. In 6.1 we introduce the controlled extension theorem and apply this in case $k = 0$. This case is simpler since $D_0 = G_0$, and serves as a model for the more complicated case when $k > 0$. We recall a space F_0 (called F in [**GT10**]) and construct ν_0 over the skeleta of F_0 as in [**GT10**]. However we choose ν_0 so that it annihilates x_m and y_m for each $m \geqslant 2$, and conclude that $\nu_0\Gamma_0$ is null homotopic.

In 6.2 we establish Statement 6.7 which depends on k. We will prove 6.7 by induction. Several important properties will be derived from this statement along the way. In particular, we construct F_k and calculate its homology. This allows for a secondary induction over the skeleta of F_k.

In 6.3 the calculations are made to construct $\nu_k = \gamma_k\eta_k$ so that it annihilates the level k obstructions. At this point it is necessary to show that the level $k-1$ obstructions do not reappear. This requires a careful analysis of the congruence class of the level $k-1$ obstructions. Theorem 6.28 and Corollary 6.37 provide the necessary decompositions, and the issue is resolved by 6.40. The induction is completed by 6.41 and 6.43

6.1. Controlled Extension and the Case $k = 0$

In this section we will introduce the controlled extension theorem and apply it to the simplest case: the construction of a retraction map

$$\nu_0 \colon E_0 \longrightarrow BW_n$$

such that the composition

$$\Sigma(\Omega G_0 \wedge \Omega G_0) \xrightarrow{\Gamma_0} E_0 \xrightarrow{\nu_0} BW_n$$

is null homotopic. This case is considerably simpler than the case $k > 0$ and will serve as a model for the later cases. The controlled extension theorem is an enhancement of the extension theorem (2.5), and our construction of ν_0 is a controlled version of the construction of $\nu_0 = \nu^E$ of section 3 of [**GT10**].

Theorem 6.1 (Controlled extension theorem). *Suppose that all spaces are localized at $p > 2$ and we have a diagram of principal fibrations induced by a map*

$\varphi\colon B \cup_\theta e^m \longrightarrow X$:

$$\begin{array}{ccc}
\Omega X & = & \Omega X \\
\downarrow & & \downarrow \\
E_0 & \longrightarrow & E \\
\downarrow & & \downarrow{\scriptstyle \pi} \\
B & \longrightarrow & B \cup_\theta e^m.
\end{array}$$

Suppose $\dim B < m$ *and we are given a map* $\chi\colon P^m(p^s) \longrightarrow E$ *with* $s \geqslant 1$ *such that* $\pi\chi\colon P^m(p^s) \longrightarrow B \cup_\theta e^m$ *induces an isomorphism in* mod p *cohomology in dimension* m. *Suppose also that we are given a map* $\gamma_0\colon E_0 \longrightarrow BW_n$. *Then:*

(a) *There is an extension of* γ_0 *to* $\gamma'\colon E \longrightarrow BW_n$.
(b) *Suppose also that we are given a map* $u\colon P \longrightarrow E$ *and a subspace* $P_0 \subset P$ *such that the composition*

$$P_0 \longrightarrow P \xrightarrow{u} E \xrightarrow{\gamma'} BW_n$$

is null homotopic and such that the quotient map $q\colon P \longrightarrow P/P_0$ *factors up to homotopy as*

$$P \xrightarrow{u} E \xrightarrow{q'} E/E_0 \xrightarrow{\xi} P/P_0$$

for some map ξ. *Then there is an extension* γ *of* γ_0 *such that* $\gamma u \sim *$.

Proof. Clearly the map $\pi\chi\colon (P^m(p^s), S^{m-1}) \longrightarrow (B \cup_\theta e^m, B)$ induces an isomorphism in homology, so the existence of γ' follows from the extension theorem (2.5).

To prove part (b), we suppose γ' is given and we construct γ as the composition:

$$E \xrightarrow{\Delta} E \times E/E_0 \xrightarrow{\gamma' \times \xi} BW_n \times P/P_0 \xrightarrow{1 \times \eta} BW_n \times BW_n \xrightarrow{\div} BW_n$$

where $\eta\colon P/P_0 \longrightarrow BW_n$ is defined by the null homotopy of $\gamma' u|_{P_0}$ and $\div$ is the H-space division map. Clearly $\gamma|_{E_0} \sim \gamma_0$. To study $\gamma|_P$, consider the diagram

$$\begin{array}{ccccccccc}
E & \xrightarrow{\Delta} & E \times E/E_0 & \xrightarrow{\gamma' \times \xi} & BW_n \times P/P_0 & \xrightarrow{1 \times \eta} & BW_n \times BW_n & \xrightarrow{\div} & BW_n \\
{\scriptstyle u}\uparrow & & \uparrow{\scriptstyle u \times q'u} & & \uparrow & & \| & & \| \\
P & \xrightarrow{\Delta} & P \times P & \xrightarrow{\eta q \times q} & BW_n \times P/P_0 & \xrightarrow{1 \times \eta} & BW_n \times BW_n & \xrightarrow{\div} & BW_n
\end{array}$$

where the lower composition of the first 3 maps factors through the diagonal map of BW_n, so the lower composition is null homotopic. □

Now we will apply this in the case $k = 0$. Recall the spaces (5.2). In this case $D_0 = G_0 = P^{2n+1}$ and $J_0 = E_0$:

$$\begin{array}{ccccccc}
\Omega^2 S^{2n+1} & \longrightarrow & E_0 & \xrightarrow{\tau_0} & F_0 & \longrightarrow & \Omega S^{2n+1} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & P^{2n+1} & = & P^{2n+1} & \longrightarrow & PS^{2n+1} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & S^{2n+1}\{p^r\} & \longrightarrow & S^{2n+1} & \xrightarrow{p^r} & S^{2n+1}.
\end{array}$$

These spaces were introduced in [**CMN79a**] where E_0 is called $E^{2n+1}(p^r)$ and F_0 is called $F^{2n+1}(p^r)$.

PROPOSITION 6.2. $H^i(F_0; Z_{(p)}) = \begin{cases} Z_{(p)} & \textit{if } i = 2mn \\ 0 & \textit{otherwise.} \end{cases}$

PROOF. This is immediate from consideration of the cohomology Serre spectral sequence of the middle fibration, which is induced from the path space fibration on the right. All differentials are controlled by the path space fibration. □

We filter F_0 by setting $F_0(m)$ to be the $2mn$ skeleton of F_0 and define $E_0(m)$ to be the pullback over $F_0(m)$

$$\begin{array}{ccc}
\Omega^2 S^{2n+1} & = & \Omega^2 S^{2n+1} \\
\downarrow & & \downarrow \\
E_0(m) & \longrightarrow & E_0 \\
\downarrow & & \downarrow{\scriptstyle \eta_0} \\
F_0(m) & \longrightarrow & F_0
\end{array}$$

Since $F_0(1) = S^{2n}$, this fibration in case $m = 1$ is the fibration which defines BW_n (see 2.3)

$$\Omega^2 S^{2n+1} \longrightarrow S^{4n-1} \times BW_n \longrightarrow S^{2n} \longrightarrow \Omega S^{2n+1}.$$

We consequently define $\nu_0(1)\colon E_0(1) \to BW_n$ by retracting onto BW_n. Clearly $\nu_0(1)y_2$ is null homotopic since $y_2\colon P^{4n-1} \to E_0$ and BW_n is $2np-3$ connected. We will use 6.1 to construct $\nu_0(m)\colon E_0(m) \to BW_n$ such that $\nu_0(m)y_i$ and $\nu_0(m)x_{i-1}$ are null homotopic for $i \leqslant m + 1$.

PROPOSITION 6.3. *For each $m \geqslant 2$, there is a retraction*

$$\nu_0(m)\colon E_0(m) \to BW_n$$

extending $\nu_0(m-1)$ such that $\nu_0(m)_*$ annihilates the classes x_m and y_{m+1} from 5.22.

The proof of this result will depend on two lemmas.

LEMMA 6.4. *The composition*

$$P^{2nj} \xrightarrow{x_j} E_0 \xrightarrow{\eta_0} F_0$$

induces a cohomology epimorphism.

PROOF. To study $\eta_0 x_j$, we use the principal fibration:

$$\Omega S^{2n+1} \longrightarrow F_0 \longrightarrow P^{2n+1}$$

Clearly $\mu\colon P^{2n} \to P^{2n+1}$ lifts to a map $x_1\colon P^{2n} \to F_0$ which induces a cohomology epimorphism. Then $x_j = [\nu, x_{j-1}]_r$ for each $j \geqslant 2$ so we can apply 3.18 to evaluate x_j in cohomology. $\square$

Now let ξ_j be the composition

$$\Omega^2 S^{2n+1} \ltimes S^{2nj} \simeq S^{2nj} \vee S^{2nj} \wedge \Omega^2 S^{2n+1} \longrightarrow S^{2nj} \vee S^{2(j+1)n-1}$$

where the last map is obtained by evaluation on the double loop space.

LEMMA 6.5. *The composition*

$$P^{2n(j+1)-1} \xrightarrow{y_{j+1}} E_0(j) \longrightarrow E_0(j)/E_0(j-1)$$
$$\Omega^2 S^{2n+1} \ltimes S^{2nj} \xrightarrow{\xi_j} S^{2nj} \vee S^{2n(j+1)-1}$$

induces an integral cohomology epimorphism.

PROOF. Since $F_0(j) = F_0(j-1) \cup e^{2mj}$,

$$E_0(j)/E_0(j-1) \simeq \Omega^2 S^{2n+1} \ltimes S^{2mj}$$

by the clutching construction (2.2). From the homotopy commutative square

$$\begin{array}{ccc} E_0(j) & \longrightarrow & E_0(j)/E_0(j-1) \ \simeq S^{2nj} \ltimes \Omega^2 S^{2n+1} \\ \downarrow & & \downarrow \\ F_0(j) & \longrightarrow & S^{2nj} \end{array}$$

and 6.4 we we see that the composition

$$P^{2nj} \xrightarrow{x_j} E_0(j) \longrightarrow E_0(j)/E_0(j-1)$$
$$\simeq \Omega^2 S^{2n+1} \ltimes S^{2nj} \xrightarrow{\xi_j} S^{2nj} \vee S^{2n(j+1)-1}$$

is an integral cohomology epimorphism. Since the action of $\Omega^2 S^{2n+1}$ on $E_0(j)$ corresponds with the action of $\Omega^2 S^{2n+1}$ on

$$E_0(j)/E_0(j-1) \simeq \Omega^2 S^{2n+1} \ltimes S^{2nj},$$

we can apply 3.18 to see that the composition in question

$$P^{2n(j+1)-1} \longrightarrow \Omega^2 S^{2n+1} \ltimes S^{2nj} \xrightarrow{\xi_j} S^{2nj} \vee S^{2n(j+1)-1}$$

induces an epimorphism in integral cohomology. $\square$

PROOF OF 6.3. We apply 6.1 to the diagram

$$\begin{array}{ccc} \Omega^2 S^{2n+1} & = & \Omega^2 S^{2n+1} \\ \downarrow & & \downarrow \\ E_0(m-1) & \longrightarrow & E_0(m) \\ \downarrow & & \downarrow \\ F_0(m-1) & \longrightarrow & F_0(m) \ = F_0(m-1) \cup_\theta e^{2mn} \end{array}$$

where θ is the attaching map of the $2mn$ cell. Let $\chi = x_m \colon P^{2mn} \longrightarrow E(m)$. Choose an extension γ' of $\gamma_0 = \nu_0(m-1)$. Let $P = P^{2mn} \vee P^{2(m+1)n-1}$ and

$$u \colon P^{2mn} \vee \ P^{2(m+1)n-1} \xrightarrow{x_m \vee y_{m+1}} E_0(m);$$

let $P_0 = S^{2mn-1} \vee S^{2(m+1)n-2}$. The composition

$$P^{2mn-1} \longrightarrow S^{2mn-1} \longrightarrow P^{2mn} \xrightarrow{x_m} E_0(m)$$

is $\beta x_m \equiv m y_m$ by 5.22. Since $\nu_0(m-1) y_m$ is null homotopic, the composition

$$P^{2mn-1} \xrightarrow{\beta} P^{2mn} \xrightarrow{x_m} E_0(m) \xrightarrow{\gamma'} BW_n$$

is null homotopic. β factors: $P^{2mn-1} \longrightarrow S^{2mn-1} \longrightarrow P^{2mn}$, so the composition

$$S^{2mn-1} \longrightarrow P^{2mn} \xrightarrow{x_m} E_0(m) \xrightarrow{\gamma'} BW_n$$

is divisible by p^r. However $p \cdot \pi_*(BW_n) = 0$, so this composition is null homotopic. Similarly, since $\beta y_{m+1} \equiv 0$, the composition

$$S^{2(m+1)n-2} \longrightarrow P^{2(m+1)n-1} \xrightarrow{y_{m+1}} E_0(m) \xrightarrow{\gamma'} BW_n$$

is null homotopic. Thus the composition

$$P_0 \longrightarrow P \xrightarrow{x_m \vee y_{m+1}} E_0(m) \xrightarrow{\gamma'} BW_n$$

is null homotopic. However

$$P = P^{2mn} \vee P^{2(m+1)n-1} \xrightarrow{x_m \vee y_{m+1}} E_0(m) \longrightarrow E_0(m)/E_0(m-1) \cong \Omega^2 S^{2n+1} \ltimes S^{2mn}$$

induces an integral cohomology epimorphism by 6.4 and 6.5.

Composition with ξ_m yields a map $P \longrightarrow S^{2mn} \vee S^{2(m+1)n-1}$ as required by 6.1, so we can choose an extension $\nu_0(m)$ of $\nu_0(m-1)$ such that $\nu_0(m)_*(x_m) = 0$ and $\nu_0(m)_*(y_{m+1}) = 0$. This completes the induction. □

COROLLARY 6.6. *There is a retraction $\nu_0 \colon E_0 \longrightarrow BW_n$ such that the composition*

$$\Sigma(\Omega G_0 \wedge \Omega G_0) \xrightarrow{\Gamma_0} E_0 \xrightarrow{\nu_0} BW_n$$

is null homotopic.

PROOF. By 6.3 $(\nu_0)_*(x_m) = 0$ and $(\nu_0)_*(y_m) = 0$ for $m \geqslant 2$. By 5.9 and 5.22,

$$\{\nu, \ldots, \nu, \{\nu, \nu\}_\times\}_r \colon \Sigma G_0 \wedge \cdots \wedge\ G_0 \longrightarrow E_0 \xrightarrow{\nu_0} BW_n$$

is null homotopic for all $j \geqslant 2$. The conclusion follows from 5.7. □

6.2. Preparation for Induction

At this point we present a statement depending on k which we will prove by induction. As pointed out in section 2.4, this will be stronger than Proposition 2.12 given there. This strengthening will be a factorization of the map ν_k through the map $\tau_k \colon E_k \to J_k$. In case $k = 0$, $E_k = J_k$ and τ_k is the identity map, so this alteration only applies when $k > 0$.

STATEMENT 6.7. *There is a map $\gamma_k \colon J_k \to BW_n$ such that the composition*

$$\Omega G_k * \Omega G_k \xrightarrow{\Gamma_k} E_k \xrightarrow{\tau_k} J_k \xrightarrow{\gamma_k} BW_n$$

is null homotopic and such that the compositions

$$E_{k-1}^{2np^k-2} \longrightarrow E_{k-1} \xrightarrow{e_k} E_k \xrightarrow{\tau_k} J_k \xrightarrow{\gamma_k} BW_n$$

$$E_{k-1}^{2np^k-2} \longrightarrow E_{k-1} \xrightarrow{\tau_{k-1}} J_{k-1} \xrightarrow{\gamma_{k-1}} BW_n$$

are homotopic for $k \geq 1$, where $\gamma_0 = \nu_0$ as constructed in 6.6.

By 2.3, the composition

$$\Omega^2 S^{2n+1} \longrightarrow E_0(1) \xrightarrow{\nu_0(1)} BW_n$$

is homotopic to ν. Consequently 6.7 implies 2.12 in case $k = 0$. We will assume that we have constructed γ_i for $i < k$, Having γ_i we obtain $\nu_i = \gamma_i \tau_i$ and construct $a(k)$, $c(k)$ and β_k as in 4.4 with $\nu_{k-1}\beta_k \sim *$. We then define D_k and J_k and proceed to construct γ_k. The construction is completed with 6.43.

The procedure in section 6.1 is a model for the inductive step. To proceed, we will first need to prove:

$$H_i(F_k; Z_{(p)}) \cong \begin{cases} Z_{(p)} & \text{if } i = 2mn \\ 0 & \text{otherwise.} \end{cases} \tag{6.8}$$

We will then use an inductive procedure over the skeleta of F_k as in section 6.1. The proof of 6.8 will be by induction on k. The case $k = 0$ is 6.2. At this point we will assume 6.8 in case $k - 1$.

PROPOSITION 6.9. *Let W_{k-1} be the fiber of γ_{k-1}. Then we have a homotopy commutative diagram of vertical fibration sequences*

$$\begin{array}{ccccc}
T & = & T & \longrightarrow & \Omega S^{2n+1} \\
\downarrow & & \downarrow & & \downarrow \\
R_{k-1} & \longrightarrow & W_{k-1} & \longrightarrow & F_{k-1} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \sigma_{k-1}} \\
G_{k-1} & \longrightarrow & D_{k-1} & = & D_{k-1}
\end{array}$$

and two diagrams of fibration sequences

$$\begin{array}{ccccc}
S^{2n-1} & \longrightarrow & \Omega^2 S^{2n+1} & \longrightarrow & BW_n \\
\downarrow & & \downarrow & & \| \\
W_{k-1} & \longrightarrow & J_{k-1} & \xrightarrow{\gamma_{k-1}} & BW_n \\
\downarrow & & \downarrow & & \\
F_{k-1} & = & F_{k-1} & &
\end{array}$$

$$\begin{array}{ccccc}
S^{2n-1} & \longrightarrow & T & \longrightarrow & \Omega S^{2n+1} \\
\downarrow & & \downarrow & & \\
W_{k-1} & = & W_{k-1} & & \\
\downarrow & & \downarrow & & \\
F_{k-1} & \longrightarrow & D_{k-1} & \longrightarrow & S^{2n+1}
\end{array}$$

PROOF. We define ν_{k-1} to be the composition

$$E_{k-1} \xrightarrow{\tau_{k-1}} J_{k-1} \xrightarrow{\gamma_{k-1}} BW_n.$$

From this it follows that we have a commutative diagram of fibration sequences

$$\begin{array}{ccccc}
R_{k-1} & \longrightarrow & E_{k-1} & \xrightarrow{\nu_{k-1}} & BW_n \\
\downarrow & & \downarrow{\scriptstyle \tau_{k-1}} & & \| \\
W_{k-1} & \longrightarrow & J_{k-1} & \xrightarrow{\gamma_{k-1}} & BW_n.
\end{array}$$

Consequently the square

$$\begin{array}{ccc}
R_{k-1} & \longrightarrow & W_{k-1} \\
\downarrow & & \downarrow \\
G_{k-1} & \longrightarrow & D_{k-1}
\end{array}$$

is the composition of two pullback squares

$$\begin{array}{ccccc}
R_{k-1} & \longrightarrow & E_{k-1} & \xrightarrow{\pi_{k-1}} & G_{k-1} \\
\downarrow & & \downarrow{\scriptstyle \tau_{k-1}} & & \downarrow \\
W_{k-1} & \longrightarrow & J_{k-1} & \xrightarrow{\xi_{k-1}} & D_{k-1},
\end{array}$$

so it is a pullback and the first diagram commutes up to homotopy. The second diagram follows from the definition of W_{k-1} and the third is a combination of the first two. $\square$

PROPOSITION 6.10. $\Omega F_{k-1} \simeq S^{2n-1} \times \Omega W_{k-1}$.

PROOF. Extending the third diagram of 6.9 to the left yields a diagram

$$\begin{array}{ccc} \Omega^2 S^{2n+1} & \longrightarrow & S^{2n-1} \\ \downarrow & & \downarrow \\ * & \longrightarrow & W_{k-1} \\ \downarrow & & \downarrow \\ \Omega S^{2n+1} & \longrightarrow & F_{k-1}; \end{array}$$

both of the horizontal maps have degree p^r in dimension $2n$, so W_{k-1} is $4n-2$ connected and the map $S^{2n-1} \to W_{k-1}$ is null homotopic. From this it follows that $\Omega F_{k-1} \simeq S^{2n-1} \times \Omega W_{k-1}$. □

PROPOSITION 6.11. *The homomorphism*

$$H^*(F_{k-1}; Z_{(p)}) \to H^*(W_{k-1}; Z_{(p)})$$

is onto and

$$H^j(W_{k-1}; Z_{(p)}) = \begin{cases} Z_{(p)}/p^{r+s-1} & \text{if } j = 2np^s \quad 0 < s < k \\ Z_{(p)}/ip^r & \text{if } j = 2ni, \text{ otherwise} \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Consider the Serre spectral sequence for the p-local homology of the fibration

$$\Omega S^{2n+1} \xrightarrow{\delta_{k-1}} F_{k-1} \xrightarrow{\sigma_{k-1}} D_{k-1}$$

Since $E^2_{p,q}$ is only nonzero when p and q are divisible by $2n$, $E^2_{p,q} = E^\infty_{p,q}$. We assume the result (6.8) for the case $k-1$ by induction. Since $E^\infty_{p,q}$ has finite order when $p > 0$ and $H_*(F_{k-1}; Z_{(p)})$ is free, all extensions are nontrivial. Let $u_i \in H^{2ni}(\Omega S^{2n+1})$ be the generator dual to the i^{th} power of a chosen fixed generator in $H_{2n}(\Omega S^{2n+1})$, so

$$u_i u_j = \binom{i+j}{i} u_{i+j}.$$

Using the nontrivial extensions in the Serre spectral sequence, we can choose generators $e_i \in H^{2ni}(F_{k-1}; Z_{(p)})$ so that

$$(\delta_{k-1})^*(e_i) = \begin{cases} p^{r+d-1} u_i & \text{if } p^{d-1} \leqslant i < p^d \quad d < k \\ p^{r+k-1} u_i & \text{if } i \geqslant p^{k-1}. \end{cases}$$

Since $(\delta_{k-1})^*$ is a monomorphism, it is easy to check that

$$e_1 e_{i-1} = \begin{cases} ip^{r-1} e_i & \text{if } i = p^s \quad 0 < s < k \\ ip^r e_i & \text{otherwise.} \end{cases}$$

It now follows from the p-local cohomology Serre spectral sequence for the fibration

$$S^{2n-1} \longrightarrow W_{k-1} \longrightarrow F_{k-1}$$

that

$$d_{2n}(e_{i-1} \otimes u) = \begin{cases} ip^{r-1} e_i & \text{if } i = p^s \quad 0 < s < k \\ ip^r e_i & \text{otherwise.} \end{cases}$$

From this we can read off the cohomology of W_{k-1}. □

PROPOSITION 6.12. *The homomorphism*

$$H^{2np^k}(W_{k-1}; Z_{(p)}) \longrightarrow H^{2np^k}(T; Z_{(p)})$$

is nontrivial of order p.

PROOF. From 6.9 we have a homotopy commutative square

$$\begin{array}{ccc} T & \longrightarrow & \Omega S^{2n+1} \\ \downarrow & & \downarrow \delta_{k-1} \\ W_{k-1} & \longrightarrow & F_{k-1} \end{array}$$

to which we apply cohomology

$$\begin{array}{ccc} H^{2np^k}(T) & \longleftarrow & H^{2np^k}(\Omega S^{2n+1}) \\ \uparrow & & \uparrow \delta^*_{k-1} \\ H^{2np^k}(W_{k-1}) & \longleftarrow & H^{2np^k}(F_{k-1}) \end{array}$$

which we evaluate

$$\begin{array}{ccc} Z/p^{r+k} & \longleftarrow & Z_{(p)} \\ \uparrow & & \uparrow p^{r+k-1} \\ Z/p^{r+k} & \longleftarrow & Z_{(p)} \end{array}$$

where the two horizontal arrows are epimorphisms. The result follows. □

PROPOSITION 6.13. *The map* $T \to R_{k-1}$ *extends to a map*

$$T/T^{2np^k-2} \to R_{k-1}$$

such that the composition

$$P^{2np^k}(p^{r+k}) = T^{2np^k}/T^{2np^k-2} \longrightarrow T/T^{2np^k-2} \longrightarrow R_{k-1} \longrightarrow R_k$$

is null homotopic.

PROOF. Since the fibration

$$T \longrightarrow R_{k-1} \longrightarrow G_{k-1}$$

is induced from the fibration over G_k, we have a homotopy commutative square

$$\begin{array}{ccc} T/\Omega G_{k-1} & \longrightarrow & R_{k-1} \\ \downarrow & & \downarrow \\ T/\Omega G_k & \longrightarrow & R_k. \end{array}$$

By 2.13(a) the inclusion $T^{2np^k-2} \longrightarrow T$ factors through ΩG_{k-1}, this gives a homotopy commutative square

$$\begin{array}{ccc} T/T^{2np^k-2} & \longrightarrow & R_{k-1} \\ \downarrow & & \downarrow \\ T/T^{2np^{k+1}-2} & \longrightarrow & R_k \end{array}$$

The result follows by restriction to T^{2np^k}/T^{2np^k-2}. □

PROPOSITION 6.14. *Let $\widetilde{\alpha_k}\colon P^{2np^k}(p^{r+k}) \longrightarrow R_{k-1}$ be the composition of the first two maps in* 6.13*. Then the composition*

$$P^{2np^k}(p^{r+k}) \xrightarrow{\widetilde{\alpha}_k} R_{k-1} \longrightarrow W_{k-1}$$

is nonzero in p local cohomology.

PROOF. This follows from 6.12 using the diagram

$$\begin{array}{ccccc} P^{2np^k}(p^{r+k}) & \longrightarrow & T/T^{2np^k-2} & \longleftarrow & T \\ & \searrow^{\widetilde{\alpha}_k} & \downarrow & & \downarrow \\ & & R_{k-1} & \longrightarrow & W_{k-1} \end{array}$$

where the three spaces on the top have isomorphic cohomology in dimension $2np^k$. □

PROOF OF 6.8. We assume the result for F_{k-1} by induction. Since F_k is the total space of a principal fibration over $D_k = D_{k-1} \cup CP^{2np^k}(p)$ whose restriction to D_{k-1} is F_{k-1}, we have by 2.2

$$F_k/F_{k-1} = P^{2np^k+1}(p) \rtimes \Omega S^{2n+1};$$

and consequently we have a short exact sequence

$$0 \longrightarrow H_{2nm}(F_{k-1}; Z_{(p)}) \longrightarrow H_{2nm}(F_k; Z_{(p)}) \longrightarrow Z/p \longrightarrow 0$$

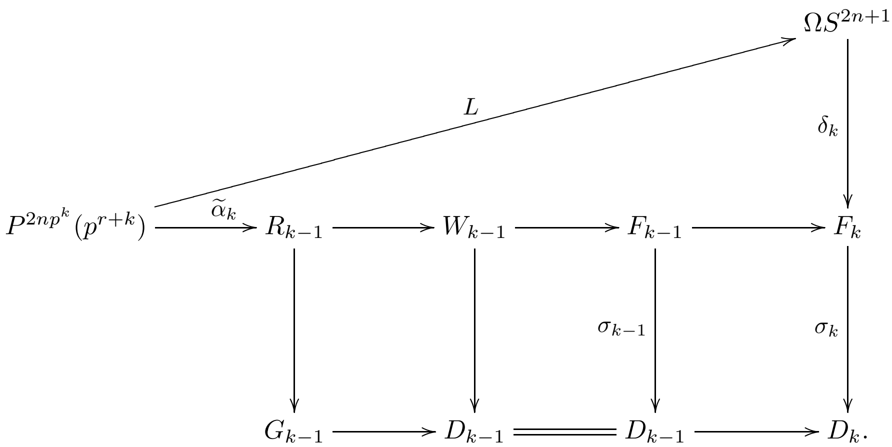
for $m \geqslant p^k$, while

$$H_{2nm}(F_{k-1}; Z_{(p)}) \simeq H_{2nm}(F_k; Z_{(p)})$$

for $m < p^k$. We will prove that the extension is nontrivial. It suffices to show that $H_{2np^k}(F_k; Z_{(p)}) \simeq Z_{(p)}$ since the module action of $H_*(\Omega S^{2n+1}; Z_{(p)})$ on both $H_*(F_{k-1}; Z_{(p)})$ and $H_*(F_k; Z_{(p)})$ implies the result for all $m > p^k$. If this failed we would conclude that $H_{2np^k}(F_k; Z_{(p)}) \cong Z_{(p)} \oplus Z/p$. This would imply that the homomorphism

$$H^{2np^k}(F_k; Z_{(p)}) \longrightarrow\!\!\!\!\!\rightarrow H^{2np^k}(F_{k-1}; Z_{(p)})$$

is onto. We will show that this is impossible. Suppose then that this homomorphism is onto and consider the homotopy commutative diagram



The map L exists since the composition into D_k factors through the composition

$$R_{k-1} \longrightarrow R_k \longrightarrow G_k \longrightarrow D_k;$$

thus this composition is null homotopic by 6.13. Now if

$$H^{2np^k}(F_k; Z_{(p)}) \longrightarrow H^{2np^k}(F_{k-1}; Z_{(p)})$$

is onto then the entire horizontal composition

$$H^{2np^k}(F_k; Z_{(p)}) \longrightarrow H^{2np^k}(P^{2np^k}(p^{r+k}); Z_{(p)})$$

is nonzero by 6.11 and 6.14. But δ_k factors:

$$\Omega S^{2n+1} \xrightarrow{\delta_{k-1}} F_{k-1} \longrightarrow F_k$$

and $(\delta_{k-1})^* \colon H^{2np^k}(F_{k-1}; Z_{(p)}) \to H^{2np^k}(\Omega S^{2n+1}; Z_{(p)})$ is divisible by p^{r+k-1}. Consequently the image of δ_k^* is divisible by p^{r+k-1}; since $L^*\delta_k^*$ is nonzero and

$$H^{2np^k}(P^{2np^k}(p^{r+k}); Z_{(p)}) \cong Z/p^{r+k},$$

we conclude that L^* is onto. This is impossible for then the composition

$$P^{2np^k}(p^{r+k}) \xrightarrow{L} \Omega S^{2n+1} \xrightarrow{H_{p^{k-1}}} \Omega S^{2np^{k-1}+1}$$

would be onto, where $H_{p^{k-1}}$ is the James Hopf invariant. But there is never a map

$$P^{2mp}(p^{r+k}) \to \Omega S^{2m+1}$$

which is onto in cohomology when $r + k > 1$ since the adjoint

$$P^{2np-1}(p^{r+k}) \longrightarrow \Omega^2 S^{2m+1}$$

would also be onto. Such a map would not commute with the Bockstein. Consequently the extension is nontrivial and the cohomology is free. $\square$

Corollary 6.15. *The induced homomorphism*

$$H_{2ni}(F_{k-1}; Z_{(p)}) \longrightarrow H_{2ni}(F_k; Z_{(p)})$$

is an isomorphism when $i < p^k$ and has degree p if $i \geqslant p^k$. Furthermore the principal action defines an isomorphism

$$H_{2ni}(\Omega S^{2n+1}; Z_{(p)}) \otimes H_{2np^k}(F_k; Z_{(p)}) \longrightarrow H_{2np^k+2ni}(F_k; Z_{(p)}) \qquad \square$$

This completes the first task of this section. Our second task will be to give a sharper understanding of the spaces R_{k-1} and, in particular, W_{k-1}. Recall that by 2.13(a), G_{k-1} is a retract of ΣT^{2np^k-2}. Consequently we have a sequence of induced fibrations from 6.9

$$\begin{array}{ccccc}
T & = & T & = & T \\
\downarrow & & \downarrow & & \downarrow \\
R_{k-1} & \longrightarrow & Q_{k-1} & \longrightarrow & R_{k-1} \\
\downarrow & & \downarrow & & \downarrow \\
G_{k-1} & \longrightarrow & \Sigma T^{2np^k-2} & \longrightarrow & G_{k-1}
\end{array}$$

from which we see that R_{k-1} is a retract of Q_{k-1}. Using the clutching construction (2.1), we see that Q_{k-1} is homotopy equivalent to a pushout with $E = Q_{k-1}$ and $E_0 = T = F$

$$\begin{array}{ccc}
T & \longrightarrow & Q_{k-1} \\
\varphi \uparrow & & \uparrow \\
T \times T^{2np^k-2} & \longrightarrow & T \times CT^{2np^k-2}
\end{array}$$

where φ is the restriction of the action map:

$$T \times T^{2np^k-2} \longrightarrow T \times \Omega G_{k-1} \xrightarrow{\ a\ } T.$$

Since the inclusion $T^{2np^k-2} \to T$ factors through ΩG_{k-1}, the composition

$$T^{2np^k-2} \longrightarrow T \longrightarrow Q_{k-1}$$

is null homotopic; by applying 2.2 we have an equivalence $Q_{k-1}/T^{2np^k-2} \simeq Q_{k-1} \vee \Sigma T^{2np^k-2}$. However, from the pushout diagram, we have

$$Q_{k-1}/T \simeq T \ltimes \Sigma T^{2np^k-2}.$$

Restricting to the $2np^k - 2$ skeleton, we get

$$Q_{k-1}^{2np^k-2} \vee \Sigma T^{2np^k-2} \simeq (T \ltimes \Sigma T^{2np^k-2})^{2np^k-2}$$

so $Q_{k-1}^{2np^k-2} \simeq \left(T \wedge \Sigma T^{2np^k-2}\right)^{2np^k-2}$. Now $T \wedge \Sigma T$ is a wedge of Moore spaces by 2.13(d) in section 2.3 and only has cells in dimensions congruent to -1, 0, or 1 mod $2n$. Consequently $Q_{k-1}^{2np^k-2}$ is a wedge of Moore spaces, and the largest exponent is the same as the largest exponent in ΣT^{2np^k-2}, which is p^{r+k-1}. Since R_{k-1} is a retract of Q_{k-1}, we have proved

PROPOSITION 6.16. *$R_{k-1}^{2np^k-2}$ is a wedge of modp^s Moore spaces $P^m(p^s)$ for $r \leqslant s < r+k$.*

REMARK. There are no spheres in this wedge as there are no Moore spaces in $\Sigma T^{2np^k-2} \wedge T$ of dimension $2np^k - 1$.

PROPOSITION 6.17. *The homomorphism in integral homology*

$$H_i(R_{k-1}; Z_{(p)}) \longrightarrow H_i(W_{k-1}; Z_{(p)})$$

is onto for all i and split for $i < 2np^k - 1$.

PROOF. Since D_{k-1} is the mapping cone of the composition

$$C_{k-1} = \bigvee_{i=1}^{k-1} P^{2np^k-1}(p^{r+i-1}) \xrightarrow{c} E_{k-1} \xrightarrow{\pi_{k-1}} G_{k-1},$$

we apply 2.2 to the fibrations in 6.9

$$\begin{array}{ccc} T & = & T \\ \downarrow & & \downarrow \\ R_{k-1} & \longrightarrow & W_{k-1} \\ \downarrow & & \downarrow \\ G_{k-1} & \longrightarrow & D_{k-1} \end{array}$$

and we can then describe W_{k-1} by a pushout diagram

$$\begin{array}{ccc} T \times C(C_{k-1}) & \longrightarrow & W_{k-1} \\ \uparrow & & \uparrow \\ T \times C_{k-1} & \longrightarrow & R_{k-1}. \end{array}$$

This leads to a long exact sequence

$$\ldots \longrightarrow \widetilde{H}_i(T \times C_{k-1}; Z_{(p)}) \longrightarrow \widetilde{H}_i(R_{k-1}; Z_{(p)}) \otimes \widetilde{H}_i(T; Z_{(p)})$$
$$\longrightarrow \widetilde{H}_i(W_{k-1}; Z_{(p)}) \longrightarrow \widetilde{H}_{i-1}(T \times C_{k-1}; Z_{(p)}).$$

We assert that the homomorphism

$$\widetilde{H}_i(W_{k-1}; Z_{(p)}) \longrightarrow \widetilde{H}_{i-1}(T \times C_{k-1}; Z_{(p)})$$

is trivial. By 6.11, $\widetilde{H}_i(W_{k-1}; Z_{(p)})$ is only nontrivial when $i = 2sn - 1$ for some $s \geqslant 2$. But $H_{2sn-2}(T \times C_{k-1}; Z_{(p)}) = 0$ since there are no cells in these dimensions. Now since $\pi_2 \colon T \times C_{k-1} \to T$ is onto in homology, we conclude that $H_i(R_{k-1}; Z_{(p)}) \longrightarrow H_i(W_{k-1}; Z_{(p)})$ is onto. To show that this is split when $i < 2np^k - 1$, we note that since $H_i(W_{k-1}; Z_{(p)})$ is cyclic by 6.11, it suffices to show that the exponent of $H_i(R_{k-1}; Z_{(p)})$ is not larger than the exponent of $H_i(W_{k-1}; Z_{(p)})$ for $i < 2np^k - 1$. By 6.11, we have

$$\exp\left(H_{i-1}(W_{k-1}; Z_{(p)})\right) = \begin{cases} r + \nu_p(i) & i \neq p^s \\ r + s - 1 & i = p^s \quad 0 < s < k. \end{cases}$$

But

$$\begin{aligned}\exp\left(H_{i-1}(R_{k-1};Z_{(p)})\right) &\leqslant \exp\left(H_{i-1}(Q_{k-1};Z_{(p)})\right)\\ &\leqslant \exp\left(H_{i-1}\left(T\wedge\Sigma T^{2np^k-2};Z_{(p)}\right)\right)\end{aligned}$$

when $i-1\leqslant 2np^k-2$. However

$$ip^r H_{2ni-1}(\Sigma T\wedge T;Z_{(p)})=0$$

and

$$p^{r+s-1}H_{2np^s-1}(\Sigma T\wedge T;Z_{(p)})=0. \qquad \square$$

PROPOSITION 6.18. $W_{k-1}^{2np^k-2}$ *is a wedge of Moore spaces.*

PROOF. Since $H_{i-1}\big(R_{k-1}^{2np^k-2};Z_{(p)}\big)\longrightarrow H_{i-1}\big(W_{k-1}^{2np^k-2};Z_{(p)}\big)$ is split onto, we can find a Moore space in the decomposition of $R_{k-1}^{2np^k-2}$ for each i representing a given generator. This constructs a subcomplex of $R_{k-1}^{2np^k-2}$ which is homotopy equivalent to $W_{k-1}^{2np^k-2}$. $\square$

COROLLARY 6.19. $W_{k-1}^{2np^k-2}\simeq\bigvee_{i=2}^{p^k-1}P^{2ni}(p^{r+n_i})$ *where*

$$n_i=\begin{cases}\nu_p(i) & \text{if } i\neq p^s, \quad 0<s<k\\ s-1 & \text{if } i=p^s \quad 0<s<k.\end{cases}$$

6.3. The Inductive Construction

In this section we perform the inductive step of constructing a retraction $\gamma_k\colon J_k\longrightarrow BW_n$ for $k\geqslant 1$. As in the proof of 6.3, we will apply 6.1 to the fibration

$$\Omega^2S^{2n+1}\longrightarrow J_k\longrightarrow F_k$$

and do an induction over the cells of F_k. At each stage in this secondary induction we will make choices to eliminate the obstructions from 5.20.

We will construct a map $\gamma_k\colon J_k\longrightarrow BW_n$ which will annihilate the level k obstructions. However, γ_{k-1} is not homotopic to the composition

$$J_{k-1}\longrightarrow J_k\xrightarrow{\gamma_k}BW_n$$

so we will need an extra argument to show that γ_k annihilates the obstructions of level less than k. This is accomplished by some general results (6.28 and 6.37) which decompose certain relative Whitehead products. This is applied in 6.40 to control the obstructions of a lower level.

We presume that γ_{k-1} has been constructed such that the composition

$$\Sigma(\Omega G_{k-1}\wedge\Omega G_{k-1})\xrightarrow{\Gamma_{k-1}}E_{k-1}\xrightarrow{\tau_{k-1}}J_{k-1}\xrightarrow{\gamma_{k-1}}BW_n$$

is null homotopic. This defines the fiber R_{k-1} of $\nu_{k-1}=\gamma_{k-1}\tau_{k-1}$ and we construct β_k, $a(k)$ and $c(k)$ in accordance with 4.4, and D_k, J_k and F_k as in 5.2.

We next construct a modification of 4.4 in this context.

PROPOSITION 6.20. *There is a homotopy commutative ladder of cofibration sequences:*

$$
\begin{array}{ccccccc}
P^{2np^k}(p) & \longrightarrow & P^{2np^k}(p^{r+k}) & \xrightarrow{\sigma} & P^{2np^k}(p^{r+k-1}) & \longrightarrow & P^{2np^k+1}(p) \\
\Big\uparrow & & \Big\downarrow \beta_k & & \Big\downarrow a(k) & & \Big\uparrow \\
 & & E_{k-1} & & E_k & & \\
 & & \Big\downarrow \tau_{k-1} & & \Big\downarrow \tau_k & & \\
= & & J_{k-1} & \xrightarrow{\iota} & J_k & & = \\
 & & \Big\downarrow \eta_{k-1} & & \Big\downarrow \eta_k & & \\
 & & F_{k-1} & \longrightarrow & F_k & & \\
 & & \Big\downarrow \sigma_{k-1} & & \Big\downarrow \sigma_k & & \\
\Big\downarrow & & & & & & \Big\downarrow \\
P^{2np^k}(p) & \longrightarrow & D_{k-1} & \longrightarrow & D_k & \longrightarrow & P^{2np^k+1}(p)
\end{array}
$$

PROOF. The upper central square commutes up to homotopy by 4.4 and 5.3 and the lower central squares follow from (5.2). By a cohomology calculation, the right hand square commutes up to homotopy. For the left hand region, observe that the $2np^k$ skeleton of the fiber of the inclusion of D_{k-1} into D_k is homotopy equivalent to $P^{2np^k}(p)$; a standard argument with cofibration sequences shows that the left hand vertical map can be taken to be the identity. □

COROLLARY 6.21. *The compositions*

$$P^{2np^k}(p^{r+k-1}) \xrightarrow{a(k)} E_k \xrightarrow{\tau_k} J_k \xrightarrow{\eta_k} F_k$$

$$P^{2np^k}(p^{r+k}) \xrightarrow{\beta_k} E_{k-1} \xrightarrow{\tau_{k-1}} J_{k-1} \xrightarrow{\eta_{k-1}} F_{k-1}$$

induce integral cohomology epimorphisms.

PROOF. The first composition is handled by applying integral cohomology to the right hand region of 6.20. For the second composition we consider the upper two parts of the middle region. The map σ has degree p in H^{2np^k} as does the map $F_{k-1} \to F_k$ by 6.15. Since $r + k \geqslant 2$, this is enough to imply the result. □

PROPOSITION 6.22. $W_{k-1}^{2np^k}$ *is a wedge of Moore spaces.*

PROOF. By 4.4, β_k factors through W_{k-1}, and by 6.11 and 6.21 the map:

$$P^{2np^k}(p^{r+k}) \xrightarrow{\beta_k} W_{k-1}^{2np^k}$$

induces an isomorphism in H_{2np^k-1}. The result follows from 6.18. □

We now filter F_k by skeleta and apply 6.8. As in section 6.1, let $F_k(m)$ be the $2mn$ skeleton of F_k, so

$$F_k(m) = F_k(m-1) \cup e^{2mn}.$$

Let $J_k(m)$ be the pullback of J_k to $F_k(m)$, so we have a map of principal fibrations

$$\begin{array}{ccc} \Omega^2 S^{2n+1} & = & \Omega^2 S^{2n+1} \\ \downarrow & & \downarrow \\ J_k(m-1) & \longrightarrow & J_k(m) \\ \downarrow & & \downarrow \\ F_k(m-1) & \longrightarrow & F_k(m) \end{array} \tag{6.23}$$

and using the clutching construction (2.1) we see that

$$J_k(m)/J_k(m-1) \simeq \Omega^2 S^{2n+1} \ltimes S^{2mn}$$

The obstructions that we need to consider at level k are the elements $\nu^i \cdot a(k)$ and $\mu\nu^{i-1} \cdot a(k)$ for $i \geqslant 1$ where ν^i and $\mu\nu^{i-1}$ generate $Z/p(\nu) \otimes \wedge(\mu) \subset A_*(D_k)$. (See 5.18 and 5.20).

PROPOSITION 6.24. *The compositions*

$$P^{2np^k+2ni} \xrightarrow{\nu^i \cdot \overline{a(k)}} J_k \xrightarrow{\eta_k} F_k$$

$$P^{2np^k+2ni-1} \xrightarrow{\mu\nu^{i-1} \cdot \overline{a(k)}} J_k(p^k+i-1) \xrightarrow{\quad q \quad} \Omega^2 S^{2n+1} \ltimes S^{2(p^k+i-1)n}$$

induce integral cohomology epimorphisms where q is the quotient map.

PROOF. The first composition is evaluated by 6.21 when $i = 0$. In case $i > 0$, we use induction on i. We apply 3.11(d) to the diagram

$$\begin{array}{ccc} J_k & \xrightarrow{\eta_k} & F_k \\ \downarrow & & \downarrow \\ D_k & = & D_k \\ {\scriptstyle \varphi'_k}\downarrow & & \downarrow \\ S^{2n+1}\{p^r\} & \longrightarrow & S^{2n+1} \end{array}$$

to see that $\eta_k(\nu^i \cdot \overline{a(k)}) \equiv [\nu, \eta_k \nu^{i-1} \cdot \overline{a(k)}]_r$. The result then follows from 3.18 and 6.15. The second composition is evaluated by using 3.18 again since $\mu\nu^{i-1} \cdot \overline{a(k)} \equiv [\mu, \nu^{i-1}\overline{a(k)}]_r$. □

COROLLARY 6.25. *The composition*

$$P^{2np^j+2ni} \xrightarrow{\nu^i \cdot \overline{a(j)}} W_j \longrightarrow W_{k-1}$$

induces an integral cohomology epimorphism when $0 \leqslant i < p^{j+1} - p^j$ and $j < k$; likewise the composition

$$P^{2np^j+2n(i+1)-1} \xrightarrow{\mu\nu^i \cdot \overline{a(j)}} W_j \longrightarrow W_{k-1}$$

is nonzero in mod p cohomology in dimension $2np^j + 2n(i+1) - 1$.

PROOF. By the induction hypothesis, $\nu^i \cdot \overline{a(j)}$ and $\mu\nu^{i-1} \cdot \overline{a(j)}$ are in the kernel of γ_j for $j < k$, so they factor through W_j. We then construct the diagram

$$\begin{array}{ccccc} P^{2np^j+2ni} & \xrightarrow{\nu^i \cdot \overline{a(j)}} & W_j & \longrightarrow & W_{k-1} \\ & & \downarrow & & \downarrow \\ & & F_j & \longrightarrow & F_{k-1} \end{array}$$

when $i < p^{j+1} - p^j$. Since the map $F_j \to F_{k-1}$ induces an isomorphism in cohomology in dimensions less than $2np^{j+1}$, the first result follows from 6.24. The second result follows directly from the first since there is a map of fibrations

$$\begin{array}{ccc} S^{2n-1} & \longrightarrow & \Omega^2 S^{2n+1} \\ \downarrow & & \downarrow \\ W_{k-1} & \longrightarrow & J_{k-1} \\ \downarrow & & \downarrow \\ F_{k-1} & = & F_{k-1} \end{array}$$

□

At this point we introduce a simplified notation analogous to the notation in case $k = 0$. We define mod p^r homotopy classes

$$x_i(k)\colon P^{2ni} \to J_k$$
$$y_i(k)\colon P^{2ni-1} \to J_k$$

for $i \geqslant 2$ by the formulas

$$(6.26)\qquad x_i(k) = \begin{cases} x_i & \text{if } k = 0 \\ \iota x_i(k-1) & \text{if } i < p^k \\ \nu^{i-p^k} \cdot \overline{a(k)} & \text{if } i \geqslant p^k \end{cases}$$
$$y_i(k) = \mu \cdot x_{i-1}(k).$$

Consequently, if $p^j \leqslant i < p^{j+1} \leqslant p^k$, $x_i(k) = x_i(j)$.

We will often not distinguish between $x_i(j)\colon P^{2ni} \to J_j$ and its composition with $J_j \to J_k$ for $k \geqslant j$. However

$$\nu \cdot x_i(k) = \begin{cases} x_{i+1}(k) & \text{if } i \neq p^t - 1 \quad t < k \\ x_{i+1}(t-1) & \text{if } i = p^t - 1 \quad t < k \end{cases}$$

We will write $\overline{x}_i(k)$ for $x_i(\ell)$ with ℓ unspecified but $\ell \leqslant k$, so $\nu^d \cdot x_i(k) = \overline{x}_{i+d}(k)$ and similarly for $\overline{y}_i(d)$.

COROLLARY 6.27. *The compositions*

$$P^{2ni} \xrightarrow{x_i(k)} J_k \longrightarrow F_k$$

$$P^{2ni-1} \xrightarrow{y_i(k)} J_k(i-1) \longrightarrow J_k(i-1)/J_k(i-2) \simeq \Omega^2 S^{2n+1} \ltimes S^{2n(i-1)}$$

induce integral cohomology epimorphisms for all $i \geqslant 2$.

PROOF. This follows from 6.25 and 6.11. □

The main technical tool in relating the $x_i(k)$ and $y_i(k)$ with $x_i(k-1)$ and $y_i(k-1)$ will be the following

THEOREM 6.28. *Suppose* $\zeta \in \pi_{m+1}(D_{k-1}; Z/p^r) \simeq \pi_m(\Omega D_{k-1}; Z/p^r)$ *with* $m > 0$ *and the composition*

$$\Sigma^2 X \xrightarrow{\varphi} W_{k-1}^{2np^k-2} \longrightarrow J_{k-1}$$

has order p^r*. Then* $\{\zeta, \varphi\}_r \colon P^{m+2} \wedge X \to J_{k-1}$ *is congruent to the sum*

$$\{\zeta, \varphi\}_r \equiv \sum_{i=2}^{p^k-1} \{\zeta, y_i(k-1)\}_r \alpha_i + \{\zeta, x_i(k-1)\}_r \beta_i$$

where $\alpha_i \colon P^m \wedge \Sigma^2 X \to P^m \wedge P^{2ni-1}$ *and* $\beta_i \colon P^m \wedge \Sigma^2 X \to P^m \wedge P^{2ni}$*.*

There are several steps in the proof of 6.28. Under the inductive hypothesis, $x_i(k-1)$ and $y_i(k-1)$ factor through W_{k-1}.

PROPOSITION 6.29. *The map*

$$\Xi \colon \bigvee_{i=2}^{p^{k-1}} P^{2ni-1} \vee P^{2ni} \xrightarrow{y_i(k-1) \vee x_i(k-1)} W_{k-1}^{2np^k-2}$$

induces a monomorphism modp cohomology.

PROOF. $H^m(W_{k-1}^{2np^k-2})$ is trivial unless $m = 2ni$ or $m = 2ni-1$ for $2 \leqslant i < p^k$, in which case it is Z/p by 6.19. Each of these classes is nontrivial under either $x_i(k-1)$ or $y_i(k-1)$. □

We seek to compare the maps $x_i(k-1)$ and $y_i(k-1)$ to a natural basis for $W_{k-1}^{2np^k-2}$. Choose maps $e_i \colon P^{2ni}(p^{r+n_i}) \to W_{k-1}^{2np^k-2}$ for $2 \leqslant i < p^k - 1$ which define the splitting of 6.19.

$$e \colon \bigvee_{i=2}^{p^k-1} P^{2ni}(p^{r+n_i}) \xrightarrow{\cong} W_{k-1}^{2np^k-2}$$

where $n_i = \nu_p(i)$ if $i \neq p^s$ and $n_i = s-1$ if $i = p^s$. Now define a map

$$\Lambda \colon \bigvee_{i=2}^{p^k-1} P^{2ni} \vee S^{2ni-1} \longrightarrow W_{k-1}^{2np^k-2}$$

with components $e_i \rho^{n_i} \colon P^{2ni} \to W_{k-1}$ and $e_i \iota_{2ni-1} \colon S^{2ni-1} \to W_{k-1}$.

PROPOSITION 6.30. *Suppose* $\varphi \colon \Sigma^2 X \to W_{k-1}^{2np^k-2}$ *has order* p^r*. Then there is a congruence*

$$\varphi \equiv \sum_{i=2}^{p^k-1} e_i \rho^{n_i} \alpha_i + e_i i_{2ni-1} \beta_i$$

for some maps $\alpha_i \colon \Sigma^2 X \to p^{2ni}$ *and* $\beta_i \colon \Sigma^2 X \to S^{2ni-1}$*.*

PROOF. This follows directly from 5.12 and 5.15, since $W_{k-1}^{2np^k-2}$ is a wedge of even dimensional Moore spaces by 6.19. □

Corollary 6.31. *If $n > 1$, there is a diagram*

$$\begin{array}{ccc} \bigvee_{i=2}^{p^k-1} P^{2ni} \vee P^{2ni-1} & & \\ \Big\downarrow F & \overset{\Xi}{\searrow} \quad \equiv & W_{k-1}^{2np^k-2} \\ & \underset{\Lambda}{\nearrow} & \\ \bigvee_{i=2}^{p^k-1} P^{2ni} \vee S^{2ni-1} & & \end{array}$$

which commutes up to congruence, for some map F.

Proof. If $n > 1$, $\bigvee_{i=2}^{p^k-1} P^{2ni} \vee P^{2ni-1}$ is a double suspension whose identity map has order p^r. Thus 6.31 follows from 6.30. □

In particular, we obtain a congruence formula by restricting 6.31 to P^{2ni}:

$$x_i(k-1) \equiv e_i\rho^{n_i} + \sum_{2 \leqslant j < i} e_j\rho^{n_j}\alpha_j + e_j\iota_{2nj-1}\beta_j$$

for some maps $\alpha_j \colon P^{2ni} \longrightarrow P^{2nj}$ and $\beta_j \colon P^{2ni} \longrightarrow S^{2nj-1}$. Actually, the coefficient of $e_i\rho^{n_i}$ in this formula is a unit by a cohomology calculation. We can safely assume it is the identity by adjusting the basis $\{e_i\}$. We intend to use this formula to replace the term $e_i\rho^{n_i}$ in 6.30 by $x_i(k-1)$ plus lower dimensional terms. This is a matter of linear substitutions, and we explain this more clearly in a general context. Observe that all the spaces in these formulas are co-H spaces and 5.14 applies.

Lemma 6.32. *In an additive category, the formulas*

$$x = \sum_{i=1}^{N} a_i\varphi_i + b_i\theta_i$$

$$x_i = a_i + \sum_{j=1}^{i-1} a_j\varphi_{ij} + b_j\theta_{ij}$$

imply that there is a formula:

$$x = \sum_{i=1}^{N} x_i\overline{\varphi}_i + b_i\overline{\theta}_i.$$

Proof. Use downward induction beginning with replacing a_N with x_N. □

Comparing 6.30 with the formula for $x_i(k-1)$ above and applying 6.32, we get

Corollary 6.33. *Suppose $\varphi \colon \Sigma^2 X \longrightarrow W_{k-1}^{2np^k-2}$ has order p^r and $n > 1$. Then φ is congruent to a sum*

$$\sum_{i=2}^{p^k-1} x_i(k-1)\overline{\varphi}_i + e_i\iota_{2ni-1}\overline{\theta}_i$$

where $\overline{\varphi}_i\colon \Sigma^2 X \longrightarrow P^{2ni}$ *and* $\overline{\theta}_i\colon \Sigma^2 X \longrightarrow S^{2ni-1}$. □

Proof of 6.28. We apply 5.19(a) to 6.33 to obtain a congruence

$$\{\zeta,\varphi\}_r \equiv \sum_{i=2}^{p^k-1} \{\zeta, x_i(k-1)\}_r \Sigma^m \overline{\varphi}_i + \{\zeta, e_i \iota_{2ni-1}\}_r \Sigma^m \overline{\theta}_i.$$

But by 3.29,

$$\{\zeta, e_i\iota_{2ni-1}\}_r = \{\zeta, e_i \iota_{2ni-1}\pi_{2ni-1}\}_r = \{\zeta, d_i\}_r$$

where

$$d_i = e_i\iota_{2ni-1}\pi_{2ni-1} = e_i\beta p^{n_i}\colon P^{2ni-1} \longrightarrow P^{2ni}(p^{r+n_i}).$$

Substituting we get

$$\{\zeta,\varphi\}_r \equiv \sum_{i=2}^{p^k-1} \{\zeta, x_i(k-1)\}_r \Sigma^m \overline{\varphi}_i + \{\zeta, d_i\}_r \Sigma^m \overline{\theta}_i. \tag{6.34}$$

We also apply 6.33 with $\varphi = y_i(k-1)$ to get

$$y_i(k-1) \equiv d_i + \sum_{2\leqslant j<i} x_j(k-1)\overline{\varphi}'_j + e_j e_{2nj-1}\overline{\theta}'_j$$

and apply 5.19, we get

$$\{\zeta, y_i(k-1)\}_r \equiv \{\zeta, d_i\}_r + \sum_{2\leqslant j<i} \{\zeta, x_j(k-1)\}_r \overline{\varphi}''_j + \{\zeta, d_j\}\overline{\varphi}''_j. \tag{6.35}$$

We now apply 6.32 to 6.34 and 6.35 with $x_i = \{\zeta, y_i(k-1)\}_r$, $a_j = \{\zeta, d_j\}_r$ and $b_j = \{\zeta, x_j(k-1)\}_r$ to obtain

$$\{\zeta,\varphi\}_r \equiv \sum_{i=2}^{p^k-1} \{\zeta, y_i-(k-1)\}_r \alpha_i + \{\zeta, x_i(k-1)\}_r \beta_i. \tag{6.28}$$ □

In case $\Sigma^2 X = P^\ell$ we can precompose with Δ

$$P^{m+\ell} \longrightarrow P^m \wedge P^\ell \xrightarrow{\{\zeta,\varphi\}_r} J_{k-1}$$

to obtain

$$[\zeta,\varphi]_r \equiv \sum_{i=2}^{p^k-1} \{\zeta, y_i(k-1)\}_r \alpha'_i + \{\zeta, x_i(k-1)\}_r \beta'_i$$

and apply 5.19(b) to obtain

Corollary 6.36. *Suppose* $\zeta\colon P^m \longrightarrow \Omega D_{k-1}$ *and* $\varphi\colon P^\ell \longrightarrow W_{k-1}^{2np^k-2}$. *Then* $[\zeta,\varphi]_r$ *is congruent to a sum*

$$\sum_{i=2}^{p^k-1} [\zeta, y_i(k-1)]_r\alpha_i + [\beta(\zeta), y_i(k-1)]_r\beta_i + [\zeta, x_i(k-1)]_r\gamma_i + [\beta(\zeta), x_i(k-1)]_r\delta_i$$

for some maps

$$\alpha_i\colon P^{m+\ell} \longrightarrow P^{m+2ni-\ell},$$
$$\beta_i\colon P^{m+\ell} \longrightarrow P^{m+2ni-2},$$
$$\gamma_i\colon P^{m+\ell} \longrightarrow P^{m+2ni}$$

and

$$\delta\colon P^{m+\ell} \longrightarrow P^{m+2ni-1}.$$

COROLLARY 6.37. *Suppose* $\varphi\colon P^{\ell} \longrightarrow W_{k-1}^{2np^k-2}$. *Then*

$$\nu^d \cdot \varphi \equiv \sum_{i=2}^{p^k-1} x_{i+d}(k-1)\alpha_i + y_{i+d}(k-1)\beta_i$$

and $\mu\nu^{d-1} \cdot \varphi \equiv \sum_{i=2}^{p^k-1} y_{i+d}(k-1)\gamma_i$.

PROOF. In case $d = 1$ we apply 6.36. The formula simplifies since $\beta(\nu) = \mu$ and $\mu \cdot y_i(k-1) \equiv 0$, while $\mu \cdot x_i(k-1) = y_i(k-1)$ and $\nu \cdot x_i(k-1) = x_{i+1}(k-1)$. In case $d > 1$ we apply induction and 5.19(b) with $\alpha = \nu$ and either

$$\delta = x_{i+d-1}(k-1) \text{ and } \xi = \alpha_{i-1}$$

or

$$\delta = y_{i+d-1}(k-1) \text{ and } \xi = \beta_{i-1}.$$

□

We will use 6.37 to compare the obstructions at adjacent levels. Recall (4.4), the map

$$P^{2np^k}(p^{r+k}) \xrightarrow{\beta_k} R_{k-1} \longrightarrow W_{k-1};$$

β_k induces a cohomology epimorphism by 6.21. We apply 6.28 where φ is one of the two maps:

$$\begin{aligned} \Delta_1 &= \beta_k\rho^k - x_{p^k}(k-1)\colon P^{2np^k} \longrightarrow W_{k-1}^{2np^k-2} \\ \Delta_2 &= \beta_k\delta_k - y_{p^k}(k-1)\colon P^{2np^k-1} \longrightarrow W_{k-1}^{2np^k-2}. \end{aligned} \tag{6.38}$$

The maps Δ_1 and Δ_2 are uniquely defined as maps to W_{k-1} since each term lies in the kernel of γ_{k-1}. The fact that they factor through $W_{k-1}^{2np^k-2}$ follows from 6.22 and 6.24 (In the case $k = 1$ apply 6.4 and 6.5 in place of 6.24). Note that by 6.20 $\iota\beta_k = a(k)\sigma$ where $\iota\colon J_{k-1} \longrightarrow J_k$, so

$$\begin{aligned} \iota\beta_k\rho^k &= a(k)\sigma\rho^k = pa(k)\rho^{k-1} = p\overline{a(k)} \\ \iota\beta_k\delta_k &= a(k)\sigma\delta_k = a(k)\delta_{k-1} = \overline{b(k)}. \end{aligned}$$

Thus we have

$$\begin{aligned} \iota\Delta_1 &= p\overline{a(k)} - \iota x_{p^k}(k-1) \\ \iota\Delta_2 &= \overline{b(k)} - \iota y_{p^k}(k-1) \end{aligned} \tag{6.39}$$

We will filter J_k by spaces $J_k(m)$ as in 6.23 and construct maps

$$\gamma_k(m)\colon J_k(m) \longrightarrow BW_n$$

by induction first on k and then on m. By design the map $\gamma_k(m)$ will annihilate the classes $x_m(k)$ and $y_{m+1}(k)$. However $\gamma_k(m)$ will not be an extension of $\gamma_{k-1}(m)$ and we need to know that the classes $x_m(k-1)$ and $y_{m+1}(k-1)$ are also in the kernel of $\gamma_k(m)_*$. For this purpose we establish the following

Lemma 6.40. *Suppose $m \geqslant p^k$ and we have constructed*

$$\gamma' \colon J_k(m) \to BW_n$$

such that the kernel of $(\gamma')_*$ contains the classes $x_i(s)$ and $y_{i+1}(s)$ for $i < m$ and $s < k$. Then the kernel of $(\gamma')_*$ also contains the classes

$$\begin{gathered} x_m(k-1) \\ y_{m+1}(k-1) \\ \nu^j \cdot \overline{b(k)} \\ \mu\nu^j \cdot \overline{b(k)} \end{gathered}$$

when $j + p^k \leqslant m + 1$.

Proof. Let $d = m - p^k$. Then we have

$$\begin{aligned} px_m(k) &= p(\nu^d \cdot \overline{a(k)}) \\ &\equiv \nu^d \cdot (\iota x_{p^k}(k-1) + \iota\Delta_1) \\ &\equiv \iota x_m(k-1) + \iota\nu^d \cdot \Delta_1 \\ &\equiv \iota x_m(k-1) + \iota \left(\sum_{i=2}^{p^k-1} \overline{x}_{i+d}(k-1)\alpha_i + \overline{y}_{i+d}(k-1)\beta_i \right) \end{aligned}$$

by 6.26, 6.39 and 6.37. Since the identity map of BW_n has order p and each of the classes $\overline{x}_{i+d}(k-1)$ and $\overline{y}_{i+d}$ is equal to $x_{i+d}(s)$ and $y_{i+d}(s)$ respectively with $s < k$, we can conclude that $(\gamma')_*(x_m(k-1)) = 0$. Similarly,

$$\begin{aligned} py_{m+1}(k) &= p\left(\mu\nu^d \cdot \overline{a(k)}\right) \\ &\equiv \mu\nu^d \cdot (\iota x_{p^k}(k-1) + \iota\Delta_1) \\ &\equiv \iota y_{m+1}(k-1) + \iota\mu\nu^d \cdot \Delta_1 \\ &\equiv \iota y_{m+1}(k-1) + \sum_{i=2}^{p^k-1} \overline{y}_{i+d+1}(k-1)\gamma_i \end{aligned}$$

by 6.37. Consequently $(\gamma')_*(y_{m+1}(k-1)) = 0$. Likewise, by 6.39 we obtain

$$\begin{aligned} \nu^j \cdot \overline{b(k)} &\equiv \nu^j \cdot (\iota y_{p^k}(k-1) + \iota\Delta_2) \\ &\equiv \iota y_{p^k+j}(k-1) + \Sigma \left(\sum_{i=2}^{p^k-1} \overline{x}_{i+j}(k-1)\alpha_i + \overline{y}_{i+j}(k-1)\beta_i \right) \end{aligned}$$

which lies in $\ker(\gamma')_*$ when $p^k + j \leqslant m + 1$. Similarly

$$\mu\nu^j \cdot \overline{b(k)} \equiv \iota \left(\sum_{i=2}^{p^k-1} \overline{y}_{i+j+1}(k-1)\gamma_i \right)$$

which is in $\ker(\gamma')_*$ when $p^k + j \leqslant m + 1$. □

THEOREM 6.41. *There are maps $\gamma_k\colon J_k \longrightarrow BW_n$ such that γ_k restricted to $J_k(p^k-1) = J_{k-1}(p^k-1)$ is homotopic to the restriction of γ_{k-1} and such that the compositions*

$$P^{2np^k}(p^{r+k-1}) \xrightarrow{a(k)} E_k \xrightarrow{\tau_k} J_k \xrightarrow{\gamma_k} BW_n \qquad (k \geqslant 1)$$

$$P^{2ni} \xrightarrow{x_i(s)} J_s \longrightarrow J_k \xrightarrow{\gamma_k} BW_n$$

$$P^{2ni-1} \xrightarrow{y_i(s)} J_s \longrightarrow J_k \xrightarrow{\gamma_k} BW_n$$

are null homotopic for $i \geqslant 2$ and $s \leqslant k$.

PROOF. Recall (6.23) that $F_k = \bigcup F_k(m)$ where

$$F_k(m) = F_{k-1}(m) \cup e^{2mn}$$

and $F_k(m) = F_{k-1}(m)$ when $n < p^k$. We have induced principal fibrations:

$$\begin{array}{ccc} \Omega^2 S^{2n+1} & = & \Omega^2 S^{2n+1} \\ \downarrow & & \downarrow \\ J_k(m-1) & \longrightarrow & J_k(m) \\ \downarrow & & \downarrow \\ F_k(m-1) & \longrightarrow & F_k(m) \end{array}$$

We will proceed by induction first on k and then on m. The result follows from 6.3 when $k = 0$. Suppose we have constructed γ_{k-1} and $\gamma_k(m-1)$ is defined agreeing with γ_{k-1} on $J_k(p^k-1)$ and such that the classes $x_i(s)$ and $y_{i+1}(s)$ are in the kernel of $\gamma_k(m-1)$ when $i < m$ and $s \leqslant k$. By 6.24 the composition

$$P^{2mn} \xrightarrow{x_m(k)} J_k(m) \xrightarrow{\eta_k} F_k(m)$$

induces an isomorphism in $\bmod p$ cohomology in dimension $2mn$. We apply 6.1 to construct an extension $\gamma'\colon J_k(m) \longrightarrow BW_n$ of $\gamma_k(m-1)$.

We first consider the case $m = p^k$. Since γ' extends $\gamma_k(m-1) = \gamma_{k-1}(m-1)$, the classes $x_i(s)$ and $y_{i+1}(s)$ are in the kernel of γ' when $s < k$ and $i < p^k$. By 6.40, the kernel of γ' also contains the classes $x_{p^k}(k-1)$, $y_{p^k+1}(k-1)$, $\nu \cdot \overline{b(k)}$, $\mu \cdot \overline{b(k)}$ and $\mu\nu \cdot \overline{b(k)}$.

Now let $P = P^{2np^k}(p^{r+k-1}) \vee P^{2n(p^k+1)-1}$ and $u\colon P \longrightarrow J_k(p^k)$ be given by $a(k) \vee \mu \cdot \overline{a(k)}$. Let $P_0 = S^{2np^k-1} \vee S^{2n(p^k+1)-2} \subset P$. We next show that the composition

$$P_0 \longrightarrow P \xrightarrow{u} J_k(p^k) \xrightarrow{\gamma'} BW_n \tag{6.42}$$

is null homotopic. Since $a(k)\beta = a(k)\sigma\beta\rho = \iota\beta_k\beta\rho$ and β_k is in the kernel of γ_{k-1}, $\gamma'(a(k)\beta) = 0$. This implies that the composition

$$S^{2np^k-1} \longrightarrow P^{2np^k}(p^{r+k-1}) \xrightarrow{a(k)} J_k(p^k) \xrightarrow{\gamma'} BW_n$$

is divisible by $p^{r+k-1} \geqslant p$. Since $p \cdot \pi_*(BW_n) = 0$, this composition is null homotopic. Similarly

$$(\mu \cdot a(k))\,\beta = -\mu \cdot \left[\left(a(k)\rho^{k-1}\right)\beta\right] = -p^{k-1}\mu \cdot b(k)$$

which is in the kernel of γ'. Thus the composition 6.42 is null homotopic and we can apply 6.1 to construct a different extension

$$\gamma_k(p^k)\colon J_k(p^k) \longrightarrow BW_n$$

of $\gamma_l(p^k-1)$. We use 6.24 to verify that the composition

$$P \longrightarrow J_k(p^k) \longrightarrow J_k(p^k)\big/J_k(p^k-1) \simeq \Omega^2 S^{2n+1} \ltimes S^{2np^k} \xrightarrow{\xi} S^{2np^k} \vee S^{2n(p^k+1)-1}$$

induces an integral cohomology epimorphism. Thus after composing with a homotopy equivalence on the wedge of spheres, we see that it is homotopic to the quotient map $P \longrightarrow P/P_0$. We apply 6.1 to construct $\gamma_k(p^k)$ with $x_{p^k}(k)$ and $y_{p^k+1}(k)$ in the kernel. Apply 6.40 again, this time to $\gamma_k(p^k)$ to see that the classes $x_{p^k}(p-1)$ and $y_{p^k+1}(k-1)$ are in the kernel. Then we repeatedly apply 6.40 to restrictions of $\gamma_k(p^k)$ to $J_s(p^k)$ for $s < p^k$ to see that $x_{p^k}(s-1)$ and $y_{p^k+1}(s-1)$ are in the kernel for $s < k$.

The case $m > p^k$ is similar. Since γ' extends $\gamma_k(m-1)$, the classes $x_i(s)$ and $y_{i+1}(s)$ are in the kernel of γ' when $i < m$ and $s < k$. By 6.40 the kernel of γ' also contains the classes $x_m(k-1)$, $y_{m+1}(k-1)$ and the classes $\nu^j \cdot \overline{b(k)}$ and $\mu\nu^j \cdot \overline{b(k)}$ when $j + p^k \leqslant m+1$. We now define P and $u\colon P \longrightarrow J_k(m)$

$$P = P^{2mn} \vee P^{2(m+1)n-1}$$
$$u = x_m(k) \vee y_{m+1}(k)$$

and we calculate

$$\begin{aligned} x(k)\beta &= \left(\nu^{m-p^k} \cdot \overline{a(k)}\right)\beta \\ &= m\mu\nu^{m-p^k} \cdot \overline{a(k)} + \nu^{m-p^k} \cdot \left(a(k)\rho^{k-1}\beta\right) \\ &= my_m(k) + p^{k-1}\nu^{m-p^k} \cdot b(k) \end{aligned}$$

$y_m(k)$ is in the kernel of $\gamma_k(m-1)$ and hence in the kernel of γ' and $\nu^{m-p^k} \cdot b(k)$ is in the kernel since $(m-p^k) + p^k \leqslant m+1$. Similarly

$$y_{k+1}(k)\beta = p^{k-1}\mu\nu^{m-p^k} \cdot b(k)$$

is in $\ker\gamma'$. As before, this implies that the restriction of u to

$$S^{2mn-1} \vee S^{2(m+1)n-2}$$

is null homotopic and we can construct the required map ξ satisfying 6.1. This allows for the construction of $\gamma_k(m)$ which annihilates $x_m(k)$ and $y_{m+1}(k)$. As before, we apply 6.40 to conclude that all classes $x_i(s)$ and $y_{i+1}(s)$ are annihilated by $\gamma_k(m)$ when $i \leqslant m$ and $s \leqslant k$. □

THEOREM 6.43. *Suppose $n > 1$. Then the composition*

$$\Sigma(\Omega G_k \wedge \Omega G_k) \xrightarrow{\Gamma_k} E_k \xrightarrow{\eta_k} J_k \xrightarrow{\gamma_k} BW_n$$

is null homotopic

Proof. By 6.41 γ_k annihilates

$$x_m(i) = \nu^{m-p^i} \cdot \overline{a(i)}$$
$$y_m(i) = \mu\nu^{m-p^i-1} \cdot \overline{a(i)}$$

for each $m \geqslant 2$ and $i \leqslant k$. By 6.40 γ_k annihilates

$$\nu^j \cdot \overline{b(k)}$$
$$\mu\nu^j \cdot \overline{b(k)}$$

for each $j \geqslant 0$. The result follows 5.7, 5.9 and 5.20. □

As in the proof of 2.12, we have a homotopy equivalence

$$\bigcup_{k\geqslant 0} J_k^{2np^{k+1}-2} \longrightarrow \bigcup_{k\geqslant 0} J_k = J$$

and consequently we can construct $\gamma_\infty \colon J \longrightarrow BW_n$ and redefine γ_k as the restriction of γ_∞ to J_k. Similarly we define $\nu_\infty = \gamma_\infty\tau_\infty$ and we have

Compatibility Theorem 6.44. *There are maps γ_k and ν_k such that $\gamma_k\iota \sim \gamma_{k-1}$ and $\nu_k e \sim \nu_{k-1}$. Furthermore, there are homotopy commutative diagrams of fibration sequences*

$$\begin{array}{ccccc}
\Omega G_k & = & \Omega G_k & & \\
\downarrow{\scriptstyle h_k} & & \downarrow{\scriptstyle \Omega\varphi_k} & & \\
T & \longrightarrow & \Omega S^{2n+1}\{p^r\} & \xrightarrow{H} & BW_n \\
\downarrow & & \downarrow & & \| \\
R_k & \longrightarrow & E_k & \xrightarrow{\nu_k} & BW_n \\
\downarrow & & \downarrow & & \\
G_k & = & G_k & &
\end{array}
\qquad
\begin{array}{ccccc}
\Omega D_k & = & \Omega D_k & & \\
\downarrow{\scriptstyle h'_k} & & \downarrow & & \\
T & \longrightarrow & \Omega S^{2n+1}\{p^r\} & \xrightarrow{H} & BW_n \\
\downarrow & & \downarrow & & \| \\
W_k & \longrightarrow & J_k & \xrightarrow{\gamma_k} & BW_n \\
\downarrow & & \downarrow & & \\
D_k & = & D_k & &
\end{array}$$

where the left hand diagram maps into the right hand diagram. The maps h_k and h'_k are the restrictions of maps $h \colon \Omega G \longrightarrow T$ and $h' \colon \Omega D \longrightarrow T$ and there are compatible maps $g_k \colon T^{2np^{k+1}-2} \longrightarrow \Omega G_k$ with $h_k g_k$ homotopic to the inclusion and compatible maps f_k: $G_k \longrightarrow \Sigma T^{2np^k}$ with $\widetilde{g}_k f_k$ homotopic to the identity.

CHAPTER 7

Universal Properties

The aim of this chapter is to prove theorem B, corollaries C and D, and to discuss some applications. We describe an obstruction theory for the existence and uniqueness of extensions $\widehat{\alpha}$ and evaluate the obstructions in some cases. We have, however, no example of a homotopy Abelian H-space and map $\alpha\colon P^{2n} \to Z$ for which no extension to an H-map $\widehat{\alpha}\colon T \to Z$ exists.*

7.1. Statement of Results

In this section we describe the basic result relating to the existence and uniqueness of an extension of a map $\alpha\colon P^{2n} \to Z$ to an H-map $\widehat{\alpha}\colon T \to Z$. The proofs of these results are reserved for the next two sections. We note, however, that a different argument for the obstruction to uniqueness was obtained in [**Gra12**].

We begin with some notation. Throughout this chapter, Z will be an arbitrary homotopy Abelian H-space. We will call an H-map $\alpha\colon \Omega G_k \to Z$ proper if the compositions:

$$P^{2np^i-1}(p^{r+i-1}) \vee P^{2np^i}\left(p^{r+i-1}\right) \xrightarrow{a(i)\vee c(i)} \Omega G_i \longrightarrow \Omega G_k \xrightarrow{\alpha} Z$$

are null-homotopic for each i, $1 \leqslant i \leqslant k$. Let $G_k(Z)$ be the Abelian group of all homotopy classes of proper H-maps $\alpha\colon \Omega G_k \to Z$, where $0 \leqslant k \leqslant \infty$ and we write $G(Z)$ for $G_\infty(Z)$. Let

$$p_k(Z) = p^{r+k-1}\pi_{2np^k-1}(Z; Z/p^{r+k})$$

by which we mean the subgroup of all elements of $\pi_{2np^h-1}(Z; Z/p^{r+k})$ which are divisible by p^{r+k-1}. Let $[Z_1, Z_2]_H$ be the Abelian group of all H-maps from Z_1 to Z_2.

Clearly

(7.1) $$G_0(Z) = \left[P^{2n}, Z\right] = \pi_{2n}(Z; Z/p^r)$$

THEOREM 7.2. $\varprojlim G_k(Z) \cong [T, Z]_H$.

THEOREM 7.3. *There is an exact sequence:*

$$0 \longrightarrow p_k(\Omega Z) \xrightarrow{e} G_k(Z) \xrightarrow{r} G_{k-1}(Z) \xrightarrow{\beta} p_k(Z)$$

We will see by example that this sequence is not exact on the right. In fact, we have no example in which $\beta \neq 0$. But there are examples in which $p_k(Z) \neq 0$.

* *Added in proof.* Such a space Z has recently been discovered.

7.2. Inductive Analysis

In this section we will prove Theorem 7.2. It is a consequence of the following propositions:

PROPOSITION 7.4. $\varprojlim G_k(Z) \cong G(Z)$

PROPOSITION 7.5. $[T, Z]_H \cong G(Z)$.

PROOF OF 7.4: We first establish that for any space X,

$$\varprojlim [G_k, X] \cong [G, X] \tag{7.6}$$

The argument here is a special case of the results in [**Gra66**]. First we observe that the restrictions define an epimorphism:

$$[G, X] \longrightarrow \varprojlim [G_k, X]$$

by the homotopy extension property applied inductively for each k. Suppose, however, that $\alpha \in [G, X]$ lies in the kernel; i.e., the restrictions:

$$G_k \longrightarrow G \xrightarrow{\alpha} X$$

are all null homotopic. We construct a homotopy commutative diagram in which the horizontal sequence is a cofibration sequence:

$$\begin{array}{ccccccccc} \bigvee_{k\geqslant 0} G_k & \longrightarrow & G & \longrightarrow & \mathcal{C}(G) & \longrightarrow & \bigvee_{k\geqslant 0} \Sigma G_k & \longrightarrow & \Sigma G \\ & & {\scriptstyle\alpha}\downarrow & \swarrow {\scriptstyle\alpha'} & & & & & \\ & & X & & & & & & \end{array}$$

However by 2.13(i), ΣG is a wedge of Moore spaces, so the map

$$\bigvee_{k\geqslant 0} \Sigma G_k \longrightarrow \Sigma G$$

has a right homotopy inverse. This implies that

$$\bigvee_{k\geqslant 0} \Sigma G_k \cong \Sigma G \vee \mathcal{C}(G)$$

and consequently the map $G \to \mathcal{C}(G)$ is null homotopic. It follows that α is null homotopic. □

To complete the proof of 7.4, consider the diagram:

$$\begin{array}{ccc} [\Omega G, Z]_H & \xrightarrow{L} & \varprojlim [\Omega G_k, Z]_H \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ [G, \Sigma Z] & \xrightarrow{\cong} & \varprojlim [G_k, \Sigma Z] \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi} \\ [\Omega G, Z]_H & \xrightarrow{L} & \varprojlim [\Omega G_k, Z]_H \end{array}$$

where $\varphi(\alpha) = (\Sigma\alpha)\nu$ and $\varphi(\beta) = \mu(\Omega\beta)$. Clearly $\psi\varphi = 1$. The middle horizontal homomorphism is an isomorphism by 7.6. Since ψ is an epimorphism, L is an epimorphism and since φ is a monomorphism, L is a monomorphism. Clearly proper H-maps in $[\Omega G, Z]_H$ correspond to proper H-maps in $[\Omega G_k, Z]_H$ for each k. □

The proof of 7.5 will depend on an analysis of R. We define spaces A and C and maps a and c as follows:

$$a\colon A = \bigvee_{k\geqslant 1} P^{2np^k}(p^{r+k-1}) \to R \to E$$

$$c\colon C = \bigvee_{k\geqslant 1} P^{2np^k+1}(p^{r+k-1}) \to R \to E$$

by the maps $a(k)$ and $c(k)$ from (4.4) on the respective factors. The maps a and c factor through R by 5.3 and 6.41.

PROPOSITION 7.7. *$R \simeq A \vee C \vee \Sigma P$ where the inclusion of ΣP in R factors through $\Gamma\colon \Omega G * \Omega G \to R$.*

PROOF OF 7.5 (BASED ON 7.7): Given an H-map $\alpha\colon T \to Z$, the composition $\beta = \alpha h$:

$$\Omega G \xrightarrow{h} T \xrightarrow{\alpha} Z$$

is a proper H-map since h is proper. We construct an inverse:

$$T \xrightarrow{g} \Omega G \xrightarrow{\beta} Z$$

However, since g is not an H-map, we need an extra argument to show that βg is an H-map. Consider the diagram:

$$\begin{array}{ccccc} T\times T & \xrightarrow{g\times g} & \Omega G\times\Omega G & \xrightarrow{h\times h} & T\times T \\ \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\mu} \\ T & \xrightarrow{g} & \Omega G & \xrightarrow{h} & T \end{array}$$

in which the left-hand square is not homotopy commutative. Since the right-hand square and the rectangle are homotopy commutative, the difference between the two sides of the left-hand square

$$\Delta = (g\mu)^{-1}\mu(g\times g)\colon T\times T \to \Omega G$$

factors through the fiber of h:

$$\Omega R \longrightarrow \Omega G \xrightarrow{h} T \longrightarrow R \longrightarrow G.$$

However since β is proper and Z is homotopy Abelian the composition:

$$\Omega R \longrightarrow \Omega G \xrightarrow{\beta} Z$$

is null homotopic by 7.7. Thus $\beta\Delta$ is null homotopic and βg is an H-map. □

The remainder of this section will be devoted to a proof of 7.7. We begin by clarifying the relationship between R and W.

PROPOSITION 7.8. *$R \simeq (T \ltimes C) \vee W$.*

PROOF. By 6.17, the homomorphism

$$H_*(R; Z_{(p)}) \to H_*(W; Z_{(p)})$$

is a split epimorphism. Since R is a wedge of Moore spaces by 6.16 and W is a wedge of Moore spaces by 6.19, the map $R \to W$ has a right homotopy inverse. In the proof of 6.17, a map of fibration sequences was studied; in the limit, this is of the form

$$\begin{array}{ccc} T & = & T \\ \downarrow & & \downarrow \\ R & \longrightarrow & W \\ \downarrow & & \downarrow \\ G & \longrightarrow & D \end{array}$$

where D is the mapping cone of the map $c\colon C \to G$. It follows from 2.1 that there is a homotopy pushout diagram

$$\begin{array}{ccc} T \times \mathcal{C}(C) & \longrightarrow & W \\ \uparrow & & \uparrow \\ T \times C & \longrightarrow & R \end{array}$$

where $\mathcal{C}(C)$ is the cone on C. Since the inclusion of T in R is null homotopic, there is an induced map

$$T \ltimes C \simeq \mathcal{C}T \cup T \times C \to R$$

whose cofiber is W;

$$T \ltimes C \to R \to W.$$

Since the map $R \to W$ has a right homotopy inverse, the result follows. □

PROPOSITION 7.9. *There is a homotopy commutative diagram*

$$\begin{array}{ccc} T \ltimes C & \longrightarrow & R \\ {\scriptstyle g \ltimes c}\downarrow & & \downarrow \\ \Omega G \ltimes E & \xrightarrow{\pi\Gamma'} & G. \end{array}$$

PROOF. The map $T \ltimes C \simeq T \times C \cup \mathcal{C}T \to R$ is given as follows. The restriction to $T \times C$ comes from a trivialization of the pullback over C in the diagram

$$\begin{array}{ccccc} T & = & T & = & T \\ \downarrow & & \downarrow & & \downarrow \\ T \times C & \longrightarrow & R & \longrightarrow & J \\ \downarrow & & \downarrow & & \downarrow \\ C & \xrightarrow{c} & G & \longrightarrow & D \end{array}$$

so the composition $T \times C \to T \times C \cup \mathcal{C}T \to R \to G$ is given by the map

$$T \times C \xrightarrow{\pi_2} C \xrightarrow{c} G.$$

To understand the map $\mathcal{C}T \to R$, consider a general fibration

$$F \longrightarrow E \xrightarrow{\pi} B.$$

Using the homotopy lifting property, one can construct a map

$$(PB, \Omega B) \xrightarrow{L} (E, F)$$

extending the connecting map $\Omega B \to F$ and such that $\pi L \colon PB \to B$ is endpoint evaluation. In our case we use the map $L \colon (PG, \Omega G) \to (R, T)$ together with $g \colon T \to \Omega G$ to obtain the composition

$$(\mathcal{C}T, T) \xrightarrow{g} (C\Omega G, \Omega G) \xrightarrow{\xi} (PG, \Omega G) \xrightarrow{L} (R, T),$$

where ξ is defined in the proof of 2.6. This defines the map $\mathcal{C}T \to R$ and the composition

$$\mathcal{C}T \longrightarrow R \longrightarrow G$$

is given by

$$\mathcal{C}T \longrightarrow \mathcal{C}(\Omega G) \xrightarrow{\epsilon} G.$$

According to the definition of Γ' (see 3.8), the map

$$\Omega G \times E \cup PG \xrightarrow{\Gamma'} E \xrightarrow{\pi} G$$

is given by

$$\Omega G \times E \xrightarrow{\pi_2} E \xrightarrow{\pi} G$$

$$PG \xrightarrow{\epsilon} G$$

□

Corollary 7.10. *$T \ltimes C \simeq (T \wedge C) \vee C$ and the composition*

$$T \wedge C \xrightarrow{\zeta} T \ltimes C \longrightarrow R \longrightarrow G$$

factors through $\Omega G * \Omega G \xrightarrow{\Gamma} E \xrightarrow{\pi} G$.

Proof. Since C is a suspension, $T \ltimes C \simeq (T \wedge C) \vee C$. By 7.9, the composition in question factors up to homotopy as

$$T \wedge C \longrightarrow T \ltimes C \xrightarrow{g \ltimes c} \Omega G \ltimes E \xrightarrow{\Gamma'} E \xrightarrow{\pi} G$$

We construct a homotopy commutative diagram:

$$\begin{array}{ccccc}
T \wedge C & \longrightarrow & T \ltimes C & \longrightarrow & C \\
\downarrow & & \downarrow{\scriptstyle g \ltimes c} & & \downarrow{\scriptstyle c} \\
\Omega G * \Omega E & \longrightarrow & \Omega G \ltimes E & \longrightarrow & E \\
\downarrow & & \downarrow{\scriptstyle \Gamma'} & & \\
\Omega G * \Omega G & \xrightarrow{\Gamma} & E & & \\
 & & \downarrow{\scriptstyle \pi} & & \\
 & & G & &
\end{array}$$

as follows: the upper right hand square commutes by naturality, and the middle horizontal sequence is a fibration sequence by 3.17. This allows for the construction of the upper left hand square. Since the lower square commutes up to homotopy by 3.15, we see that the composition in question is homotopic to the left edge of the diagram which finishes the proof. □

At this point we have $R \simeq (T \wedge C) \vee C \vee W$, where $T \wedge C \longrightarrow R \longrightarrow G$ factors through

$$\Omega G * \Omega G \xrightarrow{\Gamma} E \xrightarrow{\pi} G.$$

The remainder of this section will be focused on proving

PROPOSITION 7.11. *$W \simeq A \vee W'$ where*

$$W' = \bigvee_{i \neq p^s} P^{2ni}(p^{r+\nu_p(i)}).$$

Furthermore, there is a factorization:

$$\begin{array}{ccc} W' & \longrightarrow & R \\ \downarrow & & \downarrow \\ \Omega G * \Omega G & \xrightarrow{\Gamma} & G \end{array}$$

The main ingredient for the proof of 7.11 is the following result:

PROPOSITION 7.12. *Suppose $p^k < m < p^{k+1}$ and $s = \nu_p(m)$.Then there is a map $f(m)\colon P^{2mn}(p^{r+s}) \longrightarrow J_k$ such that*

(a) *The composition*

$$P^{2mn}(p^{r+s}) \xrightarrow{f(m)} J_k \xrightarrow{\eta_k} F_k$$

induces a cohomology epimorphism.

(b) *There is a factorization:*

$$\begin{array}{ccc} P^{2mn}(p^{r+s}) & \xrightarrow{f(m)} & J_k \\ {\scriptstyle w}\downarrow & & \uparrow{\scriptstyle \tau_k} \\ \Omega G_k * \Omega G_k & \xrightarrow{\Gamma_k} & E_k \end{array}$$

PROOF OF 7.11. Suppose $p^k < m < p^{k+1}$. By 7.12, $\gamma_k f(m)$ is null homotopic, so there is a factorization:

$$\begin{array}{ccccc} & & J_k & & \\ & \overset{f(m)}{\nearrow} & \uparrow & \overset{\eta_k}{\searrow} & \\ P^{2mn}(p^{r+s}) & & \vert & & F_k \\ & \underset{\overline{f}(m)}{\searrow} & \vert & \nearrow & \\ & & W_k & & \end{array}$$

By 6.11, $\overline{f}(m)$ induces an isomorphism in integral homology in dimension $2mn-1$. Assembling the maps $f(m)$ together for all k together with the maps $a(k)$ we get a map

$$A \vee \bigvee_{m \neq p^k} P^{2mn}(p^{r+s}) \longrightarrow W$$

which induces an isomorphism in homology. Thus $W \simeq A \vee W'$ and W' factors through $\Omega G * \Omega G$ by 7.12. $\square$

The construction of $f(m)$ when $s = 0$ is immediate. We simply set $f(m) = x_m(k)$. In case $s > 0$ we need to construct relative Whitehead products using G_k. In the special case that $m = 2p^k$ we will need to use an H-space based Whitehead product.

The proof of 7.12 will rely on four lemmas. We will call an integer m acceptable if there is a map $f(m)\colon P^{2mn}(p^{r+s}) \to J_k$ with $\nu_p(m) = s$, satisfying 7.12(a) and (b).

LEMMA 7.13. *Suppose $p^k < m < p^{k+1}$ and $x\colon P^{2mn}(p^{r+t}) \to E_k$ is a map such that the composition*

$$P^{2mn}(p^{r+t}) \xrightarrow{x} E_k \xrightarrow{\tau_k} J_k \xrightarrow{\eta_k} F_k$$

induces an integral cohomology epimorphism. Suppose $i \leqslant t \leqslant k$. Then there is a map $f(m+p^i)\colon P^{2mn+2np^i}(p^{r+i}) \to J_k$ satisfying 7.12(a) *and* (b). *Consequently if $\nu_p(m+p^i) = i$, $m + p^i$ is acceptable.*

REMARK. This applies in particular when $m > p^k$ is acceptable with $t = \nu_p(m)$.

PROOF. Write $\Sigma P = P^{2mn}(p^{r+t})$ and let $\lambda_i\colon G_i \to G_k$ be the inclusion when $i \leqslant k$. We will define $f(m+p^i)$ using the relative Whitehead product:

$$G_i \circ \Sigma P \xrightarrow{\{\lambda_i, x\}_r} E_k \xrightarrow{\tau_k} J_k \xrightarrow{\eta_k} F_k.$$

Our first task will be to evaluate this in $H_{2n(m+p^i)}$. Recall (3.10) that $\{\lambda_i, x\}_r$ is the composition:

$$G_i \circ \Sigma P \xrightarrow{\psi} \Sigma(\Omega G_i * \Omega\Sigma P) \simeq \Omega G_i * \Omega\Sigma P \xrightarrow{\zeta} \Omega G_i \ltimes \Sigma P \longrightarrow \Omega G_k \ltimes E_k \xrightarrow{\Gamma'} E_k$$

For the first part of this composition, consider the diagram

$$(7.14)\qquad \begin{array}{ccccccc} G_i \circ \Sigma P & \xrightarrow{\psi} & \Sigma\Omega G_i \wedge \Omega\Sigma P & \simeq & \Omega G_i * \Omega\Sigma P & & \\ \uparrow \simeq & & \uparrow 1 \wedge i & & \uparrow 1 * i & \searrow^{\zeta} & \Omega G_i \ltimes \Sigma P \\ G_i \wedge P & \xrightarrow{\nu \wedge 1} & \Sigma\Omega G_i \wedge P & \simeq & \Omega G_i * P & \nearrow_{\zeta'} & \end{array}$$

where ν is the co-H space structure map on G_i. The left hand square is homotopy commutative by 3.2 and the right hand triangle defines ζ' (see footnote to 3.17).

Choose a generator $b_i \in H_{2np^i+1}(G_i) \cong Z/p$ and let $\sigma b_i \in H_{2np^i}(\Omega G_i)$ be the image of this generator under ν_*. Choose a generator $f \in H_{2mn-1}(P)$. Then by the above diagram we have

$$\zeta_*\psi_*(b_i \otimes f) = \sigma b_i \otimes 1 \otimes f \in H_{2mn+2np^i}(\Omega G_i \ltimes \Sigma P),$$

by applying the commutative square in the proof of 3.17. We now construct a diagram where the two right hand squares are homotopy commutative by 3.8, 3.11(b) and 3.11(d):

$$
(7.15)\qquad
\begin{array}{ccccc}
\Omega G_i \ltimes \Sigma P & \xrightarrow{\Omega\lambda_i \ltimes x} & \Omega G_k \ltimes E_k & \xrightarrow{\Gamma'} & E_k \\
 & & \downarrow & & \downarrow{\scriptstyle \eta_i\tau_k} \\
 & & \Omega D_k \ltimes F_k & \xrightarrow{\Gamma'} & F_k \\
 & & \uparrow & & \uparrow{\scriptstyle a} \\
 & & \Omega D_k \times F_k & \longrightarrow & \Omega S^{2n+1} \times F_k
\end{array}
$$

To evaluate $\eta_k\tau_k\{\lambda_i, x\}_r$ in $\bmod p$ homology we observe that the image of $\sigma b_i \otimes | \otimes f$ is $a_*(\alpha \otimes \beta)$ where $\alpha \in H_{2np^i}(\Omega S^{2n+1})$ is the image of Ωb_i under the homomorphism induced by the composition

$$\Omega G_i \longrightarrow \Omega G_k \longrightarrow \Omega D_k \longrightarrow \Omega S^{2n+1}$$

and β is the image of $1 \otimes f$ under the homomorphism

$$\Sigma P \xrightarrow{x} E_k \xrightarrow{\eta_k\tau_k} F_k.$$

By hypothesis $\beta \in H_{2mn}(F_k)$ is a generator.

LEMMA 7.16. *The image of $\sigma b_i \in H_{2np^i}(\Omega G_k)$ under the homomorphism*

$$\Omega G_k \to \Omega D_k \to \Omega S^{2n+1}$$

is a unit multiple of the generator.

PROOF. By 5.2, the composition in question is homotopic to the composition:

$$\Omega G_k \xrightarrow{\Omega\varphi_k} \Omega S^{2n+1}\{p^r\} \longrightarrow \Omega S^{2n+1}.$$

By 6.44, this factors as

$$\Omega G_k \xrightarrow{h_k} T \longrightarrow \Omega S^{2n+1}.$$

However the composition

$$G_k \xrightarrow{\nu} \Sigma\Omega G_k \xrightarrow{\Sigma h_k} \Sigma T$$

induces a monomorphism in $\bmod p$ homology, so $(h_k)_*(\sigma b_i)$ is a nonzero generator which is mapped to a unit multiple of v^{p^i} under the map $T \to \Omega S^{2m-1}$. $\square$

We now complete the proof of 7.13. Since $i \leqslant t$, we can find a map

$$P^{2mn+2np^i}(p^{r+i}) \to G_i \circ P^{2mn}(p^{r+t})$$

which induces an isomorphism in $\bmod p$ homology in dimension $2mn + 2np^i$ using 2.13(i) and 3.25. Then let $f(m+p^i)$ be the composition:

$$P^{2mn+2np^i}(p^{r+i}) \longrightarrow G_i \circ P^{2mn}(p^{r+t}) \xrightarrow{\{\lambda_i, x\}_r} E_k \xrightarrow{\tau_k} J_k.$$

By 7.14, 7.15, and 7.16, $\eta_k f(m+p^i)$ induces an epimorphism in mod p cohomology and consequently in integral cohomology as well. By 3.15, we have a homotopy commutative diagram

$$\begin{array}{ccccc}
\Omega G_i * \Omega\Sigma P & \xrightarrow{\zeta} & \Omega G_i \ltimes \Sigma P & & \\
\downarrow & & \downarrow & & \\
\Omega G_k * \Omega E_k & \xrightarrow{\zeta} & \Omega G_k \ltimes E_k & \longrightarrow & \Omega D_k \ltimes J_k \\
\downarrow & & \Gamma' \downarrow & & \Gamma' \downarrow \\
\Omega G_k * \Omega G_k & \xrightarrow{\Gamma} & E_k & \longrightarrow & J_k
\end{array}$$

so $f(m+p^i)$ factors through Γ. □

We need to consider the case $m = 2p^k$ separately. We will construct

$$P^{4np^k}(p^{r+k}) \xrightarrow{f(2p^k)} W.$$

It is the first example of a mod p^{r+k} Moore space in W by 6.19.

PROPOSITION 7.17. *$2p^k$ is acceptable.*

We will construct $f(2p^k)$ as an H-space based Whitehead product. Let $\overline{G}_k$ be the $2np^k$ skeleton of G_k. Then $\overline{G}_k \circ G_k$ has dimension $4np^k$.

LEMMA 7.18. *$\overline{G}_k \circ G_k$ is a wedge of Moore spaces.*

PROOF. Consider the cofibration sequence

$$\overline{G}_k \circ G_k \longrightarrow G_k \circ G_k \xrightarrow{Q} S^{2np^k+1} \circ G_k \simeq \Sigma^{2np^k} G_k.$$

Since $G_k \circ G_k$ is a wedge of Moore spaces by 2.13(l) and 3.21, it suffices to show that this cofibration splits. Choose a basis $\{a_i, b_i\}$ for $H_*(G_k)$ with $0 \leqslant i \leqslant k$ and $\beta^{(r+k)}(b_i) = a_i \neq 0$. Consider the classes $b_k \circ b_i$ corresponding to $b_k \otimes b_i$. We can construct a basis for $H_*(G_k \circ G_k)$ which contains the elements $b_k \circ b_i$ and $\beta^{(r+i)}(b_k \circ b_i)$. Since $G_k \circ G_k$ is a wedge of Moore spaces we can construct maps of Moore spaces into $G_k \circ G_k$ realizing these basis elements and from this a right homotopy inverse to Q. Thus

$$G_k \circ G_k \simeq \overline{G}_k \circ G_k \vee \Sigma^{2np^k} G_k.$$ □

Now consider the principal fibration sequence:

$$\Omega\overline{G}_k \times \Omega G_k \longrightarrow \Omega\overline{G}_k * \Omega G_k \longrightarrow \overline{G}_k \vee G_k.$$

We will study the integral homology Serre spectral sequence of this fibration. The principal action defines a module structure

$$E^r_{o,q'} \otimes E^r_{p,q} \to E^r_{p,q+q'}.$$

For $i = 1, 2$, let $a(i) \in H_{2np^k}(\overline{G}_k \vee G_k; Z)$ be the image of a_k in the i^{th} axis. Let $\sigma a \in H_{2np^k-1}(\Omega\overline{G}_k; Z)$ be the desuspended image under ν_* of a_k, and $\sigma a(i)$ the image of σa in the i^{th} axis in $H_{2np^k-1}(\Omega\overline{G}_k \times \Omega G_k; Z)$. Using the universal coefficient theorem, we define a monomorphism

$$H_p(\overline{G}_k \vee G_k; Z) \otimes H_q(\Omega\overline{G}_k \times \Omega G_k; Z) \to E^2_{p,q}.$$

Then $d_{2np^k}(a(i)\otimes 1) = 1\otimes \sigma a(i)$. Define $\xi \in E^2_{2np^k,2np^k-1}$ by

$$\xi = a(1)\otimes \sigma a(2) + a(2)\otimes \sigma a(1).$$

ξ has order p^{r+k} and $d_{2np^k}(\xi) = 0$. Since $E^2_{p,q} = 0$ when $p > 2np^k + 1$, ξ survives to an element of order p^{r+k} in E^∞. By 7.18, 3.21 and 2.13(l), $\Omega\overline{G}_k * \Omega G_k$ is a wedge of Moore spaces and by an easy homology calculation, every element in the homology has order at most p^{r+k}. Consequently ξ converges to an element $[\xi]$ of order p^{r+k} in $H_{4np^k-1}(\Omega\overline{G}_k * \Omega G_k; Z)$. Since $\Omega\overline{G}_k * \Omega G_k$ is a wedge of Moore spaces, there is a map

$$\varphi\colon P^{4np^k}(p^{r+k}) \longrightarrow \Omega\overline{G}_k * \Omega G_k$$

whose homology image contains a class $\eta \in H_{4np^k}(\Omega\overline{G}_k * \Omega G_k)$ with $\beta^{(r+k)}(\eta)$ equal to the mod p reduction of $[\xi]$.

Now let γ be the composition:

$$\Omega\overline{G}_k * \Omega G_k \longrightarrow \Omega G_k * \Omega G_k \xrightarrow{\Gamma_k} E_k \xrightarrow{\tau_k} J_k \xrightarrow{\eta_k} F_k$$

LEMMA 7.19. *The composition*

$$P^{4np^k}(p^{r+k}) \xrightarrow{\varphi} \Omega\overline{G}_k * \Omega G_k \xrightarrow{\gamma} F_k$$

induces an isomorphism in mod p *homology in dimension* $4np^k$.

REMARK. This implies that $\gamma\varphi$ induces an epimorphism in integral cohomology.

PROOF. Consider the two diagrams of principal fibrations

$$\begin{array}{ccc}
\Omega\overline{G}_k \times \Omega G_k & = & \Omega\overline{G}_k \times \Omega G_k \\
\downarrow & & \downarrow \\
L & \longrightarrow & \Omega\overline{G}_k * \Omega G_k \\
\downarrow & & \downarrow \\
G_{k-1}\vee G_{k-1} & \longrightarrow & \overline{G}_k \vee G_k
\end{array}
\qquad\qquad
\begin{array}{ccc}
\Omega^2 S^{2n+1} & = & \Omega^2 S^{2n+1} \\
\downarrow & & \downarrow \\
F_{k-1} & \longrightarrow & F_k \\
\downarrow & & \downarrow \\
D_{k-1} & \longrightarrow & D_k
\end{array}$$

where L is the total space of the induced fibration. The homotopy commutative square

$$\begin{array}{ccc}
\overline{G}_k \vee G_k & \longrightarrow & D_k \\
\downarrow & & \downarrow \\
\overline{G}_k \times G_k & \longrightarrow & S^{2n+1}
\end{array}$$

induces a map from the left hand pair of fibrations to the right hand pair. We apply 2.1 to obtain the following homotopy commutative diagram:

$$\begin{array}{ccc}
\Omega\overline{G}_k * \Omega G_k \big/ L & \xrightarrow{\quad\gamma\quad} & F_k \big/ F_{k-1} \\
\simeq\downarrow & & \simeq\downarrow \\
(S^{2np^k} \vee P^{2np^k+1}(p^{r+k})) \rtimes \Omega\overline{G}_k \times \Omega G_k & \xrightarrow{\gamma'} & \Omega S^{2n+1} \ltimes P^{2np^k+1}(p)
\end{array}$$

Let $e \in H_{2np^k}(S^{2np^k})$ be the image of $a(1)$ and $f \in H_{2np^k}(P^{2np^k+1}(p^{r+k}))$ be the image of $a(2)$ under the quotient map

$$\overline{G}_k \vee G_k \longrightarrow \overline{G}_k \vee G_k \Big/ G_{k-1} \vee G_{k-1} \simeq S^{2np^k} \vee P^{2np^k+1}(p^{r+k}).$$

Choose g with $\beta^{(r+k)}(g) = f$. Then the image of $[\xi]$ under the left hand equivalence is $e \otimes \sigma a(2) + f \otimes \sigma a(1)$. The image of η is a class η' such that

$$\beta^{(r+k)}(\eta') = e \otimes \sigma a(2) + f \otimes \sigma a(1).$$

At this point in the Bockstein spectral sequence there are very few classes left and the only possibility for η' is

$$\eta' = e \otimes \sigma b(2) + g \otimes \sigma a(1)$$

where $\beta^{(r+k)} b(2) = a(2)$.

The components of the map γ' are the inclusion $S^{2np^k} \longrightarrow P^{2np^k+1}(p)$ and

$$\sigma^{r+k-1} \colon\ P^{2np^k+1}(p^{r+k}) \longrightarrow P^{2np^k+1}(p).$$

By 7.16, the image of $\sigma b(2)$ is $v^{p^k} \neq 0$. We conclude that the image of η' is nonzero

$$P^{4np^k}(p^{r+k}) \xrightarrow{\varphi} \Omega\overline{G}_k * \Omega G_k \xrightarrow{\gamma} F_k \to F_k\big/F_{k-1} \simeq P^{2np^k+1}(p) \rtimes \Omega S^{2n+1}$$

in mod p homology, from which the conclusion follows. □

Proof of 7.17. Let $f(2p^k)$ be the composition

$$P^{4np^k}(p^{r+k}) \xrightarrow{\varphi} \Omega\overline{G}_k * \Omega G_k \xrightarrow{\Gamma_k} E_k \xrightarrow{\eta_k} J_k.$$

The result follows from 7.19. □

Proof of 7.12. Write $m = e_0 + e_1 p + \cdots + e_k p^k$ where $0 \leqslant e_i < p$. Let $\ell(m)$ bet the number of coefficients e_i which are nonzero. We first deal with the case $\ell(m) = 1$. Then $m = e_k p^k$ with $1 < e_k < p$ since $p^k < m < p^{k+1}$. The case $e_k = 2$ is 7.17. If $e_k > 2$ we apply 7.13 with $x = f((e_k - 1)p^k)$ and $i = t = k$, to establish this case by induction. In case $\ell(m) = 2$, we first consider the case $m = p^i + e_k p^k$ for $i < k$. In case $e_k \geqslant 2$ we apply 7.13 with $x = f(e_k p^k)$ and $t = k$. In case $e_k = 1$ we apply 7.13 with $x = a(k)$. In this case $t = k - 1 \geqslant i$. We now consider the general case with $\ell(m) = 2$. In this case $m = e_i p^i + e_k p^k$ with $i < k$. We do this by induction on e_i with $e_i < p$ as before. The general case is by induction on $\ell(m)$ and then induction on the coefficient of the least power of p in the expansion using 7.13 repeatedly. □

Proof of 7.7. By 7.8 and 7.10 $R = T \ltimes C \vee W \simeq T \wedge C \vee C \vee W$ where $T \wedge C$ is a wedge of Moore spaces which factors through Γ. By 7.11 $W \simeq A \vee W'$ where W' is a wedge of Moore spaces which factors through Γ. Set $\Sigma P = W' \vee C \wedge W$. □

7.3. The exact sequence

In this section we will define the homomorphisms e, r, and β and prove Theorem 7.3.

We define

$$e\colon p_k(\Omega Z) \to G_k(Z)$$

as follows. Let $\phi\colon P^{2np^k-1}(p^{r+k}) \to \Omega Z$, and extend the adjoint

$$\widetilde{\phi}\colon P^{2np^k}(p^{r+k}) \to Z$$

to an H-map

$$\widehat{\phi}\colon \Omega P^{2np^k+1}(p^{r+k}) \to Z.$$

Since ϕ is divisible by p^{r+k-1}, so is $\widehat{\phi}$ and consequently the composition

$$\Omega G_k \xrightarrow{\Omega\pi'} \Omega P^{2np^k+1}(p^{r+k}) \xrightarrow{\widehat{\phi}} Z$$

is a proper H-map. We define $e(\phi) = \widehat{\phi}\Omega\pi'$. Then e is clearly a homomorphism. To see that e is a monomorphism, we suppose $e(\phi)$ is null homotopic. Since π' is a co-H map, we have a homotopy commutative diagram:

$$\begin{array}{ccccc}
G_k & \xrightarrow{\nu} & \Sigma\Omega G_k & & \\
\downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \Sigma\Omega\pi'} & \searrow^{\Sigma e(\phi)} & \\
 & & & & \Sigma Z \\
 & & & \nearrow_{\Sigma\widehat{\phi}} & \\
P^{2np^k+1}(p^{r+k}) & \longrightarrow & \Sigma\Omega P^{2np^k+1}(p^{r+k}) & &
\end{array}$$

Since $e(\phi)$ is null homotopic, the upper composition is null homotopic, so the lower composition factors over the cofiber of π':

$$G_k \xrightarrow{\pi'} P^{2np^k+1}(p^{r+k}) \longrightarrow \Sigma G_{k-1} \longrightarrow \Sigma G_k.$$

But since ΣG_k splits as a wedge of Moore spaces, the map

$$P^{2np^k+1}(p^{r+k}) \to \Sigma G_{k-1}$$

is null homotopic. It follows that the composition:

$$P^{2np^k+1}(p^{r+k}) \longrightarrow \Sigma\Omega P^{2np^k+1}(p^{r+k}) \xrightarrow{\Sigma\widehat{\phi}} \Sigma Z$$

is null homotopic. Since Z in an H space, we conclude that

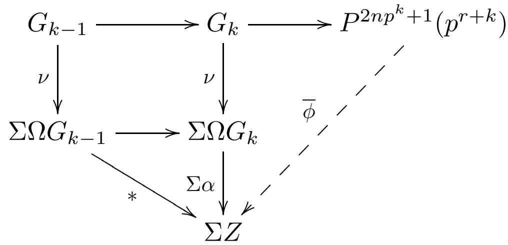
$$P^{2np^k}(p^{r+k}) \longrightarrow \Omega P^{2np^k+1}(p^{r+k}) \xrightarrow{\widehat{\phi}} Z$$

is null homotopic. Since $\widehat{\phi}$ is an H-map, $\widetilde{\phi}$ is null homotopic and consequently ϕ is as well. Thus e is a monomorphism.

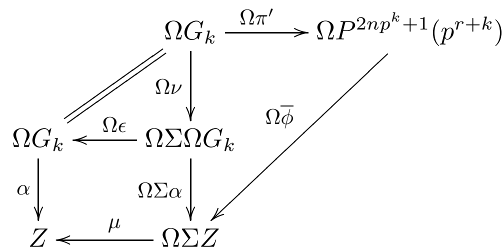
The map

$$r\colon G_k(Z) \to G_{k-1}(Z)$$

is given by restriction. Clearly $re = 0$. Suppose $r\alpha = 0$ for some proper H-map $\alpha\colon \Omega G_k \to Z$. We construct an extension $\overline{\phi}$ in the diagram:



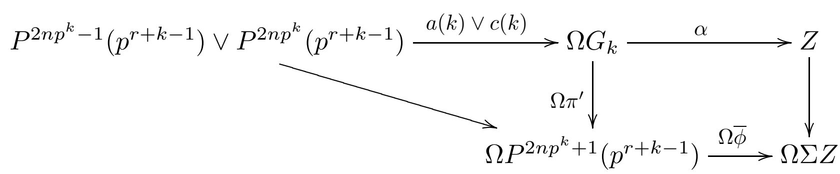
We include the loops on the right-hand triangle into the diagram:



to see that α is homotopic to $\mu(\Omega\overline{\phi})(\Omega\pi') \sim \widehat{\phi}\Omega\pi'$ where $\widehat{\phi}$ is the composition $\mu\Omega\overline{\phi}$. Restricting $\widehat{\phi}$ to $P^{2np^k}(p^{r+k})$ defines $\widetilde{\phi}$ whose adjoint is

$$\phi\colon P^{2np^k-1}(p^{r+k}) \to \Omega Z.$$

To see that ϕ is divisible by p^{r+k-1} , it suffices to show that $\overline{\phi}$ is divisible by p^{r+k-1}. However since α is proper, the upper composition in the diagram:



is null homotopic. Consequently the lower composition and its adjoint are null homotopic:

$$P^{2np^k}(p^{r+k-1}) \vee P^{2np^k+1}(p^{r+k-1}) \xrightarrow{-\delta_1 \vee p} P^{2np^k+1}(p^{r+k}) \xrightarrow{\overline{\phi}} \Sigma Z$$

It follows from 1.8 that $\overline{\phi}$ is divisible by p^{r+k-1}. Thus $\widetilde{\phi}$ and ϕ are divisible as well.

Finally we define

$$\beta\colon G_k(Z) \to p_{k+1}(Z)$$

as the composition

$$P^{2np^{k+1}-1}(p^{r+k+1}) \xrightarrow{\widetilde{\beta_{k+1}}} \Omega E_k \longrightarrow \Omega G_k \xrightarrow{\alpha} Z$$

At one point it was thought that this composition would always be null homotopic when Z is homotopy Abelian ([**The01**, 5.1]). What we will prove is that the compositions:

$$P^{2np^{k+1}-1}(p^{r+k}) \xrightarrow{\rho} P^{2np^{k+1}-1}(p^{r+k+1}) \xrightarrow{\widetilde{\beta}_{k+1}} \Omega G_k \xrightarrow{\alpha} Z$$

$$P^{2np^{k+1}-2}(p^{r+k}) \xrightarrow{\delta_1} P^{2np^{k+1}-1}(p^{r+k+1}) \xrightarrow{\widetilde{\beta}_{k+1}} \Omega G_k \xrightarrow{\alpha} Z$$

are null homotopic, which implies

THEOREM 7.20. *$\beta(\alpha)$ is divisible by p^{r+k}.*

We accomplish this by constructing maps:

$$r\colon P^{2np^{k+1}}(p^{r+k}) \to R_k \to W_k$$
$$d\colon P^{2np^{k+1}-1}(p^{r+k}) \to R_k \to W_k$$

which differ from $\beta_{k+1}p$ and $\beta_{k+1}\delta_1$ by maps which factor through $A \vee C \vee \Sigma P$.

LEMMA 7.21. *The homomorphism induced by the inclusion*

$$H_*(W_k) \to H_*(J_k)$$

is a monomorphism.

PROOF. By 6.11, W_k has one cell in each dimension of the form $2ni$ or $2ni-1$ for each $i \geqslant 2$; consequently

$$H_j(W_k) = \begin{cases} Z/p & \text{if } j = 2ni \text{ or } 2ni-1, \quad i \geqslant 2 \\ 0 & \text{otherwise.} \end{cases}$$

By 6.27, the maps $x_i(k)\colon P^{2ni} \to J_k$ and $y_i(k)\colon P^{2ni-1} \to J_k$ are nonzero in mod p homology in dimensions $2ni$ and $2ni-1$ respectively. By 6.41, the maps $x_i(k)$ and $y_i(k)$ factor through W_k up to homotopy when $i \geqslant 2$. The result follows. □

We will write $x_i \in H_{2ni}(J_k)$ and $y_i \in H_{2ni-1}(J_k)$ for the images of the generators in the homology of the respective Moore spaces. Note that in the congruence homotopy of J_k we have

$$\nu \cdot x_i(k) \equiv x_{i+1}(k)$$
$$\mu \cdot x_i(k) \equiv y_{i+1}(k)$$

when $i \geqslant p^k$ by 6.26. The action of the principal fibration defines an action

$$H_*(\Omega D_k) \otimes H_*(J_k) \to H_*(J_k).$$

Let u, v be the Hurewicz images of ν and μ (see 5.20). Then $Z/p[v] \otimes \Lambda(u) \subset H_*(\Omega D_k)$ acts on $H_*(J_k)$.

LEMMA 7.22. *If $i \geqslant p^k$, $vx_i = x_{i+1}$ and $ux_i = y_{i+1}$.*

PROOF. Apply 5.16. □

LEMMA 7.23. *There are maps*

$$r\colon P^{2np^{k+1}}(p^{r+k}) \to W_k$$

and

$$d\colon P^{2np^{k+1}-1}(p^{r+k}) \to W_k$$

which are nonzero in mod p *homology in dimensions* $2np^{k+1}$ *and* $2np^{k+1}-1$ *respectively and whose image in* J_k *factors through the map*

$$\Omega G_k * \Omega G_k \xrightarrow{\Gamma_k} E_k \xrightarrow{\tau_k} J_k.$$

REMARK. Both $\beta_{k+1}\rho$ and $\beta_{k+1}\delta_1$ satisfy the homological condition of 7.23 and factor through W_k. However β_{k+1} does not factor through Γ_k since it has order p^{r+k+1}, and there are no elements in the homotopy of $\Omega G_k * \Omega G_k$ in that dimension of that order.

PROOF. Let $x = f(p^k(p-1))\colon P^{2np^k(p-1)}(p^{r+k}) \to J_k$ be the map constructed in 7.12. Apply 7.13 with $i = k$ to construct

$$r = f(p^{k+1})\colon P^{2np^{k+1}}(p^{r+k}) \to J_k.$$

The class $x_{p^{k+1}} \in H_{2np^{k+1}}(J_k)$ is in the homology image of r since r factors through W_k and projects to a nonzero class in $H_{2np^{k+1}}(F_k)$.

To construct d, we return to the class x above and observe that the homomorphism induced by x:

$$H_*(P^{2np^k(p-1)}(p^{r+k})) \to H_*(J_k)$$

contains both $x_{p^k(p-1)}$ and $y_{p^k(p-1)}$ in its image since x factors through W_k which has $P^{2np^k(p-1)}(p^{r+k})$ as a retract.

Consequently $x\beta\colon P^{2np^k(p-1)-1}(p^{r+k}) \to J_k$ has $y_{p^k(p-1)}$ in its homology image. Now consider

$$[\lambda_k, x\beta]_r\colon G_k \circ P^{2np^k(p-1)-1}(p^{r+k}) \to J_k$$

and apply 7.14 and 7.15. It follows that $v^{p^k} y_{p^k(p-1)} = y_{p^{k+1}}$ is in the homology image by 7.16. We then choose a map

$$P^{2np^{k+1}-1}(p^{r+k}) \to G_k \circ P^{2np^k(p-1)-1}$$

which induces an isomorphism in dimension $2np^{k+1}-1$ to construct d. ☐

PROOF OF 7.20. Since W is a retract of R by 7.8 we can assume that r and d factor through R. Choose units u_1 and u_2 so that the maps

$$\Delta_1 = u_1\beta_{k+1}\rho - r\colon P^{2np^{k+1}}(p^{r+k}) \to R_k \to W_k$$

$$\Delta_2 = u_2\beta_{k+1}\delta_1 - d\colon P^{2np^{k+1}-1}(p^{r+k}) \to R_k \to W_k$$

are trivial in mod p homology. This can be done since the relevant factor of W_k is $P^{2np^{k+1}}(p^{r+k+1})$ and both $\beta_{k+1}\rho$ and r are nontrivial in dimension $2np^{k+1}$ while $\beta_{k+1}\delta_1$ and d are nontrivial in dimension $2np^{k+1}-1$. It follows from 7.7 that both Δ_1 and Δ_2 factor through

$$\bigvee_{i=1}^{k} P^{2np^i}(p^{r+i-1}) \vee P^{2np^i-1}(p^{r+i-1}) \vee \Sigma P.$$

Since α is proper, we conclude that the compositions

$$P^{2np^{k+1}-1}(p^{r+k}) \xrightarrow{\widetilde{\Delta}_1} \Omega R_k \longrightarrow \Omega G_k \xrightarrow{\alpha} Z$$

$$P^{2np^{k+1}-2}(p^{r+k}) \xrightarrow{\widetilde{\Delta}_2} \Omega R_k \longrightarrow \Omega G_k \xrightarrow{\alpha} Z$$

are both null homotopic. Since the maps r and d factor through $\Omega G_k * \Omega G_k$, these terms are null homotopic and we conclude that the compositions

$$P^{2np^{k+1}-1}(p^{r+k}) \xrightarrow{\widetilde{\beta}_{k+1}\rho} \Omega R_k \longrightarrow \Omega G_k \xrightarrow{\alpha} Z$$

$$P^{2np^{k+1}-2}(p^{r+k}) \xrightarrow{\widetilde{\beta}_{k+1}\delta_1} \Omega R_k \longrightarrow \Omega G_k \xrightarrow{\alpha} Z$$

are null homotopic. Now consider the cofibration sequence

$$P^{2np^{k+1}-2}(p^{r+k}) \vee \rho^{2np^{k+1}-1}(p^{r+k}) \xrightarrow{-\delta_1 \vee \rho} P^{2np^{k+1}-1}(p^{r+k+1}) \xrightarrow{p^{r+k}} P^{2np^{k+1}-1}(p^{r+k+1}).$$

From this we see that the composition

$$P^{2np^{k+1}-1}(p^{r+k+1}) \xrightarrow{\widetilde{\beta}_{k+1}} \Omega R_k \longrightarrow \Omega G_k \xrightarrow{\alpha} Z$$

is divisible by p^{r+k}. □

PROPOSITION 7.24. $\beta r = 0$

PROOF. Since α is proper, the composition on the right in the diagram:

$$\begin{array}{ccccc}
P^{2np^k-1}(p^{r+k}) & \xrightarrow{\sigma \vee \sigma\beta} & P^{2np^k-1}(p^{r+k-1}) \vee P^{2np^k}(p^{r+k-1}) & & \\
\downarrow{\scriptstyle \widetilde{\beta}_k} & & \downarrow{\scriptstyle a(k) \vee c(k)} & & \\
\Omega G_{k-1} & \longrightarrow & \Omega G_k & \xrightarrow{\alpha} & Z
\end{array}$$

is null homotopic. The diagram commutes up to homotopy by 4.4 from which the result follows. □

PROPOSITION 7.25. *If* $\beta(\alpha)=0$, $\alpha \sim r\alpha'$ *for some proper* H*-map* $\alpha'\colon \Omega G_k \to Z$.

PROOF. By 4.4, $G_k = G_{k-1} \cup_{\alpha_k} CP^{2np^k}(p^{r+k})$. Since G is a retract of ΣT, G_{k-1} and G_k are co-H spaces and there is a homotopy commutative diagram

$$\begin{array}{ccc}
G_{k-1} & \longrightarrow & G_k \\
\downarrow & & \downarrow \\
G_{k-1} \vee G_{k-1} & \longrightarrow & G_k \vee G_k
\end{array}$$

The map $\alpha_k \vee \alpha_k\colon P^{2np^k}(p^{r+k}) \vee P^{2np^k}(p^{r+k}) \to G_{k-1} \vee G_{k-1}$ factors through the fiber of the lower horizontal map in the diagram and defines a homotopy equivalence with the $2np^k$ skeleton of the fiber of that map. However the composition

$$P^{2np^k}(p^{r+k}) \xrightarrow{\alpha_k} G_{k-1} \longrightarrow G_{k-1} \vee G_{k-1}$$

factors through this fiber, from which we get a homotopy commutative square

$$\begin{array}{ccc} P^{2np^k}(p^{r+k}) & \xrightarrow{\alpha_k} & G_{k-1} \\ \downarrow & & \downarrow \\ P^{2np^k}(p^{r+k}) \vee P^{2np^k}(p^{r+k}) & \xrightarrow{\alpha_k \vee \alpha_k} & G_{k-1} \vee G_{k-1}; \end{array}$$

that is, α_k is a co-H map. Consequently there is also a homotopy commutative diagram:

$$\begin{array}{ccc} P^{2np^k}(p^{r+k}) & \xrightarrow{\alpha_k} & G_{k-1} \\ \downarrow & & \downarrow \nu \\ \Sigma\Omega P^{2np^k}(p^{r+k}) & \longrightarrow & \Sigma\Omega G_{k-1} \end{array}$$

However, the composition on the left and the bottom is $\Sigma\widetilde{\alpha}_k$, where

$$\widetilde{\alpha}_k \colon P^{2np^k-1}(p^{r+k}) \to \Omega G_{k-1}$$

is the adjoint of α_k. Let $b = \pi_{k-1}\beta_k \colon P^{2np^k}(p^{r+k}) \to G_{k-1}$. By 4.4, $p^{r+k-1}b$ is homotopic to α_k. This leads to a homotopy commutative diagram:

$$\begin{array}{ccccc} P^{2np^k}(p^{r+k}) & = & P^{2np^k}(p^{r+k}) & \xrightarrow{p^{r+k-1}} & P^{2np^k}(p^{r+k}) \\ \alpha_k \downarrow & & \Sigma\widetilde{\alpha}_k \downarrow & & \Sigma\widetilde{b} \downarrow \\ G_{k-1} & \xrightarrow{\nu} & \Sigma\Omega G_{k-1} & \longrightarrow & \Sigma\Omega G_{k-1} \end{array}$$

Taking cofibers vertically, we get a composition

$$G_k \longrightarrow \Sigma(\Omega G_{k-1} \cup_{\widetilde{\alpha}_k} CP^{2np^k-1}(p^{r+k})) \longrightarrow \Sigma(\Omega G_{k-1} \cup_{\widetilde{b}} CP^{2np^k-1}(p^{r+k})).$$

By hypothesis, α extends to a map

$$\Omega G_{k-1} \cup_{\widetilde{b}} CP^{2np^k-1}(p^{r+k}) \xrightarrow{\overline{\alpha}} Z.$$

Composing these maps together defines a map α''

$$G_k \longrightarrow \Sigma(\Omega G_{k-1} \cup_{\widetilde{b}} CP^{2np^k-1}(p^{r+k})) \xrightarrow{\Sigma\overline{\alpha}} \Sigma Z$$

whose restriction to G_{k-1} is the composition

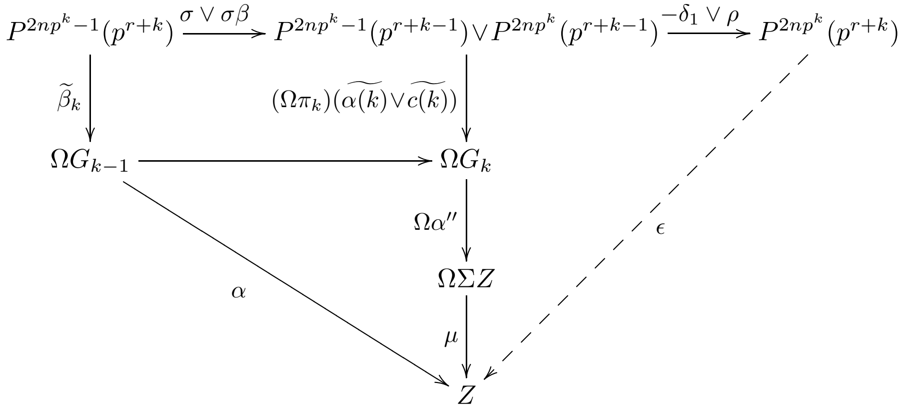
$$G_{k-1} \xrightarrow{\nu} \Sigma\Omega G_{k-1} \xrightarrow{\Sigma\alpha} \Sigma Z.$$

We now form the homotopy commutative diagram

$$(7.26) \qquad \begin{array}{ccccc} \Omega G_k & \xrightarrow{\Omega\alpha''} & \Omega\Sigma Z & \xrightarrow{\mu} & Z \\ \uparrow & & \uparrow \Omega\Sigma\alpha & & \uparrow \alpha \\ \Omega G_{k-1} & \xrightarrow{\Omega\nu} & \Omega\Sigma\Omega G_{k-1} & \xrightarrow{\Omega\epsilon} & \Omega G_{k-1} \end{array}$$

where the lower composition is homotopic to the identity. The upper composition is an H-map extending α. We will modify this slightly to satisfy our requirements.

In the diagram below, the left hand triangle is homotopy commutative due to 7.26 and the left hand square follows from 4.4.



By hypothesis $\beta(\alpha) = \alpha\widetilde{\beta}_k$ is null homotopic. Since the upper horizontal sequence is a cofibration sequence, there is an extension

$$\epsilon\colon P^{2np^k}(p^{r+k}) \longrightarrow Z.$$

Extend ϵ to an H-map $\epsilon'\colon \Omega P^{2np^k+1}(p^{r+k}) \longrightarrow Z$ and define

$$\alpha' = \mu\Omega\alpha'' - \epsilon'\Omega\pi',$$

where π' is the projection of G_k onto $P^{2np^k+1}(p^{r+k})$. Since Z is homotopy-Abelian and α' is the difference between two H-maps, α' is an H-map. Since the restriction of $\epsilon'\Omega\pi'$ to ΩG_{k-1} is null homotopic, α' extends α by 7.26. From 4.4 we construct a homotopy commutative square:

$$\begin{array}{ccc}
P^{2np^k}(p^{r+k}) & \xrightarrow{\quad i \quad} & \Omega P^{2np^k+1}(p^{r+k}) \\
\Big\uparrow{\scriptstyle -\delta_1 \vee \rho} & & \Big\uparrow{\scriptstyle \Omega\pi'} \\
P^{2np^k-1}(p^{r+k-1}) \vee P^{2np^k}(p^{r+k-1}) & \xrightarrow{(\Omega\pi_k)(\widetilde{a(k)} \vee \widetilde{c(k)})} & \Omega G_k
\end{array}$$

Consequently,

$$\begin{aligned}
\alpha'(\Omega\pi_k)(\widetilde{a(k)} \vee \widetilde{c(k)} &\sim (\mu\Omega\alpha'' - \epsilon'\Omega\pi')(\Omega\pi_k)(\widetilde{a(k)} \vee \widetilde{c(k)}) \\
&\sim \epsilon(\delta_1 \vee \rho) - \epsilon' i(-\delta_1 \vee \rho) \sim *
\end{aligned}$$

□

This completes the proof of 7.3 □

7.4. Applications

In this section we will discuss various applications of the results developed in the previous sections.

PROPOSITION 7.27. *Suppose $p^{r+1}\pi_*(Z) = 0$. Then there is a natural exact sequence:*

$$0 \longrightarrow p^r\left[P^{2np}(p^{r+1}), Z\right] \xrightarrow{\;e\;} [T, Z]_H \xrightarrow{\;r\;} \left[P^{2n}(p^r), Z\right] \xrightarrow{\;\beta\;} p^r\left[P^{2np-1}(p^{r+1}), Z\right].$$

In particular, if $p^r\pi_i(Z) = 0$ for $2np - 2 \leqslant i \leqslant 2np$, r is an isomorphism.

NOTE. We are not asserting that β is onto.

PROOF. This is immediate from 7.2 and 7.3 since $p_k(Z) = 0$ and $p_k(\Omega Z) = 0$ for all $k \geqslant 2$. □

COROLLARY 7.28. *Suppose T and T' are two homotopy Abelian Anick spaces for the same values of n, r and $p > 3$. Then there is a homotopy equivalence via an H-map and the H-space expondent is p^r.*

PROOF. We apply 7.27 with $Z = T'$. It suffices to show that $p^r\pi_i(T') = 0$ for $2np - 2 \leqslant i \leqslant 2np$. We apply the fibration sequence:

$$W_n \longrightarrow T'_{2n-1} \xrightarrow{E} \Omega T_{2n}(p^r) \xrightarrow{H} BW_n$$

from [**GT10**]. According to [**Nei83**], $p^r\pi_*(T_{2n}) = 0$ and according to [**CMN79c**] $p\pi_*(W_n) = 0$, so $p^{r+1}\pi_*(T') = 0$. However, since $p \geqslant 3$, $\pi_i(W_n) = 0$ when $2np-2 \leqslant i \leqslant 2np$, so $p^r\pi_i(T') = 0$ in this range. Thus

$$[T, T']_H \simeq \left[P^{2n}(p^r), T'\right] = Z/p^r$$

and the result follows. In particular, $p^r\pi_i(T) = 0$ for all i as a consequence. □

COROLLARY 7.29. *Suppose $\alpha\colon P^{2n}(p^r) \longrightarrow P^{2n}(p^s)$ with $s \leqslant r$ then there is a unique H-map $\widehat{\alpha}$ such that the diagram:*

$$\begin{array}{ccc} T_{2n-1}(p^r) & \xrightarrow{\widehat{\alpha}} & T_{2m-1}(p^s) \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} \\ P^{2n}(p^r) & \xrightarrow{\alpha} & P^{2m}(p^s) \end{array}$$

homotopy commutes.

NOTE. In case $r = s$, this result was the original motivation for these conjectures, leading to a secondary composition theory [**Gra93a**].

PROOF. This follows from 7.27 and 7.28. □

PROPOSITION 7.30. *There is an H-map $\theta_1\colon T_{2n-1}(p^r) \longrightarrow T_{2np-1}(p^{r+1})$ which induces a homomorphism of degree p^r in H^{2np}. Furthermore, the map e in 7.27, evaluated on on $p^r f$ is the composition:*

$$T_{2n-1}(p^r) \xrightarrow{\theta_1} T_{2np-1}(p^{r+1}) \xrightarrow{\widehat{f}} Z$$

where $\widehat{f}$ is the unique extension of f to an H-map.

PROOF. By 7.28, $p^{r+1}\pi_*(T_{2np-1}(p^{r+1})) = 0$, so we may apply 7.27 with $Z = T_{2np-1}(p^{r+1})$ and use naturality under $\widehat{f}$. This leads to a commutative square

$$\begin{array}{ccccc} 0 & \longrightarrow & p^r\left[P^{2np}(p^{r+1}), Z\right] & \xrightarrow{e} & [T,Z]_H \\ & & \uparrow \widehat{f}_* & & \uparrow \widehat{f}_* \\ 0 & \longrightarrow & p^r\left[P^{2np}(p^{r+1}), T_{2np-1}(p^{r+1})\right] & \xrightarrow{e} & \left[T, T_{2np-1}(p^{r+1})\right]_H \\ & & \wr\wr & & \\ & & Z/p & & \end{array}$$

The image of the generator in the lower left hand corner under e is an H-map θ_1 which is nonzero and of order p. To evaluate $(\theta_1)^*$ in cohomology, use the diagram:

$$\begin{array}{ccc} T_{2n-1}(p^r) & \xrightarrow{\theta_1} & T_{2np-1}(p^{r+1}) \\ h_1 \uparrow & & \uparrow \\ \Omega G_1 & \xrightarrow{p^r(\Omega\pi')} & \Omega P^{2np+1}(p^{r+1}) \end{array}$$

based on the definition of e. □

Note that a similar construction can be made in case $p^{r+k}\pi_*(Z) = 0$. In this case

$$[T, Z]_H \simeq G_k(Z)$$

and

$$e\colon\ p^{r+k-1}\left[P^{2np^k}(p^{r+k}), Z\right] \longrightarrow G_k(Z)$$

can be evaluated on $p^{r+k-1}f$ as a composition:

$$T_{2n-1} \xrightarrow{\theta_k} T_{2np^k-1} \xrightarrow{\widehat{f}} Z$$

where θ_k is an H-map of order p inducing p^{r+k-1} in H^{2np^k}.

In section 1.5, certain coefficient maps were labeled for use:

$$\beta\colon P^m(p^s) \to P^{m+1}(p^s)$$
$$\rho\colon P^m(p^s) \to P^m(p^{s+1})$$
$$\sigma\colon P^m(p^s) \to P^m(p^{s-1})$$

Analogs of these maps were implicitly defined and used in section 5.2:

$$\rho\colon T_{2n}(p^s) \to T_{2n}(p^{s+1})$$
$$\sigma\colon T_{2n}(p^s) \to T_{2n}(p^{s-1})$$

and one can easily define β as the compositions:

$$T_{2n}(p^r) \to S^{2n+1} \to T_{2n+1}(p^r)$$
$$T_{2n-1}(p^r) \to \Omega S^{2n+1} \to T_{2n}(p^r)$$

using 7.29, we can define

$$\sigma\colon T_{2n-1}(p^r) \to T_{2n-1}(p^{r-1})$$

We apply 7.27 to construct ρ, but it is not unique in general

$$\rho\colon T_{2n-1}(p^r) \to T_{2n-1}(p^{r+1})$$

PROPOSITION 7.31. *There is a split short exact sequence:*

$$0 \longrightarrow p_1(\Omega T_{2n-1}(p^{r+1})) \longrightarrow \left[T_{2n-1}(p^r), T_{2n-1}(p^{r+1})\right]_H \xrightarrow{r} Z/p^r \longrightarrow 0$$

and

$$p_1(\Omega T_{2n-1}(p^{r+1})) \cong p^r \left\{\left[P^{2np+1}(p^{r+1}), S^{2n+1}\right] \oplus \left[P^{2np+2}(p^{r+1}), S^{2n+1}\right]\right\}$$

NOTE. $p_1(\Omega T_{2n-1}(p^{r+1}))$ is known to be nonzero when p^r divides n ([**Gra69**]) and is known to be zero when $r \geqslant n$ ([**CMN79c**]).

PROOF. In order to establish this exact sequence we show that the map β in fact is zero in this case. Since β_1 factors through E_0, by 4.4, the composition:

$$P^{2np}(p^{r+1}) \xrightarrow{\beta_1} P^{2n+1}(p) \xrightarrow{\varphi_0} S^{2n+1}\{p^r\}$$

is null homotopic. Let j be composition

$$P^{2n+1}(p^r) \xrightarrow{\varphi_0} S^{2n+1}\{p^r\} \xrightarrow{\sigma} S^{2n+1}\{p^{r+1}\} = T_{2n}(p^{r+1});$$

then Ωj is homotopic to the composition:

$$\Omega P^{2n+1}(p^r) \xrightarrow{\Omega\rho} \Omega P^{2n+1}(p^{r+1}) \xrightarrow{h_0} T_{2n-1}(p^{r+1}) \xrightarrow{E} \Omega T_{2n}(p^{r+1})$$

since both compositions are H-maps which agree on $P^{2n}(p^r)$. In the diagram below, the upper composition is null homotopic and the lower sequence is a fibration sequence:

$$\begin{array}{ccccc}
P^{2np-1}(p^{r+1}) & \xrightarrow{\widetilde{\beta_1}} & \Omega P^{2n+1}(p^r) & \xrightarrow{\Omega j} & \Omega T_{2n}(p^r) \\
\Big\downarrow{\scriptstyle \xi} & & \Big\downarrow{\scriptstyle h_0\Omega\rho} & & \Big\| \\
W_n & \longrightarrow & T_{2n-1}(p^{r+1}) & \xrightarrow{E} & \Omega T_{2n}(p^{r+1}).
\end{array}$$

It follows that the map ξ exists forming a homotopy commutative square. But

$$\left[P^{2np-1}(p^{r+1}), W_n\right] = *,$$

so the composition:

$$P^{2np-1}(p^{r_*}) \xrightarrow{\widetilde{\beta_1}} \Omega P^{2n+1}(p^r) \xrightarrow{\Omega\rho} \Omega P^{2n+1}(p^{r+1}) \xrightarrow{h_0} T_{2n-1}(p^{r+1})$$

is null homotopic. However $h_0\Omega\rho$ generates

$$\left[\Omega P^{2n+1}(p^r), T_{2n-1}(p^{r+1})\right]_H \cong \left[P^{2n}(p^r), T_{2n-1}(p^{r+1})\right] \cong Z/p^r$$

and consequently $\beta = 0$. Finally

$$p_1\left(\Omega T_{2n-1}(p^{r+1})\right) = p^r \left[P^{2np}(p^{r+1}), T_{2n-1}P^{r+1})\right].$$

But since $[P^{2np}(p^{r+1}), W_n] = 0 = [P^{2np}(p^{r+1}), BW_n]$,

$$\begin{aligned}
\left[P^{2np}(p^{r+1}), T_{2n-1}(p^{r+1})\right] &= \left[P^{2np}(p^{r+1}), \Omega T_{2n}(p^{r+1})\right] \\
&= \left[P^{2np+1}(p^{r+1}), S^{2n+1}\{p^{r+1}\}\right] \\
&= \left[P^{2np+1}(p^{r+1}), S^{2n+1}\right] \oplus \left[P^{2np+2}(p^{r+1}), S^{2n+1}\right],
\end{aligned}$$

so

$$p_1\left(\Omega T_{2n-1}(p^{r+1})\right) = p^r\left\{\left[P^{2np+1}(p^{r+1}), S^{2n+1}\right] \oplus \left[P^{2np+2}(p^{r+1}), S^{2n+1}\right]\right\}.$$

These groups are stable and trivial if $r \geqslant n$. However, if p^r divides n, there is an element of $\pi_{2np}(S^{2n+1})$ of order p^{r+1} and consequently $p_1(\Omega T_{2n-1}(p^{r+1})) \neq 0$ in this case. The exact sequence is split since $T_{2n-1}(p^r)$ has exponent p^r. □

Finally we note that for every choice of ρ, $\rho\sigma = \sigma\rho = p$, $\beta = \sigma\beta\rho$ and $\beta\sigma^t = p^t\sigma^t\beta$, as in section 1.5.

Proposition 7.32. *There is a unique H-map*

$$T_{2n-1} \xrightarrow{f} \Omega^2 P^{2n+2}(p^r)$$

up to a unit whose double adjoint has a right homotopy inverse.

Proof. By [**CMN79b**], $p^{r+1}\pi_*\left(\Omega^2 P^{2n+2}(p^r)\right) = 0$, so we apply 7.27. We have

$$p^r\left[P^{2np}(p^{r+1}), \Omega^2 P^{2n+1}(p^r)\right] = p^r\left[P^{2np+2}(p^{r+1}), P^{2n+2}(p^r)\right]$$
$$p^r\left[P^{2np-1}(p^{r+1}), \Omega^2 P^{2n+1}(p^r)\right] = p^r\left[P^{2np+1}(p^{r+1}), P^{2n+2}(p^r)\right]$$

However $p^r\pi_i\left(P^{2n+2}(p^r)\right) = 0$ for $i < (4n+2)p - 1$. □

Since $P^{2n+2}(p^r)$ is a retract of $\Sigma^2 T_{2n-1}$, the double adjoint has a right homotopy inverse.

Corollary 7.33. *If Z is H-equivalent to the loop space on an H-space, every map $P^{2n}(p^r) \xrightarrow{\alpha} Z$ has an extension to an H-map $T_{2n-1} \xrightarrow{\widehat{\alpha}} Z$.*

Proof. If $Z = \Omega W$ the adjoint of α extends

$$\begin{array}{ccc} \Omega P^{2n+2}(p^r) & & \\ \uparrow & \searrow^{\alpha'} & \\ P^{2n+1}(p^r) & \xrightarrow{\widetilde{\alpha}} & W \end{array}$$

and we construct $\widetilde{\alpha}$ as the composition:

$$T_{2n-1}(p^r) \longrightarrow \Omega^2 P^{2n+2}(p^r) \xrightarrow{\Omega\alpha'} Z$$

using 7.32. □

APPENDIX A

The Case $n = 1$ and the Case $p = 3$

In section 4.2, we applied index p approximation to reduce the obstructions to a homotopy-Abelian H-space structure to a family of elements in the homotopy of E_k with mod p^s coefficients. This reduction only works when $n > 1$. We will use a different method in this case. However, the material in sections 5.1 and 6.2 on D_k, J_k and F_k does not depend on 4.2, and we can still construct $\gamma_k \colon J_k \longrightarrow BW_n$ (see for example [**Gra08**]).

THEOREM A.1. *For $p > 2$, $r \geqslant 1$ and $n = 1$, the Anick space is homotopy equivalent to a double loop space and hence has a homotopy-Abelian H-space structure.*

PROOF. Let $e \in H^4(BS^3; Z_{(p)})$ be a generator and $\kappa = p^r e$. Let X be the homotopy fiber of κ. Then we have a homotopy commutative diagram of fibration sequences

$$\begin{array}{ccccc} \Omega X & \longrightarrow & S^3 & \xrightarrow{\Omega\kappa} & K(Z;3) \\ \gamma \uparrow & & \| & & \uparrow \Omega e \\ S^3\{p^r\} & \longrightarrow & S^3 & \xrightarrow{p^r} & S^3 \end{array}$$

with γ uniquely determined. Using γ, we construct a diagram of vertical fibration sequences:

$$\begin{array}{ccccc} S^1 & \longrightarrow & \Omega^2 S^3 & \longrightarrow & S^1 \\ \downarrow & & \downarrow & & \downarrow \\ T_1 & \longrightarrow & \Omega S^3\{p^r\} & \xrightarrow{\Omega\gamma} & \Omega^2 X \\ \downarrow & & \downarrow & & \downarrow \\ \Omega S^3 & = & \Omega S^3 & = & \Omega S^3 \end{array}$$

Since the upper horizontal composition is a homotopy equivalence, $T_1 \simeq \Omega^2 X$. Note that the right hand fibration is an H-fibration and is an Anick fibration. $\square$

THEOREM A.2. *If $T_{2n-1}(3^r)$ is homotopy associative, $n = 3^k$ with $k \geqslant 0$. Furthermore if $n > 1$, then $r = 1$.*

PROOF. For any homotopy associative space T, there is a map:

$$T * T * T \longrightarrow ST \cup_{H(\mu)} C(T * T)$$

building the third stage of the classifying space construction ([**Sug57**], [**Sta63**]). The mapping cone X of this map has the cohomology of the bar construction on the homology of T through dimension $8n - 1$. In particular the mod p

cohomology of the $6n$ skeleton of X has as a basis, classes u, v, u^2, uv, u^3 where $|u| = 2n$ and $|v| = 2n + 1$. The $6n$ skeleton of the subspace $ST \cup_{H(\mu)} CT * T$ has cohomology generated by u, v, u^2, uv. Now since the map $R \to G$ is a retract of the map $H(\mu)\colon T * T \to \Sigma T$, the $4n + 1$ skeleton of $ST \cup_{H(\mu)} CT * T$ contains the $4n + 1$ skeleton of $G \cup CR$ as a retract (See the proof of 6.15). But $[G \cup CR]^{4n} = P^{2n+1} \cup_{x_2} CP^{4n}$; consequently

$$X^{6n} \simeq P^{2n+1} \cup_{x_2} CP^{4n} \cup e^{6n}.$$

Note that $\mathcal{P}^n u = u^3$ generates $H^6(X; Z/3)$. Since Σx_2 is inessential we can pinch the middle cells to a point after one suspension, and obtain a space with cell structure

$$S^{2n+1} \cup_{p^r} e^{2n+2} \cup e^{6n+1}$$

with $\mathcal{P}^n \neq 0$. However, $\mathcal{P}^n$ is decomposable unless $n = p^k = 3^k$. Furthermore, the decomposition of $\mathcal{P}^{p^k}$ by secondary operations ([**Liu62**]) implies that if $n > 1$, we must have $r = 1$. □

Note that such a space for $n > 1$ would imply that the "mod 3 Arf invariant class" survives the Adams spectral sequence. This does happen when $n = p$ with $T_5(3) = \Omega S^3\langle 3 \rangle$, but not when $n = p^2$.

Bibliography

[AG95] David Anick and Brayton Gray, *Small H spaces related to Moore spaces*, Topology **34** (1995), no. 4, 859–881, DOI 10.1016/0040-9383(95)00001-1. MR1362790

[Ani92] David J. Anick, *Single loop space decompositions*, Trans. Amer. Math. Soc. **334** (1992), no. 2, 929–940, DOI 10.2307/2154489. MR1145728

[Ani93] David Anick, *Differential algebras in topology*, Research Notes in Mathematics, vol. 3, A K Peters, Ltd., Wellesley, MA, 1993. MR1213682

[Bar61] M. G. Barratt, *Note on a formula due to Toda*, J. London Math. Soc. **36** (1961), 95–96. MR0125582

[CMN79a] F. R. Cohen, J. C. Moore, and J. A. Neisendorfer, *Decompositions of loop spaces and applications to exponents*, Algebraic topology, Aarhus 1978 (Proc. Sympos., Univ. Aarhus, Aarhus, 1978), Lecture Notes in Math., vol. 763, Springer, Berlin, 1979, pp. 1–12. MR561210

[CMN79b] F. R. Cohen, J. C. Moore, and J. A. Neisendorfer, *Torsion in homotopy groups*, Ann. of Math. (2) **109** (1979), no. 1, 121–168, DOI 10.2307/1971269. MR519355

[CMN79c] F. R. Cohen, J. C. Moore, and J. A. Neisendorfer, *The double suspension and exponents of the homotopy groups of spheres*, Ann. of Math. (2) **110** (1979), no. 3, 549–565, DOI 10.2307/1971238. MR554384

[Coh83] Frederick R. Cohen, *The unstable decomposition of* $\Omega^2\Sigma^2 X$ *and its applications*, Math. Z. **182** (1983), no. 4, 553–568, DOI 10.1007/BF01215483. MR701370

[DL59] Albrecht Dold and Richard Lashof, *Principal quasi-fibrations and fibre homotopy equivalence of bundles*, Illinois J. Math. **3** (1959), 285–305. MR0101521

[Gan70] Tudor Ganea, *Cogroups and suspensions*, Invent. Math. **9** (1969/1970), 185–197. MR0267582

[Gra66] Brayton I. Gray, *Spaces of the same n-type, for all n*, Topology **5** (1966), 241–243. MR0196743

[Gra69] Brayton Gray, *On the sphere of origin of infinite families in the homotopy groups of spheres*, Topology **8** (1969), 219–232. MR0245008

[Gra71] Brayton Gray, *A note on the Hilton-Milnor theorem*, Topology **10** (1971), 199–201. MR0281202

[Gra88] Brayton Gray, *On the iterated suspension*, Topology **27** (1988), no. 3, 301–310, DOI 10.1016/0040-9383(88)90011-0. MR963632

[Gra93a] Brayton Gray, *EHP spectra and periodicity. I. Geometric constructions*, Trans. Amer. Math. Soc. **340** (1993), no. 2, 595–616, DOI 10.2307/2154668. MR1152323

[Gra93b] Brayton Gray, *EHP spectra and periodicity. II.* Λ*-algebra models*, Trans. Amer. Math. Soc. **340** (1993), no. 2, 617–640, DOI 10.2307/2154669. MR1152324

[Gra98] Brayton Gray, *The periodic lambda algebra*, Stable and unstable homotopy (Toronto, ON, 1996), Fields Inst. Commun., vol. 19, Amer. Math. Soc., Providence, RI, 1998, pp. 93–101. MR1622340

[Gra99] Brayton Gray, *On the homotopy type of the loops on a 2-cell complex*, Homotopy methods in algebraic topology (Boulder, CO, 1999), Contemp. Math., vol. 271, Amer. Math. Soc., Providence, RI, 2001, pp. 77–98, DOI 10.1090/conm/271/04351. MR1831348

[Gra06] Brayton Gray, *On decompositions in homotopy theory*, Trans. Amer. Math. Soc. **358** (2006), no. 8, 3305–3328 (electronic), DOI 10.1090/S0002-9947-05-03964-4. MR2218977

[Gra08] Brayton Gray, *Decompositions involving Anick's spaces*, 2008, `arXiv 0804.0777v1`.

[Gra11] Brayton Gray, *On generalized Whitehead products*, Trans. Amer. Math. Soc. **363** (2011), no. 11, 6143–6158, DOI 10.1090/S0002-9947-2011-05392-4. MR2817422

[Gra12] Brayton Gray, *Universal abelian H-spaces*, Topology Appl. **159** (2012), no. 1, 209–224, DOI 10.1016/j.topol.2011.09.002. MR2852964

[GT10] Brayton Gray and Stephen Theriault, *An elementary construction of Anick's fibration*, Geom. Topol. **14** (2010), no. 1, 243–275, DOI 10.2140/gt.2010.14.243. MR2578305

[Liu62] Arunas Liulevicius, *The factorization of cyclic reduced powers by secondary cohomology operations*, Mem. Amer. Math. Soc. No. **42** (1962), 112. MR0182001

[Mah75] Mark Mahowald, *On the double suspension homomorphism*, Trans. Amer. Math. Soc. **214** (1975), 169–178. MR0438333

[Nei80] Joseph Neisendorfer, *Primary homotopy theory*, Mem. Amer. Math. Soc. **25** (1980), no. 232, iv+67, DOI 10.1090/memo/0232. MR567801

[Nei81] Joseph A. Neisendorfer, *3-primary exponents*, Math. Proc. Cambridge Philos. Soc. **90** (1981), no. 1, 63–83, DOI 10.1017/S0305004100058539. MR611286

[Nei83] Joseph Neisendorfer, *Properties of certain H-spaces*, Quart. J. Math. Oxford Ser. (2) **34** (1983), no. 134, 201–209, DOI 10.1093/qmath/34.2.201. MR698206

[Nei10a] Joseph Neisendorfer, *Algebraic methods in unstable homotopy theory*, New Mathematical Monographs, vol. 12, Cambridge University Press, Cambridge, 2010. MR2604913

[Nei10b] Joseph A. Neisendorfer, *Homotopy groups with coefficients*, J. Fixed Point Theory Appl. **8** (2010), no. 2, 247–338, DOI 10.1007/s11784-010-0020-1. MR2739026

[Sta63] James Dillon Stasheff, *Homotopy associativity of H-spaces. I, II*, Trans. Amer. Math. Soc. 108 (1963), 275-292; ibid. **108** (1963), 293–312. MR0158400

[Str72] Arne Strøm, *The homotopy category is a homotopy category*, Arch. Math. (Basel) **23** (1972), 435–441. MR0321082

[Sug57] Masahiro Sugawara, *A condition that a space is group-like*, Math. J. Okayama Univ. **7** (1957), 123–149. MR0097066

[The01] Stephen D. Theriault, *Properties of Anick's spaces*, Trans. Amer. Math. Soc. **353** (2001), no. 3, 1009–1037, DOI 10.1090/S0002-9947-00-02623-4. MR1709780

[The03] Stephen D. Theriault, *Homotopy decompositions involving the loops of coassociative co-H spaces*, Canad. J. Math. **55** (2003), no. 1, 181–203, DOI 10.4153/CJM-2003-008-5. MR1952331

[The08] Stephen D. Theriault, *The 3-primary classifying space of the fiber of the double suspension*, Proc. Amer. Math. Soc. **136** (2008), no. 4, 1489–1499, DOI 10.1090/S0002-9939-07-09249-0. MR2367123

[The11] Stephen D. Theriault, *2-primary Anick fibrations*, J. Topol. **4** (2011), no. 2, 479–503, DOI 10.1112/jtopol/jtr008. MR2805999

[Tod56] Hirosi Toda, *On the double suspension E^2*, J. Inst. Polytech. Osaka City Univ. Ser. A. **7** (1956), 103–145. MR0092968

[Tod62] Hirosi Toda, *Composition methods in homotopy groups of spheres*, Annals of Mathematics Studies, No. 49, Princeton University Press, Princeton, N.J., 1962. MR0143217

List of Symbols

Editorial Information

To be published in the *Memoirs*, a paper must be correct, new, nontrivial, and significant. Further, it must be well written and of interest to a substantial number of mathematicians. Piecemeal results, such as an inconclusive step toward an unproved major theorem or a minor variation on a known result, are in general not acceptable for publication.

Papers appearing in *Memoirs* are generally at least 80 and not more than 200 published pages in length. Papers less than 80 or more than 200 published pages require the approval of the Managing Editor of the Transactions/Memoirs Editorial Board. Published pages are the same size as those generated in the style files provided for AMS-LaTeX or AMS-TeX.

Information on the backlog for this journal can be found on the AMS website starting from `http://www.ams.org/memo`.

A Consent to Publish is required before we can begin processing your paper. After a paper is accepted for publication, the Providence office will send a Consent to Publish and Copyright Agreement to all authors of the paper. By submitting a paper to the *Memoirs*, authors certify that the results have not been submitted to nor are they under consideration for publication by another journal, conference proceedings, or similar publication.

Information for Authors

Memoirs is an author-prepared publication. Once formatted for print and on-line publication, articles will be published as is with the addition of AMS-prepared frontmatter and backmatter. Articles are not copyedited; however, confirmation copy will be sent to the authors.

Initial submission. The AMS uses Centralized Manuscript Processing for initial submissions. Authors should submit a PDF file using the Initial Manuscript Submission form found at `www.ams.org/submission/memo`, or send one copy of the manuscript to the following address: Centralized Manuscript Processing, MEMOIRS OF THE AMS, 201 Charles Street, Providence, RI 02904-2294 USA. If a paper copy is being forwarded to the AMS, indicate that it is for *Memoirs* and include the name of the corresponding author, contact information such as email address or mailing address, and the name of an appropriate Editor to review the paper (see the list of Editors below).

The paper must contain a *descriptive title* and an *abstract* that summarizes the article in language suitable for workers in the general field (algebra, analysis, etc.). The *descriptive title* should be short, but informative; useless or vague phrases such as "some remarks about" or "concerning" should be avoided. The *abstract* should be at least one complete sentence, and at most 300 words. Included with the footnotes to the paper should be the 2010 *Mathematics Subject Classification* representing the primary and secondary subjects of the article. The classifications are accessible from `www.ams.org/msc/`. The Mathematics Subject Classification footnote may be followed by a list of *key words and phrases* describing the subject matter of the article and taken from it. Journal abbreviations used in bibliographies are listed in the latest *Mathematical Reviews* annual index. The series abbreviations are also accessible from `www.ams.org/msnhtml/serials.pdf`. To help in preparing and verifying references, the AMS offers MR Lookup, a Reference Tool for Linking, at `www.ams.org/mrlookup/`.

Electronically prepared manuscripts. The AMS encourages electronically prepared manuscripts, with a strong preference for AMS-LaTeX. To this end, the Society has prepared AMS-LaTeX author packages for each AMS publication. Author packages include instructions for preparing electronic manuscripts, samples, and a style file that generates the particular design specifications of that publication series. Though AMS-LaTeX is the highly preferred format of TeX, author packages are also available in AMS-TeX.

Authors may retrieve an author package for *Memoirs of the AMS* from `www.ams.org/journals/memo/memoauthorpac.html` or via FTP to `ftp.ams.org` (login as `anonymous`, enter your complete email address as password, and type `cd pub/author-info`). The

AMS Author Handbook and the *Instruction Manual* are available in PDF format from the author package link. The author package can also be obtained free of charge by sending email to tech-support@ams.org or from the Publication Division, American Mathematical Society, 201 Charles St., Providence, RI 02904-2294, USA. When requesting an author package, please specify $\mathcal{AMS}$-LaTeX or $\mathcal{AMS}$-TeX and the publication in which your paper will appear. Please be sure to include your complete mailing address.

After acceptance. The source files for the final version of the electronic manuscript should be sent to the Providence office immediately after the paper has been accepted for publication. The author should also submit a PDF of the final version of the paper to the editor, who will forward a copy to the Providence office.

Accepted electronically prepared files can be submitted via the web at www.ams.org/submit-book-journal/, sent via FTP, or sent on CD to the Electronic Prepress Department, American Mathematical Society, 201 Charles Street, Providence, RI 02904-2294 USA. TeX source files and graphic files can be transferred over the Internet by FTP to the Internet node ftp.ams.org (130.44.1.100). When sending a manuscript electronically via CD, please be sure to include a message indicating that the paper is for the *Memoirs*.

Electronic graphics. Comprehensive instructions on preparing graphics are available at www.ams.org/authors/journals.html. A few of the major requirements are given here.

Submit files for graphics as EPS (Encapsulated PostScript) files. This includes graphics originated via a graphics application as well as scanned photographs or other computer-generated images. If this is not possible, TIFF files are acceptable as long as they can be opened in Adobe Photoshop or Illustrator.

Authors using graphics packages for the creation of electronic art should also avoid the use of any lines thinner than 0.5 points in width. Many graphics packages allow the user to specify a "hairline" for a very thin line. Hairlines often look acceptable when proofed on a typical laser printer. However, when produced on a high-resolution laser imagesetter, hairlines become nearly invisible and will be lost entirely in the final printing process.

Screens should be set to values between 15% and 85%. Screens which fall outside of this range are too light or too dark to print correctly. Variations of screens within a graphic should be no less than 10%.

Any graphics created in color will be rendered in grayscale for the printed version unless color printing is authorized by the Managing Editor and the Publisher. In general, color graphics will appear in color in the online version.

Inquiries. Any inquiries concerning a paper that has been accepted for publication should be sent to memo-query@ams.org or directly to the Electronic Prepress Department, American Mathematical Society, 201 Charles St., Providence, RI 02904-2294 USA.

Editors

This journal is designed particularly for long research papers, normally at least 80 pages in length, and groups of cognate papers in pure and applied mathematics. Papers intended for publication in the *Memoirs* should be addressed to one of the following editors. The AMS uses Centralized Manuscript Processing for initial submissions to AMS journals. Authors should follow instructions listed on the Initial Submission page found at `www.ams.org/memo/memosubmit.html`.

1155 **Hans Lundmark and Jacek Szmigielski,** An Inverse Spectral Problem Related to the Geng–Xue Two-Component Peakon Equation, 2016

1154 **Béla Csaba, Daniela Kühn, Allan Lo, Deryk Osthus, and Andrew Treglown,** Proof of the 1-Factorization and Hamilton Decomposition Conjectures, 2016

1153 **Toshihiko Masuda and Reiji Tomatsu,** Rohlin Flows on von Neumann Algebras, 2016

1152 **Xin-Rong Dai and Qiyu Sun,** The *abc*-Problem for Gabor Systems, 2016

1151 **Reiner Hermann,** Monoidal Categories and the Gerstenhaber Bracket in Hochschild Cohomology, 2016

1150 **J. P. Pridham,** Real Non-Abelian Mixed Hodge Structures for Quasi-Projective Varieties: Formality and Splitting, 2016

1149 **Ariel Barton and Svitlana Mayboroda,** Layer Potentials and Boundary-Value Problems for Second Order Elliptic Operators with Data in Besov Spaces, 2016

1148 **Joseph Hundley and Eitan Sayag,** Descent Construction for GSpin Groups, 2016

1147 **U. Meierfrankenfeld, B. Stellmacher, and G. Stroth,** The Local Structure Theorem for Finite Groups With a Large p-Subgroup, 2016

1146 **Genni Fragnelli and Dimitri Mugnai,** Carleman Estimates, Observability Inequalities and Null Controllability for Interior Degenerate Nonsmooth Parabolic Equations, 2016

1145 **Bart Bories and Willem Veys,** Igusa's p-Adic Local Zeta Function and the Monodromy Conjecture for Non-Degenerate Surface Singularities, 2016

1144 **Atsushi Moriwaki,** Adelic Divisors on Arithmetic Varieties, 2016

1143 **Wen Huang, Song Shao, and Xiangdong Ye,** Nil Bohr-Sets and Almost Automorphy of Higher Order, 2015

1142 **Roland Donninger and Joachim Krieger,** A Vector Field Method on the Distorted Fourier Side and Decay for Wave Equations with Potentials, 2015

1141 **Su Gao, Steve Jackson, and Brandon Seward,** Group Colorings and Bernoulli Subflows, 2015

1140 **Michael Aschbacher,** Overgroups of Root Groups in Classical Groups, 2015

1139 **Mingmin Shen and Charles Vial,** The Fourier Transform for Certain HyperKähler Fourfolds, 2015

1138 **Volker Bach and Jean-Bernard Bru,** Diagonalizing Quadratic Bosonic Operators by Non-Autonomous Flow Equations, 2015

1137 **Michael Skeide,** Classification of E_0-Semigroups by Product Systems, 2015

1136 **Kieran G. O'Grady,** Moduli of Double EPW-Sextics, 2015

1135 **Vassilios Gregoriades,** Classes of Polish Spaces Under Effective Borel Isomorphism, 2015

1134 **Hongzi Cong, Jianjun Liu, and Xiaoping Yuan,** Stability of KAM Tori for Nonlinear Schrödinger Equation, 2015

1133 **P. Cannarsa, P. Martinez, and J. Vancostenoble,** Global Carleman Estimates for Degenerate Parabolic Operators with Applications, 2015

1132 **Weiwei Ao, Chang-Shou Lin, and Juncheng Wei,** On Non-Topological Solutions of the A_2 and B_2 Chern-Simons System, 2015

1131 **Bob Oliver,** Reduced Fusion Systems over 2-Groups of Sectional Rank at Most 4, 2015

1130 **Timothy C. Burness, Soumaïa Ghandour, and Donna M. Testerman,** Irreducible Geometric Subgroups of Classical Algebraic Groups, 2015

1129 **Georgios Daskalopoulos and Chikako Mese,** On the Singular Set of Harmonic Maps into DM-Complexes, 2015

For a complete list of titles in this series, visit the
AMS Bookstore at **www.ams.org/bookstore/memoseries/**.

Memoirs of the American Mathematical Society Number 1162 • March 2017

ISBN 978-1-4704-2308-7

9 781470 423087

MEMO/246/1162

AMS *on the* Web
www.ams.org