K-Theory **5**: 295–332, 1992.

Calculus II: Analytic Functors★

THOMAS G. GOODWILLIE
Department of Mathematics, Box 1917, Brown University, Providence, RI 02912, U.S.A.

(Received: November 1988)

Abstract. Homotopy functors (for example, from spaces to spaces) are called analytic if, when evaluated on certain n-cubical diagrams, they satisfy certain connectivity estimates. Tools for verifying these estimates include certain generalizations of the triad connectivity theorem. Waldhausen's functor A is analytic. Analyticity has strong consequences, when combined with the concept 'derivative of a homotopy functor' that was introduced in the previous article in this series. In particular, any analytic functor with derivative zero is, in a sense, locally constant.

Key words. Homotopy functor, cubical diagram, excision.

Introduction

The calculus of functors is a method for proving certain kinds of statements in homotopy theory. This article is concerned more with developing the general method than with making specific applications, but it is motivated largely by applications to $A(X)$, Waldhausen's algebraic K-theory of spaces. In particular, some results proved here are used in [2].

In [5], we began explaining the theory of 'calculus' by introducing the concept 'first derivative of a homotopy functor'. We also laid some of the groundwork for applications to K-theory, by 'calculating' the derivative $\partial_x A(X)$ of Waldhausen's functor A at any based space (X, x).

Here we show how such calculations can be used; we show that (very roughly speaking) the derivative of an *analytic* functor determines the functor, up to a constant. A little more precisely (for exact statements, see (5.3) below and its variants (5.7)–(5.10)), we have the following *first-derivative criterion*: If $F(X)$ and $G(X)$ are ρ-*analytic* functors of the space X, and if there is a natural map $F(X) \to G(X)$ such that for every (X, x) the induced map $\partial_x F(X) \to \partial_x G(X)$ is a weak homotopy equivalence, then the homotopy fiber of $F(X) \to G(X)$ is a locally constant functor of X, in the sense that its weak homotopy type depends only on the ρ-homotopy type of X for some number ρ. This criterion is used in [2], with $F = A$ and $\rho = 1$.

A functor from spaces to spaces (or to spectra) is *analytic* if it preserves weak homotopy equivalences, and if it satisfies certain connectivity estimates related to

★ Research partially supported by NSF grant DMS-8806444 and a Sloan Fellowship.

commutative diagrams in the shape of n-dimensional cubes, for all $n \geqslant 2$. An analytic functor is always ρ-analytic for some number ρ, which is smaller or larger according as the estimates are stronger or weaker.

Our chief aims here are to explain what is meant by analyticity, to show that the functor A is 1-analytic, and to prove the first-derivative criterion.

Section 1 is concerned with some simple facts about cubical diagrams of spaces. We claim no originality here.

Section 2 offers some more serious statements about cubical diagrams – generalizations of the Blakers–Massey triad connectivity theorem from triads (or square diagrams) to $(n+1)$-ads (or n-cubical diagrams). These results, Theorems 2.3 through 2.6, are the basic tools for proving that functors are analytic. The first of them has also been obtained by Ellis and Steiner [4], and in the special case of 1-connected spaces it goes back to Barratt and Whitehead [1].

In Section 4 we define 'ρ-analytic' and prove that Waldhausen's A is 1-analytic.

Section 5 is about the first-derivative criterion.

Section 3 introduces, and begins to examine, the concept of 'nth order excision'. It is not a logical prerequisite for Sections 4 and 5 and, in a way, it belongs more to [6] than to the present work. We have included it here since 'nth order excision' is a concept related to, but more elementary than, 'analyticity'.

An appendix is devoted to strengthening a result from [5]. (The strengthened version is needed in [2].) Theorem 2.1 of [5] identified the first derivative of the functor $Q(\Lambda X_+)$, stable homotopy of the free loopspace, up to natural weak homotopy equivalence. The stronger version takes into account the action of the self-maps of the circle on ΛX, and identifies the corresponding action of these on the derivative.

0. Prerequisites and Points of View

We make the following conventions: $\mathscr{U}$ is the category of unbased spaces. $\mathscr{T}$ is the category of based spaces. $\mathscr{S}$ is the category of spectra. These are what were called prespectra in [5]; thus a spectrum is a sequence $\{E(n) \mid n \geqslant 0\}$ of based spaces, equipped with based maps $E(n) \to \Omega E(n+1)$. A morphism $D \to E$ in $\mathscr{S}$ is a sequence of based maps $D(n) \to E(n)$ strictly respecting these structure maps. The homotopy groups of a spectrum are defined by $\pi_i(E) = \operatorname{colim} \pi_{i+n} E(n)$ for $i \in \mathbb{Z}$.

The following remarks may help orient the reader to a point of view which is adopted both here and in [5] and [6]:

We have to work with various categories in which there is a class of maps called 'equivalences'. In the category $\mathscr{U}$, a map is called an equivalence if it is a weak homotopy equivalence in the usual sense (it induces a bijection on π_0 and, for every basepoint in the domain, a bijection on all π_i, $i > 0$). In $\mathscr{T}$ a map is an equivalence if the forgetful functor $\mathscr{T} \to \mathscr{U}$ takes it to an equivalence. In $\mathscr{S}$ a map is an equivalence if it induces isomorphisms on spectrum homotopy groups.

Much of our work involves homotopy functors, by which we always mean those functors (from one given category to another) which take equivalences (whatever that might mean) to equivalences. The homotopy functors from, for example, $\mathscr{T}$ to $\mathscr{T}$, are the objects of a category whose morphisms are the natural maps.

We will often have to work with a category $\mathscr{F}$ of functors $\mathscr{C} \to \mathscr{D}$ in a situation where the category $\mathscr{D}$ has a class of maps called equivalences. (For example, $\mathscr{F}$ might be the category of all homotopy functors from $\mathscr{U}$ to $\mathscr{T}$, or it might be the category of all functors from some small category to spectra.) The morphisms of $\mathscr{F}$ are then all of the natural maps between these functors, and such a map is called an *equivalence* in $\mathscr{F}$ if it is an equivalence 'pointwise', that is, for every object of $\mathscr{C}$.

If a category $\mathscr{C}$ has a chosen class of morphisms w (called 'equivalences'), then by formally inverting these, we obtain the *homotopy category* $w^{-1}\mathscr{C}$. It has the same objects as $\mathscr{C}$. It is the target category of the universal functor $\mathscr{C} \to \mathscr{D}$ taking all elements of w to invertible morphisms.

The conscientious reader will note that, in general, a set-theoretic difficulty arises in defining $w^{-1}\mathscr{C}$ if $\mathscr{C}$ is not small. There are well-known ways around this in many cases, for example if $\mathscr{C}$ is equivalent to a small category, or if $\mathscr{C}$ admits a closed model structure with w as the weak equivalences. (In particular, $w^{-1}\mathscr{T}$ exists and is equivalent to the usual category of based CW complexes and homotopy classes of maps.) In any case, these difficulties need not concern us here, as we will not be using homotopy categories outside of these introductory remarks.

Note that a homotopy functor from $\mathscr{T}$ to $\mathscr{T}$ is much more than a functor $w^{-1}\mathscr{T} \to w^{-1}\mathscr{T}$, although of course it determines one. Functors from $w^{-1}\mathscr{T}$ to $w^{-1}\mathscr{T}$ are relatively useless.

We frequently use homotopy limits (=homotopy inverse limits) and homotopy colimits (=homotopy direct limits) of diagrams of spaces. These are defined just as in Bousfield–Kan [3], except that 'space' here means 'topological space' rather than 'simplicial set'. We now review the definitions and state some standard facts, all of which are either easy consequences of analogous results in [3] or else more or less immediate from the definitions.

First consider diagrams of unbased spaces, that is, functors $\mathscr{X} : \mathscr{C} \to \mathscr{U}$, where $\mathscr{C}$ is a small category. The diagram is said to be *indexed* by $\mathscr{C}$.

The homotopy limit is denoted $\operatorname{holim}(\mathscr{X})$; we also sometimes use notation like $\operatorname{holim}\{\mathscr{X}(C) : C \in \mathscr{C}\}$, leaving the morphisms in $\mathscr{C}$ implicit. (Bousfield and Kan write 'homotopy inverse limit' and use the symbol $\underleftarrow{\operatorname{holim}}$.) The space $\operatorname{holim}(\mathscr{X})$ is the cosimplicial realization (='Tot') of a certain cosimplicial space, the 'cosimplicial replacement' of $\mathscr{X}$ ([3], XI.5.1). The latter has as its pth space ($p \geqslant 0$) the product, over all p-simplices $C_0 \to \cdots \to C_p$ in the nerve of $\mathscr{C}$, of $\mathscr{X}(C_p)$.

Thus, from the definition of cosimplicial realization, a point in $\operatorname{holim}(\mathscr{X})$ can be described as a collection of continuous maps $\Delta^p \to \mathscr{X}(C_p)$, one for each choice of $p+1$ objects and p morphisms $C_0 \to \cdots \to C_p$, subject to certain compatibility conditions. This description can be reformulated as (0.1) below. If C is an object of $\mathscr{C}$, let $\mathscr{C} \downarrow C$ be the category whose objects are maps in $\mathscr{C}$ with target C, and in which a map from $f': C' \to C$ to $f'': C'' \to C$ is a map $g: C' \to C''$ such that $f' = f'' \circ g$. The space $|\mathscr{C} \downarrow C|$ (realization of the nerve) is a functor of C. It is easy to verify the following proposition.

0.1. PROPOSITION. *For any diagram* $\mathscr{X}: \mathscr{C} \to \mathscr{U}$, holim$(\mathscr{X})$ *is homeomorphic to the space of all natural maps* $|\mathscr{C} \downarrow C| \to \mathscr{X}(C)$, *topologized as a subspace of a product of (compact-open) function spaces.*

There is a canonical map $\lim(\mathscr{X}) \to \operatorname{holim}(\mathscr{X})$ from the (categorical) limit of $\mathscr{X}$ to the homotopy limit; in fact, $\lim(\mathscr{X})$ is the equalizer of the two coface maps from 0 to 1 in the cosimplicial replacement of $\mathscr{X}$. In terms of (0.1), $\lim(\mathscr{X})$ is the subspace of $\operatorname{holim}(\mathscr{X})$ given by constant maps.

Any (natural) map $\mathscr{X} \to \mathscr{Y}$ of $\mathscr{C}$-diagrams induces a map $\operatorname{holim}(\mathscr{X}) \to \operatorname{holim}(\mathscr{Y})$; the latter is a (weak homotopy) equivalence if for each object $C \in \mathscr{C}$ the map $\mathscr{X}(C) \to \mathscr{Y}(C)$ was an equivalence. That is, 'holim' is a homotopy functor from $\mathscr{C}$-diagrams of spaces to spaces.

Homotopy limit is functorial in the indexing category $\mathscr{C}$; in particular, if $\mathscr{D} \subset \mathscr{C}$ is a subcategory, then there is a restriction map from $\operatorname{holim}(\mathscr{X})$ to $\operatorname{holim}(\mathscr{X} \mid \mathscr{D})$. This is always a (Serre) fibration, as one easily checks. There is a useful sufficient condition ([3], XI.9.2) for a functor $\mathscr{D} \to \mathscr{C}$ to induce an equivalence from holim of a $\mathscr{C}$-diagram to holim of the composed $\mathscr{D}$-diagram. The condition is called 'left cofinality' in [3]; it says that for every object in $\mathscr{C}$ the fiber product category $\mathscr{D} \times_{\mathscr{C}} (\mathscr{C} \downarrow C)$ has contractible nerve. (In other words, it is the hypothesis of Quillen's 'Theorem A' for the opposite categories.) For us, a typical use will involve a pair of posets $\mathscr{D} \subset \mathscr{C}$ such that for every $C \in \mathscr{C}$ the set $\{D \in \mathscr{D} : D \leqslant C\}$ has a final element. A simple but important case deserves special mention: if the object D is initial in $\mathscr{C}$, then the restriction map $\operatorname{holim}(\mathscr{X}) \to \mathscr{X}(D)$ is an equivalence. In this case, $\lim(\mathscr{X})$ is $\mathscr{X}(D)$ and the canonical map $\lim(\mathscr{X}) \to \operatorname{holim}(\mathscr{X})$ is an equivalence, a section of the restriction map.

Suppose that a small category $\mathscr{A}$ is covered by two subcategories $\mathscr{A}_1$ and $\mathscr{A}_2$ in the sense that the nerve of $\mathscr{A}$ is the union of their nerves. This is the case, for example, when a poset $\mathscr{A}$ is the union of subsets which are either both concave (every element of $\mathscr{A}$ which is greater than an element of $\mathscr{A}_i$ is in $\mathscr{A}_i$) or both convex (every element of $\mathscr{A}$ which is less than an element of $\mathscr{A}_i$ is in $\mathscr{A}_i$). The following proposition is clear:

0.2. PROPOSITION. *If* $\mathscr{A}$ *is covered by* $\mathscr{A}_1$ *and* $\mathscr{A}_2$ *as above, then for any functor* $F: \mathscr{A} \to \mathscr{U}$, *the diagram of fibrations*

$$\begin{tikzcd}
\operatorname{holim}(F) \arrow[r] \arrow[d] & \operatorname{holim}(F \mid \mathscr{A}_1) \arrow[d] \\
\operatorname{holim}(F \mid \mathscr{A}_2) \arrow[r] & \operatorname{holim}(F \mid \mathscr{A}_1 \cap \mathscr{A}_2)
\end{tikzcd}$$

is a pullback square.

Let us examine the holim of a diagram of the form

$$X \xrightarrow{f} Y \xleftarrow{g} Z.$$

In this case, the homotopy limit can be identified (using (0.1)) with the space of all triples $(x, \varphi, z), x \in X, \varphi : I \to Y, z \in Z, \varphi(0) = f(x), \varphi(1) = g(z)$. The categorical limit is the fiber product of X and Z over Y. We recall the standard fact:

0.3. PROPOSITION. *If either f or g is a fibration, then the canonical map from* lim *to* holim *is an equivalence.*

As an important special case, when Z is a point we have the homotopy fiber of f, the homotopy inverse limit of the diagram

$$X \xrightarrow{f} Y \longleftarrow *.$$

The homotopy fiber of $f: X \to Y$ is defined as soon as Y is based; it is based itself if X and Y are both based and f is a based map. We will sometimes write 'fiber' for homotopy fiber, and on those occasions when we need to refer to a fiber in the strict sense (the preimage of a point), we will say so explicitly.

Dually, the homotopy colimit of a diagram $\mathscr{X}$ indexed by $\mathscr{C}$ is denoted hocolim($\mathscr{X}$) or hocolim$\{\mathscr{X}(C) : C \in \mathscr{C}\}$. (Bousfield and Kan write 'homotopy direct limit' and use the symbol $\overrightarrow{\text{holim}}$.) It is the realization of a certain simplicial space, the 'simplicial replacement' of $\mathscr{X}$ ([3], XII.5.1). The latter has as its pth space ($p \geqslant 0$) the disjoint union, over all p-simplices $C_0 \to \cdots \to C_p$ in the nerve of $\mathscr{C}$, of $\mathscr{X}(C_0)$.

The hocolim construction is again a homotopy functor from $\mathscr{C}$-diagrams of spaces to spaces. It has properties dual to those of holim: There is a canonical map from the hocolim to the (categorical) colimit of a diagram. It is an equivalence if $\mathscr{C}$ has a final object. A functor $\mathscr{D} \to \mathscr{C}$ induces a map to hocolim of a $\mathscr{C}$-diagram from hocolim of the composed $\mathscr{D}$-diagram. This is an equivalence if $\mathscr{D} \to \mathscr{C}$ satisfies a right cofinality condition (for every object C, the fiber product category $\mathscr{D} \times_{\mathscr{C}} (C \downarrow \mathscr{C})$ has contractible nerve). It is a cofibration if $\mathscr{D} \to \mathscr{C}$ is the inclusion of a subcategory, and in the situation of (0.2) the diagram of cofibrations

$$\begin{array}{ccc} \text{hocolim}(F \mid \mathscr{A}_1 \cap \mathscr{A}_2) & \longrightarrow & \text{hocolim}(F \mid \mathscr{A}_2) \\ \downarrow & & \downarrow \\ \text{hocolim}(F \mid \mathscr{A}_1) & \longrightarrow & \text{hocolim}(F) \end{array}$$

is a pushout square.

The hocolim of a diagram of the form

$$X \xleftarrow{f} Y \xrightarrow{g} Z$$

is the quotient of $X \amalg (Y \times I) \amalg Z$ by $(y, 0) \sim f(y)$ and $(y, 1) \sim g(y)$. It is equivalent to the categorical colimit (pushout) if either f or g is a cofibration. If Z is a point, we obtain the homotopy cofiber ('cofiber' for short) of a map $Y \to X$, also known as the mapping cone.

We also need homotopy limits and colimits for diagrams of *based* spaces. Here the correct definition of hocolim involves a based version of simplicial replacement, with wedge instead of disjoint union. This makes the hocolim a based space. If this construction is to take (pointwise) equivalences of diagrams to equivalences, then it should only be applied to diagrams of nondegenerately based spaces.

A warning: homotopy limits and homotopy colimits are *not* categorical limits and colimits in some homotopy category. For example, the weak homotopy type of the hocolim of $\mathscr{X}: \mathscr{C} \to \mathscr{U}$ is *not* determined by the composed functor $\mathscr{C} \to w^{-1}\mathscr{U}$, and colimits in $w^{-1}\mathscr{U}$ do not in general exist. (On the other hand, although we will not use this, it is true that on the level of homotopy categories 'holim' and 'hocolim' have certain universal properties. See [3], XI.8.1 and XII.2.4.)

1. Cubical Diagrams

We will be using (commutative) diagrams in the shape of cubes. Formally these are functors $\mathscr{X}: \mathscr{P}(S) \to \mathscr{C}$, where S is a finite set, $\mathscr{P}(S)$ is the poset of all subsets of S, and $\mathscr{C}$ is some category (usually a category of spaces or spectra). We call $\mathscr{X}$ an S-cube or, if the cardinality $|S|$ is n, an n-cube. Often S will be the standard set $\underline{n} = \{1, \ldots, n\}$.

Thus, 0-cubes correspond to objects of $\mathscr{C}$, and 1-cubes to morphisms. There is evidently a category whose objects are n-cubes and whose morphisms are $(n+1)$-cubes. An n-cube can be viewed as a map of $(n-1)$-cubes in n essentially distinct ways. A T-cube of U-cubes can be viewed as an S-cube, if S is the disjoint union of T and U.

Generalizing the homotopy fiber of a map of based spaces, we have the *total fiber* $\tilde{f}\mathscr{X}$ of an n-cube $\mathscr{X}$ of based spaces. This is a based space. It can be defined by induction: View $\mathscr{X}$ as a map $\mathscr{Y} \to \mathscr{Z}$ of $(n-1)$-cubes, and define $\tilde{f}\mathscr{X}$ as the (homotopy) fiber of $\tilde{f}\mathscr{Y} \to \tilde{f}\mathscr{Z}$. To begin the induction, set $\tilde{f}\mathscr{X} = \mathscr{X}(\emptyset) = \mathscr{X}$ for a 0-cube.

It is useful to have other descriptions of the total fiber. We take the following one as a definition:

1.1. DEFINITION. Let $\mathscr{X}$ be an S-cube of based spaces. A point $\Phi \in \tilde{f}\mathscr{X}$ is a collection of continuous maps $\Phi_T: I^T \to \mathscr{X}(T)$, one for each subset $T \subset S$, satisfying (i) and (ii) below. (The space I^T is a topological cube, the product of T copies of I.) As a topological space, $\tilde{f}\mathscr{X}$ is a subset of a product of compact-open function spaces.

(i) Φ is natural with respect to T. That is, for $U \subset T \subset S$ the diagram commutes:

$$\begin{tikzcd} I^U \arrow[r] \arrow[d, "\Phi_U"'] & I^T \arrow[d, "\Phi_T"] \\ \mathscr{X}(U) \arrow[r] & \mathscr{X}(T) \end{tikzcd}$$

where the upper arrow is the map which takes a function $U \to I$ and extends it to a function $T \to I$ by making it zero on $T - U$.

(ii) For each $T \subset S$, Φ_T takes the set $(I^T)_1 = \{u \in I^T : \exists_{s \in T}\, u_s = 1\}$ to the base-point in $\mathscr{X}(T)$.

It is easy to see that (1.1) agrees with the inductive description of $\mathfrak{f}\mathscr{X}$, up to natural homeomorphism. One advantage of (1.1) is that it gives $\mathfrak{f}\mathscr{X}$ more structure: it makes it functorial with respect to bijections in the 'S' variable.

There is an alternative definition using homotopy (inverse) limits. The homotopy limit of $\mathscr{X} : \mathscr{P}(S) \to \mathscr{T}$ itself is homotopy equivalent to $\mathscr{X}(\emptyset)$, because $\emptyset$ is initial in $\mathscr{P}(S)$. Let $\mathscr{P}_0(S) \subset \mathscr{P}(S)$ be the poset consisting of all nonempty $T \subset S$, and let $h_0(\mathscr{X})$ be $\operatorname{holim}(\mathscr{X} \mid \mathscr{P}_0(S))$, the homotopy limit of the composed functor $\mathscr{P}_0(S) \to \mathscr{P}(S) \to \mathscr{T}$. The restriction map

$$\operatorname{holim}(\mathscr{X}) \longrightarrow h_0(\mathscr{X})$$

is a fibration.

1.1a. ALTERNATIVE DESCRIPTION. The total fiber $\mathfrak{f}\mathscr{X}$ is the fiber in the strict sense (not the homotopy fiber) of this restriction map

$$\operatorname{holim}(\mathscr{X}) \longrightarrow h_0(\mathscr{X}).$$

To see that (1.1) and (1.1a) are really the same (up to natural homeomorphism), we use (0.1): Observe that the topological pair $(|\mathscr{P}(S) \downarrow T|, |\mathscr{P}_0(S) \downarrow T|) = (|\mathscr{P}(T)|, |\mathscr{P}_0(T)|)$ is, as a functor of T, isomorphic to $(I^T, (I^T)_1)$. Thus, $\operatorname{holim}(\mathscr{X})$ is identified with the space of all collections $\{\Phi_T\}$ satisfying (i) above (but not necessarily (ii)), and $h_0(\mathscr{X})$ is identified with the space of all collections $\{\Phi_T : (I^T)_1 \to \mathscr{X}(T) : T \subset S\}$ satisfying (i). The strict fiber of the restriction map can be identified with $\mathfrak{f}\mathscr{X}$.

We offer yet another description of $\mathfrak{f}\mathscr{X}$. This involves a map

$$a(\mathscr{X}) : \mathscr{X}(\emptyset) \longrightarrow h_0(\mathscr{X}).$$

1.2. DEFINITION. If $\mathscr{X}$ is a cubical diagram of spaces, then $a(\mathscr{X})$ is the composition from upper left to lower right in the canonical diagram:

$$\begin{tikzcd}
\mathscr{X}(\emptyset) = \lim \mathscr{X} \arrow[r] \arrow[d] & \operatorname{holim} \mathscr{X} \arrow[d] \\
\lim(\mathscr{X} | \mathscr{P}_0(S)) \arrow[r] & \operatorname{holim}(\mathscr{X} | \mathscr{P}_0(S)) = h_0(\mathscr{X})
\end{tikzcd}$$

(The upper and right-hand maps here are the homotopy equivalence and the fibration mentioned in the discussion preceding (1.1a).)

1.1b. ALTERNATIVE DESCRIPTION. $\mathfrak{f}\mathscr{X}$ is homeomorphic to the *homotopy* fiber of $a(\mathscr{X})$.

To see that (1.1b) is the same as (1.1), think of a point in the homotopy fiber of $a(\mathscr{X})$ as consisting of a point $x \in \mathscr{X}(\emptyset)$ and a collection of paths $t \mapsto \Phi_{T,t}$ $(0 \leqslant t \leqslant 1)$

of maps $u \mapsto \Phi_{T,t}(u)$ from $(I^T)_1$ to $\mathscr{X}(T)$, compatible as in (i), such that $\Phi_{T,0}(u)$ is the image of x in $\mathscr{X}(T)$ for all u and $\Phi_{T,1}(u)$ is always the basepoint. This yields a point Φ in $\tilde{f}\mathscr{X}$ (as defined in (1.1)), by putting $\Phi_T(tu) = \Phi_{T,t}(u)$.

Of course, Definition (1.2) makes sense even if $\mathscr{X}$ is a cube of unbased spaces. We leave for the reader to check that $a(\mathscr{X})$ has an alternative, inductive, definition which begins with the case of a 2-cube:

1.2a. ALTERNATIVE DESCRIPTION. If the n-cube $\mathscr{X}$ is the map of $(n-1)$-cubes $\mathscr{Y} \to \mathscr{Z}$, then $a(\mathscr{X})$ is 'a' for the 2-cube

$$\begin{tikzcd} \mathscr{Y}(\emptyset) \arrow[r] \arrow[d, "a(\mathscr{Y})"'] & \mathscr{Z}(\emptyset) \arrow[d, "a(\mathscr{Z})"] \\ h_0(\mathscr{Y}) \arrow[r] & h_0(\mathscr{Z}) \end{tikzcd}$$

1.3. DEFINITION. The cubical diagram $\mathscr{X}$ is *Cartesian* if $a(\mathscr{X})$ is a weak homotopy equivalence. It is *k-Cartesian* if $a(\mathscr{X})$ is a k-connected map.

(A map of spaces is called k-connected if each of its homotopy fibers is $(k-1)$-connected. The convention here is that every space is (-2)-connected, nonempty spaces are (-1)-connected, path-connected spaces are 0-connected, ...)

For any n-cubical diagram $\mathscr{X}$ of based spaces we can define $\pi_i(\mathscr{X}) = \pi_{i-n}(\tilde{f}\mathscr{X})$, although we will have little or no occasion to use this definition. Note that for a map $\mathscr{X} \to \mathscr{Y}$ of n-cubes viewed as an $(n+1)$-cube, there is a long exact sequence

$$\cdots \longrightarrow \pi_i(\mathscr{X}) \longrightarrow \pi_i(\mathscr{Y}) \longrightarrow \pi_i(\mathscr{X} \longrightarrow \mathscr{Y}) \longrightarrow \pi_{i-1}(\mathscr{X}) \longrightarrow \cdots$$

ending in the based set $\pi_n(\mathscr{Y})$. If $\mathscr{X}$ is k-Cartesian, then $\pi_i(\mathscr{X})$ is trivial for $i < k + n$. This implication is not quite reversible in general because of difficulty with π_0; if it were reversible, then some proofs below could be shortened.

We call a map $\mathscr{X} \to \mathscr{Y}$ of n-cubes an *equivalence* if, for every T, the map $\mathscr{X}(T) \to \mathscr{Y}(T)$ of spaces is a (weak homotopy) equivalence. This implies (but of course is not implied by) the statement that the $(n+1)$-cube $\mathscr{X} \to \mathscr{Y}$ is Cartesian. An equivalence $\mathscr{X} \to \mathscr{Y}$ induces an equivalence $\tilde{f}\mathscr{X} \to \tilde{f}\mathscr{Y}$, because holim is a homotopy functor. For the same reason, any cube which admits an equivalence to or from a k-Cartesian cube is itself k-Cartesian.

The discussion above can be 'dualized'. The main point is to introduce the poset $\mathscr{P}_1(S) \subset \mathscr{P}(S)$ of all $T \subsetneq S$, and the map

$$b(\mathscr{X}) : h_1(\mathscr{X}) \longrightarrow \mathscr{X}(S),$$

composition from upper left to lower right in the canonical diagram:

$$\begin{tikzcd} h_1(\mathscr{X}) = \operatorname{hocolim}(\mathscr{X} \mid \mathscr{P}_1(S)) \arrow[r] \arrow[d] & \operatorname{colim}(\mathscr{X} \mid \mathscr{P}_1(S)) \arrow[d] \\ \operatorname{hocolim}(\mathscr{X}) \arrow[r] & \operatorname{colim}(\mathscr{X}) = \mathscr{X}(S) \end{tikzcd}$$

(The lower map is a homotopy equivalence and the left-hand map is a cofibration.)

1.4. DEFINITION. The *total cofiber* $\tilde{c}\mathscr{X}$ is the homotopy cofiber of $b(\mathscr{X})$. The cube $\mathscr{X}$ is *co-Cartesian* if $b(\mathscr{X})$ is a weak homotopy equivalence. It is *k-co-Cartesian* if $b(\mathscr{X})$ is a k-connected map.

One can define the homology of a cubical diagram to be the reduced homology of the total cofiber, $H_i\mathscr{X} = \tilde{H}_i(\tilde{c}\mathscr{X})$. Then, any map of cubes gives an exact sequence

$$\cdots \longrightarrow H_i(\mathscr{X}) \longrightarrow H_i(\mathscr{Y}) \longrightarrow H_i(\mathscr{X} \longrightarrow \mathscr{Y}) \longrightarrow H_{i-1}(\mathscr{X}) \longrightarrow \cdots$$

If $\mathscr{X}$ is k-co-Cartesian, then $H_i\mathscr{X} = 0$ for $i \leqslant k$. This implication is reversible if, for example, all of the spaces $\mathscr{X}(T)$ are 1-connected.

The following is an easy exercise (and more or less well-known):

1.5. PROPOSITION. *For any maps of spaces $X \to Y \to Z$:*

(i) *$X \to Z$ is k-connected if $X \to Y$ and $Y \to Z$ are both k-connected.*
(ii) *$X \to Y$ is k-connected if $Y \to Z$ is $(k+1)$-connected and $X \to Z$ is k-connected.*
(iii) *$Y \to Z$ is k-connected if $X \to Z$ is k-connected and $X \to Y$ is $(k-1)$-connected.*

It implies:

1.6. PROPOSITION. *For any map $\mathscr{X} \to \mathscr{Y}$ of cubes:*

(i) *$\mathscr{X}$ is k-Cartesian if $\mathscr{Y}$ is k-Cartesian and $\mathscr{X} \to \mathscr{Y}$ is k-Cartesian.*
(ii) *$\mathscr{X} \to \mathscr{Y}$ is k-Cartesian if $\mathscr{X}$ is k-Cartesian and $\mathscr{Y}$ is $(k+1)$-Cartesian.*

Proof. If $\mathscr{X}$ and $\mathscr{Y}$ are S-cubes, let $\mathscr{Z}$ be the $(S \cup *)$-cube $\mathscr{X} \to \mathscr{Y}$. Thus $\mathscr{X}(T) = \mathscr{Z}(T)$ and $\mathscr{Y}(T) = \mathscr{Z}(T \cup *)$ for $T \subset S$. Write $\mathscr{P}_0(S \cup *)$ as the union of $\mathscr{A} = \{T \subset S \cup * \mid T \cap S \neq \emptyset\}$ and $\mathscr{B} = \{T \subset S \cup * \mid * \in T\}$. Consider the diagram

$$\begin{tikzcd}
\mathscr{X}(\emptyset) \arrow[r, "a(\mathscr{Z})"] & h_0(\mathscr{Z}) \arrow[r] \arrow[d] & \operatorname{holim}(\mathscr{Z} \mid \mathscr{A}) \arrow[r, "\sim"] \arrow[d] & \operatorname{holim}(\mathscr{Z} \mid \mathscr{P}_0(S)) = h_0(\mathscr{X}) \\
\mathscr{Y}(\emptyset) \arrow[r, "\sim"'] & \operatorname{holim}(\mathscr{Z} \mid \mathscr{B}) \arrow[r] & \operatorname{holim}(\mathscr{Z} \mid \mathscr{A} \cap \mathscr{B}) &
\end{tikzcd}$$

in which all arrows except the two on the left are restriction maps from holim indexed by a poset to holim indexed by a smaller poset. The arrows marked '$\sim$' are equivalences, because $\mathscr{P}_0(S)$ is left cofinal in $\mathscr{A}$ and $*$ is initial in $\mathscr{B}$. The composition in the lower row is $a(\mathscr{Y})$. The square is a pullback square of fibrations, by (0.2), so that the composed map $a'\colon h_0(\mathscr{Z}) \to h_0(\mathscr{X})$ is at least as highly connected as $a(\mathscr{Y})$.

To obtain the conclusion, apply (1.5) to the maps

$$\mathscr{X}(\emptyset) \xrightarrow{a(\mathscr{Z})} h_0(\mathscr{Z}) \xrightarrow{a'} h_0(\mathscr{X})$$

and note that $a' \circ a(\mathscr{Z}) = a(\mathscr{X})$.

The reason why (1.6) has no part (iii) is that a' can be more highly connected than $a(\mathscr{Y})$. (The space $h_0(\mathscr{Y})$ might not be 0-connected.) $\square$

There is also a dual statement, with a dual proof which we omit:

1.7. PROPOSITION. *For any map $\mathscr{X} \to \mathscr{Y}$ of cubes*:

(i) *$\mathscr{Y}$ is k-co-Cartesian if $\mathscr{X} \to \mathscr{Y}$ is k-co-Cartesian and $\mathscr{X}$ is k-co-Cartesian.*
(ii) *$\mathscr{X} \to \mathscr{Y}$ is k-co-Cartesian if $\mathscr{X}$ is $(k-1)$-co-Cartesian and $\mathscr{Y}$ is k-co-Cartesian.*

Some parts of (1.5) generalize to cubes:

1.8. PROPOSITION. *For maps of cubes of spaces $\mathscr{X} \to \mathscr{Y} \to \mathscr{Z}$:*

(i) *$\mathscr{X} \to \mathscr{Z}$ is k-Cartesian if $\mathscr{X} \to \mathscr{Y}$ and $\mathscr{Y} \to \mathscr{Z}$ are k-Cartesian.*
(ii) *$\mathscr{X} \to \mathscr{Z}$ is k-co-Cartesian if $\mathscr{X} \to \mathscr{Y}$ and $\mathscr{Y} \to \mathscr{Z}$ are k-co-Cartesian.*
(iii) *$\mathscr{X} \to \mathscr{Y}$ is k-Cartesian if $\mathscr{X} \to \mathscr{Z}$ is k-Cartesian and $\mathscr{Y} \to \mathscr{Z}$ is $(k+1)$-Cartesian.*
(iv) *$\mathscr{Y} \to \mathscr{Z}$ is k-co-Cartesian if $\mathscr{X} \to \mathscr{Y}$ is $(k-1)$-co-Cartesian and $\mathscr{X} \to \mathscr{Z}$ is k-co-Cartesian.*

Proof. For (i) apply (1.6) twice to

$$\begin{tikzcd} \mathscr{X} \arrow[r] \arrow[d] & \mathscr{Y} \arrow[d] \\ \mathscr{Z} \arrow[r, phantom, "="] & \mathscr{Z} \end{tikzcd}$$

First view the square as $(\mathscr{X} \to \mathscr{Y}) \to (\mathscr{Z} \to \mathscr{Z})$ and apply (1.6.ii); then view it as $(\mathscr{X} \to \mathscr{Z}) \to (\mathscr{Y} \to \mathscr{Z})$ and apply (1.6.i).

The proofs of the other statements are similar. □

The next result generalizes (0.2). Suppose that a small category $\mathscr{A}$ is covered by a finite collection of subcategories $\{\mathscr{A}_s : s \in S\}$ in the sense that (1) the union $\bigcup_s N\mathscr{A}_s$ of their nerves is the nerve $N\mathscr{A}$ and (2) for each subset $T \subset S$ the union $\bigcup_{s \in T} N\mathscr{A}_s$ is the nerve of a subcategory $\mathscr{A}_T = \bigcup_{s \in T} \mathscr{A}_s$. This is the case, for example, when a poset $\mathscr{A}$ is the union of subsets $\mathscr{A}_s$ which are either all concave or all convex.

1.9. LEMMA. *Let $(\mathscr{A}, \{\mathscr{A}_s\})$ be as above and let F be a functor from $\mathscr{A}$ to spaces. Then the S-cube defined by*

$$T \mapsto \operatorname{holim}\left(F \mid \bigcap_{s \in T} \mathscr{A}_s\right),$$

$$\emptyset \mapsto \operatorname{holim}(F)$$

is Cartesian.

Proof. In the case of only two subcategories $\{\mathscr{A}_1, \mathscr{A}_2\}$, the assertion is (0.2). The general case follows by induction: Write S as $T \cup 0$ and consider the diagram

$$\begin{tikzcd}
\operatorname{holim}(F) \arrow[r] \arrow[d] & \operatorname{holim}(F \mid \mathscr{A}_0) \arrow[d] \\
\operatorname{holim}(F \mid \mathscr{A}_T) \arrow[r] \arrow[d, "\sim"'] & \operatorname{holim}(F \mid \mathscr{A}_T \cap \mathscr{A}_0) \arrow[d, "\sim"] \\
\operatorname{holim}\left\{U \mapsto \operatorname{holim}\left(F \mid \bigcap_{s \in U} \mathscr{A}_s\right)\right\} \arrow[r] & \operatorname{holim}\left\{U \mapsto \operatorname{holim}\left(F \mid \bigcap_{s \in U} (\mathscr{A}_s \cap \mathscr{A}_0)\right)\right\}
\end{tikzcd}$$

where the spaces in the two lowest corners are holims indexed by $U \in \mathscr{P}_0(T)$. The upper square is Cartesian by (0.2). The marked arrows are equivalences by induction. Thus, by (1.8.i), the outer square is Cartesian. This is the assertion to be proved, in view of (1.2.a). □

1.10. LEMMA. *Let $(\mathscr{A}, \{\mathscr{A}_s\})$ be as above and let F be a functor from $\mathscr{A}$ to spaces. Then the S-cube defined by*

$$T \mapsto \operatorname{hocolim}\left(F \mid \bigcap_{s \in S - T} \mathscr{A}_s\right),$$

$$S \mapsto \operatorname{hocolim}(F)$$

is co-Cartesian.

Proof. Dual to the proof of (1.9). □

1.11. *Remark.* Any functor $\mathscr{X}$ from $\mathscr{P}_0(S)$ to spaces determines a Cartesian S-cube $\mathscr{Y}$ such that $\mathscr{Y}(\emptyset) = \operatorname{holim}(\mathscr{X})$, and such that for $T \neq \emptyset$ there is an equivalence, natural in T, from $\mathscr{X}(T)$ to $\mathscr{Y}(T)$. Namely, define $\mathscr{Y}(T)$ as the homotopy limit of the restriction of $\mathscr{X}$ to $\{W : \emptyset \neq W \supset T\} \subset \mathscr{P}_0(S)$. There is an obvious equivalence $e : \mathscr{X}(T) \to \mathscr{Y}(T)$ for $T \in \mathscr{P}_0(S)$, since T is the initial W; and e is natural in T. The cube $\mathscr{Y}$ is Cartesian by (1.9). This remark has a dual which we will not write out.

1.12. *Notation.* For an S-cube $\mathscr{X}$ and subsets $U \subset T \subset S$, $\partial_U^T \mathscr{X}$ is the $(T - U)$-cube $\{V \mapsto \mathscr{X}(V \cup U) : V \subset T - U\}$. We call these cubes the *faces* of $\mathscr{X}$. We abbreviate $\partial_\emptyset^T \mathscr{X}$ by $\partial^T \mathscr{X}$ and $\partial_U^S \mathscr{X}$ by $\partial_U \mathscr{X}$.

1.13. DEFINITION. The S-cube of spaces $\mathscr{X}$ is a *fibration cube* if for every $U \subset S$ the restriction map $\mathscr{X}(U) = \lim\{\mathscr{X}(T) : U \subset T \subset S\} \to \lim\{\mathscr{X}(T) : U \subsetneq T \subset S\}$ is a fibration. It is a *cofibration cube* if for every $T \subset S$ the map $\operatorname{colim}\{\mathscr{X}(U) : U \subsetneq T\} \to \operatorname{colim}\{\mathscr{X}(U) : U \subset T\} = \mathscr{X}(T)$ is a cofibration.

1.14. *Remark.* Any S-cube $\mathscr{X}$ of spaces is equivalent to a fibration S-cube $\mathscr{Y}$, for example the one given by $\mathscr{Y}(U) = \operatorname{holim}(\partial_U \mathscr{X})$: The canonical map from $\mathscr{X}(U) = \lim(\partial_U \mathscr{X})$ to $\mathscr{Y}(U)$ is an equivalence, natural in U, and the restriction map $\mathscr{Y}(U) \to \lim\{\mathscr{Y}(T) : U \subset T \subset S, U \subsetneq T\}$ is a fibration because it is isomorphic to the restriction map from a holim indexed by $\mathscr{P}(S - U)$ to a holim indexed by $\mathscr{P}_0(S - U)$. Dually, $\mathscr{X}$ is equivalent to the cofibration cube $T \mapsto \operatorname{hocolim}(\partial^T \mathscr{X})$.

1.15. PROPOSITION. *When $\mathscr{X}$ is a fibration cube, then the canonical map $lim(\mathscr{X} \mid \mathscr{P}_0(S)) \to h_0(\mathscr{X})$ is an equivalence. Thus, a fibration cube $\mathscr{X}$ is k-Cartesian if and only if $\mathscr{X}(\emptyset) \to \lim(\mathscr{X} \mid \mathscr{P}_0(S))$ is a k-connected map.*

Proof. This is by induction on the cardinality $|S|$. We use the evident fact that the subcubes $\partial_U^T\mathscr{X}$ are also fibration cubes.

When $|S| \leqslant 1$ the map is a homeomorphism.

When $|S| = 2$ the assertion follows from (0.3).

If $|S| > 2$ write $\mathscr{X}$ as a map of T-cubes $\mathscr{Y} \to \mathscr{Z}$; the assertion is then that the diagram

$$\begin{array}{ccccc}
\lim(\mathscr{Y} \mid \mathscr{P}_0(T)) & \longrightarrow & \lim(\mathscr{Z} \mid \mathscr{P}_0(T)) & \longleftarrow & \mathscr{Z}(\emptyset) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle 1} \\
\operatorname{holim}(\mathscr{Y} \mid \mathscr{P}_0(T)) & \longrightarrow & \operatorname{holim}(\mathscr{Z} \mid \mathscr{P}_0(T)) & \longleftarrow & \mathscr{Z}(\emptyset)
\end{array}$$

induces an equivalence from lim of the top row to holim of the bottom row. But lim(top) $\to$ holim(top) $\to$ holim(bottom) are both equivalences, the latter because the vertical maps are (by induction) equivalences, the former by (0.3) again because the upper right map is a fibration. □

1.16. PROPOSITION. *When $\mathscr{X}$ is a cofibration cube then the canonical map $h_1(\mathscr{X}) \to \operatorname{colim}(\mathscr{X} \mid \mathscr{P}_1(S))$ is an equivalence. Thus, a cofibration cube $\mathscr{X}$ is k-co-Cartesian if and only if $\operatorname{colim}(\mathscr{X} \mid \mathscr{P}_1(S)) \to \mathscr{X}(S)$ is a k-connected map.*

Proof. Dual to the proof of (1.15). □

1.17. *Remark.* The concept of n-cubical diagram is of course related to the concept of $(n+1)$-ad. An $(n+1)$-ad $(X; \{X_s\}_{s\in S})$ consists of a space and n subspaces, and it determines an S-cube of spaces $\mathscr{X}$ by putting $\mathscr{X}(S) = X$, $\mathscr{X}(S - s) = X_s$, $\mathscr{X}(S - T) = \bigcap_{s\in T} X_s$. Thus, an $(n-1)$-ad corresponds to an S-cube $\mathscr{X}$ of a special kind, namely one in which (1) all of the maps $\mathscr{X}(T) \to \mathscr{X}(S)$ are inclusions of subspaces, and (2) $\mathscr{X}(T \cap U) = \mathscr{X}(T) \cap \mathscr{X}(U)$. The cofibration n-cubes are the same (up to homeomorphism) as those $(n+1)$-ads in which "every inclusion map that you can think of" is a cofibration. In particular, the cofibration cubes include the cubes determined by 'CW $(n+1)$-ads', that is, those for which X is a CW complex and each X_s is a subcomplex. The next statement is about a different class of $(n+1)$-ads, those for which each X_s is open in X. These do not correspond to cofibration cubes, but they still satisfy the conclusion of (1.16):

1.16a. PROPOSITION. *The conclusion of* (1.16) *also holds when the cube $\mathscr{X}$ has the following form: For each $T \subset S$, $\mathscr{X}(T) \to \mathscr{X}(S)$ is the inclusion of an open subset of $\mathscr{X}(S)$, and for each T and U, $\mathscr{X}(T \cap U) = \mathscr{X}(T) \cap \mathscr{X}(U)$.*

Proof. The general case can be obtained from the case $|S| = 2$ by an induction which we leave to the reader. In the case $|S| = 2$ the statement is that if A and B are open subsets of a space, then the natural map from $\operatorname{hocolim}(A \leftarrow A \cap B \to B)$ to $\operatorname{colim}(A \leftarrow A \cap B \to B) = A \cup B$ is an equivalence. Replacing A, B, $A \cap B$, and $A \cup B$ by the realizations of their singular complexes $|S(A)|, \ldots,$ and using the fact that

$|S(\ )|$ takes inclusions to cofibrations, and also using (1.16), we reduce to showing that the inclusion $|S(A)| \cup |S(B)| = |S(A) \cup S(B)| \to |S(A \cup B)|$ is an equivalence. This follows from the Seifert–van-Kampen theorem and the excision property of singular homology (which both depend on the openness). □

The next result is useful in getting around some of the confusion that can surround basepoints and π_0 in this subject. The proof uses the following obvious remark: In a map $(E \to B) \to (E' \to B')$ of fibrations, if the map $B \to B'$ of bases is an equivalence, then the map $E \to E'$ of total spaces is k-connected if and only if, for every point in B, the map of (strict) fibers is k-connected.

Note that if $\mathscr{X} \to \mathscr{Y}$ is a map of n-cubes of spaces then, for each point $y \in \mathscr{Y}(\emptyset)$, the spaces

$$\mathscr{F}_y(T) = \text{homotopy fiber of } \mathscr{X}(T) \longrightarrow \mathscr{Y}(T)$$

form another n-cube.

1.18. PROPOSITION. *A map $\mathscr{X} \to \mathscr{Y}$ of n-cubes of spaces is k-Cartesian (as an $(n+1)$-cube) if and only if, for every $y \in \mathscr{Y}(\emptyset)$, the n-cube $\mathscr{F}_y$ as defined above is k-Cartesian.*

Proof. Call the $(n+1)$-cube $\mathscr{Z}$, and consider the square diagram

$$\begin{array}{ccc} \mathscr{X}(\emptyset) & \xrightarrow{a(\mathscr{Z})} & h_0(\mathscr{Z}) \\ \downarrow & & \downarrow \\ \mathscr{Y}(\emptyset) & \xrightarrow[\sim]{} & \operatorname{holim}(\mathscr{Y}) \end{array}$$

The lower map is the canonical equivalence. The right-hand map is a fibration, the restriction to holim indexed by a subcategory. From the diagram in the proof of (1.6), its fiber (not homotopy fiber) over the image of any $y \in \mathscr{Y}(\emptyset)$ is $h_0(\mathscr{F}_y)$. Without loss of generality, the map on the left is also a fibration. Then the upper map $a(\mathscr{Z})$ is k-connected if and only if, for every y, the square induces a k-connected map of strict fibers of vertical maps. For any y, the map of fibers is isomorphic to the composition

$$[\text{strict fiber of } \mathscr{X}(\emptyset) \longrightarrow \mathscr{Y}(\emptyset)] \xrightarrow{\sim} \mathscr{F}_y(\emptyset) \xrightarrow{a(\mathscr{F}_y)} h_0(\mathscr{F}_y),$$

of $a(\mathscr{F}_y)$ with the inclusion of the strict fiber in the homotopy fiber. Thus it is k-connected if and only if $a(\mathscr{F}_y)$ is. □

1.19. *Remark.* Most of the results above, about cubical diagrams of spaces, carry over to cubical diagrams of spectra if we make the following conventions: For a cube $T \mapsto \mathscr{X}(T) = \{\mathscr{X}(T, n)\}$ of spectra, the total fiber $\tilde{f}\mathscr{X}$ is a spectrum whose nth space is the total fiber of $T \mapsto \mathscr{X}(T, n)$. (Structure maps are defined because 'Ω' commutes with $\tilde{f}$.) The total cofiber $\tilde{c}\mathscr{X}$ is a spectrum whose nth space is the total cofiber of $T \mapsto \mathscr{X}(T, n)$. (Structure maps are defined because 'Σ' commutes with $\tilde{c}$.) Cubes of spectra have all the good properties of cubes of spaces, and more: A map of spectra is an equivalence if and only if its (homotopy) fiber is equivalent to a point, and if and only if its (homotopy) cofiber is equivalent to a point. Moreover, fiber is naturally equivalent to Ω(cofiber), so 'Cartesian' = 'co-Cartesian' and 'k-Cartesian' = '$(k+n-1)$-co-Cartesian' for n-cubes.

Furthermore, the suspension spectrum functor $X \mapsto \Sigma^\infty X$ from (good) based spaces to spectra preserves k-co-Cartesian cubes, and the functor $E \mapsto \Omega^\infty E$ from spectra to based spaces (taking a spectrum to the 0th space of an equivalent Ω-spectrum) preserves k-Cartesian cubes. In particular, the composed functor $Q = \Omega^\infty\Sigma^\infty$ from spaces to spaces takes k-co-Cartesian n-cubes to $(k+1-n)$-Cartesian n-cubes.

The next four results will be needed in [6].

1.20. PROPOSITION. *Let $\mathscr{X}$ be an S-cube $\{U \mapsto \mathscr{X}(U)\}$ of T-cubes $\mathscr{X}(U) = \{V \mapsto \mathscr{X}(U, V)\}$ of spaces. Assume that $\mathscr{X}$, viewed as $(S \amalg T)$-cube, is k-Cartesian and that for each $U \neq \emptyset$, the T-cube $\mathscr{X}(U)$ is $(k+|U|-1)$-Cartesian. Then the T-cube $\mathscr{X}(\emptyset)$ is k-Cartesian.*

Proof. Induction with respect to $|S|$, using (1.6). □

1.21. PROPOSITION. *Let $\mathscr{X}$ be an S-cube $\{U \mapsto \mathscr{X}(U)\}$ of T-cubes $\mathscr{X}(U) = \{V \mapsto \mathscr{X}(U, V)\}$ of spaces. Assume that $\mathscr{X}$, viewed as $(S \amalg T)$-cube, is k-co-Cartesian and that for each $U \subsetneq S$ in S, the T-cube $\mathscr{X}(U)$ is $(k+1+|U|-|S|)$-co-Cartesian. Then the T-cube $\mathscr{X}(S)$ is k-co-Cartesian.*

Proof. Induction with respect to $|S|$, using (1.7). □

1.22. PROPOSITION. *Let $\mathscr{X}$ be a functor from $\mathscr{P}_0(S)$ to T-cubes of spaces (or spectra), and write $\mathscr{X}(U, V) = (\mathscr{X}(U))(V)$. (So $\mathscr{X}(U, V)$ is defined for $\emptyset \neq U \subset S$ and $V \subset T$.) Assume that for all $U \neq \emptyset$ the T-cube $\mathscr{X}(U)$ is k_U-Cartesian. Then the T-cube $V \mapsto \operatorname{holim}(U \mapsto \mathscr{X}(U, V))$ is k-Cartesian with $k = \min\{1-|U|+k_U\}$.*

Proof. Apply (1.11) to each of the functors $U \mapsto \mathscr{X}(U, V)$ to make an S-cube $U \mapsto \mathscr{Y}(U, V)$. Apply (1.20) to the resulting $(S \amalg T)$-cube $\mathscr{Y}$. (Each cube $V \mapsto \mathscr{Y}(U, V)$ is k_U-Cartesian, being equivalent to $\mathscr{X}(U)$. The entire $(S \amalg T)$-cube is Cartesian because each cube $U \mapsto \mathscr{Y}(U, V)$ is so.) □

1.23. PROPOSITION. *Let $\mathscr{X}$ be a functor from $\mathscr{P}_1(S)$ to T-cubes of spaces (or spectra), and write $\mathscr{X}(U, V) = (\mathscr{X}(U))(V)$. (So $\mathscr{X}(U, V)$ is defined for $U \subsetneq S$ and $V \subset T$.) Assume that for all $U \subsetneq S$ the T-cube $\mathscr{X}(U)$ is k_U-co-Cartesian. Then the T-cube $V \mapsto \operatorname{hocolim}(U \mapsto \mathscr{X}(U, V))$ is k-co-Cartesian with $k = \min\{|S|-|U|-1+k_U\}$.*

Proof. Dual to the proof of (1.22). □

2. Pushouts, Pullbacks, and Connectivity

We now compare the properties 'k-Cartesian' and 'k-co-Cartesian' for an n-cubical diagram of spaces.

For a 1-cube (=map) of spaces, the two notions coincide. (They both mean 'k-connected'.)

For a 2-cube, they do not coincide but, according to the 'triad connectivity theorem' or 'homotopy excision theorem' of Blakers and Massey, they coincide in a stable range: Any square

$$\begin{tikzcd} \mathscr{X}(\emptyset) \arrow[r, "f_1"] \arrow[d, "f_2"'] & \mathscr{X}(1) \arrow[d, "g_2"] \\ \mathscr{X}(2) \arrow[r, "g_1"'] & \mathscr{X}(1,2) \end{tikzcd}$$

which is co-Cartesian, and in which the map f_s is k_s-connected for $s = 1, 2$, is $(k_1 + k_2 - 1)$-Cartesian. There is also a dual statement, as we shall see: Any square which is Cartesian, and in which the map g_s is k_s-connected for $s = 1, 2$, is $(k_1 + k_2 + 1)$-co-Cartesian.

Theorem 2.3 below is a generalization of the Blakers–Massey theorem from 2-cubes (or triads) to n-cubes (or $(n+1)$-ads), and (2.4) is a statement dual to (2.3).

In order to state the results, we make a definition:

2.1. DEFINITION. An S-cube $\mathscr{X}$ of spaces is *strongly (co-)Cartesian* if each face of dimension $\geqslant 2$, i.e. each cube $\partial_U^T\mathscr{X}$, $U \subset T \subset S$, $|T - U| \geqslant 2$, is (co-)Cartesian.

For $\mathscr{X}$ to be strongly co-Cartesian, it is enough (by (1.7.ii)) if every two-dimensional face is co-Cartesian. Alternatively, it is enough (by (1.7.i)) if every face $\partial^T\mathscr{X}$, $|T| \geqslant 2$, is co-Cartesian. Dually, for $\mathscr{X}$ to be strongly Cartesian it is enough if either every two-dimensional face or every face $\partial_U\mathscr{X}$, $|S - U| \geqslant 2$, is Cartesian.

An example of a strongly co-Cartesian S-cube is any collection $\{\mathscr{X}(T) : T \subset S\}$ of subcomplexes of a CW complex such that $\mathscr{X}(T \cup U) = \mathscr{X}(T) \cup \mathscr{X}(U)$ and $\mathscr{X}(T \cap U) = \mathscr{X}(T) \cap \mathscr{X}(U)$ for all $T \subset S$ and $U \subset S$. In the language of (1.17), such cubes are determined by certain CW $(n+1)$-ads. The additional condition $\mathscr{X}(T \cup U) = \mathscr{X}(T) \cup \mathscr{X}(U)$ on the cube means that the corresponding $(n+1)$-ad $(X, \{X_s\}_{s \in S})$ satisfies $X = X_s \cup X_t$ for every distinct s and t in S.

More generally, given a space $\mathscr{X}(\emptyset)$ and spaces $\{\mathscr{X}(s) : s \in S\}$ and a cofibration $\mathscr{X}(\emptyset) \to \mathscr{X}(s)$ for each s, one can make a strongly co-Cartesian S-cube by letting $\mathscr{X}(T)$ be the union of the $\{\mathscr{X}(s) : s \in T\}$ along $\mathscr{X}(\emptyset)$. Call such a cube $\mathscr{X}$ a *pushout cube*.

Dually, given a space $\mathscr{X}(S)$ and spaces $\{\mathscr{X}(S - s) : s \in S\}$ and a Serre fibration $\mathscr{X}(S - s) \to \mathscr{X}(S)$ for each s, one can make a strongly Cartesian S-cube by letting $\mathscr{X}(T)$ be the fiber product of the $\{\mathscr{X}(S - s) : s \in S - T\}$ over $\mathscr{X}(S)$. Call such a cube $\mathscr{X}$ a *pullback cube*.

2.2. PROPOSITION. *Every strongly co-Cartesian cube of spaces admits an equivalence from a pushout cube. Every strongly Cartesian cube of spaces admits an equivalence to a pullback cube.*

(Recall that a map $\mathscr{X} \to \mathscr{Y}$ of cubes is called an equivalence if for every T the map $\mathscr{X}(T) \to \mathscr{Y}(T)$ of spaces is a (weak) equivalence.)

Proof. Suppose that $\mathscr{Y}$ is strongly co-Cartesian. Let $\mathscr{X}(\emptyset)$ be $\mathscr{Y}(\emptyset)$. Factor the map $\mathscr{Y}(\emptyset) \to \mathscr{Y}(s)$ as a cofibration followed by an equivalence, and call the intermediate space $\mathscr{X}(s)$:

$$\mathscr{Y}(\emptyset) = \mathscr{X}(\emptyset) \xrightarrow{f_s} \mathscr{X}(s) \xrightarrow{w_s} \mathscr{Y}(s).$$

Let $\mathscr{X}$ be the pushout cube built from the cofibrations f_s. Then $\mathscr{X}$ has a map to $\mathscr{Y}$, uniquely specified by choosing the identity as the map $\mathscr{X}(\emptyset) \to \mathscr{Y}(\emptyset)$ and w_s as the map $\mathscr{X}(s) \to \mathscr{Y}(s)$. By construction, the map $\mathscr{X}(T) \to \mathscr{Y}(T)$ is an equivalence for all $T \subset S$ with $|T| \leqslant 1$. The same is true for general $T \subset S$, by induction on $|T|$, using the fact that both $\mathscr{X}$ and $\mathscr{Y}$ are strongly co-Cartesian.

The proof of the dual statement is similar. $\square$

An obvious variation on the proof of (2.2) allows us to choose the pushout cube to be of the special CW type mentioned above:

2.2a. PROPOSITION. *Any strongly co-Cartesian S-cube $\mathscr{Y}$ of spaces admits an equivalence $\mathscr{X} \to \mathscr{Y}$ from a pushout cube which is based on cellular inclusions $\mathscr{X}(\emptyset) \to \mathscr{X}(s)$. If the map $\mathscr{Y}(\emptyset) \to \mathscr{Y}(s)$ is k_s-connected then we can arrange for all cells in $\mathscr{X}(s) - \mathscr{X}(\emptyset)$ to have dimension $\geqslant k_s + 1$.*

2.3. THEOREM (Ellis-Steiner, [4]). *If $\mathscr{X}$ is a strongly co-Cartesian S-cube with $|S| = n \geqslant 1$, and if for each $s \in S$ the map $\mathscr{X}(\emptyset) \to \mathscr{X}(s)$ is k_s-connected, then $\mathscr{X}$ is k-Cartesian with $k = 1 - n + \Sigma_{s \in S} k_s$.*

2.4. THEOREM. *If $\mathscr{X}$ is a strongly Cartesian S-cube with $|S| = n \geqslant 1$, and if for each $s \in S$ the map $\mathscr{X}(S - s) \to \mathscr{X}(S)$ is k_s-connected, then $\mathscr{X}$ is k-co-Cartesian with $k = n - 1 + \Sigma_{s \in S} k_s$.*

Before beginning the proofs, we will state some related results.

Recall that the Blakers–Massey theorem has consequences even for squares that are not co-Cartesian: If the square

$$\begin{tikzcd}
\mathscr{X}(\emptyset) \arrow[r] \arrow[d] & \mathscr{X}(1) \arrow[d] \\
\mathscr{X}(2) \arrow[r] & \mathscr{X}(1,2)
\end{tikzcd}$$

is ℓ-co-Cartesian for some ℓ, and if the map $\mathscr{X}(\emptyset) \to \mathscr{X}(s)$ is k_s-connected for $s \in \{1, 2\}$, then the same square is k-Cartesian where $k = \min(k_1 + k_2 - 1, \ell - 1)$. This fact has the following generalization for n-cubes:

2.5. THEOREM. *Let $\mathscr{X}$ be an S-cube with $|S| = n \geqslant 1$. Suppose that*

(i) *for each nonempty $T \subset S$ the T-cube $\partial^T \mathscr{X}$ (as defined in* (1.12)) *is $k(T)$-co-Cartesian, and*

(ii) *$k(U) \leqslant k(T)$, whenever $U \subset T$.*

Then $\mathscr{X}$ is k-Cartesian, where k is the minimum of $(1-n)+\Sigma_\alpha k(T_\alpha)$ *over all partitions* $\{T_\alpha\}$ *of S by nonempty sets.*

For example, if $S=\{1,2,3\}$ then $k+2$ is the minimum of $k(1,2,3)$, $k(1,2)+k(3)$, $k(1,3)+k(2)$, $k(2,3)+k(1)$, and $k(1)+k(2)+k(3)$.

There is a dual statement:

2.6. THEOREM. *Let $\mathscr{X}$ be an S-cube with $|S|=n\geqslant 1$. Suppose that*

(i) *for each $T\neq\emptyset$, the T-cube $\partial_{S-T}\mathscr{X}$ is $k(T)$-Cartesian and*
(ii) *$k(U)\leqslant k(T)$ whenever $U\subset T$.*

Then $\mathscr{X}$ is k-co-Cartesian, where k is the minimum of $n-1+\Sigma_\alpha k(T_\alpha)$ *over all partitions* $\{T_\alpha\}$ *of S by nonempty sets.*

In (2.5) and (2.6), each $k(T)$ may be either an integer or $+\infty$. Note that (2.5) [respectively, (2.6)] contains (2.3) [respectively, (2.4)] as a special case, namely the case in which $k(T)=+\infty$ for all T such that $|T|\geqslant 2$.

Theorem 2.3, under the restriction that the numbers k_s are all at least 2, is due to Barratt and J. H. C. Whitehead [1]. The more general version stated here was proved by Ellis and Steiner [4]. A weak version of Theorem 2.5 was deduced from [1] in [7].

The main theorem of [1] or [4] is stronger than our Theorem 2.3; in addition to giving the range in which the homotopy groups $\pi_*(\mathscr{X})$ of a suitable cube must vanish, it computes the first nonvanishing group of such a cube in terms of the first nonvanishing groups of the maps $\mathscr{X}(\emptyset)\to\mathscr{X}(s)$.

We will give a different proof of (2.3), using a method which does not involve the computation of any nontrivial homotopy groups. The pattern of the argument is a bit complicated; it proceeds by induction on n and requires introducing the more general statement (2.5). We need (2.5) in any case, and those readers who have seen (2.3) proved in [4] will find (2.5) deduced from it below.

We begin with a lemma. The proof of the lemma contains some geometry (specifically, a transversality argument); the rest of the proof of (2.3) and (2.5) is quite formal.

For the lemma we suppose that $\mathscr{X}$ is a pushout n-cube of spaces, formed by attaching cells e_s, $1\leqslant s\leqslant n$, to a space $X=\mathscr{X}(\emptyset)$. That is, $\mathscr{X}(T)=X\cup\{e_s:s\in T\}$ for each $T\subset\underline{n}$, for some choice of attaching maps $\partial e_s\to X$, $1\leqslant s\leqslant n$.

2.7. LEMMA. *Let $\mathscr{X}$ be as above, and let d_s be the dimension of the cell e_s. Choose a point $x\in\mathscr{X}(n)$, and for $T\subset\underline{n-1}=\{1,\dots,n-1\}$ let $\mathscr{F}(T)$ be fiber $(\mathscr{X}(T)\to\mathscr{X}(T\cup n))$, the homotopy fiber over x. Then the $(\underline{n-1})$-cube $\mathscr{F}$ is $(-1+\Sigma_{s\in S}(d_s-1))$-co-Cartesian.*

Proof. Choose an interior point $p_s\in e_s$ in each cell, taking $p_n\neq x$. (We ignore the trivial case $d_n=0$.)

The space $\mathscr{X}(T)$ is equivalent to the larger space $\mathscr{X}^*(T) = \mathscr{X}(\underline{n}) - \{p_s \mid s \in \underline{n} - T\}$. Thus, we may replace the cube $\mathscr{X}$ by the cube $\mathscr{X}^*$, and replace $\mathscr{F}(T)$ by $\mathscr{F}^*(T) = \mathrm{fiber}(\mathscr{X}^*(T) \to \mathscr{X}^*(T \cup n))$.

Let C be the contractible space $C = \mathrm{fiber}(\mathscr{X}^*(\underline{n-1}) \to \mathscr{X}^*(\underline{n-1})$. For any $T \subset \underline{n-1}$, the inclusion $\mathscr{F}^*(T) \subset C \cup \mathscr{F}^*(T)$ is an equivalence by (1.16a), since both C and $C \cap \mathscr{F}^*(T) = \mathrm{fiber}(\mathscr{X}^*(T) \to \mathscr{X}^*(T))$ are contractible. Therefore, we may replace the cube $\mathscr{F}^*$ by the cube $C \cup \mathscr{F}^*$.

By (1.16a) again, applied now to the cube $C \cup \mathscr{F}^*$, it suffices to show that the pair $(A, B) = (C \cup \mathscr{F}^*(\underline{n-1}), C \cup (\bigcup_s \mathscr{F}^*(\underline{n-1} - s)))$ is $(-1 + \Sigma_s(d_s - 1))$-connected. (The union is indexed by all $s \in \underline{n-1}$.) This means showing that, for $\ell < \Sigma_s(d_s - 1)$, every map of pairs

$$\Phi\colon (D^\ell, \partial D^\ell) \longrightarrow (A, B)$$

is homotopic to a map $(D^\ell, \partial D^\ell) \to (B, B)$.

The map Φ corresponds, by adjointness, to a map

$$\Psi\colon I \times D^\ell \longrightarrow \mathscr{X}(\underline{n})$$

satisfying the boundary conditions

$$\begin{array}{lll}
\text{(i)} & \forall z \in D^\ell, & \Psi(0, z) = x, \\
\text{(ii)} & \forall z \in D^\ell, & \Psi(1, z) \neq p_n, \\
\text{(iii)} & \forall z \in \partial D^\ell\ \exists s \in \underline{n}\ \forall t \in I, & \Psi(t, z) \neq p_s
\end{array}$$

We need to change Ψ, by a homotopy preserving these conditions, into a map satisfying the stronger condition

$$\text{(iv)} \quad \forall z \in D^\ell\ \exists s \in \underline{n}\ \forall t \in I \qquad \Psi(t, z) \neq p_s.$$

This can be done in two successive homotopies, by general-position arguments. It is convenient, before doing so, to extend Ψ to $[0, 1 + \varepsilon) \times D^\ell_r$, the product of a longer interval and a larger open disk, in order to avoid dealing with boundaries and corners.

The first homotopy makes Ψ smooth near the preimage of the point p_s, and transverse to p_s, for all $s \in \underline{n}$. (This has meaning, since $\mathscr{X}(\underline{n})$ is a manifold near p_s.) Now the sets $N_s = \Psi^{-1}(p_s)$ are disjoint submanifolds of $(0, 1 + \varepsilon) \times D^\ell_r$ with $\dim(N_s) = \ell + 1 - d_s$.

The second homotopy is defined by considering the map

$$\prod_{1 \leqslant s \leqslant n} N_s \longrightarrow ((0, 1 + \varepsilon) \times D^\ell_r)^n \longrightarrow (D^\ell_r)^n$$

(inclusion followed by projection). A small isotopy of $[0, 1 + \varepsilon) \times D^\ell_r$ will arrange for this map to be transverse to the diagonal copy of D^ℓ_r. Make a homotopy of Ψ by composing Ψ with this isotopy. By hypothesis $\Sigma_s \dim(N_s) < (n - 1)\ell$, so that the diagonal is in fact not hit at all; and this is what is required by (iv).

The homotopies can be chosen so as to preserve (i), and if they are uniformly small enough then they will preserve (ii) and (iii). □

We now prove (2.3) and (2.5) together, by induction on n. They are both trivial for 1-cubes.

Proof of (2.3) for n-cubes ($n \geqslant 2$), assuming (2.5) (and, in particular, (2.3)) for $(n-1)$-cubes:

Let $\mathscr{X}$ be a strongly co-Cartesian S-cube, $|S| = n$, let the map $\mathscr{X}(\emptyset) \to \mathscr{X}(s)$ be k_s-connected for each $s \in S$, and put $k = (1-n) + \Sigma k_s$. The plan is to reduce to the situation of (2.7) and to use (1.18).

There is no harm in assuming that $k_s \geqslant 1$ for all s. In fact, suppose $k_t \leqslant 0$. Write $\mathscr{X}$ as a map $\mathscr{Y} \to \mathscr{Z}$ of $(S-t)$-cubes. Note that, by (2.3) for $(n-1)$-cubes, both $\mathscr{Y}$ and $\mathscr{Z}$ are $(k+1)$-Cartesian, since

$$1-(n-1) + \sum_{s \neq t} k_s \geqslant \left(1 - n + \sum_s k_s\right) + 1 = k+1.$$

By (1.6.ii), $\mathscr{X}$ is k-Cartesian.

By (2.2.a) we may assume that $\mathscr{X}$ is a pushout cube, and that for each $s \in S$ the pair $(\mathscr{X}(s), \mathscr{X}(\emptyset))$ is a relative CW complex whose cells have dimension $\geqslant k_s + 1$. By a direct-limit argument we may assume that the number of these cells for each s is finite. By an induction using (1.8.i), we may assume that there is only one such cell for each s.

We are now in the situation of (2.7), with $d_s \geqslant k_s + 1 \geqslant 2$. (We may as well take $S = \underline{n}$.) Choose any basepoint in $\mathscr{X}(n)$. We apply (2.5) to the $(\underline{n-1})$-cube $\mathscr{F}$ given by $\mathscr{F}(T) = \text{fiber}(\mathscr{X}(T) \to (T \cup n))$. By (2.7), $\mathscr{F}$ is $(-1 + \Sigma_{1 \leqslant s \leqslant n} k_s)$-co-Cartesian and more generally each face $\partial^T \mathscr{F}$, $\emptyset \neq T \subset \underline{n-1}$, is $k(T)$-co-Cartesian, where

$$k(T) = -1 + k_n + \sum_{s \in T} k_s.$$

These numbers $k(T)$ satisfy hypothesis (ii) of (2.5), because $k_s \geqslant 0$ for all s. The sum $\Sigma_\alpha k(T_\alpha)$ is minimized by the one-set partition of $\underline{n-1}$, because $k_n \geqslant 1$. Thus (2.5) makes $\mathscr{F}$ k-Cartesian:

$$(1-(n-1)) + \left(-1 + k_n + \sum_{s \in \underline{n-1}} k_s\right) = (1-n) + \sum_{s \in \underline{n}} k_s = k.$$

The conclusion follows by (1.18). □

Proof of (2.5) for n-cubes, assuming (2.3) for n-cubes and (2.5) for cubes of lower dimension:

Using (1.14), take $\mathscr{X}$ to be a cofibration cube. Now hypothesis (i) of (2.5) can be restated, using (1.16): for each nonempty $T \subset S$, the cofibration

$$\operatorname{colim}(\mathscr{X} \mid \mathscr{P}_1(T)) \longrightarrow \mathscr{X}(T)$$

is $k(T)$-connected.

In the course of the argument, we will be forming colimits of $\mathscr{X}$ indexed by various convex subsets $\mathscr{A} \subset P(S)$. ('Convex' was defined just before (0.2).) Write $\mathscr{X}(\mathscr{A})$ for $\operatorname{colim}(\mathscr{X} \mid \mathscr{A})$.

This is a functor; if $\mathscr{B} \subset \mathscr{A}$ are convex sets in $\mathscr{P}(S)$, then there is an obvious map $\mathscr{X}(\mathscr{B}) \to \mathscr{X}(\mathscr{A})$. In particular, $\mathscr{X}(\mathscr{P}_1(T)) \to \mathscr{X}(\mathscr{P}(T))$ is the $k(T)$-connected cofibration above.

2.8. *Claim.*

(i) For each inclusion $\mathscr{B} \subset \mathscr{A}$ of convex subsets of $\mathscr{P}(S)$, the map $\mathscr{X}(\mathscr{B}) \to \mathscr{X}(\mathscr{A})$ is a cofibration.

(ii) For any convex subsets $\mathscr{B}$ and $\mathscr{C}$ of $\mathscr{P}(S)$, the diagram

$$\begin{array}{ccc} \mathscr{X}(\mathscr{B} \cap \mathscr{C}) & \longrightarrow & \mathscr{X}(\mathscr{C}) \\ \downarrow & & \downarrow \\ \mathscr{X}(\mathscr{B}) & \longrightarrow & \mathscr{X}(\mathscr{B} \cup \mathscr{C}) \end{array}$$

of cofibrations is a pushout.

Proof of (2.8). That the square is a pushout follows from the convexity of $\mathscr{B}$ and $\mathscr{C}$. We prove (i). Since a composition of cofibrations is again a cofibration, it suffices to consider the case when $\mathscr{A} - \mathscr{B}$ has just one element A (which must then be maximal in $\mathscr{A}$). In that case, $\mathscr{A}$ is the union of the convex sets $\mathscr{B} = \mathscr{A} - A$ and $\mathscr{P}(A)$, and $\mathscr{P}_1(A)$ is the intersection. The square

$$\begin{array}{ccc} \mathscr{X}(\mathscr{P}_1(A)) & \longrightarrow & \mathscr{X}(\mathscr{P}(A)) \\ \downarrow & & \downarrow \\ \mathscr{X}(\mathscr{A} - A) & \longrightarrow & \mathscr{X}(\mathscr{A}) \end{array}$$

shows that the map $\mathscr{X}(\mathscr{A} - A) \to \mathscr{X}(\mathscr{A})$ is a cofibration, as it is obtained by pushout from the cofibration $\mathscr{X}(\mathscr{P}_1(A)) \to \mathscr{X}(\mathscr{P}(A))$. □

The preceding argument also implies:

(2.9) If A is maximal in $\mathscr{A}$, then the cofibration $\mathscr{X}(\mathscr{A} - A) \to \mathscr{X}(\mathscr{A})$ is $k(A)$-connected.

Now, for each convex set $\mathscr{A} \subset \mathscr{P}(S)$, define a new S-cube $\mathscr{X}_{\mathscr{A}}$ by $\mathscr{X}_{\mathscr{A}}(T) = \mathscr{X}(\mathscr{A} \cap \mathscr{P}(T))$. Note that

(2.10) $\operatorname{colim}(\mathscr{X}_{\mathscr{A}} \mid \mathscr{P}_1(T)) = \mathscr{X}(\mathscr{A} \cap \mathscr{P}_1(T))$.

We will show that $\mathscr{X}_{\mathscr{A}}$ is k-Cartesian, for all $\mathscr{A}$ between $\mathscr{A}_{\min} = \{T \subset S : |T| \leqslant 1\}$ and $\mathscr{A}_{\max} = \mathscr{P}(S)$, by induction, beginning with $\mathscr{A}_{\min}$. Since $\mathscr{X} = \mathscr{X}_{\mathscr{A}_{\max}}$, that will complete the proof of (2.5).

Before beginning the argument, we observe that the cube $\mathscr{X}_{\mathscr{A}}$ shares some properties with $\mathscr{X}$: By (2.8.i) and (2.10), it is a cofibration cube. By (2.9), (2.10), and (1.16), the face $\partial^T \mathscr{X}_{\mathscr{A}}$ is $k(T)$-co-Cartesian if $T \in \mathscr{A}$ and co-Cartesian otherwise, so in any case $k(T)$-co-Cartesian. This implies the more general statement:

(2.11) For every $V \subsetneqq T \subset S$, the cube $\partial_V^T \mathscr{X}_{\mathscr{A}}$ is $k(T - V)$-co-Cartesian.

The proof is by induction on $|V|$: (2.11) for (T, V) follows from (2.11) for $(T, V - v)$ and for $(T - v, V - v)$, by (1.7.i) and hypothesis (ii) of (2.5).

We now show that the cubes $\mathcal{X}_{\mathcal{A}}$ are k-Cartesian (where k is as in (2.5)).

When $\mathcal{A} = \mathcal{A}_{\min}$, $\mathcal{X}_{\mathcal{A}}$ is strongly co-Cartesian (it is a pushout cube, by (2.8.ii)), so that (2.3) applies. The map $\mathcal{X}_{\mathcal{A}}(\emptyset) \to \mathcal{X}_{\mathcal{A}}(s)$ is the map $\mathcal{X}(\emptyset) \to \mathcal{X}(s)$, and so is $k(s)$-connected. By (2.3), $\mathcal{X}_{\mathcal{A}}$ is $(1 - n + \Sigma k(s))$-Cartesian and, therefore, k-Cartesian.

It remains to handle the induction with respect to $\mathcal{A}$, by proving:

2.12. *Claim.* Suppose $\mathcal{A} \subset \mathcal{P}(S)$ convex, $A \in \mathcal{A}$ maximal, and $|A| \geqslant 2$. Let $\mathcal{B} = \mathcal{A} - A$. If $\mathcal{X}_{\mathcal{B}}$ is k-Cartesian, then $\mathcal{X}_{\mathcal{A}}$ is k-Cartesian.

Proof of (2.12). Consider the map of cubes $\mathcal{X}_{\mathcal{B}} \to \mathcal{X}_{\mathcal{A}}$. The most interesting part of this is the map $\partial_A \mathcal{X}_{\mathcal{B}} \to \partial_A \mathcal{X}_{\mathcal{A}}$, because:

(2.13) $\quad \mathcal{X}_{\mathcal{B}}(U) = \mathcal{X}_{\mathcal{A}}(U)$ if U does not contain A.

Indeed, in this case $\mathcal{B} \cap \mathcal{P}(U) = \mathcal{A} \cap \mathcal{P}(U)$. In the other cases (2.8.ii) yields:

(2.14) $\quad$ For $A \subset U \subset T$, the following square is a pushout:

$$\begin{array}{ccc} \mathcal{X}_{\mathcal{B}}(U) & \longrightarrow & \mathcal{X}_{\mathcal{A}}(U) \\ \downarrow & & \downarrow \\ \mathcal{X}_{\mathcal{B}}(T) & \longrightarrow & \mathcal{X}_{\mathcal{A}}(T). \end{array}$$

Regard $\partial_A \mathcal{X}_{\mathcal{B}} \to \partial_A \mathcal{X}_{\mathcal{A}}$ as an $((S - A) \cup *)$-cube, and call it $\mathcal{Y}$:

$$\mathcal{Y}(U) = \mathcal{X}_{\mathcal{B}}(A \cup U), \quad U \subset S - A,$$
$$\mathcal{Y}(U \cup *) = \mathcal{X}_{\mathcal{A}}(A \cup U), \quad U \subset S - A.$$

Apply (2.5) to $\mathcal{Y}$; this is allowed, by induction on n, since $|(S - A) \cup *| < |S| = n$. We verify hypothesis (i) of (2.5) with the numbers:

$$\ell(U) = k(U), \quad \emptyset \neq U \subset S - A,$$
$$\ell(*) = k(A),$$
$$\ell(U \cup *) = +\infty, \quad \emptyset \neq U \subset S - A.$$

This means checking, for each nonempty $V \subset (S - A) \cup *$, that $\partial^V \mathcal{Y}$ is $\ell(V)$-co-Cartesian. There are three cases:

For $V \subset S - A$, the cube $\partial_A^{A \cup V} \mathcal{X}_{\mathcal{B}}$ is $k(V)$-co-Cartesian, by (2.11).
For $V = \{*\}$, the map $\mathcal{X}_{\mathcal{B}}(A) \to \mathcal{X}_{\mathcal{A}}(A)$ is $k(A)$-connected, by (2.9).
For $V = U \cup *$, $\emptyset \neq U \subset S - A$, the map $\partial_A^{A \cup U} \mathcal{X}_{\mathcal{B}} \to \partial_A^{A \cup U} \mathcal{X}_{\mathcal{A}}$ is a co-Cartesian cube, by (2.14) and (1.7.ii).

We conclude that $\mathcal{Y}$ is $(k + |A| - 1)$-Cartesian, because the sum $\Sigma_\beta \ell(V_\beta)$ for a partition of $(S - A) \cup *$ is always either $+\infty$, or the sum $\Sigma_\alpha k(T_\alpha)$ for some partition of S (in which some T_α is A).

We next prove that

(2.15) $\quad \partial_T \mathcal{X}_{\mathcal{B}} \to \partial_T \mathcal{X}_{\mathcal{A}}$ is a $(k + |T| - 1)$-Cartesian $(n - |T| + 1)$-cube, for every $T \subset A$.

We have proved it for $T = A$. We argue by downward induction on $|T|$. To go from T to $T - t$, write the map $\partial_{T-t}\mathscr{X}_{\mathscr{B}} \to \partial_{T-t}\mathscr{X}_{\mathscr{A}}$ as a square (of $(n - |T|)$-cubes)

$$\begin{tikzcd}
\partial^{S-t}_{T-t}\mathscr{X}_{\mathscr{B}} \arrow[r] \arrow[d] & \partial^{S-t}_{T-t}\mathscr{X}_{\mathscr{A}} \arrow[d] \\
\partial_T\mathscr{X}_{\mathscr{B}} \arrow[r] & \partial_T\mathscr{X}_{\mathscr{A}}
\end{tikzcd}$$

The upper arrow is an isomorphism, by (2.13), and the lower arrow is $(k + |T| - 1)$-Cartesian. Therefore, by (1.6.ii), the square is $(k + |T - t| - 1)$-Cartesian.

To prove that $\mathscr{X}_{\mathscr{A}}$ is k-Cartesian, choose an element $a \in A$ and use (2.15) with $T = \{a\}$. Consider (as S-cubes) the two maps of $(S - a)$-cubes and their composition:

$$\partial^{S-a}\mathscr{X}_{\mathscr{B}} \longrightarrow \partial_{\{a\}}\mathscr{X}_{\mathscr{B}} \longrightarrow \partial_{\{a\}}\mathscr{X}_{\mathscr{A}}$$

The second is k-Cartesian, by (2.15). The first is $\mathscr{X}_{\mathscr{B}}$, and so is k-Cartesian. The composition (which, by (2.13), is $\mathscr{X}_{\mathscr{A}}$), is therefore k-Cartesian, by (1.8.i). This completes the proof of (2.12), and of (2.5). □

2.16. *Remark.* Hypothesis (ii) in (2.5) cannot be eliminated in general, but when $n = 2$ it can be eliminated and, in general, it can be weakened. When $n = 3$, it is enough if $k(U) \leqslant k(T)$ for a suitable set of pairs (T, U); for example, the pairs $(\{1, 2\}, \{1\})$, $(\{1, 2\}, \{2\})$, and $(\{1, 3\}, \{1\})$ would suffice. We will not pursue such refinements here. A similar remark applies to (2.6).

The proof of (2.4) is a bit simpler than that of (2.3); in particular it is not mixed up with the proof of (2.6). It requires a lemma about quasifibrations.

A map $f\colon Y \to X$ of spaces is called a quasifibration if, for each point x in X, the canonical map from the fiber of f to the homotopy fiber is a (weak homotopy) equivalence. It is clear that every fibration has the following property (P), and that every map $f\colon Y \to X$ having property (P) is a quasifibration.

(P) For every continuous map $D \to X$, D a closed disk of any dimension, the projection $D \times_X Y \to D$ is a quasifibration. (That is, for every point $* \in D$, the inclusion $* \times_X Y \to D \times_X Y$ is an equivalence.)

2.17. LEMMA. *Let $\mathscr{X}$ be any S-cube of spaces such that for each $T \subset S$ the map $\mathscr{X}(T) \to \mathscr{X}(S)$ is a fibration. Then the map $b(\mathscr{X})\colon h_1(\mathscr{X}) \to \mathscr{X}(S)$ (as defined in* (1.4)) *is a quasifibration.*

Proof. It is enough if property (P) for the maps $\mathscr{X}(T) \to \mathscr{X}(S)$ implies property (P) for $b(\mathscr{X})$. We show more generally that, if $T \mapsto \mathscr{X}(T)$ is any functor from a small category to the category of spaces over a space X, then property (P) for all the maps $\mathscr{X}(T) \to X$ implies property (P) for the resulting map $\operatorname{hocolim}(\mathscr{X}(T)) \to \mathscr{X}$.

The key point is that, for every map $D \to X$ from a disk, the fiber product $D \times_X \operatorname{hocolim}(\mathscr{X}(T))$ is naturally homeomorphic to $\operatorname{hocolim}(D \times_X \mathscr{X}(T))$. (This uses the fact that product with a compact space commutes with geometric realization.) Thus, for any $* \in D$ we have a diagram

$$\begin{array}{ccc} \operatorname{hocolim}(* \times_X \mathscr{X}(T)) & \longrightarrow & \operatorname{hocolim}(D \times_X \mathscr{X}(T)) \\ \downarrow & & \downarrow \\ * \times_X \operatorname{hocolim}(\mathscr{X}(T)) & \longrightarrow & D \times_X \operatorname{hocolim}(\mathscr{X}(T)) \end{array}$$

in which the vertical maps are homeomorphisms. The lower map is then an equivalence, because by hypothesis the upper map is a hocolim of equivalences. □

Proof of (2.4). We may assume, by (2.2), that $\mathscr{X}$ is a pullback cube made from a collection of fibrations $\mathscr{X}(S-s) \to \mathscr{X}(S)$. In the present proof, 'fiber' means preimage of a point, not homotopy fiber. Fix any point in $\mathscr{X}(S)$, and let F_s be the fiber of $\mathscr{X}(S-s) \to \mathscr{X}(S)$. Thus, the fiber of the fibration $\mathscr{X}(T) \to \mathscr{X}(S)$ is the product

$$F(T) = \prod_{s \in S-T} F_s.$$

The map $b(\mathscr{X})\colon h_1(\mathscr{X}) \to \mathscr{X}(S)$ is a quasifibration, by (2.17). The fiber of $b(\mathscr{X})$ is the hocolim of the diagram $T \mapsto F(T)$ indexed by $T \in \mathscr{P}_1(S)$. This is the join of the spaces F_s, as one sees by induction, beginning with the fact that the join $X * Y$ of two spaces is the hocolim of $X \leftarrow X \times Y \to Y$. By hypothesis, F_s is $(k_s - 1)$-connected. The join of a p-connected space and a q-connected space is $(p+q+2)$-connected (recall that by convention the empty set is (-2)-connected), so that the fiber, hence, also the homotopy fiber, of $b(\mathscr{X})$ is $(n-2+\Sigma k_s)$-connected. Since this is true for every point in $\mathscr{X}(S)$, $\mathscr{X}$ is $(n-1+\Sigma k_s)$-co-Cartesian. □

Proof of (2.6). We outline the argument, which is precisely dual to the proof of (2.5) from (2.3). Assume (2.6) for smaller values of n.

We may assume that $\mathscr{X}$ is a fibration cube.

We use limits of $\mathscr{X}$ indexed by concave subsets $\mathscr{A} \subset \mathscr{P}(S)$, writing $\mathscr{X}(\mathscr{A})$ for $\lim(\mathscr{X} \mid \mathscr{A})$.

If $\mathscr{B} \subset \mathscr{A}$, then there is an obvious map $\mathscr{X}(\mathscr{A}) \to \mathscr{X}(\mathscr{B})$. It is a fibration, and for any concave sets $\mathscr{B}$ and $\mathscr{C}$ we have a pullback square:

$$\begin{array}{ccc} \mathscr{X}(\mathscr{B} \cup \mathscr{C}) & \longrightarrow & \mathscr{X}(\mathscr{C}) \\ \downarrow & & \downarrow \\ \mathscr{X}(\mathscr{B}) & \longrightarrow & \mathscr{X}(\mathscr{B} \cap \mathscr{C}) \end{array}$$

If A is minimal in $\mathscr{A}$, then the fibration $\mathscr{X}(\mathscr{A}) \to \mathscr{X}(\mathscr{A} - A)$ is $k(S-A)$-connected.

For each concave set $\mathscr{A} \subset \mathscr{P}(S)$, define a new S-cube $\mathscr{X}^{\mathscr{A}}$ by $\mathscr{X}^{\mathscr{A}}(T) = \mathscr{X}(\{U \in \mathscr{A} : T \subset U\})$. Like $\mathscr{X}$, this is a fibration cube. The face $\partial_U \mathscr{X}^{\mathscr{A}}$ is $k(S-U)$-

Cartesian. More generally, for every $U \subsetneq V \subset S$, the cube $\partial_U^V \mathscr{X}^{\mathscr{A}}$ is $k(V-U)$-Cartesian.

We show that $\mathscr{X}^{\mathscr{A}}$ is k-co-Cartesian, where k is as in (2.6), for all $\mathscr{A}$ between $\mathscr{A}_{\min} = \{T \subset S : |S-T| \leqslant 1\}$ and $\mathscr{A}_{\max} = \mathscr{P}(S)$, by induction, beginning with $\mathscr{A}_{\min}$.

When $\mathscr{A} = \mathscr{A}_{\min}$, $\mathscr{X}^{\mathscr{A}}$ is strongly Cartesian (it is a pullback cube), so that (2.4) applies. The map $\mathscr{X}^{\mathscr{A}}(S-s) \to \mathscr{X}^{\mathscr{A}}(S)$ is the map $\mathscr{X}(S-s) \to \mathscr{X}(S)$, and so is $k(s)$-connected. By (2.4), $\mathscr{X}^{\mathscr{A}}$ is $(n-1+\Sigma k(s))$-co-Cartesian and, therefore, k-co-Cartesian.

The analogue of (2.12) is this: Suppose $\mathscr{A} \subset \mathscr{P}(S)$ concave, $A \in \mathscr{A}$ minimal, and $|S-A| \geqslant 2$. Let $\mathscr{B} = \mathscr{A} - A$. If $\mathscr{X}^{\mathscr{B}}$ is k-co-Cartesian, then $\mathscr{X}^{\mathscr{A}}$ is k-co-Cartesian.

To prove it, consider the map of cubes $\mathscr{X}^{\mathscr{A}} \to \mathscr{X}^{\mathscr{B}}$. We have

$$\mathscr{X}^{\mathscr{A}}(U) = \mathscr{X}^{\mathscr{B}}(U), \quad \text{if } U \text{ is not contained in } A,$$

while, for $U \subset T \subset A$, the following square is a pullback:

$$\begin{array}{ccc} \mathscr{X}^{\mathscr{A}}(U) & \longrightarrow & \mathscr{X}^{\mathscr{B}}(U) \\ \downarrow & & \downarrow \\ \mathscr{X}^{\mathscr{A}}(T) & \longrightarrow & \mathscr{X}^{\mathscr{B}}(T). \end{array}$$

Regard $\partial^A \mathscr{X}^{\mathscr{A}} \to \partial^A \mathscr{X}^{\mathscr{B}}$ as an $(A \cup *)$-cube, and call it $\mathscr{Z}$:

$$\mathscr{Z}(U) = \mathscr{X}^{\mathscr{A}}(U), \quad U \subset A$$

$$\mathscr{Z}(U \cup *) = \mathscr{X}^{\mathscr{B}}(U), \quad U \subset A.$$

Apply (2.6) to $\mathscr{Z}$, noting that $|A \cup *| < |S| = n$. Verify hypothesis (i) of (2.6) with the numbers

$$\ell(U) = k(U), \qquad \emptyset \neq U \subset A,$$
$$\ell(*) = k(S-A),$$
$$\ell(U \cup *) = +\infty, \qquad \emptyset \neq U \subset A.$$

This means checking, for each nonempty $V \subset A \cup *$, that $\partial_{(A \cup *)-V} \mathscr{Z}$ is $\ell(V)$-Cartesian. There are three cases

For $V \subset A$, the cube $\partial_{A-V}^A \mathscr{X}^{\mathscr{B}}$ is $k(V)$-Cartesian.
For $V = \{*\}$, the map $\mathscr{X}^{\mathscr{A}}(A) \to \mathscr{X}^{\mathscr{B}}(A)$ is $k(S-A)$-connected.
For $V = U \cup *$, $\emptyset \neq U \subset A$, the map $\partial_{A-U}^A \mathscr{X}^{\mathscr{A}} \to \partial_{A-U}^A \mathscr{X}^{\mathscr{B}}$ is a Cartesian cube.

Now (2.6) implies that $\mathscr{Z}$ is $(k+1-n+|A|)$-co-Cartesian, because each sum $\Sigma_\beta \ell(V_\beta)$ for a partition of $A \cup *$ is either $+\infty$ or a sum $\Sigma_\alpha k(T_\alpha)$ for a partition of S (in which some T_α is $S-A$).

We next prove that

(2.18) $\partial^T \mathscr{X}^{\mathscr{A}} \to \partial^T \mathscr{X}^{\mathscr{B}}$ is a $(k+1-n+|T|)$-co-Cartesian $(|T|+1)$-cube, for every $T \supset A$.

We have proved it for $T = A$. We argue by upward induction on $|T|$. To go from T to $T \cup t$, write the map $\partial^{T \cup t}\mathscr{X}^{\mathscr{A}} \to \partial^{T \cup t}\mathscr{X}^{\mathscr{B}}$ as a square (of T-cubes)

$$\begin{tikzcd} \partial^T\mathscr{X}^{\mathscr{A}} \arrow[r] \arrow[d] & \partial^T\mathscr{X}^{\mathscr{B}} \arrow[d] \\ \partial_t^{T \cup t}\mathscr{X}^{\mathscr{A}} \arrow[r] & \partial_t^{T \cup t}\mathscr{X}^{\mathscr{B}} \end{tikzcd}$$

The lower arrow is an isomorphism, and the upper arrow is $(k + 1 - n + |T|)$-co-Cartesian. Therefore, by (1.7.ii), the square is $(k + 1 - n + |T \cup t|)$-co-Cartesian.

To prove that $\mathscr{X}^{\mathscr{A}}$ is k-co-Cartesian, choose an element $b \in S - A$. Consider (as S-cubes) the two maps of $(S - b)$-cubes and their composition:

$$\partial^{S-b}\mathscr{X}^{\mathscr{A}} \longrightarrow \partial^{S-b}\mathscr{X}^{\mathscr{B}} \longrightarrow \partial_b\mathscr{X}^{\mathscr{B}}.$$

The first is k-co-Cartesian, by (2.18). The second is $\mathscr{X}^{\mathscr{B}}$, and so is k-co-Cartesian. The composition (which is $\mathscr{X}^{\mathscr{A}}$) is, therefore, k-co-Cartesian. □

3. Higher-Order Excision

We now begin examining the behavior of homotopy functors on cubical diagrams.

The main point of the present work is to introduce the concept of 'analyticity' of a homotopy functor and to demonstrate one of its uses. We digress here to explain the concept of 'nth order excision', since analytic functors will by definition be those which, for all n, 'almost' satisfy nth order excision. The examples presented here may shed some light on the concept of analyticity, but logically this section can be skipped; these definitions and results will not be needed until the next article in this series.

We work with functors $F: \mathscr{C} \to \mathscr{D}$ where $\mathscr{D}$ is unbased spaces ($\mathscr{U}$) or based spaces ($\mathscr{T}$) or spectra ($\mathscr{S}$) and $\mathscr{C}$ is either $\mathscr{U}$ or $\mathscr{T}$. F is always assumed to be a homotopy functor; it preserves (weak homotopy) equivalences.

3.1. DEFINITION. F is *n-excisive*, or satisfies nth *order excision*, if for every strongly co-Cartesian $(n + 1)$-cubical diagram $\mathscr{X} : \mathscr{P}(S) \to \mathscr{C}$ the diagram $F(\mathscr{X}) = F \circ \mathscr{X} : \mathscr{P}(S) \to \mathscr{D}$ is Cartesian.

3.2. PROPOSITION. $(n - 1)$st-order excision implies nth order excision.

Proof. Viewing an $(n + 1)$-cube $\mathscr{X}$ as a map of n-cubes $\mathscr{Y} \to \mathscr{Z}$, we have:

$$\begin{aligned} \mathscr{X} \text{ strongly co-Cartesian} &\Rightarrow \mathscr{Y} \text{ and } \mathscr{Z} \text{ strongly co-Cartesian} \\ &\Rightarrow F(\mathscr{Y}) \text{ and } F(\mathscr{Z}) \text{ Cartesian} \\ &\Rightarrow F(\mathscr{X}) \text{ Cartesian.} \end{aligned}$$ □

Note that first-order excision is excision in the sense of [5], (1.1.ii), while zeroth order excision means that F takes all maps to equivalences. (Every 1-cube is strongly co-Cartesian.)

In (3.5) below, we will use 1-excisive functors to make examples of n-excisive functors.

3.3. PROPOSITION. *If $F: \mathscr{C} \to \mathscr{D}$ is n-excisive, then for any strongly co-Cartesian m-cube $\mathscr{Y}: \mathscr{P}(S) \to \mathscr{C}$ the natural map*

$$F(\mathscr{Y}(\emptyset)) \longrightarrow \operatorname{holim}\{F(\mathscr{Y}(U)) : U \subset S, |S - U| \leqslant n\}$$

is an equivalence.

(The target here is a holim indexed by a subposet of $\mathscr{P}(S)$. The map is the canonical equivalence $F(\mathscr{Y}(\emptyset)) \to \operatorname{holim}\{F(\mathscr{Y}(U)) : U \subset S\}$ followed by a restriction map.)

Proof. When $m = n + 1$, this is the definition of 'n-excisive'. When $m \leqslant n$, the statement is trivially true. For the case when $m > n + 1$, we argue by induction on m.

Define a cube $\mathscr{X}: \mathscr{P}(S) \to \mathscr{D}$ by

$$\mathscr{X}(T) = \operatorname{holim}\{F(\mathscr{Y}(U)) : T \subset U \subset S, |S - U| \leqslant n\}, \quad T \subset S.$$

There is an obvious map of cubes $F(\mathscr{Y}) \to \mathscr{X}$, given by the canonical equivalence $F(\mathscr{Y}(T)) \to \operatorname{holim}\{F(\mathscr{Y}(U)) : T \subset U \subset S\}$ followed by a restriction to $\mathscr{X}(T)$. We must show that $F(\mathscr{Y}(\emptyset)) \to \mathscr{X}(\emptyset)$ is an equivalence. The inductive hypothesis implies that the map $F(\mathscr{Y}(T)) \to \mathscr{X}(T)$ is an equivalence for each nonempty $T \subset S$. The conclusion will therefore follow if both $F(\mathscr{Y})$ and $\mathscr{X}$ are Cartesian. This is true for $F(\mathscr{Y})$ by (3.2) and for $\mathscr{X}$ by (1.9). □

The next result concerns functors $(X_1, \ldots, X_r) \mapsto M(X_1, \ldots, X_r)$ of several variables; we assume that M is a homotopy functor in each variable separately.

3.4. PROPOSITION. *If $M: \mathscr{C}^r \to \mathscr{D}$ is n_i-excisive in the* ith *variable for all $1 \leqslant i \leqslant r$, then the composition with the diagonal inclusion $\Delta: \mathscr{C} \to \mathscr{C}^r$ is n-excisive with $n = n_1 + \cdots + n_r$.*

Proof. Let $\mathscr{Y}: \mathscr{P}(S) \to \mathscr{C}$ be any strongly co-Cartesian cube with $|S| > n$. To show that the cube $(M \circ \Delta)(\mathscr{Y})$ is Cartesian, we produce a map from it to another cube $\mathscr{X}$:

$$\begin{aligned}(M \circ \Delta)(\mathscr{Y}(T)) &= M(\mathscr{Y}(T), \ldots, \mathscr{Y}(T)) \\ &\longrightarrow \operatorname{holim}\{M(\mathscr{Y}(U_1), \ldots, \mathscr{Y}(U_r)) : T \subset U_i \subset S, |S - U_i| \leqslant n_i\} \\ &= \mathscr{X}(T).\end{aligned}$$

The map can be seen to be an equivalence (for all T) by using (3.3) r times. The cube $\mathscr{X}$ is Cartesian by (1.9), since the poset

$$\mathscr{A} = \{(U_1, \ldots, U_r) : \forall_i \ U_i \subset S, |S - U_i| \leqslant n_i\}$$

is the union $\mathscr{A}_1 \cup \cdots \cup \mathscr{A}_r$ where the concave subset $\mathscr{A}_j \subset \mathscr{A}$ is defined by the additional condition $\forall_i \ j \in U_i$. □

As a special case, if the functor $M(X_1, \ldots, X_n)$ of n variables is excisive ($=1$-excisive) in each variable separately, then $M(X, \ldots, X)$ is an n-excisive functor of X. In particular, we have

3.5. EXAMPLE. For any spectrum $\underline{C}$ the functors $X \mapsto \underline{C} \wedge (X^n_+) = \underline{C} \wedge (X_+)^{\wedge n}$ and $X \mapsto \underline{C} \wedge X^{\wedge n}$ from spaces or based spaces to spectra are n-excisive, as are the functors $X \mapsto \Omega^\infty(\underline{C} \wedge X^n_+)$ and $X \mapsto \Omega^\infty(\underline{C} \wedge X^{\wedge n})$ from spaces or based spaces to based spaces.

Here the subscript '+' adds a disjoint basepoint, and the superscript '$\wedge n$' means nth smash power.

In (4.4), we will see a second method for proving statements like (3.5).

4. Analytic Functors

Again let $F: \mathscr{C} \to \mathscr{D}$ be a homotopy functor.

4.1. DEFINITION. F is *stably n-excisive*, or satisfies *stable* nth *order excision*, if the following is true for some numbers c and κ:

$E_n(c, \kappa)$: If $\mathscr{X}: \mathscr{P}(S) \to \mathscr{C}$ is any strongly co-Cartesian $(n+1)$-cube such that for all $s \in S$ the map $\mathscr{X}(\emptyset) \to \mathscr{X}(s)$ is k_s-connected and $k_s \geqslant \kappa$, then the diagram $F(\mathscr{X})$ is $(-c + \Sigma k_s)$-Cartesian.

The number c is allowed to be negative, but in interesting examples it is usually positive. The smaller c is, the stronger the condition.

Likewise, the smaller the number κ, the stronger the condition. If $E_n(c, \kappa)$ holds for all κ (i.e. for $\kappa = -1$), then we simply say $E_n(c)$.

Stable first-order excision was called stable excision in [5], (1.8).

4.2. DEFINITION. F is *ρ-analytic* if there is some number q such that F satisfies $E_n(n\rho - q, \rho + 1)$ for all $n \geqslant 1$.

4.3. EXAMPLE. *The identity functor from spaces to spaces is* 1-*analytic.*

Proof. Theorem 2.3 asserts precisely that the identity functor satisfies $E_n(n)$ for all n. □

4.4. EXAMPLE. *The functors $X \mapsto \Sigma^\infty(X^m_+)$ from spaces to spectra and $X \mapsto Q(X^m_+)$ from spaces to spaces, satisfy $E_n(0)$ for all $n \geqslant 0$. (This is only new information for $n < m$; for $n \geqslant m$ these functors are n-excisive, by* (3.5) *and* (3.2), *and so satisfy $E_n(c)$ for all c.)*

Proof. Let $\mathscr{X}$ be a strongly co-Cartesian S-cube of spaces, and assume that the map $\mathscr{X}(\emptyset) \to \mathscr{X}(s)$ is k_s-connected for all $s \in S$. We will show that $\mathscr{X}^m: T \mapsto \mathscr{X}(T)^m$ is an $(|S| - 1 + \Sigma k_s)$-co-Cartesian S-cube of spaces. By (1.19) it will follow that $\Sigma^\infty(\mathscr{X}^m_+)$ is a (Σk_s)-Cartesian S-cube of spectra, and that $Q(\mathscr{X}^m_+)$ is a (Σk_s)-Cartesian S-cube of spaces.

Using (2.2.a), we may assume that $\mathscr{X}(S)$ is CW, each $\mathscr{X}(T)$ is a subcomplex, $\mathscr{X}(T \cup U) = \mathscr{X}(T) \cup \mathscr{X}(U)$, $\mathscr{X}(T \cap U) = \mathscr{X}(T) \cap \mathscr{X}(U)$, and all cells in $\mathscr{X}(S) - \mathscr{X}(S - s) = \mathscr{X}(s) - \mathscr{X}(\emptyset)$ have dimension $\geqslant k_s + 1$. The cube $\mathscr{X}^m: T \mapsto \mathscr{X}(T)^m$ is then determined by a CW $|S|$-ad. In particular, it is a cofibration cube and (1.16) applies. We have to estimate the connectivity of the CW pair $(\mathscr{X}(S)^m, \bigcup_s \mathscr{X}(S-s)^m)$. We do so by examining the dimensions of cells. A cell of $\mathscr{X}(S)^m$ has the form $e = e_1 \times \cdots \times e_m$, where each e_j is a cell of $\mathscr{X}(S)$. Therefore, for each j, either e_j is a cell of $\mathscr{X}(\emptyset)$, or for a unique $s_j \in S$, e_j is a cell of $\mathscr{X}(s_j) - \mathscr{X}(\emptyset)$. If e is not in $\bigcup_s \mathscr{X}(S-s)^m$, then every s is s_j for at least one j and we have $\dim(e) = \Sigma_j \dim(e_j) \geqslant \Sigma_s(k_s+1) = |S| + \Sigma_s k_s$. This proves that the pair is $(|S| - 1 + \Sigma_s k_s)$-connected.

Note that when $|S| > m$ there are no such cells because there are more s's than j's. This gives a different proof that the functors are m-excisive. □

Let Map(K, X) be the space of all continuous maps from K to X.

4.5. EXAMPLE. *For any finite* CW *complex K, the functors* $X \mapsto \Sigma^\infty \mathrm{Map}(K, X)_+$ *and* $X \mapsto Q\,\mathrm{Map}(K, X)_+$ *are ρ-analytic, where ρ is the dimension of K.*

Proof. The case $\rho = 0$ was covered in Example 4.4 above. We assume $\rho > 0$ and show that the functor satisfies $E_{n-1}(n\rho, 1)$.

Let $\mathscr{X}$ be any strongly co-Cartesian S-cube of spaces with $|S| = n \geqslant 1$ and $\mathscr{X}(\emptyset) \to \mathscr{X}(s)$ k_s-connected, $k_s \geqslant 1$, for all $s \in S$. As in (4.4), it will be enough if we show that the n-cube Map$(K, \mathscr{X})$ is $(n - 1 - n\rho + \Sigma k_s)$-co-Cartesian.

The functor $X \mapsto \mathrm{Map}(K, X)$ takes k-connected maps to $(k-\rho)$-connected maps. Since it also commutes with holim, it takes k-Cartesian cubes to $(k-\rho)$-Cartesian cubes. Therefore, by (2.3), the cube Map$(K, \mathscr{X})$ is $(1 - n - \rho + \Sigma k_s)$-Cartesian. The same reasoning applied to the face $\partial_{S-T}\mathscr{X}$, for each $T \neq \emptyset$ in S, shows that the T-cube $\partial_{S-T}\mathrm{Map}(K, \mathscr{X}) = \mathrm{Map}(K, \partial_{S-T}\mathscr{X})$ is $k(T)$-Cartesian with $k(T) = 1 - |T| - \rho + \Sigma_{t \in T} k_t$. Apply (2.6). For any partition $\{T_\alpha\}$ the sum of $k(T_\alpha)$ is $-n + \Sigma k_s + \Sigma_\alpha(1-\rho)$. This is minimized when each T_α has one element, because $1 - \rho \leqslant 0$. The minimum is $-n\rho + \Sigma k_s$. We conclude that the S-cube Map$(K, \mathscr{X})$ is $(n - 1 - n\rho + \Sigma k_s)$-co-Cartesian. □

4.6. THEOREM. *Waldhausen's functor A is* 1*-analytic.*

Proof. This statement, reasonably interpreted, is true for any of the various versions of A, but let us be more definite about which one we mean.

Theorem 2.1.5 of [9] provides a choice of four naturally equivalent homotopy functors from simplicial sets to spaces. We mean any one of them, say $X \mapsto \Omega|hS_\cdot R_f(X)|$, composed with the singular complex functor from spaces to simplicial sets. Thus, A is a homotopy functor from (unbased) spaces to spaces. We claim that it satisfies $E_{n-1}(n-1, 2)$.

Let $\mathscr{X}$ be a strongly co-Cartesian S-cube of spaces, with $|S| = n \geqslant 1$ and $\mathscr{X}(\emptyset) \to \mathscr{X}(s)$ k_s-connected, $k_s \geqslant 2$, for all $s \in S$. We show that the cube $A(\mathscr{X})$ is $(1 - n + \Sigma k_s)$-Cartesian.

As an easy consequence of its definition, A takes finite disjoint union to product, and infinite disjoint union to 'weak product' (=direct limit of finite products). Therefore, we can reduce easily to the case in which $\mathscr{X}(S)$ is path-connected. In this case every $\mathscr{X}(T)$ is path-connected, since the map $\mathscr{X}(T) \to \mathscr{X}(S)$ is 1-connected (in fact, 2-connected). Choosing a point in $\mathscr{X}(\emptyset)$, we can consider $\mathscr{X}$ as a cube of based, path-connected spaces.

Now we can switch to a plus-construction definition of A, using [9], Theorems 2.2.1 and 2.1.5. In fact, for based, path-connected X, $A(X)$ is naturally equivalent to the product $\mathbb{Z} \times B\mathscr{H}(X)^+$ where $\mathscr{H}(X)$ (a sort of 'space of matrices') is a certain simplicial monoid which is a functor of X. Specifically, $\mathscr{H}(X)$ is the direct limit of $\mathscr{H}^n_k(G(X))$, as defined on page 385 of [9] or page 359 of [8], $G(X)$ being the Kan loop group of the singular complex of X. It will be enough if the cube $B\mathscr{H}(\mathscr{X})^+$ is $(1-n+\Sigma k_s)$-Cartesian. The monoid of components $\pi_0\mathscr{H}(X)$ is naturally isomorphic to $\mathrm{GL}(\mathbb{Z}[\pi_1 X])$. In particular it is a group, which implies that $\mathscr{H}(X)$ is equivalent to $\Omega B\mathscr{H}(X)$.

We will need to replace the spaces $B\mathscr{H}(\mathscr{X}(T))^+$ by their universal covers. The group $\pi_1(B\mathscr{H}(X)^+) \cong (\pi_0\mathscr{H}(X))^{ab} \cong K_1(\mathbb{Z}[\pi_1 X])$ depends only on $\pi_1 X$, and the universal cover of $B\mathscr{H}(X)^+$ is naturally equivalent to $B\mathscr{E}(X)^+$ where $\mathscr{E}(X) \subset \mathscr{H}(X)$ is the union of those components which correspond to elements of the commutator subgroup $E \subset \mathrm{GL}$. Let $\mathscr{C}(X) \subset \mathscr{E}(X) \subset \mathscr{H}(X)$ be the component of the identity. As a space, $\mathscr{C}(X)$ is naturally equivalent to the weak product (indexed by a fixed infinite countable set $\mathbb{N} \times \mathbb{N}$) of copies of the zero component of $Q(G(X)_+)$, or $Q(\Omega X_+)$. (See [8].)

All of the spaces $\mathscr{X}(T)$ have the same fundamental group, since the map $\mathscr{X}(T) \to \mathscr{X}(S)$ is 2-connected. Therefore, it will be enough if the cube formed by the universal covers or, equivalently, the cube $B\mathscr{E}(\mathscr{X})^+$, is $(1-n+\Sigma k_s)$-Cartesian.

We now make a series of connectivity estimates, applying the theorems of Section 2 several times:

$\mathscr{X}$ is $(1-n+\Sigma k_s)$-Cartesian by (2.3).

$\Omega\mathscr{X}$ is $(-n+\Sigma k_s)$-Cartesian.

$\Omega\mathscr{X}$ is $(-1+\Sigma k_s)$-co-Cartesian by (2.6). (Apply the previous statement to the faces of $\Omega\mathscr{X}$.)

$Q(\Omega\mathscr{X}_+)$ is $(-n+\Sigma k_s)$-Cartesian by (1.19).

$\mathscr{C}(\mathscr{X})$ is $(-n+\Sigma k_s)$-Cartesian.

$\mathscr{E}(\mathscr{X})$ is $(-n+\Sigma k_s)$-Cartesian. This uses the fact that $\pi_0\mathscr{E}(\mathscr{X}(T))$ is independent of T.

$B\mathscr{E}(\mathscr{X})$ is $(1-n+\Sigma k_s)$-Cartesian. This is not as obvious as it may look; we prove it by induction on n, and begin by noting the consequence that the total fiber $\tilde{f}B\mathscr{E}(\mathscr{X})$ is connected, in fact n-connected since $-n+\Sigma k_s \geqslant n$. We want the map $a(B\mathscr{E}(\mathscr{X}))\colon B\mathscr{E}(\emptyset) \to h_0 B\mathscr{E}(\mathscr{X})$ to be $(1-n+\Sigma k_s)$-connected. We know by the previous step that it becomes $(-n+\Sigma k_s)$-connected after looping, so it is enough if $h_0 B\mathscr{E}(\mathscr{X})$ is connected. Write $\mathscr{X}$ as a map $\mathscr{Y} \to \mathscr{Z}$ of $(n-1)$-cubes. By (1.2.a), $h_0 B\mathscr{E}(\mathscr{X})$ fibers over $h_0 B\mathscr{E}(Y)$ with fiber $\tilde{f}B\mathscr{E}(\mathscr{Z})$. Both base and fiber are connected by induction on n.

$B\mathscr{E}(\mathscr{X})$ is (Σk_s)-co-Cartesian by (2.6). (Apply the previous statement to the faces.)

$\Sigma^\infty(B\mathscr{E}(\mathscr{X})^+) \sim \Sigma^\infty(B\mathscr{E}(\mathscr{X}))$ is a (Σk_s)-co-Cartesian cube of spectra, by (1.19).

$B\mathscr{E}(\mathscr{X})^+$ is a (Σk_s)-co-Cartesian cube of spaces. (Note that the map $b(B\mathscr{E}(\mathscr{X})^+): h_1(B\mathscr{E}(\mathscr{X})^+) \to B\mathscr{E}(\mathscr{X}(S))^+$ is a map of 1-connected spaces, so that the vanishing of its relative stable homotopy groups in a range of low dimensions implies the vanishing of its relative homotopy groups in the same range.)

$B\mathscr{E}(\mathscr{X})^+$ is $(1 - n + \Sigma k_s)$-Cartesian by (2.4). (Apply the previous statement to faces.) □

The theorem can be extended to other versions of the theory. For example, let $\mathbf{A}(X)$ be the spectrum $\{\Omega^m|hS^{(m)}_{\bullet}R_f(X)|\}$ (see [9], 1.3 and 1.5); this is a (-1)-connected spectrum with $A(X) \sim \Omega^\infty\mathbf{A}(X)$. The homotopy functor $\mathbf{A}$ from spaces to spectra again satisfies $E_{n-1}(n-1, 2)$. To prove this, let $\mathscr{X}$ be a strongly co-Cartesian S-cube of spaces, with $|S| = n \geqslant 1$ and $\mathscr{X}(\emptyset) \to \mathscr{X}(s)$ k_s-connected, $k_s \geqslant 2$, for all $s \in S$. We have to show that the total fiber $\tilde{f}\mathbf{A}(\mathscr{X})$ is $[\Sigma(k_s - 1)]$-connected. By (4.6) we know that its homotopy groups vanish in dimensions from 0 through $\Sigma(k_s - 1)$. We must also check that it is (-1)-connected. This is easy, using induction on n: Write $\mathscr{X}$ as a map $\mathscr{Y} \to \mathscr{Z}$ of $(n-1)$-cubes. Then $\tilde{f}\mathbf{A}(\mathscr{Y})$ and $\tilde{f}\mathbf{A}(\mathscr{Z})$ are $(n-1)$-connected by induction, therefore 0-connected, and their fiber $\tilde{f}\mathbf{A}(\mathscr{X})$ is (-1)-connected.

5. Some Consequences of Analyticity

We continue to work with homotopy functors from spaces to (based or unbased) spaces or spectra. The theme of this section is that functors which are ρ-analytic tend to behave quite rigidly on the class of ρ-connected spaces, and more generally on the class of all spaces Y having a $(\rho + 1)$-connected map to a fixed space X. Here is a very easy result in that direction:

5.1. PROPOSITION (Uniqueness of analytic continuation). *Let F and G be ρ-analytic functors, and let $\vartheta: F \to G$ be a natural map. Suppose that, for some k, the map $\vartheta: F(Y) \to G(Y)$ is an equivalence for every Y admitting a k-connected map to X. Then this is so for $k = \rho + 1$.*

The proof of (5.1) uses the following *fiberwise join* construction (which will also prove very useful in [6]). Let $f: Y \to X$ be a map of spaces. If T is any finite set, define the space $Y *_X T$ as the homotopy colimit of $Y \leftarrow Y \times T \to X \times T$. This is the union along Y of several copies of the mapping cylinder of f, one for each element of T. (If T is empty, it is Y). We view $- *_X T$ as a functor from $\mathscr{U}_X$, the category of all spaces over X, to itself.

When T has just one element, then $Y *_X T$ is what was called the *fiberwise cone* of Y in [5]. When T has two elements, it is the *fiberwise unreduced suspension* $S_X Y$. In general, it is equivalent in $\mathscr{U}_X$ to the union along X of $|T| - 1$ copies of $S_X Y$. Clearly:

(5.2) If $Y \to X$ is $(k-1)$-connected, then $Y *_X T \to X$ is k-connected for nonempty T.

The fiberwise join is also a functor of T; as T runs through the subsets of a fixed finite set S, it yields an S-cube of spaces over X, $Y *_X \mathscr{P}(S) : \mathscr{P}(S) \to \mathscr{U}_X$. This is a pushout cube and, therefore, strongly co-Cartesian.

Proof of (5.1). We may assume $k = \rho + 2$. Let q satisfy the requirements of (4.2) for both F and G. Let f: $Y \to X$ be a $(\rho + 1)$-connected map.

Choose a large finite set S, say with $|S| = n + 1$, and consider the map of S-cubes of spaces

$$\vartheta : F(Y *_X \mathscr{P}(S)) \longrightarrow G(Y *_X \mathscr{P}(S))$$

In the square diagram

$$\begin{tikzcd} F(Y) \arrow[r, "\vartheta"] \arrow[d, "a"'] & G(Y) \arrow[d, "a"] \\ h_0(F(Y *_X \mathscr{P}(S)) \arrow[r] & h_0(G(Y *_X \mathscr{P}(S)) \end{tikzcd}$$

the lower map is an equivalence, by (5.2) and the hypothesis. Each of the vertical maps is $(q - n\rho + (n + 1)(\rho + 1))$-connected, since F and G satisfy $E_n(n\rho - q, \rho + 1)$. It follows that the upper map is $(q + \rho + n)$-connected. This tends to $+\infty$ with n, which was arbitrary. □

The remaining results in this section use the *derivative*, the *differential*, and the *1-jet* of an analytic functor as defined in [5]. We recall roughly what these are:

The functor F: $\mathscr{U} \to \mathscr{D}$ determines, for each space X, a functor $P_X F$ (the 1-jet of F at X) whose domain is the category $\mathscr{U}_X$ of spaces over X. There are natural maps $F(Y) \to (P_X F)(Y) \to F(X)$. Up to homotopy, $P_X F$ is the universal example of an excisive functor from $\mathscr{U}_X$ to $\mathscr{D}$ that is equipped with a map from F.

The differential $D_X F$ is defined if $\mathscr{D}$ is spectra or based spaces; it is the homotopy fiber of the canonical map from $P_X F$ to the constant functor $F(X)$ on $\mathscr{U}_X$; it is a linear (=excisive and reduced) functor from $\mathscr{U}_X$ to $\mathscr{D}$.

For a space X and point $x \in X$, the differential $D_X F$ yields a linear functor L from based spaces to $\mathscr{D}$ by putting $L(Z) = (D_X F)(X \vee Z)$. Thus there is a spectrum $\underline{C}$ such that $L(Z)$ is naturally equivalent to $\underline{C} \wedge X$ (if $\mathscr{D} = \mathscr{S}$) or $\Omega^\infty(\underline{C} \wedge Z)$ (if $\mathscr{D} = \mathscr{T}$), at least when Z is finite CW. This spectrum can be made functorial in (X, x) and F; it is called the derivative of F at (X, x) and written $\partial_x F(X)$.

We may refer to any of the theorems below as the 'first-derivative criterion'.

5.3. THEOREM. *Let F and G be ρ-analytic functors from spaces to spectra. Let $\vartheta : F \to G$ be a natural map. If the induced map $D_X F \to D_X G$ is an equivalence for all spaces X, then for every $(\rho + 1)$-connected map $Y \to X$ the diagram is Cartesian*:

$$\begin{tikzcd} F(Y) \arrow[r] \arrow[d] & G(Y) \arrow[d] \\ F(X) \arrow[r] & G(X) \end{tikzcd}$$

The meaning of the hypothesis on $D_X F \to D_X G$ is that for every space Y and map $Y \to X$ the map $(D_X F)(Y) \to (D_X G)(Y)$ is an equivalence.

In particular, taking $G = *$, we have:

5.4. COROLLARY (Functors with 'derivative zero' are locally constant). *Let F be a ρ-analytic functor from spaces to spectra. If $D_X F$ is contractible for all spaces X, then $F(X)$ depends only on the ρ-homotopy type of X: every $(\rho + 1)$-connected map $Y \to X$ induces an equivalence $F(Y) \to F(X)$.*

Proof of (5.3). We actually prove (5.4). This, with 'F' equal to the homotopy fiber of $F \to G$, yields (5.3), since $\mathrm{fiber}(D_X F \to D_X G)$ is naturally equivalent to $D_X\, \mathrm{fiber}(F \to G)$. By hypothesis, there is some q_0 such that, for all $n \geqslant 1$, F satisfies $E_n(n\rho - q_0, \rho + 1)$.

Let $Y \to X$ be a k-connected map, $k > \rho$. Consider the diagram:

$$F(Y) \longrightarrow (P_X F)(Y) \longrightarrow F(X).$$

The hypothesis $D_X F \sim *$ means that the second map is an equivalence. Therefore, using [5] (1.15.ii), $E_1(\rho - q_0, \rho + 1)$ implies

(5.5) $F(Y) \to F(X)$ is $(q_0 - \rho + 2k)$-connected whenever $Y \to X$ is k-connected and $k > \rho$.

We will show that F satisfies $E_1(\rho - q, \rho + 1)$ for *all* q, so that (5.5) becomes the statement to be proved. We will do it by proving, inductively with respect to q, that F satisfies $E_n(n\rho - q, \rho + 1)$ for all q and all n.

We assume $E_{n+1}((n+1)\rho - (q-1), \rho + 1)$ and prove $E_n(n\rho - q, \rho + 1)$.

Let $\mathscr{X}$ be a strongly co-Cartesian $(n+1)$-cube of spaces. Suppose that, for each s, the map $\mathscr{X}(\emptyset) \to \mathscr{X}(s)$ is k_s-connected, $k_s > \rho$. Call that map e_s. We show that $F(\mathscr{X})$ is $(q - n\rho + \Sigma_s k_s)$-Cartesian.

Without loss of generality, by (2.2), the maps $(e_0, \dots, e_n)$ are cofibrations and $\mathscr{X}$ is a pushout $(n+1)$-cube. Let $\mathscr{Z}$ be the pushout $(n+2)$-cube defined by the cofibrations $(e_0, e_0, e_1, \dots, e_n)$. Then $\mathscr{Z}$ may be viewed as a map from $\mathscr{X}$ to another $(n+1)$-cube $\mathscr{Y}$. By (1.6.i) it will be enough if $F(\mathscr{Y})$ and $F(\mathscr{Z})$ are $(q - n\rho + \Sigma_s k_s)$-Cartesian.

For $\mathscr{Z}$ this follows from the inductive hypothesis $E_{n+1}((n+1)\rho - (q-1), \rho + 1)$, since

$$\begin{aligned}
&q - 1 - (n+1)\rho + (2k_0 + k_1 + k_2 + \cdots + k_n) \\
&= q + (k_0 - \rho - 1) - n\rho + (k_0 + k_1 + \cdots + k_{n-1} + k_n) \\
&\geqslant q - n\rho + (k_0 + k_1 + \cdots + k_{n-1} + k_n).
\end{aligned}$$

Notice that $\mathscr{Y}$ is also a pushout cube. The defining cofibrations here are $(f_0, f_1, \dots, f_n)$, where f_s is defined by the pushout square:

$$\begin{tikzcd}
\mathscr{X}(\emptyset) \arrow[r, "e_0"] \arrow[d, "e_s"'] & \mathscr{X}(0) = \mathscr{Y}(\emptyset) \arrow[d, "f_s"] \\
\mathscr{X}(s) \arrow[r] & Y(s)
\end{tikzcd}$$

The best we can assert about f_s is that it is still k_s-connected, so we seem to have made no progress, having simply traded $\mathscr{X}$ for $\mathscr{Y}$.

However, $\mathscr{Y}$ has a special feature: One of the cofibrations f_s from which it is built, namely f_0, has a left inverse $g:\mathscr{Y}(0)\to\mathscr{Y}(\emptyset)$. It follows that $\mathscr{Y}$, viewed as a map of n-cubes, has a left inverse $\mathscr{Y}'$:

$$(5.6)\quad \mathscr{Y}(T)\xrightarrow{\mathscr{Y}}\mathscr{Y}(T\cup 0)\xrightarrow{\mathscr{Y}'}\mathscr{Y}(T),\quad T\subset \underline{n}=\{1,\ldots,n\}$$

View $\mathscr{Y}'$ as an $(n+1)$-cube. It is strongly co-Cartesian. The conclusion of $E_n(n\rho-q,\rho+1)$ will hold for the cube $\mathscr{Y}$ if it holds for $\mathscr{Y}'$. Indeed, g is (k_0+1)-connected (by (1.5.iii) applied to $\mathscr{Y}(\emptyset)\to\mathscr{Y}(0)\to\mathscr{Y}(\emptyset)$), and if $F(\mathscr{Y}')$ is $(q-n\rho+(k_0+1)+k_1+k_2+\cdots+k_n)$-Cartesian then $F(\mathscr{Y})$ is $(q-n\rho+k_0+k_1+k_2+\cdots+k_n)$-Cartesian (by (1.8.iii)).

Now a downward induction with respect to k_0 (with $k_1,\ldots,k_n$ fixed) completes the argument. To begin the induction, use (5.5). It implies that for each $T\subset\underline{n}$ the map $F(\mathscr{X}(T))\to F(\mathscr{X}(T\cup 0))$ is $(q_0-\rho+2k_0)$-connected, which implies by (1.6.ii) that the $(n+1)$-cube $F(\mathscr{X})$ is $(q_0-\rho-n+2k_0)$-Cartesian. For large k_0, this exceeds $q-n\rho+\Sigma_s k_s$. □

We indicate some variants of (5.3).

In trying to extend (5.3) to space-valued functors, we must be careful to avoid a fallacy: It is not true that a map is an equivalence [resp., a map of cubes is Cartesian] if its fiber is contractible [resp., Cartesian]. Thus we are led to a statement involving the map $(P_XF)(Y)\to F(X)$ rather than its fiber. We are also led to study the map $F\to G$ itself rather than its fiber; we cannot deduce the new version of (5.3) from a new version of (5.4):

5.7. THEOREM. *Let F and G be ρ-analytic functors from spaces to (based or unbased) spaces. Let $\vartheta: F\to G$ be a natural map. If the square*

$$\begin{array}{ccc}(P_XF)(Y) & \longrightarrow & (P_XG)(Y)\\ \downarrow & & \downarrow\\ F(X) & \longrightarrow & G(X)\end{array}$$

is Cartesian for every X and every $Y\to X$, then for every $(\rho+1)$-connected map $Y\to X$, the diagram is Cartesian:

$$\begin{array}{ccc}F(Y) & \longrightarrow & G(Y)\\ \downarrow & & \downarrow\\ F(X) & \longrightarrow & G(X)\end{array}$$

Proof (sketch). The inductive hypothesis in the proof of (5.3), that F satisfies $E_n(n\rho-q,\rho+1)$ for all n, should be replaced by the statement:

For every n, for every strongly co-Cartesian $(n+1)$-cube of spaces, if the map $\mathscr{X}(\emptyset)\to\mathscr{X}(s)$ is k_s-connected, $k_s>\rho$, for all s, then the $(n+2)$-cube $F(\mathscr{X})\to G(\mathscr{X})$ is $(q-n\rho+\Sigma_s k_s)$-Cartesian.

The analyticity means that both $F(\mathscr{X})$ and $G(\mathscr{X})$ are $(q_0 - n\rho + \Sigma_s k_s)$-Cartesian for some q_0 (independent of n). This begins the induction with respect to q.

To go from $q-1$ to q, we follow the pattern of the proof of (5.3), reducing to proving that $F(\mathscr{X}) \to G(\mathscr{X})$ is $(q - n\rho + \Sigma_s k_s)$-Cartesian when k_0 is large. This case is handled by showing that the outer square in

$$\begin{array}{ccc} F(Y) & \longrightarrow & G(Y) \\ \downarrow & & \downarrow \\ (P_X F)(Y) & \longrightarrow & (P_X G)(Y) \\ \downarrow & & \downarrow \\ F(X) & \longrightarrow & G(X) \end{array}$$

is $(q_0 - \rho + 2k_0)$-Cartesian when $Y \to X$ is k_0-connected. This is true for the upper square by ([5], (1.15.ii)); the lower square is Cartesian by hypothesis. □

5.8. THEOREM. *For functors to* connected, based *spaces* (5.3) *applies exactly as written.*

Proof. The hypothesis of (5.3) implies that of (5.7) in this case. □

Recall from [5] that a homotopy functor F is said to satisfy the *limit axiom* if for every CW complex X, the homotopy groups of $F(X)$ are colimits of homotopy groups of $F(X_\alpha)$, indexed by the finite subcomplexes $X_\alpha \subset X$. If the functors F and G satisfy the limit axiom, then the hypothesis of (5.3) or (5.8) can be altered so as to refer to the derivative instead of the differential:

5.9. THEOREM. *Let F and G be ρ-analytic functors from spaces to spectra, or to connected based spaces. Assume that they satisfy the limit axiom. Let $\vartheta: F \to G$ be a natural map. If the induced map $\partial_x F(X) \to \partial_x G(X)$ is an equivalence for all based spaces (X, x) then for every $(\rho + 1)$-connected map $Y \to X$ the diagram is Cartesian:*

$$\begin{array}{ccc} F(Y) & \longrightarrow & G(Y) \\ \downarrow & & \downarrow \\ F(X) & \longrightarrow & G(X) \end{array}$$

Proof. See [5], (1.3.v). □

Here is the variant needed in [2]. Let p be a prime number.

Define the p-completion of a spectrum E to be the homotopy limit of the tower $E \wedge M(p^n)$ of smash products with Moore spaces. This is the same as the Bousfield localization of E with respect to mod p homotopy theory; the canonical map from E to its completion induces an isomorphism of mod p homotopy groups, and a map of spectra induces an equivalence of p-completions if and only if it induces isomorphisms of mod p homotopy groups. Call a spectrum p-complete if the canonical map to the p-completion is an equivalence. If F is a homotopy functor from spaces to p-complete spectra, then we will say that it satisfies the *p-limit axiom* if, for every CW complex X, the mod p homotopy groups of $F(X)$ are colimits of mod p homotopy groups of $F(X_\alpha)$, indexed by the finite subcomplexes $X_\alpha \subset X$. For example, the p-completion of a functor that satisfies the limit axiom always satisfies the p-limit axiom.

5.10. THEOREM. *Let F and G be ρ-analytic functors from spaces to p-complete spectra. Assume that they satisfy the p-limit axiom. Let $\vartheta: F \to G$ be a natural map. If the induced map $\partial_x F(X) \to \partial_x G(X)$ is an equivalence for all based spaces (X, x), then for every $(\rho + 1)$-connected map $Y \to X$, the square is Cartesian*:

$$\begin{tikzcd} F(Y) \arrow[r] \arrow[d] & G(Y) \arrow[d] \\ F(Y) \arrow[r] & G(X) \end{tikzcd}$$

Proof. Smashing both functors with a mod p Moore space, we obtain functors F/p and G/p satisfying the hypotheses of (5.9). By (5.9) we conclude that the square

$$\begin{tikzcd} (F/p)(Y) \arrow[r] \arrow[d] & (G/p)(Y) \arrow[d] \\ (F/p)(X) \arrow[r] & (G/p)(X) \end{tikzcd}$$

is Cartesian, whence the preceding square is Cartesian after p-completion of the spectra in it. But they are p-complete. □

Appendix: Operators on the Free Loopspace

In [5], Corollary 2.5, we determined the derivative of the functor $X \mapsto Q(\Lambda X_+)$, stable homotopy of the free loopspace. Here we obtain an improved, equivariant, version of that result. The equivariant version is required in [2].

According to Corollary 2.5 of [5], the derivative of the functor $X \mapsto Q_+ \Lambda X$ is given by

(A.1) $\quad \partial_x Q_+(\Lambda X) \sim \Lambda \Sigma_+^\infty(\Omega X).$

The topological monoid of all continuous maps $f: S^1 \to S^1$ acts (continuously) on ΛX and on $Q_+(\Lambda X)$. Because 'differentiation' of functors is functorial, the monoid acts on the left-hand side of (A.1) and in fact it acts continuously, as one verifies by examining the definitions. In [2], we need to know the corresponding action on the right-hand side of (A.1). More precisely, (A.1) is shorthand for a chain of natural weak homotopy equivalences between homotopy functors of X, and what we need is a continuous action on the right-hand side such that, in the chain of equivalences, each functor of X has a natural continuous action and each equivalence is an equivariant map.

We only really need this for the submonoid generated by the rotations and the power maps. The answer is simple to state in that case. Write $S^1 = \mathbb{R}/\mathbb{Z}$, and define the rotations a_u ($u \in \mathbb{R}/\mathbb{Z}$) and the power maps P_n ($n \in \mathbb{Z}$) by $a_u(t) = t + u$ and $P_n(t) = nt$. Let $P_n^*: \Omega X \to \Omega X$ be composition with P_n.

A.2. PROPOSITION. *The equivalence* (A.1) *respects the action of the rotations and power maps if the rotation a_u acts on $\Lambda\Sigma^\infty_+(\Omega X)$ by $\lambda \mapsto \lambda \circ a_u$ and the power map P_n acts by $\lambda \mapsto [\Sigma^\infty(P_n^*)_+] \circ \lambda \circ P_n$. In other words, to make formula* (A.1) *equivariantly correct, the rotations should act only on the 'Λ' in $\Lambda\Sigma^\infty_+(\Omega X)$, while the power maps act both on the 'Λ' and on the 'Ω'.*

The proof of (A.2) really consists of re-reading the proof of (A.1) with the monoid in mind.

In [5], (A.1) was derived from a more general result (Theorem 2.4) in which S^1 was replaced by an arbitrary finite complex K. It is no harder, and maybe easier, to work out an equivariant version of that more general result. This we now do.

Theorem 2.4 of [5] identifies the differential and the derivative of the functor $F(X) = Q_+ \operatorname{Map}(K, X)$. For the differential, it gives:

$$\text{(A.3)} \quad (D_X F)(Y \to X) \sim \Theta_K(Y \to X).$$

(See section 2 of [5] for notation.) Of course $F(X)$ is a functor of K as well as X. It is clear in re-examining the proof of (2.4) in [5] that the equivalence (A.3) is natural in K. We must verify this, making explicit the way in which $\Theta_K(Y \to X)$ is a (contravariant) functor of K.

Let $Y \to X$ be a map of spaces. Any map $f: K \to L$ determines a map $\Theta_f(Y \to X) : \Theta_L(Y \to X) \to \Theta_K(Y \to X)$ as follows: It determines a map $\hat{f}$ of fibrations over $K \times X$:

$$\begin{tikzcd}
(k, g \circ f) & K \times \operatorname{Map}(K, X) = & E(K, X) \arrow[r] & K \times X \\
(k, g) \arrow[u] & K \times \operatorname{Map}(L, X) \cong & (f \times 1)^{-1} E(L, X) \arrow[r] \arrow[u, "\hat{f}"] \arrow[d] & K \times K \arrow[u, "1"'] \arrow[d, "f \times 1"] \\
& L \times \operatorname{Map}(L, X) = & E(L, X) \arrow[r] & L \times X
\end{tikzcd}$$

By pullback with $K \times Y \to K \times X$ this $\hat{f}$ determines a map $\hat{f}_Y$: $(f \times 1)^{-1} E_Y(L, X) \to E_Y(K, X)$ of fibrations over $K \times Y$. By fiberwise suspension over K, $\hat{f}_Y$ yields for each $i \geqslant 0$ a map $\hat{f}_{i,Y} : f^{-1} E_{i,Y}(L, X) \to E_{i,Y}(K, X)$ of fibrations over K with distinguished sections. The resulting maps

$$\begin{array}{ccccc}
\Gamma_L(E_{i,Y}(L, X)) & \longrightarrow & \Gamma_K(f^{-1} E_{i,Y}(L, X)) & \longrightarrow & \Gamma_K(E_{i,Y}(K, X)) \\
\gamma & \longmapsto & f^{-1}(\gamma) & \longmapsto & \hat{f}_{i,Y} \circ f^{-1}(\gamma)
\end{array}$$

of spaces of sections, taken together for all i, give a map of spectra. The associated map of Ω-spectra is $\Theta_f(Y \to X)$, and the resulting map $\bar{\Theta}_f(Y \to X)$: $\bar{\Theta}_L(Y \to X) \to \bar{\Theta}_K(Y \to X)$ corresponds in (2.4) to the map of differentials induced functorially by f.

Having obtained a form of (A.3) which is natural in K, we now restrict to a point $x \in X$ to obtain a similarly natural form of the formula

$$\text{(A.4)} \quad \partial_x Q_+ \operatorname{Map}(K, X) \sim \Gamma_K(E_x(K, X)).$$

The map f determines a map $\mathbf{\Gamma}_f\colon \mathbf{\Gamma}_L(E_x(L, X)) \to \mathbf{\Gamma}_K(E_x(K, X))$ as follows: It determines a map $\hat{f}_x$ of fibrations over K:

$$\begin{array}{cccc}
(k, g \circ f) & K \times \mathrm{Map}(K, X) \supset E_x(K, X) & \longrightarrow & K \\
\uparrow & \hat{f}_x \uparrow & & \uparrow 1 \\
(k, g) & K \times \mathrm{Map}(L, X) \supset f^{-1}E_x(L, X) & \longrightarrow & K \\
& \downarrow & & \downarrow f \\
& L \times \mathrm{Map}(L, X) \supset E_x(L, X) & \longrightarrow & L
\end{array}$$

and this yields for each $i \geqslant 0$ a map $\hat{f}_{i,x}\colon f^{-1}E_{i,x}(L, X) \to E_{i,x}(K, X)$ of fibrations over K with distinguished section. The resulting maps

$$\text{(A.5)} \quad \begin{array}{ccccc}
\Gamma_L(E_{i,x}(L, X)) & \longrightarrow & \Gamma_K(f^{-1}E_{i,x}(L, X)) & \longrightarrow & \Gamma_K(E_{i,x}(K, X)) \\
\gamma & \longmapsto & f^{-1}(\gamma) & \longmapsto & \hat{f}_{i,x} \circ f^{-1}(\gamma)
\end{array}$$

of spaces of sections constitute a map of spectra. The resulting map $\mathbf{\Gamma}_f$: $\mathbf{\Gamma}_L(E_x(L, X)) \to \mathbf{\Gamma}_K(E_x(L, X))$ corresponds in (A.4) to the map $\partial_x Q_+ \mathrm{Map}(L, X) \to \partial_x Q_+ \mathrm{Map}(K, X)$ induced functorially by f.

Now specialize to the case $K = L = S^1 = \mathbb{R}/\mathbb{Z}$. Identify $E_x(S^1, X) = \{(\vartheta, g) \in S^1 \times \Lambda X : g(\vartheta) = x\}$ with $S^1 \times \Omega X$ by $(\vartheta, g) \mapsto (\vartheta, h = g \circ a_\vartheta)$, so that the fibration $E_x(S^1, X) \to S^1$ is identified with the trivial bundle $S^1 \times \Omega X \to S^1$. The fibration $f^{-1}E_x(S^1, X) \to S^1$ can be identified with the same trivial bundle (since it is obtained by base change from a trivial bundle). In these coordinates the map $\hat{f}_x$ is given by $(\vartheta, h) \mapsto (\vartheta, f_\vartheta^*(h))$ where $f_\vartheta^*\colon \Omega X \to \Omega X$ is composition with $f_\vartheta = a_{-f(\vartheta)} \circ f \circ a_\vartheta$. The fibration $E_{i,x}(S^1, X) \to S^1$ (ith fiberwise suspension of $E_x(S^1, X)$) becomes the projection $S^1 \times S^i(\Omega X_+) \to S^1$, with $\hat{f}_{i,x}$ acting by $(\vartheta, z) \mapsto (\vartheta, S^i(f_{\vartheta +}^*)(z))$. Thus, the space of sections $\Gamma_{S^1}(E_{i,x}(S^1, X))$ becomes $\mathrm{Map}(S^1, S^i(\Omega X_+))$, and the map (A.5) becomes

$$\begin{aligned}
&\Lambda S^i(\Omega X_+) \longrightarrow \Lambda S^i(\Omega X_+) \\
&\lambda \mapsto (\vartheta \mapsto S^i(f_{\vartheta +}^*)(\lambda(f(\vartheta)))).
\end{aligned}$$

Note that when f is the rotation a_u or the nth power map P_n, we have, respectively,

$$\begin{aligned}
f_\vartheta(t) &= a_u(t + \vartheta) - a_u(\vartheta) = t, \\
f_\vartheta(t) &= P_n(t + \vartheta) - P_n(\vartheta) = P_n(t).
\end{aligned}$$

This proves (A.2). □

References

1. Barratt, M. and Whitehead, J. H. C.: The first non-vanishing group of an $(n + 1)$-ad, *Proc. London Math. Soc. (3)* **6** (1956), 417–439.
2. Bökstedt, M., Carlsson, G., Cohen, R., Goodwillie, T., Hsiang, W.-c. and Madsen, I.: On the algebraic K-theory of simply-connected spaces, preprint.

3. Bousfield, D. and Kan, D.: *Homotopy Limits, Completions, and Localizations*, Lecture Notes in Mathematics **304**, Springer-Verlag, New York (1972).
4. Ellis, G. and Steiner, R.: Higher-dimensional crossed modules and the homotopy groups of $(n+1)$-ads, *J. Pure Appl. Algebra* **46** (1987), 117–136.
5. Goodwillie, T.: Calculus I, The first derivative of pseudoisotopy theory, *K-Theory* **4** (1990), 1–27.
6. Goodwillie, T.: Calculus III, The Taylor series of a homotopy functor, in preparation.
7. Goodwillie, T.: A multiple disjunction lemma for smooth concordance embeddings, *Mem. Amer. Math. Soc.* **431** (1990), 1–317.
8. Waldhausen, F.: Algebraic K-theory of topological spaces. II, in *Lecture Notes in Mathematics* **763**, Springer-Verlag, New York (1979), pp. 356–394.
9. Waldhausen, F.: Algebraic K-theory of spaces, in *Lecture Notes in Mathematics* **1126**, Springer-Verlag, New York (1985), pp. 318–419.