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Periodicity, compositions and EHP sequences

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Abstract. In this work we describe the techniques used in the EHP method for calculation of the homotopy groups of spheres pioneered by Toda. We then seek to find other contexts where this method can be applied. We show that the Anick spaces form a refinement of the secondary suspension and describe EHP sequences and compositions converging to the homotopy of Moore space spectra at odd primes. Lastly we give a framework of how this may be generalized to Smith–Toda spectra $V(m)$ and related spectra when they exist.

0. Introduction

A central question in the chromatic approach to stable homotopy has been the existence and properties of the Smith–Toda complexes $V(m)$, whose mod p homology is isomorphic to the subalgebra $\Lambda(\tau_0, \ldots, \tau_m)$ of the dual of the Steenrod algebra. These spaces do not exist for all m at any prime ([**Nav10**]), but when $V(m-1)$ does exist, a modification of $V(m)$ can be constructed from a non nilpotent self map of $V(m-1)$ ([**DHS88**]).

We will approach spectra of this type via an unstable development. In [**Gra93a**], the question was raised of whether spectra other than the sphere spectrum could have an unstable approximation through EHP sequences, together with all the features in the classical case. It appeared that these spectra were suitable candidates for such a treatment.

We report on some recent work establishing these features for the spectrum $S^0 \cup_{p^r} e^1$, and indicate some initial steps for $V(1)$.

This work is divided into three parts. In sections 1–6 we recall the methods and tools for self-referential calculation with the EHP sequences for the homotopy groups of spheres. In particular, we explain the role of compositions pioneered by Toda [**Tod62**]. Sections 7 and 8 reviews the work in [**Gra93a**], leading to the conjectures for $V(0)$. In sections 9 and 10 we report on recent results ([**Ani93**], [**AG95**], [**Gra**], [**Gra12**], and [**GT10**]) resolving these conjectures for $S^0 \cup_{p^r} e^1$ when $p > 3$. In section 11, we take the early steps in studying $V(1)$.

1. EHP Sequences

The EHP sequences are both a historically important calculation tool and an organizing scheme for understanding the homotopy groups of spheres. Localized

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at 2, these are exact sequences

$$\text{(1.1)}\qquad \pi_{k+n+2}(S^{2n+1}) \xrightarrow{P} \pi_{k+n}(S^n) \xrightarrow{E} \pi_{k+n+1}(S^{n+1}) \xrightarrow{H} \pi_{k+n+1}(S^{2n+1}) \xrightarrow{P} \pi_{k+n-1}(S^n)$$

and are defined by a 2-local fibration sequence:

$$\Omega^2 S^{2n+1} \xrightarrow{P} S^n \xrightarrow{E} \Omega S^{n+1} \xrightarrow{H} \Omega S^{2n+1}.$$

Here E is the suspension and H is the second James–Hopf invariant. The map H is an important organizing tool since each element in the stable homotopy groups of spheres is "born" on a given sphere with non trivial Hopf invariant; a sequence of alternately desuspensions and Hopf invariants provides a complete genealogical record for each homotopy class, and siblings often have similar features.

If $\alpha \in \pi_{k+n}(S^n)$ we will say that α is in the k stem on S^n. The bootstrapping method is to first do induction on the stem, and within that do induction on the dimension of the target sphere. To calculate $\pi_{k+n+1}(S^{n+1})$ in (1.1), note that the first, fourth and fifth terms are in stems less than k when $n \geqslant 2$, and the second term is in the k stem on a lower sphere. So we can assume by induction that all terms except the middle term are known. We are left with two problems:

(1) Calculate P between known groups;
(2) Solve the extension problem

$$0 \longrightarrow \operatorname{coker} P \longrightarrow \pi_{k+n+1}(S^{n+1}) \longrightarrow \ker P \longrightarrow 0.$$

In addressing these problems, it is helpful to keep a list of all possible compositions in the k stem. This is useful because of:

Proposition 1.2. (a) $P(\alpha \circ E^2\beta) = P(\alpha) \circ \beta$
(b) $H(\alpha \circ E\beta) = H(\alpha) \circ E\beta$
(c) $H(E\alpha \circ \beta) = (E\alpha \wedge \alpha) \circ H(\beta)$

These formulas work even when the middle space is not a sphere. For example, if $\alpha \colon \Sigma X \to S^{n+1}$ and $\beta \colon S^{n+k} \to X$, then $H(\alpha \circ E\beta) = H(\alpha) \circ E\beta$.

The induction begins with $\pi_{k+1}(S^1) = 0$ for $k \geqslant 1$, and it is well known that $\pi_n(S^n) \cong \mathbb{Z}$ for all $n \geqslant 1$. We write $\iota_n \colon S^n \to S^n$ for the identity map which is a generator. Let $w_n = P(\iota_{2n+1}) \in \pi_{2n-1}(S^n)$.

Proposition 1.3. $H(\omega_n) = (1 + (-1)^n)\iota_{2n-1}$

These propositions are prehistoric and easy to prove (see for example [**Tod62**]).

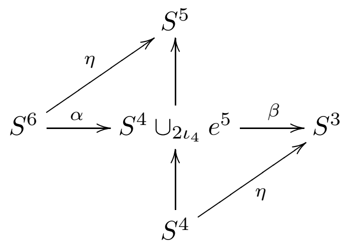
2. EHP Magic

In this section we will describe some sample calculations. If $\alpha \in \pi_k(S^n)$ we will say that α is "on S^n". We sill also use α to denote $E\alpha \in \pi_{k+1}(S^{n+1})$ and call this element "α on S^{n+1}". This should not lead to confusion. However α can have a different order "on S" than "on S^{n+1}".

It is easy to check from (1.1) that $\pi_3(S^2) \cong \mathbb{Z}$, and we will label a generator η. Because of 1.3, η has order 2 on S^n for $n \geqslant 3$. It is also easy to check that η^2 generates $\pi_{n+2}(S^n)$ for each $n \geqslant 2$ and has order 2 on each S^n using 1.2(b). The first interesting problem occurs in the 3-stem. The group $\pi_5(S^2) \cong \mathbb{Z}/2$ generated by η^3, but for $\pi_6(S^3)$ we have a short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_5(S^2) & \xrightarrow{E} & \pi_6(S^3) & \xrightarrow{H} & \pi_6(S^5) & \longrightarrow & 0 \\ & & \wr\| & & & & \wr\| & & \\ & & \mathbb{Z}/2 & & & & \mathbb{Z}/2 & & \end{array}$$

Consequently $\pi_6(S^3)$ is either $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ or $\mathbb{Z}/4$. To solve this we need to construct an element $\nu' \in \pi_6(S^3)$ with $H(\nu') = \eta$. We define $\nu' = \beta\alpha$, the horizontal composition in the homotopy commutative diagram:



The maps α and β exist because η has order 2 on S^3. The map α is in fact a suspension, so we can use 1.2(b) to calculate $H(\beta\alpha) = H(\beta)\alpha$. $H(\beta) \neq 0$, since otherwise η would have order 2 on S^2. Thus $H(\beta)$ is the projection $S^4 \cup_{2\iota_4} e^5 \longrightarrow S^5$ and $H(\nu') = \eta$. Since α is a suspension, 2α is homotopic to the composition:

$$S^6 \xrightarrow{\alpha} S^4 \cup_{2\iota_4} e^5 \simeq S^2 \wedge (S^2 \cup_{2\iota_2} e^3) \xrightarrow{2\wedge\iota} S^2 \wedge (S^2 \cup_{2\iota_2} e^3).$$

However the map $2 \wedge \iota$ has a mapping cone with $\mathrm{Sq}^2 \neq 0$ by the Cartan formula. Consequently $2 \wedge \iota$ is essential and is homotopic to the composition:

$$S^4 \cup_{2\iota_4} e^5 \longrightarrow S^5 \xrightarrow{\eta} S^4 \longrightarrow S^4 \cup_{2\iota_4} e^5.$$

It follows that $2\nu' = \eta^3$. Since η^3 generates $\pi_5(S^3) \cong \mathbb{Z}/2$, ν' has order 4. In the notation of Toda brackets, we have

$$\nu' = \{\eta, 2\iota, \eta\}.$$

This construction is what Toda calls a secondary composition and everyone else calls a Toda bracket.

Since the Hopf map $\nu\colon S^7 \to S^4$ has Hopf invariant one, $\pi_7(S^4) \cong \mathbb{Z} \oplus \mathbb{Z}/4$ generated by ν and ν'. Finally $\pi_{n+3}(S^n) \cong \mathbb{Z}/8$ for $n \geqslant 5$ generated by ν with $2\nu = \nu'$. This follows from 1.3 since 2ν is a multiple of ν', and if it were not ν' or $3\nu'$, $\nu - \nu'$ would have order 2 and have Hopf invariant one. Applying the Adem relation $\mathrm{Sq}^2\,\mathrm{Sq}^3 = \mathrm{Sq}^1\,\mathrm{Sq}^4 + \mathrm{Sq}^4\,\mathrm{Sq}^1$ to the space

$$S^n \cup_{\nu-\nu'} e^{n+1} \cup_{2\iota} e^{n+2}$$

proves that this is impossible.

The 4 and 5 stems are easier as there are no extensions, and each group is generated by compositions except for $\pi_{11}(S^6) \cong \mathbb{Z}$ which is generated by ω_6. These calculations all follow directly from 1.1, 1.2 and 1.3. For example, in the 4 stem there are elements $\eta\nu'$, $\nu'\eta$, $\eta\nu$ and $\nu\eta$, which generate the entire 4 stem by 1.2. It turns out that $\nu'\eta = \eta\nu$, and this element of order 2 generates $\pi_7(S^3)$. However

$\eta\nu$ is null homotopic on S^5 while $\nu\eta$ is non zero of order 2 on S^5. Since the stable homotopy groups are graded commutative, we must have $\omega_5 = \nu\eta$ and the stable 4 stem is trivial. We see that commutators are unstably non trivial. In the next section we will see how commutators work with the EHP sequence.

3. Commutators in the EHP sequence

The following useful results can be found in [**BH53**], [**Bar61**] and [**Tod57**]. We will use the notation $X * Y$ for the join, so $S^m * S^n \simeq S^{m+n+1}$.

THEOREM 3.1. *Suppose $\alpha \in \pi_{k+n}(S^n)$ and $\beta \in \pi_{\ell+m}(S^m)$. Then the following formulas hold:*

(a) (Barratt–Hilton) $\alpha\beta = (-1)^{k\ell}\beta\alpha$ *on* S^{m+n}*;*
(b) (Barratt–Toda) *Let* $\Delta = \alpha\beta = \alpha\beta - (-1)^{k\ell}\beta\alpha$ *on* S^{m+n-1}*.*
Then $\Delta = \pm P(H(\alpha) * H(\beta))$ *where*

$$S^{m+n+k+\ell+1} \simeq S^{m+k} * S^{n+\ell} \xrightarrow{H(\alpha)*H(\beta)} S^{2m-1} * S^{2n-1} \simeq S^{2m+2n-1}$$

Note: By $\alpha\beta$ on S^{m+n}, we mean the composition

$$S^{k+\ell+m+n} \xrightarrow{\Sigma^{k+n}\beta} S^{k+m+n} \xrightarrow{\Sigma^m\alpha} S^{m+n}$$

and similarly for the other terms. Thus, for example, since $\eta \in \pi_3(S^2)$ has Hopf invariant ι_3 and $\nu \in \pi_7(S^4)$ has Hopf invariant ι_7, we have

$$\omega_5 = \eta\nu + \nu\eta$$

on S^5. Since $\eta\nu = 0$ on S^5, this agrees with our previous calculation.

4. Odd primary EHP Sequences

There are also EHP sequences for the p-local sphere when $p > 2$ [**Tod56**]. The even spheres play a different role because of the Serre splitting

$$\Omega S^{2n} \cong S^{2n-1} \times \Omega S^{4n-1}.$$

The factor ΩS^{4n-1} is included in ΩS^{2n} by a map $S^{4n-1} \to S^{2n}$ whose suspension is null homotopic. Consequently the sphere S^{2n} does not contribute to the development of the stable homotopy. There is a substitute however. We define $\widehat{S}^{2n}$

$$\widehat{S}^{2n} = J_{p-1}(S^{2n}) = S^{2n} \cup e^{4n} \cup \cdots \cup e^{2n(p-1)} \subset J(S^{2n})$$

where $J(S^{2n})$ is the James construction. $\widehat{S}^{2n}$ replaces S^{2n} and there is a bifurcation of the EHP sequences which collapses when $p = 2$. The relevant fibrations are:

$$\Omega^2 S^{2np-1} \xrightarrow{P} S^{2n-1} \xrightarrow{E} \Omega\widehat{S}^{2n} \xrightarrow{H} \Omega S^{2np-1}$$

(4.1)

$$\Omega^2 S^{2np+1} \xrightarrow{P} \widehat{S}^{2n} \xrightarrow{E} \Omega S^{2n+1} \xrightarrow{H} \Omega S^{2np+1}$$

The analog of 1.2 holds. For 1.3 we have 2 cases:

$$\omega_n = P(\iota_{2np-1}) \in \pi_{2np-3}(S^{2n-1}) \text{ and } H(\omega_n) = 0$$
$$\omega'_n = P(\iota_{2np+1}) \in \pi_{2np-1}(\widehat{S}^{2n}) \text{ and } H(\omega'_n) = p\iota_{2np-1}.$$

Theorem 3.1(a) holds ([**Gra01**]) while 3.1(b) only makes sense if one or both of m and n is even.

5. Λ-algebra Approximations

In [**BK72**], the authors constructed a dga called the lambda algebra, which is an E^1 term for the Adams spectral sequence for S^0 localized at 2. They also constructed subalgebras $\Lambda(n)$ for each n with the property that $\Lambda(n)$ is an E^1 term for the unstable Adams spectral sequence for S^n. In particular, there are short exact sequences:

$$0 \longrightarrow \Lambda(n) \xrightarrow{E} \Lambda(n+1) \xrightarrow{H} \Lambda(2n+1) \longrightarrow 0$$

where H has degree $-n$. The resulting long exact sequence in homology gives an EHP sequence for the E^2 terms of the unstable Adams spectral sequences. Likewise there are Λ-algebra models for the EHP sequences at odd primes:

$$0 \longrightarrow \Lambda(2n-1) \xrightarrow{E} \Lambda(2n) \xrightarrow{H} \Lambda(2np-1) \longrightarrow 0$$

$$0 \longrightarrow \Lambda(2n) \xrightarrow{E} \Lambda(2n+1) \xrightarrow{H} \Lambda(2np+1) \longrightarrow 0$$

where the chain maps labeled H have degrees $1-2n(p-1)$ and $-2n(p-1)$ respectively.

When $p > 2$, the Λ-algebra is complex and can be replaced by another dga called the periodic lambda algebra $\overline{\Lambda}$ ([**Gra98**]). It has the feature that the elements v_n which figure in the periodic development are apparent and replace the μ's, and the relations are simpler. Furthermore there are EHP sequences in $\overline{\Lambda}$ as well.

6. The Kahn–Priddy Theorem

In [**KP78**] the authors described a transfer map in the stable category

$$B\Sigma_p \xrightarrow{\lambda} S^0$$

and proved that it is onto in stable homotopy localized at p. The p-localization of $B\Sigma_p$ has one cell in each dimension of the form nq and $nq-1$ and no others, and we define spaces B^{nq} and B^{nq-1} to be the appropriate skeleta of the localization. In case $p = 2$, $B^{nq} = \mathbb{R}P^{2n}$ and $B^{nq-1} = \mathbb{R}P^{2n-1}$. Unstably, the map λ has approximations

$$\Sigma^{2n+1} B^{nq} \xrightarrow{\lambda_{2n+1}} S^{2n+1}$$

$$\Sigma^{2n} B^{nq-1} \xrightarrow{\lambda_{2n}} S^{2n}.$$

It is an open question as to whether these maps are onto in unstable homotopy. Nevertheless, they have a rich image. In [**Gra84**] the stable and unstable v_1 periodic homotopy of spheres was constructed by factoring through these maps.

The maps λ_n are compatible with the EHP fibrations in the sense that there are homotopy commutative diagrams where the upper horizontal sequence is a cofibration sequence and the lower sequence is a fibration sequence

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & S^{2np-1} & \longrightarrow & \Sigma^{2n}B^{nq-1} & \longrightarrow & \Sigma^{2n}B^{nq} & \longrightarrow & S^{2np} \\
& \downarrow{\scriptstyle E^2} & & \downarrow{\scriptstyle \lambda_{2n}} & & \downarrow{\scriptstyle \widetilde{\lambda}_{2n+1}} & & \downarrow{\scriptstyle E} \\
\cdots \longrightarrow & \Omega^2 S^{2np+1} & \xrightarrow{P} & \widehat{S}^{2n} & \xrightarrow{E} & \Omega S^{2n+1} & \xrightarrow{H} & \Omega S^{2np+1}
\end{array}$$

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & S^{2np-3} & \longrightarrow & \Sigma^{2n-1}B^{(n-1)q} & \longrightarrow & \Sigma^{2n-1}B^{nq-1} & \longrightarrow & S^{2np-2} \\
& \downarrow{\scriptstyle E^2} & & \downarrow{\scriptstyle \lambda_{2n-1}} & & \downarrow{\scriptstyle \widetilde{\lambda}_{2n}} & & \downarrow{\scriptstyle E} \\
\cdots \longrightarrow & \Omega^2 S^{2np-1} & \xrightarrow{P} & S^{2n-1} & \xrightarrow{E} & \Omega\widehat{S}^{2n} & \xrightarrow{H} & \Omega S^{2np-1}
\end{array}$$

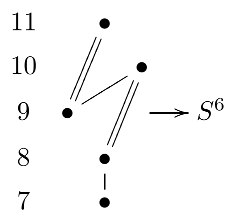
which collapse when $p = 2$ to a single diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & S^{2n-1} & \longrightarrow & \Sigma^{n}\mathbb{R}P^{n-1} & \longrightarrow & \Sigma^{n}\mathbb{R}P^{n} & \longrightarrow & S^{2n} \\
& \downarrow{\scriptstyle E^2} & & \downarrow{\scriptstyle \lambda_{n}} & & \downarrow{\scriptstyle \widetilde{\lambda}_{n+1}} & & \downarrow{\scriptstyle E} \\
\cdots \longrightarrow & \Omega^2 S^{2n+1} & \xrightarrow{P} & S^{n} & \xrightarrow{E} & \Omega S^{n+1} & \xrightarrow{H} & \Omega S^{2n+1}
\end{array}$$

As an example of how these maps behave unstably, we can consider the 3-stem on S^6. We have a map

$$\Sigma^6\mathbb{R}P^5 \xrightarrow{\lambda_6} S^6$$

which is an extension of $\eta\colon S^7 \to S^6$. In the chart below, the dots represent cells of $\Sigma\mathbb{R}P^5$. The single lines represent Sq^1 and the double lines represent Sq^2. This corresponds to relative attaching maps of 2ι and η.



The 7 cell represents η and the 8 cell the relation $2\eta = 0$ on S^3. The map η on[1] the 8 cell represents ν' and the 9 cell represents ν. The 10 cell gives the relation $2\nu = \nu'$ on S^5 and the 11 cell gives the relation $\nu\eta = 0$ on S^6.

7. EHP Spectra

It is natural to ask whether there are EHP sequences for the unstable homotopy groups of other spaces. For each space X there is a James–Hopf invariant

$$\Omega\Sigma X \xrightarrow{H} \Omega\Sigma X \wedge X;$$

however H is not a good invariant for detecting whether a map desuspends. In fact, the homotopy groups of the spaces $\Sigma^n X$ are often not a good approximation to the stable homotopy outside the stable range. The best example to see this phenomenon is in approximating the Moore space spectrum $S^0 \cup_{p^r} e^1$ by the Moore

[1]If $\alpha\colon S^m \to S^n$, we say that $\overline{\alpha}\colon S^m \to X \cup e^n$ is "α on e" if the composition $S^m \xrightarrow{\overline{\alpha}} X \cup e^n \longrightarrow S^n$ is homotopic to α.

spaces $P^n = S^{n-1} \cup_{p^r} e^n$. According to the seminal work of Cohen, Moore and Neisendorfer ([**CMN79b**])

$$\Omega P^n \simeq \Omega\Big(\bigvee_\alpha P^{n_\alpha}\Big) \times X$$

where $\bigvee_\alpha P^{n_\alpha}$ is an infinite wedge of Moore spaces, and the suspension of the map $\bigvee_\alpha P^{n_\alpha} \to P^n$ is null homotopic. The Euler Poincaré polynomial of $\Omega(\vee P^{n_\alpha})$ is considerably larger than that of X, so much of the homotopy of P^n has nothing to do with the stable homotopy. We seek spaces T_n and maps $E\colon \Sigma T_{n-1} \to T_n$ representing the Moore space spectrum where the spaces T_n are better approximations.

In general, suppose we are given a connective spectrum X and we wish to approximate it with $(n-1)$-connected spaces X_n. One way to control the situation is to suppose that there are some sort of Hopf invariants that lie in the homotopy of something closely related to X. A simple approach to this was investigated in ([**Gra93a**]): We suppose that in favorable cases there are spaces X_n, a function $f(n)$ and maps H such that there are EHP fibrations

$$X_n \xrightarrow{E} \Omega X_{n+1} \xrightarrow{H} \Omega X_{f(n)} \ .$$

An important key to the self referential calculations in the sphere spectrum was the ability to form compositions and the result 3.1 for the commutators. Suppose then that X is a ring spectrum and the ring structure is given by "composition". To make sense of this, we will assume that there is a functorial construction of an extension $\widehat{\alpha}$ for each α:

$$\begin{array}{ccc} X_k & & \\ {\scriptstyle k}\uparrow & \searrow^{\widehat{\alpha}} & \\ S^k & \xrightarrow{\alpha} & X_n. \end{array}$$

If we also assume that the ring spectrum is homotopy commutative, we can add the condition that the formulas in 3.1 hold when one or both of m and n are even (as was the case for the sphere spectrum when $p > 2$). These assumptions put a condition on the function $f(n)$ which leads to the following:

Definition 7.1 ([**Gra93a**]). A *reflexive EHP structure* on a p-local spectrum X is a presentation $\{X_n\}$ together with fibration sequences:

$$\Omega^2 X_{2nk-1} \xrightarrow{P} X_{2n-1} \xrightarrow{E} \Omega X_{2n} \xrightarrow{H} \Omega X_{2nk-1},$$

$$\Omega^2 X_{2nk+2d+1} \xrightarrow{P} X_{2n} \xrightarrow{E} \Omega X_{2n+1} \xrightarrow{H} \Omega X_{2nk+2d+1}.$$

The number $2d$ is called the period.

The indices are controlled in such a way that both sides of 3.1(b) are in the same set. One could also consider a more general situation where the targets of the Hopf invariants are spaces $\mathbb{Z}_n$ and Y_n such that there are pairings $\Sigma^2\mathbb{Z}_n \wedge \mathbb{Z}_m \to \mathbb{Z}_{m+n}$ and $\Sigma^2\mathbb{Z}_n \wedge Y_m \to Y_m$. We will see examples of this in section 9. We will refer to these as nonreflexive EHP structures.

Comparing with (4.1), we see that the sphere spectrum has a reflexive EHP structure with $k = p$ and $d = 0$ for $p > 2$. It is likely that any interesting examples will have $k = p^s$ in order to avoid dissonance with p local units.

It is important to note that this is an unstable development of X and is not necessarily unique if it even exists. If $\{X_n\}$ is an EHP structure on X, then $\{\Omega X_n\}$ is an EHP structure on $\Sigma^{-1}X$, although $\Sigma^{-1}X$ is not connective and is not a ring spectrum.

In the case of the sphere spectrum, the composition

$$\Omega^3 S^{2np+1} \xrightarrow{P} \Omega\widehat{S}^{2n} \xrightarrow{H} \Omega S^{2np-1}$$

is actually the loops on a map $\varphi_n\colon \Omega^2 S^{2np+1} \to S^{2np-1}$ constructed in [**Gra88**], where φ_n restricted to S^{2np-1} is the map of degree p. Richter recently proved:

Proposition 7.2 (Richter [**Ricar**]). *For $p \geqslant 2$ the composition*

$$\Omega^2 S^{2np+1} \xrightarrow{\varphi_n} S^{2np-1} \xrightarrow{E^2} \Omega^2 S^{2np+1}$$

is the p^{th} power map.

Cohen, Moore and Neisendorfer constructed maps

$$\Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1}$$

when $p \geqslant 3$ with this same property [**CMN79a**]; i.e., the composition

$$\Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} \longrightarrow \Omega^2 S^{2n+1}$$

is the p^{th} power map. It is not known whether $\varphi_n \sim \pi_{np}$.

Conjecture 7.3 (Reflexivity Conjecture). $\varphi_n \sim \pi_{np}$.

This has consequences as we will see in section 9.

Definition 7.4. An EHP structure satisfies *condition CMN* if there are maps

$$\pi_n\colon \Omega^2 X_{2n+2d+1} \to X_{2n-1}$$

for each n such that the square

$$\begin{array}{ccc} \Omega^4 X_{2n+2d+3} & \xrightarrow{\Omega^2\pi_{n+1}} & \Omega^2 X_{2n+1} \\ {\scriptstyle \Omega^2E^2}\uparrow & & \uparrow{\scriptstyle E^2} \\ \Omega^2 X_{2n+2d+1} & \xrightarrow{\pi_n} & X_{2n-1} \end{array}$$

homotopy commutes, and the composition

$$\Omega^3 X_{2nk+2d+1} \xrightarrow{P} \Omega X_{2n} \xrightarrow{H} \Omega X_{2nk-1}$$

is homotopic to $\Omega\pi_{nk}$.

Thus by 7.2, the sphere spectrum satisfies CMN where π_n is the map of Cohen, Moore and Neisendorfer when $(n, p) = 1$ and $\pi_{np} = \varphi_n$.

Supposing that $X = \{X_n\}$ satisfies condition CMN, we define a self map v as the composition

$$\Sigma^{2d} X_{2n-1} \xrightarrow{E'} X_{2n+2d-1} \xrightarrow{E^2} \Omega^2 X_{2n+2d+1} \xrightarrow{\pi_n} X_{2n-1},$$

where E' is the adjoint of E^{2d}. Using the telescope on the adjoint of v allows us to construct $v^{-1}X_{2n-1}$ and $v^{-1}X$ and it is easy to see that

$$v^{-1}X \cong v^{-1}X_{2n-1} \quad n \geqslant 1. \tag{7.5}$$

In the case of the sphere spectrum this is a rational equivalence between S^0 and S^{2n+1} for each $n \geqslant 0$.

The main consequence of the condition CMN, however, is the following:

THEOREM 7.6 ([**Gra93a**, 1.3]). *Suppose X has an EHP structure $\{X_n\}$ of period $2d$ which satisfies condition CMN. Let X' be the cofiber of $v\colon \Sigma^{2n}X \to X$. Then $\Sigma^{-1}X'$ has an EHP structure of period $2d'$ where $d' = (d+1)k - 1$.*

If we consider iterating this construction, beginning with the sphere spectrum, the periods we obtain are of the form $2(p^m - 1)$. It is harder to construct an EHP structure on X'. In case such a structure exists, we will call this a derived structure. We will consider the existence of a derived structure for the sphere spectrum in section 9.

8. An Ideal Development

At this point we will assume that p is large and m is small and we will examine the form of iterated derived structures of the sphere spectrum. We will see that in an algebraic sense these do exist, but of course a geometric realization will only exist in limited cases since the spectrum of the $(m+1)$st derived EHP spectrum would be the Smith–Toda Specturm $V(m)$. The EHP sequences associated with such a spectrum would be of the form

$$\begin{gathered} \xrightarrow{P} V(m)_{2n-1} \xrightarrow{E} \Omega V(m)_{2n} \xrightarrow{H} \Omega V(m)_{2np-1} \\ \xrightarrow{P} V(m)_{2n} \xrightarrow{E} \Omega V(m)_{2n+1} \xrightarrow{H} \Omega V(m)_{2np+q_{m+1}+1} \end{gathered} \tag{8.1}$$

where $q_m = 2(p^m - 1)$. These spaces would be inductively constructed from fibration sequences:

$$\begin{gathered} \Omega V(m-1)_{2n+1} \longrightarrow V(m)_{2n} \longrightarrow V(m-1)_{2n+q_m+1} \xrightarrow{v_m} V(m-1)_{2n+1} \\ \Omega^2 V(m-1)_{2n+q_m+1} \xrightarrow{\pi_n} V(m-1)_{2n-1} \longrightarrow V(m)_{2n-1} \longrightarrow \Omega V(m-1)_{2n+q_m+1} \end{gathered} \tag{8.2}$$

There will also be a key relationship between $V(m)$ and the double suspension in $V(m-1)$ given by a fibration sequence

$$V(m-1)_{2n-1} \xrightarrow{E^2} \Omega^2 V(m-1)_{2n+1} \xrightarrow{\nu} \Omega V(m)_{2np-1}. \tag{8.3}$$

In the case $m = 0$ we consider $V(-1)$ to be the sphere spectrum.

Although such a development is unlikely except in very limited cases, we can more easily investigate the existence of Λ-algebra analogs as in section 5. The main result of [**Gra93b**] is that there are short exact sequences of dga's of exactly this form which converges to $\mathrm{Ext}_{A_p}(H^*(V(m); \mathbb{Z}/p), \mathbb{Z}/p)$ where A_p is the Steenrod algebra. In this model, we think of $\Lambda_{(m)}(n)$ as being the nth approximation to $\Lambda_{(m)}$.

THEOREM 8.4 ([**Gra93b**]). *There are short exact sequences of dga's*

$$0 \longrightarrow \Lambda_{(m)}(2n-1) \xrightarrow{E} \Lambda_{(m)}(2n) \xrightarrow{H} \Lambda_{(m)}(2np-1) \longrightarrow 0$$

$$0 \longrightarrow \Lambda_{(m)}(2n) \xrightarrow{E} \Lambda_{(m)}(2n+1) \xrightarrow{H} \Lambda_{(m)}(2np+q_{m+1}+1) \longrightarrow 0$$

$$0 \longrightarrow \Lambda_{(m-1)}(2n+1) \longrightarrow \Lambda_{(m)}(2n) \longrightarrow \Lambda_{(m-1)}(2n+q_m+1) \longrightarrow 0$$

$$0 \longrightarrow \Lambda_{(m-1)}(2n-1) \longrightarrow \Lambda_{(m)}(2n-1) \longrightarrow \Lambda_{(m-1)}(2n+q_m+1) \longrightarrow 0$$

$$0 \longrightarrow \Lambda_{(m-1)}(2n-1) \xrightarrow{E^2} \Lambda_{(m-1)}(2n+1) \longrightarrow \Lambda_{(m)}(2np-1) \longrightarrow 0$$

and a "composition pairing" $\Lambda_{(m)}(n)_\sigma \cdot \Lambda_{(m)}(n+\sigma) \subset \Lambda_{(m)}(n)$. *Furthermore,* $\Lambda_{(m)}(0)$ *is acyclic and* $H_*(\Lambda_{(m)}(\infty)) = \mathrm{Ext}_{A_p}(H^*(V(m);\mathbb{Z}/p), \mathbb{Z}/p)$.

The first two exact sequences are models for (8.1) and the next two for (8.2). The fifth exact sequence corresponds to (8.3). The product formula is a model for a hypothetical composition

$$\pi_k(V(m)_{n+\sigma}) \otimes \pi_{n+\sigma}(V(m)_n) \to \pi_k(V(m)_n).$$

The calculation that $\Lambda_{(m)}(0)$ is acyclic corresponds to $V(m)_0 = \mathbb{Z}/p$ with the discrete topology for $m \geqslant 0$.

There is also an unstable model using the periodic lambda algebra $\overline{\Lambda}$ [**Gra93c**] which is considerably smaller than the lambda algebra and displays the classes v_n as generators. This is defined when $p > 2$ and has simpler relations. In fact $\overline{\Lambda}_{(m)}$ is a quotient of $\overline{\Lambda}$ while $\Lambda_{(m)}$ is a submodule of a free Λ-module.

9. EHP Structure for $S^0 \cup_{p^r} e^1$

Applying 7.6 to the sphere spectrum we obtain an EHP structure on $S^{-1} \cup_p e^0 = \Sigma^{-1}V(0)$ where $X_{2n} = \Omega S^{2n+1}\{p\}$ is the fiber of the map $\Omega S^{2n+1} \xrightarrow{p} \Omega S^{2n+1}$ and X_{2n-1} the fiber of the map $\pi_n \colon \Omega^2 S^{2n+1} \to S^{2n-1}$ of degree p of Cohen, Moore and Neisendorfer when $(n,p) = 1$ and $X_{2np-1} = BW_n$ which is the fiber of $\varphi_n \colon \Omega^2 S^{2np+1} \to S^{2np-1}$ from ([**Gra88**]). (This simplifies if the reflexive conjecture (7.3) is valid since then $\pi_{np} \sim \varphi_n$.)

More interesting is that the EHP structure on $S^0 \cup_{p^r} e^1$, is only reflexive when $r = 1$ and the reflexivity conjecture holds. To see this, recall the conditions (8.2) when $m = 0$

$$V(0)_{2n} \longrightarrow S^{2n+1} \xrightarrow{p} S^{2n+1}$$

$$\Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{m-1} \longrightarrow V(0)_{2n-1} \longrightarrow \Omega S^{2n+1}$$

where π_n is the Cohen–Moore–Neisendorfer map of degree p. Replacing p by p^r, we define spaces T_n by fibration sequences

$$T_{2n} \longrightarrow S^{2n+1} \xrightarrow{p^r} S^{2n+1}$$

$$\Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} \longrightarrow T_{2n-1} \longrightarrow \Omega S^{2n+1}$$

where in this case π_n is the Cohen–Moore–Neisendorfer map of degree p^r. Clearly $T_{2n} = S^{2n+1}\{p^r\}$ is defined by the first fibration. It is not immediately clear that the second fibration exists.

THEOREM 9.1 (Anick [**Ani93**]). *If $p \geqslant 5$ there is a fibration*

$$\Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} \longrightarrow T_{2n-1} \longrightarrow \Omega S^{2n+1}$$

where the composition

$$\Omega^2 S^{2n+1} \xrightarrow{\pi_n} S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1}$$

p^rth power map on $\Omega^2 S^{2n+1}$.

REMARK. The p^rth power map is the same as the double loops on the map of degree p^r when $p > 2$. Any space T_{2n-1} satisfying the conclusion of 9.1 will be called an Anick space.

THEOREM 9.2 ([**AG95**]). *For $p \geqslant 3$ there is an Anick space T_{2n-1} which admits an H-space structure. This space is unique up to homotopy and the fibrations in* 9.1 *are H-fibrations.*

The construction of T_{2n-1} by Anick in 1993 was the culmination of a 274 page book. It was a complex and lengthy result and serious efforts were made to simplify the construction [**The01**]. These early efforts failed.[2] The first successful attempt was published in 2010.

THEOREM 9.3 ([**GT10**]). *For $p \geqslant 3$, the Anick space exists and there is an H-space structure on the defining fibration. Furthermore, there is an EHP fibration*

$$T_{2n-1} \xrightarrow{E} \Omega T_{2n} \xrightarrow{H} BW_n$$

where $T_{2n} = S^{2n+1}\{p^r\}$.

The restriction that $p \geqslant 5$ was removed as there was no use of the Jacobi identity for Whitehead products. The construction was reduced to 13 pages and as long again for the H-space structure. Note that the EHP sequence is non reflexive as the Hopf invariant is in BW_n. If the reflexivity conjecture were true, this would be reflexive when $r = 1$. The other EHP sequence is the middle row of the homotopy commutative diagram:

$$\begin{array}{ccccc}
\Omega S^{2n+1} & = & \Omega S^{2n+1} & & \\
\downarrow & & \downarrow & & \\
T_{2n} & \xrightarrow{E} & \Omega T_{2n+1} & \xrightarrow{H} & BW_{n+1} \\
\downarrow & & \downarrow & & \| \\
S^{2n+1} & \xrightarrow{E^2} & \Omega^2 S^{2n+1} & \xrightarrow{\nu} & BW_{n+1} \\
{\scriptstyle p^r}\downarrow & & {\scriptstyle \pi_{n+1}}\downarrow & & \\
S^{2n+1} & = & S^{2n+1} & &
\end{array}$$

Consequently, we have:

[2][**The01**] contained incorrect statements and statements for which no explanation was given. It relied on a previous paper the author listed as "to appear" which was later withdrawn.

THEOREM 9.4. *For $p \geqslant 3$ there is a (non reflexive) EHP structure on $S^0 \cup_{p^r} e^1$ with EHP sequences:*

$$T_{2n-1} \xrightarrow{E} \Omega T_{2n} \xrightarrow{H} BW_n$$

$$T_{2n} \xrightarrow{E} \Omega T_{2n+1} \xrightarrow{H} BW_{n+1}$$

Clearly the reflexivity conjecture is precisely that this is reflexive when $r = 1$, since then $BW_n = \Omega T_{2np-1}$. In general this is a non reflexive EHP structure since there is a map $\Sigma^2 BW_n \wedge BW_m \to BW_{n+m}$ obtained from a retraction $\Sigma^3 BW_m \to P^{2np+2}(p)$. (See the discussion after 7.1.)

We next ask whether there is a composition theory for the spaces T_n. Specifically, we ask to describe a functorial extension of a map α to a map $\widehat{\alpha}$ in the diagram:

$$\begin{array}{ccc} T_n & & \\ {\scriptstyle i}\uparrow & \searrow^{\widehat{\alpha}} & \\ S^n & \xrightarrow{\alpha} & T_m \end{array}$$

PROPOSITION 9.5. *If $p \geqslant 5$, the H-space T_n has H-space exponent p^r.*

PROOF. This is due to Neisendorfer ([**Nei83**]) if n is even and is in [**Gra93a**] if n is odd. ☐

Consequently we can choose and fix a null homotopy of the map p^rth power map for each n. Such a null homotopy determines an extension of α to a map $\overline{\alpha}\colon P^{2n+1} \to T_m$. In the case that n is even, we have two results which determine $\widehat{\alpha}$.

PROPOSITION 9.6 ([**Nei83**]). *If $p \geqslant 5$, T_{2n} is homotopy associative and homotopy commutative.*

PROPOSITION 9.7 ([**Gra93a**]). *If X is a homotopy associative and homotopy commutative H-space and $\overline{\alpha}\colon P^{2n+1} \to X$, there is a unique extension to an H-map $T_{2n} \to X$ up to homotopy.*

These results prompted the conjecture in [**Gra93a**] that the spaces T_{2n-1} would enjoy analogous properties to 9.6 and 9.7.

PROPOSITION 9.8 ([**Gra**]). *If $p \geqslant 5$, the space T_{2n-1} is homotopy associative and homotopy commutative.*[3]

With regard to the universal property, it was established in [**AG95**] that under certain conditions on the torsion in the target space, extensions exist but they may not be H-maps and may not be unique. In [**The01**], the author claimed to prove the conjectured universal property, but the details were never published. Subsequently it was established that the universal property failed in the generality claimed ([**Gra12**]). The torsion conditions in [**AG95**] appeared to be necessary.

DEFINITION 9.9. Fix n and r. A space X is said to have the (n, r) growth condition if for all $k \geqslant 1$

$$p^{r+k-1}\pi_{2np^k-1}(X; \mathbb{Z}/p^{r+k}) = 0.$$

[3]This result was also asserted in [**The01**] without proof.

The condition for the existence of an extension in [**AG95**] was precisely this condition. We now have:

PROPOSITION 9.10 ([**Gra12**]). *Suppose that X and ΩX satisfy the (n, r) growth condition. Then any map $\overline{\alpha}\colon P^{2n} \to X$ extends uniquely to an H map $\widehat{\alpha}\colon T_{2n-1} \to X$.*

Since the spaces $T_m(p^r)$ and $\Omega T_m(p^r)$ satisfy the (n, r) growth condition by 9.5, we have:

COROLLARY 9.11. *Every map $\overline{\alpha}\colon P^{2n}1 \to T_m(p^r)$ has a unique extension to an H-map $\widehat{\alpha}\colon T_{2n-1}(p^r) \to T_m(p^r)$ up to homotopy.*

In particular, there is a composition theory for the homotopy of the T_n spaces when $p \geqslant 5$.

10. A Kahn–Priddy Map

The spaces B^{nq} and B^{nq-1} from section 6 have an analogue for the EHP structure $\{T_n\}$ on $S^0 \cup_{p^r} e^1$.

PROPOSITION 10.1. *There is a map $F_n\colon \Sigma B^{nq} \to \Sigma B^{(n-1)q}$ which is unique up to homotopy such that the diagram*

$$\begin{array}{ccc} \Sigma B^{nq} & \xrightarrow{\ p\ } & \Sigma B^{nq} \\ \uparrow & \searrow^{F_n} & \uparrow \\ \Sigma B^{(n-1)q} & \xrightarrow{\ p\ } & \Sigma B^{(n-1)q} \end{array}$$

commutes up to homotopy, and there is a factorization

$$\begin{array}{ccc} \Sigma B^{nq} & \xrightarrow{\ F_n\ } & \Sigma B^{(n-1)q} \\ {\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \pi_{n-1}} \\ P^{nq+1}(p) & \xrightarrow{\ \nu_1\ } & P^{(n-1)q+1}(p) \end{array}$$

where π_n and π_{n-1} are quotient maps and v_1 is an Adams map.

PROOF. It is easy to see that F_n exists in such a way that the upper triangle exists since $\Sigma B^{(n-1)q}$ is equivalent to the fiber of the projection π_n in this range, and uniqueness follows as well. The lower triangle also follows from a cellular argument. Since $\widetilde{K}(B^{nq}) \cong \mathbb{Z}/p^n$, F_n is essential and does not factor through $\Sigma B^{(n-1)q}$. It follows that the lower diagram can be constructed and $v_1 \neq 0$. Since every map from $P^{nq+1}(p)$ to $P^{(n-1)q+1}(p)$ is homotopic to a multiple of the Adams map ν_1 is also an Adams map. $\square$

We also have a homotopy commutative diagram ([**Gra84**, 2.1])

$$\begin{array}{ccc} \Omega^2 S^{2n+1} & \xrightarrow{\ \pi_n\ } & S^{2n-1} \\ {\scriptstyle \widetilde{\lambda}_{2n+1}}\uparrow & & \uparrow{\scriptstyle \lambda_{2n+1}} \\ \Sigma^{2n-1} B^{nq} & \xrightarrow{\ \Sigma^{2n-2} F_n\ } & \Sigma^{2n-1} B^{(n-1)q} \end{array}$$

since the difference between the two sides factors through $W_n \to S^{2n-1}$ up to homotopy and $[\Sigma^{2n-1}B^{nq}, W_n] = *$ when $p > 2$. Now define $C(2n-1)$ to be the cofiber of F_n and we get a homotopy commutative ladder:

$$\begin{array}{ccccccc}
\Omega^2 S^{2n+1} & \xrightarrow{\pi_n} & S^{2n-1} & \longrightarrow & T_{2n-1} & \longrightarrow & \Omega S^{2n+1} \\
\uparrow{\scriptstyle \widetilde{\lambda}_{2n+1}} & & \uparrow{\scriptstyle \lambda_{2n-1}} & & \uparrow{\scriptstyle \mu_{2n-1}} & & \uparrow{\scriptstyle \overline{\lambda}_{2n+1}} \\
\Sigma^{2n-1}B^{nq} & \xrightarrow{\Sigma^{2n-2}F_n} & \Sigma^{2n-1}B^{(n-1)q} & \longrightarrow & \Sigma^{2n-2}C(2n-1) & \longrightarrow & \Sigma^{2n}B^{nq}
\end{array}$$

Furthermore $C(2n+1)/C(2n-1) \simeq \Sigma^{(n+1)q}V(1)$, and we define $C(2n) = C(2n-1) \cup CP^{(n+1)q}(p) \subset C(2n+1)$. Then we have a homotopy commutative diagram:

$$\begin{array}{ccccc}
T_{2n-1} & \xrightarrow{E} & \Omega T_{2n} & \xrightarrow{E} & \Omega^2 T_{2n+1} \\
\uparrow{\scriptstyle \mu_{2n-1}} & & \uparrow{\scriptstyle \widetilde{\mu}_{2n}} & & \uparrow{\scriptstyle \widetilde{\lambda}_{2n+1}} \\
\Sigma^{2n-2}C(2n-1) & \longrightarrow & \Sigma^{2n-2}C(2n) & \longrightarrow & \Sigma^{2n-2}C(2n+1)
\end{array}$$

These maps are useful in constructing the stable and unstable v_2 periodic homotopy classes ([**Gra93c**]).

11. The case of $V(1)$

The projected derived EHP development for $V(1)$ would consist of $(n-1)$ connected spaces U_n together with EHP sequences

$$\xrightarrow{P} U_{2n-1} \xrightarrow{E} \Omega U_{2n} \xrightarrow{H} \Omega U_{2np-1}$$

$$\xrightarrow{P} U_{2n} \xrightarrow{E} \Omega U_{2n+1} \xrightarrow{H} \Omega U_{2np+q_2+1}$$

where the spaces are defined by fibrations:

$$U_{2n} \longrightarrow T_{2n+q+1} \xrightarrow{v_1} T_{2n+1}$$

$$\Omega^2 T_{2n+q+1} \xrightarrow{\pi_n} T_{2n-1} \longrightarrow U_{2n-1} \longrightarrow \Omega T_{2n+q+1}$$

where the compositions

$$T_{2n+q-1} \xrightarrow{E^2} \Omega^2 T_{2n+q+1} \xrightarrow{\pi_n} T_{2n-1}$$

$$\Omega_2 T_{2n+q+1} \xrightarrow{\pi_n} T_{2n-1} \longrightarrow \Omega^2 T_{2n+1}$$

are homotopic to v_1 and $\Omega^2 v_1$. It is easy to see that there is a unique H-map

$$v_1 \colon T_{2n+q-1} \to T_{2n-1}$$

extending the Adams map for all n when $p \geqslant 5$ by 9.6 and 9.7, so U_{2n} is well defined as this fiber. Constructing U_{2n-1} is an ongoing project. The defining fibration sequence is a secondary version of the Anick fibration.

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