Pergamon

Topology Vol. 37, No. 1, pp. 109–114, 1998

0040-9383/97 $19.00 + 0.00

PII: S0040-9383(97)00005-0

ON THE MACLANE COHOMOLOGY FOR THE RING OF INTEGERS†

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(*Received* 12 *December* 1995)

1. INTRODUCTION

Eilenberg and MacLane introduce, for each abelian group A, a chain complex $Q_*(A)$ which has the homology of an Eilenberg–MacLane spectrum $H(A)$. Moreover, for a given ring R, the complex $Q_*(R)$ carries a differential graded ring structure. MacLane [10] then defines the (co)homology of a ring R, with coefficient in an R-R-bimodule M, to be the Hochschild (co)homology of $Q_*(R)$ with coefficient in M:

$$\mathrm{HML}(R, M) := \mathrm{HH}(Q_*(R), M).$$

The MacLane cohomology has the following nice description in terms of extensions groups. For a ring R, let $\mathscr{F}(R)$ be the category whose objects are the functors from the category of finitely generated free left R-modules to the category of left R-modules, and whose maps are the natural transformations; one such functor is the inclusion functor I. The MacLane cohomology of the ring R with coefficient in a functor T of $\mathscr{F}(R)$ is defined [8] by

$$\mathrm{HML}^*(R, T) := \mathrm{Ext}^*_{\mathscr{F}(R)}(I, T).$$

When T is just tensoring with a R-R-bimodule M, that is $T = -\otimes_R M$, we write $\mathrm{HML}^*(R, M)$ for $\mathrm{HML}^*(R, -\otimes_R M)$. By [8], this is compatible with MacLane's original definition via the Q construction.

It is known [6, 12] that the MacLane homology is isomorphic to the topological Hochschild homology in the sense of Bökstedt [2] and to stable K-theory in the sense of Waldhausen [9]. Using sophisticated topological methods, Bökstedt has calculated $\mathrm{THH}(\mathbb{Z})$ and $\mathrm{THH}(\mathbb{Z}/p\mathbb{Z})$, for all primes p [3]. Bökstedt's result for $\mathbb{Z}/p\mathbb{Z}$ is closely related to Breen's [4] calculations. Later, Franjou *et al.* [7] have given algebraic computations of $\mathrm{HML}^*(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ and $\mathrm{HML}^*(\mathbb{Z}/p\mathbb{Z}, \mathrm{Sym}^n)$, where Sym^n denotes the nth symmetric power; their results include the module structure over $\mathrm{HML}^*(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ defined by the Yoneda product.

The purpose of this paper is to give a short and elementary algebraic proof of Bökstedt's result on $\mathrm{HML}^*(\mathbb{Z}, \mathbb{Z})$ which includes the Yoneda ring structure. We prove:

† Research supported by URA 1169 du CNRS and Volkswagen-Stiftung (RiP-program at MFO). The second author is partially supported by an International Science Foundation Research Grant, # MXH200 and by Grant INTAS-93-2618.
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THEOREM. *Let* $\Gamma(x)$ *denote the free divided-powers algebra on a generator* x *of degree* 2, *that is the subring of* $\mathbb{Q}[x]$ *generated by the classes* $x^{[i]} = x^i/i!$, $i \geqslant 1$.

The algebra $\mathrm{HML}^*(\mathbb{Z}, \mathbb{Z})$ *is isomorphic to the quotient of* $\Gamma(x)$ *by its ideal generated by the class* x.

This theorem is proved in Section 4, using a spectral sequence whose first line is the desired $\mathrm{HML}^*(\mathbb{Z}, \mathbb{Z})$, and which converges to zero. This spectral sequence is constructed in Section 3. Its E_2-term (but the first line) is computed by Corollary 2.3. The module structure on the spectral sequence then gives the multiplicative structure.

2. MACLANE COHOMOLOGY WITH COEFFICIENT IN SYMMETRIC OR EXTERIOR POWERS

For an abelian group A, we denote by $\mathrm{Sym}^* A$ the symmetric algebra generated by A. Our first task is the computation of $\mathrm{HML}^*(\mathbb{Z}, \mathrm{Sym}^n)$ as a right module over $\mathrm{HML}^*(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ for each $n > 1$. This relies on a change of ring type of argument.

For a given prime p, and for a functor T in $\mathscr{F}(\mathbb{Z}/p\mathbb{Z})$, we let $\hat{T}$ denote the functor in $\mathscr{F}(\mathbb{Z})$ given by $\hat{T}(X) := T(X/pX)$. Let $\mathscr{C}$ be the category of finitely generated vector spaces over $\mathbb{Z}/p\mathbb{Z}$ and $\mathscr{C} - \mathbb{Z}$ be the category of functors from $\mathscr{C}$ to the category of abelian groups. We let iT denote the same functor T, when considered as an object of $\mathscr{C} - \mathbb{Z}$. There is an isomorphism [7, Paragraph 9.2]:

$$\mathrm{HML}^*(\mathbb{Z}, \hat{T}) = \mathrm{Ext}^*_{\mathscr{C}-\mathbb{Z}}(iI, iT).$$

Taking $T = I$, it shows that the Yoneda product gives rise to a ring structure on $\mathrm{HML}^*(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$, and to a right $\mathrm{HML}^*(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$-module structure on $\mathrm{HML}^*(\mathbb{Z}, \hat{T})$. The rings maps

$$\mathrm{HML}^*(\mathbb{Z}/\mathbb{Z}) \to \mathrm{HML}^*(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \leftarrow \mathrm{HML}^*(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$$

will be used to relate the module structures over these different rings. Let us recall from [7, Paragraph 9] the computation of the right-hand side map.

We let Λ denote the following graded algebra:

$$\Lambda := \mathbb{Z}/p\mathbb{Z}[e_0, \ldots, e_h, \ldots]/(e_h^p;\, h \geqslant 0),$$

where e_h is a class of degree $2p^h$. Let Λ_k be the quotient of Λ by the ideal generated by $e_0, \ldots, e_{k-1}$. There exist isomorphisms of graded algebras:

$$\mathrm{HML}^*(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \cong \Lambda \quad [7, \text{Théorème } 7.3]$$

$$\mathrm{HML}^*(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \cong \Lambda_1 \otimes \Lambda(\xi_1) \quad [7, \text{Paragraph } 9.2]$$

where ξ_1 is a class of degree $2p - 1$ and $\Lambda(\xi)$ denotes the exterior algebra on a generator ξ. Under these isomorphisms, the map

$$\mathrm{HML}^*(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{HML}^*(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$$

is the composite: $\Lambda \twoheadrightarrow \Lambda_1 \hookrightarrow \Lambda_1 \otimes \Lambda(\xi_1)$.

PROPOSITION 2.1. *Let* n *be an integer,* $n \geqslant 2$. *If* n *is not a prime power, then* $\mathrm{HML}^*(\mathbb{Z}, \mathrm{Sym}^n) = 0$. *If* $n = p^h$ *is a power of a prime* p, *the right module structure over* $\mathrm{HML}^*(\mathbb{Z}, \mathbb{Z})$ *factors through* $\mathrm{HML}^*(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$, *making* $\mathrm{HML}^*(\mathbb{Z}, \mathrm{Sym}^n)$ *a free* Λ_h*-module on* *a class of degree* 1. *In particular,*

$$\mathrm{HML}^i(\mathbb{Z}, \mathrm{Sym}^n) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } i \equiv 1 \ (2n) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We use the following vanishing result [11, Proposition 2.15] (or [7, Lemme 0.4], see also the appendix of [1]). Say that a functor F from an additive category $\mathscr{A}$ to another additive category $\mathscr{B}$ is *diagonalizable* if it is the composite $F = T \circ \Delta$ of the diagonal $\Delta : \mathscr{A} \to \mathscr{A} \times \mathscr{A}$ and of a bifunctor $T : \mathscr{A} \times \mathscr{A} \to \mathscr{B}$ which satisfies $T(0, X) = 0 = T(X, 0)$ for every object X in $\mathscr{A}$.

LEMMA 2.2. *Let R be a ring and let F be a functor in $\mathscr{F}(R)$. If F is diagonalizable, then*

$$\mathrm{HML}^*(R, F) = 0.$$

By Lemma 2.2 we know that $\mathrm{HML}^*(\mathbb{Z}, \mathrm{Sym}^i \otimes \mathrm{Sym}^j)$ is 0, for $i, j \geqslant 1$.

Thus, by [5, 10.9], $\mathrm{HML}^*(\mathbb{Z}, \mathrm{Sym}^n)$ is annihilited by the binomial coefficient $\binom{n}{i}$, for $1 \leqslant i \leqslant n - 1$.

As in [5, 10.10], we first get $\mathrm{HML}^*(\mathbb{Z}, \mathrm{Sym}^n) = 0$ if $n \geqslant 2$ is not a prime power; now fixing a prime p and setting $n = p^h \geqslant 2$ we get as well that $\mathrm{HML}^*(\mathbb{Z}, \mathrm{Sym}^n)$ is annihilated by p.

On the other hand, for any T in $\mathscr{F}(\mathbb{Z}/p\mathbb{Z})$, one has an exact couple of right $\mathrm{HML}^*(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$-modules [7, Proposition 9.1]

$$\begin{array}{ccc} \mathrm{HML}^*(\mathbb{Z}/p\mathbb{Z}, T) & \xrightarrow{\quad i \quad} & \mathrm{HML}^*(\mathbb{Z}, \hat{T}) \\ & {}_{\pi}\nwarrow \qquad \swarrow_{\delta} & \\ & \mathrm{HML}^*(\mathbb{Z}/p\mathbb{Z}, T) & \end{array}$$

where the map δ is of degree -1, and $\mathrm{HML}^*(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ acts on $\mathrm{HML}^*(\mathbb{Z}, \hat{T})$ via the ring-map

$$\mathrm{HML}^*(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{HML}^*(\mathbb{Z}; \mathbb{Z}/p\mathbb{Z}).$$

The map π, of degree 2, is the product by the class e_0. When T is the symmetric power Sym^n, $n = p^h$, we know [7, Corollaire 7.4] that $\mathrm{HML}^i(\mathbb{Z}/p\mathbb{Z}, \mathrm{Sym}^n)$ is isomorphic to Λ_h, so that $\pi = 0$. The exact couple reduces to a short exact sequence

$$0 \to \Lambda_h \xrightarrow{i} \mathrm{HML}^*(\mathbb{Z}, \mathrm{Sym}^n \otimes \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\delta} \Lambda_h \to 0$$

which splits for degree reasons.

Finally, since the multiplication by p yields the zero map on $\mathrm{HML}^*(\mathbb{Z}, \mathrm{Sym}^n)$, the statement follows from the cohomology long exact sequence associated with the following short exact sequence of coefficients:

$$0 \to \mathrm{Sym}^n \xrightarrow{p} \mathrm{Sym}^n \to \mathrm{Sym}^n \otimes \mathbb{Z}/p\mathbb{Z} \to 0. \qquad \square$$

We deduce a similar result for the exterior power functors Λ^n.

COROLLARY 2.3. *Let n be an integer, $n \geqslant 2$. If n is not a prime power, then $\mathrm{HML}^*(\mathbb{Z}, \Lambda^n) = 0$. If $n = p^h$ is a power of a prime p, the right module structure over $\mathrm{HML}^*(\mathbb{Z}, \mathbb{Z})$ factors through $\mathrm{HML}^*(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$, making $\mathrm{HML}^*(\mathbb{Z}, \Lambda^n)$ a free Λ_h-module on* *a class of degree n. In particular,*

$$\mathrm{HML}^i(\mathbb{Z}, \Lambda^n) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } i = tn,\ t \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Apply Lemma 2.2 to the Koszul complex

$$0 \to \Lambda^n \to \cdots \Lambda^{n-i} \otimes \mathrm{Sym}^i \to \cdots \to \mathrm{Sym}^n \to 0. \qquad \square$$

3. A SPECTRAL SEQUENCE

In this section, we set up the spectral sequence that will enable us to compute $\mathrm{HML}^*(\mathbb{Z}, \mathbb{Z})$ as an algebra.

PROPOSITION 3.1. *There is a first quadrant spectral sequence of right* $\mathrm{HML}^*(\mathbb{Z}, \mathbb{Z})$. *modules with*

$$II_2^{s,t} = \mathrm{HML}^s(\mathbb{Z}, \Lambda^{t+1})$$

converging to $\mathbb{Z}$ *concentrated in degree* 0.

Proof. For a free abelian group A, let $\mathscr{I}(A)$ be the augmentation ideal of the group ring of A and let $B^*(A)$ be the bar-complex:

$$\cdots \to \mathscr{I}(A)^{\otimes i} \to \cdots \mathscr{I}(A)^{\otimes 2} \to \mathscr{I}(A).$$

For any integer k, there is a natural isomorphism from its homology $Tor_k^{\mathbb{Z}[A]}(\mathbb{Z}, \mathbb{Z})$ to $\Lambda^k(A)$.

Our spectral sequence arises from the "dual" of the bar-complex. For an abelian group A, define $DA := \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z})$. By applying the functor $\mathrm{Hom}_{\mathscr{F}(\mathbb{Z})}(I, -)$ on the complex DB^*D, one gets two hypercohomology spectral sequences with

$$I_1 = \mathrm{HML}^*(\mathbb{Z}, DB^*D), \qquad II_2^{s,t} = \mathrm{HML}^s(\mathbb{Z}/\Lambda^{t+1}).$$

The first one collapses at I_1 (though $\mathscr{I}^{\otimes i}$ is a projective in $\mathscr{F}(\mathbb{Z})$, this is not the case that $D\mathscr{I}^{\otimes i}D$ is an injective in $\mathscr{F}(\mathbb{Z})$; however, $D\mathscr{I}^{\otimes i}D$ is a direct factor of $\mathbb{Z}^{\mathrm{Hom}(-,\mathbb{Z}^i)}$ and $\mathrm{Ext}^k_{\mathscr{F}(\mathbb{Z})}(I, \mathbb{Z}^{\mathrm{Hom}(-,\mathbb{Z}^i)}) = \mathrm{Ext}^k_{\mathbb{Z}}(I(\mathbb{Z}^i), \mathbb{Z}) = 0$ for $k > 0$). The result follows. $\square$

COROLLARY 3.2 (Bökstedt [3]). *For each non-negative integer i, the group* $\mathrm{HML}^{2i}(\mathbb{Z}, \mathbb{Z})$ *is isomorphic to* $\mathbb{Z}/i\mathbb{Z}$ *and the group* $\mathrm{HML}^{2i+1}(\mathbb{Z}, \mathbb{Z})$ *is* 0.

Proof. Looking at the spectral sequence and Corollary 2.3, we first see that the group $\mathrm{HML}^i(\mathbb{Z}, \mathbb{Z})$ is finite for every positive integer i, and is zero for odd i.

Furthermore, for degree reasons, all the differentials land at the line $t = 0$. Hence, all of them are monomorphisms.

Let p be a prime, and let i be an even positive integer. Write $i = 2p^k t$ for an integer t prime to p. In total degree $i - 1$, only the k terms $\mathrm{HML}^{2p^k t - p^l}(\mathbb{Z}, \Lambda^{p^l})$, $1 \leqslant l \leqslant k$, have a non-trivial p-component, and each one has p elements. We conclude that the p-component of $\mathrm{HML}^i(\mathbb{Z}, \mathbb{Z})$ has p^k elements.

By the Bockstein exact sequence, the kernel of the multiplication by p

$$\mathrm{HML}^i(\mathbb{Z}, \mathbb{Z}) \to \mathrm{HML}^i(\mathbb{Z}, \mathbb{Z})$$

is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ or is 0; hence the p-component of $\mathrm{HML}^i(\mathbb{Z}, \mathbb{Z})$ is a cyclic group or is 0; it is therefore isomorphic to $\mathbb{Z}/p^k\mathbb{Z}$. □

4. PROOF OF THE THEOREM

Both algebras are connected and have finite underlying groups in every positive dimension. For any such algebra H, define, for short, its *p-component* to be the connected algebra which reduces in positive degrees to the p-component of H. The algebra H is clearly isomorphic to the tensor product, over all primes p, of its p-components. This makes it enough to prove the isomorphism on each p-component.

Let p be a prime. From Corollary 3.2 it follows that both algebras in question have isomorphic underlying abelian groups. In order to construct an isomorphism of rings for their p-components, we use the following.

LEMMA 4.1. *Let p be a prime. For each positive integer k, a generator a_k can be chosen in* $\mathrm{HML}^{2p^k}(\mathbb{Z}, \mathbb{Z})$ *to satisfy*

$$a_k^p = pa_{k+1}, \quad k \geqslant 1$$

To prove the lemma, we consider the spectral sequence of Proposition 3.1, and we use that its differentials are $\mathrm{HML}^*(\mathbb{Z}, \mathbb{Z})$-linear. As it was shown in the proof of Corollary 3.2, all the differentials are mono. Thanks to Corollary 2.3, for every odd t, the differential

$$d_p := II_2^{tp,p-1} \cong II_p^{tp,p-1} \to II_p^{(t+1)p,0} \cong II_2^{(t+1)p,0}$$

is an injection

$$\mathbb{Z}/p\mathbb{Z} \simeq \mathrm{HML}^{tp}(\mathbb{Z}, \Lambda^p) \to \mathrm{HML}^{(t+1)p}(\mathbb{Z}, \mathbb{Z}).$$

Starting with a generator x in $\mathrm{HML}^p(\mathbb{Z}, \Lambda^p)$, we choose $a_1 := d_p(x)$. Working by induction, suppose now that $a_1, \dots, a_k$ are chosen, $k \geqslant 1$. Then, in the group $\mathrm{HML}^{2p^{k+1}}(\mathbb{Z}, \mathbb{Z}) \simeq \mathbb{Z}/p^{k+1}\mathbb{Z}$, one has

$$p^{k-1}a_k^p = p^{k-2}a_{k-1}^p a_k^{p-1} = \cdots = a_1^p a_2^{p-1} \dots a_k^{p-1} = d_p(x)a_1^{p-1} \dots a_k^{p-1}.$$

It follows from the cohomology long exact sequence, that

$$\mathbb{Z}/p^k\mathbb{Z} \cong \mathrm{HML}^{2p^k}(\mathbb{Z}, \mathbb{Z}) \to \mathrm{HML}^{2p^k}(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$$

is an epimorphism, hence the image under this map of a generator a_h, $1 \leqslant h \leqslant k$, is a non-zero multiple of e_h. As d_p is a right $\mathrm{HML}^*(\mathbb{Z}, \mathbb{Z})$-module map, we find that $p^{k-1}a_k^p$ is, up to a unit, equal to $d_p(xe_1^{p-1} \dots e_k^{p-1})$, which is non-zero, because d_p is mono. However, $p^k a_k^p$ is zero, and there exists a generator a_{k+1} to complete the induction step.

One then can uniquely extend the correspondence $x^{[p^k]} \mapsto a_k$, $k \geqslant 1$, to a ring homomorphism between the p-components, which is an isomorphism. □

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