C_{2^n}-EQUIVARIANT RATIONAL STABLE STEMS AND CHARACTERISTIC CLASSES

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ABSTRACT. In this short note, we compute the rational C_{2^n}-equivariant stable stems and give minimal presentations for the $RO(C_{2^n})$-graded Bredon cohomology of the equivariant classifying spaces $B_{C_{2^n}}S^1$ and $B_{C_{2^n}}\Sigma_2$ over the rational Burnside functor $A_{\mathbb{Q}}$. We also examine for which compact Lie groups L the maximal torus inclusion $T \to L$ induces an isomorphism from $H^*_{C_{2^n}}(B_{C_{2^n}}L; A_{\mathbb{Q}})$ onto the fixed points of $H^*_{C_{2^n}}(B_{C_{2^n}}T; A_{\mathbb{Q}})$ under the Weyl group action. We prove that this holds for $L = U(m)$ and any $n, m \geq 1$ but does not hold for $L = SU(2)$ and $n > 1$.

CONTENTS

1. INTRODUCTION

This note is the followup to [Geo21c]. We start by computing the C_{2^n}-equivariant rational stable stems; this is done in section 4. While the method employed here is the one used in [Geo21c] and goes back to [GM95], the result is quite a bit more complicated to state and requires the notation set up in sections 2 and 3.

We then attempt to generalize the results in [Geo21c] to groups C_{2^n}. In [Geo21c], we obtained minimal descriptions of the C_2-equivariant Chern, Pontryagin and symplectic characteristic classes associated with genuine (Bredon) cohomology using coefficients in the rational Burnside Green functor $A_{\mathbb{Q}}$. The idea was based on the maximal torus isomorphism: if L is any one of $U(m), Sp(m), SO(m), SU(m)$, T is a maximal torus in L and W is the associated Weyl group then the inclusion $B_{C_2}T \to B_{C_2}L$ induces an isomorphism $H^{\bigstar}_{C_2}(B_{C_2}L; A_{\mathbb{Q}}) \to H^{\bigstar}_{C_2}(B_{C_2}T; A_{\mathbb{Q}})^W$. We then computed $H^{\bigstar}_{C_2}(B_{C_2}T; A_{\mathbb{Q}})$ from $H^{\bigstar}_{C_2}(B_{C_2}S^1; A_{\mathbb{Q}})$ and the Kunneth formula, which reduced us to the algebraic problem of computing a minimal presentation of the fixed points $H^{\bigstar}_{C_2}(B_{C_2}T; A_{\mathbb{Q}})^W$.

In section 5, we generalize the maximal torus isomorphism to groups $G = C_{2^n}$ when $L = U(m)$, but show that the maximal torus isomorphism is not true for

$G = C_{2^n}$ and $L = SU(2)$ when $n > 1$. We also compute the Green functor $H_G^{\bigstar}(B_G S^1; A_{\mathbb{Q}})$ which turns out to be algebraically quite a bit more complex compared to the C_2 case of [Geo21c]. For that reason, we do not attempt to follow the program in [Geo21c] and get minimal descriptions of $H_G^{\bigstar}(B_G U(m); A_{\mathbb{Q}})$ from the maximal torus isomorphism.

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2. Rational Mackey functors

The rational Burnside Green functor $A_{\mathbb{Q}}$ over a group G is defined on orbits as $G/H \mapsto A(H) \otimes \mathbb{Q}$ where $A(H)$ is the Burnside ring of H. A rational G-Mackey functor is by definition an $A_{\mathbb{Q}}$ module.

We shall use G-equivariant *unreduced* co/homology in $A_{\mathbb{Q}}$ coefficients. So if X is an unbased G-space, $H^G_{\bigstar}(X)$ is the rational G-Mackey functor defined on orbits as

$$H^G_{\bigstar}(X)(G/H) = [S^{\bigstar}, X_+ \wedge HA_{\mathbb{Q}}]^H$$

where $HA_{\mathbb{Q}}$ is the equivariant Eilenberg-MacLane spectrum associated to $A_{\mathbb{Q}}$ and the index $\bigstar$ ranges over the real representation ring $RO(G)$.

We warn the reader of differing conventions that can be found in the literature: $H_{\bigstar}(X)$ is sometimes used to denote the *reduced* homology Mackey functor (the group G being implicit), with $H^G_{\bigstar}(X)$ denoting the value of this Mackey functor on the top level (i.e. the G/G orbit). In this paper, $H^G_{\bigstar}(X)$ always denotes the Mackey functor and $H^G_{\bigstar}(X)(G/G)$ always denotes the top level. This convention also applies when $\bigstar = *$ ranges over the integers, in which case $H_*(X)$ denotes the nonequivariant rational homology of X.

All these conventions apply equally for cohomology $H_G^{\bigstar}(X)$.

The $RO(G)$-graded Mackey functor $H^G_{\bigstar}(X)$ is a module over the homology of a point $H^G_{\bigstar} := H^G_{\bigstar}(*)$. This Green functor agrees with the equivariant rational stable stems:

$$\pi^G_{\bigstar} S \otimes \mathbb{Q} = H^G_{\bigstar}$$

Two facts about rational Mackey functors that we shall liberally use ([GM95]):

- All rational Mackey functors are projective and injective, so we have the Kunneth formula:
$$H^G_{\bigstar}(X \times Y) = H^G_{\bigstar}(X) \boxtimes_{H^G_{\bigstar}} H^G_{\bigstar}(Y)$$
and duality formula:
$$H_G^{\bigstar}(X) = \mathrm{Hom}_{H^G_{\bigstar}}(H^G_{\bigstar}(X), H^G_{\bigstar})$$
- For a G-Mackey functor M and a subgroup H of G consider the $W_G H$ module $M(G/H)/Im(Tr)$ where $W_G H = N_G H/H$ is the Weyl group and $Im(Tr)$ is the submodule spanned by the images of all transfer maps Tr_K^H for $K \subseteq H$. If we let H vary over representatives of conjugacy classes of subgroups of G then we get a sequence of $W_G H$ modules. This functor from rational G-Mackey functors

to sequences of $\mathbb{Q}[W_G H]$-modules is an equivalence of symmetric monoidal categories.

From now on, we specialize to the case $G = C_{2^n}$.

There are two 1-dimensional $\mathbb{Q}[G]$ modules up to isomorphism: $\mathbb{Q}$ with the trivial action and $\mathbb{Q}$ with action $g \cdot 1 = -1$ where $g \in G$ is a generator. We shall denote the two modules by $\mathbb{Q}$ and $\mathbb{Q}_-$ respectively; every other module splits into a sum of these.

The representatives of conjugacy classes of $G = C_{2^n}$ are $H = C_{2^i}$ for every $0 \leq i \leq n$ thus the datum of a rational G-Mackey functor is equivalent to a sequence of rational $W_G H = C_{2^n}/C_{2^i}$ modules.

We let $M_i^+, 0 \leq i \leq n$, and $M_i^-, 0 \leq i < n$, be the Mackey functors corresponding to the sequences $C_{2^n}/C_{2^j} \mapsto \delta_{ij}\,\mathbb{Q}$ and $C_{2^n}/C_{2^j} \mapsto \delta_{ij}\,\mathbb{Q}_-$ respectively.

For example, M_0^+, M_0^- are the constant Mackey functors corresponding to the modules $\mathbb{Q}$ and $\mathbb{Q}_-$ respectively.

Observe that:

- The $M_i^{\pm}$ are self-dual.
- $M_i^{\pm} \boxtimes M_j^{\pm} = 0$ if $i \neq j$.
- $M_i^{\alpha} \boxtimes M_i^{\beta} = M_i^{\alpha\beta}$ where $\alpha, \beta \in \{-1, 1\}$.

Henceforth we shall write M_i for M_i^+.

The notation $M_i\{a\}$ shall mean a copy of M_i with a choice of generator $a \in M_i(C_{2^n}/C_{2^i}) = \mathbb{Q}$. The element a generates $M_i\{a\}$ through its transfers:

$$M_i\{a\}(C_{2^n}/C_{2^j}) = \begin{cases} \mathbb{Q}\{\mathrm{Tr}_{2^i}^{2^j}(a)\} & \text{if, } j \geq i \\ 0 & \text{if, } j < i \end{cases}$$

We analogously define $M_i^-\{a\}$.

The rational Burnside G-Green functor is

$$A_{\mathbb{Q}}(C_{2^n}/C_{2^i}) = \frac{\mathbb{Q}[x_{i,j}]}{x_{i,j}x_{i,k} = 2^{i-\max(j,k)}x_{i,\min(j,k)}}$$

where $x_{i,j} = [C_{2^i}/C_{2^j}] \in A(C_{2^i})$ for $0 \leq j < i$. To complete the Mackey functor description, we note that:

$$\mathrm{Tr}_{2^i}^{2^{i+1}}(x_{i,j}) = x_{i+1,j} \text{ , } \mathrm{Tr}_{2^i}^{2^{i+1}}(1) = x_{i+1,i}$$

Let

$$y_i = \begin{cases} 1 - \frac{x_{i,i-1}}{2} & \text{if, } i \geq 1 \\ 1 & \text{if, } i = 0 \end{cases}$$

living in $A_{\mathbb{Q}}(C_{2^n}/C_{2^i})$. We can see that y_i spans a copy of M_i in $A_{\mathbb{Q}}$ and:

$$A_{\mathbb{Q}} = \oplus_{i=0}^{n} M_i\{y_i\}$$

This is an isomorphism of Green functors, where the RHS becomes a Green functor by setting the product of elements from different summands to be 0 and furthermore setting the y_i to be idempotent ($y_i^2 = y_i$).

3. Euler and orientation classes

The real representation ring $RO(C_{2^n})$ is spanned by the irreducible representations $1, \sigma, \lambda_{s,k}$ where σ is the 1-dimensional sign representation and $\lambda_{s,m}$ is the 2-dimensional representation given by rotation by $2\pi s(m/2^n)$ degrees for $1 \leq m$ dividing 2^{n-2} and odd $1 \leq s < 2^n/m$. Note that 2-locally, $S^{\lambda_{s,m}} \simeq S^{\lambda_{1,m}}$ as C_{2^n}-equivariant spaces, by the s-power map. Therefore, to compute $H_\bigstar^{C_{2^n}}(X)$ it suffices to only consider $\bigstar$ in the span of $1, \sigma, \lambda_k := \lambda_{1,2^k}$ for $0 \leq k \leq n-2$ ($\lambda_{n-1} = 2\sigma$ and $\lambda_n = 2$).

We shall now define generating classes for $H_\bigstar^G$.

We first have Euler classes $a_\sigma : S^0 \hookrightarrow S^\sigma$ and $a_{\lambda_k} : S^0 \hookrightarrow S^{\lambda_k}$ given by the inclusion of the north and south poles; under the Hurewicz map these classes are $a_\sigma \in H_{-\sigma}^G(G/G)$ and $a_{\lambda_k} \in H_{-\lambda_k}^G(G/G)$.

There are also orientation classes $u_\sigma \in H_{1-\sigma}^G(C_{2^n}/C_{2^{n-1}})$, $u_{2\sigma} \in H_{2-2\sigma}^G(G/G)$ and $u_{\lambda_k} \in H_{2-\lambda_k}^G(G/G)$ but we shall need a small computation in order to define them.

Using the cofiber sequence $C_{2^n}/C_{2^{n-1}+} \to S^0 \xrightarrow{a_\sigma} S^\sigma$ we get:

$$\tilde{H}_0^G(S^\sigma) = M_n\{a_\sigma\}$$
$$\tilde{H}_1^G(S^\sigma) = \oplus_{i=0}^{n-1} M_i^-$$

where $\tilde{H}_*^G(X)$ denotes the reduced homology of a based G-space X. We can further see that $\tilde{H}_1^G(S^\sigma)$ is generated as a Green functor module by a class $u_\sigma \in \tilde{H}_{1-\sigma}^G(C_{2^n}/C_{2^{n-1}})$. So we get

$$\tilde{H}_*^G(S^\sigma) = M_n\{a_\sigma\} \oplus \oplus_{i=0}^{n-1} M_i^-\{y_i \operatorname{Res}_{2^i}^{2^{n-1}}(u_\sigma)\}$$

Using that $S^{2\sigma} = S^\sigma \wedge S^\sigma$ and the Kunneth formula, we get a class $u_{2\sigma}$ restricting to u_σ^2 and:

$$\tilde{H}_*^G(S^{2\sigma}) = M_n\{a_\sigma^2\} \oplus \oplus_{i=0}^{n-1} M_i\{y_i \operatorname{Res}_{2^i}^{2^n}(u_{2\sigma})\}$$

For $0 \leq k \leq n-2$ we have a G-CW decomposition $S^0 \subseteq X \subseteq S^{\lambda_k}$ where X consists of the points $(x_1, x_2, x_3) \in S^{\lambda_k} \subseteq \mathbb{R}^3$ with $x_1 = 0$ or $x_2 = 0$. From this decomposition we can see that:

$$\tilde{H}_0^G(S^{\lambda_k}) = \oplus_{i=k+1}^n M_i\{y_i \operatorname{Res}_{2^i}^{2^n}(a_{\lambda_k})\}$$
$$\tilde{H}_2^G(S^{\lambda_k}) = \oplus_{i=0}^k M_i\{y_i \operatorname{Res}_{2^i}^{2^n}(u_{\lambda_k})\}$$

for a class $u_{\lambda_k} \in H_{2-\lambda_k}^G(G/G)$. This also works for $k = n-1$ and $\lambda_{n-1} = 2\sigma$ giving a different way of obtaining $a_{2\sigma} = a_\sigma^2$ and $u_{2\sigma}$.

The classes u_σ, u_{λ_k}, $0 \leq k \leq n-1$, have not been canonically defined so far. Once we fix orientations for the spheres S^{λ_k}, the u_{λ_k} are uniquely determined by the following two facts:

- A G-self-equivalence of S^{λ_k} induces the identity map on the Mackey functor $\tilde{H}_2^G(S^{\lambda_k})$ if it does so on its bottom level $\tilde{H}_2^G(S^{\lambda_k})(G/e)$.
- An orientation of S^{λ_k} determines a generator for $\mathbb{Z} = \tilde{H}_2(S^2; \mathbb{Z})$ and consequently a generator for $\mathbb{Q} = \tilde{H}_2(S^2; \mathbb{Q}) = \tilde{H}_2^G(S^{\lambda_k})(G/e)$.

The first fact is proven using that the Mackey functor $\tilde{H}^G_2(S^{\lambda_k})$ is generated by the transfers of $y_i \operatorname{Res}^{2^n}_{2^i}(u_{\lambda_k})$ where $i \le k$, so we only need to check that the induced map is the identity on $\tilde{H}^G_2(S^{\lambda_k})(G/C_{2^i}) = \tilde{H}^{C_{2^i}}_2(S^{\lambda_k})(C_{2^i}/C_{2^i})$ which follows from the fact that C_{2^i} acts trivially on S^{λ_k} when $i \le k$.

We can similarly uniquely determine u_σ upon fixing an orientation of S^σ that is compatible with the orientation for $S^{\lambda_{n-1}} = S^{2\sigma}$, meaning that $\operatorname{Res}^{2^n}_{2^{n-1}}(u_{2\sigma}) = u_\sigma^2$.

The discussion regarding orientation classes can also be performed integrally, defining $A_{\mathbb{Z}}$-orientation classes $u_\sigma, u_{2\sigma}, u_{\lambda_k}$ upon fixing orientations for $S^\sigma, S^{2\sigma}$, S^{λ_k} as above. The $A_{\mathbb{Z}}$-orientation classes map to the corresponding $\mathbb{Z}$-orientation classes of [HHR16] under the map $HA_{\mathbb{Z}} \to H\underline{\mathbb{Z}}$ where $\underline{\mathbb{Z}}$ is the constant Green functor corresponding to the trivial G-module $\mathbb{Z}$.

4. Rational stable stems

In this section we shall give a presentation of the Green functor $H^G_\bigstar$ with generators and relations.

The generators are elements $r_k \in H^G_{V_k}(C_{2^n}/C_{2^{i_k}})$ spanning $M^{\epsilon_k}_{i_k}$, where $\epsilon_k = +$ or $-$, such that every element of $\coprod_{H\subseteq G, \bigstar\in RO(G)} H^G_\bigstar(G/H)$ can be obtained from the r_k using the operations of addition, multiplication, restriction, transfer and scalar multiplication (where the scalars are elements of $\coprod_{H\subseteq G} A_{\mathbb{Q}}(G/H)$).

The fact that the r_k span $M^{\epsilon_k}_{i_k}$ gives all the additive (Mackey functor) relations, but also implies certain multiplicative relations by means of the Kunneth formula: If $i_k < i_l$ then $r_k \cdot \operatorname{Res}^{2^{i_l}}_{2^{i_k}}(r_l) = 0$ and if $i_k = i_l$ then $r_k r_l$ spans $M^{\epsilon_k\epsilon_l}_{i_k}$.

Finally, if $r \in H^G_{V_k}(C_{2^n}/C_{2^i})$ and there exists a unique $r' \in H^G_{-V_k}(C_{2^n}/C_{2^i})$ with $rr' = y_i$, then we shall use the notation y_i/r to denote r'. If $r, y_i/r$ are generators then we have the implicit relation $r \cdot (y_i/r) = y_i$.

Proposition 4.1. *The Green functor $H^G_\bigstar$ has a presentation whose generating set is the union of the following four families:*

- $y_i \operatorname{Res}^{2^{n-1}}_{2^i}(u_\sigma)$ *and* $y_i/\operatorname{Res}^{2^{n-1}}_{2^i}(u_\sigma)$ *spanning* M^-_i, *where* $0 \le i < n$.
- $y_i \operatorname{Res}^{2^n}_{2^i}(u_{\lambda_k})$ *and* $y_i/\operatorname{Res}^{2^n}_{2^i}(u_{\lambda_k})$ *spanning* M_i, *where* $0 \le i \le k$ *and* $0 \le k \le n-2$.
- $y_i \operatorname{Res}^{2^n}_{2^i}(a_{\lambda_k})$ *and* $y_i/\operatorname{Res}^{2^n}_{2^i}(a_{\lambda_k})$ *spanning* M_i, *where* $k < i \le n$ *and* $0 \le k \le n-2$.
- a_σ $(= y_n a_\sigma)$ *and* y_n/a_σ *spanning* M_n.

We have implicit relations of the form $(y_i\gamma)\cdot(y_i/\gamma) = y_i$ in each of the four families. The remaining multiplicative relations can be obtained using the Kunneth formula.

Two observations:

- For $0 \le i < n$, the square of $y_i \operatorname{Res}^{2^{n-1}}_{2^i}(u_\sigma)$ is $y_i \operatorname{Res}^{2^n}_{2^i}(u_{2\sigma})$ and spans M_i.
- The ring $H^G_\bigstar(G/G)$ has multiplicative relations: $a_\sigma u_{2\sigma} = 0$, $a_\sigma u_{\lambda_k} = 0$ and $a_{\lambda_k} u_{\lambda_s} = 0$ for $s \le k$.

The Green functor presentation also gives us an additive decomposition of $H^G_\bigstar$ into M_i, M_i^- but to state it explicitly, we'll need some notation: For each integer tuple $t = (j_0, ..., j_{n-1}, j'_0, ..., j'_{n-1})$ let

$$k(t) = \begin{cases} n & \text{if } j_k = 0 \text{ for all } k \\ \min\{k : j_k \neq 0\} & \text{otherwise} \end{cases}$$

and

$$k'(t) = \begin{cases} -1 & \text{if } j'_{k'} = 0 \text{ for all } k' \\ \max\{k' : j'_{k'} \neq 0\} & \text{otherwise} \end{cases}$$

and consider the representation

$$V_t^{\pm} = \sum_{k=0}^{n-2} (j_k(2-\lambda_k) - j'_k\lambda_k) + j_{n-1}(1-\sigma) - j'_{n-1}\sigma$$

where the sign $\pm$ in $V_t^{\pm}$ is $+$ if j_{n-1} is even and $-$ if j_{n-1} is odd.

Let T be the set of all tuples t with $k'(t) < k(t)$; as t ranges over T, the $V_t^{\pm}$ are pairwise non-isomorphic virtual representations. We can now state the additive description:

Proposition 4.2. *The C_{2^n} equivariant rational stable stems are:*

$$H^G_{\bigstar} = \begin{cases} \oplus_{k'(t)<i\leq k(t)} M_i & \text{if } \bigstar = V_t^+ \text{ for } t \in T \\ \oplus_{k'(t)<i\leq k(t)} M_i^- & \text{if } \bigstar = V_t^- \text{ for } t \in T \\ 0 & \text{otherwise} \end{cases}$$

Proof. (Of Proposition 4.1). Any representation sphere S^V is the smash product of $S^{\sigma}, S^{\lambda_k}$ and their duals $S^{-\sigma}, S^{-\lambda_k}$. By duality,

$$\tilde{H}^G_*(S^{-\sigma}) = \tilde{H}_G^{-*}(S^{\sigma}) = M_n \oplus \oplus_{i=0}^{n-1} M_i^- \{y_i \operatorname{Res}^{2^{n-1}}_{2^i}(u_\sigma^{-1})\}$$

Let t be a generator for this copy of M_n; then

$$\tilde{H}^G_0(S^0) = \tilde{H}^G_0(S^{\sigma}) \boxtimes \tilde{H}^G_0(S^{-\sigma}) \oplus \tilde{H}^G_1(S^{\sigma}) \boxtimes \tilde{H}^G_{-1}(S^{-\sigma})$$

On the left hand side we have a factor $M_n\{y_n\}$ and on the right hand side we have $M_n\{a_\sigma\} \boxtimes M_n\{t\} = M_n\{a_\sigma t\}$ so $y_n = \lambda a_\sigma t$ for $\lambda \in \mathbb{Q}^\times$. Thus we can pick $t = y_n/a_\sigma$. The result then follows from the Kunneth formula. □

We note that taking geometric fixed points inverts all Euler classes, annihilating all orientation classes and setting $y_i = 1$. Therefore:

$$\Phi^{C_{2^n}}(HA_{\mathbb{Q}})_{\bigstar} = \mathbb{Q}[a_\sigma^{\pm}, a_{\lambda_k}^{\pm}]_{0\leq k\leq n-2}$$

hence $\Phi^{C_{2^n}} HA_{\mathbb{Q}} = H\mathbb{Q}$ as nonequivariant spectra. The homotopy fixed points, homotopy orbits and Tate fixed points are computed using that $HA_{\mathbb{Q}} \to H\underline{\mathbb{Q}}$ is a nonequivariant equivalence, where $\underline{\mathbb{Q}} = M_0$ is the constant Green functor. Thus:

$$(HA_{\mathbb{Q}})_{hC_{2^n}\bigstar} = (HA_{\mathbb{Q}})^{hC_{2^n}}_{\bigstar} = \mathbb{Q}[u_{2\sigma}^{\pm}, u_{\lambda_k}^{\pm}]_{0\leq k\leq n-2}$$

and $(HA_{\mathbb{Q}})^{tC_{2^n}} = *$.

5. C_{2^n} RATIONAL CHARACTERISTIC CLASSES

Proposition 5.1. *As a Green functor algebra over the homology of a point:*

$$H_G^{\bigstar}(B_GS^1) = \frac{H_G^{\bigstar}[u, \alpha_{m,j}]_{1\le m\le n, 1\le j<2^m}}{\alpha_{m,j}\alpha_{m',j'} = \delta_{mm'}\delta_{jj'}\alpha_{m,j}\ ,\ \mathrm{Res}^{2^n}_{2^{m-1}}(\alpha_{m,j}) = 0}$$

for $|u| = 2$ *and* $|\alpha_{m,j}| = 0$.

Proof. Note that

$$H_G^{\bigstar}(X) = H_G^*(X) \boxtimes_{A_{\mathbb{Q}}} H^G_{\bigstar}$$

so it suffices to describe the integer graded cohomology.

For an explicit model of B_GS^1 we take $\mathbb{C}P^\infty$ with a G action that can be described as follows: Let $V_1, ..., V_{2^n}$ be an ordering on the irreducible complex G-representations and set $V_{k+2^nm} = V_k$ for any $m \in \mathbb{Z}$, $1 \le k \le 2^n$. The action of $g \in G$ on homogeneous coordinates is $g(z_1 : z_2 : \cdots) = (gz_1 : gz_2 : \cdots)$ where g acts on z_i as it does on V_i.

Fix a subgroup $H = C_{2^m}$ of G. The fixed points under the H-action are:

$$(B_GS^1)^H = \coprod_{j=1}^{2^m} \mathbb{C}P^\infty$$

To understand the indexing, let $W_1, ..., W_{2^m}$ be an ordering on the irreducible complex C_{2^m}-representations; the j-th $\mathbb{C}P^\infty$ in $(B_GS^1)^H$ corresponds to the set of points with homogeneous coordinates $(z_1 : z_2 : \cdots)$ such that $z_k = 0$ if $\mathrm{Res}^{2^n}_{2^m}(V_k) \neq W_j$.

By [GM95] we have:

$$H_G^*(B_GS^1) = \oplus_{m=0}^n H^*((B_GS^1)^{C_{2^m}})^{C_{2^n}/C_{2^m}}$$

where $H^*(X)$ is nonequivariant cohomology in $\mathbb{Q}$ coefficients. The action of C_{2^n}/C_{2^m} on nonequivariant cohomology is trivial since it's determined in degree $* = 2$ and thus on the 2-skeleton, which itself is the disjoint union of copies of $S^2 = \mathbb{C}P^1$ and for each S^2 the action is a rotation hence has degree 1. Thus

$$H_G^*(B_GS^1) = \oplus_{m=0}^n \oplus_{j=1}^{2^m} H^*(\mathbb{C}P^\infty) = \oplus_{m=0}^n \oplus_{j=1}^{2^m} \mathbb{Q}[e_{m,j}]$$

where each $e_{m,j}$ spans M_m. Set $\alpha_{m,j} = e^0_{m,j}$ and $u = \sum_{m,j} e_{m,j}$; then

$$\sum_{j=1}^{2^m} \alpha_{m,j} = \frac{\mathrm{Tr}^{2^n}_{2^m}(y_m)}{2^m}$$

so the $\alpha_{m,2^m}$ are superfluous. Thus we can take $1 \le m \le n$ and $1 \le j < 2^m$ in the indexing for $\alpha_{m,j}$. □

We can similarly prove that:

Proposition 5.2. *We have an isomorphism of Green functor algebras over* $H_G^{\bigstar}$*:*

$$H_G^{\bigstar}(B_G\Sigma_2) = \frac{H_G^{\bigstar}(B_GS^1)}{u}$$

where the quotient map $H_G^{\bigstar}(B_GS^1) \to H_G^{\bigstar}(B_G\Sigma_2)$ *is induced by complexification:* $B_G\Sigma_2 = B_GO(1) \to B_GU(1) = B_GS^1$.

The set of generators $\{u, \alpha_{m,j}\}$ for $H^*_G(B_G S^1)$ is not minimal. Indeed, whenever we have generators $e_1, ..., e_s$ with $e_i e_j = \delta_{ij} e_i$, we can replace them by a single generator defined by $e = e_1 + 2e_2 + \cdots + se_s$:

$$\frac{\mathbb{Q}[e_1, ..., e_s]}{e_i e_j = \delta_{ij} e_i} = \frac{\mathbb{Q}[e]}{e(e-1)\cdots(e-s)}$$

This isomorphism follows from the fact that any polynomial f on $e_1, ..., e_n$ satisfies:

$$f(e) = f(0) + (f(1) - f(0))e_1 + \cdots + (f(s) - f(0))e_s$$

and thus

$$e_i = \frac{f_i(e)}{f_i(i)} \text{ where } f_i(x) = \frac{x(x-1)\cdots(x-s)}{x-i}$$

In this way, $H^*_G(B_G S^1)$ is generated as an $A_{\mathbb{Q}}$ algebra by two elements u, α but now with α satisfying some rather complicated relations. If $n = 1$ i.e. $G = C_2$, we only have one $\alpha_{m,j}$ element, namely $\alpha = \alpha_{1,1}$ satisfying $\alpha^2 = \alpha$.

Proposition 5.3. *The inclusion $B_G U(1)^m \to B_G U(m)$ induces an isomorphism of Green functor algebras over $H^\bigstar_G$:*

$$H^\bigstar_G(B_G U(m)) = (\otimes^m H^\bigstar_G(B_G U(1)))^{\Sigma_m}$$

Proof. Let V_i be the complex G-representation corresponding to the root of unity $e^{2\pi i/n}$. The Grassmannian model for $B_G U(m)$ uses complex m-dimensional subspaces of $\mathbb{C}^{\infty\rho_G}$; a G-fixed point W of $B_G U(m)$ is then a G-representation and thus as splits as $W = \oplus_{i=1}^{2^n} k_i V_i$ for $k_i = 0, 1, ...$ with $\sum_i k_i = m$. An automorphism of W is made out automorphisms for each $k_i V_i$ hence

$$B_G U(m)^G = \coprod_{\sum k_i = m} \prod_{i=1}^{2^n} BU(k_i)$$

Following [Geo21c] and inducting on the n in $G = C_{2^n}$, it suffices to show that

$$H^*((B_G U(m))^G) \to H^*(\prod^m (B_G S^1)^G)$$

is an isomorphism after taking Σ_m fixed points on the RHS. Spelling this out, we have:

$$\prod_{\sum k_i = m} \otimes_{i=1}^{2^n} H^*(BU(k_i)) \to \prod^{2^{nm}} \otimes^m H^*(BS^1)$$

where the product on the right is indexed on configurations $(V_{r_1}, ..., V_{r_m})$. If we fix k_i with $\sum_i k_i = m$ then we get

$$\otimes_{i=1}^{2^n} H^*(BU(k_i)) \to \prod \otimes^m H^*(BS^1)$$

where the product on the right is indexed on configurations $(V_{r_1}, ..., V_{r_m})$ where k_i many of the r_j's are equal to i. Taking Σ_m fixed points is equivalent to fixing a configuration and then taking $\Sigma_{k_1} \times \cdots \times \Sigma_{k_{2^n}}$ fixed points, where each Σ_{k_i} permutes the k_i many coordinates that are V_i in the configuration. Thus we are reduced to the nonequivariant isomorphism:

$$H^*(BU(k_i)) = (\otimes^{k_i} H^*(BS^1))^{\Sigma_{k_i}}$$

□

For $n=1$, $G=C_2$ and $H^*_G(B_GS^1)$ has a simple enough description to allow the computation of an explicit minimal presentation of $H^*_G(B_GU(m)) = H^*_G(B_GU(1)^m)^{\Sigma_m}$. Due to the greater algebraic complexity of $H^*_G(B_GS^1)$ for $n\ge 2$ ($G=C_{2^n}$), we do not attempt to generalize this and the rest of [Geo21c] to groups $G=C_{2^n}$ for $n\ge 2$.

We note that the maximal torus isomorphism does not work C_{2^n} equivariantly for the Lie group $L=SU(2)=Sp(1)$ and $n\ge 2$. The reason is that a C_{2^n} representation in $SU(2)$ is $2, 2\sigma$ or $V_i\oplus V_{-i}$, $1\le i<2^{n-1}$, using the notation of the proof above. Thus:

$$B_GSU(2)^G = BSU(2)\coprod BSU(2)\coprod \coprod_{i=1}^{2^{n-1}-1} BS^1$$

so $H^0(B_GSU(2)^G)$ has dimension $2^{n-1}+1$. On the other hand, the maximal torus is $U(1)\subseteq SU(2)$ with Weyl group C_2 and

$$B_GU(1)^G = \coprod_{i=1}^{2^n} BS^1$$

The C_2 action does not affect $H^0(B_GU(1)^G)$ which has dimension 2^n. Finally, $2^{n-1}+1=2^n$ only when $n=1$.

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