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On the general linear group and Hochschild homology

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Introduction

Our main result here is a rational computation of the homology of the adjoint action of the infinite general linear group of an arbitrary ring. Before stating the result we establish some notation and conventions.

Rings are associative and with unit. If A is a ring then $\mathrm{GL}(A) = \bigcup_{k \geq 0} \mathrm{GL}_k(A)$ is its infinite general linear group. An A-bimodule is an abelian group B which is both a left A-module and a right A-module and satisfies $(a_1 b)a_2 = a_1(ba_2)$ for $a_i \in A$, $b \in B$ (for example $B = A$). If B is an A-bimodule, then $M(B) = \bigcup_{k \geq 0} M_k(B)$ is the infinite additive group of matrices with entries in B. Conjugation defines an action (the adjoint action) of $\mathrm{GL}(A)$ on $M(B)$. Note that an $A \otimes \mathbf{Q}$-bimodule is just an A-bimodule which is also a rational vector space. If B is an $A \otimes \mathbf{Q}$-bimodule, then $H_n(A \otimes \mathbf{Q}; B)$ denotes the Hochschild homology of $A \otimes \mathbf{Q}$ with coefficients in B.

Our main result (it appears in slightly more detailed form as Theorem V.3) is:

Main theorem. *Let A be a ring and B an $A \otimes \mathbf{Q}$-bimodule. Then*

$$H_n(\mathrm{GL}(A); M(B)) \cong \bigoplus_{p+q=n} H_p(\mathrm{GL}(A)) \otimes H_q(A \otimes \mathbf{Q}; B). \tag{1}$$

Moreover, the projection

$$\begin{aligned} H_n(\mathrm{GL}(A); M(B)) &\to H_n(\mathrm{GL}(A)) \otimes H_0(A \otimes \mathbf{Q}; B) \\ &= H_n(\mathrm{GL}(A); H_0(A \otimes \mathbf{Q}; B)) \end{aligned}$$

is induced by the trace $M(B) \to B \to H_0(A \otimes \mathbf{Q}; B)$.

This theorem is very useful for making relative rational calculations in the algebraic K-theory of simplicial rings. In fact, it can be interpreted as a fact

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about "stable K-theory" in the sense of Waldhausen and Kassel:

$$K_*^S(A;B) \otimes \mathbf{Q} \cong H_*(A;B) \otimes \mathbf{Q}$$

(see [K]). In a future paper we will use this to prove the formula

$$K_n(f) \otimes \mathbf{Q} \cong HC_{n-1}(f) \otimes \mathbf{Q} \tag{2}$$

when $f\colon R \to S$ is a one-connected map of simplicial rings. Here $K_n(f)$ is the relative algebraic K-group $\pi_{n-1}(\text{fiber}(B\,\widehat{\mathrm{GL}}(R)^+ \to B\,\widehat{\mathrm{GL}}(S)^+))$ and $HC_{n-1}(f)$ is a suitable relative version of Connes' cyclic homology. (It seems likely that (2) is true more generally whenever the map of rings $\pi_0 R \to \pi_0 S$ is surjective with nilpotent kernel.)

Formula (2) or something resembling it has already been obtained in certain special cases: the case $S = Z$ ([D-H-S 1], [H-S 1], [H-S 2], [B]); the case $S = Z[G]$, G a finite group ([D-H-S 2]); and the case when $R \to S = \mathcal{O}_K$ is a surjective map of discrete rings with nilpotent kernel and $\mathcal{O}_K$ the maximal order in a finite extension field of $\mathbf{Q}$ ([St]). In each instance a key role has been played by a result like the Main Theorem above (with $A = \pi_0(S)$). For example, the special case $A = \mathbf{Z}$, $B = \mathbf{Q}$ of our theorem, which says that the trace map $M(\mathbf{Q}) \to \mathbf{Q}$ induces an isomorphism

$$H_n(\mathrm{GL}(\mathbf{Z}); M(\mathbf{Q})) \to H_n(\mathrm{GL}(\mathbf{Z}); \mathbf{Q}),$$

is Lemma 2.3 of [F-H] and is used in [D-H-S 1].

In cases where the Main Theorem is already known the proof has used algebraic geometry. For the general case a different method is required. Here is an outline of our proof.

In order to construct a map from the left-hand side of (1) to the right-hand side we observe that the trace map $M(B) \to H_0(A \otimes \mathbf{Q}; B)$ can be realized by a map of chain complexes of $\mathrm{GL}(A)$-modules. In fact, consider the direct limit as $k \to \infty$ of the standard Hochschild complex for the ring $M_k(A \otimes \mathbf{Q})$ and bimodule $M_k(B)$. On the one hand, this complex has a $\mathrm{GL}(A)$-action; on the other hand, its homology is just $H_*(A \otimes \mathbf{Q}; B)$. What's more, its $\mathrm{GL}(A)$-hyperhomology is the right-hand side of (1) (Proposition V.2). The inclusion of $M(B)$ as the 0-chains in the complex thus induces a map from the left-hand side of (1) to the right-hand side.

To prove that the map is an isomorphism it is enough to consider the case in which B is a free bimodule F of rank one. In this case $H_n(A \otimes \mathbf{Q}; F) = 0$ for $n > 0$, so the statement to be proved is that the trace map

$$\mathrm{tr}\colon M(F) \to H_0(A \otimes \mathbf{Q}; F)$$

induces an isomorphism in $H_n(\mathrm{GL}(A); -)$. That is, we must prove

$$H(\mathrm{GL}(A); V) = 0, \tag{3}$$

where $V = \ker(\mathrm{tr})$. Strangely enough, we prove (3) by first proving its analogue in Lie algebra homology:

$$H(\mathfrak{gl}(A \otimes \mathbf{Q}); V) = 0. \tag{4}$$

The proof of (4) (= Lemma V.4 below) is an application of classical invariant theory in the style of ([L-Q], Proposition 6.6). It takes up most of Section V.

The most unusual feature of the proof of the Main Theorem is the way in which (3) is deduced from (4). As intermediaries between $\mathrm{GL}(A)$ and $\mathfrak{gl}(A \otimes \mathbf{Q})$ we use the triangular groups $T^\sigma(A) \subset \mathrm{GL}(A)$ and triangular Lie algebras $\mathfrak{t}^\sigma(A \otimes \mathbf{Q}) \subset \mathfrak{gl}(A \otimes \mathbf{Q})$. (These are defined at the beginning of §I and §II respectively.) These nilpotent groups and nilpotent Lie algebras are much more intimately related to each other than $\mathrm{GL}(A)$ and $\mathfrak{gl}(A \otimes \mathbf{Q})$ are.

Now on the group side we consider Volodin's space

$$X(A) = \bigcup_\sigma BT^\sigma(A) \subset B\mathrm{GL}(A).$$

There is a fibration up to homotopy

$$X(A) \to B\mathrm{GL}(A) \to B\mathrm{GL}(A)^+;$$

so by a Serre spectral sequence (3) will follow from

$$H(X(A); V) = 0. \tag{3'}$$

On the Lie algebra side we define a chain complex $X_*(A \otimes \mathbf{Q}; V)$ (for any $\mathfrak{gl}(A \otimes \mathbf{Q})$-module V) which is a "Lie analogue" of $X(A)$, or rather of the chains on $X(A)$ with coefficients in V. Namely in the Koszul complex $C_*(\mathfrak{gl}(A \otimes \mathbf{Q}); V)$, let $X_*(A \otimes \mathbf{Q}; V)$ be the subomplex generated by the Koszul complexes $C_*(\mathfrak{t}^\sigma(A \otimes \mathbf{Q}); V)$. We show (Theorem II.3) that $H(\mathfrak{gl}(A \otimes \mathbf{Q}); V)$ is related to $H(X_*(A \otimes \mathbf{Q}; V))$ by a spectral sequence analogous to the Serre spectral sequence which relates $H(\mathrm{GL}(A); V)$ to $H(X(A); V)$. Using this we show that (4) implies

$$HX_*(A \otimes \mathbf{Q}; V) = 0. \tag{4'}$$

Finally, (3′) and (4′) are equivalent; in fact, by methods of rational homotopy theory $H_n(X(A); V) \cong H_n X_*(A \otimes \mathbf{Q}; V)$ for *any* rational vector space V on which both $\mathrm{GL}(A)$ and $\mathfrak{gl}(A \otimes \mathbf{Q})$ act, provided the two actions are "the same" on triangular matrices (Proposition III.5).

I. Volodin's Space $X(A)$

Let A be an associative ring with identity.

A partial ordering σ of the natural numbers $\mathbf{N}$ is *supported* in a set $J \subset \mathbf{N}$ if $i \overset{\sigma}{<} j \Rightarrow (i, j) \in J \times J$; we write $\mathrm{supp}(\sigma) \subset J$. Every σ which we consider will have finite support, i.e. $\mathrm{supp}(\sigma) \subset J$ for some finite set $J \subset \mathbf{N}$. The ordering σ determines the *triangular subgroup*

$$T^{\sigma}(A) = \left\{ U \in \mathrm{GL}(A) | U_{ij} = I_{ij} \text{ unless } i \overset{\sigma}{<} j \right\}.$$

Note that $T^{\sigma}(A) \subset \mathrm{GL}_n(A)$ if $\mathrm{supp}(\sigma) \subset \{1, \dots, n\} = \underline{n}$.

For any (discrete) group G let BG be its classifying space constructed in the standard simplicial manner (the realization of the nerve of the one-object category with morphisms G). Thus for each ordering σ we have $BT^{\sigma}(A) \subset B\mathrm{GL}(A)$.

Definition I.1. The space $X(A)$ is the subcomplex

$$X(A) = \bigcup_{\sigma} BT^{\sigma}(A) \subset B\mathrm{GL}(A)$$

The following result is proved in [Su].

Proposition I.2(a). *$X(A)$ is connected, $\pi_1 X(A)$ is isomorphic to the Steinberg group $St(A)$, and the inclusion $X(A) \hookrightarrow B\mathrm{GL}(A)$ induces the usual homomorphism*

$$St(A) \to \mathrm{GL}(A)$$

with cokernel $K_1(A)$ and kernel $K_2(A)$.

(b) *$X(A)$ is acyclic*, i.e., $\tilde{H}_* X(A) = 0$.

(c) *$X(A)$ is simple*, i.e., *$\pi_1 X(A)$ acts trivially on $\pi_k X(A)$ for $k > 1$.*

From I.2 it follows that there is a pushout diagram

$$\begin{array}{ccc} X(A) & \subset & B\mathrm{GL}(A) \\ \cap & & \cap \\ X(A)^+ & \subset & B\mathrm{GL}(A)^+ \end{array}$$

which is homotopy-cartesian and in which $X(A)^+$ is contractible. This implies the equivalence of Volodin K-theory and Quillen K-theory ([Va]): The fiber product $V(A) = X(A) \times_{B\mathrm{GL}(A)} E\mathrm{GL}(A)$ is homotopy-equivalent to $\Omega B\mathrm{GL}(A)^+$. It also implies that $X(A)$ is homotopy-equivalent to the homotopy fiber of $B\mathrm{GL}(A) \to B\mathrm{GL}(A)^+$, and hence:

Proposition I.3. *Any $\mathrm{GL}(A)$-module V (viewed as a locally trivial coefficient system on $B\mathrm{GL}(A)$) determines an abelian action of $\mathrm{GL}(A)$ on $H_*(X(A); V)$ and a spectral sequence*

$$E^2_{p,q} = H_p\big(\mathrm{GL}(A); H_q(X(A); V)\big) \Rightarrow H_{p+q}(\mathrm{GL}(A); V).$$

We will also need a variant $X^S(A)$ which depends on the choice of a finite set $S \subset \mathbf{N} \times \mathbf{N}$. Call an ordering σ *orthogonal* to S and write $\sigma \perp S$ if $i \overset{\sigma}{<} j \Rightarrow (i, j) \notin S$.

Definition I.4. $X^S(A) = \bigcup_{\sigma \perp S} BT^\sigma(A) \subset X(A)$.

PROPOSITION I.5. *For any finite set* $S \subset \mathbf{N} \times \mathbf{N}$ *the inclusion* $X^S(A) \hookrightarrow X(A)$ *is a homotopy equivalence.*

Proof of I.5. Let $S \subset \mathbf{N} \times \mathbf{N}$ be finite. For $n \gg 0$ we have $S \subset \underline{n} \times \underline{n}$. Write

$$X_n(A) = X(A) \cap B\mathrm{GL}_n(A) = \bigcup_{\mathrm{supp}(\sigma) \subset \underline{n}} BT^\sigma(A),$$

$$X_n^S(A) = X^S(A) \cap B\mathrm{GL}_n(A) = \bigcup_{\substack{\mathrm{supp}(\sigma) \subset \underline{n} \\ \sigma \perp S}} BT^\sigma(A).$$

It will be enough if the inclusion $(X_n(A), X_n^S(A)) \hookrightarrow (X_{2n}(A), X_{2n}^S(A))$ is null-homotopic as a map of pairs.

Any element $U \in G$ of a group determines a homotopy from the identity map $BG \to BG$ to the map $B\,\mathrm{Inn}(U)$ induced by the inner automorphism $\mathrm{Inn}(U)\colon G \to G$. (To see this, view G as a category with one object; U provides a natural transformation from the identity functor to the functor $\mathrm{Inn}(U)$.) Taking $G = \mathrm{GL}_{2n}(A)$ and $U =$ either $\begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$ or $\begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}$, we see that the associated homotopy carries $X_n(A)$ into $X_{2n}(A)$ and $X_n^S(A)$ into $X_{2n}^S(A)$ for all time. Indeed, for any σ with $\mathrm{supp}(\sigma) \subset \underline{n}$ there exists $\tau \supset \sigma$ with $\mathrm{supp}(\tau) \subset \underline{2n}$ such that $U \in T^\tau(A)$; and if $\sigma \perp S$ then we can choose $\tau \perp S$ as well. It follows that for $U = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ -I & I \end{pmatrix}\begin{pmatrix} I & I \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ the map $B\,\mathrm{Inn}\,U\colon B\mathrm{GL}_{2n}(A) \to B\mathrm{GL}_{2n}(A)$ is homotopic to the identity by a homotopy which when restricted to $X_n(A)$ (respectively $X_n^S(A)$) takes place in $X_{2n}(A)$ (respectively $X_{2n}^S(A)$). But $B\,\mathrm{Inn}\,U$ takes $X_n(A)$ into $X_{2n}^S(A)$. □

II. A Lie analogue of Volodin's construction

We will state here and prove in Section IV an analogue of Proposition I.3 for Lie algebra homology (Theorem II.3 below). Thus we are concerned with a $\mathfrak{gl}(A)$-module V and its homology groups $H_*(\mathfrak{gl}(A); V)$. Our ultimate goal in Section V is only a rational homology computation (we can do no better because of the method used in §III below) and therefore we may as well restrict ourselves here to the case in which A and V are rational vector spaces. This saves some trouble in getting the definition of $H_*(\mathfrak{gl}(A); V)$ right and also makes it possible to take shortcuts by appealing to the results of Section III. (It may well be, however, that with a suitable definition, Theorem II.3, like Proposition I.3, is true "over **Z**.")

Note. In our main result (Theorem V.3) the ring A is *not* assumed to be a **Q**-algebra. Rather, in the proof of V.3 we at one point apply I.3 to A and at another point apply II.3 to $A \otimes \mathbf{Q}$.

Recall the definition of Lie algebra homology. If $\mathfrak{g}$ is a Lie algebra over **Q**, then a (right) $\mathfrak{g}$-module is a rational vector space V equipped with a $\mathfrak{g}$-action, i.e., a linear map

$$V \otimes \mathfrak{g} \to V$$
$$v \otimes u \mapsto [v, u]$$

satisfying

$$[v, [u_1, u_2]] = [[v, u_1], u_2] - [[v, u_2], u_1].$$

This is the same as a Lie algebra antihomomorphism

$$\mathfrak{g} \xrightarrow{\theta} \mathrm{End}(V)$$
$$u \mapsto (v \mapsto [v, u]).$$

It is also the same as a right (associative unital) action of the universal enveloping algebra $U\mathfrak{g}$. Similar remarks apply to left actions. We sometimes write $[u, v] = -[v, u]$ (thus implicitly changing a right action into a left action). The *Lie algebra homology* of the right $\mathfrak{g}$-module V is

$$H_*(\mathfrak{g}; V) = \mathrm{Tor}_*^{U\mathfrak{g}}(V, \mathbf{Q}),$$

where **Q** has the trivial left action $[\ ,\ ] \equiv 0$. The standard chain complex for computing $H_*(\mathfrak{g}; V)$ is the *Koszul complex* $C_*(\mathfrak{g}; V)$. This has as its n-th chain group

$$C_n(\mathfrak{g}; V) = V \otimes \Lambda^n \mathfrak{g},$$

and if we write $(v|u_1| \cdots |u_n)$ for $v \otimes (u_1 \wedge \cdots \wedge u_n)$, then the boundary is given by

(II.1)

$$d(v|u_1| \cdots |u_n) = \sum_{1 \le i \le n} (-1)^{i+1}\big([v, u_i]|u_1| \cdots |\widehat{u_i}| \cdots |u_n\big) + \sum_{1 \le i < j \le n} (-1)^{i+j}\big(v|[u_i, u_j]|u_1| \cdots |\widehat{u_i}| \cdots |\widehat{u_j}| \cdots |u_n\big).$$

(The Koszul complex is based on a certain standard free resolution for **Q** as left $U\mathfrak{g}$-module.)

Now if $A = A \otimes \mathbf{Q}$ is a ring, then $\mathfrak{gl}(A)$ denotes the Lie algebra of all $\mathbf{N} \times \mathbf{N}$ matrices over A with only finitely many nonzero entries. For each finitely supported partial ordering σ of $\mathbf{N}$ define the *triangular Lie algebra*

$$\mathfrak{t}^{\sigma}(A) = \left\{ u \in \mathfrak{gl}(A) | u_{ij} = 0 \text{ unless } i \overset{\sigma}{<} j \right\}.$$

If V is a $\mathfrak{gl}(A)$-module then by analogy with Definition I.1 we make:

Definition II.2. $X_*(A; V)$ is the chain complex

$$\sum_{\sigma} C_*(\mathfrak{t}^{\sigma}(A); V) \subset C_*(\mathfrak{gl}(A); V).$$

Our main result concerning $X_*(A; V)$ is the following analogue of Proposition I.3.

THEOREM II.3. *Let $A = A \otimes \mathbf{Q}$ be a ring. Any $\mathfrak{gl}(A)$-module V determines an abelian action of $\mathfrak{gl}(A)$ on $H_q X_*(A; V)$ and a spectral sequence*

$$E^2_{p,q} = H_p\big(\mathfrak{gl}(A); H_q X_*(A; V)\big) \Rightarrow H_{p+q}(\mathfrak{gl}(A); V).$$

Proof. Deferred until Section IV.

Remark II.4. "Abelian" means that the action factors through the abelianized Lie algebra $\mathfrak{gl}(A)/[\mathfrak{gl}(A), \mathfrak{gl}(A)]$. It is easy to prove the "additive Whitehead Lemma": The commutator subalgebra $[\mathfrak{gl}(A), \mathfrak{gl}(A)]$ is equal to its own commutator subalgebra and is generated (as a Lie algebra) by matrices with a single, non-diagonal, entry different from zero. It consists of all matrices whose traces are in the additive subgroup $[A, A] \subset A$ generated by commutators.

III. Rational equivalence of the Volodin construction and its Lie analogue

Let A be a ring.

Our aim here is to show (Proposition III.5 below) that if a rational vector space V has both a GL(A)-action and a $\mathfrak{gl}(A \otimes \mathbf{Q})$-action and if the two actions are compatible in a certain obvious sense (Definition III.3 below), then

$$H_n(X(A); V) \cong H_n X_*(A \otimes \mathbf{Q}; V).$$

The key idea is that for any σ (finitely supported ordering of $\mathbf{N}$) the group algebra $\mathbf{Q}T^{\sigma}(A)$ and the universal enveloping algebra $U\mathfrak{t}^{\sigma}(A \otimes \mathbf{Q})$ become isomorphic after completion with respect to powers of the augmentation ideal. This yields isomorphisms

$$H_n(T^{\sigma}(A); V) \cong H_n(\mathfrak{t}^{\sigma}(A \otimes \mathbf{Q}); V)$$

for each σ. Considering all σ at once and working on the chain level, it is then not hard to obtain the stated conclusion.

Definition III.1. An action Θ: $G \to \mathrm{Aut}(V)$ of a group on a vector space is *nilpotent* if for some $m \geq 0$, for all $U_1, \dots, U_m \in G$ we have

$$\prod_{i=1}^{m} (\Theta(U_i) - I) = 0.$$

Definition III.2. An action θ: $\mathfrak{g} \to \mathrm{End}(V)$ of a Lie algebra on a vector space is *nilpotent* if for some $m \geq 0$, for all $u_1, \dots, u_m$, we have

$$\prod_{i=1}^{m} \theta(u_i) = 0.$$

Definition III.3. A $(\mathrm{GL}, \mathfrak{gl})$-module for a ring A is a rational vector space V with a $\mathrm{GL}(A)$-action Θ and a $\mathfrak{gl}(A \otimes \mathbf{Q})$-action θ such that the following conditions hold for every ordering σ:

(i) Θ restricted to $T^\sigma(A)$ is a nilpotent action;
(ii) θ restricted to $\mathfrak{t}^\sigma(A \otimes \mathbf{Q})$ is a nilpotent action;
(iii) For every $U \in T^\sigma(A)$ we have $\theta \log(U \otimes 1) = \log \Theta(U)$.

Remark III.4. In (iii) the two "logarithms" are both defined by the series

$$\log X = \sum_{j \geq 1} (-1)^{j+1} (X - 1)^j / j.$$

This makes sense because in each case the series is really a finite sum: If $U \in T^\sigma(A)$ then on the one hand $U - I$ and $U \otimes 1 - I \in \mathfrak{t}^\sigma(A \otimes \mathbf{Q})$ are nilpotent matrices, while on the other hand by (i), $\Theta(U) - I$ is a nilpotent endomorphism of V. (Thus some condition such as (i) is necessary if (iii) is to make sense. Also, as will become clear shortly, (i) and (iii) imply (ii) while on the other hand (ii) and an "exponential" reformulation of (iii) imply (i).)

Proposition III.5. *If V is a* $(\mathrm{GL}, \mathfrak{gl})$*-module for A, then* $H_n(X(A); V) \cong H_n X_*(A \otimes \mathbf{Q}; V)$.

Proof. This will occupy almost all of Section III. It relies on Quillen's equivalences of categories between Malcev groups, complete Hopf algebras over $\mathbf{Q}$, and Malcev Lie algebras over $\mathbf{Q}$. For relevant definitions and proofs see [Q], Appendix A.

For any nilpotent group G let $\mathfrak{g}(G) = \mathscr{P}\hat{\mathbf{Q}}G$. This is a nilpotent Lie algebra. If G is finitely generated then $\mathfrak{g}(G)$ is finite-dimensional. In general $\mathfrak{g}(G)$ can be identified with the direct limit of $\mathfrak{g}(\Gamma)$ as Γ runs through all finitely generated subgroups of G. The next lemma identifies $\mathfrak{g}(T^\sigma(A))$ with $\mathfrak{t}^\sigma(A \otimes \mathbf{Q})$.

Lemma III.6. $\hat{\mathbf{Q}}T^\sigma(A)$ *and* $\hat{U}\mathfrak{t}^\sigma(A \otimes \mathbf{Q})$ *are isomorphic complete Hopf algebras.*

Proof. We first claim that the natural group homomorphism $\phi: T^\sigma(A) \to T^\sigma(A \otimes \mathbf{Q})$ induces an isomorphism $\hat{\mathbf{Q}}T^\sigma(A) \to \hat{\mathbf{Q}}T^\sigma(A \otimes \mathbf{Q})$. It is enough ([Q], p. 275, Theorem 3.3) if ϕ induces an isomorphism of Malcev completions, since the Malcev completion of a group G is the group $\mathscr{G}\hat{\mathbf{Q}}G$ of "grouplike elements" in $\hat{\mathbf{Q}}G$. In fact, $T^\sigma(A \otimes \mathbf{Q})$ is the Malcev completion both of $T^\sigma(A)$ and of itself. To see this it suffices (by [Q], p. 278, Corollary 3.8) to verify that $T^\sigma(A)$ is nilpotent, $T^\sigma(A \otimes \mathbf{Q})$ is nilpotent and uniquely divisible, every element of $\ker(\phi)$ has finite order, and every element of $T^\sigma(A \otimes \mathbf{Q})$ has some positive power in the image of ϕ. We leave these verifications to the reader.

For the rest of the proof of the lemma we may suppose $A = A \otimes \mathbf{Q}$. Thus $T^\sigma(A)$ is a (discrete) Malcev group which we identify with the group $\mathscr{G}\hat{\mathbf{Q}}T^\sigma(A)$ of grouplike elements in $\hat{\mathbf{Q}}T^\sigma(A)$ and $\mathfrak{t}^\sigma(A)$ is a (discrete) Malcev Lie algebra which we identify with the Lie algebra $\mathscr{P}\hat{U}\mathfrak{t}^\sigma(A)$ of primitive elements in $\hat{U}\mathfrak{t}^\sigma(A)$.

To prove that $\hat{\mathbf{Q}}T^\sigma(A) \cong \hat{U}\mathfrak{t}^\sigma(A)$ it will suffice to give an isomorphism $\mathscr{G}\hat{\mathbf{Q}}T^\sigma(A) \cong \mathscr{G}\hat{U}\mathfrak{t}^\sigma(A)$ of topological groups. But there are homeomorphisms (of discrete spaces)

$$\mathscr{G}\hat{\mathbf{Q}}T^\sigma(A) = T^\sigma(A) \xrightarrow{\log} \mathfrak{t}^\sigma(A) = \mathscr{P}\hat{U}\mathfrak{t}^\sigma(A) \xleftarrow{\log} \mathscr{G}\hat{U}\mathfrak{t}^\sigma(A)$$

([Q], p. 270, Proposition 2.6). Moreover they both impose the same group structure on $\mathfrak{t}^\sigma(A)$, namely the one defined by the Baker-Campbell-Hausdorff formula. □

LEMMA III.7. *Let R be either $\mathbf{Q}G$ where G is a finitely generated nilpotent group, or $U\mathfrak{g}$ where $\mathfrak{g}$ is a finite-dimensional nilpotent Lie algebra over $\mathbf{Q}$. Let I be the kernel of the augmentation $R \to G$ and let $\hat{R}$ be the I-adic completion of R. Then*

(i) *R is a (left and right) Noetherian ring.*

(ii) *(Artin-Rees property) If $M \supset N$ are finitely generated R- modules then the I-adic topology of N coincides with the relative topology for the I-adic topology of M.*

(iii) *The I-adic completion functor from finitely generated R-modules to $\hat{R}$-modules is exact.*

(iv) *I-adic completion of finitely generated R-modules is the same as tensor product with $\hat{R}$ over R.*

(v) *$\hat{R}$ is a flat R-module.*

(vi) *For any left $\hat{R}$-module V the natural map* $\mathrm{Tor}_*^R(\mathbf{Q}, V) \to \mathrm{Tor}_*^{\hat{R}}(\mathbf{Q}, V)$ *is an isomorphism.*

Proof. For (i) the hypothesis is stronger than it needs to be. We prove (i) when $R = \mathbf{Q}G$, G a polycyclic group. By induction it is enough to show that

$R = \mathbf{Q}G$ is Noetherian if G is an extension of a cyclic group C by a group H such that the ring $S = \mathbf{Q}H$ is Noetherian. We can also assume C infinite: If it is not, then form the fiber product of $G \twoheadrightarrow C \twoheadleftarrow \mathbf{Z}$ and use the fact that a ring admitting a surjection from a Noetherian ring is Noetherian.

Thus R is a twisted Laurent extension of S:

$$R = \bigoplus_{n \in \mathbf{Z}} SX^n,$$
$$Xs = \alpha(s)X \qquad \text{for all } s \in S,$$

for some automorphism α of S. The twisted polynomial ring

$$R^+ = \bigoplus_{n \geq 0} SX^n \subset R$$

is Noetherian if S is, which we get by generalizing the usual proof in the commutative case ([A-M], p. 81). Since every one-sided ideal in R is generated by its intersection with R^+ it follows that R is Noetherian.

The conclusion (i) is well-known when $R = U\mathfrak{g}$ for any finite-dimensional Lie algebra $\mathfrak{g}$ ([Bo], p. 18, Prop. 6).

In general if I is a two-sided ideal in any Noetherian ring then in order to conclude (ii) it is enough to know that I has a generating set $(x_1, \ldots, x_r)$ such that for all $i = 1, \ldots, r$ the image of x_i in the quotient ring $R/(x_1, \ldots, x_{i-1})$ is central ([N-G], 2.7-2.8). This condition clearly holds in the cases considered here.

By a standard argument ([A-M]) (iii) and (iv) follow from (i) and (ii). Of course (v) follows from (iii) and (iv).

For (vi) use a free $\hat{R}$-resolution F_* of V. By (v), F_* is also a flat R-resolution of V. Thus since Tor can be computed using flat resolutions, it will be enough if $\mathbf{Q} \otimes_R F_n \to \mathbf{Q} \otimes_{\hat{R}} F_n$ is an isomorphism for all n, i.e., if $\mathbf{Q} \otimes_R \hat{R} \to \mathbf{Q} \otimes_{\hat{R}} \hat{R}$ is an isomorphism. But this follows by application of (iv) to the R-module $\mathbf{Q}$. □

Now let V be a (GL, $\mathfrak{gl}$)-module for A. For each σ this means that (i) the $QT^\sigma(A)$-module V is discrete (in the topology of the augmentation ideal), (ii) the $U\mathfrak{t}^\sigma(A \otimes \mathbf{Q})$-module V is discrete, and (iii) the two resulting actions of $\hat{\mathbf{Q}}T^\sigma(A) = \hat{U}\mathfrak{t}^\sigma(A \otimes \mathbf{Q})$ on V are equal. Fix σ and let $\varinjlim$ denote direct limit over all finitely generated subgroups G of $T^\sigma(A)$. We have

$$\begin{aligned} H_*(T^\sigma(A); V) &= \operatorname{Tor}_*^{\mathbf{Q}T^\sigma(A)}(\mathbf{Q}, V) \\ &\cong \varinjlim \operatorname{Tor}_*^{\mathbf{Q}G}(\mathbf{Q}, V) \\ &\cong \varinjlim \operatorname{Tor}_*^{\hat{\mathbf{Q}}G}(\mathbf{Q}, V) \text{ (by III.7)} \\ &= \varinjlim \operatorname{Tor}_*^{\hat{U}\mathfrak{g}(G)}(\mathbf{Q}, V) \\ &= \varinjlim \operatorname{Tor}_*^{U\mathfrak{g}(G)}(\mathbf{Q}, V) \text{ (by III.7)} \\ &= \operatorname{Tor}_*^{U\mathfrak{g}(T^\sigma(A))}(\mathbf{Q}, V) \\ &= H_*(\mathfrak{t}^\sigma(A \otimes \mathbf{Q}); V) \text{ (by III.6)}. \end{aligned}$$

It is not hard to see that this chain of isomorphisms comes from a finite sequence of chain complexes starting with $C_*(T^\sigma(A);V)$ and ending with $C_*(\mathfrak{t}^\sigma(A\otimes \mathbf{Q});V)$, each with a quasi-isomorphism (QI) to or from the next. Moreover, it is easily arranged for the chain complexes all to be functorial in σ and the QI's natural.

All that remains to prove III.5 is to piece this all together somehow as σ varies. Let X_*^σ be any one of the chain complexes referred to above and let $X_\sigma^\tau\colon X_*^\sigma \to X_*^\tau$ $(\sigma \subset \tau)$ be the maps which make it a functor. Define the "homotopy colimit" $\underset{\vec{\sigma}}{\operatorname{holim}} X_*^\sigma$ to be the total complex of the following double complex (X_{**}, d_1, d_2):

$$X_{p,q} = \bigoplus_{\sigma_0 \subset \cdots \subset \sigma_p} X_q^{(\sigma_0,\dots,\sigma_p)} \quad \text{for } p \geq 0,\ q \geq 0, \text{ where } X_q^{(\sigma_0,\dots,\sigma_p)} = X_q^{\sigma_0}.$$

$$d_1 x = \sum_{i=0}^{p} (-1)^i \partial_i x \in X_{p-1,q} \quad \text{for } x \in X_{p,q}, \quad \text{where}$$

$$\partial_i x = \begin{Bmatrix} x \in X_q^{(\sigma_0,\dots,\hat{\sigma}_i,\dots,\sigma_p)}, & 0 < i \leq p \\ X_{\sigma_0}^{\sigma_1} x \in X_q^{(\sigma_1,\dots,\sigma_p)}, & i = 0 \end{Bmatrix} \quad \text{for } x \in X_q^{(\sigma_0,\dots,\sigma_p)}.$$

$$d_2 x = dx \in X_{q-1}^{\sigma_0} = X_{q-1}^{(\sigma_0,\dots,\sigma_p)} \subset X_{p,q-1}$$
$$\text{for } x \in X_q^{\sigma_0} = X_q^{(\sigma_0,\dots,\sigma_p)} \subset X_{p,q}.$$

It is clear that $\underset{\vec{\sigma}}{\operatorname{holim}}$ takes QI's which are natural in σ to QI's, since a map of double complexes which in each column is a QI induces a QI of total complexes. Thus

$$H_n \underset{\vec{\sigma}}{\operatorname{holim}}\, C_*(T^\sigma(A);V) \cong H_n \underset{\vec{\sigma}}{\operatorname{holim}}\, C_*(\mathfrak{t}^\sigma(A\otimes \mathbf{Q});V).$$

It remains to prove that

$$H_n \underset{\vec{\sigma}}{\operatorname{holim}}\, C_*(T^\sigma(A);V) \cong H_n C_*(X(A);V),$$
$$H_n \underset{\vec{\sigma}}{\operatorname{holim}}\, C_*(\mathfrak{t}^\sigma(A\otimes \mathbf{Q});V) = H_n X_*(A\otimes \mathbf{Q};V)$$

(that is, that these two $\overrightarrow{\operatorname{holim}}$'s are quasi-isomorphic to the corresponding $\overrightarrow{\lim}$'s).

One argument covers both cases. Let $C_*^\sigma = C_*(T^\sigma(A);V)$ (respectively $C_*(\mathfrak{t}^\sigma(A\otimes \mathbf{Q});V)$). The complexes C_*^σ are all contained in the larger complex $C_*(\mathrm{GL}(A);V)$ (respectively $C_*(\mathfrak{gl}(A);V)$) and the subcomplex $\Sigma_\sigma C_*^\sigma$ which they generate is $C_*(X(A);V)$ (respectively $X_*(A\otimes \mathbf{Q};V)$). Also $C_*^\sigma \cap C_*^\tau = C_*^{\sigma\cap\tau}$.

Define a chain map $\alpha\colon \underset{\vec{\sigma}}{\operatorname{holim}} C_*^\sigma \to \Sigma_\sigma C_*^\sigma$ by making it zero on $C_{p,q}$ if $p > 0$ and setting $\alpha(x) = x \in \Sigma_\sigma C_q^\sigma$ for $x \in C_q^{\sigma_0} \subset C_{0,q}$. We will prove that α is a QI.

The proof is inductive. Let I be any nonempty finite set of orderings such that

(III.8) $$\sigma \subset \tau \in I \Rightarrow \sigma \in I.$$

We can express $\underrightarrow{\operatorname{holim}}\, C_*^\sigma$ and $\Sigma_\sigma C_*^\sigma$ as unions

$$\bigcup_I \underset{\sigma \in I}{\underrightarrow{\operatorname{holim}}}\, C_*^\sigma, \qquad \bigcup_I \sum_{\sigma \in I} C_*^\sigma.$$

Since homology commutes with filtered colimits of chain complexes, it suffices to prove:

Claim III.9. *For each I (satisfying* (III.8), *α defines a* QI: $\underset{\sigma \in I}{\underrightarrow{\operatorname{holim}}}\, C_*^\sigma \to \Sigma_{\sigma \in I} C_*^\sigma$.

Proof of Claim. Use induction on card(I). If I can be written $I = J \cup K$ where J and K are strictly smaller sets satisfying III.8, then $L = J \cap K$ also satisfies III.8. The chain complexes $\Sigma_{\sigma \in J} C_*^\sigma$ and $\Sigma_{\sigma \in K} C_*^\sigma$ have sum $\Sigma_{\sigma \in I} C_*^\sigma$ and intersection $\Sigma_{\sigma \in L} C_*^\sigma$, and likewise with "$\underrightarrow{\operatorname{holim}}$" instead of "$\Sigma$". This yields two Mayer-Vietoris sequences and a map between them, so that the 5-lemma and the inductive hypothesis complete the argument.

Otherwise I has a final object τ, i.e. $I = \{\sigma \subset \tau\}$ for some τ. In this case the chain map

$$\underset{\sigma \subset \tau}{\underrightarrow{\operatorname{holim}}}\, C_*^\sigma \to \sum_{\sigma \subset \tau} C_*^\sigma = C_*^\tau$$

has an obvious right inverse which is easily seen to be a chain homotopy inverse. □

This completes the proof of III.5.

Corollary III.10. *If V is an abelian $\mathfrak{gl}(A \otimes \mathbf{Q})$-module, then*

$$H_n X_*(A \otimes \mathbf{Q}; V) = \begin{cases} V, & n = 0 \\ 0, & n > 0 \end{cases}.$$

Proof. Give V the trivial GL(A)-action; this makes it a (GL, $\mathfrak{gl}$)-module. The result now follows from III.5 and I.2.b. □

In analogy with Definition I.4 we make:

Definition III.11. If $S \subset \mathbf{N} \times \mathbf{N}$ is finite and $A = A \otimes \mathbf{Q}$ then

$$X_*^S(A; V) = \sum_{\sigma \perp S} C_*(\mathfrak{t}^\sigma(A; V)) \subset X_*(A; V).$$

Corollary III.12. *If V is a* (GL, $\mathfrak{gl}$)*-module for A then the inclusion $X_*^S(A \otimes \mathbf{Q}; V) \hookrightarrow X_*(A \otimes \mathbf{Q}; V)$ is a quasi-isomorphism.*

Proof. Repeat the proof of III.5 using only those σ for which $\sigma \perp S$. This yields the left-hand isomorphism in the commutative diagram

$$\begin{array}{ccc} H_* X^S(A \otimes \mathbf{Q}; V) & \to & H_* X(A \otimes \mathbf{Q}; V) \\ \downarrow \cong & & \downarrow \cong \\ H_*(X^S(A); V) & \to & H_*(X(A); V). \end{array}$$

The lower arrow is an isomorphism by I.5. □

IV. Proof of Theorem II.3

As in Section II we now assume $A = A \otimes \mathbf{Q}$.

It will be convenient to have some special notation and terminology. An *edge* is a pair $(i, j) \in \mathbf{N} \times \mathbf{N}$. If $\varepsilon = (i, j)$ is an edge and $a \in A$ is a ring element, then εa denotes the matrix whose (i, j) entry is a and whose other entries are all zero. A *sequence* (of *length* $n \geq 0$) is a finite sequence of edges $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$, $\varepsilon_k = (i_k, j_k)$. The sequence $\underline{\varepsilon}$ is a *path* (or a *path from* i_1 *to* j_n) if $n \geq 1$ and $j_1 = i_2$, $j_2 = i_3, \dots, j_{n-1} = i_n$. It is a *loop* if in addition $j_n = i_1$. A sequence $(\varepsilon_1, \dots, \varepsilon_n)$ *contains* any sequence of the form $(\varepsilon_{k_1}, \dots, \varepsilon_{k_m})$ where $\{k_1, \dots, k_m\} \subset \{1, \dots, n\}$. A sequence is *good* if it contains no loop.

Of course in the Koszul complex $C_*(\mathfrak{gl}(A); V)$ the chain group $C_n(\mathfrak{gl}(A); V)$ is generated by the elements $(v|\varepsilon_1 a_1| \cdots |\varepsilon_n a_n)$, where $(\varepsilon_1, \dots, \varepsilon_n)$ is any sequence, $v \in V$, and $a_k \in A$. The reader may check that the subgroup $X_n(A; V)$ is generated by only those $(v|\varepsilon_1 a_1| \cdots |\varepsilon_n a_n)$ for which $(\varepsilon_1, \dots, \varepsilon_n)$ is good. The plan is to filter the complex $C_*(\mathfrak{gl}(A); V)$ according to "how bad" such sequences are.

Definition IV.1(a). The *badness* $\beta(\underline{\varepsilon})$ of a sequence $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$ is the number of k $(1 \leq k \leq n)$ such that ε_k belongs to some loop contained in $\underline{\varepsilon}$.

(b) $F_pC_n = F_pC_n(\mathfrak{gl}(A); V)$ is the subgroup of $C_n(\mathfrak{gl}(A); V)$ generated by all $(v|\varepsilon_1 a_1| \cdots |\varepsilon_n a_n)$ such that $\beta(\varepsilon_1, \dots, \varepsilon_n) \leq p$.

For example $\beta(\underline{\varepsilon}) = 0$ if and only if $\underline{\varepsilon}$ is good, and $\beta(\underline{\varepsilon}) = 1$ if and only if exactly one ε_k belongs to a loop contained in $\underline{\varepsilon}$ (whence ε_k must be a diagonal pair (i, i)). At the other extreme a loop of length n has badness n. The sequence

$$\underline{\varepsilon} = ((1,2),(1,2),(2,3),(2,5),(4,6),(4,7),(5,1),(6,6),(6,7))$$

of length 9 has badness 5 because the loops which it contains are $((1,2),(2,5),(5,1))$ and its cyclic permutations and $(6,6)$, and these involve the edges ε_1, ε_2, ε_4, ε_7, and ε_8.

The definition implies that

$$F_pC_n \subset F_{p+1}C_n, \quad F_nC_n = C_n(\mathfrak{gl}(A); V), \quad F_0C_n = X_n(A; V), \text{ and } F_{-1}C_n = 0.$$

Also, the Koszul differential II.1 preserves the filtration; that is,

$$dF_p C_n \subset F_pC_{n-1}.$$

(This follows from the statement: Any sequence of length $n-1$ which is obtained from a sequence $\underline{\varepsilon}$ of length n by either (1) deleting an edge or (2) replacing two edges (i, j) and (j, k) by a single edge (i, k) must have badness $\leq \beta(\underline{\varepsilon})$.)

The spectral sequence of the theorem will be the one associated to the filtered chain complex $\{F_p C_*\}$. We must analyze E^0, E^1, and E^2 of this spectral sequence.

We can write

$$C_n(\mathfrak{gl}(A); V) = \bigoplus_{\underline{\varepsilon}} V \otimes \Lambda^n_{\underline{\varepsilon}} A, \tag{IV.2}$$

where $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)$ runs through a system of representatives for the action of the symmetric group Σ_n on $(\mathbf{N} \times \mathbf{N})^n$ and $\Lambda^n_{\underline{\varepsilon}} A$ is the quotient of $A^{\otimes n}$ obtained by "partial antisymmetrization" using only the subgroup of Σ_n which fixes $\underline{\varepsilon}$. Explicitly the inclusion of the "$\underline{\varepsilon}$" summand in IV.2 is given by

$$v \otimes \{a_1 \otimes \cdots \otimes a_n\} \mapsto (v|\varepsilon_1 a_1| \cdots |\varepsilon_n a_n)$$

($\{\ \}$ denotes the class in $\Lambda^n_{\underline{\varepsilon}} A$ of an element of $A^{\otimes n}$.) In terms of the identification IV.2 we have

$$F_p C_n = \bigoplus_{\beta(\underline{\varepsilon}) \leq p} V \otimes \Lambda^n_{\underline{\varepsilon}} A$$

and

$$\begin{aligned} E^0_{p,q} &= F_p C_{p+q} / F_{p-1} C_{p+q} \\ &= \bigoplus_{\beta(\underline{\varepsilon}) = p} V \otimes \Lambda^{p+q}_{\underline{\varepsilon}} A. \end{aligned}$$

If $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)$ is any sequence, let $\underline{\varepsilon}'$ be the sequence obtained from $\underline{\varepsilon}$ by deleting each ε_k which does not belong to any loop contained in $\underline{\varepsilon}$. Clearly $\beta(\underline{\varepsilon}') = \beta(\underline{\varepsilon}) =$ length of $\underline{\varepsilon}'$. In the last expression for $E^0_{p,q}$ choose the representative $\underline{\varepsilon}$ such that $\underline{\varepsilon}'$ is a final segment of $\underline{\varepsilon}$. Then we obtain

$$E^0_{p,q} = \bigoplus_{\beta(\underline{\varepsilon}') = p} \ \bigoplus_{\beta(\underline{\varepsilon}'', \underline{\varepsilon}') = p} V \otimes \Lambda^q_{\underline{\varepsilon}''} A \otimes \Lambda^p_{\underline{\varepsilon}'} A.$$

Here $\underline{\varepsilon}' = (\varepsilon'_1, \ldots, \varepsilon'_p)$ runs through a system of representatives for orbits of the action of Σ_p on $(\mathbf{N} \times \mathbf{N})^p$ with $\beta(\underline{\varepsilon}') = p$, while for fixed $\underline{\varepsilon}'$ the sequence $\underline{\varepsilon}'' = (\varepsilon''_1, \ldots, \varepsilon''_q)$ runs through a system of representatives for the action of Σ_q on only those elements of $(\mathbf{N} \times \mathbf{N})^q$ such that the sequence $(\underline{\varepsilon}'', \underline{\varepsilon}') = (\varepsilon''_1, \ldots, \varepsilon''_q, \varepsilon'_1, \ldots, \varepsilon'_p)$ has $\beta(\underline{\varepsilon}'', \underline{\varepsilon}') = p$. Explicitly for each $\underline{\varepsilon}'$ and $\underline{\varepsilon}''$, $V \otimes \Lambda^q_{\underline{\varepsilon}''} A \otimes \Lambda^p_{\underline{\varepsilon}'} A$ is included into $E^0_{p,q}$ by

$$v \otimes \left\{a''_1 \otimes \cdots \otimes a''_q\right\} \otimes \left\{a'_1 \otimes \cdots \otimes a'_p\right\} \mapsto \left(v|\varepsilon''_1 a''_1| \cdots |\varepsilon'_p a'_p\right).$$

For a fixed sequence $\underline{\varepsilon}'$ of length p with $\beta(\underline{\varepsilon}') = p$ let us examine the condition on a sequence $\underline{\varepsilon}''$ of length q: $\beta(\underline{\varepsilon}'', \underline{\varepsilon}') = p$. It says that $\underline{\varepsilon}''$ should contain no loops *and* that if $T = \{(i_1, j_1), \ldots, (i_r, j_r)\}$ is a set of edges ($r > 0$) such that $\underline{\varepsilon}'$ contains paths from j_1 to i_2, j_2 to $i_3, \ldots,$ and j_r to i_1, then for some edge $(i_k, j_k) \in T$, $\underline{\varepsilon}''$ should fail to contain any path from i_k to j_k. This may be restated by giving some finite list $\{S_\nu(\underline{\varepsilon}')\}$ of finite sets $S_\nu(\underline{\varepsilon}') \subset \mathbf{N} \times \mathbf{N}$ and requiring that $\underline{\varepsilon}''$ be good and that for some ν, for all $(i, j) \in S_\nu(\underline{\varepsilon}')$, $\underline{\varepsilon}''$ not contain any path from i to j. Namely for each way of choosing one edge from each set T let the set of chosen edges be one of the sets $S_\nu(\underline{\varepsilon}')$. Thus, recalling Definition III.11, we have

$$V \otimes \bigoplus_{\beta(\underline{\varepsilon}'', \underline{\varepsilon}')=p} \Lambda^q_{\underline{\varepsilon}''}(A) \cong \sum_\nu X_q^{S_\nu(\underline{\varepsilon}')}(A;V) \subset X_q(A;V).$$

Hence we can write

$$E^0_{p,q} = \bigoplus_{\beta(\underline{\varepsilon}')=p} \left\{ \sum_\nu X_q^{S_\nu(\underline{\varepsilon}')}(A;V) \right\} \otimes \Lambda^p_{\underline{\varepsilon}'}(A). \tag{IV.3}$$

Moreover the differential $d^0_{p,q}\colon E^0_{p,q} \to E^0_{p,q-1}$ is given in terms of (IV.3) by

$$d^0_{p,q}(x \otimes y) = dx \otimes y$$

where d is the differential in $X_*(A;V)$. Indeed taking $x = (v|\varepsilon_1''a_1''| \cdots |\varepsilon_q''a_q'')$ and $y = \{\varepsilon_1'a_1' \otimes \cdots \otimes \varepsilon_p'a_p'\}$ and computing

$$d\big(v|\varepsilon_1''a_1''| \cdots |\varepsilon_q''a_q''|\varepsilon_1'a_1'| \cdots |\varepsilon_p'a_p'\big)$$

by (II.1) one finds five kinds of terms, involving respectively $[v, \varepsilon_k''a_k'']$, $[v, \varepsilon_k'a_k']$, $[\varepsilon_k''a_k'', \varepsilon_l''a_l'']$, $[\varepsilon_k''a_k'', \varepsilon_l'a_l']$, and $[\varepsilon_k'a_k', \varepsilon_l'a_l']$. Terms of the second, fourth and fifth kinds involve sequences with badness $< p$ and so do not appear in $E^0_{p,q-1}$. The remaining terms add up to (a representative for the element of $E^0_{p,q-1}$ corresponding to) $dx \otimes y$.

We next use the following result, which was already proved (Corollary III.16) in the case when V is a $(\mathrm{GL}, \mathfrak{gl})$-module.

Lemma IV.4. *For any finite set* $S \subset \mathbf{N} \times \mathbf{N}$ *the inclusion* $X^S_*(A;V) \subset X_*(A;V)$ *induces an isomorphism in homology.*

Proof. Deferred to the end of Section IV.

Note. In proving the main result V.3 we will only use II.3 in the case of a $(\mathrm{GL}, \mathfrak{gl})$-module. Thus for the purpose of proving V.3 we may consider Lemma IV.4 to be proved.

The lemma implies the following more general statement:

Lemma IV.5. *For any finite collection* $\{S_\nu\}$ *of finite sets* $S_\nu \subset \mathbf{N} \times \mathbf{N}$ *the inclusion*

$$\sum_\nu X^{S_\nu}(A;V) \subset X_*(A;V)$$

induces an isomorphism in homology.

Proof of IV.5. We use induction on the number of S_ν's. Choose one ν_0. There is a short exact sequence of complexes

$$0 \to X_* \Big/ \sum_{\nu \neq \nu_0} X_*^{S_{\nu_0} \cup S_\nu} \to X_*/X_*^{S_{\nu_0}} \oplus X_* \Big/ \sum_{\nu \neq \nu_0} X_*^{S_\nu} \to X_* \Big/ \sum_\nu X_*^{S_\nu} \to 0$$

because

$$X_*^{S_{\nu_0}} \cap \sum_{\nu \neq \nu_0} X_*^{S_\nu} = \sum_{\nu \neq \nu_0} X_*^{S_{\nu_0}} \cap X_*^{S_\nu} = \sum_{\nu \neq \nu_0} X_*^{S_{\nu_0} \cup S_\nu}.$$

(Use the fact that each X_*^S is a direct sum of some of the summands in IV.2.) The resulting long exact sequence, with the inductive hypothesis, finishes the proof. $\square$

Using IV.3 and Lemma IV.5 we have

$$E^1_{p,q} = H_q X(A;V) \otimes \bigoplus_{\beta(\underline{\varepsilon}')=p} \Lambda^p_{\underline{\varepsilon}'} A. \tag{IV.6}$$

Before identifying the differential $d^1_{p,q}$ we must define the action of $\mathfrak{gl}(A)$ on $H_q X(A;V)$ which is mentioned in II.3. Each $u \in \mathfrak{gl}(A)$ determines an endomorphism λ_u of the Koszul complex $C_*(\mathfrak{gl}(A);V)$:

$$\lambda_u(v|u_1|\cdots|u_n) = ([v,u]|u_1|\cdots|u_n) + \sum_{i=1}^n (v|u_1|\cdots|[u_i,u]|\cdots|u_n).$$

The induced map on homology is zero because $\lambda_u = d\mu_u + \mu_u d$, where $\mu_u(v|u_1|\cdots|u_n) = (v|u|u_1|\cdots|u_n)$.

The chain map λ_u does not preserve the subcomplex $X_*(A;V)$, but it nearly does so. In fact, λ_u carries $X_*^S(A;V)$ into $X_*(A;V)$ for any finite set $S \subset \mathbf{N} \times \mathbf{N}$ which contains all pairs (j,i) such that $u_{ij} \neq 0$. Call the restricted map $\lambda_u^S\colon X_*^S(A;V) \to X_*(A,V)$. It induces a map $\bar\lambda_u^S$ and hence a map $\bar\lambda_u$:

$$\begin{tikzcd} H_q X_*(A;V) \arrow[r, "\bar\lambda_u"] & H_q X_*(A;V) \\ & H_q X_*^S(A;V) \arrow[ul, "\cong"', "\text{by IV.4}"] \arrow[u, "\bar\lambda_u^S"'] \end{tikzcd}$$

It is clear that $\bar\lambda_u$ is independent of the choice of S and satisfies $\bar\lambda_{[u,v]} = [\bar\lambda_v, \bar\lambda_u]$, i.e. gives a right action of $\mathfrak{gl}(A)$ on $H_* X(A;V)$. To see that the action is abelian it suffices (Remark II.4) to check that $\bar\lambda_u = 0$ when $u = \varepsilon a$, $\varepsilon = (i,j)$, $i \neq j$. To do so, note that in this case the nullhomotopy μ_u takes $X_n^S(A;V)$ into $X_{n+1}(A;V)$.

We use the following definition in writing down $d^1_{p,q}$.

Definition IV.7. For an abelian action of $\mathfrak{gl}(A)$ on a $\mathbf{Q}$-vector space Y, $C_*^+(\mathfrak{gl}(A);Y)$ is the quotient complex

$$C_n^+(\mathfrak{gl}(A);Y) = F_nC_n(\mathfrak{gl}(A);Y)/F_{n-1}C_n(\mathfrak{gl}(A);Y)$$

of $C_*(\mathfrak{gl}(A);Y)$.

We leave it for the reader to check that the differential in $C_*(\mathfrak{gl}(A);Y)$ is well-defined in this quotient. (The hypothesis that the action is abelian is necessary.)

Note that by IV.6 we may identify $E^1_{p,q}$ with $C_p^+(\mathfrak{gl}(A); H_qX(A;V))$.

CLAIM IV.9. *The map* $(-1)^q d^1_{p,q}$: $E^1_{p,q} \to E^1_{p-1,q}$ *is the same as the differential in* $C_*^+(\mathfrak{gl}(A); H_qX(A;V))$.

Proof. Note that $C_p^+(\mathfrak{gl}(A); H_qX(A;V))$ is generated by images of elements $z = (\overline{w}|u_1'|\cdots|u_p') \in C_p(\mathfrak{gl}(A); H_qX(A;V))$ where $u_i' = (\varepsilon_i'a_i')$, $\beta(\varepsilon_1', \dots, \varepsilon_p') = p$, and $\overline{w} \in H_qX(A;V)$ has a representative cycle $w \in \sum X_q^{S_\nu(\varepsilon')}(A;V)$. Write

$$w = \sum\left(v|u_1''|\dots|u_q''\right),$$

a sum of several terms. We have to compute the Koszul differential of

$$\tilde{z} = \sum\left(v|u_1''|\cdots|u_q''|u_1'|\cdots|u_p'\right).$$

By II.1 we have

$$\begin{aligned}
d\tilde{z} = {} & \sum\sum_{k=1}^{q}(-1)^{k+1}\left([v,u_k'']|u_1''|\cdots|\widehat{u_k''}|\cdots|u_p'\right)\\
&+\sum\sum_{l=1}^{p}(-1)^{q+l+1}\left([v,u_l']|u_1''|\cdots|\widehat{u_l'}|\cdots|u_p'\right)\\
&+\sum\sum_{1\le k<l\le q}(-1)^{k+l}\left(v|[u_k'',u_l'']|u_1''|\cdots|\widehat{u_k''}|\cdots|\widehat{u_l''}|\cdots|u_p'\right)\\
&+\sum\sum_{1\le k\le q}\sum_{1\le l\le p}(-1)^{k+q+l}\left(v|[u_k'',u_l']|u_1''|\cdots|\widehat{u_k''}|\cdots|\widehat{u_l'}|\cdots|u_p'\right)\\
&+\sum\sum_{1\le k<l\le p}(-1)^{q+k+q+l}\left(v|[u_k',u_l']|u_1''|\cdots|\widehat{u_k'}|\cdots|\widehat{u_l'}|\cdots|u_p'\right).
\end{aligned}$$

The first and third terms sum to zero because w is a cycle. Rewriting the remaining terms yields

$$\begin{aligned}
(-1)^q\,dz = {} & \sum_{l=1}^{p}(-1)^{l+1}\left(\overline{\lambda}_{u_l'}(w)|u_1'|\cdots|\widehat{u_l'}|\cdots|u_p'\right)\\
&+\sum_{1\le k<l\le p}(-1)^{k+l}\left(w|[u_k',u_l']|u_1'|\cdots|\widehat{u_k'}|\cdots|\widehat{u_l'}|\cdots|u_p'\right)
\end{aligned}$$

in $C_{p-1}(\mathfrak{gl}(A); H_qX(A; V))$ and in particular in the quotient $C^+_{p-1}(\mathfrak{gl}(A); H_qX(A; V))$ as asserted. $\square$

From IV.9 we conclude

$$E^2_{p,q} = H_pC^+\big(\mathfrak{gl}(A); H_qX(A; V)\big). \tag{IV.10}$$

To obtain the conclusion of the theorem we now only need:

LEMMA IV.11. *For any abelian* $\mathfrak{gl}(A)$*-module* Y *the quotient map* $C_*(\mathfrak{gl}(A); Y) \to C^+_*(\mathfrak{gl}(A); Y)$ *induces an isomorphism in homology.*

Proof. Use IV.10, taking $V = Y$. The homology map in question is the left-hand arrow in a commutative triangle

$$\begin{array}{ccc} H_n(\mathfrak{gl}(A); Y) \to E^2_{n,0} = H_nC^+(\mathfrak{gl}(A); H_0X_*(A; Y)) \\ \searrow \qquad \nearrow \\ H_nC^+(\mathfrak{gl}(A); Y) \end{array}$$

where the upper arrow is an edge homomorphism and the right-hand arrow is induced by $Y = X_0(A; Y) \to H_0X_*(A; Y)$. But since Y is an abelian $\mathfrak{gl}(A)$-module Corollary III.10 applies and shows that these other two arrows are isomorphisms. $\square$

Proof of Lemma IV.4. We show that $H_n(X_*/X^S_*) = 0$, assuming this for smaller n and all S. Note that the proof that IV.4 $\Rightarrow$ IV.5 did not "lose" any dimensions; that is, by induction we may assume that in the situation of IV.5, $H_p(X_*/\Sigma_\nu X^{S_\nu}_*) = 0$ for $p < n$.

The proof of IV.4 has much in common with the proof of II.3. We start by filtering X_*/X^S_*. If $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$ is good then let the S-badness $\beta^S(\underline{\varepsilon})$ be the number of k $(1 \le k \le n)$ such that for some $(i, j) \in S$ the sequence $\underline{\varepsilon}$ contains some path from i to j involving ε_k.

Definition IV.12. $F^S_pX_n$ is the subgroup of $X_n = X_n(A; V)$ generated by all $(v|\varepsilon_1a_1| \cdots |\varepsilon_na_n)$ with $\beta(\varepsilon_1, \dots, \varepsilon_n) = 0$ and $\beta^S(\varepsilon_1, \dots, \varepsilon_n) \le p$.

Much as before, we have $F^S_pX_n \subset F^S_{p+1}X_N$, $F^S_nX_n = X_n$, $F^S_0X_n = X^S_n$, $dF^S_pX_n \subset F^S_pX_{n-1}$. Consider the spectral sequence associated to the filtered complex $\{F^S_pX_*/F^S_0X_*\}$. Note that $E^0_{p,q} = 0$ if $p \le 0$ or $q < 0$. We must show that $E^\infty_{p,q} = 0$ for $p + q = n$.

Arguing as in the proof of II.3 we obtain an expression

$$E^0_{p,q} = \bigoplus_{\substack{\beta(\underline{\varepsilon}')=0 \\ \beta^S(\underline{\varepsilon}')=p}} \Big\{\sum_\nu X^{S_\nu}_q\Big\} \otimes \Lambda^p_{\underline{\varepsilon}'}(A), \tag{IV.13}$$

with $d^0_{p,q} = d \otimes 1$. When $q < n - 1$ the inductive hypothesis implies

$$E^1_{p,q} = H_qX_* \otimes Z^S_p(A) \qquad (q < n-1), \tag{IV.14}$$

where we have written $Z_p^S(A)$ for

$$\bigoplus_{\substack{\beta(\underline{\varepsilon}')=0 \\ \beta^S(\underline{\varepsilon}')=p}} \Lambda^p_{\underline{\varepsilon}'}(A).$$

Note that $Z_*^S(A)$ can be viewed as a chain complex, a quotient complex of $X_*(A;\mathbf{Q})/X_*^S(A;\mathbf{Q})$. (We have $Z_n^S(A) \cong F_n^SX_n(A;\mathbf{Q})/F_{n-1}^SX_n(A;\mathbf{Q})$.)

CLAIM IV.15. *For $q < n-1$ the differential $d^1_{p,q}$ is given (in terms of IV.14) by $(-1)^q d^1_{p,q} = 1 \otimes d$, where d is the differential in $Z_*^S(A)$.*

Proof. This follows from the same computation that proved IV.9. □

We now have

(IV.16) $$E^2_{p,q} = H_qX_*(A;V) \otimes H_pZ_*^S(A), \qquad q < n-1.$$

But $H_pZ_*^S(A) = 0$, because this same spectral sequence in the case $V = \mathbf{Q}$ must on the one hand satisfy IV.14 and IV.16 for *all* q (by Corollary III.12) and on the other hand must have $E^\infty = 0$ (again by III.12). Thus (returning to the case of general V) we have $E^2_{p,q} = 0$; hence $E^\infty_{p,q} = 0$, for $q < n-1$.

It remains to prove that $E^\infty_{1,n-1} = 0$, i.e. that every n-dimensional cycle in $F_1^SX_*/F_0^SX_*$ is a boundary in $X_*/F_0^SX_*$. This can in fact be done directly for all n, without using induction. Any cycle in $F_1^SX_n/F_0^SX_n$ is by IV.13 a sum of cycles each represented by an element $(w|\varepsilon a) \in X_n$ such that $\varepsilon = (i,j)$, $i \neq j$, $a \in A$, and $w \in \sum_\nu X_{n-1}^{S_\nu}$ is a cycle. For any such ε, a, and w, choose $l \in \mathbf{N}$ having nothing to do with ε, w, or any S_ν. The element

$$(w|(i,l)a|(l,j)1) \in X_{n+1}$$

has boundary

$$(dw|(i,l)a|(l,j)1) + (-1)^{n-1}(w|(i,j)a) + \text{terms in } F_0^SX_n.$$

Since $dw = 0$, $(w|\varepsilon a)$ is a boundary in X_*/X_*^S. □

V. The homology of the adjoint action

Let A be a ring (associative, with unit). Recall ([C-E], p. 175) that if A is torsion-free as an additive group and B is an A-bimodule then the following chain complex $C_*(A;B)$ computes the Hochschild homology groups $H_*(A;B)$:

(V.1) $$C_n(A;B) = B \otimes A^{\otimes n}, \qquad n \geq 0;$$

$$\begin{aligned} d(b \otimes a_1 \otimes \cdots \otimes a_n) &= ba_1 \otimes a_2 \otimes \cdots \otimes a_n \\ &\quad + \sum_{i=1}^{n-1} (-1)^i b \otimes a_1 \otimes \cdots \otimes a_ia_{i+1} \otimes \cdots \otimes a_n \\ &\quad + (-1)^n a_nb \otimes a_1 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

Hochschild homology is *Morita-invariant*; that is, the homology groups remain the same if A and B are replaced respectively by the $k \times k$ matrix ring $M_k(A)$ and the bimodule $M_k(B)$. In fact, $B \to M_k(B)$ is an equivalence of categories from A-bimodules to $M_k(A)$-bimodules, under which the two abelian-group-valued functors $H_0(A; -)$ and $H_0(M_k(A); -)$ correspond, which implies the same for their derived functors $H_n(A; -)$ and $H_n(M_k(A); -)$. Moreover the "inclusion"

$$C_*(A; B) \hookrightarrow C_*(M_k(A); M_k(B))$$

given by viewing an element of A or B as a 1×1 matrix is in fact a quasi-isomorphism. By a direct limit argument the same is true "when $k = \infty$": if $M(A)$ is the (non-unital) ring $\bigcup_k M_k(A)$ then the complex $C_*(M(A); M(B))$ (defined by V.1) has the same homology as the subcomplex $C_*(A; B)$.

Now let A be any associative ring with unit and let B be an $A \otimes \mathbf{Q}$-bimodule. We want to compute the homology groups $H_*(\mathrm{GL}(A); M(B))$, where $\mathrm{GL}(A)$ acts on $M(B)$ by conjugation. We will do this by comparing $M(B)$ with the chain complex $C_*(M(A \otimes \mathbf{Q}); M(B))$, which we abbreviate $C_*(B)$.

The complex $C_*(B)$ has an action of $\mathrm{GL}(A \otimes \mathbf{Q})$ and hence of $\mathrm{GL}(A)$; the matrix $U \in \mathrm{GL}(A \otimes \mathbf{Q})$ acts on $M(B) \otimes M(A \otimes \mathbf{Q})^{\otimes n}$ by

$$N \otimes M_1 \otimes \cdots \otimes M_n \mapsto U^{-1}NU \otimes U^{-1}M_1U \otimes \cdots \otimes U^{-1}M_nU.$$

Whenever a group G acts on a chain complex K_* the bar construction gives a double complex $C_*(G; K_*)$, whose (total) homology we will call the hyperhomology $\mathbf{H}_n(G; K_*)$. Of course if K_* is concentrated in dimension zero then $\mathbf{H}_n(G; K_*) = H_n(G; K_0)$.

PROPOSITION V.2.

$$\mathbf{H}_n(\mathrm{GL}(A); C_*(B)) \cong \bigoplus_{p+q=n} H_p(\mathrm{GL}(A)) \otimes H_q(A \otimes \mathbf{Q}; B).$$

Proof. In general an action of G on K_* determines an action of G on $H_q(K_*)$ and a spectral sequence

$$E^2_{p,q} = H_p\big(G; H_q(K_*)\big) \Rightarrow \mathbf{H}_{p+q}(G; K_*).$$

In the case at hand $H_q K_* = H_q(A \otimes \mathbf{Q}; B)$ and we must prove that

(i) G acts trivially on $H_q K_*$;

(ii) $E^2 = E^\infty$;

(iii) $\mathbf{H}_n(G; K_*)$ splits as $\oplus_{p+q=n} E^\infty_{p,q}$.

If G acts trivially on K_* then (i)–(iii) always hold. In our case the subgroup $\mathrm{GL}_k(A) \subset \mathrm{GL}(A)$ acts trivially on the subcomplex $C_*(M'_k(A \otimes \mathbf{Q}); M'_k(B)) \subset C_*(B)$, where

$$M'_k(-) = \left\{ M \in M(-) | M_{ij} = 0 \text{ if } i \le k \text{ or } j \le k \right\}.$$

Therefore (i)–(iii) hold for $\mathrm{GL}_k(A)$ acting on this subcomplex. But the subcomplex has the same homology as all of $C_*(B)$ by Morita invariance; so a comparison argument proves (i)–(iii) for $\mathrm{GL}_k(A)$ acting on $C_*(B)$. Now as $k \to \infty$ a direct limit argument completes the proof. □

Here is our main result.

THEOREM V.3. *Let A be a ring and B an $A \otimes \mathbf{Q}$-bimodule. The inclusion*

$$M(B) = C_0(M(A \otimes \mathbf{Q}); M(B)) \hookrightarrow C_*(M(A \otimes \mathbf{Q}); M(B))$$

induces an isomorphism

$$H_n(\mathrm{GL}(A); M(B)) \cong \mathbf{H}_n(\mathrm{GL}(A); C_*(M(A \otimes \mathbf{Q}); M(B))).$$

Therefore

$$H_n(\mathrm{GL}(A); M(B)) \cong \bigoplus_{p+q=n} H_p(\mathrm{GL}(A)) \otimes H_q(A \otimes \mathbf{Q}; B).$$

Moreover the projection

$$H_n(\mathrm{GL}(A); M(B)) \to H_n(\mathrm{GL}(A)) \otimes H_0(A \otimes \mathbf{Q}; B)$$

$$= H_n(\mathrm{GL}(A); H_0(A \otimes \mathbf{Q}; B))$$

is induced by the trace

$$M(B) \to B \to B/[B, A \otimes \mathbf{Q}] = H_0(A \otimes \mathbf{Q}; B).$$

Proof. It suffices to prove the first statement. The second then follows from V.2 and the third is clear.

We first reduce to the case of a free bimodule. This is easy: Any bimodule B admits a surjection $F \to B$ from a free bimodule. Let R be the kernel. Each $C_n(M(A \otimes \mathbf{Q}); M(-))$ is an exact functor, so a five-lemma argument applies; if the conclusion holds for F and holds through dimension $n-1$ for R, then it holds through dimension n for B.

It is sufficient to consider the free bimodule of rank one $F_A = A \otimes \mathbf{Q} \otimes A$. In this case we have

$$H_0(A \otimes \mathbf{Q}; F_A) \cong A \otimes \mathbf{Q},$$

$$H_n(A \otimes \mathbf{Q}; F_A) = 0, \qquad n > 0.$$

Therefore the problem is to show that the trace

$$\mathrm{tr}\colon M(F_A) \to A \otimes \mathbf{Q}$$

induces an isomorphism in $H_*(\mathrm{GL}(A); -)$ or equivalently that $H_*(\mathrm{GL}(A); \ker(\mathrm{tr})) = 0$. By Proposition I.3 it will be enough if $H_*(X(A); \ker(\mathrm{tr})) = 0$.

At this point we observe that all these $\mathrm{GL}(A)$-modules are $(\mathrm{GL}, \mathfrak{gl})$-modules. In fact, for an arbitrary $A \otimes \mathbf{Q}$-bimodule B the whole complex $C_*(B)$ has a $\mathfrak{gl}(A \otimes \mathbf{Q})$-action given by

$$[N \otimes M_1 \otimes \cdots \otimes M_n, u] = (Nu - uN) \otimes M_1 \otimes \cdots \otimes M_n + \sum_{i=1}^{n} N \otimes M_1 \otimes \cdots \otimes (M_i u - uM_i) \otimes \cdots \otimes M_n$$

and it is easy to check that for each n, $C_n(B)$ satisfies the conditions of Definition III.3. Moreover any trivial $\mathrm{GL}(A)$-module becomes a $(\mathrm{GL}, \mathfrak{gl})$-module when given the trivial $\mathfrak{gl}(A \otimes \mathbf{Q})$-action. Thus tr is a map of $(\mathrm{GL}, \mathfrak{gl})$-modules and its kernel is a $(\mathrm{GL}, \mathfrak{gl})$-module. Theorem V.3 will follow from:

Lemma V.4. *If $A = A \otimes \mathbf{Q}$ is a ring and* tr: $M(A \otimes A) \to A$ *is the trace then $H_*(\mathfrak{gl}(A); \ker(\mathrm{tr})) = 0$.*

Proof that V.4 *implies the theorem.* Consider the spectral sequence of II.3, with $V = \ker(\mathrm{tr})$. By V.4 we have $E^\infty = 0$. Assuming for the moment that the action of $\mathfrak{gl}(A)$ on $H_q X_*(A; \ker(\mathrm{tr}))$ is trivial, we have

$$E^2_{p,q} \cong H_p(\mathfrak{gl}(A); \mathbf{Q}) \otimes H_q X_*(A; \ker(\mathrm{tr})),$$

so that $E^\infty = 0 \Rightarrow E^2 = 0$. But this implies $H_q X_*(A; \ker(\mathrm{tr})) = 0$, which with III.5 gives what we want.

To see that the action is trivial note that any cycle in $X_q(A; \ker(\mathrm{tr}))$ is "supported" on a finite subset of $\mathbf{N}$. Any element of $\mathfrak{gl}(A)$ which is "supported" on a disjoint set must act trivially on the cycle and hence on its class. But the action of $\mathfrak{gl}(A)$ on $H_q X_*(A; \ker(\mathrm{tr}))$ is abelian, and modulo $[\mathfrak{gl}(A), \mathfrak{gl}(A)]$ the "support" of an element of $\mathfrak{gl}(A)$ can be shifted off any finite subset of $\mathbf{N}$ (in fact onto any one-element set—see Remark II.4). $\square$

Proof of V.4. To begin let B be any A-bimodule. The Koszul complex $C_*(\mathfrak{gl}(A); M(B))$ has a $\mathfrak{gl}(A)$-action and in particular a $\mathfrak{gl}(\mathbf{Q})$-action. As in [L-Q, §6] we can replace the complex by its complex of $\mathfrak{gl}(\mathbf{Q})$-coinvariants $C_*(\mathfrak{gl}(A); M(B))_{\mathfrak{gl}(\mathbf{Q})}$ without changing its homology. We omit the details.

Claim V.5. *$C_*(\mathfrak{gl}(A); M(B))_{\mathfrak{gl}(\mathbf{Q})}$ is isomorphic as a complex to the tensor product*

$$C_*(A; B) \otimes C_*(\mathfrak{gl}(A); \mathbf{Q})_{\mathfrak{gl}(\mathbf{Q})}.$$

Proof of claim. We analyze the coinvariants as in [L-Q]. Classical invariant theory gives the following description of $(\mathfrak{gl}(\mathbf{Q})^{\otimes n})_{\mathfrak{gl}(\mathbf{Q})}$. Let π be any permutation of $\{1, \dots, n\}$. Define a linear map $\mathfrak{gl}(\mathbf{Q})^{\otimes n} \to \mathbf{Q}$ by

$$u_1 \otimes \cdots \otimes u_n \mapsto \prod_i \mathrm{Trace}(u_i u_{\pi(i)} \cdots u_{\pi^{a_i - 1}(i)})$$

where i runs through a system of representatives for the orbits of π acting on $\{1, \ldots, n\}$ and a_i is the cardinality of the orbit of i. (The choice of representatives is immaterial because $\mathrm{Trace}(uv - vu) = 0$.) These functionals form a basis for the space of all functionals that factor through the coinvariants, and so they give an isomorphism from $(\mathfrak{gl}(\mathbf{Q})^{\otimes n})_{\mathfrak{gl}(\mathbf{Q})}$ to a rational vector space of dimension $n!$

Applying this to $\mathfrak{gl}(A)^{\otimes n} \cong \mathfrak{gl}(\mathbf{Q})^{\otimes n} \otimes A^{\otimes n}$ we obtain as in [L-Q],

$$\begin{aligned}\left(\mathfrak{gl}(A)^{\otimes n}\right)_{\mathfrak{gl}(\mathbf{Q})} &\cong \left(\mathfrak{gl}(\mathbf{Q})^{\otimes n}\right)_{\mathfrak{gl}(\mathbf{Q})} \otimes A^{\otimes n} \\ &\cong \bigoplus_{\pi \in \Sigma_n} A^{\otimes n};\end{aligned}$$

and antisymmetrizing, we have

$$C_n(\mathfrak{gl}(A); \mathbf{Q})_{\mathfrak{gl}(\mathbf{Q})} \cong \bigoplus_{\pi} \Lambda^n_\pi(A),$$

where π now ranges over a system of representatives for the conjugacy classes of Σ_n, and $\Lambda^n_\pi(A)$ is the partial antisymmetrization of $A^{\otimes n}$ with respect to the centralizer of π. Explicitly, for any $\pi \in \Sigma_n$ the projection of $C_n(\mathfrak{gl}(A); \mathbf{Q})$ to $\Lambda^n_\pi(A)$ is given by

$$(1|u_1| \cdots |u_n) \to \bigotimes_i \mathrm{trace}\left(u_i \otimes u_{\pi(i)} \otimes \cdots \otimes u_{\pi^{a_i-1}(i)}\right)$$

where the "trace" of a tensor product of matrices is defined by

$$\mathrm{trace}(u^1 \otimes \cdots \otimes u^n) = \sum u^1_{i_1 i_2} \otimes u^2_{i_2 i_3} \otimes \cdots \otimes u^n_{i_n i_1} \in A^{\otimes n}.$$

The same approach applied to $C_*(\mathfrak{gl}(A); M(B))$ yields

$$\left(M(B) \otimes \mathfrak{gl}(A)^{\otimes n}\right)_{\mathfrak{gl}(\mathbf{Q})} \cong \bigoplus_{\pi \in \mathrm{Aut}\{0, \ldots, n\}} B \otimes A^{\otimes n};$$

and after antisymmetrizing with respect to

$$\Sigma_n = \mathrm{Aut}\{1, \ldots, n\} \subset \mathrm{Aut}\{0, \ldots, n\}$$

we have

$$C_n(\mathfrak{gl}(A); M(B))_{\mathfrak{gl}(\mathbf{Q})} \cong \bigoplus_{\pi} B \otimes \Lambda^n_\pi A$$

where π ranges over a system of representatives for the conjugation action of Σ_n on $\mathrm{Aut}\{0, \ldots, n\}$ and Λ^n_π antisymmetrizes with respect to the centralizer of π in $\mathrm{Aut}(\{0, \ldots, n\})$. We may as well choose each π in such a way that the orbit of 0 looks like $0 \to 1 \to 2 \to \cdots \to p \to 0$ for some $p \geq 0$. Then the expression becomes

$$C_n(\mathfrak{gl}(A); M(B))_{\mathfrak{gl}(\mathbf{Q})} \cong \bigoplus_{0 \leq p \leq n} (B \otimes A^{\otimes p}) \otimes \bigoplus_{\pi'} \Lambda^{n-p}_{\pi'}(A)$$

where π' ranges through representatives for conjugacy classes in $\mathrm{Aut}(p+1,\dots,n)\cong\Sigma_{n-p}$. In other terms,

$$C_n(\mathfrak{gl}(A);M(B))_{\mathfrak{gl}(\mathbf{Q})}\cong\bigoplus_{p+q=n}C_p(A;B)\otimes C_q(\mathfrak{gl}(A);\mathbf{Q})_{\mathfrak{gl}(\mathbf{Q})}.$$

This is the isomorphism which Claim V.5 refers to. We still have to show that it is a chain map.

The inverse isomorphism is given by

$$C_p(A;B)\otimes C_q(\mathfrak{gl}(A);\mathbf{Q})_{\mathfrak{gl}(\mathbf{Q})}\to C_{p+q}(\mathfrak{gl}(A);M(B))_{\mathfrak{gl}(\mathbf{Q})}$$

$$(b\otimes a_1\otimes\cdots\otimes a_p)\otimes\{(1|u_1|\cdots|u_q)\}\mapsto\{(\varepsilon_0 b|\varepsilon_1 a_1|\cdots|\varepsilon_p a_p|u_1|\cdots|u_q)\}$$

where the edges $\varepsilon_k=(i_k,j_k)$ are chosen to form a "non-self-intersecting loop" disjoint from the "support" of the matrices u_k; i.e., $j_0=i_1$, $j_1=i_2,\dots,$ $j_p=i_0$ are distinct natural numbers such that the corresponding rows and columns of the matrices u_k are all zero. (It is straightforward to check that this is well-defined and is a right inverse, hence an inverse, to the isomorphism.) Moreover, this inverse is easily seen to be a chain map. This proves the claim. □

The claim implies the Lie analogue of Theorem V.3. That is,

$$H_n(\mathfrak{gl}(A);M(B))\cong\bigoplus_{p+q=n}H_p(A;B)\otimes H_q(\mathfrak{gl}(A);\mathbf{Q}).$$

Moreover the projection

$$\begin{aligned}H_n(\mathfrak{gl}(A);M(B))&\to H_0(A;B)\otimes H_n(\mathfrak{gl}(A);\mathbf{Q})\\&\cong H_n(\mathfrak{gl}(A);H_0(A;B))\end{aligned}$$

is clearly induced by the trace $M(B)\to H_0(A;B)$. Now specializing to the case $B=F_A$ we have that this projection is an isomorphism, which proves Lemma V.4. □

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