Kervaire invariant and Whitehead square

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1 Introduction

In this note, we try to shed some light on the relationships between the Hopf invariant, Kervaire invariant, and Whitehead square. More specifically, we prove two classical results (theorems 2.1 and 3.1) in a way that seems more transparent (at least to the author) than what is found in the literature. No new results are claimed, only a different exposition. In addition to the references cited, most of the ideas in section 3 come from discussions with Michael Hopkins, for which the author is grateful.

Notation. *We will use the standard notation* $\iota_n \in \pi_n(S^n)$ *for the class of the identity map, and by Whitehead square we mean the Whitehead product* $[\iota_n, \iota_n] \in \pi_{2n-1}(S^n)$.

2 Hopf invariant and Whitehead square

In this section we tackle the Hopf invariant problem as a warm-up to the Kervaire invariant problem.

Theorem 2.1. *For* $n \geq 2$*, there is an element of Hopf invariant one in* $\pi_{2n-1}(S^n)$ *iff the Whitehead square* $[\iota_{n-1}, \iota_{n-1}]$ *is zero.*

Here is the more streamlined proof which uses facts about the EHP sequence ([7], section XII.2). It is essentially Whitehead's proof in [6] (claim 3.49, proved at the end of section 7). First recall what the EHP long exact sequence looks like:

$$\pi_{3m-2}(S^m) \to \cdots$$
$$\cdots \to \pi_q(S^m) \xrightarrow{E} \pi_{q+1}(S^{m+1}) \xrightarrow{H} \pi_{q+1}(S^{2m+1}) \xrightarrow{P} \pi_{q-1}(S^m) \to \cdots$$

Proof. Assume $n \geq 3$. The relevant part of the EHP sequence is

$$\cdots \to \pi_{2n-2}(S^{n-1}) \xrightarrow{E} \pi_{2n-1}(S^n) \xrightarrow{H} \pi_{2n-1}(S^{2n-1}) \xrightarrow{P} \pi_{2n-3}(S^{n-1}) \xrightarrow{E} \pi_{2n-2}(S^n) \to \cdots$$

From this we get the following chain of equivalent statements.

There is an element of Hopf invariant one in $\pi_{2n-1}(S^n)$
iff $H : \pi_{2n-1}(S^n) \to \pi_{2n-1}(S^{2n-1})$ is surjective
iff $P : \pi_{2n-1}(S^{2n-1}) \cong \mathbb{Z} \to \pi_{2n-3}(S^{n-1})$ is zero
iff $P(\iota_{2n-1}) = [\iota_{n-1}, \iota_{n-1}]$ is zero.

Alternate to this last line: iff $\ker\big(E : \pi_{2n-3}(S^{n-1}) \to \pi_{2n-2}(S^n)\big)$ is zero. But this kernel is generated by $[\iota_{n-1}, \iota_{n-1}]$. □

Remark 2.2. *Perhaps With enough care, we can extract more information from the EHP sequence to cover the case $n = 2$. Alternately, one easily checks that $[\iota_1, \iota_1]$ is zero and that the Hopf map $\eta \in \pi_3(S^2)$ has Hopf invariant 1.*

Now we give a slightly different proof, reformulating the problem in terms of cohomological properties. The purpose is to use ideas that can be generalized or adapted to the Kervaire invariant problem. First, we present the theorem in a different way.

Theorem 2.3. *For $n \geq 2$, the following are equivalent.*

1. *There is an element of Hopf invariant one in $\pi_{2n-1}(S^n)$.*
2. *There exists a two-cell complex K with an m-cell, an $(m+n)$-cell, and*

$$\mathrm{Sq}^n : \mathrm{H}^m(K; \mathbb{Z}/2) \to \mathrm{H}^{m+n}(K; \mathbb{Z}/2)$$

   is non-zero, i.e. is an iso $\mathbb{Z}/2 \xrightarrow{\cong} \mathbb{Z}/2$.

3. *The Whitehead square $[\iota_{n-1}, \iota_{n-1}]$ is zero.*

For a thorough overview of the Hopf invariant problem, see section 1.1 of Adams's paper [1].

Remark 2.4. *For $n = 1$, part (3) doesn't make sense, but the statement still holds in that (1) and (2) are both true: $2\iota \in \pi_1(S^1)$ has Hopf invariant 1 and its mapping cone $\mathbb{R}\mathrm{P}^2$ satisfies (2). We exclude the case $n = 1$ from now on, since the arguments don't work for it (and we've just settled it!).*

Remark 2.5. *Part (1) is equivalent to there being an element of odd Hopf invariant. If n is even, then all elements of* $\pi_{2n-1}(S^n)$ *have Hopf invariant zero, and if n is odd, there are elements of any even Hopf invariant. In fact,* $[\iota_n, \iota_n]$ *has Hopf invariant* ± 2.

Proof. ($1 \Rightarrow 2$) If $f \in \pi_{2n-1}(S^n)$ has Hopf invariant 1, then its mapping cone C_f satisfies (2). Indeed, for some generators $u \in \mathrm{H}^n(C_f; \mathbb{Z})$ and $v \in \mathrm{H}^{2n}(C_f; \mathbb{Z})$, we have $u^2 = v$. Reducing mod 2, we obtain:

$$\mathrm{Sq}^n(\overline{u}) = \overline{u}^2 = \overline{v} \neq 0.$$

($2 \Rightarrow 1$) Let $f : S^{m+n-1} \to S^m$ be the attaching map of the $(m+n)$-cell of K. Let's desuspend K (or equivalently f) to produce the map we're looking for. Going to the bottom of the stable range, we can uniquely desuspend f to $\underline{f} : S^{2n} \to S^{n+1}$. By Freudenthal, we can desuspend once more (non-uniquely) to $\widetilde{f} : S^{2n-1} \to S^n$, since that's the critical dimension where suspension is surjective but in general not an iso. Since Steenrod squares are stable operations, we have an iso

$$\mathrm{Sq}^n : \mathrm{H}^n(C_{\widetilde{f}}; \mathbb{Z}/2) \xrightarrow{\cong} \mathrm{H}^{2n}(C_{\widetilde{f}}; \mathbb{Z}/2)$$

Hence $\widetilde{f}$ has odd Hopf invariant.

($2 \Rightarrow 3$) Let K be our complex and f the attaching map as above. By desuspending, we can assume $m = n$ and $f : S^{2n-1} \to S^n$ has Hopf invariant 1 (so really, we're showing $1 \Rightarrow 3$). We know f does not desuspend further, since its mapping cone supports a Sq^n, but more is true. Let's see why it doesn't desuspend, i.e. look at the adjoint map f':

$$\begin{array}{ccccc} S^{2n-2} & \xrightarrow{f'} & \Omega S^n & \xrightarrow{H} & \Omega S^{2n-1} \\ & \searrow_{\nexists} & \nearrow & & \\ & & S^{n-1} & & \end{array}$$

Saying that f has Hopf invariant 1 is saying that the top composite is adjoint to the identity of S^{2n-1}. Here, the Hopf invariant is equal to the "degree" of that composite, i.e. its effect on $\mathrm{H}^{2n-2}(-; \mathbb{Z})$ (or $\mathrm{H}_{2n-2}(-; \mathbb{Z})$). Up to homotopy, f' factors through the $(2n-2)$-skeleton of ΩS^n, which is $S^{n-1} \cup_{[\iota,\iota]} e^{2n-2}$. Hence we get:

$$\begin{array}{ccccc} S^{2n-2} & \xrightarrow{f'} & S^{n-1} \cup_{[\iota,\iota]} e^{2n-2} & \xrightarrow{\text{collapse}} & S^{2n-2} \\ & & \uparrow \text{incl} & & \\ & & S^{n-1} & & \end{array}$$

and the top composite has degree 1 i.e. is an equivalence. Hence the mapping cone $S^{n-1} \cup_{[\iota,\iota]} e^{2n-2}$ is actually a wedge [1], so the attaching map $[\iota_{n-1}, \iota_{n-1}]$ is null.

(3 $\Rightarrow$ 1) Do the steps in reverse. If $[\iota_{n-1}, \iota_{n-1}]$ is zero, use the summand inclusion

$$S^{2n-2} \hookrightarrow S^{n-1} \cup_{[\iota,\iota]} e^{2n-2} \to \Omega S^n$$

and take the adjoint map:

$$S^{2n-1} \to S^n.$$

As remarked above, this map has Hopf invariant 1. □

In this proof, we haven't used the reformulation in terms of cohomological properties (part 2) in an essential way. However, the analogous strategy for the Kervaire invariant problem will be more useful.

3 Kervaire invariant and Whitehead square

Recall the **strong** form of the Kervaire invariant problem [4]: Is there an element $\theta_j \in \pi^S_{2^{j+1}-2}$ of Kervaire invariant 1 and **of order 2**, i.e. such that $2\theta_j = 0$? This problem is related to the Whitehead square.

Notation. *From now on, write N for* $2^{j+1} - 2$, *and* $j \geq 1$ *throughout.*

Theorem 3.1. *There is a Kervaire class* θ_j *of order 2 iff* $[\iota_{N+1}, \iota_{N+1}]$ *is divisible by 2, i.e. is equal to* 2α *for some element* $\alpha \in \pi_{2N+1}(S^{N+1})$.

More precisely:

- ($\Rightarrow$) desuspending a Kervaire class θ_j of order 2 yields an α which is half the Whitehead square;
- ($\Leftarrow$) If α is non-zero, then stably it is a Kervaire class θ_j of order 2. (We exclude the cases $\alpha = 0$ by hand.)

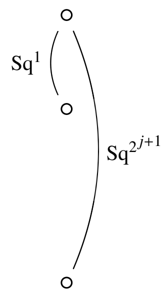
The theorem is proved in [4], corollary 3.2.

[1]Indeed, assuming $n \geq 3$, the induced map $S^{n-1} \vee S^{2n-2} \to S^{n-1} \cup_{[\iota,\iota]} e^{2n-2}$ is a homology iso between simply-connected CW-complexes, hence a homotopy equivalence. A more subtle argument could cover the case $n = 2$, but as mentioned in (2.2), we can treat it separately and assume $n \geq 3$.

Remark 3.2. *Though it's not clear in the statement, theorem 3.1 holds in other dimensions as well, in the sense that the Whitehead square* $[\iota_{2m+1}, \iota_{2m+1}]$ *cannot be halved unless* $2m + 2$ *is a power of* 2*. We already know it is zero iff* $2m + 2$ *is* 2,4, *or* 8.

To prove theorem 3.1, we reformulate the strong Kervaire invariant problem in terms of cohomological properties, like we did for the Hopf invariant problem in section 2.

Theorem 3.3. *There is a Kervaire class* θ_j *of order 2 iff there exists a 3-cell complex (spectrum) with the following mod* 2 *cohomology:*



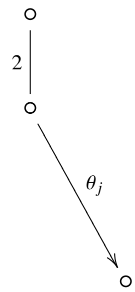
i.e. the three non-zero cohomology groups are $\mathbb{Z}/2$ *and the indicated Steenrod squares are isomorphisms.*

More precisely, the attaching map from the middle cell to the bottom cell is the Kervaire class. For a detailed exposition of 3.3, see notes by Haynes Miller [5]. Let us only mention the key ingredient: in the 2-local Adams spectral sequence for the sphere, there is a differential

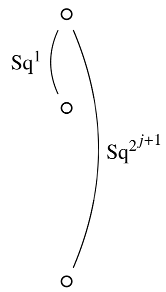
$$d_2(h_{j+1}) = h_0 h_j^2$$

when $j \geq 3$. Now we prove 3.1.

Proof. ($\Rightarrow$) Assume there is a Kervaire class $\theta_j \in \pi^S_{2^{j+1}-2}$ of order 2. This means the map θ_j extends to a Moore spectrum and we get a map between complexes:



Taking its mapping cone, the resulting 3-cell complex has the following cohomology:

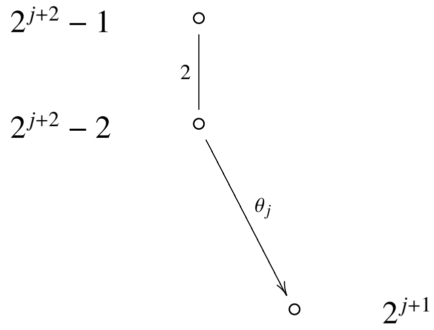


(1)

by the more precise formulation of theorem 3.3. Now let's try to realize this complex as a space, which is where the Whitehead square will appear. Going to the bottom of the stable range, we have:

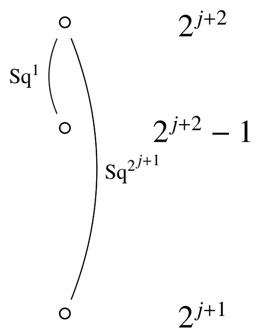
$$\theta_j \in \pi^S_{2^{j+1}-2} = \pi_{2^{j+2}-2}(S^{2^{j+1}})$$

so we get a map between spaces:



(2)

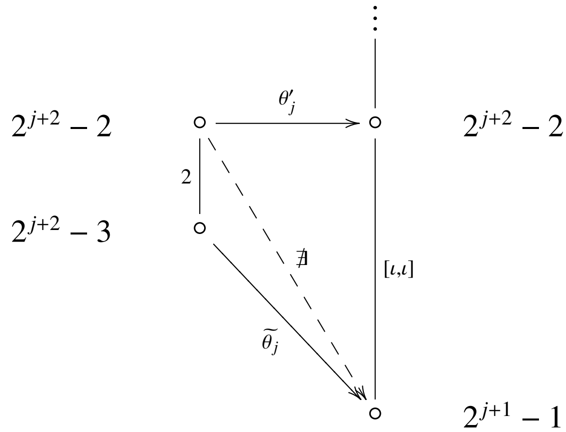
and its mapping cone is a space whose stabilization is the 3-cell complex above, with similar cohomology:



Observe that we can't desuspend further since the bottom class in dimension 2^{j+1} supports a $\mathrm{Sq}^{2^{j+1}}$. Let's identify the obstruction to desuspending, by looking at the adjoint map

$$\theta'_j : M \to \Omega S^{2^{j+1}} = \Omega\Sigma S^{2^{j+1}-1}$$

where the left-hand side is still a Moore space but shifted down one dimension. Remembering the cell structure of the right-hand side, we can write this schematically as



The restriction of θ'_j to the bottom cell on the LHS lands into the bottom cell in the RHS, and that factorization is a desuspension $\widetilde{\theta_j}$ of θ_j (which we knew existed, by Freudenthal). The fact that $\widetilde{\theta_j}$ does not extend to the Moore space means $2\widetilde{\theta_j}$ is non-zero. However, after including the bottom cell $S^{2^{j+1}-1}$ into $\Omega\Sigma S^{2^{j+1}-1}$, the map

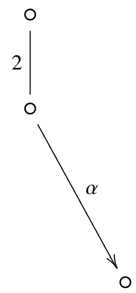
$$S^{2^{j+2}-3} \xrightarrow{\widetilde{\theta_j}} S^{2^{j+1}-1} \hookrightarrow \Omega\Sigma S^{2^{j+1}-1}$$

does extend to the Moore space, hence twice it is zero. But this composite is precisely $E(\widetilde{\theta_j})$, where

$$E : \pi_{2^{j+2}-3}(S^{2^{j+1}-1}) \to \pi_{2^{j+2}-2}(S^{2^{j+1}})$$

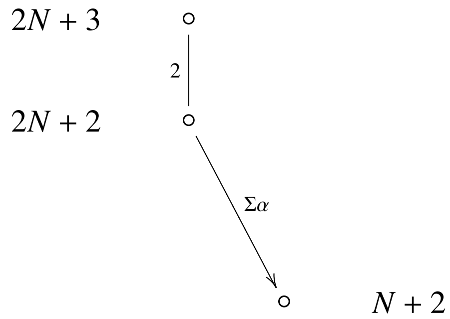
is the suspension map. Thus we have $2E(\widetilde{\theta_j}) = E(2\widetilde{\theta_j}) = 0$, i.e. $2\widetilde{\theta_j} \in \ker E \cong \mathbb{Z}/2$, generated by $[\iota,\iota]$ (see [6], claim 3.49). We conclude $[\iota,\iota] = 2\widetilde{\theta_j}$.

($\Leftarrow$) Recall that $[\iota_{N+1},\iota_{N+1}]$ is zero iff j is 0, 1, or 2. If j is 1 or 2, then there are Kervaire classes $\theta_1 = \eta^2$ and $\theta_2 = \nu^2$ of order 2, where $\eta \in \pi_1^S$ and $\nu \in \pi_3^S$ are the Hopf maps. Now assume $j \geq 3$ and $[\iota_{N+1},\iota_{N+1}] = 2\alpha$ for some α (and 2α is not zero). Stably, we have $2\alpha = 0$, so there is a map of spectra:



We want to show that its mapping cone has the desired cohomology (1). Then theorem 3.3 would imply that α is a Kervaire class (of order 2).

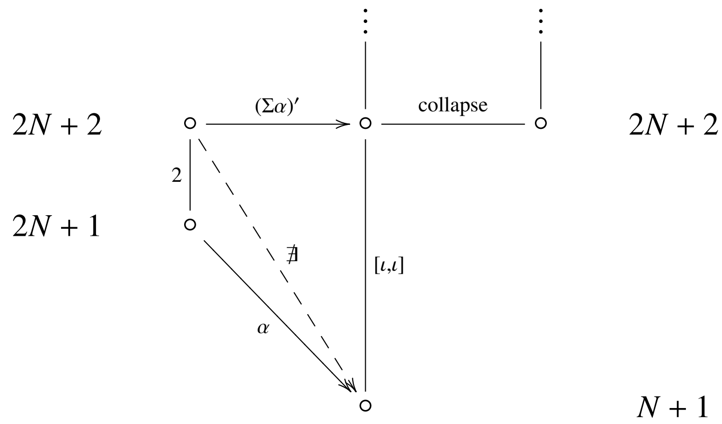
Going to the bottom of the stable range, we realize this diagram as a map of spaces:



Following an argument of Cohen ([2], proposition 11.4, (3) is equivalent to (4)), it suffices to check that the adjoint map

$$(\Sigma\alpha)' : M(2N + 1) \to \Omega S^{N+2}$$

is non-zero in mod 2 homology (or cohomology). As above, write the map schematically as



The non-existent arrow comes from the fact that 2α is non-zero. In this range, the cofiber sequence given by collapsing the bottom cell of ΩS^{N+2} acts as a fiber sequence, so the non-existent arrow tells us the composite is non-null. By the Hopf-Whitney theorem, (pointed) homotopy classes of maps from $M(2N+1)$ to $\Omega S^{N+2}/S^{N+1}$ are parametrized by

$$\begin{aligned} &\mathrm{H}^{2N+2}\left(M(2N+1);\pi_{2N+2}(\Omega S^{N+2}/S^{N+1})\right)\\ =&\,\mathrm{H}^{2N+2}\left(M(2N+1);\mathbb{Z}\right)\\ =&\,\mathbb{Z}/2 \end{aligned}$$

and hence the map $(\Sigma\alpha)'$ is non-zero on $\mathrm{H}^{2N+2}(-;\mathbb{Z}/2)$. □

4 Conclusion

We presented the fact that the strong Kervaire invariant problem is equivalent to the divisibility of the Whitehead square by 2. In fact, the Whitehead square is a classic object of study in homotopy theory and is related to various other problems. For more information, see for example [2], sections 11 and 12, or [3], section 9.

References

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