Fields Institute Communications
Volume 19, 1998

The Periodic Lambda Algebra

Brayton Gray
Department of Mathematics
University of Illinois at Chicago
850 S. Morgan
Chicago, Illinois USA 60607-7045

Abstract. The lambda algebra has been a useful tool both for making *Ext* calculations in a limited range and for understanding *Ext* on the chain level, both stably and unstably. For odd primes, the lambda algebra is inherently more complex and much less intuitive. This article presents a replacement for the lambda algebra at odd primes which is smaller, simpler, yields unstable analogs, and displays v_m periodicity both stably and unstably.

1 Introduction

The object of this work is to present a differential graded algebra (dga) $\overline{\Lambda}$ at primes $p > 2$ together with unstable approximations $\overline{\Lambda}(n)$. These objects are a significant improvement over the classical Λ algebra [B]. The advantages are:

(1) It is considerably smaller.

(2) The relations, and consequently the description of a basis is simpler.

(3) There is a natural generalization to a dga $\overline{\Lambda}_{(m)}$ for $m \geq -1$ ($\overline{\Lambda} = \overline{\Lambda}_{(-1)}$) whose homology is $\mathrm{Ext}^{**}_{A_*}(H_*(V(m)))$, where $V(m)$ is the Smith-Toda complex.[1] $\overline{\Lambda}_{(m)}$ is actually smaller than $\overline{\Lambda}$, as is its homology. This is contrary to what happens when a standard approach is used to calculate the Ext groups of a finite complex [B].

(4) The classes v_n which figure prominently in the periodic approach to homotopy theory are clearly visible as generators in $\overline{\Lambda}$.

Some important features of the classical lambda algebra are preserved. In particular:

1991 *Mathematics Subject Classification.* Primary 55Q40; Secondary 55T15.

[1] Although $V(m)$ only exists under certain restrictions on m and $p, H_*(V(m)) \simeq E(\tau_0, \dots, \tau_m)$ always exists.

(1) There is a multiplicative epimorphic chain equivalence

$$\overline{C}^*(A_*) \xrightarrow{\iota^*} \overline{\Lambda}$$

where $\overline{C}^*(A_*)$ is the cobar construction on the dual of the Steenrod algebra. Consequently, $\overline{\Lambda}$ can be used to calculate Massey products.

(2) There is a natural filtration by subcomplexes $\overline{\Lambda}(2n+1)$ whose homology is the appropriate unstable Ext group for S^{2n+1}.

(3) $\overline{\Lambda}_{(m)}$ is likewise filtered by subcomplexes $\overline{\Lambda}_{(m)}(2n+1)$ in a way suggestive of unstable analogs to the spectra $V(m)$ with corresponding EHP sequences, periodicity operators, and unstable compositions as in [G 2]. (The complexes described in [G2] are a generalization of results of Mahowald [M] and Harper and Miller [HM]).

We now state our result:

Theorem 3.2. There is a trigraded dga $\overline{\Lambda}^{s_1,s_2,t}$ and a multiplicative epimorphic chain equivalence:

$$\iota^* : \overline{C}^*(A_*) \longrightarrow \overline{\Lambda}$$

where $\overline{C}^*(A_*)$ is the cobar construction on the dual of the Steenrod algebra. Furthermore $\overline{\Lambda}$ is generated by classes $\lambda_k \in \overline{\Lambda}^{0,1,kq}$ for $k \geq 1$ and $v_n \in \overline{\Lambda}^{1,0,2p^n-1}$ for $n \geq 0$ subject to the relations:

$$\text{(3C)}\quad
\begin{array}{ll}
\text{(i)} & \lambda_i\lambda_{pi+k} = \Sigma(-1)^{j+1}\binom{(p-1)(k-j)-1}{j}\lambda_{k+i-j}\lambda_{pi+j} \\
\text{(ii)} & v_n v_k = v_k v_n \\
\text{(iii)} & \lambda_k v_n = v_n\lambda_k + v_{n-1}\lambda_{k+p^{n-1}} \text{ if } n > 0 \\
\text{(iv)} & \lambda_k v_0 = v_0\lambda_k \\
\text{(v)} & \partial v_n = v_{n-1}\lambda_{p^{n-1}} \text{ if } n > 0 \\
\text{(vi)} & \partial v_0 = 0 \\
\text{(vii)} & \partial\lambda_k = \Sigma(-1)^{j+1}\binom{(p-1)(k-j)-1}{j}\lambda_{k-j}\lambda_j.
\end{array}$$

Furthermore, let $\overline{\Lambda}_{(m)} = \overline{\Lambda}/(v_0, v_1, \ldots, v_m)$; write $A_{(m)}$ for the subalgebra of A generated by the P^n and Q_k with $k > m \geq -1$. Then there is an epimorphic multiplicative chain equivalence:

$$\overline{C}^*(A_{(m)*}) \xrightarrow{\iota_m^*} \overline{\Lambda}_{(m)}.$$

Finally, if L is a right $A_{(m)}$ module ($m \geq -1$), $\mathrm{Ext}_A^{**}(E(\tau_0, \ldots, \tau_m) \otimes L)$ is isomorphic with $H_*(\Lambda_{(m)} \otimes L)$ where the differential is determined by

$$\partial(1 \otimes x) = \sum \lambda_j \otimes xP^j + (-1)^{|x|}\sum v_n \otimes xQ_n.$$

Stably the complex $\overline{\Lambda}$ appears in Haynes Miller's thesis [Mi 1] without the algebra structure. In fact the multiplicative structure can be derived by applying the results of [Mi 2]. It does not appear explicitly in the literature, however. We choose an alternative route which is, in some sense, more elementary: we show that the Milnor basis for the Steenrod algebra [Miln] is a Koszul basis and apply the results of Priddy [P 2].

This material was developed after conversations with S. Pemmaraju [P 1] who, without the aid of the multiplicative structure, has been making calculations in Ext for $V(1)$ when $p = 3$ for use in his thesis. Our interest in this is as a theoretical and practical calculational tool, both stably and unstably. We believe that a systematic use of these results, using a downward induction on m, is a considerable improvement over standard methods of obtaining the Ext groups. For example, Tangora [T] calculates the homology of Λ in dimensions less than 94 and the homology of the subalgebra Λ' generated by all λ_i, in dimensions less than 242 when $p = 3$.

2 Koszul Resolutions

We begin by paraphrasing and summarizing the results in [P 2]. Let A be a pre-Koszul algebra; i.e., a connected augmented algebra over $\mathbf{Z}/p$ together with a presentation as an algebra with generators a_i, $i \in I$ modulo relations of the form:

$$\sum f_i a_i + \sum f_{jk} a_j a_k = 0.$$

Thus A has a $\mathbf{Z}/p$ basis consisting of words in the a_i. Filter A by minimal word length. Then E^0A, the associated graded, is generated by $a_i \in E^0_1 A$ subject to the relations:

$$\sum f_{jk} a_j a_k = 0.$$

If $J = (j_1, \dots, j_n)$ with $j_k \in I$, write a_J for the word $a_{j_1} \dots a_{j_n}$. Suppose that we have choosen a $\mathbf{Z}/p$ basis B for E^0A consisting of all words a_J for $J \in S$, where S is a distinguished set of sequences of indices from I. We will call the sequences $J \in S$ and the corresponding words a_J admissible. We define coefficients $f\begin{pmatrix} k & \ell \\ i & j \end{pmatrix}$ by the rule:

$$a_k a_\ell = \sum_{(i,j)\in S} f\begin{pmatrix} k & \ell \\ i & j \end{pmatrix} a_i a_j \quad \text{in} \quad E^0 A$$

and $f\binom{k,\ell}{m}$ by the consequent formula:

$$a_k a_\ell = \sum_m f\binom{k,\ell}{m} a_m + \sum_{(i,j)\in S} f\begin{pmatrix} k & \ell \\ i & j \end{pmatrix} a_i a_j \quad \text{in} \quad A.$$

We suppose that the indexing set I is countable and totally ordered. We order the set of sequences $J \in S$ lexicographically. If $J_1 = \{j_1 \dots j_s\}$ and $J_2 = \{j_{s+1}, \dots, j_k\}$ we will write $J = J_1 J_2$ for the sequence $\{j_1, \dots, j_k\}$. We will now say that E^0A is a Poincaré-Birkhoff-Witt (PBW) algebra if the following criteria are met:

(1) Cutting: If $J = J_1 J_2 \in S$, then $J_1 \in S$ and $J_2 \in S$,

(2) Expansion: If J_1, $J_2 \in S$ and $J = J_1 J_2$, then either $J \in S$ or each term of the expansion of a_J in the basis of admissible sequence, involves only a_K with $K > J$.

With this in mind, Priddy [P 2] shows that each PBW algebra is a Koszul algebra (which we needn't define) and concludes with the following result, which we summarize and paraphrase.

Theorem 2.1 [P 2]. Suppose A is a pre-Koszul algebra and E^0A is a PBW algebra. Then there is a dga $\overline{K}^*(A)$ called the co-Koszul complex satisfying the following conditions:

a) There is an epimorphism of dga's

$$\iota^* : \overline{C}(A^*) \longrightarrow \overline{K}^*(A)$$

where $\overline{C}(A^*)$ is the cobar construction on A^*.

b) ι^* induces an isomorphism in homology, and hence $H^*(A) \cong H_*(\overline{K}^*(A))$.

c) $\overline{K}^*(A)$ is generated by classes $\beta_i \in E_1^0A$ for $i \in I$ corresponding to $[a_i^*]$ in the cobar construction. Write $|\beta_i| = 1 + \deg a_i$. Then $\overline{K}^*(A)$ has defining relations:

$$(-1)^{\nu_{i,j}}\beta_i\beta_j + \sum_{(k,\ell)\notin S} (-1)^{\nu_{k,\ell}} f\begin{pmatrix} k & \ell \\ i & j \end{pmatrix} \beta_k\beta_\ell = 0$$

for each $(i,j) \in S$, where $\nu_{u,v} = |\beta_u| + (|\beta_u| - 1)(|\beta_v| - 1)$.

d) The differential in $\overline{K}^*(A)$ is defined by:

$$\delta\beta_m = \sum_{(k,\ell)\notin S} (-1)^{\nu_{k,\ell}} f\begin{pmatrix} k, \ell \\ m \end{pmatrix} \beta_k\beta_\ell.$$

e) Suppose L is a left A module. Then $H^*(A; L)$ is the homology of the complex $\overline{K}^*(A) \otimes L^*$, where L^* has the induced right A module structure, and the differential is determined by d) above and

$$\delta(\iota \otimes \lambda) = \sum (-1)^{|\lambda||\beta_j|} \beta_j \otimes \lambda a_j.$$

Proof Everything except the last statement can be found in [P 2]. The last statement follows since if $\mu_L : A \otimes L \longrightarrow L$ is the left action with dual $\mu_L^* : L^* \longrightarrow A^* \otimes L^*$, then

$$\mu_L^*(\lambda) = \sum_{\xi\in B} (-1)^{|\xi|(|\xi|+|\lambda|)} \xi^* \otimes \lambda\xi$$

using the sign conventions in [P 2]. Thus the projection onto E_1^0A, $\overline{\mu}_L^*$, is given by

$$\overline{\mu}_L^*(\lambda) = \sum (-1)^{|\beta_j|(|\beta_j|+|\lambda|)} \beta_j \otimes \lambda a_j.$$

The formula then follows from [P 2; 4.2]. □

3 Application to the Milnor basis

We now apply the above theory to some subalgebras of the Steenrod algebra. Milnor [Miln] has described the Steenrod algebra as having generators P^n, Q_k for $n > 0$ and $k \geq 0$ where $|Q_k| = 2p^k - 1$ and $|P^n| = nq$ subject to the following relations:

$$\text{(3A)}\quad \begin{aligned} &\text{(i)}\quad \text{if } a < pb,\ P^aP^b = (-1)^a\binom{(p-1)b-1}{a}P^{a+b} \\ &\qquad + \sum_{t=1}^{[a/p]}(-1)^{a+t}\binom{(p-1)(b-t)-1}{a-pt}P^{a+b-t}P^t \\ &\text{(ii)}\quad Q_kQ_\ell = -Q_\ell Q_k \\ &\text{(iii)}\quad P^nQ_k = \begin{cases} Q_kP^n + Q_{k+1}P^{n-p^k}, & n > p^k \\ Q_kP^n + Q_{k+1}, & n = p^k \\ Q_kP^n, & n < p^k. \end{cases} \end{aligned}$$

We define the subalgebra $A_{(m)}$ to be the algebra generated by all P^n and all Q_k with $k > m \geq -1$. Clearly $\mathrm{Ext}_{A_{(m)*}}(\mathbf{Z}/p, L) \cong \mathrm{Ext}_{A_*}(\mathbf{Z}/p,\, E(\tau_0, \ldots, \tau_m) \otimes L)$ for each left $A_{(m)*}$ comodule L.

It is clear from (3A) that $A_{(m)}$ is a pre-Koszul algebra for each $m \geq -1$. Now let I be the ordered set $2\omega = \{1, 2, \ldots, \omega, \omega+1, \ldots\}$ and define $a_n = P^n$ for $n < \omega$ and $a_{\omega+n} = Q_{m+n+1}$. We now write any monomial in $A_{(m)}$ as a sum of monomials in which:

(3B)
- (a) all the Q_k's occur prior to all the P^n's using (3A iii),
- (b) the Q_k's are ordered in decreasing order using (3A ii),
- (c) the P^n's are in standard admissible ordering using (3A i).

We will call such a monomial an admissible monomial. For a sequence $\{i_1, \ldots, i_n\}$ to be admissible, $i_j \in I$ we have $i_k > i_{k+1}$ if $i_{k+1} \geq \omega$ and $i_k \geq pi_{k+1}$ if $i_k < \omega$. Let S be the set of sequences satisfying these conditions and $B_{(m)}$ the basis of admissible monomials. Then we have:

Proposition 3.1. $E^0A_{(m)}$ is a PBW algebra.

Proof Clearly if $J = J_1J_2$ is admissible, J_1 and J_2 are as well. Suppose now that J_1 and J_2 are admissible, $J = J_1J_2$ and a_J is not admissible. Then we can write a_J as a sum of admissible monomials as described in (3B). Each move writes a term as a sum of terms with a higher order in the lexicographic ordering. □

Consequently, we may describe the co-Koszul complex whose homology is the sought after Ext group. As in the usual case, we obtain the opposite algebra structure. Thus we define $\overline{\Lambda}_{(m)} = \overline{K}^*(A_{(m)})^{op}$. This has no consequence since $H_*(A_{(m)})$ is commutative.

Proof of 3.2 (see Section 1). Write $v_n = [Q_n]$ and $\lambda_k = [P^k]$. The relations in (3C) then follow immediately from the corresponding relations in (3A) and 2.1c and d. Since $A_{(m)}$ is a subalgebra of A, $A_{(m)*}$ is a quotient co-algebra of A_* and hence $\overline{\Lambda}_{(m)}$ is a quotient of $\overline{\Lambda}$. The Koszul generators correspond, excepting that $v_i = 0$ in $\overline{\Lambda}_{(m)}$ if $i \leq m$. The formula for the boundary in $\Lambda_{(m)} \otimes L$ follows from 2.1c. □

Notes

(1) The trigrading is possible since all the formulas in (3C) respect the polynomial degree in the v_i's. This induces a trigrading in the corresponding Ext groups. This trigrading is probably the same as that given by the number of μ's.

(2) By the description in (3B), $\overline{\Lambda}_{(m)}$ has a basis consisting of elements of the form $f(v)\lambda_I$ where $f(v)$ is a monomial in the v_m's, $n > m$ and λ_I is an admissible monomial in Λ'. We could have just as well written as a basis, all $\lambda_I f(v)$, since by (3C)

$$v_n\lambda_k = \lambda_k v_n - \lambda_{k+p^{n-1}}v_{n-1} + \lambda_{k+p^{n-1}+p^{n-2}}v_{n-2} - \dots$$

(3) It is easy to see that there is no multiplicative chain equivalence either from Λ to $\overline{\Lambda}$ or from $\overline{\Lambda}$ to Λ, since in Λ, $\lambda_1\mu_0 \neq 0$ while $\mu_0\lambda_1 = 0$. This suggests an interesting question. Define a partial ordering of dga's by $\Lambda_1 \leq \Lambda_2$ if there is a multiplicative chain equivalence $\gamma : \Lambda_2 \longrightarrow \Lambda_1$. Then $\Lambda < \overline{C}^*(A_*)$ and $\overline{\Lambda} < \overline{C}^*(A_*)$, but Λ and $\overline{\Lambda}$ are not comparable. What are the minimal elements in the set of all M with $M < \overline{C}^*(A_*)$?

Let $v_m : \overline{\Lambda}_{(m-1)} \longrightarrow \overline{\Lambda}_{(m-1)}$ be left multiplication by the central cycle v_m. Then there is an exact sequence:

$$0 \longrightarrow \overline{\Lambda}_{(m-1)} \xrightarrow{v_m} \overline{\Lambda}_{(m-1)} \xrightarrow{\pi} \overline{\Lambda}_{(m)} \longrightarrow 0. \tag{3E}$$

This induces a long exact sequence in homology, and in particular, a connecting homomorphism $\delta : H_*(\Lambda_{(m)}) \longrightarrow H_*(\Lambda_{(m-1)})$ of tridegree $(-1, 1, -(2p^m - 1))$. Using this we can define the rth Bockstein $\beta^{(r)} : H_*(\overline{\Lambda}_{(m)}) \longrightarrow H_*(\overline{\Lambda}_{(m)})$ of tridegree $(-r, 1, -r(2p^m - 1))$ by $\beta^{(r)}(\pi x) = [v_m^{-r}x]$. One can easily see for example that $\beta^{(p^r)}(v_{m+1}^{p^r}) = \lambda_{p^{r+n-1}}$. These Bocksteins are the differentials in the spectral sequence of a filtered dga.

Theorem 3.3. There is a Bockstein spectral sequence which is a spectral sequence of graded rings such that:

$$E^1_{s,t} = Z/p\langle v_m^s\rangle \otimes H_t(\overline{\Lambda}_{(m)})$$
$$d^r_{s,t}(v_m^s \otimes c) = v_m^{s+r} \otimes \beta^{(r)}(c)$$
$$d^r_{s,t} : E^r_{s,t} \longrightarrow E^r_{s+r,t-rq_m-1} \text{ where } q_m = 2p^m - 2$$
$E^\infty_{s,t}$ is the associated graded to the filtration
$$F^s = v_m^s H_t(\overline{\Lambda}_{(m-1)}) \subset H_{sq_m+t}(\overline{\Lambda}_{(m-1)}).$$

Proof Filter $\overline{\Lambda}_{(m-1)}$ by $F^s\overline{\Lambda}_{(m-1)} = v_m^s\overline{\Lambda}_{(m-1)}$. This makes $\overline{\Lambda}_{(m-1)}$ into a filtered dga. The spectral sequence is then the standard one in this situation. □

4 An unstable filtration of $\overline{\Lambda}_{(m)}$

In [G 2] we gave a filtration of a dga $\Lambda_{(m)}$ whose homology for each $m \geq -1$ is $\mathrm{Ext}_{A_*}(\mathbf{Z}/p\{\tau_0, \dots, \tau_m\})$. The filtration $F^n\Lambda_{(m)} = \Lambda_{(m)}(n)$ generalized to the context of $V(m)$ the chain complex $\Lambda_{(-1)}(n) = \Lambda(n)$ whose homology is the E^2 term for the unstable Adams spectral sequence converging to $\pi_*(S^n)$. The case $m = 0$ had been considered by Mahowald [Ma] and Harper and Miller [HM]. This filtration provided an algebraic glimpse of unstable periodic homotopy. These constructions used the full lambda algebra Λ, and it is our purpose here to show that $\overline{\Lambda}_{(m)}$ has a similar filtration. In particular, in case $m = -1$, we construct subcomplexes $\overline{\Lambda}(2n+1) \subset \overline{\Lambda}$ whose homology is the appropriate Ext group for $\pi_*(S^{2n+1})$.

Let us write Λ' for the subalgebra of Λ generated by the λ_i for $i > 0$. Write $\Lambda'(2n+1) = \Lambda' \cap \Lambda(2n+1)$; this is spanned by all admissible $\lambda_I = \lambda_{i_1} \dots \lambda_{i_k}$ with $i_1 \le n$. Define $H_n : \Lambda'(2n+1) \longrightarrow \Lambda'(2np-1)$ by[2] $H_n(\lambda_I) = 0$ if $\lambda_I \in \Lambda'(2n-1)$ and I is admissible, and $H_n(\lambda_n \lambda_I) = \lambda_I$ if $\lambda_I \in \Lambda'(2np-1)$ and I is admissible. This covers all cases and we have a short exact sequence of chain complexes:

$$0 \longrightarrow \Lambda'(2n-1) \longrightarrow \Lambda'(2n+1) \xrightarrow{H_n} \Lambda'(2np-1) \longrightarrow 0. \tag{5A}$$

By [HM; 1.6], we have

$$\lambda_k \Lambda'(2n+1) \subset \Lambda'(2n-kq+1) \quad \text{if} \quad n \ge pk. \tag{5B}$$

Now let $PV_{(m)} = \mathbf{Z}/p[v_{m+1}, v_{m+2}, \dots]$ for $m \ge -1$. This is bigraded with $v_i \in PV_{(m)}^{1,2p^i-1}$. For each $n \ge 0$ we define $\overline{\Lambda}_{(m)}(2n+1)$ as a subspace of $\overline{\Lambda}_{(m)}$ by:

$$\overline{\Lambda}_{(m)}(2n+1) = \oplus PV_{(m)}^{s,t} \otimes \Lambda'(2n+s+t+1). \tag{5C}$$

It is easy to check that for each $k > m$:

$$x \in \overline{\Lambda}_{(m)}(2n+1) \quad \text{iff} \quad v_k x \in \overline{\Lambda}_{(m)}(2n-2p^k+1). \tag{5D}$$

Notice that when $m = -1$ and $k = 0$, this corresponds to the geometric fact, due to Cohen, Moore, and Neisendorfer, that $p \cdot \pi_*(S^{2n+1}) \subset \pi_*(S^{2n-1})$.

By induction on the first degree $s(\lambda_I)$ = length of I, and use of (5B) one can easily see that:

$$\lambda_k \overline{\Lambda}_{(m)}(2n+1) \subset \overline{\Lambda}_{(m)}(2n-kq+1) \quad \text{if} \quad n \ge pk. \tag{5E}$$

Combining (5D), (5E), and induction, one can then prove:

$$\overline{\Lambda}_{(m)}^{s_1,s_2,t}(2n+1) \cdot \Lambda_{(m)}(2n+t+s_1+1) \subset \overline{\Lambda}_{(m)}(2n+1). \tag{5F}$$

This is reminiscent of unstable composition and is slightly stronger than [G 2; 6.6].

Consider $x = f(v)\lambda_I \in \overline{\Lambda}_{(m)}(2n+1)$ with $f(v) \in PV_{(m)}^{s,t}$ and $\lambda_I \in \Lambda'(2n+s+t+1)$. Then

$$\begin{aligned} \partial x = \big[\partial f(v)\big]\lambda_I + f(v)\partial\lambda_I &\in \big[\partial f(v)\big]\Lambda'(2n+s+t+1) \\ &+ f(v)\Lambda'(2n+s+t+1) \subset \overline{\Lambda}_{(m)}(2n+1) \end{aligned}$$

when $n \ge 0$ by (5D) and (5E). Hence

$$\overline{\Lambda}_{(m)}(2n+1) \text{ is a subcomplex of } \overline{\Lambda}_{(m)} \text{ when } n \ge 0. \tag{5G}$$

We now define $\phi_n : \overline{\Lambda}_{(m)}(2n+1) \longrightarrow \overline{\Lambda}_{(m+1)}(2np-1)$ for $n > 0$ by

$$\phi_n(v_{i_1} \dots v_{i_s} \lambda_I) = v_{v_1+1} \dots v_{i_s+1} H_{n+p^f}(\lambda_I) \tag{5H}$$

where $p^f = p^{i_1} + \dots + p^{i_s}$; ϕ_n has tridegree $(0, -1, -nq)$. It is straightforward but somewhat tedious to check that $\phi_n \partial = -\partial \phi_n$, and there is a short exact sequence[3]:

$$0 \longrightarrow \overline{\Lambda}_{(m)}(2n-1) \longrightarrow \overline{\Lambda}_{(m)}(2n+1) \xrightarrow{\phi_n} \overline{\Lambda}_{(m+1)}(2np-1) \longrightarrow 0. \tag{5I}$$

[2] see [G 2; 3.1] where H_n is called H'_n

[3] Compare to [G 2; 3.5]

In case $m = -1$, this corresponds to the fibration sequence:

$$S^{2n-1} \longrightarrow \Omega^2 S^{2n+1} \longrightarrow BW_n$$

of [G1].

Using (5D) we see that there is a short exact sequence of chain complexes:

$$0 \longrightarrow \overline{\Lambda}_{(m-1)}(2n+2p^m+1) \xrightarrow{v_m} \overline{\Lambda}_{(m-1)}(2n+1) \xrightarrow{\pi} \overline{\Lambda}_{(m)}(2n+1) \longrightarrow 0 \qquad (5J)$$

which is a desuspension of (3E). In particular, we can desuspend the Bockstein spectral sequence in Theorem 3.3 by setting $F^s\overline{\Lambda}_{(m-1)}(2n+1) = v_m^s\overline{\Lambda}_{(m)}(2n+2sp^m+1)$. This spectral sequence has $E^1_{s,t} = \mathbf{Z}/p\langle v_m^s\rangle \otimes H_t\big(\overline{\Lambda}_{(m)}(2n+2sp^m+1)\big)$ and converges to $H_*\big(\overline{\Lambda}_{(m-1)}(2n+1)\big)$.

In [G 2], we defined $\Lambda_{(m)}(n)$ for all n and it would be desireable to have a suitable definition for $\overline{\Lambda}_{(m)}(2n)$. Such a definition can be obtained in the form of a pull back diagram:

$$\begin{array}{ccccc}
\overline{\Lambda}_{(m)}(2n) & \longrightarrow & \overline{\Lambda}_{(m)}(2n+1) & & \\
\big\downarrow & & \big\downarrow \phi & & \\
\overline{\Lambda}_{(m)}(2np-1) & \longrightarrow & \overline{\Lambda}_{(m+1)}(2np-1) & \longrightarrow & 0 \\
 & & \big\downarrow & & \\
 & & 0. & &
\end{array} \qquad (5K)$$

Although $\overline{\Lambda}_{(m)}(2n)$ contains $\overline{\Lambda}_{(m)}(2n-1)$ as a subcomplex, the natural map $\overline{\Lambda}_{(m)}(2n) \longrightarrow \overline{\Lambda}_{(m)}(2n+1)$ is an epimorphism. Thus $\overline{\Lambda}_{(m)}(2n)$ sits more awkwardly between $\overline{\Lambda}_{(m)}(2n-1)$ and $\overline{\Lambda}_{(m)}(2n+1)$ than in the corresponding case in [G2], where both suspensions are monomorphisms. Nevertheless, there are EHP sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{\Lambda}_{(m)}(2n-1) & \longrightarrow & \overline{\Lambda}_{(m)}(2n) & \longrightarrow & \overline{\Lambda}_{(m)}(2np-1) & \longrightarrow & 0 \\
0 & \longrightarrow & \overline{\Lambda}_{(m)}(2np+2p^{m+1}-1) & \longrightarrow & \overline{\Lambda}_{(m)}(2n) & \longrightarrow & \overline{\Lambda}_{(m)}(2n+1) & \longrightarrow & 0
\end{array} \qquad (5L)$$

corresponding to [G2; 4.2].

References

[B] Bousfield, A. K. and Curtis, E. B. [1972], *The homotopy spectral sequence of a space with coefficients in a ring*, Topology **11**, 79-106.

[CMN] Cohen, F. R., Moore, J. C. and Neisenderfer, J. A. [1979], *Torsion in homotopy groups*, Ann. Math. **109**, 121-168.

[G 1] Gray, B. [1988], *On the Iterated Suspension*, Topology **27**, 301-310.

[G 2] Gray, B. [1993], *EHP Spectra and Periodicity II: Λ-algebra models*, TAMS **340**, 617-640.

[HM] Harper, J. R. and Miller, H. R. [1982], *On the double suspension homomorphism at odd primes*, TAMS **273**, 319-331.

[Ma] Mahowald, M., *On the double suspension homomorphism*, TAMS **214**, 169-178.

[Mi 1] Miller, H. R. [1974], *Some apsects of the Adams-Novikov spectral sequence*, Thesis, Princeton Univ.

[Mi 2] Miller, H. R. [1978], *A localization theorem in homological algebra*, Math. Proc. Phil. Soc. **84**, 73-84.

[Miln] Milnor, J. [1958], *The Steenrod algebra and its dual*, Ann Math. **67**, 150-171.

[P 1] Pemmaraju, S., private conversations, 1994-5.

[P 2] Priddy, S. [1970], *Koszul resolutions*, TAMS **152**, 39-60.

[R] Ravenel, D. [1986], *Complex Cobordism and stable homotopy groups of spheres*, Academic Press.

[T] Tangora, M. C. [1985], *Computing the homology of the lamdba algebra*, Memoirs AMS #337.