C_2 EQUIVARIANT CHARACTERISTIC CLASSES OVER THE RATIONAL BURNSIDE RING

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ABSTRACT. We give minimal presentations for the $RO(C_2)$-graded Bredon cohomology of the equivariant classifying spaces $B_{C_2}U(n), B_{C_2}SO(n)$ and $B_{C_2}Sp(n)$ with coefficients in the rational Burnside Green functor $A_{\mathbb{Q}}$. This results in an efficient description of rational C_2 equivariant Chern, Pontryagin and symplectic characteristic classes. These classes are then related to each other using the inclusions of maximal tori.

CONTENTS

1. Introduction

Characteristic classes are classical and invaluable tools for understanding and distinguishing bundles over spaces. If we have a compact Lie group G acting on a space X, there is a corresponding theory of G-equivariant bundles and G-equivariant characteristic classes.

May proves in [May87] that when Borel cohomology

$$H^*_{G,Borel}(X) = H^*(X \times_G EG)$$

is used, the theory of Borel equivariant characteristic classes reduces to the non-equivariant one, in the sense that $H^*_{G,Borel}(B_G L) = H^*(BG) \otimes H^*(BL)$ for any compact Lie group L (which can be $L = U(n), SO(n), Sp(n)$ and so on).

Equivariant characteristic classes in genuine (Bredon) equivariant cohomology are much less understood, owing to the significant complexity involved in computing it.

Recall that for a G-space X, unreduced G-equivariant (Bredon) cohomology $H^\bigstar_G(X)$ is not just a ring, but a Green functor: for every orbit G/H we have a ring $H^\bigstar_G(X)(G/H)$ with an action from the Weyl group $W_G H = N_G H / H$ (where $N_G H$ is the normalizer of H in G) and these rings are related to each other via restriction and transfer maps satisfying certain axioms. In more detail, for any subgroup inclusion $K \subseteq H$ we have a corresponding restriction and transfer maps:

$$\mathrm{Res}^H_K : H^\bigstar_G(X)(G/H) \to H^\bigstar_G(X)(G/K)$$
$$\mathrm{Tr}^H_K : H^\bigstar_G(X)(G/K) \to H^\bigstar_G(X)(G/H)$$

Moreover, the index $\bigstar$ is not just an integer, but an element of the real representation ring $RO(G)$. The coefficients used in $RO(G)$-graded cohomology are also Green functors and the initial ring $\mathbb{Z}$ is supplanted by the initial Burnside Green functor $A_\mathbb{Z}$. So $H^\bigstar_G(X)$ is by definition $H^\bigstar_G(X; A_\mathbb{Z})$ and we can more generally consider $H^\bigstar_G(X; R)$ for a G-Green functor R.

Computing the coefficients of $RO(G)$-graded cohomology, namely the Green functor $H^\bigstar_G(*; A_\mathbb{Z})$, is a non-trivial undertaking on its own. The reader can consult [Lew88] for the rather complicated answer when $G = C_p$ is the cyclic group of prime order p. Even when we replace the coefficients $A_\mathbb{Z}$ by the constant Green functors corresponding to trivial G-modules $\mathbb{Z}$ and $\mathbb{F}_2$, the computations remain quite involved (see [Geo19] and [Geo21a] for the case of $G = C_4$).

For characteristic classes, we further need to compute the $RO(G)$-graded cohomology of equivariant classifying spaces such as $B_G U(n), B_G SO(n)$ and $B_G Sp(n)$. Such calculations for $n \leq 3$, $G = C_2$ and using $A_\mathbb{Z}$ coefficients are performed in [Shu14], [Cho18]. For $n = 1$, $G = C_2$ and using constant $\mathbb{F}_2$ coefficients, the cohomology of $B_{C_2} O(1) = B_{C_2} \Sigma_2$ is the test module used in the determination of the dual Steenrod algebra ([HK96]) and equivariant Dyer-Lashof operations ([Wil19]). The same calculation for $G = C_4$ is significantly more complicated ([Geo21a]).

A way to simplify the algebra involved is to use coefficients in the rational Burnside Green functor $A_\mathbb{Q}$. Indeed, a result by Greenlees-May reduces the computation of the $RO(G)$-graded cohomology of a space X in $A_\mathbb{Q}$ coefficients to nonequivariant rational cohomology of the fixed points X^H where H ranges over the subgroups of G ([GM95]). This allows us to compute explicit descriptions

of the Green functors $H_G^\bigstar(B_G U(n); A_\mathbb{Q})$, $H_G^\bigstar(B_G SO(n); A_\mathbb{Q})$, $H_G^\bigstar(B_G Sp(n); A_\mathbb{Q})$ and so on.

However, those explicit descriptions are rather inefficient: For $G = C_2$, the ring $H_G^\bigstar(B_G U(n))(G/G)$, according to the Greenlees-May decomposition, has $n^2 + 2n$ many algebra generators over the homology of a point, which is just under double the minimal amount $\frac{n^2+2n}{2} + 1$ of generators that we can obtain (see the remarks after Proposition 4.2). Part of the goal of this paper is to systematically obtain such minimal explicit descriptions; said another way, we are producing only the essential characteristic classes upon which all the others are built.

Our method rests on equivariant generalizations of the following nonequivariant arguments: By a classical Theorem of Borel ([BCM]), if L is a connected compact Lie group, $T \subseteq L$ a maximal torus and $W_L T = N_L T / T$ is the Weyl group then, at least rationally,

$$H^*(BL) = H^*(BT)^{W_L T}$$

Through this result, the characteristic classes in $H^*(BL)$ can be computed from $H^*(BS^1)$, as long as the Weyl group action is understood. For example, if we take $L = U(n)$ then $T = (S^1)^n$ and $W_L T = \Sigma_n$ acts on $H^*(BT; \mathbb{Q}) = \mathbb{Q}[a_1, ..., a_n]$ by permuting the generators a_i. The fixed points under this permutation action are minimally generated by the elementary symmetric polynomials on the a_i, which are by definition the Chern classes c_i. In this way, $H^*(BU(n); \mathbb{Q}) = \mathbb{Q}[c_1, ..., c_n]$.

The same method can be performed equivariantly for $G = C_2$ and coefficients in $A_\mathbb{Q}$. There is an extra degree of complexity owing to the fact that $H_G^\bigstar(B_G S^1; A_\mathbb{Q})$ is not polynomial on one generator over $H_G^\bigstar(*; A_\mathbb{Q})$, but rather on two generators, one of which is idempotent (Proposition 5.1). As such, in the $L = U(n)$ example, the elementary symmetric polynomials c_i must be replaced by a family of more complicated polynomials $\alpha, c_i, \gamma_{s,j}$ (Proposition 4.3). Moreover, while this family of generators is minimal, it is not algebraically independent i.e. there are relations within this family. It is true however that $H_G^\bigstar(B_G U(n); A_\mathbb{Q})$ is a finite module over $H_G^\bigstar(*; A_\mathbb{Q})[c_1, ..., c_n]$ where the c_i are C_2-equivariant refinements of the classical Chern classes.

We use this method to obtain explicit minimal descriptions of $H_{C_2}^\bigstar(B_{C_2} L; A_\mathbb{Q})$ where $L = U(n), SO(n), Sp(n)$. We also examine the cases of $L = O(n), SU(n)$ and of the non-compact Lie groups $L = U, SO, Sp, O, SU$. The resulting equivariant Chern, Pontryagin and symplectic classes are compared using the complexification, quaternionization and forgetful maps between the aforementioned Lie groups. We also compute the effect of these characteristic classes on the direct sum of bundles and on the tensor product of line bundles. Finally, we give an explicit formula for the C_2-equivariant Chern character that is an isomorphism between rational C_2-equivariant complex K-theory and C_2-equivariant Bredon cohomology in $A_\mathbb{Q}$ coefficients.

As for the organization of this paper, sections 2 and 3 set up the notation used throughout and contain the computation of the C_2 rational stable stems.

Section 4 contains a summary of all our results on C_2 characteristic classes. The proofs are then found in sections 5-11 for the interested reader.

Finally, appendix A contains the results on symmetric polynomials with relations that are critical for our presentation of $H_{C_2}^\bigstar(B_{C_2} U(n); A_\mathbb{Q})$. In particular,

it contains an algorithm for writing every "symmetric polynomial" in terms of the "elementary symmetric polynomials" $\gamma_{s,j}$; this also leads to an algorithm for explicitly obtaining the relations between the $\gamma_{s,j}$. We have implemented these algorithms in a computer program available here (executable files are available here for a quick demonstration). The appendix is completely self contained and independent of the rest of the paper.

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2. Conventions and Notations

Throughout this paper, the ambient group is $G = C_2$ and all our G-Mackey functors are modules over the rational Burnside Green functor $A_{\mathbb{Q}}$:

$$A_{\mathbb{Q}} = \underset{x \mapsto 2}{\left(\begin{array}{c} \frac{\mathbb{Q}[x]}{x^2 = 2x} \\ \uparrow \\ \downarrow \end{array}\right)}_{1 \mapsto x} = \underset{x \mapsto 2}{\left(\begin{array}{c} \mathbb{Q}\,x \\ \uparrow \\ \downarrow \end{array}\right)}_{1 \mapsto x} \oplus \left(\begin{array}{c} \mathbb{Q}\,y \\ \uparrow \\ \downarrow \end{array}\right)$$
$$\mathbb{Q} \qquad\qquad \mathbb{Q} \qquad\qquad 0$$

where $x = \mathrm{Tr}(1)$ and $y = 1 - x/2$.

The *unreduced* cohomology of a G-space X in $A_{\mathbb{Q}}$ coefficients is the G-Green functor defined on orbits as

$$H_G^{\bigstar}(X)(G/H) = [X_+, \Sigma^{\bigstar} HA_{\mathbb{Q}}]^H$$

where $HA_{\mathbb{Q}}$ is the Eilenberg-MacLane spectrum associated to $A_{\mathbb{Q}}$ and the index $\bigstar$ is an element of the real representation ring $RO(G) = RO(C_2)$. This ring is spanned by the trivial representation 1 and the sign representation σ so $\bigstar = n + m\sigma$ for $n, m \in \mathbb{Z}$. Moreover, $H_G^{\bigstar}(X)$ is a Green functor algebra over the cohomology of a point $H_G^{\bigstar}(*) = H_G^{\bigstar}$.

The same conventions apply to homology $H_{\bigstar}^G(X)$ (with the exception of the ring structure, which exists only when X is an equivariant H-space).

The advantage of using $A_{\mathbb{Q}}$ coefficients is twofold: [GM95] prove that
- All rational Mackey functors (i.e. $A_{\mathbb{Q}}$ modules) are projective and injective, so we have the Kunneth formula:

$$H_{\bigstar}^G(X \times Y) = H_{\bigstar}^G(X) \boxtimes_{H_{\bigstar}^G} H_{\bigstar}^G(Y)$$

and duality formula:

$$H_G^{\bigstar}(X) = \mathrm{Hom}_{H_{\bigstar}^G}(H_{\bigstar}^G(X), H_{\bigstar}^G)$$

- We have the isomorphism of graded Green functors:

$$H_G^*(X) = \left(\begin{array}{c} H^*(X)^G \\ \uparrow \\ \downarrow \end{array}\right) \oplus \left(\begin{array}{c} H^*(X^G) \\ \uparrow \\ \downarrow \end{array}\right)$$
$$H^*(X) \qquad\qquad 0$$

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The second bullet allows us to reduce equivariant computations to nonequivariant ones, as long as we use integer grading $* \in \mathbb{Z}$. Using the first bullet, integer graded cohomology together with the homology of a point recover the $RO(G)$-graded cohomology:

$$H_G^{\bigstar}(X) = H_G^*(X) \boxtimes_{A_{\mathbb{Q}}} H_G^{\bigstar}$$

As such, once $H_G^{\bigstar}$ is computed, we need only worry about integer grading.

3. The C_2 rational stable stems

The Green functor $H_{\bigstar}^G = H_G^{-\bigstar}$ agrees with the G-equivariant rational stable stems:

$$\pi_{\bigstar}^G(S) \otimes \mathbb{Q} = \pi_{\bigstar}^G(HA_{\mathbb{Q}}) = H_{\bigstar}^G$$

The generating classes for $H_{\bigstar}^G$ are the Euler and orientation classes. The Euler class a_σ is the inclusion of north-south poles $S^0 \hookrightarrow S^\sigma$ and its image in $H_G^{\bigstar}$ under the Hurewicz map generates a Mackey functor that we denote by M_1:

$$M_1\{a_\sigma\} = \begin{array}{c} \mathbb{Q}\,a_\sigma \\ \left(\begin{array}{c} \uparrow \\ \downarrow \end{array}\right) \\ 0 \end{array}$$

The orientation class u_σ is the generator of the reduced nonequivariant homology group $\tilde{H}_1(S^\sigma; \mathbb{Z}) = \mathbb{Z}$ (determined uniquely once we fix an orientation for S^σ) and generates a Mackey functor that we denote by M_0^-:

$$M_0^-\{u_\sigma\} = \begin{array}{c} 0 \\ \left(\begin{array}{c} \uparrow \\ \downarrow \end{array}\right) \\ \mathbb{Q}\,u_\sigma \end{array} \circlearrowleft C_2$$

The Weyl group action by the generator $g \in C_2$ is $gu_\sigma = -u_\sigma$.

The square of u_σ, u_σ^2, is the restriction of the orientation class $u_{2\sigma}$ generating a Mackey functor that we denote by M_0:

$$M_0\{u_{2\sigma}\} = \begin{array}{c} \mathbb{Q}\,u_{2\sigma} \\ {}_1\left(\begin{array}{c} \uparrow \\ \downarrow \end{array}\right){}_2 \\ \mathbb{Q}\,u_\sigma^2 \end{array}$$

This follows from the fact $M_0^- \boxtimes_{A_{\mathbb{Q}}} M_0^- = M_0$ and by the Kunneth formula for $S^{2\sigma} = S^\sigma \wedge S^\sigma$. Note that $a_\sigma u_{2\sigma} = 0$ since $M_1 \boxtimes_{A_{\mathbb{Q}}} M_0 = 0$.

Using the duality formula:

$$\tilde{H}_{-*}^G(S^{-\sigma}) = \tilde{H}_G^*(S^\sigma) = \mathrm{Hom}_{A_{\mathbb{Q}}}(\tilde{H}_*^G(S^\sigma), A_{\mathbb{Q}})$$

we see that there is a class generating M_1 which when multiplied with a_σ returns $y \in A_{\mathbb{Q}}$; we denote this class by y/a_σ. We similarly have classes u_σ^{-1} and $x/u_{2\sigma}$ spanning M_0^- and M_0 respectively. We have proven the following Proposition:

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Proposition 3.1. *The C_2 equivariant rational stable stems are:*

$$H^G_{k+n\sigma} = \begin{cases} M_0 & \text{if } k = n : \text{even and} \neq 0 \\ M_0^- & \text{if } k = n : \text{odd} \\ M_1 & \text{if } k = 0 , n \neq 0 \\ A_{\mathbb{Q}} & \text{if } k = n = 0 \\ 0 & \text{otherwise} \end{cases}$$

and:

- $u^j_{2\sigma}, x/u^j_{2\sigma}$ *generate a copy M_0 for each $j = 1, 2, ...$*
- u^{2j+1}_{σ} *generate a copy M_0^- for each $j \in \mathbb{Z}$.*
- $a^j_{\sigma}, y/a^j_{\sigma}$ *generate a copy of M_1 for each $j = 1, 2, ...$*
- 1 *generates $A_{\mathbb{Q}}$.*

To spell things out, as a ring, the C_2/C_2 level of $H^G_\star$ is $\mathbb{Q}[x, u_{2\sigma}, x/u_{2\sigma}, a_\sigma, y/a_\sigma]$ modulo the relations:

$$x^2 = 2x$$
$$xu_{2\sigma} = 2u_{2\sigma}$$
$$ya_\sigma = a_\sigma$$
$$u_{2\sigma}(x/u^i_{2\sigma}) = x/u^{i-1}_{2\sigma}$$
$$a_\sigma(y/a^i_\sigma) = y/a^{i-1}_\sigma$$
$$a_\sigma u_{2\sigma} = 0$$
$$a_\sigma(x/u^i_{2\sigma}) = 0$$
$$u_{2\sigma}(y_1/a^i_\sigma) = 0$$
$$(x/u^i_{2\sigma})(y/a^j_\sigma) = 0$$

4. Summary of the C_2 characteristic classes

We summarize our results on C_2 characteristic classes in $A_{\mathbb{Q}}$ coefficients that we shall prove in sections 5-11.

Slightly abusing the notation, we shall use $H^*_G(X)$ to simultaneously denote both the G-Green functor and its top level $H^*_G(X)(G/G)$. We can do that because knowledge of the top and bottom levels and of the restriction map can be used to recover the Mackey functor, as long as the restriction is surjective (the transfer is computed from $\text{Tr}(a) = xb$ where $\text{Res}(b) = a$). In all cases we encounter, restriction is indeed surjective so it suffices to describe the top level and how generators restrict to the bottom (nonequivariant) level.

4.1. Chern classes. We start with the results on Chern classes.

We view $H^*_G(B_G U(n))$ as an augmented algebra over $H^*(BU(n))$ with the augmentation being restriction.

Proposition 4.1. *The augmentation*

$$\text{Res} : H^*_G(B_G U(n)) \to H^*(BU(n))$$

is a split surjection, so the nonequivariant Chern classes have C_2 equivariant refinements.

We fix a section of the augmentation, i.e. equivariant refinements $c_1, ..., c_n$ of the Chern classes, according to Proposition 4.4.

Proposition 4.2. *There exist elements $\alpha \in H_G^0(B_G U(n))$ and $\gamma_{s,j} \in H_G^{2s}(B_G U(n))$ for $1 \leq s < n$ and $1 \leq j \leq n - s$, generating $H_G^*(B_G U(n))$ as an augmented algebra over $H^*(BU(n)) \otimes A_{\mathbb{Q}}$:*

$$H_G^*(B_G U(n)) = \frac{(H^*(BU(n)) \otimes A_{\mathbb{Q}})[\alpha, \gamma_{s,j}]}{\mathrm{Res}(\alpha), \mathrm{Res}(\gamma_{s,j}), S}$$

where the finite set of relations $S \subseteq \mathbb{Q}[\alpha, c_i, \gamma_{s,j}]$ is described in Proposition 5.4.
The c_i are algebraically independent and for each degree $*$, $H_G^*(B_G U(n))$ is a finitely generated module over $\mathbb{Q}[c_1, ..., c_n]$.
The generating family $\{\alpha, c_i, \gamma_{s,j}\}$ has cardinality $\frac{n^2 + 2n}{2} + 1$ and is a minimal generating set of $H_G^*(B_G U(n))$ as an $A_{\mathbb{Q}}$ algebra, in the sense that any other generating set has at least $\frac{n^2 + 2n}{2} + 1$ many elements.

Substituting $H^*(BU(n)) = \mathbb{Q}[c_1, ..., c_n]$ in the formula for $H_G^*(B_G U(n))$ gives:

Proposition 4.3. *As an algebra over $A_{\mathbb{Q}}$,*

$$H_G^*(B_G U(n)) = \frac{A_{\mathbb{Q}}[\alpha, c_i, \gamma_{s,j}]}{x\alpha, x\gamma_{s,j}, S}$$

Two observations:

- The relations $x\alpha = 0, x\gamma_{s,j} = 0$ are equivalent to $\alpha, \gamma_{s,j}$ having trivial restrictions (i.e. augmentations) respectively. This completes the description of the Mackey functor structure of $H_G^*(B_G U(n))$.

- The $\frac{n^2 + 2n}{2} + 1$ many generators of the generating set $\{\alpha, c_i, \gamma_{s,j}\}$ are just over half of the $n^2 + 2n$ many generators given by the idempotent decomposition ([GM95]) of the Mackey functor $H_G^*(B_G U(n))$.

For $n = 1$ the computation takes a simpler form:

$$H_G^*(B_G U(1)) = A_{\mathbb{Q}}[\alpha, c_1]/(\alpha^2 = \alpha, x\alpha)$$

To simplify the notation in the next Proposition, we set $u = c_1 \in H_G^2(B_G U(1))$.

Proposition 4.4. *The maximal torus inclusion $U(1)^n \hookrightarrow U(n)$ induces an isomorphism*

$$H_G^*(B_G U(n)) = (H_G^*(B_G U(1))^{\otimes n})^{\Sigma_n}$$

Explicitly:

$$A_{\mathbb{Q}}[\alpha, c_i, \gamma_{s,j}]/(x\alpha, x\gamma_{s,j}, S) = (A_{\mathbb{Q}}[\alpha_i, u_i]/(x\alpha_i))^{\Sigma_n}$$

under the identifications:

$$\alpha = \sigma_1(\alpha_1, ..., \alpha_n) = \sum_{1 \leq m \leq n} \alpha_m$$

$$c_i = \sigma_i(u_1, ..., u_n) = \sum_{m_* \in K_i} u_{m_1} \cdots u_{m_i}$$

$$\gamma_{s,j} = \sum_{(m_*, l_*) \in K_{s,j}} u_{m_1} \cdots u_{m_s} \alpha_{l_1} \cdots \alpha_{l_j}$$

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where K_i consists of all partitions $1 \leq m_1 < \cdots < m_i \leq n$ and $K_{s,j} \subseteq K_s \times K_j$ consists of all pairs of disjoint partitions. The polynomial σ_i is the i-th elementary symmetric polynomial.

The family of generators $\alpha, c_i, \gamma_{s,j}$ is determined upon choosing $\alpha, u = c_1$ in $H^0_G(B_GU(1)), H^2_G(B_GU(1))$ respectively, with:

$$H^*_G(B_GU(1)) = A_{\mathbb{Q}}[\alpha, u]/(\alpha^2 = \alpha, x\alpha)$$

The choice of u is unique under the additional requirement that its restriction is the nonequivariant Chern class c_1 (in this way, the equivariant c_i are all canonically determined). There are two equally good candidates for α however: α and $y - \alpha$. They can only be distinguished upon fixing a model for $B_GU(1)$, as we do in subsection 5.1. As such, there is no canonical choice of $\alpha \in H^0_G(B_GU(1))$.

Proposition 4.5. *The map $B_GU(n) \to B_GU(n+1)$ given by direct sum with a trivial complex representation induces on cohomology:*

$$\alpha \mapsto y + \alpha$$
$$c_i \mapsto c_i$$
$$\gamma_{s,j} \mapsto \gamma_{s,j} + \gamma_{s,j-1}$$

using the convention $\gamma_{s,0} = yc_s$.
The map $B_GU(n) \to B_GU(n+1)$ given by direct sum with a σ representation induces on cohomology:

$$\alpha \mapsto \alpha$$
$$c_i \mapsto c_i$$
$$\gamma_{s,j} \mapsto \gamma_{s,j}$$

For both maps we use the conventions that $c_{n+1} = 0$ and $\gamma_{s,n+1-s} = 0$ in every RHS.

Proposition 4.6. *The direct sum of bundles map $B_GU(n) \times B_GU(m) \to B_GU(n+m)$ induces on cohomology:*

$$\alpha \mapsto \alpha \otimes 1 + 1 \otimes \alpha$$
$$c_i \mapsto \sum_{j+k=i} c_j \otimes c_k$$
$$\gamma_{s,j} \mapsto \sum_{\substack{s'+s''=s \\ j'+j''=j}} \gamma_{s',j'} \otimes \gamma_{s'',j''}$$

using the conventions $c_0 = 1, \gamma_{s,0} = yc_s, \gamma_{0,j} = (j!)^{-1}\alpha(\alpha-1)\cdots(\alpha-j+1)$ in every RHS.

Proposition 4.7. *The tensor product of line bundles map $B_GU(1) \times B_GU(1) \to B_GU(1)$ induces on cohomology:*

$$\alpha \mapsto y - \alpha \otimes 1 - 1 \otimes \alpha + 2\alpha \otimes \alpha$$
$$c_1 \mapsto c_1 \otimes 1 + 1 \otimes c_1$$

The C_2-equivariant Chern character is induced by the Hurewicz map:

$$KU_{C_2} \to KU_{C_2} \wedge HA_{\mathbb{Q}}$$

Let v be the Bott element in $\pi^{C_2}_2(KU_{C_2})$.

Proposition 4.8. *Under the equivalence $KU_{C_2} \wedge HA_\mathbb{Q} = HA_\mathbb{Q}[v^\pm]$ the Chern character*

$$KU_{C_2}(X) \otimes \mathbb{Q} \to \prod_n H_{C_2}^{2n}(X; A_\mathbb{Q})$$

is the isomorphism determined on line bundles by:

$$L \mapsto \frac{x}{2}e^{c_1(L)} + (2\alpha(L) - y)e^{c_1(L)}$$

4.2. **Symplectic classes.** The theory of C_2 symplectic characteristic classes is entirely analogous to Chern classes, by replacing $B_G U(n)$ with $B_G Sp(n)$ and the generators $c_i, \gamma_{s,j}$ with generators $k_i, \kappa_{s,j}$ of double degree. Propositions 4.1-4.3 become:

Proposition 4.9. *There exist classes $\alpha, k_i, \kappa_{s,j} \in H_G^*(B_G Sp(n))$ of degrees $0, 4i, 4s$ respectively, where $1 \leq i, s \leq n$ and $1 \leq j \leq n - s$, such that*

$$H_G^*(B_G Sp(n)) = \frac{A_\mathbb{Q}[\alpha, k_i, \kappa_{s,j}]}{x\alpha, x\kappa_{s,j}, S}$$

where the relation set S is the same as that for $H_G^*(B_G U(n))$ with $c_i, \gamma_{s,i}$ replaced by $k_i, \kappa_{s,i}$.

The generators k_i restrict to the nonequivariant symplectic classes k_i, so the restriction map $H_G^*(B_G Sp(n)) \to H^*(BSp(n))$ is a split surjection.

The maximal torus inclusion $U(1)^n \hookrightarrow Sp(n)$ induces an isomorphism

$$H_G^*(B_G Sp(n)) = (H_G^*(B_G U(1))^{\otimes n})^{C_2 \wr \Sigma_n}$$

Explicitly:

$$A_\mathbb{Q}[\alpha, k_i, \kappa_{s,j}]/(x\alpha, x\kappa_{s,j}, S) = (A_\mathbb{Q}[\alpha_i, u_i]/(x\alpha_i))^{C_2 \wr \Sigma_n}$$

under the identifications:

$$\alpha = \sum_{1 \leq m \leq n} \alpha_m$$

$$k_i = \sum_{m_* \in K_i} u_{m_1}^2 \cdots u_{m_i}^2$$

$$\kappa_{s,j} = \sum_{(m_*, l_*) \in K_{s,j}} u_{m_1}^2 \cdots u_{m_s}^2 \alpha_{l_1} \cdots \alpha_{l_j}$$

where K_i and $K_{s,j}$ are as in Proposition 4.4.

Propositions 4.5-4.7 have analogous statements in the symplectic case, replacing $B_G U(n)$ by $B_G Sp(n)$ and $c_i, \gamma_{s,j}$ with $k_i, \kappa_{s,j}$ respectively; we shall not repeat them here.

Proposition 4.10. *The forgetful map $B_G Sp(n) \to B_G U(2n)$ induces on cohomology:*

$$\alpha \mapsto \alpha$$
$$c_{2i+1}, \gamma_{2s+1,j} \mapsto 0$$
$$c_{2i} \mapsto (-1)^i k_i$$
$$\gamma_{2s,j} \mapsto (-1)^s \kappa_{s,j}$$

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The quaternionization map $B_G U(n) \to B_G Sp(n)$ induces:

$$\alpha \mapsto \alpha$$

$$k_i \mapsto \sum_{a+b=2i} (-1)^{a+i} c_a c_b$$

The effect of quaternionization on the $\kappa_{s,j}$ is explained in Proposition 8.2.

4.3. **Pontryagin and Euler classes.** The results are analogous to the symplectic classes, but we need to distinguish between $B_G SO(2n)$ and $B_G SO(2n+1)$. The following Proposition contains the shared aspects of both cases:

Proposition 4.11. *The restriction map $H_G^*(B_G SO(n)) \to H^*(BSO(n))$ is a split surjection. The maximal torus inclusion $T \hookrightarrow SO(n)$ induces an isomorphism*

$$H_G^*(B_G SO(n)) = (H_G^*(B_G T))^W$$

where W is the corresponding Weyl group.

This gives us C_2 equivariant refinements p_i, χ of the Pontryagin and Euler classes respectively. Recall that for $BSO(2n)$ the characteristic classes are $p_1, ..., p_{n-1}$, χ (and $p_n = \chi^2$) while for $BSO(2n+1)$ they are $p_1, ..., p_n$.

Proposition 4.12. *There exist classes α, $\pi_{s,j}$ of degrees $0, 4s$ respectively in $H_G^*(B_G SO(2n))$, where $1 \leq s < n$ and $1 \leq j \leq n - s$ such that*

$$H_G^*(B_G SO(2n)) = \frac{A_{\mathbb{Q}}[\alpha, p_i, \pi_{s,j}, \chi]}{x\alpha, x\pi_{s,j}, S}$$

where the relation set S is the same as that for $H_G^*(B_G U(n))$ with $c_i, \gamma_{s,i}$ replaced by $p_i, \pi_{s,i}$ and using that $p_n = \chi^2$.

Under the maximal torus isomorphism:

$$\alpha = \sum_{1 \leq m \leq n} \alpha_m$$

$$p_i = \sum_{m_* \in K_i} u_{m_1}^2 \cdots u_{m_i}^2$$

$$\pi_{s,j} = \sum_{(m_*, l_*) \in K_{s,j}} u_{m_1}^2 \cdots u_{m_s}^2 \alpha_{l_1} \cdots \alpha_{l_j}$$

$$\chi = u_1 \cdots u_n$$

where K_i and $K_{s,j}$ are as in Proposition 4.4.

Proposition 4.13. *The map $B_G SO(2n) \to B_G SO(2n+1)$ induces an injection in cohomology and:*

$$H_G^*(B_G SO(2n+1)) = \frac{A_{\mathbb{Q}}[\alpha, p_i, \pi_{s,j}]}{x\alpha, x\pi_{s,j}, S}$$

where $i = 1, ..., n$.

Propositions 4.5-4.7 have analogous statements in this context. The action on the Euler class χ is the same as in the nonequivariant case; for example, under $B_G SO(n) \times B_G SO(m) \to B_G SO(n+m)$ we get:

$$\chi \mapsto \chi \otimes \chi$$

Proposition 4.14. *The complexification map $B_G SO(2n) \to B_G U(2n)$ induces on cohomology:*

$$\alpha \mapsto \alpha$$
$$c_{2i+1}, \gamma_{2s+1,j} \mapsto 0$$
$$c_{2i} \mapsto (-1)^i p_i$$
$$\gamma_{2s,j} \mapsto (-1)^s \pi_{s,j}$$

The forgetful map $B_G U(n) \to B_G SO(2n)$ induces on cohomology:

$$\alpha \mapsto \alpha$$
$$p_i \mapsto \sum_{a+b=2i} (-1)^{a+i} c_a c_b$$
$$\chi \mapsto c_n$$

and the action on $\pi_{s,j}$ is explained in Proposition 9.2.

4.4. Stable characteristic classes.

In the C_2-equivariant case, there are different notions of stability for complex bundles, represented by the following spaces:

- $B_G^+ U = \operatorname{colim}(B_G U(1) \xrightarrow{\oplus 1} B_G U(2) \xrightarrow{\oplus 1} \cdots)$. This is the usual equivariant classifying space $B_G U = E_G U / U$ and is a G-equivariant H-space using the direct sum of bundles maps $B_G U(n) \times B_G U(m) \to B_G U(n+m)$.
- $B_G^- U = \operatorname{colim}(B_G U(1) \xrightarrow{\oplus \sigma} B_G U(2) \xrightarrow{\oplus \sigma} \cdots)$. This is equivalent to $B_G^+ U$.
- $B_G^\pm U = \operatorname{colim}(B_G^+ U \xrightarrow{\oplus \sigma} B_G^+ U \xrightarrow{\oplus \sigma} \cdots) = \operatorname{colim}(B_G^- U \xrightarrow{\oplus 1} B_G^- U \xrightarrow{\oplus 1} \cdots)$. This becomes a G-equivariant H-space using the direct sum of bundles, and is the group completion of $B_G^+ U$ (and $B_G^- U$). Moreover, $B_G^\pm U \times \mathbb{Z}$ represents equivariant K-theory.

Computing $H_G^*(B_G^- U)$ in terms of the generators $\alpha, c_i, \gamma_{s,j}$ is more complicated compared to the nonequivariant case because for fixed degree $*$, the $\mathbb{Q}$-dimension of $H_G^*(B_G U(n))$ does not stabilize as $n \to +\infty$ and as a result, $H_G^*(B_G^- U)$ is infinite dimensional (dimension is $2^{\aleph_0}$). In degree $* = 0$, $H_G^0(B_G^- U)$ is linearly spanned over $A_{\mathbb{Q}}$ by series of the form

$$a_{-1} + \sum_{i \geq 0} a_i \alpha (\alpha - 1) \cdots (\alpha - i)$$

Generally, the graded algebra $H_G^*(B_G^- U)$ is generated over $H_G^0(B_G^- U)[c_1, c_2, ...]$ by series of the form

$$\sum_{j=1}^\infty a_j \gamma_{s,j} \in H_G^{2s}(B_G^- U)$$

for $a_j \in \mathbb{Q}$ and $s = 1, 2, ...$ See section 7 for more details.

For the ring $H_G^*(B_G^\pm U)$ we also have to compute the effect of the $\oplus 1$ map on the series in $H_G^*(B_G^- U)$. If we restrict our attention to finite series, we are in essence dealing with characteristic classes that are stable under addition of both the $\oplus 1$ and $\oplus \sigma$ representations. Since the $\oplus 1$ map takes the form $\gamma_{s,j} \mapsto \gamma_{s,j} + \gamma_{s,j-1}$ (and $\gamma_{s,0} = y c_s$, $\gamma_{0,1} = \alpha$) we can immediately see that for $i \geq 1$, the elements

$$c_i \ , \ \gamma_i := c_i \alpha - \gamma_{i,1}$$

are stable under both $\oplus 1$ and $\oplus \sigma$. We conjecture that all classes with this property are polynomially generated by c_i, γ_i; this is equivalent to the elements $\gamma_1, \gamma_2, \dots$ being algebraically independent over $\mathbb{Q}[c_1, c_2, \dots]$.

In any case, the elements c_i, γ_i span sub-Hopf-algebras of $H_G^*(B_G^- U)$ and $H_G^*(B_G^\pm U)$ with

$$\gamma_s \mapsto \sum_{i+j=s} (c_i \otimes \gamma_j + \gamma_i \otimes c_j)$$

using the conventions $c_0 = 1$ and $\gamma_0 = 0$.

The spaces $B_G^+ U, B_G^- U, B_G^\pm U$ are equivariant H-spaces, hence their equivariant homology is a Green functor dual to their equivariant cohomology. This homology can be expressed in terms of the classes $a_i, b_i, d \in H_*^G(B_G U(1))$ dual to $\alpha c_1^i, c_1^i, x/2 + \alpha \in H_G^*(B_G U(1))$ respectively, where $i \geq 1$. Note that the $\gamma_1^i \in H_G^*(B_G U(2))$ map to αc_1^i under $H_G^*(B_G U(2)) \xrightarrow{\oplus 1} H_G^*(B_G U(1))$ so the a_i can be thought of as duals to the γ_1^i.

Proposition 4.15. *We have:*

$$H_*^G(B_G^- U) = \frac{A_{\mathbb{Q}}[d, a_i, b_i]}{xa_i, xd = x}$$

$$H_*^G(B_G^\pm U) = \frac{A_{\mathbb{Q}}[d^\pm, a_i, b_i]}{xa_i, xd = x}$$

and for the coalgebra structure:

$$d \mapsto d \otimes d$$

$$a_i \mapsto \sum_{j+k=i} a_j \otimes a_k$$

$$b_i \mapsto \sum_{j+k=i} (b_j \otimes b_k - b_j \otimes a_k - b_k \otimes a_j + 2a_j \otimes a_k)$$

using the conventions $a_0 = d - x/2$ and $b_0 = 1$.

The case of stable symplectic classes is entirely analogous: We can distinguish between $B_G^+ Sp, B_G^- Sp$ and $B_G^\pm Sp$ and we have classes $k_i, \kappa_i = k_i \alpha - \kappa_{i,1}$ that are stable under both $\oplus 1, \oplus \sigma$ maps. Moreover,

Proposition 4.16. *The forgetful map $Sp \to U$ induces*

$$c_{2s+1}, \gamma_{2s+1} \mapsto 0$$

$$c_{2s} \mapsto (-1)^s k_s$$

$$\gamma_{2s} \mapsto (-1)^s \kappa_s$$

while quaternionization $U \to Sp$ induces

$$k_i \mapsto \sum_{a+b=2i} (-1)^{a+i} c_a c_b$$

$$\kappa_i \mapsto \sum_{a+b=2i} (-1)^{a+i} c_a \gamma_b$$

The dual homology result can be expressed in terms of the classes $a_i^{sp}, b_i^{sp}, d \in H_*^G(B_G Sp(1))$ dual to $\alpha k_1^i, k_1^i, x/2 + \alpha \in H_G^*(B_G Sp(1))$ respectively, for $i \geq 1$ (the a_i^{sp} are dual to κ_1^i). The analogue of Proposition 4.15 holds, and:

Proposition 4.17. *The forgetful map $Sp \to U$ induces*

$$d \mapsto d$$

$$a_i^{sp} \mapsto \sum_{2i=j+k} (-1)^k a_j a_k$$

$$b_i^{sp} \mapsto \sum_{2i=j+k} (-1)^k (b_j b_k - a_j b_k - a_k b_j + 2a_j a_k)$$

while quaternionization $U \to Sp$ induces

$$d \mapsto d$$

$$a_{2i+1} \mapsto 0 \,,\, a_{2i} \mapsto a_i^{sp}$$

$$b_{2i+1} \mapsto 0 \,,\, b_{2i} \mapsto b_i^{sp}$$

The case of stable Pontryagin classes is entirely analogous, replacing Sp by SO (the forgetful map $Sp \to U$ is replaced by complexification $SO \to U$ and the quaternionization map $U \to Sp$ is replaced by the forgetful map $U \to SO$). In brief, setting $\pi_i = p_i \alpha - \pi_{i,1}$ gives the analogue of 4.16. Moreover, we have classes $a_i^{so}, b_i^{so}, d \in H_*^G(B_G SO(2))$ dual to $\alpha p_1^i, p_1^i, x/2 + \alpha \in H_G^*(B_G SO(2))$ respectively, for $i \geq 1$, and the analogues of Propositions 4.15 and 4.17 also hold.

4.5. **Orthogonal groups.** Unlike their nonequivariant counterparts, the C_2 classifying spaces of the orthogonal groups $O(n)$ don't generally satisfy the maximal torus isomorphism, i.e. $H_G^*(B_G O(n)) \to H_G^*(B_G T)^W$ is not generally an isomorphism, where T is the maximal torus in $O(n)$ and W the Weyl group. Moreover, $H_G^*(B_G O(2n))$ is not isomorphic to $H_G^*(B_G O(2n+1))$, but rather, the inclusion-induced map

$$H_G^*(B_G O(2n+1)) \to H_G^*(B_G O(2n))$$

is always a surjection with nontrivial kernel. The spaces $B_G O(2n+1)$ can be put into our framework using the splitting $O(2n+1) = SO(2n+1) \times O(1)$:

Proposition 4.18. *There is a generator $\beta \in H_G^0(B_G O(1))$ such that*

$$H_G^*(B_G O(2n+1)) = \frac{A_{\mathbb{Q}}[\alpha, \beta, p_i, \pi_{s,j}]}{x\alpha, x\beta, x\pi_{s,j}, S}$$

The $H_G^*(B_G O(2n))$ can then be understood as quotients of $H_G^*(B_G O(2n+1))$ (see section 10). The stable case similarly reduces to $B_G SO$ by use of the fact that $B_G O = B_G SO \times B_G O(1)$.

4.6. **Special unitary groups.** For $SU(n)$ we have the maximal torus isomorphism equivariantly:

Proposition 4.19. *The maximal torus inclusion $U(1)^{n-1} \to SU(n)$ induces an isomorphism*

$$H_G^*(B_G SU(n)) \to H_G^*(B_G U(1)^{n-1})^{\Sigma_n}$$

We prove that for any n, the inclusion induced map

$$H_G^*(B_G U(n)) \to H_G^*(B_G SU(n))$$

is a surjection, and $c_1 = \gamma_{1,n-1} = 0$ in $H_G^*(SU(n))$. There are more relations however; for example, if $n = 2$ there is an additional relation $\alpha^2 = 2\alpha$ since $SU(2) = Sp(1)$.

In the stable case, we can distinguish between $B_G^+ SU$, $B_G^- SU$ and $B_G^\pm SU$ and we have $c_1 = \gamma_1 = 0$.

5. C_2 CHERN CLASSES

The goal of this section is to prove our results on Chern classes. In effect, we need to prove the isomorphism:

$$H_G^*(B_G U(n)) = (H_G^*(B_G U(1))^{\otimes n})^{\Sigma_n}$$

and then use the computation of $H_G^*(B_G U(1))$ to obtain the algebraic description in Proposition 4.2 and Proposition 4.4.

5.1. The $n = 1$ computation. For any C_2-space X, by [GM95],

$$H_{C_2}^*(X) = H^*(X)^{C_2} \oplus H^*(X^{C_2})$$

Now take $X = B_G S^1$ which is $\mathbb{C}P^\infty$ with the C_2 action given on complex homogeneous coordinates by:

$$g(z_0 : z_1 : z_2 : z_3 : \cdots) = (z_0 : -z_1 : z_2 : -z_3 : \cdots)$$

We have $H^*(\mathbb{C}P^\infty) = \mathbb{Q}[r]$ for a generator r of degree 2. The C_2 action is trivial as can be verified on the 2-skeleton $S^2 = \mathbb{C}P^1 \subseteq \mathbb{C}P^\infty$: the C_2 action on S^2 is a rotation hence has degree 1.

We also have $(B_G S^1)^{C_2} = \mathbb{C}P^\infty \coprod \mathbb{C}P^\infty$ spanned by $v^+ = (z_0 : 0 : z_2 : 0 : \cdots)$ and $v^- = (0 : z_1 : 0 : z_3 : \cdots)$ respectively. Thus we get

$$H_{C_2}^*(B_G S') = H^*(BS^1) \oplus H^*((B_G S^1)^{C_2}) = \mathbb{Q}[e_1] \oplus \mathbb{Q}[e_2] \oplus \mathbb{Q}[e_3]$$

where e_1 is the nonequivariant generator and e_2, e_3 correspond to v^+, v^- respectively.

Define $\alpha = e_2^0$ and $u = e_1 + e_2 + e_3$; then

$$e_1^0 = x/2 \,,\, e_2^0 = \alpha \,,\, e_3^0 = y - \alpha$$
$$e_1 = xu/2 \,,\, e_2 = u\alpha \,,\, e_3 = (y - \alpha)u$$

We have proven:

Proposition 5.1. *As an algebra over $A_{\mathbb{Q}}$:*

$$H_G^*(B_{C_2} S^1) = \frac{A_{\mathbb{Q}}[u, \alpha]}{\alpha^2 = \alpha, x\alpha = 0}$$

for $|u| = 2$ and $|\alpha| = 0$.

We should compare this with the description

$$H_G^\star(B_G S^1) = H_G^\star[c, b]/(c^2 = a_\sigma^2 c + u_{2\sigma} b)$$

obtained in [Shu14]. The correspondence of generators is:

$$\alpha = c \frac{y}{a_\sigma^2}$$
$$u = c \frac{x}{2u_{2\sigma}} + b \frac{y}{a_\sigma^2}$$
$$c = u u_{2\sigma} + \alpha a_\sigma^2$$
$$b = -u^2 u_{2\sigma} + u a_\sigma^2$$

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We also compute the Chern character for C_2-equivariant rational K-theory in terms of our C_2-Chern classes c_1, α. Let v be the Bott element in $\pi_2^{C_2}(KU_{C_2})$.

Proposition 5.2. *The Hurewicz map*

$$KU_{C_2} \to KU_{C_2} \wedge HA_{\mathbb{Q}} = HA_{\mathbb{Q}}[v^{\pm}]$$

induces the Chern character

$$KU_{C_2}(X) \to KU_{C_2}(X) \otimes \mathbb{Q} \xrightarrow{\cong} \prod_n H_{C_2}^{2n}(X)$$

which on line bundles is given by

$$L \mapsto \frac{x}{2} e^{c_1(L)} + (2\alpha(L) - y) e^{c_1(L)}$$

Proof. We have an $HA_{\mathbb{Q}}$ algebra map $HA_{\mathbb{Q}}[v^{\pm}] \to KU_{C_2} \wedge HA_{\mathbb{Q}}$ that is an equivalence by the general fact that $E \simeq \vee_n \Sigma^n H\pi_n E$ for any rational G-spectrum E, as long as G is finite ([GM95]).

For the formula of the Chern character on a line bundle L, we are reduced to the case $X = B_G U(1)$ and L being the tautological line bundle. Using the Greenlees-May identification

$$[E, F]^{C_2} = ([\Phi^e E, \Phi^e F])^{C_2} \oplus [\Phi^{C_2} E, \Phi^{C_2} F]$$

for C_2-equivariant rational spectra E, F, it suffices to compute the Chern character on the underlying spectra and on the C_2-geometric fixed points. For the former, we recall that the nonequivariant Chern character is given by $L \mapsto e^{c_1}$ and thus we get

$$\frac{x}{2} L \mapsto \frac{x}{2} e^{c_1}$$

For the map on geometric fixed points, we note that L restricts to the tautological line bundles $L^+, \sigma_{\mathbb{C}} L^-$ on the two components of $(B_{C_2} U(1))^{C_2} = BU(1) \coprod BU(1)$ respectively, where $\sigma_{\mathbb{C}}$ is the complex sign representation of C_2. By the definition of α, c_1 and the fact that $\sigma_{\mathbb{C}} \in KU_{C_2}(*) = R(C_2)$ maps to $x - 1 \in H_{C_2}^0(*; A_{\mathbb{Q}}) = A(C_2) \otimes \mathbb{Q}$ we get that the Chern character on the geometric fixed points is

$$yL \mapsto e^{c_1(L^+)} + (x - 1) e^{c_1(L^-)} = \alpha e^{c_1} + (x - 1)(y - \alpha) e^{c_1} = (2\alpha - y) e^{c_1}$$

In total,

$$L = \frac{x}{2} L + yL \mapsto \frac{x}{2} e^{c_1} + (2\alpha - y) e^{c_1}$$

$\square$

A few remarks:

- The Chern character defined above coincides with the one defined by Lück and Oliver in [LO01].
- Every natural $A(C_2)$-algebra map

$$KU_{C_2}(X) \otimes \mathbb{Q} \to \prod_n H_{C_2}^{2n}(X)$$

is given on line bundles by

$$L \mapsto \frac{x}{2} e^{k c_1(L)} + (2\alpha(L) - y) e^{k' c_1(L)}$$

for $k, k' \in \mathbb{Q}$. Indeed, by naturality and Proposition 4.7, any such map must take the tautological line bundle L to a power series

$$f(c_1, \alpha) \in A_{\mathbb{Q}}[[c_1, \alpha]] / (\alpha^2 = \alpha, x\alpha)$$

such that

$$f(c_1 + c_1', y - \alpha - \alpha' + 2\alpha\alpha') = f(c_1, \alpha) f(c_1', \alpha')$$

Substituting $f(c_1, \alpha) = \frac{x}{2} g(c_1) + y h(c_1) + \alpha t(c_1)$ gives the desired description of f.

5.2. **Maximal tori and Weyl groups.** If L is a compact connected Lie group and T a maximal torus in L, we have the inclusion-induced map

$$H_G^*(B_G L) \to H_G^*(B_G T)$$

The Weyl group $W = W_L T$ acts on L, T by conjugation hence on $H^*(B_G T), H^*(B_G L)$ and the inclusion-induced map is W-equivariant. Actually, W acts trivially on $B_G L$, which is a special case of the fact that an inner automorphism of H induces the identity map on $B_G H$ up to homotopy (see [BCM] for the classical nonequivariant case; the equivariant generalization is straightforward). Thus our map factors through the W-fixed points:

$$H_G^*(B_G L) \to H_G^*(B_G T)^W$$

This breaks into:

$$H^*(BL) \to H^*(BT)^W$$

$$H^*((B_G L)^{C_2}) \to H^*((B_G T)^{C_2})^W$$

The first map is an isomorphism ([BCM]), so if we can prove that the second map is an isomorphism then

$$H_G^*(B_G L) \to H_G^*(B_G T)^W$$

will also be an isomorphism.

If $T = \prod^n S^1$ then

$$(B_G T)^{C_2} = \coprod^{2^n} \prod^n BS^1$$

The coproduct is indexed over C_2^n i.e. sign configurations $(\pm, ..., \pm)$ (the next subsection explains why it's natural to use sign configurations). By considering the number m of $+$'s in a configuration, we can further break this into

$$(B_G T)^{C_2} = \coprod_{m=0}^{n} \coprod^{\binom{n}{m}} \prod^n BS^1$$

In cohomology:

$$H^*((B_G T)^{C_2}) = \oplus_{m=0}^{n} \oplus^{\binom{n}{m}} H^*(BS^1)^{\otimes n}$$

The Σ_n action permuting the S^1 factors in T has the effect of preserving the m, permuting the $\binom{n}{m}$ many sign configurations and permuting the factors in the tensor product.

5.3. **The maximal torus isomorphism.** We use the Grassmannian model $Gr(n, \mathbb{C}^{\infty\rho})$ for $B_G U(n)$, that consists of n-dimensional (complex) subspaces of $\mathbb{C}^{\infty\rho}$ where $\rho = 1 + \sigma$ is the complex regular representation of G. We have:

$$B_G U(n)^{C_2} = \coprod_{m=0}^{n} BU(m) \times BU(n-m)$$

Indeed, a fixed point V is a C_2 subspace and thus admits a unique decomposition $V = V^+ \oplus V^-$ where $gv = v$ for any $v \in V^+$ and $gv = -v$ for any $v \in V^-$. So a subspace V in the LHS corresponds to the pair (V^+, V^-) in the RHS.

We use the maximal torus $T = \prod^n S^1$ in $U(n)$. The Weyl group is Σ_n and the action is by permuting the S^1 factors.

Proposition 5.3. *The maximal torus inclusion induces an isomorphism:*

$$H_G^*(B_G U(n)) = H_G^*(B_G T)^{\Sigma_n}$$

Proof. The map $B_G T \to B_G U(n)$ is on the C_2-fixed points:

$$\coprod^{2^n} BU(1)^n \to \coprod_{m=0}^{n} BU(m) \times BU(n-m)$$

sending $(v_1^\pm, ..., v_n^\pm)$ to the direct sum $v_1^\pm \oplus ... \oplus v_n^\pm$ (we are implicitly using the identification $Gr_n(\mathbb{C}^{\infty\rho} \oplus \mathbb{C}^{\infty\rho}) = Gr_n(\mathbb{C}^{\infty\rho})$ through a fixed linear equivariant isomorphism $\mathbb{C}^{\infty\rho} \oplus \mathbb{C}^{\infty\rho} = \mathbb{C}^{\infty\rho}$). Here, v_i^+ denotes a 1-dimensional subspace with trivial C_2 action, while v_i^- is a 1-dimensional subspace with antipodal C_2 action. The signs in $(v_1^\pm, ..., v_n^\pm)$ correspond to the sign configuration and the index m on the RHS corresponds to the amount of $+$ signs in a configuration. Thus, the map above breaks into

$$\coprod^{\binom{n}{m}} BU(1)^n \to BU(m) \times BU(n-m)$$

for every $m = 0, ..., n$. Fixing the m, the induced map on cohomology is

$$H^*(BU(m)) \otimes H^*(BU(n-m)) \to \oplus^{\binom{n}{m}} H^*(BU(1))^{\otimes n}$$

The action of Σ_n on the right permutes the factors in the tensor product and the sign configuration. Taking Σ_n fixed points is equivalent to fixing our favorite configuration, say $(+, ..., +, -, ..., -)$, and then taking $\Sigma_m \times \Sigma_{n-m}$ fixed points, where Σ_m permutes only the $+$'s and Σ_{n-m} permutes only the $-$'s. With that in mind, our map factors through Σ_n fixed points and we get

$$H^*(BU(m)) \otimes H^*(BU(n-m)) \to (H^*(BU(1))^{\otimes m})^{\Sigma_m} \otimes (H^*(BU(1))^{\otimes(n-m)})^{\Sigma_{n-m}}$$

This is the tensor product of maps $H^*(BU(i)) \to (H^*(BU(1))^{\otimes i})^{\Sigma_i}$ where $i = m, n - m$. These maps are induced by the maximal torus inclusions $U(1)^i \to U(i)$, hence are isomorphisms by the nonequivariant case. $\square$

5.4. **The fixed point computation.** The cohomology of the maximal torus is:

$$H_G^*(B_G T) = H_G^*(\prod_n B_G S^1) = \frac{A_{\mathbb{Q}}[u_i, \alpha_i]_{1 \leq i \leq n}}{x\alpha_i = 0, \alpha_i^2 = \alpha_i}$$

with the Σ_n action permuting the u_i and α_i separately (namely $\sigma u_i = u_{\sigma(i)}$ and $\sigma \alpha_i = \alpha_{\sigma(i)}$ for $\sigma \in \Sigma_n$). In this subsection, we summarize the computation of the Σ_n-fixed points:

$$H_G^*(B_G U(n)) = (H_G^*(B_G S^1)^{\otimes n})^{\Sigma_n} = \left(\frac{A_{\mathbb{Q}}[u_i, \alpha_i]}{x\alpha_i = 0, \alpha_i^2 = \alpha_i}\right)^{\Sigma_n}$$

that is proven in a more general form in Appendix A (see Propositions A.1 and A.2). We have:

Corollary 5.4. *For $1 \leq i, s \leq n$ and $1 \leq j \leq n - s$ consider the elements of the $A_{\mathbb{Q}}$-algebra:*

$$H_G^*(B_G S^1)^{\otimes n} = \frac{A_{\mathbb{Q}}[u_i, \alpha_i]}{x\alpha_i = 0, \alpha_i^2 = \alpha_i}$$

given by:

$$\alpha = \sigma_1(\alpha_1, ..., \alpha_n) = \sum_{1 \leq m \leq n} \alpha_m$$

$$c_i = \sigma_i(u_1, ..., u_n) = \sum_{m_* \in K_i} u_{m_1} \cdots u_{m_i}$$

$$\gamma_{s,j} = \sum_{(m_*, l_*) \in K_{s,j}} u_{m_1} \cdots u_{m_s} \alpha_{l_1} \cdots \alpha_{l_j}$$

where K_i consists of partitions $1 \leq m_1 < \cdots < m_i \leq n$, $K_{s,j} \subseteq K_s \times K_j$ consists of pairs of disjoint partitions and σ_i is the i-th elementary symmetric polynomial. Then:

$$H_G^*(B_G U(n)) = \frac{A_{\mathbb{Q}}[\alpha, c_i, \gamma_{s,j}]}{x\alpha = 0, x\gamma_{s,j} = 0, S}$$

where the finite set S of relations consists of three types of relations:

- *Type I:*

$$\alpha^{n+1} = \sum_{m=1}^n r_m \alpha^m$$

where $r_1 = (-1)^n n!$ and

$$r_{m+1} = (-1)^{n+m} n! \sum_{1 \leq i_1 < \cdots < i_m \leq n} \frac{1}{i_1 \cdots i_m}$$

We can also write this relation as

$$\alpha(\alpha - 1) \cdots (\alpha - n) = 0$$

- *Type II:*

$$\alpha^s \gamma_{s,i} = \frac{s!}{(s+i)!} c_s \alpha^{s+i} + \cdots$$

where $\cdots$ denotes a homogeneous polynomial smaller than $c_s \alpha^{s+i}$ (see Appendix A for a definition of this order).

- *Type III: If $s \leq t \leq s + i$,*

$$\gamma_{s,i}\gamma_{t,j} = \binom{\min(i+j+s,n)-t}{j} c_t \gamma_{s,\min(i+j,n-s)} + \cdots$$

where $\cdots$ denotes a homogeneous polynomial smaller than $c_t\gamma_{s,\min(i+j,n-s)}$.

The polynomials $\cdots$ can be algorithmically computed in terms of $\alpha, c_i, \gamma_{s,j}$; the algorithm is described in Appendix A and has been implemented in the computer program found here.

The elements c_i are algebraically independent and $H_G^*(B_G U(n))$ is finite over $A_{\mathbb{Q}}[c_1, ..., c_n]$ hence has Krull dimension n.

A basis of $H_G^*(B_G U(n))$ over $A_{\mathbb{Q}}$ consists of the elements

$$\alpha^a \prod_{i=1}^{n} c_i^{k_i} \prod_{s=1}^{n} \prod_{i=1}^{n-s} \gamma_{s,i}^{\epsilon_{s,i}}$$

where $0 \leq a \leq n$ and $\epsilon_{s,i} = 0, 1$ are such that for any $\gamma_{s,i}$ appearing in the product we must have $a < s$ and for any two factors $\gamma_{s,i}, \gamma_{t,j}$ with $s \leq t$ we must have $s + i < t$.

Corollary 5.5. *Any set of homogeneous algebra generators of $H_G^*(B_G U(n))$ over $A_{\mathbb{Q}}$ has cardinality at least $1 + n + \binom{n}{2}$, which is the cardinality of the generating set $\{\alpha, c_s, \gamma_{s,j}\}$.*

We can extend $\gamma_{s,i}$ to $s = 0$ and $i = 0$ via:

$$\gamma_{s,0} = y c_s = y \sigma_i(u_1, ..., u_n)$$
$$\gamma_{0,i} = \sigma_i(\alpha_1, ..., \alpha_n) = (i!)^{-1} \alpha(\alpha - 1) \cdots (\alpha - i + 1)$$
$$\gamma_{0,0} = y$$

Whenever we write $\gamma_{s,i}$ it is implicit that $s, i > 0$, unless we explicitly state that we are using the convention above.

5.5. **Dimension count.** Consider the modified partition function $p(n, m)$ counting sequences $a_1 \geq \cdots \geq a_n \geq 0$ with $m = \sum_i a_i$ (the usual partition function requires $a_n \geq 1$). We have the recursion

$$p(n, m) = p(n, m - n) + p(n - 1, m)$$

Then $p(n, m)$ is the dimension of the vector space of symmetric polynomials in $\mathbb{Q}[x_1, ..., x_n]$ of degree m, which we grade as $|x_1| = \cdots = |x_n| = 1$. If R is as in subsection 5.4 with $k = \mathbb{Q}$, we have:

$$\dim(R_m^{\Sigma_n}) = \sum_{i=0}^{n} \sum_{j=0}^{m} p(i, j) p(n - i, m - j)$$

We can equivalently express these facts as:

$$\dim H^{2m}(BU(n)) = p(n, m)$$

and

$$\dim H_G^{2m}(B_G U(n)) = p(n, m) + \sum_{i=0}^{n} \sum_{j=0}^{m} p(i, j) p(n - i, m - j)$$

Note that for fixed m the dimensions $\dim H^{2m}(BU(n))$ stabilize for large enough n. That is not the case for $\dim H_G^{2m} B_G U(n)$.

6. C_2 Chern classes of sums and tensor products

Proposition 6.1. *The map $B_G U(n) \to B_G U(n+1)$ given by direct sum with a trivial complex representation induces on cohomology:*

$$\alpha \mapsto y + \alpha$$
$$c_i \mapsto c_i$$
$$\gamma_{s,i} \mapsto \gamma_{s,i} + \gamma_{s,i-1}$$

using the convention $\gamma_{s,0} = yc_s$.
The map $B_G U(n) \to B_G U(n+1)$ given by direct sum with a complex σ representation induces on cohomology:

$$\alpha \mapsto \alpha$$
$$c_i \mapsto c_i$$
$$\gamma_{s,i} \mapsto \gamma_{s,i}$$

For both maps we use the conventions that $c_{n+1} = 0$ and $\gamma_{s,n+1-s} = 0$ in every RHS.

Proof. We have a commutative diagram

$$
\begin{array}{ccc}
B_G U(n) & \xrightarrow{\ \oplus 1\ } & B_G U(n+1) \\
\uparrow & & \uparrow \\
B_G T^n = B_G T^n \times * & \longrightarrow & B_G T^n \times B_G S^1 = B_G T^{n+1}
\end{array}
$$

where the bottom map is the product of the identity map $B_G T^n \to B_G T^n$ and the inclusion map $* \to B_G S^1$ given by $* \mapsto v^+$ where $v^+ = (1 : 0 : 1 : 0 : \cdots)$ in the homogeneous coordinates of $B_G S^1$. This inclusion map induces on cohomology:

$$A_{\mathbb{Q}}[\alpha, u]/(\alpha^2 = \alpha, x\alpha) \to A_{\mathbb{Q}}$$

given by $y - \alpha \mapsto 0$ and $u \mapsto 0$ (this is verified by looking at the C_2 fixed points). Similarly, adding a σ representation induces

$$A_{\mathbb{Q}}[\alpha, u]/(\alpha^2 = \alpha, x\alpha) \to A_{\mathbb{Q}}$$

given by $\alpha \mapsto 0$ and $u \mapsto 0$. Thus the $\oplus 1$ induced map is determined by:

$$u_i \mapsto u_i \, , \, \alpha_i \mapsto \alpha_i \, , \, i < n+1$$
$$u_{n+1} \mapsto 0 \, , \, \alpha_{n+1} \mapsto y$$

The $\oplus \sigma$ induced map is determined by:

$$u_i \mapsto u_i \, , \, \alpha_i \mapsto \alpha_i \, , \, i < n+1$$
$$u_{n+1} \mapsto 0 \, , \, \alpha_{n+1} \mapsto 0$$

These descriptions imply the ones on the generators $\alpha, c_i, \gamma_{s,j}$. $\qquad\square$

Proposition 6.2. *The direct sum of bundles map $B_G U(n) \times B_G U(m) \to B_G U(n+m)$ induces on cohomology:*

$$\alpha \mapsto \alpha \otimes 1 + 1 \otimes \alpha$$

$$c_i \mapsto \sum_{j+k=i} c_j \otimes c_k$$

$$\gamma_{s,i} \mapsto \sum_{\substack{s'+s''=s \\ i'+i''=i}} \gamma_{s',i'} \otimes \gamma_{s'',i''}$$

using the convention for defining $c_0, \gamma_{s,0}, \gamma_{0,i}$ in the RHS.

Proof. The corresponding map on maximal tori

$$\prod^n B_G S^1 \times \prod^m B_G S^1 \to \prod^{n+m} B_G S^1$$

induces on cohomology:

$$u_i \mapsto \begin{cases} u_i \otimes 1 & \text{if } i \leq n \\ 1 \otimes u_{i-n} & \text{if } i > n \end{cases}, \quad \alpha_i \mapsto \begin{cases} \alpha_i \otimes 1 & \text{if } i \leq n \\ 1 \otimes \alpha_{i-n} & \text{if } i > n \end{cases}$$

This implies the formulas on $\alpha, c_i, \gamma_{s,j}$. $\qquad\qquad\square$

Proposition 6.3. *The tensor product of bundles map $B_G U(1) \times B_G U(1) \to B_G U(1)$ induces on cohomology:*

$$\alpha \mapsto y - \alpha \otimes 1 - 1 \otimes \alpha + 2\alpha \otimes \alpha$$

$$u \mapsto u \otimes 1 + 1 \otimes u$$

Proof. The map in question, induced from multiplication $S^1 \times S^1 \to S^1$, is given on the homogeneous coordinates by multiplication of polynomials:

$$(x_0 : x_1 : \cdots) \otimes (y_0 : y_1 : \cdots) = (x_0 y_0 : x_0 y_1 + x_1 y_0 : \cdots)$$

Write

$$H^*_{C_2}(B_G U(1)) = H^*(BU(1)) \oplus H^*(B_G U(1)^{C_2}) = \mathbb{Q}[e_1] \oplus \mathbb{Q}[e_2] \oplus \mathbb{Q}[e_3]$$

as in subsection 5.1. Then $B_G U(1) \times B_G U(1) \to B_G U(1)$ induces

$$H^*_G(B_G U(1)) \to H^*_G(B_G U(1)) \boxtimes_{A_{\mathbb{Q}}} H^*_G(B_G U(1))$$

which breaks into

$$H^*(BU(1)) \to H^*(B_G U(1)) \otimes H^*(B_G U(1))$$

$$H^*(B_G U(1)^{C_2}) \to H^*(B_G U(1)^{C_2}) \otimes H^*(B_G U(1)^{C_2})$$

By the nonequivariant case, the first map is

$$e_1^0 \mapsto e_1^0 \otimes e_1^0$$

$$e_1 \mapsto e_1 \otimes e_1^0 + e_1^0 \otimes e_1$$

For the second map, note that in the H-space structure, v^+, v^- multiply according to: $v^\alpha \cdot v^\beta = v^{\alpha\beta}$ for $\alpha, \beta = \pm 1$. This means that the $+$ part of $H^*(B_G U(1)^{C_2})$ maps to the $+ \otimes +$ and $- \otimes -$ parts of $H^*(B_G U(1)^{C_2}) \otimes H^*(B_G U(1)^{C_2})$ to give:

$$e_2^0 \mapsto e_2^0 \otimes e_2^0 + e_3^0 \otimes e_3^0$$

$$e_2 \mapsto e_2 \otimes e_2^0 + e_2^0 \otimes e_2 + e_3 \otimes e_3^0 + e_3^0 \otimes e_3$$

Similarly, the $-$ part of $H^*(B_G U(1)^{C_2})$ maps to the $+ \otimes -$ and $- \otimes +$ parts of $H^*(B_G U(1)^{C_2}) \otimes H^*(B_G U(1)^{C_2})$ to give:

$$e_3^0 \mapsto e_2^0 \otimes e_3^0 + e_3^0 \otimes e_2^0$$
$$e_3 \mapsto e_2 \otimes e_3^0 + e_2^0 \otimes e_3 + e_3 \otimes e_2^0 + e_3^0 \otimes e_2$$

In terms of the u, α generators, recall $e_2^0 = \alpha, e_3^0 = y - \alpha, e_1 = u e_1^0, e_2 = u e_2^0$ and $e_3 = u e_3^0$. Substituting these gives the desired formulas. $\qquad \square$

Iterating $B_G S^1 \times B_G S^1 \rightarrow B_G S^1$ gives $(B_G S^1)^{\times n} \rightarrow B_G S^1$ which induces on cohomology:

$$\alpha \mapsto \frac{(-1)^n + 1}{2} y + (-1)^{n+1} \sum_{i=1}^{n} (-2)^{i-1} \sigma_i(\alpha_1, ..., \alpha_n)$$
$$u \mapsto u_1 + \cdots + u_n$$

We will also need that the map induced from conjugation $B_G U(1) \rightarrow B_G U(1)$ is:

$$\alpha \mapsto \alpha$$
$$u \mapsto -u$$

This is verified similarly to Proposition 6.3.

7. C_2 STABLE CHERN CLASSES

Since there are two maps $B_G U(n) \rightarrow B_G U(n+1)$ (given by direct sum with the trivial or the σ representation) one can try to stabilize against, there are a few distinct notions of stable characteristic classes.

7.1. **Stabilizing against one representation.** First, we can stabilize with respect to direct sum with the trivial representation and get:

$$B_G^+ U = \mathrm{colim}(B_G U(1) \xrightarrow{\oplus 1} B_G U(2) \xrightarrow{\oplus 1} \cdots)$$

Then

$$K_G^+(X) = [X, B_G^+ U \times \mathbb{Z}]^G$$

is the semi-ring of virtual bundles $V - n$ on X modulo the equivalence relation $V \sim W \iff V + n = W + n$ for some n. For example, $K_G^+(*) = \mathbb{Z} \times \mathbb{N}$. Group completing K_G^+ gives equivariant K-theory K_G.

The fixed points of $B_G^+ U$ are:

$$(B_G^+ U)^{C_2} = BU \times \coprod_{n=0}^{\infty} BU(n)$$

We note that $B_G^+ U$ is the usual equivariant classifying space $B_G U = E_G U / U$.

An equivalent way of getting $B_G U$ is to stabilize with respect to direct sum with the σ representation:

$$B_G^- U = \mathrm{colim}(B_G U(1) \xrightarrow{\oplus \sigma} B_G U(2) \xrightarrow{\oplus \sigma} \cdots)$$

Note that $B_G^+ U \simeq B_G^- U$.

7.2. **Stabilizing with respect to all representations.** Group completing $B_G^+ U$ or equivalently $B_G^- U$, gives:

$$B_G^\pm U = \mathrm{colim}(B_G^+ U \xrightarrow{\oplus \sigma} B_G^+ U \xrightarrow{\oplus \sigma} \cdots) = \mathrm{colim}(B_G^- U \xrightarrow{\oplus 1} B_G^- U \xrightarrow{\oplus 1} \cdots)$$

Then

$$K_G(X) = [X, B_G^\pm U \times \mathbb{Z}]^G$$

is the semi-ring of virtual bundles $V - n - m\sigma$ modulo the equivalence relation $V \sim W \iff V + n + m\sigma = W + n + m\sigma$ for some n, m. Thus $K_G(X)$ is exactly the equivariant K-theory of X.

We finally have the fixed point computation:

$$(B_G^\pm U)^{C_2} = BU \times BU \times \mathbb{Z}$$

7.3. **Cohomology computations.** We prefer to work with $B_G^- U$ since the map $H_G^*(B_G U(n+1)) \xrightarrow{\oplus \sigma} H_G^*(B_G U(n))$ has a simpler expression on the generators $\alpha, \gamma_{s,j}$ compared to the map $H_G^*(B_G U(n+1)) \xrightarrow{\oplus 1} H_G^*(B_G U(n))$.

The $\mathbb{Q}$-subalgebras spanned by α in each $H_G^*(B_G U(n))$ have limit:

$$\lim \frac{\mathbb{Q}[\alpha]}{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - n)} = \prod_{n \geq 0} \mathbb{Q}$$

under the correspondence:

$$a_0 + \sum_{i \geq 0} a_{i+1} \alpha(\alpha - 1) \cdots (\alpha - i) \mapsto (a_0, a_0 + a_1, a_0 + 2a_1 + 2a_2, \ldots)$$

given by evaluating the series in the LHS at $\alpha = 0, 1, 2, \ldots$.

Under this correspondence we have:

$$H_G^0(B_G^- U) = A_\mathbb{Q} \times \prod_{n \geq 1} A_\mathbb{Q}/x$$

As a graded $H_G^0(B_G^- U)[c_1, c_2, \ldots]$-algebra, $H_G^*(B_G^- U)$ is generated by the series $\sum_{i=1}^\infty r_i \gamma_{s,i} \in H_G^{2s}(B_G^- U)$ for $r_i \in \mathbb{Q}$ and $s = 1, 2, \ldots$.

Describing $H_G^*(B_G^\pm U)$ in terms of the generators $\alpha, c_i, \gamma_{s,j}$ is even more complicated, as we need to take the limit of $H_G^*(B_G^- U)$ with respect to the $\oplus 1$ maps. We can alternatively view $H_G^*(B_G^\pm U)$ as the limit of a diagram indexed on $\mathbb{N} \times \mathbb{N}$ with $(n, m) \mapsto B_G U(n)$, horizontal maps being $\oplus 1$ and vertical maps being $\oplus \sigma$. An element of $H_G^*(B_G^\pm U)$ will then be a compatible doubly indexed sequence $s_{n,m} \in H_G^*(B_G U(n))$. The constant sequences (in both variables) consist of elements that are invariant under both the $\oplus 1$ and $\oplus \sigma$ maps. We can see that

$$c_i, \gamma_i := c_i \alpha - \gamma_{i,1}$$

have this property, and we conjecture that the subalgebra of constant sequences is generated by them. This is equivalent to the $\gamma_1, \gamma_2, \ldots$ being algebraically independent over $\mathbb{Q}[c_1, c_2, \ldots]$ and further equivalent to the subalgebra of constant sequences being

$$\frac{A_\mathbb{Q}[c_i, \gamma_i]}{x \gamma_i}$$

The coalgebra structure is

$$\gamma_s \mapsto \sum_{i+j=s} (c_i \otimes \gamma_j + \gamma_i \otimes c_j)$$

using the conventions $c_0 = 1$ and $\gamma_0 = 0$.

7.4. **Homology computations.** For homology, the idempotent decomposition gives

$$H_*^{C_2}(X) = H_*(X)^{C_2} \oplus H_*(X^{C_2})$$

as Mackey functors. For an H-space X where $X \times X \to X$ and $S^0 \to X$ are C_2 equivariant, this becomes an isomorphism of Green functors.

Setting $X = B_G^- U$ and using

$$(B_G^- U)^{C_2} = \coprod_{n=0}^{\infty} BU(n) \times BU$$

we get that $H_*^G(X)$ is the sum of

$$H_*(BU) = \mathbb{Q}[b_i^e]_{i \geq 1} ,$$

$$H_*(B_G^- U^{C_2}) = \oplus_{n \geq 1} H_*(BU(n))) \otimes H_*(BU) = \mathbb{Q}[b_0^+, b_i^+, b_i^-]_{i \geq 1}$$

where for $i \geq 1$, the b_i^e, b_i^+, b_i^- are duals of c_1^i using the first Chern class in the respective nonequivariant cohomology rings (b_0^+ indexes the components).

If we let $b_i = b_i^e + b_i^+ + b_i^-$, $a_i = b_i^+$ for $i \geq 1$ and $d = x/2 + b_0^+$ we get

$$H_*^G(B_G^- U) = A_{\mathbb{Q}}[d, a_i, b_i]_{i \geq 1}/(xd = x, xa_i = 0)$$

Group completing is localization at d:

$$H_*^G(B_G^\pm U) = A_{\mathbb{Q}}[d^\pm, a_i, b_i]_{i \geq 1}/(xd = x, xa_i = 0)$$

The a_i, b_i are dual to γ_1^i, c_1^i respectively, while d is dual to the finite series $x/2 + \alpha \in H^0(B_G^- U)$ i.e. the sequence $(x/2, 1, 2, ...) \in A_{\mathbb{Q}} \times \prod_{n \geq 1} A_{\mathbb{Q}}/x$.

The coalgebra structure is:

$$d \mapsto d \otimes d$$

$$a_i \mapsto \sum_{j+k=i} a_j \otimes a_k$$

$$b_i \mapsto \sum_{j+k=i} (b_j \otimes b_k - b_j \otimes a_k - b_k \otimes a_j + 2a_j \otimes a_k)$$

using the conventions $a_0 = d - x/2$ and $b_0 = 1$.

8. C_2 SYMPLECTIC CLASSES

Analogously to the Chern classes, we have:

Proposition 8.1. *There exist classes* $\alpha, k_i, \kappa_{s,j} \in H_G^*(B_G Sp(n))$ *of degrees* $0, 4i, 4s$ *respectively, where* $1 \leq i, s \leq n$ *and* $1 \leq j \leq n - s$, *such that*

$$H_G^*(B_G Sp(n)) = A_{\mathbb{Q}}[\alpha, k_i, \kappa_{s,j}]/(x\alpha, x\kappa_{s,j}, S)$$

The relation set S is the same as that for $H_G^*(B_G U(n))$ with $c_i, \gamma_{s,i}$ replaced by $k_i, \kappa_{s,i}$.

The generators $\alpha, \kappa_{s,j}$ restrict to 0 while the k_i restrict to the nonequivariant symplectic classes.

The maximal torus inclusion $U(1)^n \hookrightarrow Sp(n)$ induces an isomorphism

$$H_G^*(B_G Sp(n)) = (H_G^*(B_G U(1))^{\otimes n})^{C_2 \wr \Sigma_n}$$

Explicitly:

$$A_{\mathbb{Q}}[\alpha, k_i, \kappa_{s,j}]/(x\alpha, x\kappa_{s,j}, S) = (A_{\mathbb{Q}}[\alpha_i, u_i]/(x\alpha_i))^{C_2 \wr \Sigma_n}$$

Under this identification:

$$\alpha = \sum_{1 \le i \le n} \alpha_i$$

$$k_i = \sum_{m_* \in K_i} u_{m_1}^2 \cdots u_{m_i}^2$$

$$\kappa_{s,j} = \sum_{(m_*, l_*) \in K_{s,j}} u_{m_1}^2 \cdots u_{m_s}^2 \alpha_{l_1} \cdots \alpha_{l_j}$$

where K_i and $K_{s,j}$ are as in Corollary 5.4.

Proof. First, we have the fixed point computation:

$$B_G Sp(n)^{C_2} = \coprod_{m+k=n} BSp(m) \times BSp(k)$$

The maximal torus in $Sp(n)$ is $T = \prod^n S^1$, the same as in $U(n)$, but the Weyl group now is $C_2 \wr \Sigma_n$ with Σ_n permuting the S^1 factors and the i-th C_2 in $C_2 \wr \Sigma_n = C_2^n \rtimes \Sigma_n$ acting as conjugation on the i-th S^1 factor in T.

Following the $B_G U(n)$ case, the maximal torus inclusion on the fixed points breaks into

$$\coprod^{\binom{n}{m}} BU(1)^n \to BSp(m) \times BSp(n-m)$$

where the coproduct of the left is indexed on sign configurations with m many $+$'s. The induced map on cohomology is

$$H^*(BSp(m)) \otimes H^*(BSp(n-m)) \to \oplus^{\binom{n}{m}} H^*(BU(1))^{\otimes n}$$

The action of Σ_n on the right permutes the factors in the tensor product and the sign configuration, while the i-th C_2 in $C_2^n \rtimes \Sigma_n$ acts by as the conjugation-induced map $H^*(BS^1) \to H^*(BS^1)$ on the i-th factor in the tensor product (fixing the sign configuration).

Thus analogously to the $B_G U(n)$ case, we have an isomorphism into the $C_2 \wr \Sigma_n$ fixed points of the right hand side. In conclusion:

$$H_G^*(B_G Sp(n)) = (\otimes^n H_G^*(B_G S^1))^{C_2 \wr \Sigma_n}$$

To compute the C_2 action on $H_G^*(B_G S^1)$ recall that conjugation $S^1 \to S^1$ induces $\alpha \mapsto \alpha$ and $u \mapsto -u$ on cohomology. Fixing under the C_2^n action, we get that $H_G^*(B_G Sp(n)) = R'^{\Sigma_n}$ where $R' = R(u_1^2, ..., u_n^2, \alpha_1, ..., \alpha_n)$ is the ring R we used for $B_G U(n)$ but now with u_i replaced by u_i^2. So we get equivariant symplectic classes $\alpha, k_i, \kappa_{s,i}$ of degrees $0, 4i, 4s$ with the desired expressions in terms of α_i, u_i^2. $\qquad \square$

The maps $B_G Sp(n) \to B_G Sp(n+1)$ and $B_G Sp(n) \times B_G Sp(m) \to B_G Sp(n+m)$ have the same formulas as the analogous maps for $B_G U(n)$, with $c_i, \gamma_{s,i}$ replaced by $k_i, \kappa_{s,i}$ respectively.

Proposition 8.2. *The forgetful map $B_G Sp(n) \to B_G U(2n)$ induces on cohomology:*
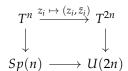
$$\alpha \mapsto \alpha$$
$$c_{2i+1}, \gamma_{2s+1,j} \mapsto 0$$
$$c_{2i} \mapsto (-1)^i k_i$$
$$\gamma_{2s,j} \mapsto (-1)^s \kappa_{s,j}$$

The quaternionization map $B_G U(n) \to B_G Sp(n)$ induces:

$$\alpha \mapsto \alpha$$
$$k_i \mapsto \sum_{a+b=2i} (-1)^{a+i} c_a c_b$$
$$\kappa_{1,j} \mapsto c_1 \gamma_{1,j} - \alpha \gamma_{2,j-1} + (j-2)\gamma_{2,j} + (j-1)\gamma_{2,j-1}$$
$$\kappa_{s,j} \mapsto c_s \gamma_{s,j} + \cdots$$

where $\cdots$ denotes a homogeneous polynomial in R^{Σ_n} smaller than $c_s \gamma_{s,j}$ (according to the order defined in appendix A). This polynomial can be computed algorithmically according to the algorithm in appendix A which has been implemented in the computer program found here.

Proof. To compute the effect of the forgetful map $B_G Sp(n) \to B_G U(2n)$ note that we have the commutative diagram

$$
\begin{array}{ccc}
T^n & \xrightarrow{z_i \mapsto (z_i, \bar{z}_i)} & T^{2n} \\
\downarrow & & \downarrow \\
Sp(n) & \longrightarrow & U(2n)
\end{array}
$$

(here $T = S^1$). The top map induces $H_G^*(B_G T^{2n}) \to H_G^*(B_G T^n)$ given by $\alpha_i \mapsto \alpha_i, \alpha_{i+n} \mapsto \alpha_i, u_i \mapsto u_i, u_{i+n} \mapsto -u_i$ for $1 \le i \le n$. $\qquad \square$

The stable symplectic classes work analogously to the stable Chern classes; see subsection 4.4 for a summary.

9. C_2 EULER AND PONTRYAGIN CLASSES

Analogously to the symplectic classes, we have:

Proposition 9.1. *There exist classes $\alpha, p_i, \pi_{s,j}, \chi$ of degrees $0, 4i, 4s, n$ respectively in $H_G^*(B_G SO(2n))$, where $1 \le i, s < n$ and $1 \le j \le n - s$ such that*

$$H_G^*(B_G SO(2n)) = A_{\mathbb{Q}}[\alpha, p_i, \pi_{s,j}, \chi]/(x\alpha, x\pi_{s,j}, S)$$

where the relation set S is the same as that for $H_G^*(B_G U(n))$ with $c_i, \gamma_{s,i}$ replaced by $p_i, \pi_{s,i}$ and using the convention $p_n = \chi^2$.

The generators $\alpha, \pi_{s,j}$ restrict to 0 while the p_i, χ restrict to the nonequivariant Pontryagin and Euler classes respectively.

The map $B_G SO(2n) \to B_G SO(2n+1)$ induces an injection in cohomology with

$$H_G^*(B_G SO(2n+1)) = A_{\mathbb{Q}}[\alpha, p_i, \pi_{s,j}]/(x\alpha, x\pi_{s,j}, S)$$

where i is allowed to be n (i.e. $p_n = \chi^2$ is included).

The maximal torus inclusion $T \hookrightarrow SO(n)$ induces an isomorphism

$$H_G^*(B_G SO(n)) = (H_G^*(B_G T))^W$$

where W is the corresponding Weyl group. Under this isomorphism,

$$\alpha = \sum_{1 \leq m \leq n} \alpha_m$$

$$p_i = \sum_{m_* \in K_i} u_{m_1}^2 \cdots u_{m_i}^2$$

$$\pi_{s,j} = \sum_{(m_*, l_*) \in K_{s,j}} u_{m_1}^2 \cdots u_{m_s}^2 \alpha_{l_1} \cdots \alpha_{l_j}$$

$$\chi = u_1 \cdots u_n$$

where $K_i, K_{s,j}$ are as in Corollary 5.4.

Proof. We use the oriented Grassmannian model for $B_G SO(n)$, consisting of n-dimensional oriented subspaces of $\mathbb{R}^{\infty \rho}$ where $\rho = 1 + \sigma$ is the real regular representation of $G = C_2$. The G action sends an oriented subspace with basis $v_1, ..., v_n$ to one with basis $gv_1, ..., gv_n$. If $V \in B_G SO(n)$ is fixed by the G action, then $V = V^+ \oplus V^-$ where V^+ is an oriented subspace with $gv = v$ for every $v \in V^+$ and V^- is an oriented subspace with $gv = -v$ for every $v \in V^-$. Moreover, since g must act by an $SO(n)$ action on V, the dimension of V^- must be even. In particular, if a 2-dimensional subspace V is fixed then $V = V^+$ or $V = V^-$. As for the uniqueness of the decomposition, note that if $W^+ \oplus W^- = V^+ \oplus V^-$ through an $SO(n)$ matrix A, then A is block diagonal with blocks $B \in O(n - 2k), C \in O(2k)$ and $\det(B) \det(C) = 1$; B gives $W^+ = V^+$ while C gives $W^- = V^-$. Thus

$$B_G SO(n)^{C_2} = \coprod_{k=0}^{n/2} B Z_{2k,n}$$

where $Z_{k,n}$ is the subgroup of $O(k) \times O(n - k)$ consisting of pairs (A, B) with $\det(A) \det(B) = 1$. Note $Z_{k,n} = Z_{n-k,n}$ and $Z_{0,n} = Z_{n,0} = SO(n)$; if $0 < k < n$ we have

$$Z_{k,n} = (SO(k) \times SO(n - k)) \rtimes C_2$$

with C_2 acting diagonally by conjugation.

The maximal torus of $Z_{2k,2n}$ is $SO(2)^n$ with Weyl group $H \rtimes (\Sigma_k \times \Sigma_{n-k}) \subseteq C_2^n \rtimes (\Sigma_k \times \Sigma_{n-k})$ where $H \subseteq C_2^n$ consists of elements with even number of coordinates equal to -1.

The maximal torus of $Z_{2k,2n+1}$ is $SO(2)^n$ with Weyl group $C_2^n \rtimes (\Sigma_k \times \Sigma_{n-k})$.

Following the $B_G U(n)$ case, the maximal torus inclusion $SO(2)^n \to SO(2n + 1)$ induces $B_G SO(2)^n \to B_G SO(2n + 1)$ which on the fixed points becomes:

$$\coprod^{2^n} BSO(2)^n \to \coprod_{k=0}^{n} B Z_{2k,2n+1}$$

with the LHS indexed over sign configurations as usual. Fixing the total amount k of $-$ signs, the coproduct breaks into

$$\coprod^{\binom{n}{k}} BSO(2)^n \to B Z_{2k,2n+1}$$

which induces
$$H^*(BZ_{2k,2n+1}) \to \oplus^{\binom{n}{k}} H^*(BSO(2))^{\otimes n}$$
Taking $C_2 \wr \Sigma_n = C_2^n \rtimes \Sigma_n$ fixed points on the right hand side is equivalent to fixing a sign configuration and then taking $C_2^n \rtimes (\Sigma_k \times \Sigma_{n-k})$ fixed points i.e. fixing under the action of the Weyl group of $BZ_{2k,2n+1}$. Therefore we are reduced to proving $H^*(BZ_{2k,2n+1}) = H^*(BSO(2)^n)^{C_2 \wr (\Sigma_k \times \Sigma_{n-k})}$. Although Borel's Theorem is stated for connected Lie groups, it nonetheless works for $Z_{2k,2n+1}$ as it does for $O(n)$. This can be seen from the covering space

$$C_2 \to BSO(2k) \times BSO(2n+1-2k) \to BZ_{2k,2n+1}$$

for which the associated transfer map

$$H^*(BZ_{2k,2n+1}) \to H^*(BSO(2k) \times BSO(2n+1-2k))^{C_2}$$

is an isomorphism. Computing the right hand side shows that it's isomorphic to $H^*(BSO(2)^n)^{C_2 \wr (\Sigma_k \times \Sigma_{n-k})}$ as desired.

Similar arguments work for $H^*_G(B_G SO(2n))$ proving the maximal torus isomorphism in that case as well.

Given the maximal torus isomorphism, we compute $H^*_G(B_G SO(2n+1))$ identically to $H^*_G(B_G Sp(n))$ so we get the desired equivariant Pontryagin classes $p_i, \pi_{s,j}$. For $H^*_G(B_G SO(2n))$ we only want to fix under an even number of sign changes on the u_i hence we get the equivariant Pontryagin classes plus the Euler class $\chi = \sigma_n(u_1, ..., u_n)$. Note that $p_n = \chi^2$ and after removing p_n there are no relations involving χ and the other generators. $\qquad\square$

In the identification $B_G SO(2) = B_G U(1)$ we have

$$\chi = c_1$$

The maps $B_G SO(n) \times B_G SO(m) \to B_G SO(n+m)$ work analogously to the symplectic case, replacing the $k_i, \kappa_{s,j}$ with $p_i, \pi_{s,j}$; the action on the Euler class is

$$\chi \mapsto \chi \otimes \chi$$

Proposition 9.2. *The complexification map $B_G SO(2n) \to B_G U(2n)$ induces on cohomology:*

$$\alpha \mapsto \alpha$$
$$c_{2i+1}, \gamma_{2s+1,j} \mapsto 0$$
$$c_{2i} \mapsto (-1)^i p_i$$
$$\gamma_{2s,j} \mapsto (-1)^s \pi_{s,j}$$

The forgetful map $B_G U(n) \to B_G SO(2n)$ induces on cohomology:

$$\alpha \mapsto \alpha$$
$$p_i \mapsto \sum_{a+b=2i} (-1)^{a+i} c_a c_b$$
$$\chi \mapsto c_n$$
$$\pi_{1,j} \mapsto c_1 \gamma_{1,j} - u\gamma_{2,j-1} + (j-2)\gamma_{2,j} + (j-1)\gamma_{2,j-1}$$
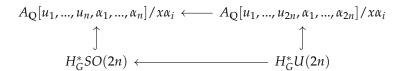$$\pi_{s,j} \mapsto c_s \gamma_{s,j} + \cdots$$

where $\cdots$ denotes the same homogeneous polynomial as the $\cdots$ in Proposition 8.2.

Proof. To understand the effect of complexification $SO(2n) \to U(2n)$ we use the nonstandard maximal torus $T^2 \to U(2)$ making the following diagram commute:

$$
\begin{array}{ccc}
S^1 & \xrightarrow{a \,\mapsto\, (a,\bar{a})} & T^2 \\
\| & & \downarrow \\
SO(2) & \longrightarrow & U(2)
\end{array}
$$

where the map in the bottom row is complexification. More generally we have the commutative diagram

$$
\begin{array}{ccc}
T^n & \xrightarrow{(a_1,\,...,\,a_n) \,\mapsto\, (a_1,\bar{a}_1,\,...,\,a_n,\bar{a}_n)} & T^{2n} \\
\downarrow & & \downarrow \\
SO(2n) & \longrightarrow & U(2n)
\end{array}
$$

Any two maximal tori are conjugate hence induce the same map in $H_G^*(B_G H)$ so we get

$$
\begin{array}{ccc}
A_{\mathbb{Q}}[u_1,...,u_n,\alpha_1,...,\alpha_n]/x\alpha_i & \longleftarrow & A_{\mathbb{Q}}[u_1,...,u_{2n},\alpha_1,...,\alpha_{2n}]/x\alpha_i \\
\uparrow & & \uparrow \\
H_G^* SO(2n) & \longleftarrow & H_G^* U(2n)
\end{array}
$$

In the top row, $u_i \mapsto u_i, u_{n+i} \mapsto -u_i, \alpha_i \mapsto \alpha_i, \alpha_{n+i} \mapsto \alpha_i$ for $i \leq n$. This implies the formulas for the effect of the complexification map. $\qquad\square$

We only have one map $B_G SO(n) \to B_G SO(n+1)$, given by direct sum with the trivial representation; direct sum with the σ representation does not result in a C_2 equivariant map. There is however a C_2-equivariant map $B_G SO(n) \to B_G SO(n+2)$ given by adding 2σ.

The map $B_G SO(n) \xrightarrow{\oplus 1} B_G SO(n+1)$ induces

$$
\begin{aligned}
\alpha &\mapsto y + \alpha \\
p_i &\mapsto p_i \\
\pi_{s,j} &\mapsto \pi_{s,j} + \pi_{s,j-1} \\
\chi &\mapsto 0
\end{aligned}
$$

(with the usual conventions on the RHS, including that $\pi_{s,0} = y p_s$).

The map $B_G SO(n) \xrightarrow{\oplus 2\sigma} B_G SO(n+2)$ induces:

$$
\begin{aligned}
\alpha &\mapsto \alpha \\
p_i &\mapsto p_i \\
\pi_{s,i} &\mapsto \pi_{s,i} \\
\chi &\mapsto 0
\end{aligned}
$$

(with the usual conventions on the RHS).

So we can distinguish between $B_G^+ SO, B_G^- SO$ and $B_G^\pm SO$ as usual. The results here are then parallel to the $B_G^+ U, B_G^- U$ and $B_G^\pm U$ cases respectively, by replacing

$\alpha, c_i, \gamma_{s,i}$ by $\alpha, p_i, \pi_{s,i}$; the Euler class χ is not stable. See subsection 4.4 for a summary.

10. Orthogonal groups

The groups $O(2n), SO(2n+1), O(2n+1)$ have the same maximal torus and Weyl group. The resulting nonequivariant classifying spaces have isomorphic cohomology rings and the maximal torus isomorphism works in all cases (even though the orthogonal groups are disconnected). In this subsection, we shall see that this observation doesn't generalize to the C_2 equivariant case.

For $B_G O(n)$ we compute the fixed points:

$$B_G O(n)^{C_2} = \coprod_{m+k=n} BO(m) \times BO(k)$$

We consider the map

$$H_G^*(B_G O(n)) \to H_G^*(B_G T)^W$$

where $T \subseteq O(n)$ is a maximal torus and W the Weyl group. We can see directly that for $n \leq 3$ the RHS has smaller dimension compared to the LHS even before taking fixed points. So this map cannot be an isomorphism for $n \leq 3$.

We can also see from the fixed point computation that $H_G^*(B_G O(n))$ is never isomorphic to $H_G^*(B_G O(n+1))$ for any n.

There is however a natural description of the characteristic classes for the group $O(2n+1)$: since $O(2n+1) = SO(2n+1) \times O(1)$ we have that

$$B_G O(2n+1) = B_G SO(2n+1) \times B_G O(1)$$

hence

$$H_G^*(B_G O(2n+1)) = H_G^*(B_G SO(2n+1)) \boxtimes_{A_{\mathbb{Q}}} H_G^*(B_G O(1))$$

and

$$H_G^*(B_G O(1)) = A_{\mathbb{Q}}[\beta]/(\beta^2 = \beta, x\beta)$$

for β in degree 0. So the C_2-characteristic classes for $O(2n+1)$ are $\alpha, p_i, \pi_{s,j}, \beta$

The map $B_G O(2n) \to B_G O(2n+1)$ is always a surjection in cohomology (this follows by looking at the fixed points) so $H_G^*(B_G O(2n))$ is a quotient of $H_G^*(B_G O(2n+1))$. For $n=1$ we get:

$$H_G^*(B_G O(2)) = \frac{A_{\mathbb{Q}}[\alpha, \beta, p_1]}{x\beta, \alpha^2 = \alpha, \beta^2 = \beta, \alpha\beta = \alpha, yp_1 = \beta p_1}$$

More generally, in degree 0:

$$H_G^0(B_G O(2n)) = H_G^0(B_G O(2n+1))/\alpha(\alpha-1)\cdots(\alpha-n+1)(1-\beta)$$

For $n=2$ the other relations in higher degrees are $p_1\alpha^2(1-\beta) = 0$ and $p_2(y - \beta - \alpha + \alpha\beta) = 0$.

Finally, stability for orthogonal groups is understood from $B_G O = B_G SO \times B_G O(1)$.

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11. Special unitary groups

Proposition 11.1. *The maximal torus inclusion $U(1)^{n-1} \to SU(n)$ induces an isomorphism*

$$H_G^*(B_G SU(n)) \to H_G^*(B_G U(1)^{n-1})^{\Sigma_n}$$

Proof. Analogously to $B_G SO(n)$,

$$B_G SU(n)^{C_2} = \coprod_{k=0}^{n/2} BZ'_{2k,n}$$

where $Z'_{k,n} \subseteq U(k) \times U(n-k)$ consists of (A, B) with $\det(A)\det(B) = 1$. So $Z'_{k,n} = Z'_{n-k,n}$ and $Z'_{0,n} = SU(n)$. If $0 < k < n$,

$$Z'_{k,n} = (SU(k) \times SU(n-k)) \rtimes S^1$$

where S^1 acts diagonally by conjugation. The maximal torus in $Z'_{2k,n}$ is $U(1)^{n-1}$ and the Weyl group is $\Sigma_{2k} \times \Sigma_{n-2k}$.

Taking fixed points in $B_G U(1)^{n-1} \to B_G SU(n)$ gives:

$$\coprod^{2^{n-1}} BU(1)^{n-1} \to \coprod_{k=0}^{n/2} BZ'_{2k,n}$$

and as in $B_G SO(n)$, the coproduct breaks into

$$\coprod^{\binom{n}{2k}} BU(1)^{n-1} \to BZ'_{2k,n}$$

In cohomology:

$$H^*(BZ'_{2k,n}) \to \oplus^{\binom{n}{2k}} H^*(BU(1))^{\otimes(n-1)}$$

The group Σ_n acts on the right by permuting the sign configuration and the tensor factors $H^*(BU(1)) = \mathbb{Q}[a_i]$ where $a_n = -(a_1 + \cdots + a_{n-1})$. Thus, taking Σ_n fixed points is equivalent to fixing a sign configuration, say $(-, ..., -, +, ..., +)$, and then taking $\Sigma_{2k} \times \Sigma_{n-2k}$ fixed points, which is exactly the Weyl group of $Z'_{2k,n}$. This establishes the maximal torus isomorphism for $B_G SU(n)$. $\qquad\square$

In conclusion, $H_G^*(B_G SU(n)) = R'^{\Sigma_n}$ where $R' = A_{\mathbb{Q}}[u_i, \alpha_i]_{1 \leq i \leq n}$ modulo the relations

$$\alpha_n = \frac{(-1)^{n+1}+1}{2}y + (-1)^n \sum_{i=1}^{n-1}(-2)^{i-1}\sigma_i(\alpha_1, ..., \alpha_{n-1})$$

$$u_n = -u_1 \cdots - u_{n-1}$$

Using the same definitions for $\alpha, c_i, \gamma_{s,j}$ in terms of α_i, u_i as in $B_G U(n)$, we can see that

$$c_1 = \gamma_{1,n-1} = 0$$

There are more relations however: for example, $SU(2) = Sp(1)$ so we need:

$$\alpha^2 = 2\alpha$$

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as an additional relation. The identification $SU(2) = Sp(1)$ then becomes:

$$\frac{A_{\mathbb{Q}}[c_2, \alpha]}{\alpha^2 = 2\alpha, x\alpha} \rightarrow \frac{A_{\mathbb{Q}}[k_1, \alpha]}{\alpha^2 = \alpha}$$

$$c_2 \mapsto k_1$$

$$\alpha \mapsto 2\alpha$$

The rest of the relations for each $B_G SU(n)$ can be computed algorithmically.

As for stability, we have the map $B_G SU(n) \rightarrow B_G SU(n+1)$ given by direct sum with a trivial representation. The map $B_G SU(n) \rightarrow B_G SU(n+1)$ given by direct sum with a σ representation is not C_2 equivariant, so we instead use the map $B_G SU(n) \rightarrow B_G SU(n+2)$ adding 2σ.

Analogously to the $B_G U$ case, we can distinguish between spaces $B_G^+ SU, B_G^- SU$ and $B_G^\pm SU$. Under the inclusion $B_G^\pm SU \rightarrow B_G^\pm U$, $c_1 = \gamma_1 = 0$.

APPENDIX A. SYMMETRIC POLYNOMIALS WITH RELATIONS

For a fixed commutative ring k consider the graded k-algebra R:

$$R = R(u_1, ..., u_n, \alpha_1, ..., \alpha_n) := k[u_i, \alpha_i] / (\alpha_i^2 = \alpha_i)$$

whose generators have degrees $|u_i| = 1$ and $|\alpha_i| = 0$. The group Σ_n acts on R by permuting the u_i and α_i separately.

Any monic monomial in R takes the unique form

$$u_1^{a_1} \cdots u_n^{a_n} \alpha_1^{\epsilon_1} \cdots \alpha_n^{\epsilon_n}$$

for $a_i \geq 0$ and $\epsilon_i = 0, 1$. We order the monic monomials of the same degree lexicographically by the powers $a_1, ..., a_n, \epsilon_1, ..., \epsilon_n$. For a homogeneous element $p \in R$, we consider the (nonzero) monomials p_i in p and let p_i' be the corresponding monic monomials; the greatest of the p_i' is the dominant term $\text{dom}(p)$ of p. We compare homogeneous polynomials using their dominant terms (ignoring their coefficients).

Proposition A.1. *We have the k-algebra presentation:*

$$R^{\Sigma_n} = \frac{k[\gamma_{s,i}]}{\gamma_{s,i}\gamma_{t,j} = \binom{\min(i+j+s,n)-t}{j}\gamma_{t,0}\gamma_{s,\min(i+j,n-s)} + p_{s,i,t,j,n}}$$

where:

- *For each pair of nonnegative indices s, i such that $s + i \leq n$ we have one generator $\gamma_{s,i}$ of degree s given by:*

$$\gamma_{s,i} = \sum_{(m_*, l_*)} u_{m_1} \cdots u_{m_s} \alpha_{l_1} \cdots \alpha_{l_i}$$

  where (m_*, l_*) ranges over pairs of disjoint partitions $1 \leq m_1 < \cdots < m_s \leq n$ and $1 \leq l_1 < \cdots < l_i \leq n$.
- *For each quadruple of indices s, i, t, j such that $0 \leq s \leq t \leq s+i$, $0 < i \leq n - s$ and $0 < j \leq n - t$, we have a relation:*

$$\gamma_{s,i}\gamma_{t,j} = \binom{\min(i+j+s,n)-t}{j}\gamma_{t,0}\gamma_{s,\min(i+j,n-s)} + p_{s,i,t,j,n}$$

  where $p_{s,i,t,j,n}$ is a homogeneous polynomial in R^{Σ_n} smaller than $\gamma_{t,0}\gamma_{s,\min(i+j,n-s)}$.

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The elements $c_i = \gamma_{i,0}$ are algebraically independent over k and R^{Σ_n} is finite over $k[c_1, ..., c_n]$ so we have the equality of Krull dimensions:

$$\dim(R) = \dim(k) + n$$

A k-module basis of R^{Σ_n} consists of the elements

$$\prod_{i=1}^{n} c_i^{r_i} \prod_{s=0}^{n} \prod_{i=1}^{n-s} \gamma_{s,i}^{\epsilon_{s,i}}$$

where the $\epsilon_{s,i} = 0, 1$ are such that whenever $\epsilon_{s,i} = \epsilon_{t,j} = 1$ for $1 \leq i, j$ and $s \leq t$ then we must also have $s + i < t$.

Three remarks about the polynomials $p_{n,s,i,t,j}$ appearing in the relations:

- The proof of Proposition A.1 provides an algorithm for computing these polynomials. This algorithm has been implemented in a computer program available here (executable files are available here for a quick demonstration).
- The coefficients of the polynomials $p_{s,i,t,j,n}$ are in the image of the initial homomorphism $\mathbb{Z} \to k$ (i.e. they are independent of k).
- If $n \geq i + j + s$ then $p_{s,i,t,j,n}$ is independent of n and moreover the relation on $\gamma_{s,i}\gamma_{t,j}$ is independent of n, taking the simpler form:

$$\gamma_{s,i}\gamma_{t,j} = \binom{i + j + s - t}{j} c_t \gamma_{s,i+j} + p_{s,i,t,j}$$

The elements $c_i = \gamma_{i,0}$ generating $k[c_1, ..., c_n]$ are the elementary symmetric polynomials on the variables u_i:

$$c_i = \sigma_i(u_1, ..., u_n) = \sum_{1 \leq m_1 < \cdots < m_i \leq n} u_{m_1} \cdots u_{m_i}$$

Similarly, the elements $\gamma_{0,i}$ generating $R_0^{\Sigma_n}$ are the elementary symmetric polynomials on the variables α_i:

$$\gamma_{0,i} = \sigma_i(\alpha_1, ..., \alpha_n) = \sum_{1 \leq l_1 < \cdots < l_i \leq n} \alpha_{l_1} \cdots \alpha_{l_i}$$

Proposition A.2. *Any set of homogeneous algebra generators of R^{Σ_n} over $R_0^{\Sigma_n}$ has cardinality at least $n + \binom{n}{2}$, which is the cardinality of the generating set $\{c_s, \gamma_{s,j}\}_{s,j>0}$.*

A minimal generating set for the k-algebra $R_0^{\Sigma_n}$ depends on which primes are invertible in k. For example, if k is a $\mathbb{Q}$-algebra, which is our case of interest, we can generate all $\gamma_{0,i}$ from the single element

$$\alpha = \gamma_{0,1} = \alpha_1 + \cdots + \alpha_n$$

via the formula:

$$\gamma_{0,i} = \frac{\alpha(\alpha - 1) \cdots (\alpha - i + 1)}{i!}$$

The disadvantage of supplanting $\gamma_{0,i}$ with α is that the relations between the generators now require additional rational coefficients, as in Corollary 5.4. For the purposes of the algorithm implemented in our computer program, it is better (in terms of speed and numerical stability) to use all the $\gamma_{0,i}$ and $\mathbb{Z}$-coefficients.

The rest of this appendix is dedicated to proving Propositions A.1 and A.2.

Proof. (Of Proposition A.1) We will need that the dominant term of $\gamma_{s,i}$ is:

$$\mathrm{dom}(\gamma_{s,i}) = u_1 \cdots u_s \alpha_{s+1} \cdots \alpha_{s+i}$$

Let us note a subtlety about dominant terms and multiplication: $\mathrm{dom}(pq)$ is $\mathrm{dom}(p)\,\mathrm{dom}(q)$ if p or q are polynomials solely on the u_i, but if both p, q contain α_i's that may not be the case: for example $\gamma_{1,1}$ has dominant term $u_1 \alpha_2$ but $\gamma_{1,1}^2$ has dominant term $u_1^2 \alpha_2 \alpha_3$.

Every Σ_n orbit of a monic monomial in R has a greatest term M that can be written as either:

$$M = u_1^{a_1} \cdots u_s^{a_s} \alpha_1^{\epsilon_1} \cdots \alpha_s^{\epsilon_s} \alpha_{s+1} \cdots \alpha_{s+i}$$

with $a_1 \geq \cdots \geq a_s > 0$ and $\epsilon_i = 0, 1$, or as:

$$M = \alpha_1 \cdots \alpha_k$$

It suffices to prove that any such M is the dominant term of a polynomial on $\gamma_{s,j}$. Note that $M = \alpha_1 \cdots \alpha_k$ is the dominant term of $\gamma_{0,k}$, so we may restrict our attention exclusively to M's of the first form. It should also be noted that we can't assume that the ϵ_i are in decreasing order, since applying a permutation to fix such an order would affect the decreasing order on the a_i.

So let

$$M = u_1^{a_1} \cdots u_s^{a_s} \alpha_1^{\epsilon_1} \cdots \alpha_s^{\epsilon_s} \alpha_{s+1} \cdots \alpha_{s+i}$$

be greatest in its Σ_n orbit, where $a_1 \geq \cdots \geq a_s > 0$ and $\epsilon_i = 0, 1$. We shall prove that $M = \mathrm{dom}(P)$ where P is a product of $\gamma_{s,j}$. To ensure that P is unique per M, we insist that if $\gamma_{t,j}, \gamma_{t',j'}$ are factors of P with $0 < j, j'$ and $t \leq t'$ then $t + j < t'$. With this extra requirement, no two distinct products P can have the same dominant term.

Further simplifying matters, note that it suffices to write

$$M = u_1^{k_1} \cdots u_s^{k_s} \mathrm{dom}(P')$$

where $k_1 \geq \cdots \geq k_s \geq 0$ and P' is a product of $\gamma_{s,i}$ with $i > 0$ satisfying the condition above. This is because $u_1^{k_1} \cdots u_s^{k_s}$ is the dominant term of a product P_c of c_i by the fundamental result on symmetric polynomials. Then we can take $P = P_c P'$ and $M = \mathrm{dom}(P)$.

We distinguish cases on the number of ϵ_i's in M that are nonzero, i.e. the number of α_i's in M with $i \leq s$. If there are none then

$$M = u_1^{a_1} \cdots u_s^{a_s} \alpha_{s+1} \cdots \alpha_{s+i}$$

can be written as

$$(u_1^{a_1-1} \cdots u_s^{a_s-1})(u_1 \cdots u_s \alpha_{s+1} \cdots \alpha_{s+i})$$

so we use $P' = \gamma_{s,i}$.

Now assume there's only one α_j with $j \leq s$:

$$M = u_1^{a_1} \cdots u_s^{a_s} \alpha_j \alpha_{s+1} \cdots \alpha_{s+i}$$

If $j > 1$ notice that $a_{j-1} > a_j$ for otherwise we can exchange $j-1, j$ and get a greater term in our order, contradicting that M is greatest in its Σ_n orbit. If

further $j < s$, we can write M as

$$(u_1^{a_1-2} \cdots u_{j-1}^{a_{j-1}-2} u_j^{a_j-1} \cdots u_s^{a_s-1})(u_1 \cdots u_{j-1}\alpha_j)(u_1 \cdots u_s\alpha_{s+1} \cdots \alpha_{s+i})$$

and use $P_{\alpha,\gamma} = \gamma_{j-1,1}\gamma_{s,i}$. If $j = s$ we instead write M as

$$(u_1^{a_1-1} \cdots u_{s-1}^{a_{s-1}-1} u_s^{a_s})(u_1 \cdots u_{s-1}\alpha_s\alpha_{s+1} \cdots \alpha_{s+i})$$

and use $P' = \gamma_{s-1,i+1}$. If $j = 1$ then

$$M = u_1^{a_1} \cdots u_s^{a_s}\alpha_1\alpha_{s+1} \cdots \alpha_{s+i}$$

is

$$(u_1^{a_1-1} \cdots u_s^{a_s-1})\alpha_1(u_1 \cdots u_s\alpha_{s+1} \cdots \alpha_{s+i})$$

and we use $P' = \gamma_{0,1}\gamma_{s+1,i}$.

Now assume there are two α_j's with $j \leq s$ in M, say α_j, α_k with $j < k \leq s$:

$$M = u_1^{a_1} \cdots u_s^{a_s}\alpha_j\alpha_k\alpha_{s+1} \cdots \alpha_{s+i}$$

If $j > 1$ then as before we must have $a_{j-1} > a_j$ and $a_{k-1} > a_k$. If furthermore $j < k-1$ and $k < s$ we can write M as

$$\left(\prod_{l=1}^{j-1} u_l^{a_l-3} \prod_{l=j}^{k-1} u_l^{a_l-2} \prod_{l=k}^{s} u_l^{a_l-1}\right)(u_1 \cdots u_{j-1}\alpha_j)(u_1 \cdots u_{k-1}\alpha_k)(u_1 \cdots u_s\alpha_{s+1} \cdots \alpha_{s+i})$$

and use $P' = \gamma_{j-1,1}\gamma_{k-1,1}\gamma_{s,i}$. If $j = k-1$ and $k < s$ then write M as

$$(u_1^{a_1-2} \cdots u_{j-1}^{a_{j-1}-2} u_j^{a_j-1} \cdots u_s^{a_s-1})(u_1 \cdots u_{j-1}\alpha_j\alpha_{j+1})(u_1 \cdots u_s\alpha_{s+1} \cdots \alpha_{s+i})$$

and use $P' = \gamma_{j-1,2}\gamma_{s,i}$. The other cases are all handled similarly. Note that for $j = 1, k = 2$ we get

$$(u_1^{a_1-1} \cdots u_s^{a_s-1})(\alpha_1\alpha_2)(u_1 \cdots u_s\alpha_{s+1} \cdots \alpha_{s+i})$$

and use $P' = \gamma_{0,2}\gamma_{s,i}$.

We proceed in this fashion to treat the case where r many α_j's with $j \leq s$ appear in M, for any $r = 0, ..., s$. $\qquad\square$

We now prove Proposition A.2 regarding the minimality of generators.

Proof. (Of Proposition A.2) For every s with $1 \leq s \leq n$, consider the equivalence relation on $R_s^{\Sigma_n}$ whereby two elements are equivalent if they have equal dominant terms. The orbit set of the equivalence relation, when ordered from least to greatest, starts with $\gamma_{s,0}, \gamma_{s,1}, ..., \gamma_{s,n-s}$ and continues with products of elements in $R_t^{\Sigma_n}$, $t < s$.

Let X be a homogeneous generating set of R^{Σ_n} over $R_0^{\Sigma_n}$ of minimum cardinality, ordered first by degree and then by dominant term. The smallest element $x \in X$ must then satisfy $\mathrm{dom}(x) = \mathrm{dom}(\gamma_{1,0})$ hence $x - r\gamma_{1,0} \in R_0^{\Sigma_n}$ for some $r \in k$; this means that we can replace x by $\gamma_{1,0}$ resulting in a new generating set X with minimum cardinality. Applying the same argument repeatedly shows that all elements of X can be replaced by $\gamma_{1,0}, \gamma_{1,1}, ..., \gamma_{n,0}$ while preserving cardinality and polynomial span. $\qquad\square$

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