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CYCLIC HOMOLOGY, DERIVATIONS, AND THE FREE LOOPSPACE†

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INTRODUCTION

CONNES has defined "cyclic homology groups" $HC_*(A)$ for any associative algebra A over a field K of characteristic zero [3] [4]. His construction has been studied and generalized by a number of authors [1] [5] [10] [11] [15]. Although Connes' original interest was in the case when A is a C^* algebra it is clear by now that cyclic homology is destined to be an important tool in many kinds of ring theory.

Our main purpose here is to introduce into cyclic homology theory a principle analogous to the homotopy-invariance of de Rham cohomology. To explain the result and the analogy we must recall two elements of the theory. A good reference for both is [11]. All rings and algebras considered here will be with unit.

(1) The groups $HC_*(A)$ are related to the Hochschild homology groups $H_*(A)$ by a natural long exact sequence

$$\ldots \to H_p(A) \xrightarrow{I} HC_p(A) \xrightarrow{S} HC_{p-2}(A) \xrightarrow{B} H_{p-1}(A) \to \ldots$$

(2) If V is a smooth affine algebraic variety over a field K of characteristic zero and $A = K[V]$ is its coordinate ring then, denoting by $\Omega^p(V)$ the module of "algebraic p-forms" (i.e. the p-th exterior power over A of the module of differentials $\Omega_{A/K}$), we have

(a) $H_p(A) \cong \Omega^p(V)$

(b) $B \circ I$: $H_p(A) \to H_{p+1}(A)$ corresponds to the exterior derivative

$$d\colon \Omega^p(V) \to \Omega^{p+1}(V)$$

(c) $HC_p(A) \cong \Omega^p(V)/d\Omega^{p-1}(V) \oplus H^{p-2}_{dR}(V) \oplus H^{p-4}_{dR}(V) \oplus \ldots$

(d) $\overset{\lim}{\underset{k}{\leftarrow}} HC_{p+2k}(A) \cong \bigoplus_{(i \equiv p \bmod 2)} H^i_{dR}(V).$

(The inverse limit is with respect to the map S of (1).)

Now returning to the general situation any derivation D of A (over K) determines endomorphisms of $H_*(A)$ and $HC_*(A)$ (see §II.4 below) and in fact of the whole sequence (1). We prove three slightly different statements (Corollaries II.4.3, II.4.4, and II.4.6 below), each one of which reduces, if $A = K[V]$ as above, to the fact that D acts trivially on $H^*_{dR}(V)$. These statements say respectively that D acts trivially on $\mathrm{Ker}(B \circ I)/\mathrm{Im}(B \circ I)$, on $HC^{per}_*(A)$ (which we define more or less as $\overset{\lim}{\underset{k}{\leftarrow}} HC_{*+2k}(A)$—actually there is a $\lim^1$ term as well), and on the image of S: $HC_{*+2}(A) \to HC_*(A)$.

We are really less interested in algebras than in chain algebras (i.e. differential graded algebras (A, d) with $\deg(d) = -1$). We therefore develop a theory of "hyperhomology"—both cyclic ($\mathbf{HC}_*(A, d)$) and Hochschild ($\mathbf{H}_*(A, d)$). (This has also been done in [1].) We extend the "homotopy-invariance" principle to this differential graded setting (Corollaries III.4.2–III.4.4).

The homotopy-invariance implies a "Poincaré Lemma" for graded algebras (proof of Claim 1 in proof of Proposition II.5.2), and this in turn implies our main result (Theorem

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IV.2.1): The groups $\mathbf{HC}_*^{per}(A,d)$ (cyclic-hyperhomology-made-periodic) depend only on A_0/dA_1 if the ground ring K contains the rational numbers.

We have good reason to believe that this result will be useful in obtaining a computation, in terms of cyclic (hyper) homology, of relative rational algebraic K-theory $K_*(f)\otimes Q$ for any one-connected map of simplicial rings $f\colon R\to S$. (In the special case $S=Z$ such computations were made by Hsiang and Staffeldt ([8] [9]) except that their answers were not in terms of cyclic homology since the latter had not been invented yet. The main result of [1] is a reformulation of their work in the light of Connes' theory and the work of Loday and Quillen. [15] is also relevant here; it represents a first step beyond the case $S=Z$.)

As a more immediate application we prove a theorem in topology: Localized rational $SO(2)$-equivariant cohomology of the free loop space of X depends only on $\pi_1(X)$. The proof of this requires, besides our main result, the fact that the equivariant homology of the free loopspace of a path-connected space X:

$$H_*^{SO(2)}(\Lambda X) \underset{\text{def}}{=} H_*(\Lambda X \underset{SO(2)}{\times} ESO(2))$$

can be identified with the cyclic hyperhomology of the singular chains on the (Moore) loopspace of X:

$$\mathbf{HC}_* S_*(MX).$$

Such identifications have also been obtained by Burghelea, Dwyer, and others.

The paper is organized as follows.

In §I we recall Connes' notion of "cyclic object in a category" (a cyclic object is a simplicial object with some additional structure) and establish notation for dealing with such objects.

In §II we develop from scratch the theory of cyclic homology for flat associative algebras over a commutative ring K and prove the "homotopy invariance". §II.1 and §II.2 contain nothing new. We follow the treatment in [11] pretty closely except that we take a more general point of view as in [5]: we define cyclic homology as a functor of cyclic K-modules (= cyclic objects in the category of K-modules) and in particular define the cyclic homology of a flat K-algebra A to be that of a certain cyclic K-module ZA. In §II.3 we make the fairly obvious definition of the periodic theory; by taking an inverse limit on the chain level we invert the "periodicity map"

$$HC_{*+2}(A)\xrightarrow{S} HC_*(A)$$

to get groups $HC_{n*}^{per}(A)$ depending only on n mod 2. It is not until §II.4 that a new idea appears: we make a derivation of A act on everything in sight and prove the three versions of homotopy-invariance mentioned above. In §II.5 this leads to the theorem:

(3) $A\to A/I$ induces isomorphisms $HC_*^{per}(A)\to HC_*^{per}(A/I)$ if $I\subset A$ is a nilpotent ideal and K is a field of characteristic zero.

In §III we extend the definitions and results of §II to the differential graded setting. That is, what we have already done for flat algebras (and more generally cyclic modules) we now do for flat chain algebras (and more generally cyclic chain complexes of modules). In particular we prove an analogue of (3) in which the role of "nilpotent ideal" is played by a homogeneous differential ideal which has no nontrivial elements of degree zero.

In §IV we use free resolutions of chain algebras to extend our results in routine ways. In particular we obtain the main result: $\mathbf{HC}_*^{per}(A,d)$ depends only on A_0/dA_1 if $K\supset Q$.

§V describes the application to the free loopspace. Thus it is mainly concerned with proving the isomorphism (valid for any pointed path-connected space X and any coefficient ring K)

$$H_*^{SO(2)}(\Lambda X;K)\cong \mathbf{HC}_* S_*(MX;K).$$

Finally, here are two remarks on notation.

(i) In our construction of a chain complex for cyclic homology (in §II.2) we call t_n what is called $(-1)^{n+1}t_n$ in [11].

(ii) We think of a cyclic object as a *contravariant* functor with domain Λ, since a simplicial object is a contravariant functor with domain $\Delta \subset \Lambda$. This differs from the convention used in [5], but in view of the isomorphism $\Lambda \cong \Lambda^0$ (Lemma 1 of [5]) it should cause no alarm.

§I. CYCLIC OBJECTS IN A CATEGORY

Following Connes [5] we define the notion of *cyclic object* in an arbitrary category C. A cyclic object is a simplicial object together with some extra structure. Namely, in addition to C-objects X_n $(n \geq 0)$ and face and degeneracy maps

$$\partial_i: X_n \to X_{n-1}, 0 \leq i \leq n, n \geq 1$$
$$s_i: X_n \to X_{n+1}, 0 \leq i \leq n, n \geq 0$$

satisfying the usual identities, there is an action of a cyclic group of order $n+1$ on X_n for each $n \geq 0$. Denote a preferred generator of this group by t_{n+1}. Then in order to define a cyclic object the group actions are required to satisfy

(I.1) $$\partial_i t_{n+1} = \begin{cases} t_n \partial_{i-1} & 0 < i \leq n \\ \partial_n & i = 0 \end{cases}$$

(I.2) $$s_i t_{n+1} = \begin{cases} t_{n+2} s_{i-1} & 0 < i \leq n \\ t_{n+2}^2 s_n & i = 0 \end{cases}$$

and of course

(I.3) $t_{n+1}^{n+1} = 1.$

Of course one can also express this by saying that a cyclic object is a contravariant functor from Λ to C, where Λ is a certain category that contains the category Δ of simplicial theory (See Connes [5] for an explicit definition of Λ.)

§II. CYCLIC HOMOLOGY AND DERIVATIONS

1. *The Cyclic Module Associated to an Algebra*

This is the main source of examples of cyclic objects. Let K be a commutative ring with identity. By an algebra over K we will always mean an associative algebra with unit, and until further notice we assume that every algebra is a flat K-module. To any K-algebra A we associate a cyclic K-module, in other words a cyclic object in the category of K-modules. We denote it by ZA (in Connes' notation $ZA = A\natural$). Namely for each $n \geq 0$ define

$$Z_n A = A^{\otimes n+1},$$

the tensor product over K of $n+1$ copies of A. Define face maps, degeneracy maps, and cyclic group actions by

(II.1.1) $$\partial_i(a_0, \ldots, a_n) = \begin{cases} (a_0, \ldots, a_i a_{i+1}, \ldots, a_n), & 0 \leq i < n \\ (a_n a_0, a_1, \ldots, a_{n-1}), & i = n \end{cases}$$

(II.1.2) $s_i(a_0, \ldots, a_n) = (a_0, \ldots, a_i, 1, a_{i+1}, \ldots, a_n), 0 \leq i \leq n$

(II.1.3) $t_{n+1}(a_0, \ldots, a_n) = (a_n, a_0, \ldots, a_{n-1}).$

It is easy to verify the necessary identities (simplicial identities and I.1 – I.3) that make this a cyclic K-module.

2. *Cyclic Homology*

Now let $X = \{X_n, \partial_i, s_i, t_{n+1}\}$ be a cyclic object in an abelian category $\mathbf{C}$ (for example, X

$= ZA$, $\mathbf{C} =$ K-modules). We associate to it a two-quadrant double complex of $\mathbf{C}$-objects

$$\begin{tikzcd}
\leftarrow C_{2p,j}(X) \arrow[d, "b"] & C_{2p+1,j}(X) \arrow[l, "\varepsilon"'] \arrow[d, "-b'"] & C_{2p+2,j}(X) \arrow[l, "N"'] \arrow[d, "b"] \leftarrow \\
\leftarrow C_{2p,j-1}(X) & C_{2p+1,j-1}(X) \arrow[l, "\varepsilon"'] & C_{2p+2,j-1}(X) \arrow[l, "N"'] \leftarrow
\end{tikzcd}$$

as follows:

$$C_{ij}(X) = X_j, \qquad \text{for } i \in \mathbf{Z} \text{ and } j \geq 0.$$

The maps b and b' from X_j to X_{j-1} are defined using the face maps:

$$b = \sum_{k=0}^{j} (-1)^k \partial_k$$

$$b' = \sum_{k=0}^{j-1} (-1)^k \partial_k.$$

The maps ε and N from X_j to X_j are defined using the group action:

$$\varepsilon = 1 - (-1)^j t_{j+1}$$

$$N = \sum_{k=1}^{j+1} ((-1)^j t_{j+1})^k.$$

One can verify, as in ([11], Lemma I.1) that $C_{**}(X)$ is indeed a double complex, i.e.,

$$b^2 = 0 = b'^2$$
$$\varepsilon N = 0 = N\varepsilon$$
$$b\varepsilon - \varepsilon b' = 0$$
$$Nb - b'N = 0.$$

The complex which occurs as every even-numbered column of $C_{**}(X)$ is called $C^h_*(X)$, the *Hochschild complex* of X:

$$X_0 \xleftarrow{b} X_1 \xleftarrow{b} X_2 \leftarrow \cdots.$$

Its homology is called $H_*(X)$, the *Hochschild homology* of X. In the case of an algebra A we write $H_*(A)$ for $H_*(ZA)$; these are indeed isomorphic to the Hochschild homology groups of A ([6], Ch. IX, §6.)

The other complex

$$X_0 \xleftarrow{b'} X_1 \xleftarrow{b'} X_2 \leftarrow \cdots$$

is called $C^a_*(X)$; its homology is zero because if we write

(II.2.1) $\quad u = (-1)^j s_j \colon X_j \to X_{j+1}$

then $b'u + ub' = 1$.

From the first quadrant of $C_{**}(X)$ we can make a chain complex $C_*(X)$ in the usual way:

$$C_n(X) = \bigoplus_{i \geq 0} C_{i,n-i}(X)$$

$$\text{boundary map} = \begin{cases} b + N & \text{in } C_{2p,j}(X) \\ -b' + \varepsilon & \text{in } C_{2p+1,j}(X) \end{cases}.$$

The homology of $C_*(X)$ is by definition $HC_*(X)$, the *cyclic homology* of X. In the case of an algebra we write $HC_*(A)$ for $HC_*(ZA)$.

The double complex $C_{**}(X)$ has a periodicity

$$C_{i,j}(X) \underset{\cong}{\overset{s}{\to}} C_{i-2,j}(X),$$

and this gives a short exact sequence of complexes

$$0 \to \ker(s) \to C_*(X) \xrightarrow{s} C_{*-2}(X) \to 0.$$

On the other hand there is a short exact sequence

$$0 \to C^h_*(X) \to \ker(s) \to C^a_{*-1}(X) \to 0.$$

since $\ker(s)$ is made out of the 0-th and 1-st columns of $C_{**}(X)$. This yields the long exact sequence of Connes:

$$\text{(II.2.2)} \quad \to H_*(X) \to HC_*(X) \xrightarrow{s} HC_{*-2}(X) \to H_{*-1}(X) \to \dots$$

By composing two maps in this sequence we get a map

$$H_*(X) \xrightarrow{B} H_{*+1}(X)$$

satisfying $B \circ B = 0$; the complex $(H_*(X), B)$ is the *de Rham complex* and its cohomology is the *de Rham homology* of X, $H^{dR}_*(X)$. We write $H^{dR}_*(A)$ for $H^{dR}_*(ZA)$. (See [11], Proposition 2.2, for an explanation of this terminology.)

A slightly different approach to the relationship between $H_*(X)$ and $HC_*(X)$ is to filter $C_*(X)$ by subcomplexes

$$\text{(II.2.3)} \quad \bigoplus_{0 \le i \le 2p} C_{i,*-i}(X), \qquad p \ge 0.$$

This yields a spectral sequence

$$\text{(II.2.4)} \quad E^1_{p,q} = \begin{Bmatrix} H_{q-p}(X), q \ge p \ge 0 \\ 0 \text{ otherwise} \end{Bmatrix} \Rightarrow HC_*(X).$$

The differential $d^1_{p,q}$: $H_{q-p}(X) \to H_{q-p+1}(X)$ is the map B above.

It is sometimes convenient to use another double complex $B_{**}(X)$ instead of $C_{**}(X)$. Roughly speaking one just eliminates the acyclic odd-numbered columns from $C_{**}(X)$. Let

$$B_{p,q}(X) = C_{2p,q-p}(X) = \begin{cases} X_{q-p}, q \ge p. \\ 0, q < p \end{cases}.$$

Define differentials by letting b: $B_{p,q}(X) \to B_{p,q-1}(X)$ be the map b already defined, and defining B: $B_{p,q}(X) \to B_{p-1,q}(X)$ to be the composite

$$\begin{tikzcd} C_{2p-2,q-p+1}(X) & C_{2p-1,q-p+1}(X) \arrow[l, "\varepsilon"'] \\ & C_{2p-1,q-p}(X) \arrow[u, "u"] & C_{2p,q-p}(X). \arrow[l, "N"'] \end{tikzcd}$$

(Here u is the chain homotopy of (II.2.1).) Then $B_{**}(X)$ is a double complex, i.e.

$$\begin{aligned} b^2 &= 0 \quad \text{(we already know this)} \\ B^2 &= 0 \quad (\text{since } N\varepsilon = 0) \\ bB + Bb &= b\varepsilon uN + \varepsilon uNb \\ &= \varepsilon(b'u + ub')N = \varepsilon N = 0. \end{aligned}$$

Collect the first quadrant of $B_{**}(X)$ into a single complex:

$$B_*(X) = \bigoplus_{\substack{p+q=*\\ p\geq 0}} B_{p,q}(X)$$

with differential $b+B$.

PROPOSITION II.2.5. $H_*(B_*(X)) \cong HC_*(X)$

Proof. Define a chain map $B_*(X) \xrightarrow{\phi} C_*(X)$ by

$$\begin{aligned} B_{p,q}(X) &\to C_{2p,q-p}(X) \oplus C_{2p-1,q-p+1}(X) \\ x &\to (x, uNx). \end{aligned}$$

(Exercise: It is a chain map.) Filter $B_*(X)$ by columns of $B_{**}(X)$ and filter $C_*(X)$ as in II.2.3. Then ϕ becomes a map of filtered complexes. It induces an isomorphism of E^1-terms, so also an isomorphism $H_*(B_*(X)) \to H_*(C_*(X))$. □

Note. Our two uses of "B" to denote homomorphisms are now seen to be compatible: The map $B: X_j \to X_{j+1}$ which is the "horizontal" differential in $B_{**}(X)$ induces the d^1 differential in the spectral sequence for $B_*(X)$, which can be identified with the map B in the de Rham complex by the remark following II.2.4.

The complex $B_{**}(X)$ can be replaced by a "reduced" complex $\bar{B}_{**}(X)$. Set

$$\bar{X}_n = X_n / \sum_{i=0}^{n-1} s_i X_{n-1}$$

$$\bar{B}_{p,q} = \bar{X}_{q-p}.$$

One checks that b and B descend to maps of the quotients

$$\bar{B}_{p,q}(X) \xrightarrow{b} \bar{B}_{p,q-1}(X)$$

$$\bar{B}_{p,q}(X) \xrightarrow{B} \bar{B}_{p-1,q}(X)$$

so that $\bar{B}_{**}(X)$ is a double complex. Set

$$\bar{B}_*(X) = \Bigg(\bigoplus_{\substack{p+q=*\\ p\geq 0}}\Bigg) \bar{B}_{p,q}(X).$$

Thus $\bar{B}_*(X)$ can be viewed as a quotient complex of $B_*(X)$.

PROPOSITION II.2.6. $H_*(\bar{B}_*(X)) \cong HC_*(X)$.

Proof. Filter both $B_*(X)$ and $\bar{B}_*(X)$ according to columns of the double complexes. The quotient map $B_*(X) \to \bar{B}_*(X)$ induces a map of spectral sequences which is an E^1-isomorphism, that is, the quotient map from the Hochschild complex to the "reduced Hochschild complex" induces an isomorphism in homology. □

Finally we make the obvious remark:

PROPOSITION II.2.7. *A short exact sequence of cyclic objects*

$$0 \to X' \to X \to X'' \to 0$$

gives rise to long exact sequences

$$\ldots \to H_n(X') \to H_n(X) \to H_n(X'') \to H_{n-1}(X') \to \ldots$$

and

$$\dots \to HC_n(X') \to HC_n(X) \to HC_n(X'') \to HC_{n-1}(X') \to \dots.$$

Proof. There are short exact sequences of complexes

$$0 \to C^h_*(X') \to C^h_*(X) \to C^h_*(X'') \to 0$$
$$0 \to C_*(X') \to C_*(X) \to C_*(X'') \to 0. \qquad \square$$

3. *Periodic Homology*

By "inverting the map s" we will define a 2-periodic variant of $HC_*(X)$ provided the abelian category **C** admits exact infinite products. Let $\bar{B}^{per}_*(X)$ be the complex obtained from the whole of $\bar{B}_{**}(X)$ rather than just the first quadrant (we could have used $B_{**}(X)$ or $C_{**}(X)$ instead of $\bar{B}_{**}(X)$ with the same result):

$$\bar{B}^{per}_*(X) = \prod_{i \in Z} \bar{B}_{i,*-i}(X)$$
$$\cong \overset{\lim}{\underset{k}{\leftarrow}} \bar{B}_{*+2k}(X).$$

(The inverse limit is with respect to the surjective map s: $\bar{B}_{*+2}(X) \to \bar{B}_*(X)$ induced by the obvious isomorphism s: $\bar{B}_{p,q}(X) \to \bar{B}_{p-1,q-1}(X)$.) The homology of $\bar{B}^{per}_*(X)$ is called *periodic homology* and denoted $HC^{per}_*(X)$, or if $X = ZA$ $HC^{per}_*(A)$. One has isomorphisms

$$\text{(II.3.1)} \quad \begin{aligned} s&: \bar{B}^{per}_*(X) \to \bar{B}^{per}_{*-2}(X) \\ s&: HC^{per}_*(X) \to HC^{per}_{*-2}(X) \end{aligned}$$

and short exact sequences

$$\text{(II.3.2)} \quad 0 \to \overset{\lim^1}{\underset{n}{\to}} HC_{*+1+2n}(X) \to HC^{per}_*(X) \to \overset{\lim}{\underset{n}{\to}} HC_{*+2n}(X) \to 0.$$

(For this last, use a short exact sequence of complexes

$$0 \to \bar{B}^{per}_*(X) \to \prod_{k \geq 0} \bar{B}_*(X) \to \prod_{k \geq 0} \bar{B}_*(X) \to 0.)$$

We also have:

Proposition II.3.3. *Proposition II.2.7 applies to HC^{per}_* as well as H_* and HC_*.*

Proof. Clear. $\square$

4. *Derivations*

The key idea here is that a derivation D: $A \to A$ of an algebra acts on the associated cyclic module ZA, and that the induced action on "de Rham homology" is zero.

Let A be a (flat associative unital) K-algebra and D a derivation of A, i.e. a K-linear map $A \to A$ satisfying $D(ab) = (Da)b + aDb$.

Definition II.4.1. *L_D: $ZA \to ZA$ is the endomorphism*

$$L_D(a_0, \dots, a_n) = \sum_{i=0}^{n} (a_0, \dots, Da_i, \dots, a_n).$$

One checks easily that this is indeed a map of cyclic K-modules, i.e., $L_D \partial_i = \partial_i L_D$, $L_D s_i = s_i L_D$, and $L_D t_{n+1} = t_{n+1} L_D$.

THEOREM II.4.2. *Given A and D there are natural K-linear maps*

$$e_D: Z_nA \to Z_{n-1}A$$
$$E_D: Z_nA \to Z_{n+1}A$$

which descend to maps

$$e_D: \bar{Z}_nA \to \bar{Z}_{n-1}A$$
$$E_D: \bar{Z}_nA \to \bar{Z}_{n+1}A$$

and satisfy

(i) $[e_D, b] = 0$ *in* Z_nA

(ii) $[e_D, B] + [E_D, b] = L_D$ *in* Z_nA

(iii) $[E_D, B] = 0$ *in* $\bar{Z}_nA$.

("Natural" refers to the category of pairs (A, D), in which a morphism $(A, D) \to (A', D')$ is an algebra homomorphism $f: A \to A'$ such that $D'f = fD$. The brackets are graded commutators $[e_D b] = e_D b + b e_D$, etc., since b, B, e_D, and E_D all have odd degree.) Before proving the theorem we deduce three corollaries.

COROLLARY II.4.3. *L_D acts like zero on $H_*^{dR}(A)$.*

Proof. The maps L_D, B and (by (i)) e_D are chain endomorphisms of the Hochschild complex $C_*^h(A)$ of degrees 0, 1, and -1 respectively. Denote their respective actions on $H_*(A)$ by the same three symbols again. Then viewing E_D as a chain homotopy we see from (ii) that

$$[e_D, B] = L_D \quad \text{in } H_*(A).$$

Now view e_D as a cochain homotopy to complete the proof. □

COROLLARY II.4.4. L_D acts like zero on $HC_*^{per}(A)$.

Proof. Combine (i), (ii), and (iii) into:

(II.4.5) $[e_D + E_D, b + B] = L_D$ in $\bar{B}_{**}(A)$;

here e_D, E_D, b, and B have bidegrees (1,0), (0,1), $(0, -1)$, and $(-1, 0)$ respectively. Interpret (II.4.5) as an equation in $\bar{B}_*^{per}(A)$, and view $e_D + E_D$ as a chain homotopy. □

COROLLARY II.4.6. $L_D \circ s = 0$: $HC_*A \to HC_{*-2}A$.

Proof. Note that (II.4.5) does not make sense in the quotient complex $\bar{B}_*(A)$ of $\bar{B}_*^{per}(A)$, because the map e_D does not preserve the kernel of the quotient map. However, if s: $\bar{B}_*^{per}(A) \to \bar{B}_{*-2}^{per}(A)$ is the isomorphism of (II.3.1) then the equation

(II.4.7) $$[e_D \circ s + E_D \circ s, b + B] = L_D \circ s,$$

which follows from (II.4.5), does make sense (and is true) in the quotient. The result then follows like the preceding one. □

Proof of the Theorem: Define e_D: $Z_nA \to Z_{n-1}A$ by the formula:

(II.4.8) $$e_D(a_0, \ldots, a_n) = (-1)^{n+1}((Da_n)a_0, a_1, \ldots, a_{n-1}).$$

The reader can verify (i).

Instead of defining E_D: $Z_nA \to Z_{n+1}A$ by an explicit formula we will construct it by something like the "acyclic models" method of algebraic topology.

We begin by thinking generally about natural K-linear maps F_D: $Z_nA \to Z_mA$ (natural with respect to (A, D)). Of course these correspond exactly to K-multilinear natural maps from A^{n+1} to Z_mA. Without the multilinearity condition such maps correspond exactly to elements $x \in Z_mA(n)$, where $A(n)$ is the tensor algebra over K in variables $a_0, \ldots, a_n$,

$Da_0, \ldots, Da_n, D^2a_0, \ldots, D^2a_n, \ldots$. The multilinearity of F_D means that x should have multidegree $(1, \ldots, 1)$ in the multigrading of $Z_mA(n) = A(n)^{\otimes m+1}$ induced by the multigrading of $A(n)$ which assigns to each generator D^ka_i the multidegree

$$\underset{\substack{\uparrow \\ i\text{-th place}}}{(0, \ldots, 1, \ldots 0)}$$

In other words, any natural map F_D must be given by a formula, and the allowable formulae are the K-linear combinations of the expressions

$$F_D(a_0, \ldots, a_n) = (M_0, \ldots, M_m),$$

where each M_j is a (noncommutative) monomial in the expressions D^ka_i and

(II.4.9) for each $i (0 \le i \le n)$ the total number of occurrences of all the D^ka_i in all the M_j is 1.

Notice that if $m = n+1$ then any such F_D necessarily descends to a map $\bar{Z}_nA \to \bar{Z}_{n+1}A$: Assume (without loss of generality) that the formula for F_D is a single term $(M_0, \ldots, M_{n+1})$ as above. By II.4.9 there are only two possibilities: *Case* 1. For some $j > 0$ $M_j = 1$. Then the composed map

$$Z_nA \xrightarrow{F_D} Z_{n+1}A \to \bar{Z}_{n+1}A$$

is zero. ($F_D(a_0, \ldots, a_n)$ is in the image of the degeneracy $s_{j-1}: Z_nA \to Z_{n+1}A$.) *Case* 2. The formula is $F_D(a_0, \ldots, a_n) = (1, D^{k_0}a_{\sigma(0)}, \ldots, D^{k_n}a_{\sigma(n)})$, where $k_j \ge 0$ and σ is a permutation of $\{0, \ldots, n\}$. The kernel of $Z_nA \to \bar{Z}_nA$ is generated by the elements $(a_0, \ldots, a_n)$ where some a_i $(1 \le i \le n)$ equals 1. But if $a_i = 1$ then $D^{k_{\sigma^{-1}(i)}}a_i = 1$ or 0, so $F_D(a_0, \ldots, a_n)$ goes to zero in $\bar{Z}_{n+1}A$.

Similarly if $m = n+2$ then any natural K-linear map from Z_nA to Z_mA (for example, $[E_D, B]$) descends to the zero map $\bar{Z}_nA \to \bar{Z}_mA$. (Case 1 always holds.) Thus (iii) will take care of itself. It only remains to find a natural E_D satisfying (ii).

Assume that E_D has already been defined in $Z_0A, \ldots, Z_{n-1}A$, and that so far (ii) holds. Our task is to find $x \in Z_{n+1}A(n)$ having multidegree $(1, \ldots, 1)$ and satisfying

(II.4.10) $bx = (L_D - [e_D, B] - E_Db)(a_0, \ldots, a_n) \in Z_nA(n)$.

We have:

$$b(L_D - [e_D, B] - E_Db)(a_0, \ldots, a_n) =$$
$$(L_D - [e_D, B] - [E_D, b])b(a_0, \ldots, a_n) = 0 \in Z_{n-1}A(n)$$

by induction on n. Thus we will be done if in $Z_nA(n)$ the kernel of b equals the image of b. (The multidegree condition is no trouble because the maps L_D, b, B, e_D and the previously constructed E_D's all preserve multidegree, so that the right-hand side of II.4.10 has multidegree $(1, \ldots, 1)$). That is, we need to know that the n-th Hochschild homology of a tensor algebra is zero. This is true if $n \ge 2$, by [11], Lemma 5.2.

To begin the induction we need formulae for E_D in Z_0A and Z_1A. In Z_0A we have

$$\begin{aligned}(L_D - [e_D, B])(a_0) &= (Da_0) - e_D((a_0, 1) + (1, a_0)) \\ &= (Da_0) - 0 - (Da_0) = 0,\end{aligned}$$

so we define

(II.4.11) $E_D(a_0) = 0$.

In Z_1A we have

$$\begin{aligned}(L_D-[e_D,B]-E_Db)(a_0,a_1) &= (Da_0,a_1)+(a_0,Da_1)\\ &\quad -e_D(-(a_0,a_1,1)+(a_1,a_0,1)+(1,a_0,a_1)\\ &\quad -(1,a_1,a_0))\\ &\quad -B((Da_1)a_0)\\ &\quad -0\\ &= (Da_0,a_1)+(a_0,Da_1)\\ &\quad +(Da_1,a_0)-(Da_0,a_1)\\ &\quad -((Da_1)a_0,1)-(1,(Da_1)a_0)\\ &= b(1,Da_1,a_0)-b(1,(Da_1)a_0,1),\end{aligned}$$

so we define

(II.4.12) $\quad E_D(a_0,a_1)=(1,Da_1,a_0)-(1,(Da_1)a_0,1).$ □

With a little extra care the theorem can be refined:

Addendum II.4.13. The maps e_D and E_D can be chosen to depend linearly on D.

Proof. Note that we have defined L_D to satisfy this requirement. That is, the letter "D" occurs exactly once in each term of the right-hand side of II.4.1. The same is true of e_D (by II.4.8) and of E_D in dimensions ≤ 1 (by II.4.11) and II.4.12). The inductive construction of E_D can be carried out so that this holds for E_D in all dimensions: Just refine the multigrading of $A(n)$ by adding one new grading in which the generator D^ka_i has degree k. The induced grading of $Z_mA(n)$ is such that the maps b and B have degree zero and L_D and e_D have degree one. We can inductively arrange for E_D to have degree one as well. □

We also record the following for future reference:

PROPOSITION II.4.14. *If $\Delta\colon A\to A$ is another derivation and $[\Delta,D]=0$ then $[e_D,L_\Delta]=0$ $=[E_D,L_\Delta]$.*

Proof. In fact any natural K-linear map $F_D\colon Z_nA\to Z_mA$ commutes with L_Δ if D commutes with Δ: Without loss of generality $F_D(a_0,\ldots,a_n)=(M_0,\ldots,M_m)$ satisfying (II.4.9). Say

$$M_j=\coprod_{l=1}^{r(j)} D^{k(j,l)}a_{i(j,l)},$$

where in the collection $\{i(j,l)\,|\,0\leq j\leq m,\ 1\leq l\leq r(j)\}$ each i $(0\leq i\leq n)$ occurs exactly once. Then

$$\begin{aligned}L_\Delta F_D(a_0,\ldots,a_n) &= L_\Delta(M_0,\ldots,M_m)\\ &= \sum_{j=0}^{m}(M_0,\ldots,\Delta M_j,\ldots,M_m)\\ &= \sum_{j=0}^{m}\sum_{l=1}^{r(j)}(M_0,\ldots,D^{k(j,1)}a_{i(j,1)}\ldots\Delta D^{k(j,l)}a_{i(j,l)}\ldots D^{k(j,r(j))}a_{i(j,r(j))},\\ &\quad \ldots,M_m)\\ &= \sum_{j=0}^{m}\sum_{l=1}^{r(j)}(M_0,\ldots,D^{k(j,1)}a_{i(j,1)}\ldots D^{k(j,l)}\Delta a_{i(j,l)}\ldots D^{k(j,r(j))}a_{i(j,r(j))},\\ &\quad \ldots,M_m)\\ &= \sum_{i=0}^{n}F_D(a_0,\ldots,\Delta a_i,\ldots,a_n)\\ &= F_DL_\Delta(a_0,\ldots,a_n).\end{aligned}$$

□

5. *An Application*

THEOREM II.5.1. *Let K be a field of characteristic zero, A a K-algebra, and $I \subset A$ a nilpotent ideal. Then the quotient map $A \to A/I$ induces isomorphisms*

$$HC_*^{per}(A) \to HC_*^{per}(A/I).$$

Proof. Filter A by the powers of I:

$$A = I^0 \supset I^1 \supset I^2 \supset \ldots \supset I^m \supset I^{m+1} = 0.$$

The quotients $gr_k(A) = I^k/I^{k+1}$ form an algebra

$$gr(A) = \bigoplus_{k \geq 0} gr_k(A).$$

Filter the vector spaces $Z_n A$ by:

$$F_n^k = \sum_{k_0 + \ldots + k_n = k} I^{k_0} \otimes \ldots \otimes I^{k_n} \subset A^{\otimes n+1} = Z_n A.$$

The structure maps ∂_i, s_i, t_{n+1} of ZA preserve the filtration, so we actually have a cyclic vector space F^k for each $k \geq 0$, with

$$ZA = F^0 \supset F^1 \supset F^2 \supset \ldots$$

PROPOSITION II.5.2. *$Z\, gr(A) \cong \bigoplus_{k \geq 0} F^k/F^{k+1}$ as cyclic vector spaces.*

Proof. Easy exercise. □

We have to show that $HC_*^{per}(F^0) \to HC_*^{per}(F^0/F^1)$ is an isomorphism, i.e. (*Proposition* II.3.3) that $HC_*^{per}(F^1) = 0$. We will do this by using II.3.2 and showing that the map

$$HC_{*+2k}(F^1) \xrightarrow{s^k} HC_*(F^1)$$

is zero for $k \geq m(*+1)$. This follows from two facts:

Claim 1. The map $HC_{*+2k}(F^1/F^{k+1}) \xrightarrow{s^k} HC_*(F^1/F^{k+1})$ is zero for *all* $*$.

Claim 2. $HC_*(F^k) = 0$ for $* < \frac{k}{m} - 1$ (and hence $HC_*(F^1) \to HC_*(F^1/F^{k+1})$ is injective for $* < \frac{k+1}{m} - 1$).

Proof of Claim 1: It will suffice to show that $s = 0$ in $HC_*(F^k/F^{k+1})$ for $k > 0$. The graded algebra $gr(A)$ has a derivation D defined by

$$Da = ka \quad \text{for } a \in gr_k(A).$$

Clearly the endomorphism L_D of $Z\,gr(A)$ acts like k on the summand F^k/F^{k+1} (see Proposition II.5.2). Therefore by Corollary II.4.6 we have

$$ks = L_D \circ s = 0 \quad \text{on } HC_*(F^k/F^{k+1}).$$

Since $\frac{1}{k} \in K$ we are done. □

Proof of Claim 2: From the definition of F_n^k and the fact that $I^{k'} = 0$ for $k' > m$ we have $F_n^k = 0$ for $n < \frac{k}{m} - 1$. Thus $C_{ij}(F^k) = 0$ for $j < \frac{k}{m} - 1$, and the complex $C_*(F^k)$ has no nonzero chains below dimension $\frac{k}{m} - 1$. □

§III. CYCLIC HYPERHOMOLOGY AND DERIVATIONS

We indicate how the definitions and results of Section II can be extended to the case of chain algebras. We will be brief, because we have set up the theory in Section II so as to make this extension as easy as possible.

Again let K be a commutative ring with 1.

DEFINITION. *A chain algebra (A, d) over K is a nonnegatively graded associative K-algebra A with identity, equipped with a K-linear map $d\colon A \to A$ of degree -1 satisfying*

$$d(ab) = (da)b + (-1)^{|a|}adb$$
$$d^2 = 0$$

Throughout Section III we will continue to work only with flat algebras. We will relax this condition in Section IV.

1. *The Cyclic Chain Complex Associated to a Chain Algebra*

If (A, d) is a chain algebra over K and flat as a K-module then we define a cyclic object $Z(A, d)$ in the category of chain complexes (of K-modules, nonnegatively graded). Set

$$Z_n(A, d) = (A, d)^{\otimes n+1}.$$

This is a tensor product of chain complexes. Thus its differential (which we call "d") is given by:

$$\mathrm{d}(a_0, \ldots, a_n) = \sum_{i=0}^{n} (-1)^{|a_0| + \cdots + |a_{i-1}|}(a_0, \ldots, \mathrm{d}a_i, \ldots, a_n).$$

Face and degeneracy maps and cyclic group actions are again given by II.1.1–II.1.3, except that

(III.1.1) In the formulae for ∂_n and t_{n+1} we insert the customary sign $(-1)^{|a_n|(|a_0| + \cdots + |a_{n-1}|)}$.

The maps

$$Z_n(A, d) \xrightarrow{\partial_i} Z_{n-1}(A, d)$$

$$Z_n(A, d) \xrightarrow{s_i} Z_{n+1}(A, d)$$

$$Z_n(A, d) \xrightarrow{t_{n+1}} Z_n(A, d)$$

are then chain maps, and the identities I.1–I.3 hold, so we have a cyclic chain complex.

2. *Cyclic Hyperhomology*

We consider cyclic chain complexes (X, d) in general and $Z(A, d)$ in particular. (Chain complexes are always nonnegatively graded chain complexes of K-modules unless the contrary is explicitly stated.) The construction of complexes $C^h_*(X, d)$, $C^a_*(X, d)$, $C_*(X, d)$, $B_*(X, d)$, and $\bar{B}_*(X, d)$ goes through as in II.2 because there we worked in an arbitrary abelian category. However, these complexes are now complexes of complexes, i.e. *hypercomplexes*, so it makes good sense to replace them by single complexes (their *total complexes*) and then take homology.

For example, the Hochschild complex $C^h_*(X, d)$

$$X_0 \xleftarrow{b} X_1 \xleftarrow{b} X_2 \leftarrow \ldots$$

is really a hypercomplex:

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \\ \downarrow & & \downarrow & & \downarrow & & \\ X_{01} & \xleftarrow{b} & X_{11} & \xleftarrow{b} & X_{21} & \leftarrow & \dots \\ {\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow & & \\ X_{00} & \xleftarrow{b} & X_{10} & \xleftarrow{b} & X_{20} & \leftarrow & \dots . \end{array} \qquad (db = bd)$$

Its total complex $\mathrm{Tot}(C_*^h(X,d))$ is $\bigoplus_{n\geq 0} X_{n,*-n}$ with differential $\mathbf{b} = b + (-1)^n d$ in $X_{n,k}$. The *Hochschild hyperhomology* is by definition

$$\mathbf{H}_*(X,d) = H_*(\mathrm{Tot}(C_*^h(X,d))).$$

Likewise *cyclic hyperhomology* is

$$\mathbf{HC}_*(X,d) = H_*(\mathrm{Tot}(C_*(X,d))).$$

Just as in II.2 one can prove that $H_*(\mathrm{Tot}(C_*^a(X,d))) = 0$ and deduce:
(III.2.2) A natural long exact sequence

$$\dots \to \mathbf{H}_*(X,d) \to \mathbf{HC}_*(X,d) \xrightarrow{s} \mathbf{HC}_{*-2}(X,d) \to \mathbf{H}_{*-1}(X,d) \to \dots$$

(III.2.3) A natural spectral sequence

$$E^1_{p,q} = \left\{ \begin{matrix} \mathbf{H}_{q-p}(X,d), & q \geq p \geq 0 \\ 0 & \text{otherwise} \end{matrix} \right\} = > \mathbf{HC}_*(X,d)$$

with $d^1 = B =$ composition of two maps in III.2.2.
One also has as before:

(III.2.4) $\mathbf{HC}_*(X,d) \cong H_*(\mathrm{Tot}(B_*(X,d)))$
$\cong H_*(\mathrm{Tot}(\bar{B}_*(X,d)))$

and

(III.2.5) A short exact sequence of cyclic chain complexes gives rise to long exact sequences in $\mathbf{H}_*$ and $\mathbf{HC}_*$.

For what it's worth one can define $\mathbf{H}_*^{dR}(X,d)$, "de Rham hyperhomology" of (X,d), to be the cohomology of the complex

$$\mathbf{H}_0(X,d) \xrightarrow{B} \mathbf{H}_1(X,d) \xrightarrow{B} \mathbf{H}_2(X,d) \to \dots .$$

If $(X,d) = Z(A,d)$ then we write $\mathbf{H}_*(A,d)$, $\mathbf{HC}_*(A,d)$, $\mathbf{H}_*^{dR}(A,d)$ for $\mathbf{H}_*(X,d)$, $\mathbf{HC}_*(X,d)$, $\mathbf{H}_*^{dR}(X,d)$.

DEFINITION III.2.6 *A map $(X,d) \to (Y,d)$ of cyclic chain complexes is an equivalence if for each $n \geq 0$ the chain map $(X_n,d) \to (Y_n,d)$ is a quasi-isomorphism (QI), i.e. induces isomorphisms in homology.*

PROPOSITION III.2.7. *An equivalence induces isomorphisms in $\mathbf{H}_*$, $\mathbf{HC}_*$, and $\mathbf{H}_*^{dR}$.*

Proof. For $\mathbf{H}_*$ filter the complex Tot C_*^h by columns of the hypercomplex C_*^h and use a comparison of spectral sequences. For $\mathbf{HC}_*$ use III.2.2 or III.2.3. For $\mathbf{H}_*^{dR}$ use the definition. □

DEFINITION III.2.8. *A map $(A,d) \to (B,d)$ of chain algebras is an equivalence if as a chain map it is QI.*

PROPOSITION III.2.9. *Any equivalence $f\colon (A,d)\to(B,d)$ of flat chain algebras induces isomorphisms of $\mathbf{H}_*$, $\mathbf{HC}_*$, and $\mathbf{H}_*^{dR}$.*

Proof. By III.2.7 it will suffice to know that the map $Z(f)\colon Z(A,d)\to Z(B,d)$ is an equivalence, i.e. that for each $n\geq 0$ the chain map

$$f^{\otimes n+1}\colon (A,d)^{\otimes n+1}\to(B,d)^{\otimes n+1}$$

is QI. This can be proved by induction with respect to n by factoring $f^{\otimes n+1}=(f^{\otimes n}\otimes 1)\circ(1\otimes f)$ and using flatness. □

It will be convenient to have long exact sequences as in III.2.5 for any map of cyclic chain complexes $f\colon (X,d)\to(Y,d)$. We therefore define relative groups $\mathbf{H}_*(f)$ and $\mathbf{HC}_*(f)$ as follows: Construct a new cyclic chain complex M^f by setting

$$M_n^f = \text{algebraic mapping cone of the chain map } f_n\colon (X_n,d)\to(Y_n,d).$$

There are obvious chain maps ∂_i, s_i, and t_{n+1} making the $\{M_n^f\}$ into a cyclic chain complex M^f, simply because "algebraic mapping cone" is a functor from chain maps to chain complexes. Set $\mathbf{H}_*(f)=\mathbf{H}_*(M^f)$, and likewise for $\mathbf{HC}_*$. One obtains a natural diagram, in which the row is exact, the triangle commutes, and the vertical map is an equivalence:

$$\begin{tikzcd} 0 \arrow[r] & (X,d) \arrow[r] \arrow[dr, "f"'] & Y' \arrow[r] \arrow[d, "\cong"] & M^f \arrow[r] & 0 \\ & & (Y,d) & & \end{tikzcd}$$

Here Y' is the algebraic mapping cylinder of f. Using III.2.5 and III.2.7 this yields exact sequences

(III.2.10) $$\ldots\to\mathbf{H}_*(X,d)\to\mathbf{H}_*(Y,d)\to\mathbf{H}_*(f)\to\mathbf{H}_{*-1}(X,d)\to\ldots$$

$$\ldots\to\mathbf{HC}_*(X,d)\to\mathbf{HC}_*(Y,d)\to\mathbf{HC}_*(f)\to\mathbf{HC}_{*-1}(X,d)\to\ldots$$

If $f\colon (A,d)\to(B,d)$ is a map of flat chain algebras then we write $\mathbf{H}_*(f)$ (resp. $\mathbf{HC}_*(f)$) for $\mathbf{H}_*(Z(f))$ (resp. $\mathbf{HC}_*(Z(f))$), where $Z(f)\colon Z(A,d)\to Z(B,d)$ is induced by f.

Finally, there are spectral sequences arising from the filtrations of $\mathrm{Tot}(C_*^h(X,d))$ and $\mathrm{Tot}(C_*(X,d))$ by rows of $C_*^h(X,d)$ and $C_*(X,d)$:

$$E^1=\mathbf{H}_*(X,0)\Rightarrow\mathbf{H}_*(X,d)$$
$$E^1=\mathbf{HC}_*(X,0)\Rightarrow\mathbf{HC}_*(X,d).$$

More precisely, $(X,0)$ is a cyclic graded K-module, and as such it has Hochschild (resp. cyclic) homology groups which are themselves graded K-modules. Denote the k-th graded part of the n-th group by $H_n(X,0)_k$ (resp. $HC_n(x,0)_k$). Then this is $E^1_{n,k}$.

Warning. In the case $(X,d)=Z(A,d)$, so $(X,0)=Z(A,0)$, the graded module $\mathbf{H}_n(A,0)$ does *not* have for its underlying module $H_n(A)$. The same warning applies to $\mathbf{HC}_n$. The reason is that the underlying cyclic module of the cyclic graded module $Z(A,0)$ is not isomorphic to $Z(A)$, because of the signs in III.1.1.

3. *Periodic Hyperhomology*

We continue to work with a cyclic object (X,d) in the category of nonnegatively graded chain complexes over k.

Define a complex (graded by all of $\mathbf{Z}$)

$$\overline{B}_*^{per}(X,d)=\varprojlim_k \mathrm{Tot}(\overline{B}_{*+2k}(X,d))$$

($\varprojlim_k$ is with respect to the surjection

$$s\colon \mathrm{Tot}\,\overline{B}_{*+2}(X,d)\to\mathrm{Tot}\,\overline{B}_*(X,d)$$

induced by the surjection of hypercomplexes

$$s\colon \bar{B}_{*+2}(X,d) \to \bar{B}_*(X,d).)$$

Define *periodic hyperhomology* by $\mathbf{HC}_n^{per}(X,d) = H_n\bar{B}^{per}(X,d)$. We write $\mathbf{HC}_n^{per}(A,d)$ if $(X,d) = Z(A,d)$. Again, as in II.3, s gives a periodicity isomorphism

(III.3.1) $\mathbf{HC}_*^{per}(X,d) \underset{\cong}{\overset{s}{\to}} \mathbf{HC}_{*-2}^{per}(X,d)$

and again there is an exact sequence

(III.3.2) $0 \to \varprojlim_k{}^1 \mathbf{HC}_{*+2k+1}(X,d) \to \mathbf{HC}_*^{per}(X,d) \to \varprojlim_k \mathbf{HC}_{*+2k}(X,D) \to 0.$

Note also that III.2.5 holds for $\mathbf{HC}_*^{per}$.

4. *Derivations*

We now adapt the definitions and results of II.4 to the setting of chain algebras.

Consider *graded* derivations $D\colon (A,d) \to (A,d)$. We may as well allow D to have arbitrary integer degree $|D|$, although we will only use the case $|D| = 0$. Thus D is a K-linear map $A_* \to A_{*+|D|}$. It is a derivation if it satisfies

$$D(ab) = (Da)b + (-1)^{|D||a|}aDb.$$

We also require it to be a graded chain map, i.e.,

$$[D,d] = Dd - (-1)^{|D|}dD = 0.$$

Define $L_D\colon Z(A,d) \to Z(A,d)$ by

$$L_D(a_0 \ldots, a_n) = \sum_{i=0}^{n} (-1)^{|D|(|a_0| + \cdots + |a_{i-1}|)}(a_0, \ldots, Da_i, \ldots, a_n).$$

Thus L_D is a graded map $Z_n(A,d) \to Z_n(A,d)$ of degree $|D|$ for each n. One can check that it is a (graded) chain map:

$$[L_D, d] = L_D d - (-1)^{|D|} dL_D = 0.$$

(In other notation this says $[L_D, L_d] = 0$.) Also L_D is a map of cyclic objects, i.e.

$$L_D\partial_i = \partial_i L_D$$
$$L_D s_i = s_i L_D$$
$$L_D t_{n+1} = t_{n+1} L_D$$

just as in II.4.

THEOREM III.4.1. *Given a chain algebra (A,d) and derivation D as above, there exist natural chain maps (of degree $|D|$)*

$$e_D\colon Z_n(A,d) \to Z_{n-1}(A,d)$$
$$E_D\colon Z_n(A,d) \to Z_{n+1}(A,d)$$

which descend to chain maps

$$e_D\colon \bar{Z}_n(A,d) \to \bar{Z}_{n-1}(A,d)$$
$$E_D\colon \bar{Z}_n(A,d) \to \bar{Z}_{n+1}(A,d)$$

and satisfy:

(i) $[e_D, b] = 0$ *in* $Z_n(A,d)$

(ii) $[e_D, B] + [E_D, b] = L_D$ *in* $Z_n(A,d)$

(iii) $[E_D, B] = 0$ *in* $\bar{Z}_n(A,d)$.

("Natural" refers to the category of chain-algebras-with-derivation (A, d, D). The brackets are graded commutators with respect to *total* degree, e.g.,

$$[e_D, b] = e_D b - (-1)^{|D|-1} b e_D.$$

(Total degree of a chain map $Z_n \to Z_m$ is $m - n +$ (degree as a chain map).))

Proof. Define e_D by inserting an appropriate sign in formula II.4.8. Thus

$$e_D(a_0, \ldots, a_n) = (-1)^{n+1}(-1)^{(|a_n|)(|a_0| + \cdots + |a_{n-1}|)}((Da_n)a_0, a_1, \ldots, a_{n-1})$$

The sign rule that we have used in extending to the graded case the definitions of the maps $\partial_i, s_i, t_{n+1}, L_D$, and now e_D is this: Each letter (a_i or D) appearing as an argument on the left-hand side of the defining equation has a degree associated with it, and a factor of -1 is introduced on the right-hand side each time a pair of letters on the left-hand side, both of odd degree, appears in reverse order on the right-hand side. In order for this rule to make sense it is essential that every letter on the left-hand side should appear exactly once on the right-hand side. This is the case in the definitions of ∂_i, s_i, and t_{n+1} (II.1.1–II.1.3), L_D (II.4.1), and e_D (II.4.8). Thanks to Addendum II.4.13 it is also the case with E_D. Thus we may extend the definition of E_D to the graded setting by the same sign rule.

Having defined e_D and E_D we ask whether they satisfy (i)–(iii). The answer is yes; there is a general principle at work here, which guarantees, for example, that the graded formula for $e_D b$ can be arrived at by applying the sign rule to the ungraded formula for $e_D b$. The same principle applies to $be_D, e_D B, \ldots$, and allows (i), (ii) and (iii) to be deduced from their counterparts in Theorem II.4.2.

We must also check that e_D and E_D are chain maps. This is what Proposition II.4.14 is for: It implies its graded analogue, which we then use with $\Delta = d$. □

The theorem has three corollaries exactly analogous to those of Theorem II.4.2.

Corollary III.4.2. *L_D acts like zero on $\boldsymbol{H}^{dR}_*(A, d)$.*

Proof. Mimic the proof of Corollary II.4.3. Thus L_D, B, and e_D are chain endomorphisms of $\operatorname{Tot} C^h_*(A, d)$, of degrees $|D|$, 1, and $-|D| - 1 \ldots$. (One needs here that (i) and (ii) still hold when b is replaced by $\mathbf{b} = b + (-1)^n d$, the total differential of $\operatorname{Tot} C^h_*(A, d)$. They do, because $[e_D, d] = 0 = [E_D, d]$.) □

Corollary III.4.3. *L_D acts like zero on $\boldsymbol{HC}^{per}_*(A, d)$.*

Proof. Mimic the proof of Corollary II.4.4. Thus II.4.5 is replaced by

$$[e_D + E_D, \mathbf{b} + B] = L_D$$

which is then interpreted as an equation in $\overline{B}^{per}_*(A, d)$. □

Corollary III.4.4. *$L_D \circ s$ acts like zero on $\mathbf{HC}_*(A, d)$.*

Proof. Clear, by now. □

5. *An Application*

In close analogy with Theorem II.5.1 we have

Theorem III.5.1. *Let (A, d) be a chain algebra over a field K of characteristic zero, and let $I \subset A$ be a chain ideal, i.e., a graded ideal satisfying $dI \subset I$. Assume $I_0 = 0$. Then the quotient map $(A, d) \to (A/I, d)$ induces an isomorphism*

$$\boldsymbol{HC}^{per}_*(A, d) \to \boldsymbol{HC}^{per}_*(A/I, d).$$

Proof. This is just like the proof of II.5.1. Set

$$gr(A,d) = \bigoplus_{k \geq 0} I^k/I^{k+1}.$$

This is a chain algebra. (The grading by k is an additional grading, with respect to which the differential d has degree 0, not -1.) Filter each chain complex $Z_n(A,d)$ by subcomplexes

$$F_n^k = \sum_{k_0 + \ldots + k_n = k} I^{k_0} \otimes \ldots \otimes I^{k_n}.$$

Identify the two cyclic chain complexes:

$$\text{(III.5.2)} \quad Z\, gr(A,d) \cong \bigoplus_{(k \geq 0)} F^k/F^{k+1}.$$

It is enough to prove that the map

$$\mathbf{HC}_{*+2k}(F^1) \xrightarrow{s^k} \mathbf{HC}_*(F^1)$$

is zero for $* \leq k$, and this follows from two claims:

Claim 1. $\mathbf{HC}_{*+2k}(F^1/F^{k+1}) \xrightarrow{s^k} \mathbf{HC}_*(F^1/F^{k+1})$ is zero for *all* $*$.

Proof. Just as in the proof of II.5.1, but using Corollary III.4.4 instead of Corollary II.4.6. □

Claim 2. $\mathbf{HC}_*(F^k) = 0$ for $* < k$.

Proof. The complex $\mathrm{Tot}(C_*(F^k))$ has no nonzero chains in dimensions $< k$, since the same is true of each complex F_n^k. □□

§IV. GENERALIZATIONS

By systematically replacing chain algebras with equivalent ones (in the sense of III.2.8) we can generalize some definitions and results of Section III. As usual, all algebras are over a fixed commutative ring K. We no longer assume that all algebras are flat K-modules.

We denote the n-th homology of the underlying chain complex of a chain algebra (A, d) by $h_n(A, d)$. Of course $h_*(A, d)$ is a graded algebra.

As usual, we call a chain complex n-connected if its homology groups vanish in dimensions $\leq n$. We call a chain map n-connected if it induces homology isomorphisms in dimensions $< n$ and a surjection in dimension n, and we call a map of chain algebras n-connected if it is n-connected as a chain map.

1. *The Non-Flat Case*

We extend the definition of cyclic hyperhomology to the case of chain algebras which are not necessarily flat.

PROPOSITION IV.1.1. *Every chain algebra (A, d) admits a natural equivalence $\varepsilon_A\colon (R_A, d) \to (A, d)$ from a chain algebra whose underlying graded algebra R_A is a graded tensor algebra.*

Proof: The method is a standard one; inductively define maps

$$\varepsilon_A\colon (R_A(n), d) \to (A, d)$$

such that $R_A(n)$ is a graded tensor algebra and ε_A is n-connected. Given $R_A(n-1)$ the construction of $R_A(n)$ is as follows. Adjoin an element x of degree n to $R_A(n-1)$ for each pair

(y, z), $y \in R_A(n-1)_{n-1}$, $z \in A_n$ such that

$$\varepsilon_A(y) = dz \quad \text{and} \quad dy = 0.$$

Extend ε_A and d from $R_A(n-1)$ to $R_A(n)$ by setting

$$dx = y \quad \text{and} \quad \varepsilon_A x = z.$$

To start the induction take $R_A(-1) = K$ with $d = 0$ and ε_A the unit map: $K \to A$. Finally, let R_A be the union $\bigcup_n R_A(n)$. □

Note that if A is flat then ε_A induces isomorphisms of Hochschild, cyclic, periodic, and de Rham hyperhomology, by III.2.9 and III.3.2.

Definition IV.1.2. *If (A, d) is any chain algebra then $HC_*(A, d) = HC_*(R_A, d)$, and likewise for H_*, HC_*^{per}, and H_*^{dR}.*

This is the "right" definition because it coincides with the old one in the flat case (as we have just seen) and is "homotopy invariant":

Proposition IV.1.3. *Proposition III.2.9 now holds in the general (non-flat) case.*

Proof: Use the commutative diagram

$$\begin{tikzcd} (R_A, d) \arrow[r, "R_f"] \arrow[d, "\varepsilon_A"] & (R_B, d) \arrow[d, "\varepsilon_B"] \\ (A, d) \arrow[r, "f"] & (B, d) \end{tikzcd}$$

□

2. *One-Connected Maps of Chain Algebras*

The following theorem says that in characteristic zero periodic homology of (A, d) depends only on $h_0(A, d)$. Notice that Theorem III.5.1 was a special case of this.

Theorem IV.2.1. *Any one-connected map $(A, d) \to (B, d)$ of chain algebras over a field K of characteristic zero induces isomorphisms*

$$HC_*^{per}(A, d) \to HC_*^{per}(B, d).$$

Proof: We reduce to the case already treated (in III.5.1) by the device of replacing chain algebras by equivalent ones. There are two steps:

Lemma IV.2.2. *Any zero-connected map of chain algebras can be factored as an equivalence followed by a surjection.*

Lemma IV.2.3. *For any one-connected surjection $(A, d) \to (A/J, d)$ of chain algebras there is a commutative diagram*

$$\begin{tikzcd} (T, d) \arrow[r] \arrow[d] & (A, d) \arrow[d] \\ (T/I, d) \arrow[r] & (A/J, d) \end{tikzcd}$$

of chain algebras such that the horizontal maps are equivalences, T is a tensor algebra, the chain ideal I is generated as an ideal by a subset of a tensor basis, and $I_0 = 0$.

Proof of IV.2.2. Let f: $(A, d) \to (B, d)$ be zero-connected. It will be enough if we construct a chain algebra (B^I, d) with equivalences

$$(B, d) \xrightarrow{\Delta} (B^I, d) \underset{p_1}{\overset{p_0}{\rightarrow}} (B, d)$$

satisfying

(IV.2.4) $p_0 \circ \Delta = 1 = p_1 \circ \Delta$

(IV.2.5) The map $B_n^I \to B_n \oplus B_n$

$$w \to (p_0 w, p_1 w)$$

is surjective if $n > 0$ and has image

$$\{(x, z) \in B_0 \oplus B_0 \,|\, x - z \in dB_1\} \quad \text{if} \quad n = 0.$$

If so, then we factor $f = g \circ h$ as follows. Let (A', d) be the fiber product

$$\begin{array}{ccc} (A', d) & \xrightarrow{f'} & (B^I, d) \\ \downarrow p_0' & & \downarrow p_0 \\ (A, d) & \xrightarrow{f} & (B, d) \end{array}$$

The diagram

$$\begin{array}{ccc} (A, d) & \xrightarrow{\Delta \circ f} & (B^I, d) \\ 1\downarrow & & \downarrow p_0 \\ (A, d) & \xrightarrow{f} & (B, d) \end{array}$$

yields h: $(A, d) \to (A', d)$ and we make g the composite

$$(A', d) \xrightarrow{f'} (B^I, d) \xrightarrow{p_1} (B, d).$$

To see that h is an equivalence observe that its left inverse p_0' is a pullback of the surjective equivalence p_0 and so is an equivalence. Surjectivity of g follows from IV.2.5 and the zero-connectedness of f.

Define B^I as follows.

$$B_n^I = \begin{cases} B_n \oplus B_{n+1} \oplus B_n & \text{if} \quad n > 0 \\ \{(x, y, z) \in B_0 \oplus B_1 \oplus B_0 \,|\, dy = z - x\} & \text{if} \quad n = 0. \end{cases}$$

$$(x, y, z)(x', y', z') = (xx', xy' + (-1)^j yz', zz')$$

$$\text{for } (x, y, z) \in B_i^I \text{ and } (x', y', z') \in B_j^I.$$

$$d(x, y, z) = (dx, (-1)^n x + dy - (-1)^n z, dz)$$

$$\text{for } (x, y, z) \in B_n^I.$$

One easily checks that this is a chain algebra and that the maps

$$\Delta(x) = (x, 0, x)$$
$$p_0(x, y, z) = x$$
$$p_1(x, y, z) = z$$

are chain algebra maps satisfying IV.2.4 and IV.2.5. To see that Δ, p_0, and p_1 are equivalences use a chain homotopy between $\Delta \circ p_0$ and 1:

$$(1 - \Delta p_0)(x, y, z) = (dH + Hd)(x, y, z)$$

where H: $B_n^I \to B_{n+1}^I$ is defined by

$$H(x, y, z) = (0, 0, (-1)^n y). \qquad \square$$

Proof of IV.2.3. This is in the same spirit as the proof of IV.1.1. We inductively construct commutative diagrams

$$\begin{array}{ccc}(T(n),(d)) & \xrightarrow{\varepsilon} & (A,d)\\ \downarrow & & \downarrow\\ (T(n)/I(n),d) & \xrightarrow{\varepsilon'} & (A/J,d)\end{array}$$

such that ε' and the map of kernels

$$(I(n),d) \xrightarrow{\varepsilon''} (J,d)$$

are both n-connected (and hence so is ε). The inductive construction from $n-1$ to n is in two steps.

Step 1: Given $(T(n-1),d)$ with chain ideal $I(n-1)$ and map ε satisfying the conditions above for $n-1$, define a graded algebra $T'(n)$ by adjoining to $T(n-1)$ an element x' in degree n for each (y',z'), $y' \in T(n-1)_{n-1}$, $z' \in A_n$, such that $\varepsilon y' = dz'$ and $dy' = 0$. Extend d and ε from $T(n-1)$ to $T'(n)$ by setting $dx' = y'$ and $\varepsilon x' = z'$, and let $I'(n)$ be the ideal in $T'(n)$ generated by $I(n-1)$. This yields

$$\begin{array}{ccc}(T'(n),d) & \xrightarrow{\varepsilon} & (A,d)\\ \downarrow & & \downarrow\\ (T'(n)/I'(n),d) & \xrightarrow{\varepsilon'} & (A/J,d)\end{array}$$

such that ε and ε' are both n-connected and hence

$$(I'(n),d) \xrightarrow{\varepsilon''} (J,d)$$

is $(n-1)$-connected.

Step 2: Define $T(n)$ by adjoining to $T'(n)$ an element x in degree n for each (y,z), $y \in I'(n)_{n-1}$, $z \in J_n$, such that $\varepsilon y = dz$ and $dy = 0$. Extend d and ε from $T'(n)$ to $T(n)$ by setting $dx = y$ and $\varepsilon x = z$, and let $I(n)$ be the ideal in $T(n)$ generated by $I'(n-1)$ and the x's. This yields

$$\begin{array}{ccc}(T(n),d) & \xrightarrow{\varepsilon} & (A,d)\\ \downarrow & & \downarrow\\ (T(n)/I(n),d) & \xrightarrow{\varepsilon'} & (A/J,d)\end{array}$$

with both ε' and

$$(I(n),d) \xrightarrow{\varepsilon''} (J,d)$$

n-connected.

To start the induction take $T(-1) = K$ and $I(-1) = 0$.

This would complete the proof by taking $T = \bigcup_n T(n)$, $I = \bigcup_n I(n)$, except that we have not arranged that $I_0 = 0$. We can do this by omitting the step in which $T(0)$ is constructed from $T'(0)$ and instead taking $T(0) = T'(0)$, $I(0) = I'(0)$. This is permissible because $(I'(0),d) \to (J,d)$ is automatically zero-connected. (By assumption $h_0(J,d) = 0$.) □

It is clear that the two lemmas imply Theorem IV.2.1; using the first we reduce to the case of a surjection $(A,d) \to (A/J,d)$, and using the second we reduce further to the case in which $J_0 = 0$, which is handled by III.5.1. □

The best result that can be proved by these methods seems to be the following.

THEOREM IV.2.6. *Let $f\colon (A, d) \to (B, d)$ be a one-connected map of chain algebras (over an arbitrary commutative ring K). The map*

$$HC_{*+2k}(f) \xrightarrow{k!s^k} HC_*(f)$$

is zero for $* < k$.

Proof. Use Lemmas IV.2.2 and IV.2.3 (they are valid for any K) to reduce to the case of a map

$$(T, d) \xrightarrow{f} (T/I, d)$$

where T is a tensor algebra, I is generated by a subset of a tensor basis, and $I_0 = 0$. Define a filtration $\{F^k\}$ of $Z(A, d)$ as in the proof of III.5.1. The graded K-modules I^n/I^{n+1} are free, and this means that even though K is not a field it is still possible to identify $I^{k_0} \otimes \cdots \otimes I^{k_n}$ with a submodule of $A \otimes \cdots \otimes A$ and to prove III.5.2. Identify $\mathbf{HC}_*(f)$ with $\mathbf{HC}_{*-1}(F^1)$.

Claim 1 and Claim 2 from the proof of III.5.1 then finish the proof, except that in Claim 1 we must substitute $k!s^k$ for s^k since we only have

$$ks = 0\colon \mathbf{HC}_{*+2}(F^k/F^{k+1}) \to HC_*(F^k/F^{k+1}). \qquad \square$$

§V. THE FREE LOOPSPACE

We prove that the $SO(2)$-equivariant homology of the free loopspace ΛX of a space X is naturally isomorphic to the cyclic hyperhomology of the chains on the based loopspace of X. Then applying IV.2.6 we conclude that the localized rational equivariant cohomology of ΛX depends only on $\pi_1(X)$.

1. *The Free Loopspace and Cyclic Homology*

For any space X let ΛX be the free loopspace of X, i.e. the space of all continuous maps from S^1 to X (with compact-open topology). The rotation group $G = SO(2)$ acts on S^1 and hence on ΛX. Form the homotopy orbit space (or associated bundle) $\Lambda X \times_G EG$. Its (co)homology with coefficients in K is called equivariant (co)homology of ΛX and denoted $H^G_*(\Lambda X; K)$ (resp. $H^*_G(\Lambda X; K)$).

THEOREM V.1.1. *For any path-connected pointed space X and ring K we have*

$$H^G_*(\Lambda X; K) \cong \mathbf{HC}_*(S_*(MX; K)),$$

where $S_*(MX; K)$ is the algebra of singular chains on the Moore loopspace of X.

Proof. At the heart of the proof is a comparison of two cyclic spaces ZMX and $X^{S^1 \times \Delta^\cdot}$, which we now define.

Let MX be the Moore loopspace of X. Thus MX has a strictly associative multiplication with unit and is homotopy-equivalent to the ordinary loopspace ΩX. By ZMX we mean the cyclic space given on objects by

$$Z_n MX = (MX)^{n+1}$$

and on morphisms by formulae (II.1.1)–(II.1.3). This is the obvious nonlinear analogue of the construction in II.1 by which an algebra A yields a cyclic module ZA; for any discrete (resp. topological, resp. simplicial) monoid M it yields a cyclic object ZM in the category of sets (resp. spaces, resp. simplicial sets).

The other cyclic space $X^{S^1 \times \Delta^\cdot}$ is defined using a *cocyclic space* $S^1 \times \Delta^\cdot$. A cocyclic space is a covariant functor from Λ (see §I) to spaces, i.e., a sequence of spaces related by structure maps which satisfy the cosimplicial identities and the duals of the identities (I.1)–(I.3). The cocyclic space $S^1 \times \Delta^\cdot$ will by definition consist of the spaces $S^1 \times \Delta^n$, $n \geq 0$, and the maps

$$\begin{aligned}
&\partial_i \colon S^1 \times \Delta^{n-1} \to S^1 \times \Delta^n, 0 \leq i \leq n, n \geq 1 \\
&s_i \colon S^1 \times \Delta^{n+1} \to S^1 \times \Delta^n, 0 \leq i \leq n \\
&t_{n+1} \colon S^1 \times \Delta^n \to S^1 \times \Delta^n
\end{aligned}$$

given by

$$\begin{aligned}
(\theta, u_0, \ldots u_{n-1})\partial_i &= (\theta, u_0, \ldots u_{i-1}, 0, u_i, \ldots u_{n-1}) \\
(\theta, u_0, \ldots u_{n+1})s_i &= (\theta, u_0, \ldots u_{i-1}, u_i + u_{i+1}, \ldots u_{n+1}) \\
(\theta, u_0, \ldots u_n)t_{n+1} &= (\theta - u_0, u_1, \ldots u_n, u_0).
\end{aligned}$$

Here θ is the coordinate in $S^1 = \mathbf{R}/\mathbf{Z}$ and $(u_0, \ldots u_n)$ are barycentric coordinates in Δ^n. We have written ∂_i, s_i, and t_{n+1} on the right so that the identities (I.1)–(I.3) apply without change. It is straightforward to check that those identities hold, so that this is in fact a cocyclic space. Notice that its underlying cosimplicial space $U(S^1 \times \Delta^\cdot)$ is the product of the constant cosimplicial space S^1 and the standard cosimplicial space $\Delta^\cdot$ which plays a fundamental role in simplicial theory. (It seems that $S^1 \times \Delta^\cdot$ should play an analogous role in "cyclic theory".) Now for any space X let $X^{S^1 \times \Delta^\cdot}$ be the cyclic space given by the spaces $X^{S^1 \times \Delta^n}$, $n \geq 0$ (function spaces with compact-open topology) and maps

$$\left.\begin{aligned}
\partial_i(f)(\theta, u) &= f((\theta, u)\partial_i) \\
s_i(f)(\theta, u) &= f((\theta, u)s_i) \\
t_{n+1}(f)(\theta, u) &= f((\theta, u)t_{n+1})
\end{aligned}\right\} f \in X^{S^1 \times \Delta^n}, (\theta, u) \in S^1 \times \Delta^n$$

Let X be a pointed space. It is straightforward if tedious to check that the following is a map of cyclic spaces.

$$ZMX \xrightarrow{\lambda} X^{S^1 \times \Delta^\cdot}$$

$$(MX)^{n+1} \to X^{S^1 \times \Delta^n}$$

$$\lambda(f_0, \ldots f_{n+1})(\theta, u_0, \ldots u_n) =$$

$$(f_0 \ldots f_n)\left(\theta \sum_{j=0}^{n} |f_j| - \sum_{0 \leq i < j \leq n} u_i |f_j|\right).$$

Here the f_i are elements of MX, $f_0 \ldots f_n$ is their product in MX, $|f|$ denotes the "length" of a Moore loop f, i.e., the length of the interval which parametrizes it, and θ is chosen so that the argument of $f_0 \ldots f_n$ lies in the correct interval $\left[0, \sum_{i=0}^{n} |f_i|\right]$.

Now, any cyclic space gives rise to a cyclic chain complex of K-modules by means of the functor S_* (singular chains with coefficients in K). The proof of Theorem V.1.1 is in three parts:

Lemma V.1.2. $HC_*(ZS_*(MX)) \cong HC_*(S_*(ZMX))$.

Lemma V.1.3. *λ induces an isomorphism* $HC_*(S_*(ZMX)) \to HC_*(S_*(X^{S^1 \times \Delta^\cdot}))$

Lemma V.1.4. $HC_*(S_*(X^{S^1 \times \Delta})) \cong H_*^G(\Lambda X)$.

Proof of V.1.2. Here MX could be replaced by any topological monoid M. By III.2.7 it will be enough to write down a map of cyclic chain complexes

$$ZS_*(M) \to S_*(ZM)$$

which is an equivalence (in the sense of III.2.6). This is easy. Recall the standard chain equivalence

$$S_*(A) \otimes S_*(B) \xrightarrow{\phi} S_*(A \times B)$$

for spaces A and B, given by the "shuffle product". The multiplication in $S_*(M)$ is by definition the composition

$$S_*(M)\otimes S_*(M) \xrightarrow{\phi} S_*(M\times M) \to S_*(M),$$

where the second map is induced by the multiplication in M. The obvious diagrams all commute:

$$\begin{array}{ccc} (S_*A\otimes S_*B)\otimes S_*C & \xrightarrow{\phi\circ(\phi\otimes 1)} & S_*((A\times B)\times C) \\ \downarrow\cong & & \downarrow\cong \\ S_*A\otimes(S_*B\otimes S_*C) & \xrightarrow{\phi\circ(1\otimes\phi)} & S_*(A\times(B\times C)) \end{array}$$

$$\begin{array}{ccc} S_*A\otimes S_*B & \xrightarrow{\phi} & S_*(A\times B) \\ \downarrow\cong & & \downarrow\cong \\ S_*B\otimes S_*A & \xrightarrow{\phi} & S_*(B\times A) \end{array}$$

$$\begin{array}{ccc} S_*(A)\otimes S_*(\text{point}) & \xrightarrow{\phi} & S_*(A\times\text{point}) \\ \downarrow\cong & & \downarrow\cong \\ S_*(A)\otimes K & \cong & S_*(A) \end{array}$$

and this easily implies that the chain equivalences

$$S_*(M)^{\otimes n+1} \to S_*(M^{n+1})$$

given by iterating ϕ n times constitute a map of cyclic chain complexes

$$ZS_*M \to S_*ZM. \qquad \square$$

Proof of V.1.3. In view of II.2.4 a comparison of spectral sequences reduces us to proving that λ induces an isomorphism

$$\mathbf{H}_*S_*(ZMX) \to \mathbf{H}_*S_*(X^{S^1\times\Delta}).$$

This is a statement about the map of underlying simplicial spaces

$$UZMX \xrightarrow{U\lambda} U(X^{S^1\times\Delta}).$$

In fact from the definitions it is clear that for any cyclic space Z we have

(V.1.5) $\mathbf{H}_*S_*(Z) \cong H_*(\|UZ\|)$,

the homology of the realization of UZ in the sense of [14].

The simplicial space $UZMX$ is isomorphic to the "cyclic bar construction" on MX (cf. [16]). The simplicial space $U(X^{S^1\times\Delta^{\cdot}})$ is $(\Lambda X)^{\Delta^{\cdot}}$, the "topological total singular complex" of the space ΛX. We will be done if we can show that the map

$$\|UZMX\| \xrightarrow{\|U\lambda\|} \|(\Lambda X)^{\Delta^{\cdot}}\|$$

is a weak homotopy-equivalence. (In other words we have to explain a Moore loop space version of an example given in [16, p. 368, last full paragraph].) We will do this by algebraically mimicking the fibration sequence

(V.1.6) $\Omega X \to \Lambda X \to X$.

(The second map is evaluation at $0 \in \mathbf{R}/\mathbf{Z}$.)

We will use the following fact, which is essentially the main result of [12].

LEMMA V.1.7. *Let $E \to B$ be a map of simplicial spaces such that*

(i) *all homotopy fibers of $E_n \to B_n$ have the homotopy type of a fixed space F, and*

(ii) *each face or degeneracy diagram*

$$\begin{array}{ccc} E_n & \to & E_m \\ \downarrow & & \downarrow \\ B_n & \to & B_m \end{array}$$

yields a homotopy equivalence from E_n to the homotopy pullback of $B_n \to B_m \leftarrow E_m$.

Then $\|E\| \to \|B\|$ also has all homotopy fibers equivalent to F, and the inclusions

$$\begin{array}{ccc} E_0 & \hookrightarrow & \|E\| \\ \downarrow & & \downarrow \\ B_0 & \hookrightarrow & \|B\| \end{array}$$

induce an equivalence of homotopy fibers.

Now consider the diagram of simplicial spaces

$$\text{(V.1.8)} \qquad \begin{array}{ccc} MX & \xrightarrow{\mu} & (\Omega X)^{\Delta^{\cdot}} \\ \downarrow & & \downarrow \\ UZMX & \xrightarrow{U\lambda} & (\Lambda X)^{\Delta^{\cdot}} \\ \downarrow & & \downarrow \\ BMX & \xrightarrow{\nu} & X^{\Delta^{\cdot}} \end{array}$$

The right hand column here is induced by V.1.6. MX is the Moore loopspace considered as a (constant) simplicial space. BMX is the bar construction on the topological monoid MX:

$$B_nMX = (MX)^n$$

$$\partial_i(f_1, \ldots f_n) = \begin{cases} (f_2, \ldots f_n), & i = 0 \\ (f_1, \ldots f_i f_{i+1}, \ldots f_n), & 0 < i < n \\ (f_1, \ldots f_{n-1}), & i = n \end{cases}$$

$$s_i(f_1, \ldots f_n) = (f_1, \ldots f_i, 1, f_{i+1}, \ldots f_n),\ 0 \leq i < n.$$

The map $UZMX \to BMX$ is

$$(f_0, \ldots f_n) \to (f_1, \ldots f_n).$$

The map ν is defined to make the lower square in V.1.8 commute:

$$\nu(f_1, \ldots f_n)(u_0, \ldots u_n) = (f_1 \ldots f_n)\Big(\sum_{1 \leq j \leq i \leq n} u_i |f_j|\Big).$$

MX is included in $UZMX$ as the space in simplicial degree zero, and μ is defined to make the upper square commute:

$$\mu(f)(u_0, \ldots u_n)(t) = f(t|f|), \quad 0 \leq t \leq 1.$$

Call a sequence of spaces $A \to B \to C$ a *fibration sequence* if the composition $A \to C$ is constant and the map from A to the homotopy fiber of $B \to C$ is a weak equivalence. If $A \to B \to C$ is a sequence of simplicial spaces, call it a fibration sequence if $\|A\| \to \|B\| \to \|C\|$ is one. Call a map $A \to B$ of simplicial spaces an equivalence if $\|A\| \to \|B\|$ is a weak equivalence.

Certainly V.1.6 is a fibration sequence, and it follows easily that the right-hand column of V.1.8 is one. The left-hand column is also a fibration sequence, by V.1.7. Therefore to prove that $U\lambda$ is an equivalence it will suffice to show that μ and ν are.

For μ this is clear; μ is essentially the standard equivalence between Moore loops and ordinary loops.

For ν one can play the same game again using the contractible path space

$$PX = \{p\colon ([0, 1], 0) \to (X, \text{basepoint})\}$$

instead of ΛX. One has

$$\begin{array}{ccc} MX & \xrightarrow{\mu} & (\Omega X)^{\Delta} \\ \downarrow & & \downarrow \\ EMX & \xrightarrow{\pi} & (PX)^{\Delta} \\ \downarrow & & \downarrow \\ BMX & \xrightarrow{\nu} & X^{\Delta} \end{array}$$

where EMX is the contractible simplicial space

$$E_nMX = (MX)^{n+1}$$

$$\partial_i(f_0, \ldots f_n) = \begin{cases} (f_0, \ldots f_i f_{i+1}, \ldots f_n), & 0 \le i < n \\ (f_0, \ldots f_{n-1}), & i = n \end{cases}$$

$$s_i(f_0, \ldots f_n) = (f_0, \ldots f_i, 1, f_{i+1}, \ldots f_n), \quad 0 \le i \le n$$

and the map to BMX is $(f_0, \ldots f_n) \to (f_1, \ldots f_n)$ and π is given by

$$\pi(f_0, \ldots f_n)(u_0, \ldots u_n)(t) = (f_0, \ldots f_n)\left(\sum_{j=0}^{n} \max\left(0, t - \sum_{i=0}^{j-1} u_i\right) |f_j|\right).$$

Since μ and π are equivalences so is ν.

Proof of V.1.4. We must somehow relate the category Λ to the topological group G. We do this by embedding them both in a larger *topological category* L. For us a topological category C will be a small category C with a topology on each morphism set $C(X, Y)$ such that the composition law

$$C(X, Y) \times C(Y, Z) \to C(X, Z)$$

is continuous for all objects X, Y, and Z. A functor $C \to D$ between topological categories is *continuous* if it maps morphism spaces continuously.

Let L be the following subcategory of the category T of spaces: the objects are the spaces $S^1 \times \Delta^n$ ($n \ge 0$) and the maps are the maps of degree one. Give $L(S^1 \times \Delta^m, S^1 \times \Delta^n)$ the compact-open topology. Thus L is a topological category.

View Λ as a topological category by giving each morphism set the discrete topology.

View the group G as a category in the usual way (one object; one morphism for each group element). Since G was a topological group it becomes a topological category.

Define continuous functors

$$\Lambda \xrightarrow{\alpha} L \xleftarrow{\beta} G$$

as follows. The composition $\Lambda \xrightarrow{\alpha} L \subsetneq T$ is the cocyclic space $S^1 \times \Delta^{\cdot}$. The usual action of $G = SO(2)$ on $S^1 \times \Delta^0 = S^1$ defines β. We defer the proof of the following result until the next section.

LEMMA V.1.9. *The continuous functors α and β induce equivalences of nerves.*

If Φ: $C \to T$ is any functor from a topological category to spaces such that

$$\text{(V.1.10)} \qquad \begin{cases} \text{the evaluation maps} \\ C(A, B) \times \Phi(A) \to \Phi(B) \\ (f, p) \to \Phi(f)(p) \\ \text{are all continuous} \end{cases}$$

then we can make a simplicial space $\amalg \Phi$:

$$\amalg_n \Phi = \coprod_{A_0, \ldots A_n \in ObC} \Phi(A_0) \times \prod_{i=1}^{n} C(A_{i-1}, A_i).$$

(If C has the discrete topology then this is essentially the same as the "simplicial replacement" of a diagram in [2, p. 337].)

Denote by F: $L^0 \to T$ the functor

$$S^1 \times \Delta^n \to X^{S^1 \times \Delta^n}.$$

("^{0}" denotes opposite category.) The proof of V.1.4 is in four steps:

$$\begin{aligned} \mathbf{HC}_* S_* (F \circ \alpha^0) &\underset{1}{\cong} H_* \| \amalg (F \circ \alpha^0) \| \\ &\underset{2}{\cong} H_* \| \amalg F \| \\ &\underset{3}{\cong} H_* \| \amalg (F \circ \beta^0) \| \\ &\underset{4}{\cong} H_*^G (\Lambda X; K). \end{aligned}$$

Isomorphism 1 holds with any functor Φ: $\Lambda^0 \to T$ in place of $F \circ \alpha^0$. It is the "cyclic" analogue of V.1.5. To prove it recall the notion [7, p. 153] of homology $H_*(\mathbf{C}; X)$ of a small category $\mathbf{C}$ with coefficients in a functor X: $\mathbf{C} \to \mathbf{A}$ to an abelian category (admitting exact infinite direct sums). This can be defined abstractly as the left derived functor of the direct limit functor (from the category of all functors from $\mathbf{C}$ to $\mathbf{A}$, to $\mathbf{A}$) or concretely by means of a chain complex $C_*(\mathbf{C}; X)$ with

$$C_n(\mathbf{C}; X) = \underset{\text{in } \mathbf{C}}{\underset{C_0 \to \cdots \to C_n}{\bigoplus}} X(C_0)$$

In the case $C = \Lambda^0$ there is a third way:

$$H_*(\Lambda^0; X) \cong HC_*(X).$$

In fact there is a natural quasi-isomorphism

(V.1.11) $\quad C_*(\Lambda^0; X) \to C_*(X)$.

(This is implicit in [5, §IV]. The construction of the complex $C_*(X)$ for any functor X: $\Lambda^0 \to A$ can be described in terms of a projective resolution which Connes constructs for the constant cocyclic object

$$\mathbf{Z}: \Lambda \to \{\text{abelian groups}\}.$$

In the same way the construction of $C_*(\mathbf{C}; X)$ can be described in terms of a certain canonical projective resolution of the constant functor

$$\mathbf{Z}: \mathbf{C}^0 \to \{\text{abelian groups}\}.$$

Taking $\mathbf{C} = \Lambda^0$ and recalling that projective resolutions are unique up to chain equivalence one can deduce V.1.11.)

In particular when $\mathbf{A}$ is {chain complexes of K-modules} and X is $S_*(\Phi)$ for some Φ we see that $\mathbf{HC}_* S_*(\Phi)$ is isomorphic to the hyperhomology of the complex of complexes $C_*(\Lambda^0; S_*(\Phi)) \cong S_*(\amalg \Phi)$, which in turn is isomorphic to $H_*(\| \amalg \Phi \|)$.

For isomorphisms 2 and 3 we reason as follows. The nerve NC of a topological category C is a simplicial space which can be defined as $\amalg(*)$, where $*$ denotes the constant functor $C \to T$ given by a one-point space. For Φ: $C \to T$ satisfying V.1.10 there is a map of simplicial spaces

$$\amalg \Phi \to NC$$

(induced by the natural map $\Phi \to *$), and if Φ takes all morphisms of C to homotopy equivalences then V.1.7 applies. In particular, consider

$$\begin{tikzcd} \amalg(F \circ \alpha^0) \arrow[r] \arrow[d] & N\Lambda^0 \arrow[d, "N\alpha^0"] \\ \amalg F \arrow[r] & NL^0 \\ \amalg(F \circ \beta^0) \arrow[r] \arrow[u] & NG^0 \arrow[u, "N\beta^0"'] \end{tikzcd}$$

The arrows on the right are equivalences by V.1.9; using V.1.7 those on the left are, too.

For isomorphism 4 note that

$$\Lambda X \underset{G}{\times} EG \cong |\amalg(F \circ \beta^0)|.$$

Thus it only remains to show that the natural map

$$\|\amalg(F \circ \beta^0)\| \to |\amalg(F \circ \beta^0)|.$$

is an equivalence. But $\amalg(F \circ \beta^0)$ is "good" (see [14]).

This concludes the proof of Theorem V.1.1, modulo Lemma V.1.9. □

2. *Proof of* V.1.9.

Recall from [5] that $|N\Lambda|$ is a $K(\mathbf{Z}, 2)$. (For an alternate proof of this apply Quillen's Theorem B ([13]) to the inclusion functor E: $\Delta^0 \to \Lambda^0$. The "under category" functor

$$\Lambda \to \mathrm{Cat}$$
$$Y \mapsto Y \backslash E$$

is such that the composite

$$\Lambda \to \mathrm{Cat} \xrightarrow{N} \text{Simplicial sets} \xrightarrow{\|\ \|} T$$

is isomorphic to our favorite cocyclic space $S^1 \times \Delta^{\cdot}$.) Of course $|NG|$ is also a $K(\mathbf{Z}, 2)$, and both maps

$$\|N\Lambda\| \to |N\Lambda|$$
$$\|NG\| \to |NG|$$

are equivalences.

To show that $N\beta$ is an equivalence just factor β as

$$G \to L_0 \hookrightarrow L$$

where L_0 has all the objects of L but the only maps

$$S^1 \times \Delta^m \to S^1 \times \Delta^n$$

in L_0 are those of the form gxf, where g is a rotation and f is continuous. The inclusion $NL_0 \to NL$ is an equivalence because this is so in each simplicial degree. (See [14] or use V.1.7.) But L_0 is the product (in an obvious sense) of G and another topological category whose nerve is contractible (it has a final object). Since nerve preserves products and $\|\ \ \|$ preserves products up to equivalence (see [14]) we are done.

Now since $\|N\Lambda\|$ and $\|NL\|$ are $K(\mathbf{Z}, 2)$'s it only remains to show that α induces an isomorphism

(V.2.1) $H^2(\|NL\|; Z) \to H^2(\|N\Lambda\|; Z)$.

We will do this by using finite cyclic groups. For each $m > 0$ there is a diagram of topological categories and continuous functors

$$\begin{tikzcd} & L & \\ \Lambda \arrow[ur, "\alpha"] & & G \arrow[ul, "\beta"'] \\ & \mathbf{Z}/m\mathbf{Z} \arrow[ul, "\gamma_m"] \arrow[ur, "\delta_m"'] & \end{tikzcd}$$

commuting up to a natural transformation (γ_m makes $\mathbf{Z}/m\mathbf{Z}$ act faithfully on the $(m-1)$-st object of Λ; δ_m is injective.) Passing to $H^2(\|N(-)\|; \mathbf{Z})$ we obtain a commutative diagram

$$\begin{tikzcd} & \mathbf{Z} \arrow[dl] \arrow[dr] & \\ \mathbf{Z} \arrow[dr] & & \mathbf{Z} \arrow[dl] \\ & \mathbf{Z}/m\mathbf{Z} & \end{tikzcd}$$

with both maps on the right surjective. It follows that V.2.1 is surjective mod m for all m, so is an isomorphism. □

3. *A Corollary*

We now work out the joint consequence of Theorems IV.2.6 and V.1.1. First we need to extend Theorem V.1.1 in two small ways:

Addendum V.3.1. Under the isomorphism of V.1.1 the map s (of III.2.2) corresponds to (plus or minus) the cap product map

$$\cap u\colon H_*^G(\Lambda X; K) \to H_{*-2}^G(\Lambda X; K)$$

where $u \in H_G^2(\Lambda X)$ is the pullback of a generator of $H^2(BG)$.

Addendum V.3.2. Theorem V.1.1 holds in the relative case: If $f\colon X \to Y$ is a map of pointed path-connected spaces then the relative homology with coefficients in K of the map

$$\Lambda f \underset{G}{\times} EG\colon \Lambda X \underset{G}{\times} EG \to \Lambda Y \underset{G}{\times} EG$$

is isomorphic to the relative cyclic hyperhomology

$$\mathbf{HC}_*(S_*(Mf; K))$$

of the map of chain algebras

$$S_*(Mf; K)\colon S_*(MX; K) \to S_*(MY; K)$$

(in the sense of III.2).

We will not prove the addenda; it is not hard to extract proofs of them from the proof of V.1.1.

Now assume $f\colon X \to Y$ is two-connected. It follows that the map of Moore loop spaces $Mf\colon MX \to MY$ is one-connected and $S_*(Mf; K)$ is a one-connected map of chain algebras (in the sense of §IV).

Thus by IV.2.6 the map

$$\cap k!u^k\colon H_{*+2k}^G(\Lambda f; K) \to H_*^G(\Lambda f; K)$$

is zero for $* < k$.

If $K = \mathbf{Q}$ we may omit the "$k!$" and use the Universal Coefficient Theorem to get that the cup product map

$$\cup u^k\colon H_G^*(\Lambda f; \mathbf{Q}) \to H_G^{*+2k}(\Lambda f; \mathbf{Q})$$

is zero. In particular

$$\varinjlim_k H_G^{*+2k}(\Lambda f; \mathbf{Q}) = 0.$$

That is,

COROLLARY V.3.3. *A two-connected map $f\colon X \to Y$ of spaces induces isomorphisms*

$$\varinjlim_k H_G^{*+2k}(\Lambda Y, Q) \xrightarrow{\cong} \varinjlim_k H_G^{*+2k}(\Lambda X; Q).$$

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